axioms

Article

Solution to Integral Equation in an *O*-Complete Branciari *b*-Metric Spaces

Menaha Dhanraj [1], Arul Joseph Gnanaprakasam [1], Gunaseelan Mani [2], Ozgur Ege [3,*] and Manuel De la Sen [4,*]

[1] Department of Mathematics, Faculty of Engineering and Technology, College of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur, Kanchipuram 603203, India
[2] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, India
[3] Department of Mathematics, Ege University, Bornova, Izmir 35100, Turkey
[4] Institute of Research and Development of Processes, University of the Basque Country, 48940 Leioa, Spain
* Correspondence: ozgur.ege@ege.edu.tr (O.E.); manuel.delasen@ehu.eus (M.D.l.S.)

Abstract: In this paper, we prove fixed point theorem via orthogonal Geraghty type α-admissible contraction map in an orthogonal complete Branciari b-metric spaces context. An example is presented to strengthen our main result. We provided an application to find the existence and uniqueness of a solution to the Volterra integral equation. We have compared the approximate solution and exact solution numerically.

Keywords: orthogonal set; orthogonal preserving; orthogonal continuous; orthogonal Branciari b-metric space; orthogonal Geraghty type α-admissible contraction; fixed point theory

MSC: 47H10; 54H25

Citation: Dhanraj, M.; Gnanaprakasam, A.J.; Mani, G.; Ege, O.; De la Sen, M. Solution to Integral Equation in an *O*-Complete Branciari *b*-Metric Spaces. *Axioms* **2022**, *11*, 728. https://doi.org/10.3390/axioms11120728

Academic Editor: Hsien-Chung Wu

Received: 11 November 2022
Accepted: 12 December 2022
Published: 13 December 2022

1. Introduction

In 1922, Banach initiated the famous fixed point result called Banach contraction principle. This is one of the most important and fundamental results in complete metric space. The generalization of b-metric space was introduced by Bakhtin and Czerwik [1,2]. It is the most widely applied fixed point result in many branches of Mathematics and Sciences. A number of authors have defined contractive type mappings on b-metric spaces in many different directions to improve the results see [3–9]. The concept of Branciari metric space was initiated by Branciari [10] in 2000. In 2015, George et al. [11] introduced the generalization of Branciari b-metric spaces and proved some fixed point results. Thereafter, many authors initiated and extended the results of Branciari metric spaces and Branciari b-metric spaces and proved fixed point theorems in such spaces, see [12,13] and the references therein.

In 1973, one of the interesting results was given by Geraghty [14] in the setting of complete metric spaces by considering an auxiliary function. Several papers have been improved to the Geraghty contraction mapping type of fixed point theory in complete b-metric spaces refers to see [15–17]. Very recently, Samet et al. [18] initiated α-admissible mapping and also proved the fixed results for α-ψ-contractive mappings. Tunç et al. [19,20] proved the existence of solutions of some fixed results of non-linear 2D integral equations and also found the stability, integrability and boundedness of systems of integro-differential equations. In 2017, Eshaghi Gordji et al. [21] has established the main idea of the orthogonality and framework to our main finding of results. And also, Eshaghi Gordji and Habibi [22] has extended and proved some fixed point theorem in generalized O-metric spaces. For other results related to orthogonal concepts, see [23–30].

In this paper, we introduce the fixed point theorem for an O-generalized O-Geraghty type α-admissible (in short, O-G-α admissible) mapping on O-complete Branciari b-metric space and also we presented an example and application to integral equation by using our main results.

2. Preliminaries

We recall the following definitions and results will be needed in the sequel.

The notion of b-metric space was introduced by Bakhtin [1] in 1989, as follows:

Definition 1 ([1]). *Let $\mathcal{V}$ be a non-void set and a constant $\delta \geq 1$. A function $\mathfrak{w}_{\mathfrak{b}} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}^+$ is called a b-metric if the below axioms hold, for all $\mathfrak{s}, v, \vartheta \in \mathcal{V}$:*

(A_1) $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = 0$ *if and only if* $\mathfrak{s} = v$,
(A_2) $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{s})$,
(A_3) $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) \leq \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \vartheta) + \mathfrak{w}_{\mathfrak{b}}(\vartheta, v)]$.

The pair $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ is said to be a b-metric space with constant $\delta \geq 1$.

Example 1. *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a metric space and let $\gamma > 1$, $E \geq 0$ and $L > 0$. For $\mathfrak{s}, v \in \mathcal{V}$, set $P(\mathfrak{s}, v) = E\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) + L\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v)^{\gamma}$. Then $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ is a b-metric space with the parameter $\delta = 2^{\gamma-1}$ and not a metric space on $\mathcal{V}$.*

Branciari [10] initiated the concept of Branciari metric space as follows:

Definition 2 ([10]). *Let $\mathcal{V}$ be a non void set. A function $\mathfrak{w}_{\mathfrak{b}} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}^+$ is called a Branciari metric if the below axioms hold, for all $\mathfrak{s}, v \in \mathcal{V}$:*

$(B1)$ $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = 0$ *if and only if* $\mathfrak{s} = v$,
$(B2)$ $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{s})$,
$(B3)$ $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) \leq \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \omega) + \mathfrak{w}_{\mathfrak{b}}(\omega, \mu) + \mathfrak{w}_{\mathfrak{b}}(\mu, v)$ *for all distinct points* $\omega, \mu \in \mathcal{V}/\{\mathfrak{s}, v\}$.

The pair $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ is said to be a Branciari metric space (in short, BMS).

The notion of Branciari b-metric space was initiated by George [11] as follows:

Definition 3 ([11]). *Let $\mathcal{V}$ be a non void set and a constant $\delta \geq 1$. A function $\mathfrak{w}_{\mathfrak{b}} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}^+$ is said to be a Branciari b-metric if the below axioms hold, for all $\mathfrak{s}, v \in \mathcal{V}$:*

$(B1)$ $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = 0$ *if and only if* $\mathfrak{s} = v$,
$(B2)$ $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{s})$,
$(B3)$ $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) \leq \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \omega) + \mathfrak{w}_{\mathfrak{b}}(\omega, \mu) + \mathfrak{w}_{\mathfrak{b}}(\mu, v)]$ *for all distinct points* $\omega, \mu \in \mathcal{V}/\{\mathfrak{s}, v\}$.

Then, the pair $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ is said to be a Branciari b-metric space (in short, B_bMS).

The following proposition was proved by Erhan [13].

Proposition 1 ([13]). *Let $\{\mathfrak{s}_{\hat{\imath}}\}$ be a Cauchy sequence in a BMS $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ such that $\lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, v) = 0$, where $\mathfrak{s} \in \mathcal{V}$. Then*

$$\lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, v) = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v), \quad for\ all\ \ v \in \mathcal{V}.$$

In particular, the $\{\mathfrak{s}_{\hat{\imath}}\}$ does not converge to v if $\mathfrak{s} \neq v$.

Geraghty [14] introduced the Geraghty type contraction mappings who have extended the results for Banach contraction theorem by the property defined as the set of all functions $\alpha : \mathbb{R}^+ \to [0, 1)$ satisfying the condition:

$$\lim_{\hat{\imath}\to\infty} \alpha(\varrho_{\hat{\imath}}) = 1 \text{ implies } \lim_{\hat{\imath}\to\infty} \varrho_{\hat{\imath}} = 0.$$

Theorem 1 ([14]). *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a complete metric space and a mapping $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ is satisfying the following:*

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \gamma(\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v))\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v), \ \ \forall \mathfrak{s}, v \in \mathcal{V},$$

where $\gamma \in \mathcal{F}_\delta$. Then $\mathfrak{P}$ has a UFP *(Briefly unique fixed point).*

Dukic et al. [15] reconsidered the above Theorem 1 for the framework of b-metric spaces in 2011.

Consider a b-metric space $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ with constant $\delta \geq 1$ and the set $\mathcal{F}_\delta$ of all functions $\gamma : \mathbb{R}^+ \to [0, \frac{1}{\delta})$, satisfying the condition:

$$\lim_{\hat{\imath}\to\infty} \gamma(\varrho_{\hat{\imath}}) = \frac{1}{\delta} \ \Rightarrow \ \lim_{\hat{\imath}\to\infty} \varrho_{\hat{\imath}} = 0. \tag{1}$$

Theorem 2 ([15])**.** *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a complete b-metric space with constant $\delta \geq 1$ and $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ be a self-map. Suppose that there exists $\gamma \in \mathcal{F}_\delta$ such that,*

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \gamma(\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v))\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v)$$

for all $\mathfrak{s}, v \in \mathcal{V}$. Then $\mathfrak{P}$ has a UFP *$\mathfrak{s}_0 \in \mathcal{V}$.*

Remark 1 ([15])**.** *If we replace BMS by B_bMS, then proposition (1) holds.*

Definition 4 ([15])**.** *A mapping $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ is called α-admissible if for all $\mathfrak{s}, v \in \mathcal{V}$ we have*

$$\alpha(\mathfrak{s}, v) \geq 1 \Rightarrow \alpha(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \geq 1,$$

where $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ is a given function.

Definition 5 ([15])**.** *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a B_bMS with a parameter $\delta \geq 1$ and let $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ and $\gamma \in \mathcal{F}_\delta$ be two functions. A generalized Geraghty type α-admissible (in short, G-α-admissible) contractive mapping $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ is of type-(1) if it is α-admissible and the following condition holds*

$$\alpha(\mathfrak{s}, v)\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \gamma(\mathcal{K}(\mathfrak{s}, v))\mathcal{K}(\mathfrak{s}, v), \ \ \forall \mathfrak{s}, v \in \mathcal{V},$$

where

$$\mathcal{K}(\mathfrak{s}, v) = \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{P}\mathfrak{s}), \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{P}v)\}.$$

Theorem 3 ([15])**.** *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a B_bMS with a parameter $\delta \geq 1$ and let $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ and $\gamma \in \mathcal{F}_\delta$ be two functions. Let $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ be an α-admissible mapping satisfying*

$$\alpha(\mathfrak{s}, v)\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \gamma(\mathcal{K}(\mathfrak{s}, v))\mathcal{K}(\mathfrak{s}, v), \ \ \forall \mathfrak{s}, v \in \mathcal{V}, \tag{2}$$

where

$$\mathcal{K}(\mathfrak{s}, v) = \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{P}\mathfrak{s}), \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{P}v)\}.$$

Then $\mathfrak{P}$ has a UFP.

We next define the Geraghty type mappings of another class on B_bMS.

Definition 6 ([15])**.** *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a B_bMS with a parameter $\delta \geq 1$ and let $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ and $\gamma \in \mathcal{F}_\delta$ be two functions. Then a G-α-admissible contractive mapping $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ is of type-(2) if it is α-admissible and the following condition holds*

$$\alpha(\mathfrak{s}, v)\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \gamma(\mathcal{N}(\mathfrak{s}, v))\mathcal{N}(\mathfrak{s}, v), \ \ \forall \mathfrak{s}, v \in \mathcal{V},$$

where

$$\mathcal{N}(\mathfrak{s}, v) = \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v), \frac{1}{2}[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{P}\mathfrak{s}), \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{P}v)]\}.$$

Remark 2 ([15]). *For all $\mathfrak{s}, \upsilon \in \mathcal{V}$ the relation $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \upsilon) \leq \mathcal{N}(\mathfrak{s}, \upsilon)) \leq \mathcal{K}(\mathfrak{s}, \upsilon)$ holds.*

Theorem 4 ([15]). *Let $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ be a B_bMS with a parameter $\delta \geq 1$ and let $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ and $\gamma \in \mathcal{F}_\delta$ be two functions. Let $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ be an α-admissible mapping satisfying*

$$\alpha(\mathfrak{s}, \upsilon)\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}\upsilon) \leq \gamma(\mathcal{N}(\mathfrak{s}, \upsilon))\mathcal{N}(\mathfrak{s}, \upsilon), \ \ \forall \mathfrak{s}, \upsilon \in \mathcal{V},$$

where

$$\mathcal{N}(\mathfrak{s}, \upsilon) = \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \upsilon), \frac{1}{2}[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{P}\mathfrak{s}), \mathfrak{w}_{\mathfrak{b}}(\upsilon, \mathfrak{P}\upsilon)]\}.$$

Then, $\mathfrak{P}$ has a UFP.

Gordji et al. [21] proposed an orthogonal sets and generalized Banach fixed point theorems in 2017. He gave the following definition in [21].

Definition 7 ([21]). *Let $\mathcal{V} \neq \phi$ and $\perp \subseteq \mathcal{V} \times \mathcal{V}$ be a binary relation. If $\perp$ satisfies the condition.*

$$\exists \ \mathfrak{s}_0 \in \mathcal{V} : (\forall \mathfrak{s} \in \mathcal{V}, \mathfrak{s} \perp \mathfrak{s}_0) \quad or \quad (\forall \mathfrak{s} \in \mathcal{V}, \mathfrak{s}_0 \perp \mathfrak{s}),$$

then, it is said to be an orthogonal set (in short, O-set). We denote this O-set by $(\mathcal{V}, \perp)$.

Example 2 ([21]). *Let $\mathcal{V} = \mathbb{R}^+$ and define $\mathfrak{s} \perp \upsilon$ if $\mathfrak{s}\upsilon \in \{\mathfrak{s}, \upsilon\}$. Then, setting $\mathfrak{s}_0 = 0$ or $\mathfrak{s}_0 = 1$, $(\mathcal{V}, \perp)$ is an orthogonal-set.*

Definition 8 ([21]). *Let $(\mathcal{V}, \perp)$ be an O-set. A sequence $\{\mathfrak{s}_{\hat{\imath}}\}$ is said to be an orthogonal sequence (in short, O-sequence) if*

$$(\forall \hat{\imath} \in \mathbb{N}, \mathfrak{s}_{\hat{\imath}} \perp \mathfrak{s}_{\hat{\imath}+1}) \quad or \quad (\forall \hat{\imath} \in \mathbb{N}, \mathfrak{s}_{\hat{\imath}+1} \perp \mathfrak{s}_{\hat{\imath}}).$$

We first introduce the concept of an O-contractions of Geraghty type mapping.

Definition 9. *The triplet $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ is called an O-B_bMS if $(\mathcal{V}, \perp)$ is an O-set and $(\mathcal{V}, \mathfrak{w}_{\mathfrak{b}})$ is Branciari b-metric $\mathfrak{w}_{\mathfrak{b}}$ on $\mathcal{V}$ with a real number $\delta \geq 1$.*

Definition 10. *Let $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ be an O-complete B_bMS. A mapping $\boldsymbol{\mathfrak{P}} : \mathcal{V} \to \mathcal{V}$ is called orthogonal continuous at $\mathfrak{s} \in \mathcal{V}$ if for each O-sequence $\{\mathfrak{s}_{\hat{\imath}}\}$ in $\mathcal{V}$ with $\mathfrak{s}_{\hat{\imath}} \to \mathfrak{s}$, we have $\boldsymbol{\mathfrak{P}}(\mathfrak{s}_{\hat{\imath}}) \to \boldsymbol{\mathfrak{P}}(\mathfrak{s})$.*

Definition 11. *Let $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ be an O-complete B_bMS. Then, $\mathcal{V}$ is called orthogonal complete if each Cauchy O-sequence is convergent.*

Definition 12. *Let $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ be an O-complete B_bMS. A mapping $\boldsymbol{\mathfrak{P}} : \mathcal{V} \to \mathcal{V}$ is called $\perp$-preserving if $\boldsymbol{\mathfrak{P}}\mathfrak{s} \perp \boldsymbol{\mathfrak{P}}\upsilon$ whenever $\mathfrak{s} \perp \upsilon$ for all $\mathfrak{s}, \upsilon \in \mathcal{V}$.*

Here, we discuss orthogonal Geraghty type-contraction on O-complete B_bMS. Let

$$\mathfrak{s} \perp \upsilon \ \text{ or } \ \upsilon \perp \mathfrak{s}, \ \mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}\upsilon) > 0 \implies \mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}\upsilon) \leq \zeta\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \upsilon), \ \ \forall \mathfrak{s}, \upsilon \in \mathcal{V},$$

where $0 < \zeta < \frac{1}{\delta}$.

Definition 13. *Let $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ be an O-complete B_bMS with an O-element $\mathfrak{s}_0$ and parameter $\delta \geq 1$. Let $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ and $\gamma \in \mathcal{F}_\delta$ be two functions. A generalized $O - G - \alpha$-admissible contractive mapping $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ is of type-(1) if it is α-admissible and holds*

$$\begin{aligned} &\mathfrak{s} \perp \upsilon \ \text{ or } \ \upsilon \perp \mathfrak{s}, \ \mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}\upsilon) > 0 \\ &\qquad \Rightarrow \alpha(\mathfrak{s}, \upsilon)\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}\upsilon) \leq \gamma(\mathcal{K}(\mathfrak{s}, \upsilon))\mathcal{K}(\mathfrak{s}, \upsilon), \ \ \forall \mathfrak{s}, \upsilon \in \mathcal{V}, \end{aligned}$$

where

$$\mathcal{K}(\mathfrak{s}, v) = \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{P}\mathfrak{s}), \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{P}v)\}.$$

3. Main Results

In this section, we use $O - G - \alpha$ admissible contraction map to demonstrate the unique fixed point results in O-complete B_bMS. The advantages of our main results are as follows:

1. The following fixed point theorem of self mapping which is defined on orthogonal b-metric spaces are given by using extensions of orthogonal Geraghty-alpha—contractions.
2. To find the existence and uniqueness solution of the integral equation based on our main results.
3. We are comparing numerical difference between an approximation solution and an exact solution.

Theorem 5. *Let $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ be an O-complete B_bMS with an O-element $\mathfrak{s}_0$ and parameter $\delta \geq 1$. Let the two given functions $\alpha : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ and $\gamma \in \mathcal{F}_\delta$. Let $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ be an α-admissible mapping satisfying:*

(i) $\mathfrak{P}$ is $\perp$- preserving.
(ii) $\mathfrak{P}$ is $O - G - \alpha$- admissible contraction.
(iii) $\mathfrak{P}$ is O-continuous.

Then, $\mathfrak{P}$ has a UFP.

Proof. Consider $(\mathcal{V}, \perp)$ is an orthogonal set, there exists

$$\mathfrak{s}_0 \in \mathcal{V} : \forall \mathfrak{s} \in \mathcal{V},\ \mathfrak{s} \perp \mathfrak{s}_0 \quad (or) \quad \forall \mathfrak{s} \in \mathcal{V},\ \mathfrak{s}_0 \perp \mathfrak{s}.$$

It follows that $\mathfrak{s}_0 \perp \mathfrak{P}\mathfrak{s}_0$ or $\mathfrak{P}\mathfrak{s}_0 \perp \mathfrak{s}_0$. Let

$$\mathfrak{s}_1 = \mathfrak{P}\mathfrak{s}_0,\ \mathfrak{s}_2 = \mathfrak{P}\mathfrak{s}_1 = \mathfrak{P}^2\mathfrak{s}_0 \cdots \mathfrak{s}_{\hat{\imath}} = \mathfrak{P}\mathfrak{s}_{\hat{\imath}-1} = \mathfrak{P}^{\hat{\imath}}\mathfrak{s}_0 \quad \forall \hat{\imath} \in \mathbb{N}.$$

For any $\mathfrak{s}_0 \in \mathcal{V}$, set $\mathfrak{s}_{\hat{\imath}} = \mathfrak{P}\mathfrak{s}_{\hat{\imath}-1}$. Now, we assume that the below cases:

(a) If $\exists\ \hat{\imath} \in \mathbb{N} \cup \{0\}$ such that $\mathfrak{s}_{\hat{\imath}} = \mathfrak{s}_{\hat{\imath}+1}$ then $\mathfrak{P}\mathfrak{s}_{\hat{\imath}} = \mathfrak{s}_{\hat{\imath}}$. It is clear that $\mathfrak{s}_{\hat{\imath}}$ is a fixed point of $\mathfrak{P}$. Hence, the proof is complete.
(b) If $\mathfrak{s}_{\hat{\imath}} \neq \mathfrak{s}_{\hat{\imath}+1}$, for any $\hat{\imath} \in \mathbb{N} \cup \{0\}$, then we have $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}}) > 0$, for each $\hat{\imath} \in \mathbb{N}$.

Since $\mathfrak{P}$ is $\perp$-preserving, we have

$$\mathfrak{s}_{\hat{\imath}} \perp \mathfrak{s}_{\hat{\imath}+1} \quad \text{(or)} \quad \mathfrak{s}_{\hat{\imath}+1} \perp \mathfrak{s}_{\hat{\imath}}.$$

This implies that $\{\mathfrak{s}_{\hat{\imath}}\}$ is an O-sequence. Since $\mathfrak{P}$ is α-admissible, from $\alpha(\mathfrak{s}_0, \mathfrak{P}\mathfrak{s}_0) \geq 1$ we have

$$\alpha(\mathfrak{s}_0, \mathfrak{s}_1) = \alpha(\mathfrak{s}_0, \mathfrak{P}\mathfrak{s}_0) \geq 1 \Rightarrow \alpha(\mathfrak{P}\mathfrak{s}_0, \mathfrak{P}\mathfrak{s}_1) = \alpha(\mathfrak{s}_1, \mathfrak{s}_2) \geq 1,$$

and inductively,

$$\alpha(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) \geq 1, \quad \forall \hat{\imath} \in \mathbb{N}. \tag{3}$$

Also, from the condition $\alpha(\mathfrak{s}_0, T^2\mathfrak{s}_0) \geq 1$ we have,

$$\alpha(\mathfrak{s}_0, \mathfrak{s}_2) = \alpha(\mathfrak{s}_0, \mathfrak{P}^2\mathfrak{s}_0) \geq 1 \Rightarrow \alpha(\mathfrak{P}\mathfrak{s}_0, \mathfrak{P}\mathfrak{s}_2) = \alpha(\mathfrak{s}_1, \mathfrak{s}_3) \geq 1, \tag{4}$$

and hence,

$$\alpha(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+2}) \geq 1, \quad \forall \hat{\imath} \in \mathbb{N}.$$

Define the O-sequences $\{\mathfrak{q}_{\hat{\imath}}\}$ and $\{\kappa_{\hat{\imath}}\}$ as

$$\mathfrak{q}_{\hat{\imath}} = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}), \quad \kappa_{\hat{\imath}} = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}).$$

We will prove that both the O-sequence $\{\mathfrak{q}_{\hat{\imath}}\}$ and $\{\kappa_{\hat{\imath}}\}$ converge to 0, that is,

$$\lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) = \lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}) = 0.$$

Regarding (3) and the fact that $0 \leq \gamma(\varrho) < \frac{1}{\delta}$, the contractive condition (2) with $\mathfrak{s} = \mathfrak{s}_{\hat{\imath}}$ and $v = \mathfrak{s}_{\hat{\imath}+1}$ becomes

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) &= \mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}), \mathfrak{P}(\mathfrak{s}_{\hat{\imath}_1}, \mathfrak{s}_{\hat{\imath}}) \\
&\leq \alpha(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}})\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) \\
&\leq \gamma(\omega(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}))\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) \\
&\leq \frac{1}{\delta}\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) \ \ \forall \hat{\imath} \geq 1,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) &= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{P}\mathfrak{s}_{\hat{\imath}-1}), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{P}\mathfrak{s}_{\hat{\imath}})\} \\
&= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1})\} \\
&= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}), \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1})\}.
\end{aligned}$$

Suppose that $\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1})$ for some $\hat{\imath} \geq 1$. Then, we have

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) &\leq \gamma(\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1})\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1})) \\
&< \frac{1}{\delta}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) \ \ \forall \hat{\imath} \in \mathbb{N},
\end{aligned}$$

which is a contradiction. Therefore, $\forall \hat{\imath} \geq 1$, $\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}})$. In this case, the inequality (5) implies

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) &\leq \gamma(\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}})\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}})) \\
&< \frac{1}{\delta}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) \\
&< \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) \ \ \forall \hat{\imath} \geq 1.
\end{aligned} \tag{6}$$

In other words, the positive and decreasing O-sequence $\{\mathfrak{q}_{\hat{\imath}}\} = \{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}})\}$ is O-convergent to some $\mathfrak{w}_{\mathfrak{b}} \geq 0$. Taking limit in (6) we get,

$$\mathfrak{w}_{\mathfrak{b}} = \lim_{\hat{\imath}\to\infty} \mathfrak{q}_{\hat{\imath}+1} \leq \lim_{\hat{\imath}\to\infty} \gamma(\mathfrak{q}_{\hat{\imath}})\mathfrak{q}_{\hat{\imath}} = \mathfrak{w}_{\mathfrak{b}} \lim_{\hat{\imath}\to\infty} \gamma(\mathfrak{q}_{\hat{\imath}}) \leq \frac{1}{\delta}\mathfrak{w}_{\mathfrak{b}}.$$

This implies $\lim_{\hat{\imath}\to\infty} \gamma(\mathfrak{q}_{\hat{\imath}}) = \frac{1}{\delta}$ and hence, by (1),

$$\lim_{\hat{\imath}\to\infty} \mathfrak{q}_{\hat{\imath}} = \lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}}) = 0. \tag{7}$$

On the other hand, we observe that repeated application of (6) leads to

$$\mathfrak{q}_{\hat{\imath}+1} < \frac{1}{\delta}\mathfrak{q}_{\hat{\imath}} < \frac{1}{\delta^2}\mathfrak{q}_{\hat{\imath}-1} < \cdots < \frac{1}{\delta^{\hat{\imath}+1}}\mathfrak{q}_0. \tag{8}$$

Now, taking into account (4), we substitute $\mathfrak{s} = \mathfrak{s}_{\hat{\imath}-1}$ and $\mathfrak{s} = \mathfrak{s}_{\hat{\imath}+1}$ in (2). This yields

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+2}) &= \mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}), \mathfrak{P}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}) \\
&\leq \alpha(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1})\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}) \\
&\leq \gamma(\omega(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}))\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}) \\
&\leq \frac{1}{\delta}\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1}, \mathfrak{s}_{\hat{\imath}+1}) \ \ \forall \hat{\imath} \in \mathbb{N},
\end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}+1}) &= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}+1}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{P}\mathfrak{s}_{\hat{\imath}-1}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1},\mathfrak{P}\mathfrak{s}_{\hat{\imath}+1})\},\\ &= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}+1}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1},\mathfrak{s}_{\hat{\imath}+2})\}.\end{aligned}$$

Regarding (6), the maximum $\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}+1})$ is either $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}+1})$ or $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}})$, that is, either $\kappa_{\hat{\imath}}$ or $\mathfrak{q}_{\hat{\imath}}$. From the inequality (9) we have,

$$\kappa_{\hat{\imath}+1} = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{s}_{\hat{\imath}+2}) < \frac{1}{\delta}\omega(\kappa_{\hat{\imath}}) = \frac{1}{\delta}\max\{\kappa_{\hat{\imath}},\mathfrak{q}_{\hat{\imath}}\} \ \ \forall \hat{\imath}\in\mathbb{N}. \tag{10}$$

In addition, from (6) we have,

$$\mathfrak{q}_{\hat{\imath}+1} < \mathfrak{q}_{\hat{\imath}} \leq \max\{\kappa_{\hat{\imath}},\mathfrak{q}_{\hat{\imath}}\}.$$

We deduce that,

$$\max\{\kappa_{\hat{\imath}+1},\mathfrak{q}_{\hat{\imath}+1}\} \leq \max\{\kappa_{\hat{\imath}},\mathfrak{q}_{\hat{\imath}}\} \ \ \forall \hat{\imath}\geq 1.$$

that is, the O-sequence $\max\{\kappa_{\hat{\imath}},\mathfrak{q}_{\hat{\imath}}\}$ is non increasing and hence, it O-converges to some $\mathrm{j}\geq 0$. Choose that $\mathrm{j}>0$. Taking limits we get

$$\begin{aligned}\mathrm{j} &= \lim_{\hat{\imath}\to\infty}\max\{\kappa_{\hat{\imath}},\mathfrak{q}_{\hat{\imath}}\}\\ &= \lim_{\hat{\imath}\to\infty}\kappa_{\hat{\imath}}.\end{aligned}$$

Alternatively, letting $\hat{\imath}\to\infty$ in (10) we get

$$\mathrm{j} = \lim_{\hat{\imath}\to\infty}\kappa_{\hat{\imath}+1} < \lim_{\hat{\imath}\to\infty}\max\{\kappa_{\hat{\imath}},\mathfrak{q}_{\hat{\imath}}\} = \mathrm{j},$$

which is a contradiction. Hence $\mathrm{j}=0$, and then we have,

$$\lim_{\hat{\imath}\to\infty}\kappa_{\hat{\imath}} = \lim_{\hat{\imath}\to\infty}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}+1}) = 0. \tag{11}$$

Next, we will prove that $\mathfrak{s}_{\hat{\imath}}\neq\mathfrak{s}_{\hat{\jmath}}$ for all $\hat{\imath}\neq\hat{\jmath}$. Assume that $\mathfrak{s}_{\hat{\imath}}=\mathfrak{s}_{\hat{\jmath}}$ for every $\hat{\imath},\hat{\jmath}\in\mathbb{N}$ with $\hat{\imath}\neq\hat{\jmath}$, we have $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{s}_{\hat{\imath}+1})>0$ for each $\hat{\imath}\in\mathbb{N}$. Without loss of generality, we may take $\hat{\jmath}>\hat{\imath}+1$. The assumption $\mathfrak{s}_{\hat{\imath}}=\mathfrak{s}_{\hat{\jmath}}$ implies

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{P}\mathfrak{s}_{\hat{\imath}}) = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}},\mathfrak{P}\mathfrak{s}_{\hat{\jmath}}).$$

Using the inequality (5) we have,

$$\begin{aligned}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{s}_{\hat{\imath}+1}) =&\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{P}\mathfrak{s}_{\hat{\imath}}) = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}},\mathfrak{P}\mathfrak{s}_{\hat{\jmath}})\\ =&\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{P}\mathfrak{s}_{\hat{\jmath}}) \leq \alpha(\mathfrak{s}_{\hat{\jmath}-1}\mathfrak{s}_{\hat{\jmath}})\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{P}\mathfrak{s}_{\hat{\jmath}})\\ \leq&\gamma(\mathcal{K}(\mathfrak{s}_{\hat{\imath}-1},\mathfrak{s}_{\hat{\imath}}))\mathcal{K}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}})\\ <&\frac{1}{\delta}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}) &= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{P}\mathfrak{s}_{\hat{\jmath}-1}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}},\mathfrak{P}\mathfrak{s}_{\hat{\jmath}})\}\\ &= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}},\mathfrak{s}_{\hat{\jmath}+1})\}\\ &= \max\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}),\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}},\mathfrak{s}_{\hat{\jmath}+1})\} = \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}).\end{aligned}$$

Then, we have

$$\begin{aligned}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}},\mathfrak{s}_{\hat{\jmath}+1}) &\leq \gamma(\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}})\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}))\\ &< \frac{1}{\delta}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}})\\ &< \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1},\mathfrak{s}_{\hat{\jmath}}) \ \ \forall\hat{\jmath}>\hat{\imath}+1.\end{aligned}$$

Continuing the process we conclude,

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}}, \mathfrak{s}_{\hat{\jmath}+1}) < \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1}, \mathfrak{s}_{\hat{\jmath}}) < \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\jmath}-1}, \mathfrak{s}_{\hat{\jmath}}) < \cdots < \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}),$$

which contradicts the assumption $\mathfrak{s}_{\hat{\imath}} = \mathfrak{s}_{\hat{\jmath}}$ for some $\hat{\imath} \neq \hat{\jmath}$. Hence $\mathfrak{s}_{\hat{\imath}} \neq \mathfrak{s}_{\hat{\jmath}}$ for all $\hat{\imath} \neq \hat{\jmath}$.

Now to prove that $\{\mathfrak{s}_{\hat{\imath}}\}$ is a O-Cauchy sequence, that is,

$$\lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+\zeta}) = 0, \ \ \forall \zeta \in \mathbb{N}. \tag{12}$$

Notice that (12) is satisfied for $\zeta = 1$ and $\zeta = 2$ due to (7) and (11). Hence, we choose $\zeta \leq 3$. Now we consider two cases, for $\zeta \in \mathbb{N}$.

Case 1. Consider $\zeta = 2\hat{\jmath} + 1$ where $\hat{\jmath} \geq 1$. We have $\mathfrak{s}_{\mathfrak{j}} \neq \mathfrak{s}_{\delta}$ *for all* $\mathfrak{j} \neq \delta$ and $\mathfrak{s}_{\mathfrak{j}} \neq \mathfrak{s}_{\mathfrak{j}+1}$ *for all* $\mathfrak{j} \geq 0$. We use the inequality $(B3)$ in (3), hence,

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \zeta_{\mathfrak{n}+\mathfrak{k}}) =& \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}+1}) \leq \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}+1})] \\
\leq& \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2})] \\
&+ \delta^2[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+3}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+3}, \mathfrak{s}_{\hat{\imath}+4}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+4}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}+1})] \\
&\vdots \\
\leq& \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2})] + \delta^2[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+3}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+3}, \mathfrak{s}_{\hat{\imath}+4})] \\
&+ \delta^3[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+4}, \mathfrak{s}_{\hat{\imath}+5}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+5}, \mathfrak{s}_{\hat{\imath}+6})] + \ldots + \delta^{\hat{\jmath}+1}[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}+1})] \\
\leq& \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \delta^2[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2}) + \delta^3[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+3}) + \cdots \\
&+ \delta^{\hat{\imath}+2\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}+1}).
\end{aligned}$$

Then, by the inequality (8) we conclude

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+\zeta}) \leq& \frac{1}{\delta^{\hat{\imath}+1}}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0, \mathfrak{s}_1) + \frac{1}{\delta^{\hat{\imath}}}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0, \mathfrak{s}_1) + \cdots + \frac{1}{\delta^{\hat{\imath}+2\hat{\jmath}}}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0, \mathfrak{s}_1) \\
=& \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0, \mathfrak{s}_1)\Big[\sum_{\zeta=0}^{\hat{\imath}+2\hat{\jmath}} \frac{1}{\delta^{\hat{\imath}}} - \sum_{\zeta=0}^{\hat{\imath}-2} \frac{1}{\delta^{\hat{\imath}}}\Big] \\
=& \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0, \mathfrak{s}_1)\Big[\frac{\delta^{\hat{\imath}+2\hat{\jmath}+1} - 1}{\delta^{\hat{\imath}+2\hat{\jmath}(\delta-1)}} - \frac{\delta^{\hat{\imath}-1} - 1}{\delta^{\hat{\imath}-2(\delta-1)}}\Big].
\end{aligned}$$

Letting $\hat{\imath} \to \infty$ in the above inequality, we get

$$0 \leq \lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+\zeta}) \leq \lim_{\hat{\imath}\to\infty} \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0, \mathfrak{s}_1)\Big[\frac{\delta^{\hat{\imath}+2\hat{\jmath}+1} - 1}{\delta^{\hat{\imath}+2\hat{\jmath}(\delta-1)}} - \frac{\delta^{\hat{\imath}-1} - 1}{\delta^{\hat{\imath}-2(\delta-1)}}\Big] = 0.$$

Case 2. Consider $\zeta = 2\hat{\jmath}$ where $\hat{\jmath} \geq 2$. Again, using the inequality (B3) in Definition 3, we obtain

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \zeta_{\mathfrak{n}+\mathfrak{k}}) =& \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}) \leq \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}})] \\
\leq& \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2})] \\
&+ \delta^2[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+3}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+3}, \mathfrak{s}_{\hat{\imath}+4}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+4}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}})] \\
\leq& \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2})] + \delta^2[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+3}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+3}, \mathfrak{s}_{\hat{\imath}+4})] \\
&+ \cdots + \delta^{\hat{\jmath}-1}[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-4}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-3}) + \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-3}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2}) \\
&+ \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}})] \\
\leq& \delta[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}}, \mathfrak{s}_{\hat{\imath}+1}) + \delta^2[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1}, \mathfrak{s}_{\hat{\imath}+2}) + \delta^3[\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2}, \mathfrak{s}_{\hat{\imath}+3}) + \cdots \\
&+ \delta^{\hat{\imath}+2\hat{\jmath}-3}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-3}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2}) + \delta^{\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2}, \mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}).
\end{aligned}$$

By the inequality in (8), we have

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{s}_{\hat{\imath}+\zeta}) &\le \frac{1}{\delta^{\hat{\imath}+1}}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0,\mathfrak{s}_1)+\frac{1}{\delta^{\hat{\imath}}}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0,\mathfrak{s}_1)+\cdots+\frac{1}{\delta^{\hat{\imath}+2\hat{\jmath}-2}}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0,\mathfrak{s}_1) \\
&\quad + \delta^{\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2},\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}) \\
&= \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0,\mathfrak{s}_1)\Big[\sum_{\zeta=0}^{\hat{\imath}+2\hat{\jmath}_2}\frac{1}{\delta^{\zeta}}-\sum_{\zeta=0}^{\hat{\imath}-2}\frac{1}{\delta^{\zeta}}\Big]+\delta^{\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2},\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}) \\
&= \mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0,\mathfrak{s}_1)\Big[\frac{\delta^{\hat{\imath}+2\hat{\jmath}+1}-1}{\delta^{\hat{\imath}+2\hat{\jmath}(\delta-1)}}-\frac{\delta^{\hat{\imath}-1}-1}{\delta^{\hat{\imath}-2(\delta-1)}}\Big]+\delta^{\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2},\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}}).
\end{aligned} \tag{13}$$

From (11) we have $\lim_{\hat{\imath}\to\infty}\delta^{\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2},\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}})=0$ and hence, using (13) we obtain

$$\begin{aligned}
0 &\le \lim_{\hat{\imath}\to\infty}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{s}_{\hat{\imath}+\zeta}) \\
&\le \lim_{\hat{\imath}\to\infty}\{\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_0,\mathfrak{s}_1)\Big[\frac{\delta^{\hat{\imath}+2\hat{\jmath}+1}-1}{\delta^{\hat{\imath}+2\hat{\jmath}(\delta-1)}}-\frac{\delta^{\hat{\imath}-1}-1}{\delta^{\hat{\imath}-2(\delta-1)}}\Big]+\delta^{\hat{\jmath}-1}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}-2},\mathfrak{s}_{\hat{\imath}+2\hat{\jmath}})\}=0,
\end{aligned}$$

$\lim_{\hat{\imath}\to\infty}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\mathfrak{s}_{\hat{\imath}+\zeta})=0.$

Therefore, O-sequence $\{\mathfrak{s}_{\hat{\imath}}\}$ in $(\mathcal{V},\mathfrak{w}_{\mathfrak{b}})$ is O-Cauchy sequence. Since $(\mathcal{V},\mathfrak{w}_{\mathfrak{b}})$ is a O-complete OB_bMS, there exists $\omega\in\mathcal{V}$ such that

$$\lim_{\hat{\imath}\to\infty}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}},\omega)=0. \tag{14}$$

Since $\mathfrak{P}$ is a O-continuous map, from (14) we get

$$\lim_{\hat{\imath}\to\infty}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}_{\hat{\imath}},\mathfrak{P}\omega)=\lim_{\hat{\imath}\to\infty}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}_{\hat{\imath}+1},\mathfrak{P}\omega)=0,$$

that is, the O-Cauchy sequence $\{\mathfrak{s}_{\hat{\imath}}\}$ is O-convergent to $\mathfrak{P}\omega$. Then the proposition (1) implies that $\mathfrak{P}\omega=\omega$, i.e., ω is a fixed point of $\mathfrak{P}$.

Since $\gamma\in\mathcal{F}_{\delta}$, we conclude $\lim_{\hat{\imath}\to\infty}\omega(\mathfrak{s}_{\hat{\imath}},\omega)=0$. Therefore, $\omega=\mathfrak{P}\omega$. We prove now the point $\omega\in\mathcal{V}$ is unique.

Assume that ω and μ are distinct fixed points of $\mathfrak{P}$. Suppose that, $\mathfrak{P}^{\hat{\imath}}\omega=\omega\ne\mu=\mathfrak{P}^{\hat{\imath}}\mu$ for all $\hat{\imath}\in\mathbb{N}$. By choice of $\mathfrak{s}_0$ in the first part of proof, we obtain

$$(\mathfrak{s}_0\perp\omega,\ \ \mathfrak{s}_0\perp\mu)\text{ or }(\omega\perp\mathfrak{s}_0,\ \ \mu\perp\mathfrak{s}_0).$$

Since $\mathfrak{P}$ is $\perp$-preserving, we have

$$(\mathfrak{P}^{\hat{\imath}}\mathfrak{s}_0\perp\mathfrak{P}^{\hat{\imath}}\omega,\ \ \mathfrak{P}^{\hat{\imath}}\mathfrak{s}_0\perp\mathfrak{P}^{\hat{\imath}}\mu)\text{ or }(\mathfrak{P}^{\hat{\imath}}\omega\perp\mathfrak{P}^{\hat{\imath}}\mathfrak{s}_0,\ \ \mathfrak{P}^{\hat{\imath}}\mu\perp\mathfrak{P}^{\hat{\imath}}\mathfrak{s}_0),$$

for all $\hat{\imath}\in\mathbb{N}$. Since $\mathfrak{P}$ is an orthogonal Geragthy contraction, we get

$$\mathfrak{w}_{\mathfrak{b}}(\omega,\mu)=\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\omega,\mathfrak{P}\mu)\le\gamma(\mathcal{K}(\omega,\mu))\mathcal{K}(\omega,\mu),$$

where

$$\begin{aligned}
\mathcal{K}(\omega,\mu)&=\max\{\mathfrak{w}_{\mathfrak{b}}(\omega,\mu),\mathfrak{w}_{\mathfrak{b}}(\omega,\mathfrak{P}\omega),\mathfrak{w}_{\mathfrak{b}}(\mu,\mathfrak{P}\mu)\} \\
&\le\mathfrak{w}_{\mathfrak{b}}(\omega,\mu).
\end{aligned}$$

Therefore, we have $\mathfrak{w}_{\mathfrak{b}}(\omega,\mu)<\frac{1}{\delta}\mathfrak{w}_{\mathfrak{b}}(\omega,\mu)$, which is a contradiction. Then, $\omega=\mu$. Hence $\mathfrak{P}$ has a UFP in $\mathcal{V}$. □

Corollary 1. *Let $(\mathcal{V},\mathfrak{w}_{\mathfrak{b}})$ be an O-complete B_bMS with a parameter $\delta\ge 1$ and let $\gamma\in\mathcal{F}_{\delta}$ be a function. Let $\mathfrak{P}:\mathcal{V}\to\mathcal{V}$ be an O-continuous self mapping satisfying*

$$\begin{aligned}
&\mathfrak{s}\perp\upsilon\ \ or\ \ \upsilon\perp\mathfrak{s},\ \mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s},\mathfrak{P}\upsilon)>0 \\
&\qquad\Rightarrow\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s},\mathfrak{P}\upsilon)\le\gamma(\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s},\upsilon))\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s},\upsilon),\ \ \forall\mathfrak{s},\upsilon\in\mathcal{V}.
\end{aligned}$$

Then, $\mathfrak{P}$ has a UFP.

Next, we give an example to support our Theorem (5).

Example 3. *Let $\mathcal{V} = \mathcal{I} \cup \mathcal{J}$ where $\mathcal{I} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}\}$ and $\mathcal{J} = [1, 2]$. Define the binary relation $\perp$ on $\mathcal{V}$ by $\mathfrak{s} \perp v$ if $\mathfrak{s}, v \geq 0$. Define the function $\mathfrak{w}_{\mathfrak{b}} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ such that $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = \mathfrak{w}_{\mathfrak{b}}(v, \mathfrak{s})$ as follows:*

For $\mathfrak{s}, v \in \mathcal{V}$ or $\mathfrak{s} \in \mathcal{I}$ and $v \in \mathcal{J}$, $\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = |\mathfrak{s} - v|$ and

$$\begin{aligned}
&\mathfrak{w}_{\mathfrak{b}}(\frac{1}{2}, \frac{1}{4}) = \mathfrak{w}_{\mathfrak{b}}(\frac{1}{6}, \frac{1}{8}) = 0.2. \\
&\mathfrak{w}_{\mathfrak{b}}(\frac{1}{2}, \frac{1}{6}) = \mathfrak{w}_{\mathfrak{b}}(\frac{1}{4}, \frac{1}{6}) = \mathfrak{w}_{\mathfrak{b}}(\frac{1}{4}, \frac{1}{8}) = 0.1. \\
&\mathfrak{w}_{\mathfrak{b}}(\frac{1}{2}, \frac{1}{8}) = 1.
\end{aligned}$$

Clearly, $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ is an O-complete B_bMS with constant $\delta = \frac{10}{3}$. Let $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ be defined as

$$\mathfrak{P}\mathfrak{s} = \begin{cases} \frac{\mathfrak{s}}{8} & if\ \mathfrak{s} \in \mathcal{J} \\ \frac{1}{6} & if\ \mathfrak{s} \in \mathcal{I}. \end{cases}$$

Clearly, $\mathfrak{P}$ is an $\perp$-preserving. Now, we verify that $\mathfrak{P}$ is an orthogonal Geraghty type α-admissible contraction. We see that

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) = \begin{cases} 0 & if\ \mathfrak{s}, v \in \mathcal{I} \\ 0.2 & if\ \mathfrak{s} \in \mathcal{I}, v = 1 \\ 0.1 & if\ \mathfrak{s} \in \mathcal{I}, v = 2 \\ 0.1 & if\ \mathfrak{s}, v \in \mathcal{J}. \end{cases}$$

Then, for all $\mathfrak{s}, v \in \mathcal{V}$ the mapping $\mathfrak{P}$ satisfies the condition

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \frac{3}{20}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v) = \frac{1 \setminus 2}{10 \setminus 3}\mathfrak{w}_{\mathfrak{b}}(\mathfrak{s}, v).$$

Consider a mapping $\mathfrak{P} : [0, \infty) \to \mathcal{R}$ defined by $\mathfrak{P}(\mathfrak{s}) = \mathfrak{s}$, it is clearly that $\gamma \in \mathcal{F}_\delta$, and we get

$$\mathfrak{w}_{\mathfrak{b}}(\mathfrak{P}\mathfrak{s}, \mathfrak{P}v) \leq \gamma(\mathcal{K}(\mathfrak{s}, v))\mathcal{K}(\mathfrak{s}, v) \ \ \forall\ \mathfrak{s}, v \in \mathcal{V}.$$

Hence, the condition of Corollary 1 holds with $\gamma(\varrho) = \frac{1}{2\delta} = \frac{3}{20}$ and $\mathfrak{P}$ has a UFP *which is $\mathfrak{s} = \frac{1}{6}$.*

4. Applications

As an application of Theorem 5, we find the existence and uniqueness of the following integral equation:

$$\omega(\varrho) = \lambda(\varrho) + \int_0^{\mathfrak{a}} \mathcal{G}(\varrho, \delta)\mathcal{H}(\varrho, \delta, \omega(\delta))d\delta, \ \ \varrho \in [0, \mathfrak{a}], \mathfrak{a} > 0. \tag{15}$$

Consider $\mathcal{V} = \mathcal{C}([0, \mathfrak{a}], \mathbb{R})$ be a real continuous functions on $[0, \mathfrak{a}]$ and $\mathfrak{P} : \mathcal{V} \to \mathcal{V}$ the mapping defined by

$$\mathfrak{w}_{\mathfrak{b}}(\omega, \mu) = \max_{0 \leq \varrho \leq \mathfrak{a}} |\omega(\varrho) - \mu(\varrho)|^2, \ \ \omega, \mu \in \mathcal{V}. \tag{16}$$

Obviously, $(\mathcal{V}, \mathfrak{w})$ is a complete b-metric space with constant $\delta = 2$ and $\omega(\varrho)$ is a solution of the Equation (15) iff $\omega(\varrho)$ is a fixed point of $\mathfrak{P}$.

Theorem 6. *Suppose that*

(R1) *The mappings $\mathcal{G} : [0,\mathfrak{a}] \times \mathbb{R} \to \mathbb{R}^+$, $\mathcal{H} : [0,\mathfrak{a}] \times \mathbb{R} \to \mathbb{R}$, and $\lambda : [0,\mathfrak{a}] \to \mathbb{R}$ are O-continuous functions.*

(R2) *There exists, for all $\varrho, \delta \in [0,\mathfrak{a}]$ and $\omega, \mu \in \mathcal{V}$ such that*

$$|\mathcal{H}(\varrho,\delta,\omega(\delta)) - \mathcal{H}(\varrho,\delta,\mu(\delta))| \leq \sqrt{\frac{e^{-\mathcal{K}(\omega,\mu)}\mathcal{K}(\omega,\mu)}{2}}. \tag{17}$$

(R3) *For all $\varrho, \delta \in [0,\mathfrak{a}]$, we have*

$$\max \int_0^{\mathfrak{a}} \mathcal{G}(\varrho,\delta)^2 d\delta \leq \frac{1}{\mathfrak{a}}.$$

Then, (15) *has a unique solution in $\mathcal{V}$.*

Proof. Consider the O-relation $\perp$ on $\mathcal{V}$ defined by

$$\omega \perp \mu \iff \omega(\varrho)\mu(\varrho) \geq \omega(\varrho) \quad or \quad \omega(\varrho)\mu(\varrho) \geq \mu(\varrho), \forall \varrho \in [0,\mathfrak{a}].$$

Then $(\mathcal{V}, \perp)$ is an O-set. In fact, if $\{\omega_{\hat{\imath}}\}$ is an arbitrary Cauchy O-sequence in $\mathcal{V}$, then there exists a subsequence $\{\omega_{\hat{\imath_{\mathfrak{n}}}}\}$ of $\{\omega_{\hat{\imath}}\}$ for which $\omega_{\hat{\imath_{\mathfrak{n}}}} = 0$ for all $\mathfrak{n} \geq 1$ or there exists a monotone subsequence $\{\omega_{\hat{\imath_{\mathfrak{n}}}}\}$ of $\{\omega_{\hat{\imath}}\}$ for which $\omega_{\hat{\imath_{\mathfrak{n}}}} \leq \frac{1}{2}$ for all $\mathfrak{n} \geq 1$. It follows that $\{\omega_{\hat{\imath_{\mathfrak{n}}}}\}$ converges to a point $\omega \in [0, \frac{1}{2}] \subset \mathcal{V}$. Therefore, $(\mathcal{V}, \perp, \mathfrak{w}_{\mathfrak{b}})$ is an O-complete B_bMS with parameter $\delta = 2$. For each $\omega, \mu \in \mathcal{V}$ with $\omega \perp \mu$ and $\varrho \in [0,\mathfrak{a}]$, we have

$$\mathfrak{P}(\omega(\varrho)) = \lambda(\varrho) + \int_0^{\mathfrak{a}} \mathcal{G}(\varrho,\delta)\mathcal{H}(\varrho,\delta,\omega(\delta))d\delta \geq 1. \tag{18}$$

Accordingly $[(\mathfrak{P}\omega)(\varrho)][(\mathfrak{P}\mu)(\varrho)] \geq (\mathfrak{P}\mu)(\varrho)$ and so $(\mathfrak{P}\omega)(\varrho) \perp (\mathfrak{P}\mu)(\varrho)$. Then, $\mathfrak{P}$ is $\perp$-preserving.

Let $\omega, \mu \in \mathcal{V}$ with $\omega \perp \mu$. Suppose that $\mathfrak{P}(\omega) \neq \mathfrak{P}(\mu)$. For each $\varrho \in [0,\mathfrak{a}]$, we have

$$\begin{aligned}
\mathfrak{w}(\mathfrak{P}\omega, \mathfrak{P}\mu) &= \max_{\varrho \in [0,\mathfrak{a}]} |\mathfrak{P}\omega(\varrho) - \mathfrak{P}\mu(\varrho)|^2 \\
&= \max_{\varrho \in [0,\mathfrak{a}]} \left\{ \left| \lambda(\varrho) + \int_0^{\mathfrak{a}} \mathcal{G}(\varrho,\delta)\mathcal{H}(\varrho,\delta,\omega(\delta))d\delta - \lambda(\varrho) - \int_0^{\mathfrak{a}} \mathcal{G}(\varrho,\delta)\mathcal{H}(\varrho,\delta,\mu(\delta))d\delta \right|^2 \right\} \\
&= \max_{\varrho \in [0,\mathfrak{a}]} \left\{ \left| \int_0^{\mathfrak{a}} \mathcal{G}(\varrho,\delta)(\mathcal{H}(\varrho,\delta,\omega(\delta)) - \mathcal{H}(\varrho,\delta,\mu(\delta)))d\delta \right|^2 \right\} \\
&\leq \max_{\varrho \in [0,\mathfrak{a}]} \left\{ \int_0^{\mathfrak{a}} \mathcal{G}(\varrho,\delta)^2 d\delta \int_0^{\mathfrak{a}} |\mathcal{H}(\varrho,\delta,\omega(\delta)) - \mathcal{H}(\varrho,\delta,\mu(\delta))|^2 d\delta \right\} \\
&\leq \frac{1}{\mathfrak{a}} \int_0^{\mathfrak{a}} \left| \sqrt{\frac{e^{-\mathcal{K}(\omega,\mu)}\mathcal{K}(\omega,\mu)}{2}} \right|^2 d\delta \\
&\leq \frac{e^{-\mathcal{K}(\omega,\mu)}}{2}\mathcal{K}(\omega,\mu).
\end{aligned}$$

Thus, $\mathfrak{w}(\mathfrak{P}\omega, \mathfrak{P}\mu) \leq \gamma(\mathcal{K}(\omega,\mu))\mathcal{K}(\omega,\mu)$, for each $\omega, \mu \in \mathcal{V}$. Therefore, all the conditions of Theorem (5) for $\gamma(\varrho) = \frac{e^{-\varrho}}{2}$, $\varrho > 0$ and $\gamma(0) \in [0, \frac{1}{2})$ are satisfied. Hence, the (15) has a unique solution. □

Example 4. *Let us consider the equation*

$$\mathfrak{g}(\mathfrak{s}) = \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \mathfrak{g}(v)\delta v, \ \ 0 \leq x \leq 1. \tag{19}$$

Clearly, above Equation (19) *satisfy the assumption of Theorem 6, that is:*

$\sin(\pi\mathfrak{s}^2) - \dfrac{\mathfrak{s}^2}{\pi}$ *is an orthogonal continuous function on* $[0,1]$.

Kernel, $\mathtt{K}(\mathfrak{s}, v)$ *is an orthogonal continuous on* $\mathbb{R} = \{(\mathfrak{s}, v), 0 < \mathfrak{s}, v < 1\}$.

We get

$$\mathfrak{g}_{\vartheta+1}(\mathfrak{s}) = \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \mathfrak{g}_\vartheta(v)\delta v, \ \ 0 \le x \le 1.$$

Choosing $\sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi}$ *as the initial function, we can apply fixed point iteration method to get numerical solution:*

$$\begin{aligned}
\mathfrak{g}_1(\mathfrak{s}) &= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \mathfrak{g}_0(v)\delta v \\
&= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \sin(\pi v^2)\delta v \\
&= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \mathfrak{s}^2 \frac{1}{2\pi}(1 - \cos(\pi\mathfrak{s}^2)). \\
\mathfrak{g}_2(\mathfrak{s}) &= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \mathfrak{g}_1(v)\delta v \\
&= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \Big(\sin(\pi v^2) - \frac{v^2}{\pi} + v^2 \frac{1}{2\pi}(1 - \cos(\pi v^2)) Big \Big) \delta v \\
&= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \frac{\mathfrak{s}^2}{8\pi^3}\Big(-4\pi^2 - 2 + 4\pi^2\cos(\pi\mathfrak{s}^2) + \pi^2\mathfrak{s}^4 + 2\mathfrak{s}^2\pi\sin(\pi\mathfrak{s}^2) + 2\cos(\pi\mathfrak{s}^2)\Big). \\
\mathfrak{g}_3(\mathfrak{s}) &= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \mathfrak{g}_2(v)\delta v \\
&= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \int_0^x \mathfrak{s}^2 v \Big(\sin(\pi v^2) - \frac{v^2}{\pi} + \frac{v^2}{8\pi^2}(-4\pi^2 - 2 + 4\pi^2\cos(\pi v^2) + v^4\pi^2 \\
&\quad + 2\mathfrak{s}^2\pi\sin(\pi\mathfrak{s}^2) + 2\cos(\pi\mathfrak{s}^2))\Big)\delta v \\
&= \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi} + \mathfrak{s}^2\Bigg\{ -\frac{1}{64\pi^6}\Bigg(-32\pi^5 - 16\pi^3 - 24\pi + 32\pi^5\cos(\pi\mathfrak{s}^2) + 8\mathfrak{s}^4\pi^5 - 4\mathfrak{s}^4\pi^3 \\
&\quad + 16\mathfrak{s}^2\pi^4\Big(\sin(\pi\mathfrak{s}^2) + 16\pi^3\cos(\pi\mathfrak{s}^2) + \mathfrak{s}^8\pi^5 - 8\mathfrak{s}^4\pi^3\cos(\pi\mathfrak{s}^2) + 24\mathfrak{s}^2\pi^2\sin(\pi\mathfrak{s}^2) \\
&\quad + 24\pi\cos(\pi\mathfrak{s}^2)\Big)\Bigg)\Bigg\}.
\end{aligned}$$

Consider that for $|\mathfrak{s}| \le 1$, *an O-sequence* $\{\mathfrak{g}_\vartheta(\mathfrak{s})\}$ *will converge to* $\mathfrak{g}(\mathfrak{s}) = \sin(\pi\mathfrak{s}^2) - \frac{\mathfrak{s}^2}{\pi}$.

Now, we find the difference between an approximation solution and an exact solution from Table 1 and Figure 1.

Table 1. Comparison between an approximation solution and an exact solution.

$\mathfrak{s}_j$	**Approximation Solution**	**Exact Solution**	**Absolute Error**
0.000	0.000	0.000	0.000
0.100	0.023	0.028	0.005
0.200	0.102	0.113	0.011
0.300	0.234	0.250	0.016
0.400	0.412	0.431	0.019
0.500	0.609	0.628	0.018
0.600	0.779	0.790	0.011
0.700	0.848	0.844	0.004
0.800	0.730	0.701	0.029
0.900	0.358	0.304	0.054
1.000	−0.251	−0.318	0.067

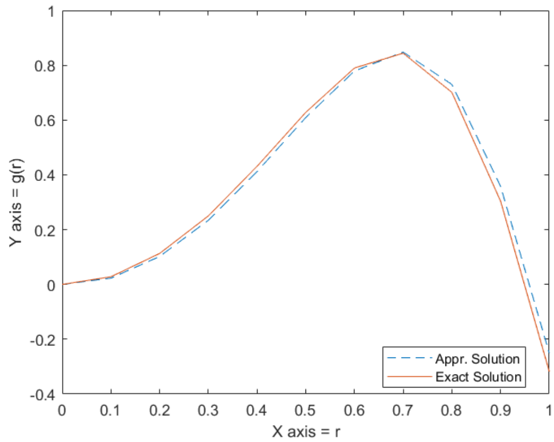

Figure 1. Graph of the comparison between an approximation and exact solution with h = 0.1.

The comparison shows that the absolute error between an approximation and an exact solution is very small.

5. Conclusions

In this paper, we proved fixed point theorem for an $O-G-\alpha$-admissible contraction mapping in an O-complete B_bMS. We have provided a non-trivial example to support our main Theorem 5. Also, we provided an application to find the existence and uniqueness of a solution to the integral equation and compared the approximate solution and exact solution.

Author Contributions: Conceptualization, M.D., A.J.G., G.M., O.E. and M.D.l.S.; methodology, M.D., A.J.G., G.M. and O.E.; validation, M.D. and A.J.G.; formal analysis, M.D. and O.E.; investigation, M.D., A.J.G., G.M. and O.E.; writing—original draft, M.D., A.J.G., G.M. and O.E.; writing—review and editing, O.E. and M.D.l.S.; visualization, A.J.G.; supervision, M.D.l.S.; project administration, O.E. and M.D.l.S.; funding acquisition, M.D.l.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Basque Government under Grant IT1207-19.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are very grateful to the Basque Government for Grant IT1207-19. They also appreciate the reviewers' time, considerations, and suggestions, all of which improved the quality of this work.

Conflicts of Interest: The authors declare that they have no competing interests concerning the publication of this article.

References

1. Bakhtin, I.A. The contraction mapping principle in almost metric space. In *Functional Analysis;* Gos Ped Inst: Ulyanovsk, Russia, 1989; pp. 26–37.
2. Czerwik, S. Contraction mappings in *b*-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
3. Aleksic, S.; Huang, H.; Mitrovic, Z.; Radenovic, S. Remarks on some fixed point results in *b*-metric space. *J. Fixed Point Theory Appl.* **2018**, *20*, 147. [CrossRef]
4. Dosenovic, T.; Pavlovic, M.; Radenovic, S. Contractive conditions in *b*-metric spaces. *Vojnotehnicki Glasnik/Mil. Tech. Cour.* **2017**, *65*, 851–865. [CrossRef]
5. Kirk, W.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer: Berlin/Heidelberg, Germany, 2014.
6. Koleva, R.; Zlatanov, B. On fixed points for Chatterjeas maps in *b*-metric spaces. *Turk. J. Anal. Number Theory* **2016**, *4*, 31–34.
7. Abbas, M.I.; Ragusa, M.A. Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel. *AIMS Math.* **2022**, *7*, 20328–20340. [CrossRef]
8. Krim, S.; Abbas, S.; Benchohra, M.; Karapinar, E. Terminal Value Problem for Implicit Katugampola Fractional Differential Equations in *b*-Metric Spaces. *J. Funct. Spaces* **2021**, *2021*, 5535178. [CrossRef]

9. Nurwahyua, B.; Khanb, M.S.; Fabianoc, S.; Radenovic, S. Common Fixed Point on Generalized Weak Contraction Mappings in Extended Rectangular b-Metric Spaces. *Filomat* **2021**, *35*, 3621–3634. [CrossRef]
10. Branciari, A. A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces. *Publ. Mathenaticae Debr.* **2000**, *57*, 31–37. [CrossRef]
11. George, R.; Redenovic, S.; Reshma, K.P.; Shukla, S. Rectangular b-metric space and contraction principles. *Olhares-Med.-Viajante Giovanni Palombini Rao Grande Sul.* **2015**, *8*, 1005–1013. [CrossRef]
12. Lukacs, A.; Kajanto, S. Fixed point theorems for various types of F-contractions in complete b-metric spaces. *Fixed Point Theory* **2018**, *19*, 321–334. [CrossRef]
13. Erhan, I.M. Geraghty type contraction mappings on Branciari b-metric spaces. *Adv. Theory Nonlinear Anal. Its Appl.* **2017**, *2*, 147–160.
14. Geraghty, M.A. On contractive mappings. *Proc. Am. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
15. Dukic, D.; Kadelburg, Z.; Radenovic, S. Fixed points of Geraghty-type mappings in various generalized metric spaces. *Abstr. Appl. Anal.* **2011**, *2011*, 561245. [CrossRef]
16. Shahkoohi, R.J.; Razani, A. Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces. *J. Inequal. Appl.* **2014**, *2014*, 373. [CrossRef]
17. Huang, H.; Paunovic, L.; Radenovic, S. On some fixed point results for rational Geraghty contractive mappings in ordered b-metric spaces. *J. Nonlinear Sci. Appl.* **2015**, *8*, 800–807. [CrossRef]
18. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for α-ϕ-contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
19. Tunç, C.; Tunç, O. On the existence of solutions of non-linear 2D Volterra integral equations in a Banach Space. *RACSAM* **2022**, *116*, 11.
20. Tunc, C.; Tunc, O. On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation. *RACSAM* **2021**, *115*, 1–17. [CrossRef]
21. Gordji, M.E.; Ramezani, M.; De La Sen, M.; Cho, Y.J. On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory (FPT)* **2017**, *18*, 569–578. [CrossRef]
22. Eshaghi Gordji, M.; Habibi, H. Fixed point theory in generalized orthogonal metric space. *J. Linear Topol. Algebra (JLTA)* **2017**, *6*, 251–260.
23. Beg, I.; Gunaseelan, M.; Arul Joseph, G. Fixed Point of Orthogonal F-Suzuki Contraction Mapping on Orthogonal Complete b-Metric Spaces with Applications. *J. Funct. Spaces* **2021**, *2021*, 6692112. [CrossRef]
24. Arul Joseph, G.; Gunaseelan, M.; Jung, R.L.; Park, C. Solving a nonlinear integral equation via orthogonal metric space. *AIMS Math.* **2021**, *7*, 1198–1210.
25. Gunaseelan, M.; Arul Joseph, G.; Nasreen, K.; Mohammad, M.; Salahuddin. Orthogonal F-Contraction Mapping on Orthogonal Complete Metric Space with Applications. *Int. J. Fuzzy Logic Intell. Syst.* **2021**, *21*, 243–250.
26. Gunaseelan, M.; Arul Joseph, G.; Park, C.; Sungsik, Y. Orthogonal F-contractions on O-complete b-metric space. *AIMS Math.* **2021**, *6*, 8315–8330.
27. Arul Joseph, G.; Gunaseelan, M.; Parvaneh, V.; Aydi, H. Solving a Nonlinear Fredholm Integral Equation via an Orthogonal Metric. *Adv. Math. Phys.* **2021**, *2021*, 1202527.
28. Mukheimer, A.; Gnanaprakasam, A.J.; Haq, A.U.; Prakasam, S.K.; Mani, G.; Baloch, I.A. Solving an Integral Equation via Orthogonal Brianciari Metric Spaces. *J. Funct. Spaces* **2022**, *2022*, 7251823. [CrossRef]
29. Gnanaprakasam, A.J.; Nallaselli, G.; Haq, A.U.; Mani, G.; Baloch, I.A.; Nonlaopon, K. Common Fixed-Points Technique for the Existence of a Solution to Fractional Integro-Differential Equations via Orthogonal Branciari Metric Spaces. *Symmetry* **2022**, *14*, 1859. [CrossRef]
30. Prakasam, S.K.; Gnanaprakasam, A.J.; Kausar, N.; Mani, G.; Munir, M. Solution of Integral Equation via Orthogonally Modified F-Contraction Mappings on O-Complete Metric-Like Space. *Int. J. Fuzzy Log. Intell. Syst.* **2022**, *22*, 287–295. [CrossRef]