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Controllability and Observability Results of an Implicit Type Fractional Order Delay Dynamical System

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Abstract: Recently, several research articles have investigated the existence of solutions for dynamical systems with fractional order and their controllability. Nevertheless, very little attention has been given to the observability of such dynamical systems. In the present work, we explore the outcomes of controllability and observability regarding a differential system of fractional order with input delay. Laplace and inverse Laplace transforms, along with the Mittage–Leffler matrix function, are applied to the proposed dynamical system in Caputo's sense, and a general solution is obtained in the form of an integral equation. Then, we set out conditions for the controllability of the underlying model, regarding the linear case. We then expound controllability conditions for the nonlinear case by utilizing the fixed point result of Schaefer and the Arzola–Ascoli theorem. Using the fixed point concept, we prove the observability of the linear case using the observability Grammian matrix. The necessary and sufficient conditions for the nonlinear case are investigated with the help of the Banach contraction mapping theorem. Finally, we add some examples to elaborate on our work.

Keywords: controllability; observability; grammian matrix; fractional differential equations; fixed point theorem

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1. Introduction

In the recent past, fractional calculus (FC) has emerged as novel tools for modeling nonlinear phenomena occurring in different branches of science and engineering fields, such as viscoelasticity [1], electronic circuits [2], modified bituminous binders [3], epidemiology mechanism [4], and stochastic models of stock market swing [5]. Models described in this way are more passable and appropriate compared with integer-order models for the investigation of nonlinear phenomena. There are many applications of FC in applied sciences. For instance, FC has used to study hidden chaotic structures in a 4D dynamical system [6]. Oscillatory and chaotic dynamics of the HIV-1 model has been studied through FC in the literature [7]. FC calculus has been used in virology [8,9]. Zeb et al. used a piecewise fractional order model of COVID-19 [10]. Zeb et al. investigated two different vaccinated fractional order models of COVID-19 by using Caputo–Fabrizio and generalized Caputo operators [11]. FC also has applications in mathematical physics [12], bioengineering [13], and agriculture [14].

Among other qualitative behaviors of dynamical systems, both controllability and observability are the two key concepts that play a vital role in the analysis of control theory [15,16]. Controllability and observability represent two major concepts of modern control system theory. For a system to be controllable, we mean that the system state can be driven to any desirable state by applying an input control function within a determinate time duration. More specifically, a system is said to be controllable if there exists an

admissible control input signal $u(t)$ that steers the system state y_0 at $t = t_0$ to any advisable state y_f at time $t = t_f$. In contrast, the system is said to be observable if the system state $y(t)$ at any time t can be figured out from the system output $z(t)$. The controllability of linear finite dimensional systems and infinite dimensional systems has been discussed in [4,17]. The controllability of nonlinear systems with input delay has been reported in [1,2]. The authors in [5] reported on the controllability of a fractional order system in finite dimensional space. The time delay systems are the fundamental precipitating factors of the performance degradation and stability of fractional order systems [18,19]. It is therefore vital to investigate such effects on the dynamical behavior of the system. For a detailed study of such situations, see the work of Yan [20], Muthukumar and Rajivganthi [21] and Valliammal et al. [22]. To establish the connection between our proposed model and the existing literature regarding the controllability of systems describing some real-world phenomena, we give here a brief history of the recent work of some authors.

In [23], the authors have investigated the controllability as well as the observability of two-dimensional thermal flow in bulk storage facilities exploiting sensitivity fields. They have considered the convection diffusion reaction (CDR) equation, which describes the dynamics of energy and mass in physical systems such as flow systems, heat exchangers, bulk food storage systems, and almost all kinds of chemical reactions; see [24,25] for details. Physical phenomena, such as the transmission of momentum, energy, mass, etc., occur either inside the system or through its boundaries. The boundary-controlled CDR systems investigated by the authors in [23] are described by the PDE given by,

$$
\begin{aligned}
\left(\frac{\partial W}{\partial t} + \varpi . \nabla W \right) &= c \Delta W + r_W \in (0, t] \times \Omega_{d1}, \\
W &= u_{\text{Dirichlet}} \text{ on } (0, t] \times \partial \Omega_{d1}, \\
\frac{\partial W}{\partial n} &= u_{\text{Neumann}} \text{ on } (0, t] \times \partial \Omega_{d2}.
\end{aligned}
$$

The last twoequations represent the Dirichlet and Neumann boundary conditions at the boundaries $\partial \Omega_{d1}$ and $\partial \Omega_{d2}$, respectively. The symbol, $W \in \mathbb{R}^n$ shows the state vector; t is the time, ϖ represents the velocity vector, the diffusion coefficient is denoted by c; and the first-order reaction vector is symbolized by r_Y. Similarly, the symbols $u_{Dirichlet}$ and $u_{Neumann}$ represent the respective input u and flux through the boundaries $\partial \Omega_{d1}$ and $\partial \Omega_{d2}$. The authors here considered ν constant in their system.

In [26], Joseph and Murthy presented some novel results regarding the controllability of LDS subject to sparsity constraints on the input. They described that by unwinding the sparsity constraint, the classical results can be easily recovered for the unconstrained system. The discrete time LDS has been proposed, whose state $y_k \in R$ at any time k is given by

$$ y_k = D y_{k-1} + H h_k. $$

Here, $D \in \mathbb{R}^{n \times n}$ represents the transfer matrix, $H \in \mathbb{R}^{n \times n}$ is the input matrix, $h_k \in \mathbb{R}^L$ is the input vector being assumed to be sparse, i.e., $\|h_k\|_0 \leq z$, $\forall\, k$ and R_D, and R_D and R_H represent the respective ranks of D and H. Their definition of sparse controllability states that their underlying LDS is controllable if for any initial and final state x_0 and x_f, respectively, there exists an input $\|h_k\|_0 \leq s$, which steers the system from the initial state x_0 to any final state $x_f = x_K$ in a finite duration of time K''.

In [27], Nawaz et al. have recently formulated the controllability conditions of an NLFS having time-delay in the state function described by two parameters, delayed Mittage–Leffler matrix functions utilizing the fixed point concept of Schauder. Their proposed system is defined as

$$
\begin{cases}
{}^c D_{0+}^{\delta} W(t) = P W(t - \nu) + Q u(t), & t \in I = [0, b], \\
W(t) = \phi(t), & -\nu \leq t \leq 0.
\end{cases}
$$

The conforming nonlinear system is:

$$\begin{cases} {}^cD_{0^+}^\delta W(t) = PW(t-\nu) + Qu(t) + f(t, W(t-\nu), u(t)), & t \in I = [0, b], \quad b \geq 0, \\ W(t) = \phi(t), -\nu \leq t \leq 0, \end{cases}$$

where $W : [-\nu, b] \to \mathbb{R}^n$ is continuously differentiable in $[0, b]$ such that $b > (n-1)\nu, n \in N$, $0 < \delta \leq 1$. Matrices P and Q have, respectively, orders of $n \times n$ and $n \times m$, while $\nu > 0$ denotes the time-delay. The state vector is represented by the symbol $z(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ is the control function. Similarly, the initial state function is symbolized as $\phi(t)$ and $f : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and nonlinear.

In [28], the authors have investigated the controllability of nonlinear FO integro-differential systems with input delay, exploiting the so-called fixed point result of Schauder. Their proposed fractional order integro-differential inclusion is:

$$\begin{cases} {}^cD^\delta z(t) = Lz(t) + MF(t) + NF(t-\mu) + F(t, z(t)) \\ \quad + h(t, z(t), \int_0^t g(t, \mathsf{c}, z(\mathsf{c}))d\mathsf{c}), t \in I = [0, c], \\ z(0) = z_0, \quad F(t) = \phi(t), -\mu \leq t \leq 0. \end{cases}$$

In the above system $0 < \delta \leq 1, L \in \mathbb{R}^{n \times n}, M, N \in \mathbb{R}^{n \times m}, F : I \times R^n \to \mathbb{R}^n, h : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : I \times I \times \mathbb{R}^n \to \mathbb{R}^n$ are all continuous functions.

In [29], Balachandran et al. reported the observability of the fractional system which is:

$$^cD^\delta W(t) = MW(t) + g(t, W(t)), t \in J = [0, T],$$

where $M \in \mathbb{R}^{n \times n}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function with linear observation

$$z(t) = HW(t).$$

where H is an appropriate order matrix.

In [30], the authors have established some results for the controllability and observability of a linear system with non-integer distinct orders. Their proposed dynamical system is as follows:

$$\begin{pmatrix} {}^cD^\nu y_1(t) \\ {}^cD^\mu y_1(t) \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} F(t),$$

where ${}^cD^\nu y_1(t)$ and ${}^cD^\mu y_1(t)$ are the Caputo derivatives of orders $0 < \nu < 1$ and $0 < \mu < 1$, respectively. Here, $y_1 \in \mathbb{R}^{n_1}$ and $y_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$ are the state vectors, $M_{ij}, B_{ij}, i, j = 1, 2$ are constant matrices, and $F \in \mathbb{R}$ is the input vectors. For some recent results on controllability and observability, we also refer interested readers to see [31,32].

As we discussed above, controllability and observability are two important qualitative aspects of dynamical systems. To the best of our knowledge, controllability and observability results for fractional order systems input delay have not been studied in the literature. Inspired by the above work, in this paper, we investigate the controllability and observability of fractional order systems input delay by using concept of fixed point result of Schaefer, the Arzela–Ascoli theorem, and the Banach contracting principle. We add some examples to support our work at the end of the paper.

2. Preliminaries

In this portion of our manuscript, we include some important definitions, lemmas, notations, and preliminary facts regarding fractional operators. Let FD, ML and FI denote the fractional derivative, Mittag–Leffler and integral, respectively.

Definition 1 ([12]). *The Riemann–Liouville FD of $f(t)$ of order $\wp > 0$ with $j - 1 < \wp \le j, j \in \mathcal{N}$ is expressed by*

$$D^{\wp} f(t) = \frac{1}{\Gamma(j - \wp)} \frac{d^j}{dt^j} \int_0^t (t - c)^{j - \wp - 1} f(c)\, dc.$$

Here, $j = 1 + [\wp]$, $[\wp]$ is the integer part of $\wp$.

Definition 2 ([12]). *The Caputo FD of $f(t)$ is expressed as*

$$^c D^{\wp} f(t) = \frac{1}{\Gamma(j - \wp)} \int_0^t (t - c)^{j - \wp - 1} f^{(j)}(c)\, dc, j - 1 < \wp \le j.$$

Here, $j = 1 + [\wp]$, $[\wp]$ denotes integer part of $\wp$. For $0 < \wp \le 1$, we have

$$^c D^{\wp} f(t) = \frac{1}{\Gamma(1 - \wp)} \int_0^t (t - c)^{-\wp} f'(c)\, dc.$$

Definition 3 ([12]). *The Riemann–Liouville FI of a function $f(t)$ is expressed as*

$$I_0^{\wp} f(t) = \frac{1}{\Gamma(\wp)} \int_0^t (t - c)^{\wp - 1} f(c)\, dc.$$

Definition 4 ([12]). *The ML matrix function for two parameters is expressed as*

$$E_{\wp, \beta}(At^{\wp}) = \sum_{k=0}^{\infty} \frac{A^k t^{k\wp}}{\Gamma(k\wp + \beta)}, \quad \wp, \beta > 0,$$

Here, A is an arbitrary square matrix. The monoparameteric ML function can be achieved by putting $\beta = 1$ in the last equation, i.e.,

$$E_{\wp, 1}(At^{\wp}) = E_{\wp}(At^{\wp}) = \sum_{k=0}^{\infty} \frac{A^k t^{k\wp}}{\Gamma(k\wp + 1)}.$$

The ML function satisfies the property: $D_t^{\wp} E_{\wp}(At^{\wp}) = A E_{\wp}(At^{\wp})$.

Definition 5 ([12]). *The jth order derivative of the two parameters ML function is defined by the following expression*

$$\frac{d^j}{dt^j}(t^{\wp - 1} E_{\wp, \beta}(At^{\wp})) = t^{\wp - j - 1} E_{\wp, \beta - j}(At^{\wp}), j \in N.$$

Definition 6 ([33]). *A mapping $T : X \to Y$ from one Banach space to another is said to be continuous if for $\epsilon > 0$ and each $x \in X$, one can find a small positive δ in such a way that for each $y \in X$*

$$\|T(y) - T(x)\|_Y < \epsilon, \text{ whenever } \| - x + y\|_X < \delta \Rightarrow .$$

The mapping T is said to be uniformly continuous on $U \subset X$ provided for every $\epsilon > 0$, there corresponds a small positive δ such that for all $x, y \in A$

$$\|T(y) - T(x)\|_Y < \epsilon \text{ whenever } \|y - x\|_X < \delta.$$

Let us suppose that $T_{\lambda} : X \to Y, \lambda \in \Lambda$ is a (finite or infinite) class of mappings from one Banach space to another. These mappings are said to be equicontinuous on the set A, where A is a subset of X if for every $\lambda > 0$, one can associate a positive δ however small, such that for any $\lambda \in A$ and every two elements $x, y \in A$, the following holds

$$\|y - x\|_X < \delta \text{ implies } \|T(y) - T(x)\|_Y < \epsilon.$$

Definition 7 ([34]). *A system is deemed to be controllable on $[t_0, t_f]$ if for an initial state $u_0 \in \mathbb{R}^n$ at $t = t_0$, and final state $u_f \in \mathbb{R}^n$ at $t = t_f$, there corresponds an input control signal $v(t) : [0, T] \to \mathbb{R}^m$, such that the corresponding solution of the system satisfies $u(t_0) = u_0$ and $u(t_f) = u_f, t \in [t_0, t_f]$.*

3. Results of the Paper

Consider the fractional-order system given below on a bounded domain,

$$\begin{cases} {}^cD^{\nu}y(t) = Lu(t) + Ky(t) + Mu(t-q) + f(t, y(t), {}^cD^{\nu-1}y(t)), t \in I = [0, d], \\ y(0) = y_0, \ y'(0) = 0, \\ u(t) = \phi(t), \ -q \le t \le 0, \end{cases} \quad (1)$$

where $1 < \nu \le 2$; K is a $n \times n$ matrix; L and M are $n \times m$ matrices and f is a nonlinear continuous function. Utilizing the Laplace transform and its inverse along with the Mittag–Leffler function, the general solution of the fractional order model (1) is:

$$\begin{aligned} y(t) \ = \ & E_{\nu}(Kt^{\nu})y_0 + \int_0^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^{\nu-1})(Lu(t) + Mu(t-q) \\ & + f(t, y(t), {}^cD^{\nu-1}y(t)))dc. \end{aligned} \quad (2)$$

Lemma 1 ([35]). *For the case $0 \le t \le q$, the solution (2) can be expressed as*

$$\begin{aligned} y(t) \ = \ & E_{\nu}(Kt^{\nu})y_0 + \int_0^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^{\nu})Lu(c)dc \\ & + \int_{-q}^{t-q} (t-q-c)^{\nu-1}E_{\nu,\nu}(K(t-q-c)^{\nu})M\phi(c)dc \\ & + \int_0^t E_{\nu,\nu}(K(t-q-c)^{\nu}) \times f(c, y(c), {}^cD^{\nu-1}y(c))dc. \end{aligned} \quad (3)$$

While for the case $t > q$, this solution can be expressed as

$$\begin{aligned} y(t) \ = \ & E_{\nu}(Kt^{\nu})y_0 + \int_{-q}^0 (t-q-c)^{\nu-1}E_{\nu,\nu}(K(t-q-c)^{\nu})M\phi(c)dc \\ & + \int_0^{t-q} [(t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^{\nu})L + (t-q-c)^{\nu-1}E_{\nu,\nu}(K(t-q-c)^{\nu})M]u(c)dc \\ & + \int_{t-q}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^{\nu})Lu(c)dc \\ & + \int_0^t [(t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^{\nu}) \times f(c, y(c), {}^cD^{\nu-1}y(c))]dc. \end{aligned} \quad (4)$$

Lemma 2 ([35]). *The fractional linear system*

$$\begin{cases} {}^c_0D^{\nu}_t y(t) = Lu(t) + Ky(t) + Mu(t-q), \ t \in I = [0, d], \\ y(0) = y_0, \quad y'(0) = 0, \\ u(t) = \psi(t), \quad -q \le t \le 0, \end{cases} \quad (5)$$

is controllable on I if and only if the controllability Grammian matrices $W(t)$ in each of the following cases are invertible.

Case (1): *When $0 \le t \le q$*

$$\begin{aligned} W(t) = \ & \int_0^d (d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^{\nu})LL^T((d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^{\nu}))^T dc \\ & + \int_{-q}^{d-q} (d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^{\nu})MM^T \\ & \times ((d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^{\nu}))^T dc. \end{aligned} \quad (6)$$

Case (2): *When $t > q$*

$$
\begin{aligned}
W(t) = \int_0^{d-q} & [(d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)L + (d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M] \\
& \times [(d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)L + (d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M]^T dc \\
+ \int_{d-q}^d & (d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)LL^T((d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu))^T dc.
\end{aligned}
\tag{7}
$$

Proof. For the case $0 \le t \le q$, the ordinary solution of Equation (5) by the previous lemma becomes

$$
\begin{aligned}
y(t) = E_\nu(Kt^\nu)y_0 &+ \int_0^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu)Lu(c)dc \\
&+ \int_{-q}^{t-q} (t-q-c)^{\nu-1}E_{\nu,\nu}(K(t-q-c)^\nu)M\phi(c)dc.
\end{aligned}
\tag{8}
$$

Sufficiency: Let the Grammian's matrix W defined by Equation (6) be invertible on $[0, d]$; then, W^{-1} must exist. The input control function for Equation (5) is then given by

$$
u(t) = \begin{cases}
((t-q-c)^{\nu-1}E_{\nu,\nu}(K(t-q-c)^\nu)M)^T W^{-1}[y_1 - \psi_0(t)y_0], & -q \le t \le d-q; \\
0, & d-q \le t \le 0; \\
((t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu)L)^T W^{-1}[y_1 - \psi_0(t)y_0], & 0 < t \le d;
\end{cases}
\tag{9}
$$

substituting Equation (9) in Equation (8), after a bit of simplification, one can easily see that $y(0) = y_0$ and $y(d) = y_1$. Hence, the linear system Equation (5) is controllable.

Necessity: Let Equation (5) be controllable, but W is not invertible; then, $\exists$ vector $v \ne 0$ such that $v^T W v = 0$, which implies that

$$
\begin{aligned}
\int_0^d & \|v^T(d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)L\|^2 dc \\
+ \int_{-q}^{d-q} & \|v^T(d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M\|^2 dc = 0.
\end{aligned}
\tag{10}
$$

From this, we have

$$
\begin{aligned}
v^T(d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)L &= 0, \\
v^T(d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M &= 0.
\end{aligned}
\tag{11}
$$

Assume there are two input control functions $u_1(t)$ and $u_2(t)$, such that

$$
\begin{aligned}
y(d) = E_\nu(Kd^\nu)y_0 &+ \int_0^d (d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)Lu_1(c)dc \\
&+ \int_{-q}^{d-q} (d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M\phi(c)dc = 0,
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
v = E_\nu(Kd^\nu)y_0 &+ \int_0^d (d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)Lu_2(c)dc \\
&+ \int_{-q}^{d-q} (d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M\phi(c)dc, \\
v - E_\nu(Kd^\nu)y_0 &- \int_0^d (d-c)^{\nu-1}E_{\nu,\nu}(K(d-c)^\nu)Lu_2(c)dc \\
&- \int_{-q}^{d-q} (d-q-c)^{\nu-1}E_{\nu,\nu}(K(d-q-c)^\nu)M\phi(c)dc = 0,
\end{aligned}
\tag{13}
$$

Equations (12) and (13) together will yield

$$v + \int_0^d (d - c)^{\nu-1} E_{\nu,\nu}(K(d-c)^\nu)L[u_1(c) - u_2(c)]dc = 0, \tag{14}$$

which implies that

$$v^T v + \int_0^d e^T (d - c)^{\nu-1} E_{\nu,\nu}(K(d-c)^\nu)L[u_1(c) - u_2(c)]dc = 0, \tag{15}$$

utilizing Equation (11), we have $v^T v = 0$, i.e., $v = 0$, which is in contradiction to our supposition that $v \neq 0$. Hence, our supposition that the system is controllable and W is not invertible was wrong, and the given statement is true. Proofs for the case $t > q$ can be tackled in a similar way as we did above. $\square$

To investigate the controllability of the system (1), we have the underlying hypotheses.

(H_1). The nonlinear function $f : I \times R^n \times R^n \to R^n$ is measurable and continuous, and a positive constant p exists such that

$$\|f(c, y(c), {}^c D^{\nu-1} y(c))\| \leq p, \forall\, t \in I. \tag{16}$$

(H_2). For brevity, we assume the following:

$$\begin{cases}
\psi_0(t) = E_\nu(Kt^\nu), t \in I, & sup\|\psi_0\| = k_0; \\
\psi_1(t, c) = (t - c)^{\nu-1} E_{\nu,\nu}(K(t-c)^\nu), t \in I, & sup\|\psi_1\| = k_1; \\
\psi_2(t, q, c) = (t - q - c)^{\nu-1} E_{\nu,\nu}(K(t-q-c)^\nu), t \in I, & sup\|\psi_2\| = k_2; \\
\psi_3(t) = KE_{\nu,1-j}(Kt^\nu), t \in I, & sup\|\psi_3\| = k_3; \\
\psi_4(t, c) = (t - c)^{\nu-j-1} E_{\nu,\nu-j}(K(t-c)^{\nu-j}), t \in I & sup\|\psi_4\| = k_4; \\
\psi_5(t, q, c) = (t - q - c)^{\nu-j-1} E_{\nu,\nu-j}(K(t-q-c)^{\nu-j}), t \in I, & sup\|\psi_5\| = k_5; \\
\psi_6(t, c) = (t - c)^{j-\nu-1}, t \in I, & sup\|\psi_6\| = k_6; \\
\psi_7(c, y) = f(c, y(c), {}^c D^{\nu-1} y(c)).
\end{cases}$$

Theorem 1. *If the model (1) is assumed to be controllable on the interval I and the hypotheses H_1 and H_2 hold, then the nonlinear system of fractional order (1) is also controllable on I.*

Proof. Case I. When $t > q$

To prove the theorem, we define the Banach space $Y = \{y : y^{(q)}, {}^c D^\nu(y) \in (I, R^n)\}$, with norm $\|y\| = \max\{\|y(t)\|, \|{}^c D^\nu y(t)\|, \|u\|\}$. Further utilizing the hypothesis H_1 and H_2, the input $u(t)$ of the system (1) for an arbitrary solution $y(.)$ is:

$$u(t) = \begin{cases}
0, & -q \leq t \leq 0; \\
(\psi_1(d, t)L + \psi_2(d, q, t)M)^T W^{-1}\Phi, & 0 \leq t \leq d - q; \\
(\psi_1(d, t)L)^T W^{-1}\Phi, & d - q \leq t \leq d;
\end{cases} \tag{17}$$

where

$$\Phi = y_1 - \psi_0(t)y_0 - \int_0^d \psi_1(t, c)\psi_7(c, y)dc.$$

We define the nonlinear operator $T : Y \to Y$, and it is expressed by

$$\begin{aligned}
Ty(t) &= \psi_0(t)y_0 + \int_{-q}^0 \psi_2(t, q, c)M\phi(c)dc + \int_0^{t-q} [\psi_1(t, c)L + \psi_2(t, q, c)M]u(c)dc \\
&+ \int_{t-q}^t \psi_1(t, c)Lu(c)dc + \int_0^t [\psi_1(t, c)\psi_7(c, y)]dc.
\end{aligned} \tag{18}$$

The operator defined above possesses a fixed point, and this fixed point comprises a particular solution of (1). Inserting (17) in (18), one reaches:

$$
\begin{aligned}
(Ty)(t) \;=\;& \psi_0(t)y_0 + \int_0^{t-q}[\psi_1(t,c)L + \psi_2(t,q,c)M] \\
\times\;& [\psi_1(d,c)L + \psi_2(d,q,c)M]^T W^{-1}\Phi dc \\
+\;& \int_{t-q}^{t}\psi_1(t,c)L(\psi_1(d,c)L)^T W^{-1}\Phi dc \\
+\;& \int_0^{t}[\psi_1(t,c)\psi_7(c,y)]dc.
\end{aligned}
\tag{19}
$$

Clearly, $Ty(d) = y_1$. Furthermore, it means that if the nonlinear operator has a fixed point, then there exists an input $u(t)$ that steers the system from the initial state y_0 to the final state y_1 in time d.

Next, we demonstrate that the operator T follows Schaefer's fixed point theorem. Our proof consists of three steps:

Step I. In the first step, we show the boundedness of the set $\xi(T) = \{y \in Y : y = \eta Ty, \eta \in [0,1]\}$, in I. For an arbitrary $y \in \xi(T)$ and $0 < \eta < 1$, one reaches

$$
\begin{aligned}
y(t) \;=\;& \eta\psi_0(t)y_0 + \eta\int_0^{t-q}[\psi_1(t,c)L + \psi_2(t,q,c)M] \\
\times\;& [\psi_1(d,c)L + \psi_2(d,q,c)M]^T W^{-1}\Phi dc \\
+\;& \eta\int_{t-q}^{t}\psi_1(t,c)K(\psi_1(d,c)L)^T W^{-1}\Phi dc \\
+\;& \eta\int_0^{t}[\psi_1(t,c)\psi_7(c,y)]dc.
\end{aligned}
\tag{20}
$$

Then, utilizing hypothesis H_1 and H_2, we have

$$
\begin{aligned}
\|\Phi\| \;\leq\;& \|y_1\| + \|\psi_0(t)\|\|y_0\| + \int_0^{d}(\|\psi_1(t,c)\|\|\psi_7\|)dc, \\
\leq\;& \|y_1\| + k_0\|x_0\| + dk_1 p.
\end{aligned}
\tag{21}
$$

and

$$
\|u(t)\| =
\begin{cases}
0, & -q \leq t \leq 0; \\
[k_1\|L\| + k_2\|M\|]^T\|W^{-1}\|\|\Phi\|, & 0 \leq t \leq d - q; \\
(k_1\|L\|)^T\|W^{-1}\|\|\Phi\|, & d - q \leq t \leq d.
\end{cases}
\tag{22}
$$

In view of (21) and (22), (20) will give

$$
\begin{aligned}
\|y(t)\| \;\leq\;& k_0\|y_0\| + \int_0^{t-q}[k_1\|L\| + k_2\|M\|][k_1\|L\| + k_2\|M\|]^T\|W^{-1}\|\|\Phi\|dc \\
+\;& \int_{t-q}^{t}k_1\|L\|(k_1\|L\|)^T\|W^{-1}\|\|\Phi\|dc + \int_0^{t}[k_1 p]dc, \\
\leq\;& k_0\|y_0\| + d\|W^{-1}\|[(k_1\|L\| + k_2\|M\|)(k_1\|L\| + k_2\|M\|)^T + K_1^2\|L\|\|L^T\|] \\
\times\;& (\|y_1\| + k_1\|y_0\| + dk_1 p) + dk_1 p = \gamma_1.
\end{aligned}
\tag{23}
$$

Furthermore, by Definition 5, one obtains

$$
\begin{aligned}
y^{(j)}(t) \;=\;& \eta\psi_3(t)y_0 + \eta\int_0^{t-q}[\psi_4(t,c)L + \psi_5(t,q,c)M] \\
\times\;& [\psi_4(d,c)L + \psi_5(d,q,c)M]^T W^{-1}\Phi dc + \eta\int_{t-q}^{t}\psi_4(t,c)L(\psi_4(d,c)L)^T W^{-1}\Phi dc \\
+\;& \eta\int_0^{t}[\psi_4(t,c)\psi_7(c,y)]dc.
\end{aligned}
\tag{24}
$$

Which further gives

$$
\begin{aligned}
\|y^{(j)}(t)\| \;\leq\;\; & k_3\|y_0\| + d[(k_4\|L\| + k_5\|M\|)(k_4\|L\| + k_5\|M\|)^T + k_4^2\|L\|\|L\|] \\
\times \;\; & \|W^{-1}\|(\|y_1\| + k_1\|y_0\| + dk_1 p) + dk_4 p = \gamma_2.
\end{aligned}
\tag{25}
$$

Utilizing Definition 2, we have

$$
\|{}^c D^\nu y(t)\| \leq \|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^t (k_6 \gamma_2) dc\|.
\tag{26}
$$

Hence, ${}^c D^\nu y(t)$ is bounded. It means that ξT is bounded as well because $\|y\| = max[\|y\|,$ $\|{}^c D^\nu y\|, \|u\|]$.

Step II. In this step, we prove that the operator T is completely continuous. To do this, we assume that $B_s = \{y \in Y; \|y\| \leq s\}$, which is mapped into an equicontinuous family by T. Then, for any $y \in B_s$ and $t_1, t_2 \in I$ with $0 < t_1 < t_2 < d$, one obtains

$$
\begin{aligned}
\|Ty(t_2) - Ty(t_1)\| \;\leq\;\; & \|\psi_0(t_2) - \psi_0(t_1)\|\|y_0\| \\
+ \;\; & \|\int_{t_1-q}^{t_2-q}[(\psi_1(t_2,c)L + \psi_2(t_2,q,c)M)] \\
\times \;\; & (\psi_1(d,c)L + \psi_2(d,q,c)M)^T W^{-1}\Phi dc\| \\
+ \;\; & \|\int_0^{t_1-q}[(\psi_1(t_2,c)L + \psi_2(t_2,q,c)M) - (\psi_1(t_1,c)L + \psi_2(t_1,q,c)M)] \\
\times \;\; & (\psi_1(d,c)L + \psi_2(d,q,c)M)^T W^{-1}\Phi dc\| \\
+ \;\; & \|\int_{t_1-q}^{t_1}[\psi_1(t_2,c)L - \psi_1(t_1,c)L](\psi_1(d,c)L)^T W^{-1}\Phi dc\| \\
+ \;\; & \|\int_0^{t_1}[\psi_1(t_2,c) - \psi_1(t_1,c)]\psi_7(c,y)dc\| \\
+ \;\; & \|\int_{t_1}^{t_2}\psi_1(t_2,c)\psi_7(c,y)dc\| \\
+ \;\; & \|\int_{t_1}^{t_2}\psi_1(t_2,c)L(\psi_1(d,c)L)^T W^{-1}\Phi\|.
\end{aligned}
\tag{27}
$$

In view of (27), (17) is expressed as

$$
\|Tu(t_2) - Tu(t_1)\| \leq
$$
$$
\begin{cases}
0, & -q \leq t \leq 0; \\
\|[(\psi_1(d,t_2)L + \psi_2(d,q,t_2)M) - \\
(\psi_1(d,t_1)L + \psi_2(d,q,t_1)M)]^T\|\|W^{-1}\|\|\Phi\|, & 0 \leq t \leq d-q; \\
\|[(\psi_1(d,t_2)L) - (\psi_1(d,t_1)L)]^T\|\|W^{-1}\|\|\Phi\|, & d-q \leq t \leq d.
\end{cases}
\tag{28}
$$

This further implies that

$$
\begin{aligned}
\|{}^c D^\nu Ty(t_2) - {}^c D^\nu Ty(t_1)\| \leq & \\
\|\frac{1}{\Gamma(j-\nu)}\|\|\int_{t_1}^{t_2}(\psi_6(t_2,c))(Ty)^{(j)} dc\| + & \\
\|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^{t_1}(\psi_6(t_2,c)(Ty)^{(j)} - \psi_6(t_1,c)(Ty)^{(j)})dc\|. &
\end{aligned}
\tag{29}
$$

Evidently,

$$
\lim_{t_2 \to t_1} \|(Ty)(t_2) - (Ty)(t_1)\| \to 0,
$$
$$
\lim_{t_2 \to t_1} \|(Tx)^{(j)}(t_2) - (Tx)^{(j)}(t_1))\| \to 0
$$
$$
\lim_{t_2 \to t_1} \|{}^c D^\nu (Ty)(t_2) - ({}^c D^\nu Ty)(t_1)\| \to 0.
$$

Hence, the equicontinuous family of functions, $\{(Ty) : y \in B_s\}$, is uniformly bounded. Now, we want to verify that the operator T is compact. For any $y \in B_s$ and a real number ϵ such that $0 < \epsilon < t$ where $t \in [0, d]$, we define

$$
\begin{aligned}
(T_\epsilon y)(t) \;=\; & \psi_0(t)y_0 + \int_0^{t-\epsilon-q} [\psi_1(t,c)L + \psi_2(t,q,c)M] \\
\times \; & (\psi_1(d,c)L + \psi_2(d,q,c)M)^T W^{-1} \Phi dc \\
+ \; & \int_{t-q}^{t-\epsilon} \psi_1(t,c)L(\psi_1(d,c)L)^T W^{-1} \Phi dc \\
+ \; & \int_0^{t-\epsilon} [\psi_1(t,c)\psi_7(c,y)]dc.
\end{aligned}
\tag{30}
$$

As above, we obtain that $\{(T_\epsilon y) : y \in B_s\}$ is an equicontinuous family of functions that fulfills the uniform-bounded condition. Therefore, one has

$$
\begin{aligned}
\|(Ty)(t) - (T_\epsilon y)(t)\| \;\leq\; & \Big\| \int_{t-\epsilon-q}^{t-q} [\psi_1(t,c)L + \psi_2(t,q,c)M] \\
\times \; & (\psi_1(d,c)L + \psi_2(d,q,c)M)^T W^{-1} \Phi dc \Big\| \\
+ \; & \Big\| \int_{t-\epsilon}^{t} \psi_1(t,c)L(\psi_1(d,c)L)^T W^{-1} \Phi dc \Big\| \\
+ \; & \Big\| \int_{t-\epsilon}^{t} [\psi_1(t,c)\psi_7(c,y)]dc \Big\|, \\
\leq\; & \epsilon \|W^{-1}\| \|\Phi\| [(k_1\|L\| + k_2\|M\|)(k_1\|L\| + k_2\|M\|)^T \\
+ \; & k_1^2 \|L\| \|L^T\|] + \epsilon k_1 p.
\end{aligned}
\tag{31}
$$

Utilizing the above, we reach

$$
\begin{aligned}
\|(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)\| \;\leq\; & \Big\| \int_{t-\epsilon-q}^{t-q} [\psi_4(t,c)L + \psi_5(t,q,c)M] \\
\times \; & (\psi_1(d,c)L + \psi_2(d,q,c)M)^T W^{-1} \Phi dc \Big\| \\
+ \; & \Big\| \int_{t-\epsilon}^{t} \psi_4(t,c)L(\psi_1(d,c)L)^T W^{-1} \Phi dc \Big\| \\
+ \; & \Big\| \int_{t-\epsilon}^{t} [\psi_4(t,c)\psi_7(c,y)]dc \Big\|, \\
\leq\; & \epsilon \|W^{-1}\| \|\Phi\| [(k_4\|L\| + k_5\|M\|)(k_1\|L\| + k_2\|M\|)^T \\
+ \; & k_1 k_4 \|L\| \|L^T\|] + \epsilon k_4 p.
\end{aligned}
\tag{32}
$$

Applying the definition of Caputo derivative, we have

$$
\begin{aligned}
\|{}^c D^v((Ty)(t_2) - {}^c D^v(T_\epsilon y)(t_1))\| \leq \\
\Big\| \frac{1}{\Gamma(j-v)} \Big\| \|\| \int_0^t \psi_6(t,c)[(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}](t)dc \Big\|.
\end{aligned}
\tag{33}
$$

Distinctly,

$$
\begin{aligned}
\lim_{\epsilon \to 0} \|(Ty)(t) - (T_\epsilon y)(t)\| &\to 0, \\
\lim_{\epsilon \to 0} \|(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)\| &\to 0, \\
\lim_{\epsilon \to 0} \|{}^c D^v(Ty)(t) - {}^c D^v(T_\epsilon y)(t)\| &\to 0.
\end{aligned}
$$

Hence, by the Arzela–Ascoli theorem, $\{(Ty)(t) : y \in B_s\}$ is compact in Y.
Step III. The last step to show that T is continuous. We make two more hypotheses:
(H_3). Let $Y = \{y_1, y_2, \ldots, y_n\}$, $\lim_{n \to \infty} \|y_n - y(t)\| = 0$.

(H_4). Let $z = max\{\|y_n\|, \|u_n\|, \|{}^cD^\nu y_n\|\}$, z be a positive constant.

Utilizing the above hypothesis, we obtain

$$f(t, y_n(t), {}^cD^{(\nu-1)}y_n(t)) \leq f(t, y(t), {}^cD^{(\nu-1)}y(t)), i.e$$
$$\psi_7(c, y_n) \leq \psi_7(c, y).$$

Now, by Fatou–Lebesgue theoerm,

$$
\begin{aligned}
\|(Ty_n)(t) - (Ty)(t)\| &\leq& \|\int_0^t k_1[(k_1\|L\| + k_2\|M\|)(k_1\|L\| + k_2\|M\|)^T + k_1^2\|L\|\|L^T\|] \\
&\times& \|W^{-1}\|\|\int_0^s((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta))))d\vartheta\|dc\| \\
&+& k_1\|\int_0^t((\psi_7(c, y_n)) - (\psi_7(c, y)))dc\|.
\end{aligned}
\tag{34}
$$

Utilizing a similar approach as the above, we also have

$$
\begin{aligned}
\|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| &\leq& \|\int_0^t k_4[(k_4\|L\| + k_5\|M\|)(k_4\|L\| + k_5\|M\|)^T + k_1k_4\|L\|\|L^T\|] \\
&\times& \|W^{-1}\|\|\int_0^s((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta))))d\vartheta\|dc\| \\
&+& k_4\|\int_0^t((\psi_7(c, y_n)) - (\psi_7(c, y)))dc\|.
\end{aligned}
\tag{35}
$$

Making use of Definition 2, one obtains

$$\|{}^cD^\nu(Ty_n)(t) - {}^cD^\nu(Ty(t))\| \leq \|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^t \psi_6(t, c)[(Ty_n)^{(j)}(t) - (Ty)^{(j)}](t)dc\|.$$

Clearly

$$\lim_{n\to\infty} \|(Ty_n)(t) - (Ty)(t)\| = 0,$$
$$\lim_{n\to 0} \|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| = 0,$$
$$\lim_{n\to\infty} \|{}^cD^\nu(Ty_n)(t) - {}^cD^\nu(Ty)(t)\| = 0.$$

This clearly indicates the continuity of T. Hence, following the Arzela–Ascoli and Schaefer's fixed point theorems, one may easily deduce that the operator T is continuous and possesses a fixed point Y in B_s. Furthermore, this fixed point Y is the solution of the (1). It is therefore concluded that (1) is controllable on I for the case $t > q$.

Case II. When $0 \leq t \leq q$:

We define the Banach space as $Y = \{y : y^{(j)}, {}^cD^{(\nu-1)}y, t \in (I, R^n)\}$, with norm $\|y\| = max\{\|y\|, \|{}^cD^\nu(y)\|, \|u\|\}$. Then, utilizing an arbitrary solution $y(.)$ of (1) and the hypotheses H_1 and H_2, the input signal $u(t)$ can be obtained as

$$
u(t) = \begin{cases}
(\psi_2(d, q, t)M)^T W^{-1}\Phi, & -q \leq t \leq d - q; \\
0, & d - q \leq t \leq 0; \\
(\psi_1(d, t)L)^T W^{-1}\Phi, & 0 < t \leq d;
\end{cases}
\tag{36}
$$

where

$$\Phi = y_1 - \psi_0(t)y_0 - \int_0^d \psi_1(d, c)\psi_7(c)dc.$$

Define the nonlinear operator $T : Y \to Y$ by

$$
\begin{aligned}
Ty(t) \;=\;& \psi_0(t)y_0 + \int_0^t \psi_1(t,c)Lu(c)dc + \int_{-q}^{t-q} \psi_2(t,q,c)M\phi(c)dc \\
& + \int_0^t [\psi_1(t,c)\psi_7(c,y)]dc.
\end{aligned} \tag{37}
$$

The operator T has a fixed point that is a particular solution of (1). Putting (36) in (37), we obtain

$$
\begin{aligned}
Ty(t) \;=\;& \psi_0(t)y_0 + \int_0^t \psi_1(t,c)L(\psi_1(d,c)L)^T W^{-1}\Phi dc \\
& + \int_{-q}^{t-q} \psi_2(t,q,c)M(\psi_2(d,q,c)M)^T W^{-1}\Phi dc + \int_0^t [\psi_1(t,c)\psi_7(c,y)]dc. \tag{38}
\end{aligned}
$$

Clearly, $Ty(d) = y_1$. Furthermore, it means that if the nonlinear operator has a fixed point, then there exists an input $u(t)$ that steers the system from the initial state y_0 to the final state y_1 in time d.

Now, we have to verify that the operator T satisfies the Schaefer's fixed point theorem. Our proof consists of three steps.

Step I. In the first step, we show boundedness of the set $\xi(T) = \{y \in Y : y = \eta Ty, \eta \in [0,1]\}$, on I.

For an arbitrary $y \in \xi(T)$ and $0 < \eta < 1$, one achieves

$$
\begin{aligned}
y(t) \;=\;& \eta\psi_0(t)y_0 + \eta \int_0^t \psi_1(t,c)L(\psi_1(d,c)L)^T W^{-1}\Phi dc \\
& + \eta \int_{-q}^{t-q} \psi_2(t,q,c)M(d,q,c)M)^T W^{-1}\Phi dc + \eta \int_0^t [\psi_1(t,c)\psi_7(c,y)]dc. \tag{39}
\end{aligned}
$$

Then, utilizing hypothesis H_1 and H_2, we have

$$
\begin{aligned}
\|\Phi\| \;\leq\;& \|y_1\| + \|\psi_0(t)\|\|y_0\| + \int_0^d (\|\psi_1(t,c)\|\|\psi_7(c)\|)dc \\
\leq\;& \|y_1\| + k_1\|y_0\| + dk_1 p, \tag{40}
\end{aligned}
$$

and

$$
\|u(t)\| = \begin{cases} k_2 M^T \|W^{-1}\|\|\Phi\|, & -q \leq t \leq d-q; \\ 0, & d-q \leq t \leq 0; \\ k_1 L^T \|W^{-1}\|\|\Phi\|, & 0 < t \leq d. \end{cases} \tag{41}
$$

In view of (40) and (41), (39) give

$$
\begin{aligned}
\|y(t)\| \;\leq\;& k_0\|y_0\| + \int_0^t k_1\|L\|\|(k_1 L)^T\|\|W^{-1}\|\|\Phi\|dc \\
& + \int_{-q}^{t-q} k_2\|M\|\|(k_2 M)^T\|\|W^{-1}\|\|\Phi\|dc + \int_0^t [k_1 p]dc, \\
\leq\;& k_0\|y_0\| + [k_1^2\|L\|\|L^T\| + k_2^2\|M\|\|M^T\|] \\
& \times d\|W^{-1}\|[\|y_1\| + k_1\|y_0\| + dk_1 p] + dk_1 p = \gamma_3. \tag{42}
\end{aligned}
$$

Furthermore, by Definition 5

$$
\begin{aligned}
y^{(j)}(t) \;=\; & \eta\psi_3(t)y_0 + \eta \int_0^t \psi_4(t,c)L(\psi_1(d,c)L)^T W^{-1}\Phi dc \\
& + \; \eta \int_{-q}^{t-q} \psi_5(t,q,c)M(d,q,c)M)^T W^{-1}\Phi dc \\
& + \; \eta \int_0^t [\psi_4(t,c)\psi_7(c,y)]dc,
\end{aligned}
\tag{43}
$$

which gives

$$
\begin{aligned}
\|y^{(j)}(t)\| \;\leq\; & k_3\|y_0\| + [dk_1 k_4\|L\|\|L^T\| + dk_2 k_5\|M\|\|M^T\|] \\
& \times \; \|W^{-1}\|(\|y_1\| + k_1\|y_0\| + dk_1 p) + dk_4 p = \gamma_4.
\end{aligned}
\tag{44}
$$

Utilizing Definition 2, we have

$$
\|{}^c D^\nu y(t)\| \leq \|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^t (k_6\gamma_4)dc\|.
\tag{45}
$$

Hence, ${}^c D^\nu y(t)$ is bounded. It means that ξT is bounded as well because $\|y\| = max[\|y\|,$ $\|{}^c D^\nu y\|, \|u\|]$.

Step II. Here, we verify that T is a completely continuous operator. Suppose $B_s = \{y \in Y; \|y\| \leq s\}$, which is mapped into an equicontinuous family by T. Then, for any $y \in B_s$ and $t_1, t_2 \in I$ with $0 < t_1 < t_2 < d$, we show that TB_s is uniformly bounded

$$
\begin{aligned}
\|Ty(t_2) - Ty(t_1)\| \;\leq\; & \|\psi_0(t_2) - \psi_0(t_1)\|\|y_0\| \\
& + \int_0^{t_1} [\psi_1(t_2,c)L - \psi_1(t_1,c)L] \times (\psi_1(d,c)L)^T W^{-1}\Phi \\
& + \int_{-q}^{t_1-q} [\psi_2(t_2,q,c)M - \psi_2(t_1,q,c)M](\psi_2(d,q,c)M)^T W^{-1}\Phi \\
& + \int_0^{t_1} [\psi_1(t_2,c) - \psi_1(t_1,c)]\psi_7(c,y)dc \\
& + \int_{t_1}^{t_2} [\psi_2(t_2,c)L(\psi_2(d,c)L)^T]W^{-1}\Phi \\
& + \int_{t_1-q}^{t_2-q} [\psi_2(t_2,q,c)M(\psi_2(d,q,c)M)^T]W^{-1}\Phi \\
& + \int_{t_1}^{t_2} \psi_1(t_2,c)\psi_7(c,y)dc.
\end{aligned}
\tag{46}
$$

In view of (46), (36) is expressed as

$$
\|(Tu)(t_2) - (Tu)(t_1)\| \leq
\begin{cases}
[(\psi_2(d,q,t_2)M) - (\psi_2(d,q,t_1)M)]^T W^{-1}\Phi, \\
\quad -q \leq t \leq d-q, \\
0, \quad d-q \leq t \leq 0, \\
[(\psi_1(d,t_2)L) - (\psi_1(d,t_1)L)]^T W^{-1}\Phi, 0 < t \leq d.
\end{cases}
\tag{47}
$$

This further implies that

$$
\begin{aligned}
& \|{}^c D^\nu Ty(t_2) - {}^c D^\nu Ty(t_1)\| \leq \\
& \|\frac{1}{\Gamma(j-\nu)}\|\|\int_{t_1}^{t_2} (\psi_6(t_2,c))(Ty)^{(j)}dc\| + \\
& \|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^{t_1} (\psi_6(t_2,c)(Ty)^{(j)} - \psi_6(t_1,c)(Ty)^{(j)})dc\|.
\end{aligned}
\tag{48}
$$

Consequently,

$$\lim_{t_2 \to t_1} \|(Ty)(t_2) - (Ty)(t_1)\| \to 0,$$

$$\lim_{t_2 \to t_1} \|(Tx)^{(q)}(t_2) - (Tx)^{(q)}(t_1))\| \to 0$$

$$\lim_{t_2 \to t_1} \|{}^c D^v (Ty)(t_2) - ({}^c D^v Ty)(t_1)\| \to 0.$$

Hence, the equicontinuous family of functions,$\{(Ty) : y \in B_s\}$ is uniformly bounded. Now, we prove that the operator T is compact. For any $y \in B_s$ and a real number ϵ such that $0 < \epsilon < t$ where $t \in [0, d]$, we define

$$
\begin{aligned}
(Ty_\epsilon)(t) \;=\;& \psi_0(t)y_0 + \int_0^{t-\epsilon} \psi_1(t,c) L(\psi_1(d,c)L)^T W^{-1}\Phi dc \\
&+ \int_{-q}^{t-\epsilon-q} \psi_2(t,q,c) M(\psi_2(d,q,c)M)^T W^{-1}\Phi dc \\
&+ \int_0^{t-\epsilon} [\psi_1(t,c)\psi_7(c,y)]dc.
\end{aligned}
\tag{49}
$$

As above, we obtain that $\{(T_\epsilon y) : y \in B_s\}$ is an equicontinuous family of functions that fulfills the uniform bounded condition. Therefore, we can infer

$$
\begin{aligned}
\|(Ty)(t) - (T_\epsilon y)(t)\| \;\leq\;& \|\int_{t-\epsilon}^{t} \psi_1(t,c) L(\psi_1(d,c)L)^T W^{-1}\Phi dc\| \\
&+ \|\int_{t-\epsilon-q}^{t-q} \psi_2(t,q,c) M(\psi_2(d,q,c)M)^T W^{-1}\Phi dc\| \\
&+ \|\int_{t-\epsilon}^{t} [\psi_1(t,c)\psi_7(c,y)]dc\|, \\
\leq\;& \epsilon[k_1^2\|L\|\|L^T\| + k_2^2\|M\|\|M^T\|]\|W^{-1}\|\|\Phi\| + \epsilon k_1 p.
\end{aligned}
\tag{50}
$$

Utilizing the above, one reaches

$$
\begin{aligned}
\|(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)\| \;\leq\;& \|\int_{t-\epsilon}^{t} \psi_4(t,c) L(\psi_1(d,c)L)^T W^{-1}\Phi dc\| \\
&+ \|\int_{t-\epsilon-q}^{t-q} \psi_5(t,q,c) M(\psi_2(d,q,c)M)^T W^{-1}\Phi dc\| \\
&+ \|\int_{t-\epsilon}^{t} [\psi_4(t,c)\psi_7(c,y)]dc\|, \\
\leq\;& \epsilon[k_1 k_4\|L\|\|L^T\| + k_2 k_5\|M\|\|M^T\|]\|W^{-1}\|\|\Phi\| + \epsilon k_4 p.
\end{aligned}
\tag{51}
$$

Applying the definition of Caputo derivative, we have

$$
\begin{aligned}
\|{}^c D^v((Ty)^{(j)}(t_2) - (T_\epsilon y)^{(j)}(t_1))\| \leq \\
\|\frac{1}{\Gamma(j-v)}\|\|\int_0^t \psi_6[(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)]dc\|.
\end{aligned}
\tag{52}
$$

Distinctly,

$$\lim_{\epsilon \to 0} \|(Ty)(t) - (T_\epsilon y)(t)\| \to 0,$$

$$\lim_{\epsilon \to 0} \|(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)\| \to 0,$$

$$\lim_{\epsilon \to 0} \|{}^c D^v(Ty)(t) - {}^c D^v(T_\epsilon y)(t)\| \to 0.$$

It follows form Arzela–Ascoli theorem that $\{(Ty)(t) : y \in B_s\}$ is compact in Y.

Step III. Next, we prove that T is continuous. Assume:

(H_{3a})Let $Y = \{y_1, y_2, \ldots, y_n\}, \lim_{n \to \infty} \|y_n - y(t)\| = 0$.

(H_{4a})Let $z = max\{\|y_n\|, \|u_n\|, \|^c D^\nu y_n\|\}$, z is a positive constant.

Utilizing H_{3a} and H_{4a}, we have

$$
\begin{aligned}
f(t, y_n(t), {}^c D^{(\nu-1)} y_n(t)) &\leq f(t, y(t), {}^c D^{(\nu-1)} y(t)), i.e \\
\psi_7(\text{c}, y_n) &\leq \psi_7(\text{c}, y).
\end{aligned}
$$

Now, by Fatou–Lebesgue theorem

$$
\begin{aligned}
\|(Ty_n)(t) - (Ty)(t)\| &\leq \int_0^t [k_1^2 \|L\| \|L^T\| + k_2^2 \|M\| \|M^T\|] \\
&\times \|W^{-1}\| k_1 \| \int_0^s ((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta)))) d\vartheta \| d\text{c} \\
&+ k_1 \| \int_0^t ((\psi_7(\text{c}, y_n)) - (\psi_7(\text{c}, y))) d\text{c} \|.
\end{aligned} \tag{53}
$$

Utilizing a similar approach as above, we also have

$$
\begin{aligned}
\|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| &\leq \int_0^t [k_1 k_4 \|L\| \|L^T\| + k_2 k_5 \|M\| \|M^T\|] \\
&\times \|W^{-1}\| k_4 \| \int_0^s ((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta)))) d\vartheta \| d\text{c} \\
&+ k_4 \| \int_0^t ((\psi_7(\text{c}, y_n)) - (\psi_7(\text{c}, y))) d\text{c} \|.
\end{aligned} \tag{54}
$$

Making use of Definition 2, one obtains

$$
\begin{aligned}
\|^c D^\nu (Ty_n)(t) - {}^c D^\nu (Ty)(t)\| &\leq \\
\|\frac{1}{\Gamma(j - \nu)}\| \| \int_0^t \psi_6(t, \text{c})((Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)) d\text{c} \|.
\end{aligned} \tag{55}
$$

Clearly

$$
\begin{aligned}
\lim_{n \to \infty} \|(Ty_n)(t) - (Ty)(t)\| &= 0, \\
\lim_{n \to 0} \|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| &= 0, \\
\lim_{n \to \infty} \|^c D^\nu (Ty_n)(t) - {}^c D^\nu (Ty)(t)\| &= 0.
\end{aligned}
$$

Hence, T is continuous. So, by Arzela–Ascoli and Schaefer's fixed point theorem, it can be concluded the operator T is continuous and has a fixed point Y in B_s. Furthermore, this fixed point Y is the solution of the system (1). We conclude that (1) is controllable for $0 \leq t \leq q$ on I. $\square$

4. Observability

Observability is a property of dynamical systems that measures how well the internal states of a system can be obtained from the information of its external outputs. For the observability of our proposed model, we assume that $u(t) = 0$, because it has been shown in [36] that the observability of a system is independent of the input signal $u(t)$. After this change is made and adding a linear observer, the system (1) obtains the form,

$$
\begin{cases}
{}^c D^\nu y(t) = Ky(t) + f(t, y(t), {}^c D^{\nu-1} y(t)), t \in I = [0, d], \\
y(0) = y_0, \quad y'(0) = 0, \\
z(t) = Hy(t).
\end{cases} \tag{56}
$$

where $1 < \nu \leq 2$; K is a $n \times n$ matrix and f is a nonlinear continuous function.

4.1. Linear Case

Definition 8. *The time-invariant linear system in (56) is said to be observable at time $t \in I$, if $z(t) = Hy(t) = 0$ implies that $y(t) = 0$.*

Theorem 2. *The linear system in (56) is observable in I if and only if the observability Grammian matrix*

$$Q(0,d) = \int_0^d E_\nu(K^* t^\nu) H^* H E_\nu(K t^\nu) dt \tag{57}$$

is positive definite.

Proof. By applying the Laplace transform, the Mittag–Leffler function, and the initial conditions, the solution of the linear system in (56) is given by

$$y(t) = E_\nu(K t^\nu) y_0. \tag{58}$$

With the help of this equation, we have $z(t) = H E_\nu(K t^\nu) y_0$, and

$$\|z(t)\|^2 = \int_0^d z^*(t) z(t) dt$$
$$= y_0^* \int_0^t E_\nu(K t^\nu) H^* H E_\nu(K t^\nu) y_0 dt$$
$$= y_0^* Q(0,d) y_0,$$

clearly, $Q(0,d)$ is symmetric, and the equation is quadratic in y_0. If $Q(0,d)$ is positive definite and $z(t) = y_0^* Q(0,d) y_0 = 0$, then $y_0 = 0$. Hence, the linear system in (56) is observable. If $Q(0,d)$ is not positive definite, then there exists some non-zero y_0 such that $y_0^* Q(0,d) y_0 = 0$. This implies that $y(t) = E_\nu(K t^\nu) y_0 \neq 0$, but $\|z\| = 0 \Rightarrow y = 0$, which in turn implies that the system is not observable. Hence the required proof. □

4.2. Nonlinear Case

For the observability of the nonlinear system (56), one needs to estimate the unidentified state $y(t)$ at the current time t from the information of the system output $z(t)$ in $[\bar{t}, t]$, where $\bar{t}$ denotes the past time.

Definition 9. *The nonlinear system (56) is called observable at time t if one can determine $\bar{t} < t$ in such a way that the state of the system at time t can be estimated from the information of the system's output through the interval $[\bar{t}, t]$. If a given system is observable for all $t \in I$, we call it completely observable.*

Let the nonlinear system (56) possess a distinctive solution for some initial condition $y = y(t_0)$, $t_0 \in (\bar{t}, t)$, and it is given by

$$y(t) = E_\nu(K(t - t_0)^\nu) y(t_0) + \int_{t_0}^t (t - c)^{\nu-1} E_{\nu,\nu}(K(t - c)^\nu) f(t, y(t), {}^c D^{\nu-1} y(c)) dc, \tag{59}$$

It is solved for $y(t_0)$ by assuming $[E_\nu(K(t - t_0)^\nu)]$ is invertible. Then, we obtain

$$y(t_0) = [E_\nu(K(t - t_0)^\nu)]^{-1} [y(t) - \int_{t_0}^t (t - c)^{\nu-1} E_{\nu,\nu}(K(t - c)^\nu) f(t, y(t), {}^c D^{\nu-1} y(c)) dc]. \tag{60}$$

This further gives

$$z(t_0) = [E_\nu(K(t - t_0)^\nu)]^{-1} [Hy(t) - H \int_{t_0}^t (t - c)^{\nu-1} E_{\nu,\nu}(K(t - c)^\nu) f(t, y(t), {}^c D^{\nu-1} y(c)) dc],$$

$$= \frac{1}{[E_\nu(K(t - t_0)^\nu)]^2} [Hy(t) - H \int_{t_0}^t (t - c)^{\nu-1} E_{\nu,\nu}(K(t - c)^\nu) f(t, y(t), {}^c D^{\nu-1} y(c)) dc] \times$$
$$E_\nu(K(t - t_0)^\nu).$$

Integrating the above equation from $\bar{t}$ to t, after multiplying it by $E_\nu(K^*(t-t_0)^\nu)H^*$, we obtain

$$
\int_{\bar{t}}^t [E_\nu(K(t-t_0)^\nu)]^2 E_\nu(K^*(t-t_0)^\nu)H^* z(t_0)dt_0
$$

$$
= \int_{\bar{t}}^t E_\nu(K^*(t-t_0)^\nu)H^* H y(t) E_\nu(K(t-t_0)^\nu)dt_0
$$

$$
- \int_{\bar{t}}^t E_\nu(K^*(t-t_0)^\nu)H^* H \times \left(\int_{t_0}^t (t-c)^{\nu-1} E_{\nu,\nu}(K(t-c)^\nu) \times f(t,y(t),{}^c D^{\nu-1}y(c))dc \right)
$$

$$
\times E_\nu(K(t-t_0)^\nu)dt_0
$$

$$
= \int_{\bar{t}}^t E_\nu(K^*(t-t_0)^\nu)H^* H E_\nu(K(t-t_0)^\nu)dt_0 y(t)
$$

$$
- \int_{\bar{t}}^t (t-c)^{\nu-1} E_{\nu,\nu}(K(t-c)^\nu) \times f(t,y(t),{}^c D^{\nu-1}y(c))
$$

$$
\times \left(\int_{\bar{t}}^s E_\nu(K^*(t-t_0)^\nu)H^* H E_\nu(K(t-t_0)^\nu)dt_0 \right)dc.
$$

This implies that

$$
\int_{\bar{t}}^t [E_\nu(K(t-t_0)^\nu)]^2 E_\nu(K^*(t-t_0)^\nu)H^* z(t_0)dt_0
$$
$$
= Q(\bar{t},t)y(t) - \int_{\bar{t}}^t (t-c)^{\nu-1} E_{\nu,\nu}(K(t-c)^\nu) \times f(t,y(t),{}^c D^{\nu-1}y(c)) \times Q(\bar{t},c)dc. \tag{61}
$$

Now, in case the matrix $Q(\bar{t},t)$ is invertible, i.e., the linear system in (56) is observable, then from the last equation, we obtain

$$
y(t) = Q^{-1}(\bar{t},t) \int_{\bar{t}}^t [E_\nu(K(t-c)^\nu)]^2 E_\nu(K^*(t-c)^\nu)H^* z(c)ds
$$
$$
+ Q^{-1}(\bar{t},t) \int_{\bar{t}}^t (t-c)^{\nu-1} E_{\nu,\nu}(K(t-c)^\nu) \times f(t,y(t),{}^c D^{\nu-1}y(c)) \times Q(\bar{t},c)dc. \tag{62}
$$

Let

$$
G_1(t,\bar{t},c) = Q^{-1}(\bar{t},t)[E_\nu(K(t-c)^\nu)]^2 E_\nu(K^*(t-c)^\nu)H^*,
$$
$$
G_2(t,\bar{t},c) = Q^{-1}(\bar{t},t)E_{\nu,\nu}(K(t-c)^\nu)Q(\bar{t},c),
$$

we obtain

$$
y(t) = \int_{\bar{t}}^t G_1(t,\bar{t},c)z(c)ds + \int_{\bar{t}}^t (t-c)^{\nu-1}G_2(t,\bar{t},c) \times f(t,y(t),{}^c D^{\nu-1}y(c))dc. \tag{63}
$$

The above equation represents the relation between the state variable $y(t)$ and the system output $z(t)$ over the interval $[\bar{t},t]$; hence, the following is deduced.

Theorem 3. *The nonlinear system (56) is (a) observable globally at time t and (b) observable completely if the conditions given below are fulfilled.*

- *$det(Q(\bar{t},t)) \geq c$, for some positive c.*
- *One can associate a unique and continuous solution for any z of (62) in $[\bar{t},t]$, for some $\bar{t} < t$,*

  1. *The situation of an observable system at time t, and*
  2. *The situation of completely observable system $\forall t$.*

The time $\bar{t}$ in (63) is not necessarily fixed, so it can be replaced by t_0. After this change is incorporated and the resultant equation is substituted in (60), one obtains

$$
y(t_0) = [E_\nu(K(t-t_0)^\nu)]^{-1} \left[\int_{t_0}^t G_1(t,t_0,c)z(c)ds + \int_{t_0}^t (t-c)^{\nu-1}G_2(t,t_0,c) \times \right.
$$
$$
\left. f(t,y(t),{}^cD^{\nu-1}y(c))dc - \int_{t_0}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu)f(t,y(t),{}^cD^{\nu-1}y(c))dc \right].
$$
(64)

Let

$$
G_3(t,t_0,c) = [E_\nu(K(t-t_0)^\nu)]^{-1}G_1(t,t_0,c),
$$
$$
G_4(t,t_0,c) = [E_\nu(K(t-t_0)^\nu)]^{-1}[G_2(t,t_0,c) - E_{\nu,\nu}(K(t-c)^\nu)].
$$

After these assumptions are made, (63) reduces to

$$
y(t_0) = \int_{t_0}^t G_3(t,t_0,c)z(c)ds + \int_{t_0}^t (t-c)^{\nu-1}G_4(t,t_0,c)f(t,y(t),{}^cD^{\nu-1}y(c))ds.
$$
(65)

This equation shows that the same results are also valid if we replace (63) by (65) in Theorem 3 with a simple change of variable. Next, we apply Banach's contraction theorem to the nonlinear system given by

$$
\begin{cases}
{}^cD^\nu y(t) = Ky(t) + \epsilon f(t,y(t),{}^cD^{\nu-1}y(t)), \\
z(t) = Hy(t).
\end{cases}
$$
(66)

where ϵ is a positive constant. Assume that $\exists$ constants $\mathcal{K} > 0$ and $0 < \mathcal{L} < 1$, such that

$$
\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \leq \mathcal{K}\|u - \bar{u}\| + \mathcal{L}\|v - \bar{v}\|.
$$
(67)

Theorem 4. *The nonlinear system (66) is (a) observable globally at time t and (b) observable completely if the conditions given below are fulfilled.*

- $det(Q(\bar{t},t)) \geq c$, *for some* $c > 0$.
- *A positive constant* $\epsilon < \frac{\nu(1-\mathcal{L}I)}{(t-\bar{t})^\nu \ell(\bar{t},t)\mathcal{K}}$ *in* $[\bar{t},t]$, *for some* $\bar{t} < t$,

    1. *The situation of an observable system at time t, and*
    2. *The situation of completely observable system for all t.*

Proof. A general solution of the nonlinear system (66) with $y = y(t_0)$ as an initial condition, utilizing the Laplace transform, inverse Laplace transform, and the Mittag–Leffler matrix function is expressed as

$$
y(t) = E_\nu(K(t-t_0)^\nu)y(t_0) + \epsilon \int_{t_0}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu)f(t,y(t),{}^cD^{\nu-1}y(c))dc, \quad (68)
$$

it is solved for $y(t_0)$, obtaining

$$
y(t_0) = [E_\nu(K(t-t_0)^\nu)]^{-1}[y(t) - \epsilon \int_{t_0}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu)f(t,y(t).{}^cD^{\nu-1}y(c))dc], \quad (69)
$$

After some calculation, just like we obtained (62) from (60), the next equation is derived from (69) is given by

$$
y(t) = Q^{-1}(\bar{t},t) \int_{\bar{t}}^t [E_\nu(K(t-c)^\nu)]^2 E_\nu(K^*(t-c)^\nu)H^*z(c)ds
$$
$$
+ \epsilon Q^{-1}(\bar{t},t) \int_{\bar{t}}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu) \times f(t,y(t),{}^cD^{\nu-1}y(c)) \times Q(\bar{t},c)dc.
$$
(70)

Using (70) in Equation (69), we obtain

$$
\begin{aligned}
y(t_0) = {} & [E_\nu(K(t-t_0)^\nu)]^{-1}[Q^{-1}(\bar{t},t)\int_{\bar{t}}^t [E_\nu(K(t-c)^\nu)]^2 E_\nu(K^*(t-c)^\nu)H^*z(c)ds \\
& + \epsilon Q^{-1}(\bar{t},t)\int_{\bar{t}}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu) \times f(c,y(c),{}^cD^{\nu-1}y(c)) \times Q(\bar{t},c)dc \\
& - \epsilon \int_{t_0}^t (t-c)^{\nu-1}E_{\nu,\nu}(K(t-c)^\nu)f(c,y(c){}^cD^{\nu-1}y(c))dc].
\end{aligned}
\tag{71}
$$

It can be concluded from the last equation that the system (66) is observable. For this, it is sufficient to prove that $Q(\bar{\cdot}, \cdot)$ is invertible and there exists a unique solution of (71). If it is assumed that there exist two such solutions, say $y, \bar{y}, y \neq \bar{y}$ of (71) for a given z then utilizing (67), we have

$$
\begin{aligned}
|y(t_0) - \bar{y}(t_0)| \leq {} & \epsilon \Big| [E_\nu(K(t-t_0)^\nu)]^{-1} \Big| \Big| Q^{-1}(\bar{t},t) \Big| \int_{\bar{t}}^t (t-c)^{\nu-1}|E_{\nu,\nu}(K(t-c)^\nu)||Q(\bar{t},c)| \\
& \times [\Big| f(c,y(c),{}^cD^{\nu-1}y(c)) - f(c,\bar{y}(c),{}^cD^{\nu-1}\bar{y}(c)) \Big|]dc \\
& + \epsilon \Big| [E_\nu(K(t-t_0)^\nu)]^{-1} \Big| \int_{t_0}^t (t-c)^{\nu-1}|E_{\nu,\nu}(K(t-c)^\nu)| \\
& \times [\Big| f(c,y(c){}^cD^{\nu-1}y(c)) - f(c,\bar{y}(c){}^cD^{\nu-1}\bar{y}(c)) \Big|]dc \\
\leq {} & \frac{\epsilon}{\nu}\ell_1(\bar{t},t) \times (t-\bar{t})^\nu \times [\frac{\mathcal{K}}{1-\mathcal{L}I}]|y-\bar{y}| \\
& + \frac{\epsilon}{\nu}\ell_2(\bar{t},t) \times (t-\bar{t})^\nu \times [\frac{\mathcal{K}}{1-\mathcal{L}I}]|y-\bar{y}|
\end{aligned}
\tag{72}
$$

where

$$
\begin{aligned}
\ell_1(\bar{t},t) &= \max_{\bar{t}<t_0<s<t} \Big| [E_\nu(K(t-t_0)^\nu)]^{-1}Q^{-1}(\bar{t},t) \Big| \times |E_{\nu,\nu}(K(t-c)^\nu)Q(\bar{t},c)|, \\
\ell_2(\bar{t},t) &= \max_{\bar{t}<t_0<s<t} \Big| [E_\nu(K(t-t_0)^\nu)]^{-1}E_{\nu,\nu}(K(t-c)^\nu) \Big|, \\
|I| &= \Big| \int_{\bar{t}}^t dt \Big|.
\end{aligned}
\tag{73}
$$

Further simplification gives

$$
\|y(t_0) - \bar{y}(t_0)\| \leq \frac{\epsilon(t-\bar{t})^\nu \ell(\bar{t},t)\mathcal{K}}{\nu(1-\mathcal{L}I)}|y-\bar{y}|.
\tag{74}
$$

where $\ell(\bar{t},t) = \ell_1(\bar{t},t) + \ell_1(\bar{t},t)$. If

$$
\frac{\epsilon(t-\bar{t})^\nu \ell(\bar{t},t)\mathcal{K}}{\nu(1-\mathcal{L}I)} < 1,
\tag{75}
$$

then Equation (71) has a unique solution and system (66) is observable. □

5. Examples

In this section, we illustrate our theoretical results with the help of examples.

Example 1. *Given a nonlinear fractional order system*

$$
\begin{cases}
{}^cD^\nu y(t) = Ky(t) + Lu(t) + Mu(t-q) + f(t,y(t),{}^cD^{\nu-1}y(t)), t \geq 0, \\
y(0) = y_0, y'(0) = 0, \\
u(t) = \phi(t), -j \leq t \leq 0,
\end{cases}
\tag{76}
$$

where $j - 1 \leq v \leq j$, $t \in I$ *and*

$$K = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & 2 \\ 5 & 1 & 7 \end{bmatrix}, L = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, M = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, y(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

And the nonlinear function f

$$f(t, y(t), {}^c D^{v-1} y(t)) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2e^{t+1}(1+|y(t)|+|^c D^{v-1} y(t)|)} \end{bmatrix}.$$

Then, by the Mittage–Leffler matrix function

$$E_{v.v}(At^v) = \sum_{k=0}^{\infty} \frac{A^k t^{kv}}{\Gamma((k+1)v)}.$$

We obtain

$$E_{v.v}(K(d-c)^v)L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}, E_{v.v}(K(d-q-c)^v)M = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix},$$

where

$$l_1 = \frac{1}{\Gamma(v)} + \frac{4(d-c)^v}{\Gamma(2v)} + \frac{6(d-c)^{2v}}{\Gamma(3v)} + \dots,$$

$$l_2 = \frac{7(d-c)^v}{\Gamma(2v)} + \frac{85(d-c)^{2v}}{\Gamma(3v)} + \dots,$$

$$l_3 = \frac{2}{\Gamma(v)} + 19\frac{(d-c)^v}{\Gamma(2v)} + \frac{160(d-c)^{2v}}{\Gamma(3v)} + \dots,$$

$$s_1 = \frac{2}{\Gamma(v)} + \frac{7(d-q-c)^v}{\Gamma(2v)} + \frac{9(d-q-c)^{2v}}{\Gamma(3v)} + \dots,$$

$$s_2 = \frac{12(d-q-c)^v}{\Gamma(2v)} + \frac{143(d-q-c)^{2v}}{\Gamma(3v)} + \dots,$$

$$s_3 = \frac{3}{\Gamma(v)} + \frac{31(d-q-c)^v}{\Gamma(2v)} + \frac{264(d-q-c)^{2v}}{\Gamma(3v)} + \dots.$$

Now, the Grammian $W(t)$ *in both of the following cases for arbitrary* $d > 0$ *is nonsingular.*
   Case I. $0 \leq t \leq q$

$$
\begin{aligned}
W(t) &= \int_0^d (d-c)^{2(v-1)} \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} dc \\
&+ \int_{-q}^{d-q} (d-q-c)^{2(v-1)} \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}^T \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} dc, \\
&= \int_0^d (d-c)^{2(v-1)} \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & l_3^2 \end{bmatrix} dc \\
&+ \int_{-q}^{d-q} (d-q-c)^{2(v-1)} \begin{bmatrix} s_1^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & s_2^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & s_3^2 \end{bmatrix} dc.
\end{aligned}
$$

Case II. $t > q$

$$
\begin{aligned}
W(t) &= \int_0^{d-q} [(d-\mathrm{c})^{\nu-1}([l_1 \quad l_2 \quad l_3]^T + (d-q-\mathrm{c})^{\nu-1}[s_1 \quad s_2 \quad s_3]^T) \\
&\quad \times ((d-\mathrm{c})^{\nu-1}[l_1 \quad l_2 \quad l_3] + (d-q-\mathrm{c})^{\nu-1}[s_1 \quad s_2 \quad s_3])]d\mathrm{c} \\
&\quad + \int_{d-q}^d [(d-\mathrm{c})^{\nu-1}[l_1 \quad l_2 \quad l_3]^T [l_1 \quad l_2 \quad l_3]d\mathrm{c}, \\
&= \int_0^{d-q} [(d-\mathrm{c})^{2(\nu-1)} \begin{bmatrix} l_1^2 & l_1l_2 & l_1l_3 \\ l_1l_2 & l_2^2 & l_2l_3 \\ l_1l_3 & l_2l_3 & l_3^2 \end{bmatrix} \\
&\quad + ((d-\mathrm{c})(d-q-\mathrm{c}))^{\nu-1} \begin{bmatrix} l_1s_1 & l_1s_2 & l_1s_3 \\ l_2s_1 & l_2s_2 & l_2s_3 \\ l_3s_1 & l_3s_2 & l_3s_3 \end{bmatrix} \\
&\quad + ((d-\mathrm{c})(d-q-\mathrm{c}))^{\nu-1} \begin{bmatrix} l_1s_1 & l_2s_1 & l_3s_1 \\ l_1s_2 & l_2s_2 & l_3s_2 \\ l_1s_3 & l_2s_3 & l_3s_3 \end{bmatrix} \\
&\quad + (d-q-\mathrm{c}))^{2(\nu-1)} \begin{bmatrix} s_1^2 & s_1s_2 & s_1s_3 \\ s_1s_2 & s_2^2 & s_2s_3 \\ s_1s_3 & s_2s_3 & s_3^2 \end{bmatrix}]d\mathrm{c} \\
&\quad + \int_{d-q}^d (d-\mathrm{c})^{2(\nu-1)} \begin{bmatrix} l_1^2 & l_1l_2 & l_1l_3 \\ l_1l_2 & l_2^2 & l_2l_3 \\ l_1l_3 & l_2l_3 & l_3^2 \end{bmatrix} d\mathrm{c}.
\end{aligned}
$$

Since the nonlinear fractional differential function f satisfies the aforementioned hypothesis, and the Grammian matrices $W(t)$ in both cases are nonsingular, hence, by Theorem 1, the system (76) is controllable on I.

Example 2. *We here construct a fractional order system (56) as follows,*

$$
K = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & 2 \\ 5 & 1 & 7 \end{bmatrix}, f(t, y(t), {}^cD^{\nu-1}y(t)) = \begin{bmatrix} 0 \\ \frac{1}{2e^{t+1}(1+|y(t)|+|{}^cD^{\nu-1}y(t))} \end{bmatrix}, t \in [0,1].
$$

Clearly, f is continuous and for any $y_1, y_2, \bar{y}_1, \bar{y}_2 \in \mathbb{R}$ and $t \in [0,1]$

$$
|f(t, y_1, y_2) - f(t, \bar{y}_1, \bar{y}_2)| \leq \frac{1}{2e}(|y_1 - y_2| + |\bar{y}_1 - \bar{y}_2|), \mathcal{K} = \mathcal{L} = \frac{1}{2e}.
$$

In addition, the Mittag–Leffler function for the given K and $\nu = 3/2$ is given by

$$
\begin{aligned}
E_{3/2}(Kt^{3/2}) &= \sum_{i=0}^{\infty} \frac{K^i t^{\frac{3}{2}i}}{\Gamma(i\wp + 1)} \\
&= \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}
\end{aligned}
$$

and

$$E_{3/2}(K^* t^{3/2}) = \sum_{i=0}^{\infty} \frac{K^{*i} t^{\frac{3}{2}i}}{\Gamma(i\wp + 1)}$$

$$= \begin{bmatrix} a_{11}(t) & a_{21}(t) & a_{31}(t) \\ a_{12}(t) & a_{22}(t) & a_{32}(t) \\ a_{13}(t) & a_{23}(t) & a_{33}(t) \end{bmatrix}$$

where

$$a_{11}(t) = \frac{8}{3} \frac{t^{3/2}}{\sqrt{\pi}} - \frac{32}{21} \frac{t^{9/2}}{\sqrt{\pi}} + 1 + \dots,$$

$$a_{12}(t) = \frac{-4t^{3/2}}{\sqrt{\pi}} \frac{-10}{3} t^3 - \frac{3104}{945} \frac{t^{9/2}}{\sqrt{\pi}} + \dots,$$

$$a_{13}(t) = \frac{4}{3} \frac{t^{3/2}}{\sqrt{\pi}} + \frac{1}{2} t^3 - \frac{608}{945} \frac{t^{9/2}}{\sqrt{\pi}} + \dots,$$

$$a_{21}(t) = \frac{4t^{3/2}}{\sqrt{\pi}} + \frac{31}{6} t^3 + \frac{8032}{945} \frac{t^{9/2}}{\sqrt{\pi}} + \dots,$$

$$a_{22}(t) = \frac{20}{3} \frac{t^{3/2}}{\sqrt{\pi}} + 3 t^3 + \frac{256}{315} \frac{t^{9/2}}{\sqrt{\pi}} + 1 + \dots,$$

$$a_{23}(t) = \frac{8}{3} \frac{t^{3/2}}{\sqrt{\pi}} + \frac{9}{2} t^3 + \frac{8192}{945} \frac{t^{9/2}}{\sqrt{\pi}} + \dots,$$

$$a_{31}(t) = \frac{20}{3} \frac{t^{3/2}}{\sqrt{\pi}} + 8 t^3 + \frac{11744}{945} \frac{t^{9/2}}{\sqrt{\pi}} + \dots,$$

$$a_{32}(t) = \frac{4}{3} \frac{t^{3/2}}{\sqrt{\pi}} - \frac{1}{2} t^3 - \frac{3296}{945} \frac{t^{9/2}}{\sqrt{\pi}} + \dots,$$

$$a_{33}(t) = \frac{28}{3} \frac{t^{3/2}}{\sqrt{\pi}} + \frac{28}{3} t^3 + \frac{1984}{135} \frac{t^{9/2}}{\sqrt{\pi}} + 1 + \dots.$$

We have

$$Q(0,1) = \int_0^1 E_\nu(K^* t^\nu) H^* H E_\nu(Kt^\nu) dt$$

$$= \int_0^1 \begin{bmatrix} a_{11}(t) & a_{21}(t) & a_{31}(t) \\ a_{12}(t) & a_{22}(t) & a_{32}(t) \\ a_{13}(t) & a_{23}(t) & a_{33}(t) \end{bmatrix} H^* H \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} dt.$$

which is positive definite for suitable H, i.e., $Q^{-1}(0,1)$ exists. Hence, by Theorem 4, the system (56) is observable.

6. Conclusions

The present article explores the dynamical aspects and qualitative study of a nonlinear fractional-order system with input delay. We found a general solution to the proposed dynamical system in the form of an integral equation and proved controllability as well as observability for the linear case. The non-linear problem has been transformed into a fixed-point problem, and a set of necessary and sufficient conditions for the controllability within two different domains, $0 \le t \le q$ and $0 > t \ge q$, utilizing Schaefer's fixed-point theorem together with the theorem of Arzela–Ascoli, have been established. We also explored the observability of the nonlinear case of our proposed dynamical system in the absence of control input $u(t)$ using the Banach contraction mapping theorem. For authentication of the method, we put an example at the end of the paper.

In the study of dynamical systems, the observability property plays an important role. In sensor networking, it is used in the controller configuration of a closed-loop feedback system as well as to reduce the number of output sensors. Both the dynamical properties, controllability, and observability assist in "Actuator and Sensor" selection. This further suggests that with the minimum number of components, we can achieve maximum stability in the system and observe a less noisy system. From a mathematical point of view, the Gramain criterion is used to check the observability of a system. Observability Gramain informs us about the order. It means that using Gramain criteria, we obtain information from the most observable to the least observable state. In a dynamical system, some states "p" can be easily observed given a state "q", or in certain situations, some states possess less noise measurement as compared to other states [37,38].

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