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Performance measures of nonstationary inventory models for perishable products under the EWA policy

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A B S T R A C T

Accurately estimating key performance indicators in inventory models for perishable items is essential in order to assess and improve the management strategy of these systems. We analyse the production of platelet concentrates at blood banks under the EWA replenishment policy. We give analytical approximations of the most important performance measures, such as the size of orders, the size of stocks, the percentage of outdating, the age distribution of stocks and the freshness of units issued, among others. The production of platelet concentrates is a prototypical example of inventory models for short life items with random demand and a weekly pattern, where a high service level is required. The methodology and the approximations presented here can be easily adapted to other inventory systems with similar characteristics. Most of the formulae in this article are new for nonstationary models under the EWA policy; indeed, formulae for the age distribution of units in stock and of units issued have not appeared in the literature even for the simpler base-stock replenishment policy. We apply our results to a real blood bank and find very close agreement between the formulae and the results of Monte Carlo simulations. The accuracy of our approximations is also tested in several scenarios, depending on the lifetime of units, safety stock levels and the probabilistic distribution of demand.

1. Introduction

1.1. Background and motivation

Inventory management of perishable items is of great importance in many sectors of the economy. Food and blood products are just a couple of examples of perishable goods. The mathematical analysis of inventory models for these products is much more difficult than for nonperishable items. Also, although there are a great many research papers on models for perishable goods they are still far fewer in number than the papers and books devoted to nonperishable products; see, e.g., Silver, Pyke, & Peterson (1998).

Much of the research on inventory management for perishable products has focused on blood products, and has been published both in medical and mathematical journals; see Atkinson, Fontaine, Goodnough, & Wein (2012), Beliën & Forcé (2012), Civelek, Karaesmen, & Scheller-Wolf (2015), Ensafian & Yaghoubi (2017), Rajendran & Ravindran (2019). Blood products are used for transfusion in most hospitals and are seen as a scarce, precious resource. They are of vital importance for patients, so a sufficient stock must be kept in order to avoid stockouts. Different components of blood, such as red blood cells, plasma and platelets, are used for transfusion. Among them, platelet concentrates are considered as a critical product since they have a short lifetime (usually 5 or 7 days). They are also expensive (for instance, Haijema, van der Wal, & van Dijk (2007) assume a cost of more than 450 Euros per patient per treatment), and overcautious policies in keeping big stocks result in a large number of outdated units, leading to an unnecessary waste of money and ethical concerns.

In this paper we focus on a periodic review model for fixed lifetime perishable goods such that stockouts must be kept to a minimum. Platelets are a clear example of such products, and we use the inventory model of the Basque Centre for Transfusion and Human Tissues (CVTTH) in Galdakao, Bizkaia, Spain, for the derivation of our formulae. This research originated in collaboration between the University of the Basque Country and the CVTTH for the implementation of a mathematical model for the management of blood products; see Pérez Vaquero, Gorria, Lezaun, López, Monge, Eguizabal, & Vesga (2016), Gorria, Labata, Lezaun, López, Pérez Aliaga, & Pérez Vaquero (2020). The paper focuses on this model for the

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production of platelet concentrates, but our analysis can easily be adapted to other perishable goods where the service level needs to be high.

1.2. Objective of the paper

In practice, it is very useful to have approximations of the performance measures of an inventory system, which can be used to validate policies and optimise the parameters of the model. When such approximations are not available, validation and optimisation must rely on simulations. This paper sets out to derive analytical approximations for the main performance measures of nonstationary models when the Estimated Withdrawal and Aging (EWA) replenishment policy is used and a high service level is needed. EWA is a modification of the base-stock replenishment policy where the inventory position is modified by subtracting an estimation of the amount of outdating, for placing a new order.

We do not assume any specific form for the probabilistic distribution of the daily demand. Among others, we obtain formulae for the expected on-hand inventory, the probability of stockout, expected outdating, the age distribution of stocks and the freshness of units issued. The models that we consider are not stationary, but weekly stationary: the weekly pattern exists in the distribution of the demand and in the operational assumptions of the system (for instance, orders are placed every weekday but not on weekends). Since our model has a weekly pattern, all the formulae are obtained for each day of the week.

We derive the approximations for a particular model to be described in detail in Section 2. This may be seen as somewhat restrictive, but we see two advantages in it. First, the model is realistic, since it is very close to the operation of a real blood bank. Second, it includes a variety of situations (e.g. not all days have the same lead or review times and units arriving on Mondays have a different remaining lifetime than units arriving on other days); thus, the reasoning and derivation of the formulae for the model can be adapted without difficulty to other systems.

To the best of our knowledge, this is the first paper where approximations for performance measures are given when the model is nonstationary and the EWA policy is used (except for the fill rate, which has already been approximated in van Donselaar & Broekmeulen, 2011). A key point in our analysis concerns the formulae for ν_t^i, the expected number of units ordered on day t which are in stock at the end of day $t+i$, derived in Section 3.3. These formulae enable us to give approximations for many performance measures of the model, such as expected outdating. Freshness, defined as the expected remaining lifetime of units issued, can also be easily computed from ν_t^i. In fact, we give more comprehensive information on units issued by deriving approximations for the values w_t^r, the expected number of units issued on day t with r days of remaining lifetime.

We also use ν_t^i to compute the age distribution of the stock, i.e. the expected number of units in stock with remaining lifetime $1, \dots, m$ at the beginning of a day in the long run. This distribution gives full information on the behaviour of the system. We point out that age distribution of stock in nonstationary models for perishable items has not appeared in the literature even under the simpler base-stock policy.

While in most instances of inventory systems demand is discrete in nature, continuous distributions are often used to model it. Throughout the paper we assume that demand is well modelled by a continuous distribution, so our formulae are expressed using integrals and probability density functions (PDF); if a discrete distribution for demand is to be used, then, PDFs must be replaced by probability mass functions and integrals by sums.

1.3. Literature review

There is a great variety of mathematical models for perishable items. They differ in many characteristics, such as deterministic or random demand, fixed or random lifetime, zero or positive lead time, stationary or time-varying demand, among others. We work with a random demand, fixed lifetime model here, so we restrict ourselves to that setting for the rest of the paper.

An excellent review of the research published on inventory models for perishable items can be found in Nahmias (1982) for early papers on the subject; in Raafat (1991) for papers up to 1991; in Goyal & Giri (2001) for publications from the early 90s to 2000; and in Bakker, Riezebos, & Teunter (2012), Janssen, Claus, & Sauer (2016) and Chaudhary, Kulshrestha, & Routroy (2018) for more recent work. The first mathematical studies on inventory systems for perishable items set out to find optimal solutions in terms of minimising cost functions. However, in contrast to what happens for nonperishable items, where optimal solutions are known in a wide variety of settings, researchers found that models for perishable items were much harder to analyse, at least when demand was random. Thus, optimal solutions were obtained only in a limited number of situations such as very short product lifetimes ($m = 1, 2$) or zero lead time; see Nahmias (1982) and references therein.

One way of finding an optimal solution is by using dynamic programming, which is a suitable tool for these models, taking a state space defined by the age distribution of the stock and a stochastic transfer function (see Nahmias, 1975). This technique solves, at least theoretically, the problem of finding a policy which minimises the cost function subject to a service constraint. However, due to the "curse of dimensionality" of dynamic programming, the state space of the problems becomes huge even for moderate values of m (lifetime) and maximum storage capacity and the problems become unsolvable in practice. In the last few decades the increasing speed and capacity of computers have led to a reemergence of this technique, although it still needs to be combined with aggregation of states or simulation to find solutions in a reasonable time. Haijema et al. (2007), who combine dynamic programming with simulation, work with a model whose state space is larger than 10^8, which implies a complexity of the order of 10^{13} for one week iteration, so a downscaling of four to one units is carried out. Algorithms based on aggregation of states in multiple levels are proposed in Voelkel, Sachs, & Thonemann (2020).

Another approach for finding good policies in inventory models is discrete event simulation. It consists of modelling the system and implementing it in simulation software. By running the simulation with different policies and in various settings, the performance of the policies can be compared with a view to choosing the best. Simulation has been widely used to model real blood banks. For instance, Rytilä & Spens (2006) compare different scenarios of production and distribution of blood components in Finland. Asllani, Culler, & Ettkin (2014) build a model for a blood bank centre supplying 50 health care facilities in the US to search for the best platelet production policy in the week, when platelets are differentiated by blood type. Dalalah, Bataineh, & Alkhaledi (2019) use a simulation-optimisation approach to find an optimal policy when demand is differentiated by the age of platelets, and apply it to Kuwait public hospitals. Gorria et al. (2020) use data from two blood banks in Spain to study the decrease in outdating when the lifetime of platelet concentrates is extended from 5 to 7 days via pathogen reduction technologies and analyse what days of the week are most appropriate for applying these technologies. An advantage of the simulation approach is that the model can be as realistic as desired. However, no analytical expressions of the optimal solution or the performance measures of the model are obtained, which prevents the parameters of the model from being interpreted directly; moreover, simulations must be run every time a change in the parameters is observed.

Yet another approach is to use heuristics to find a good solution. This approach does not seek to find the best of all feasible solutions, but rather to propose reasonable, easy-to-implement policies which perform well in practice. Many of these policies are myopic, in the sense that they make period-by-period decisions, and/or involve a simplification of the state space (for instance, using a simple function of the composition of the stock instead of its complete age distribution). Nandakumar & Morton (1993) describe heuristic solutions in the case of zero lead time which are close to optimal. For positive lead time, Chiu (1995) develops a solution based on a period-by-period optimisation of an approximation of the cost function which only takes into account the size of the stock but not its age distribution. Haijema & Minner (2019) give an overview of some of the most important stock-age dependent order policies, propose new ones and compare them in a broad set of scenarios.

One heuristic developed for periodic review and fixed lifetime models, which yields good results, is the EWA policy, introduced in Broekmeulen & van Donselaar (2009). It is known (see Broekmeulen & van Donselaar, 2009; Haijema & Minner, 2019) that the EWA policy significantly outperforms the base-stock policy. The EWA policy has been analysed by van Donselaar & Broekmeulen (2011), van Donselaar & Broekmeulen (2012), Broekmeulen & van Donselaar (2019). van Donselaar & Broekmeulen (2011) give approximations for the fill rate both when demand is stationary and when it has a weekly pattern. In the case of stationary models, van Donselaar & Broekmeulen (2012) give analytical expressions to approximate the expected outdating; these approximations are then improved by simulating a large number of scenarios and fitting a regression model. Also for stationary models, Broekmeulen & van Donselaar (2019) propose several policies for reducing waste and increasing freshness. Freshness is a very important performance measure when dealing with perishable items, since waste due to outdating occurs very frequently at customer level: e.g. households for food (Secondi, Principato, & Laureti, 2015; van Geffen, van Herpen, & van Trijp, 2020) or hospitals for platelets (Flint, McQuilten, Irwin, Rushford, Haysom, & Wood, 2020; Pérez Vaquero et al., 2016). Moreover, fresher units are usually preferred by customers in the case of both food products (Li & Teng, 2018) and platelets, since they have better properties than older ones, Caram-Deelder, Kreuger, Jacobse, van der Bom, & Middelburg (2016), Aubron, Flint, Ozier, & McQuilten (2018). Broekmeulen & van Donselaar (2019) were the first to obtain an approximation of freshness in an inventory model for perishable items. To estimate freshness, they propose an expression based on Little's formula for queuing theory; the expression uses an estimation of expected outdating, so simulations must be run and a regression model fitted as in van Donselaar & Broekmeulen (2012) in order to compute the approximation of freshness.

A more comprehensive description of the performance of an inventory system for perishable items is achieved by computing the age distribution of the stock. Formulae for the age distribution of stock are challenging to obtain even in the stationary case and under the base-stock policy. They have been computed only for continuous-review models assuming that demand follows a Poisson process, using the theory of queuing networks. See Kouki, Legros, Babai, & Jouini (2020) and references therein.

The rest of the paper is organised as follows. Section 2 describes the model used for developing our approximations and how the EWA policy applies to it. The formulae for the approximations are given in Section 3. The accuracy of the approximations is assessed via comparison with Monte Carlo simulations in a real example in Section 4. Section 5 shows the results for 72 scenarios and analyses the extent to which our approximations can be regarded as reliable. Conclusions and ideas for future work are given in Section 6. The paper has four appendices with some additional formulae and information and a Supplementary Material file with tables related to Sections 4 and 5.

2. The model

For ease of exposition, we develop our results for a particular model, i.e. the production of platelet concentrates in the CVTTH. The characteristics of the model presented are common to many blood banks. For instance, a similar model is analysed by Haijema (2013) using dynamic programming. The formulae derived here can be easily adapted to any model with periodic review, stochastic demand, fixed lifetime and a weekly pattern.

We consider a FIFO issuing policy (older items are issued first). FIFO is the most common issuing policy in the literature on perishable items, especially when dealing with blood products; it is known that the freshness of units issued is lower using a FIFO policy than with other issuing policies such as LIFO (newer items are issued first), but outdating is lower with FIFO than with LIFO; see, for instance, Cohen & Prastacos (1981), Stanger, Yates, Wilding, & Cotton (2012).

There is daily demand from hospitals from Monday to Sunday whose distribution depends on the day of the week. Let D_t be the random variable representing the demand on day $t \geq 1$. We take day $t = 1$ to be a Monday. As usual in these models, we assume that demands on different days are independent of each other. Similarly, we assume that the demand is weekly stationary, i.e. the probability distribution of D_{t+7k} is identical to D_t for all $1 \leq t \leq 7$, $k \geq 1$. We denote by F_t the cumulative distribution function (CDF) associated with D_t and its mean and variance by $\mu_t = E[D_t]$ and $\sigma_t^2 = Var[D_t]$, respectively.

For $t, i \geq 1$, the aggregated demand during the interval $[t, t+i]$ is denoted by

$$D_{t,t+i} = \sum_{j=0}^{i} D_{t+j}.$$

For $t \leq s$, let $F_{t,s}$ be the CDF of $D_{t,s}$ and $\mu_{t,s} = E[D_{t,s}]$ and $\sigma_{t,s}^2 = Var[D_{t,s}]$ the corresponding mean and variance. We also write $F_{t,s}$ with $1 \leq s < t \leq 7$ to denote the CDF of $D_{t,s+7}$, which has the same distribution as $D_{t,7} + D_{1,s}$; for instance, $F_{5,2}$ represents the distribution of $D_{5,9}$, the demand from a Friday to the following Tuesday. Accordingly, for $1 \leq s < t \leq 7$, let $\mu_{t,s} = E[D_{t,7}] + E[D_{1,s}]$ and $\sigma_{t,s}^2 = Var[D_{t,7}] + Var[D_{1,s}]$, the mean and variance of $F_{t,s}$.

In practice, historical data is used to fit a distribution for F_t and to estimate μ_t and σ_t, $t = 1, \ldots, 7$. The estimations of $F_{t,s}$, $\mu_{t,s}$ and $\sigma_{t,s}$ can be computed from the estimations of F_t, μ_t and σ_t, $t = 1, \ldots, 7$ since we assume independence of the random variables D_t.

Platelet concentrates have a fixed lifetime of m days. We derive our formulae for general m; when a specific value of m is needed we take $m = 5$ since this is the most common lifetime of platelet concentrates and the one used in Pérez Vaquero et al. (2016). Production orders can be placed every day from Monday to Friday. This is equivalent to saying that there is a review interval of one day from Monday to Thursday, $R = 1$, and of three days on Friday, $R = 3$. An order means that blood is processed immediately after the order and platelet concentrates are produced during the day. If the order is placed on any day between Monday and Thursday, the concentrates are ready for use in the morning of the following day, with a remaining lifetime of m days; orders placed on Friday are ready for use on Monday morning, with a remaining lifetime of $m - 2$ days. That means that the lead time is $L = 1$ for orders placed from Monday to Thursday and $L = 3$ for orders placed on Friday. Note that this model is not daily stationary, but weekly stationary: the distribution of demand, the review interval and the lead time depend on the day of the week. We assume that the service level is high, which is customary in blood banks, and that unsatisfied demand is not backlogged (in practice, stockouts are covered by an urgent request to a neighbouring bank). Orders are placed at the beginning of the day, taking into account the on-hand inventory and the arriving units but before the demand for the day is known. If a unit has not been used by the mth day of life it is disposed of; for instance, if $m = 5$, concentrates ordered on Monday and not used by Saturday are discarded. The order of events each day is: (1) receive the incoming order (only for weekdays); (2) place the new order (only for weekdays); (3) observe and meet the demand; (4) dispose of outdated units.

2.1. The EWA policy

Ours is a typical model for perishable items with fixed lifetimes. As commented in the Introduction, there is no known optimal policy for such a model (weekly pattern, nonzero lead time, stochastic demand and lifetime greater than 2). A heuristic approach to find a reasonable solution is the base-stock policy, which is a simple order-up-to policy. The EWA policy is an improvement of that policy. We describe both these policies here.

The base-stock policy has been widely used as a heuristic for stochastic demand, fixed lifetime inventory systems; see for instance Nahmias (1982), Cooper (2001) and references therein. It has been also used as a benchmark for comparison with more complex policies: Tekin, Gürler, & Berk (2001), Broekmeulen & van Donselaar (2009), Duan & Liao (2013), Haijema & Minner (2019).

The base-stock policy works like an order-up-to policy for non-perishable items. Its rationale is to have sufficient on-hand inventory to meet demand until the inventory replenishment corresponding to the next order. At each review point t an order of size

$$Q_t^B = \max\{SS_t + \mu_{t,t+L+R-1} - IP_t, 0\} \tag{1}$$

is placed. Here SS_t is the safety stock for day t, $\mu_{t,t+L+R-1}$ is the expected demand from the placement of the order until the arrival of the next order and IP_t is the inventory position. Note that both L and R may depend on the day of the week. When values are assigned to the subscript $t+L+R-1$, the R corresponding to day t and the L corresponding to day $t+R$ are taken. Let $S_t = SS_t + \mu_{t,t+L+R-1}$ be the order-up-to quantity of day t.

This policy takes into account the inventory position for placing an order, but not its age distribution. The EWA policy, proposed by Broekmeulen & van Donselaar (2009), works like the base-stock policy, but the inventory position is decreased by an estimation of the expected outdating between the placement of an order and day $L+R-1$ thereafter. The order quantity under this more sophisticated policy for day t is

$$Q_t = \max\{SS_t + \mu_{t,t+L+R-1} - IP_t + \hat{E}_t, 0\},$$

where $\hat{E}_t$ is an estimation of E_t, the expected outdating during the interval $[t, t+L+R-2]$. Note that the outdated quantity on day $t+L+R-1$ is not included because it does not affect the ability to meet demand that day. In the EWA policy, the estimation $\hat{E}_t$ is computed assuming that demands during the interval $[t, t+L+R-2]$ are equal to their mean values. More accurate estimations of waste can be computed by using the exact distribution of demand instead of its mean, but they show little improvement and are very time-consuming (see Haijema & Minner, 2019). There is an irrelevant shift of the index of the days to be considered for outdating in the computation of $\hat{E}_t$, when compared to Broekmeulen & van Donselaar (2009). In their paper the days to be considered are $t+1, \ldots, t+L+R-1$ while we take $t, \ldots, t+L+R-2$. This is because in their study orders are placed at the end of the day but in ours they are placed at the beginning of the following day.

To estimate E_t, we need some notation. Let B_t^r, $r = 1, \ldots, m$ be the number of units in stock (on-hand) at the beginning of day t with r days of remaining lifetime after the arrival of new items, and $B_t = B_t^1 + \cdots + B_t^m$. Note that B_t is equal to IP_t except for Saturdays and Sundays, where Friday's order is included in IP_t but not in B_t. Let W_t^r be the number of units with r days of remaining lifetime which are issued on day t. The following recursive formulae relate the outdated quantity O_t on day t to the on-hand units B_t^r, the units issued W_t^r and the demand D_t (see Broekmeulen & van Donselaar, 2009):

$$\begin{aligned} O_t &= B_t^1 - W_t^1 = \max\left\{B_t^1 - D_t, 0\right\}, \\ W_t^r &= \min\left\{B_t^r, D_t - \sum_{k=1}^{r-1} W_t^k\right\}, \quad r = 1, \ldots, m, \\ B_{t+1}^{r-1} &= B_t^r - W_t^r + A_{t+1}^{r-1}, \quad r = 2, \ldots, m, \qquad B_{t+1}^m = A_{t+1}^m, \end{aligned} \tag{2}$$

where A_t^r is the number of units arriving on day t with r days of remaining lifetime. The term A_{t+1}^{r-1} is not included in formula (6) of Broekmeulen & van Donselaar (2009) because in their case all units enter the system with m days of remaining lifetime. In our case, units arriving from Tuesday to Friday have m days of remaining lifetime while units arriving on Monday have $m-2$. Due to the recursive nature of (2), there is no simple way to express O_{t+i} as a function of $B_t^1, \ldots, B_t^m$ and $D_t, \ldots, D_{t+i}$. Moreover, since $D_t, \ldots, D_{t+i}$ are random, so are $O_t, \ldots, O_{t+i}$, but a deterministic value is needed for the latter in order to approximate the expected outdating E_t. The EWA policy assumes that $D_t, \ldots, D_{t+L+R-2}$ are equal to their expected values, uses (2) to get the estimates $\hat{O}_t, \ldots, \hat{O}_{t+L+R-2}$ of $O_t, \ldots, O_{t+L+R-2}$ and then takes $\hat{E}_t = \hat{O}_t + \cdots + \hat{O}_{t+L+R-2}$.

2.2. Application of the EWA policy to the model

We first show the application of the base-stock policy to our model, which is needed for the EWA policy. Recalling (1), and due to weekly stationarity, we need to define SS_t, $t = 1, \ldots, 5$, the safety stock for Mondays, Tuesdays, Wednesdays, Thursdays and Fridays, respectively (no orders are placed on Saturdays or Sundays). Note that the values of R and L depend on the day of the week. The values of R and L to be used for the order placed on day t are the time until the next order is placed (R) and the time between the placement of that order and its arrival (L). For Monday, $t = 1$, the review interval is $R = 1$, since a new order will be placed on Tuesday, and the lead time is $L = 1$, since the order placed on Tuesday will arrive on Wednesday. Also, $L = R = 1$ for Tuesday and Wednesday ($t = 2, 3$). For Thursday, $t = 4$, the review interval is $R = 1$, since a new order will be placed on Friday, and $L = 3$, since the order placed on Friday will arrive on Monday; for Friday, $t = 5$, we have $R = 3$, $L = 1$. In other words, the period from t to $t+L+R-1$ corresponds to Mon-Tue for orders placed on Monday, Tue-Wed for orders placed on Tuesday, Wed-Thu for orders placed on Wednesday, Thu-Sun for orders placed on Thursday and Fri-Mon for orders placed on Friday. There are different ways to determine the safety stock for day t. A common option, for both nonperishable and perishable items, is to take it as a factor of the standard deviation of the demand; see Chapter 7 in Silver et al. (1998). We take it as

$$SS_t = \begin{cases} k\sigma_{t,t+L+R-1} + k_1 & \text{for } t = 1, 2, 3, \\ k\sigma_{t,t+L+R-1} + k_2 & \text{for } t = 4, 5, \end{cases}$$

with $k, k_1, k_2 \geq 0$, where the values of L, R depend on the day of the week as explained above. That is, the safety stock is proportional to the standard deviation of the demand to be covered, plus a fixed value (k_1 or k_2); we allow different values of k_j for Mondays, Tuesdays and Wednesdays, which cover only 2 day demand,

Table 1
Random variables related to the inventory system.

B_t^r	$t \geq 1; r = 1, \ldots, m$	number of units in stock (on-hand) at the beginning of day t with r days of remaining lifetime once the incoming order has arrived
B_t	$t \geq 1$	total number of units in stock (on-hand) at the beginning of day t once the incoming order has arrived
Q_t	$t \geq 1$	order quantity of day t
S_t	$t \geq 1$	order-up-to quantity of day t
O_t	$t \geq 1$	number of units outdated on day t
E_t	$t \geq 1$	expected outdating in the interval $[t, t+L+R-2]$
W_t^r	$t \geq 1; r = 1, \ldots, m$	number of units issued on day t with r days of remaining lifetime
V_t^i	$t \geq 1, i = 1, \ldots, m$	number of units ordered on day t which are in stock (on-hand) at the end of day $t+i$ before outdated units are discarded
H_t	$t \geq 1$	number of units in stock (on-hand) at the end of day t once outdated units are discarded
U_t	$t \geq 1$	unsatisfied demand on day t

and for Thursdays and Fridays, which cover 4 day demand. We decided to use only two parameters, k_1 and k_2, instead of five different parameters $k_t,\ t = 1, \ldots, 5$, one for each weekday, for reasons of simplicity and efficiency. This assertion is based on the conclusions of a simulation-optimisation model for the management of platelets used in Pérez Vaquero et al. (2016), where the optimal solutions in terms of few outdatings and freshness of units issued were found for the empirical data of 52 weeks in the CVTTH. In any event, the choice of this form of safety stocks does not affect the derivation of our formulae, and they can be straightforwardly adapted to any other form of safety stocks, such as taking a different parameter k_t for $t = 1, \ldots, 5$.

We now turn to the application of the EWA policy. To compute $\hat{E}_t$, we consider several cases, depending on the day of the week. When day t is a Monday, from (2), $O_t = (B_t^1 - D_t)^+$, where $a^+ = \max\{a, 0\}$. Since the EWA policy assumes $D_t = \mu_1$, it follows that $\hat{O}_t = (B_t^1 - \mu_1)^+$. As $L + R - 2 = 0$ in this case, $\hat{E}_t = (B_t^1 - \mu_1)^+$. The same expression is valid when day t is a Tuesday or a Wednesday (replacing μ_1 by μ_2, μ_3, respectively).

When day t is a Thursday or a Friday, $L + R - 2 = 2$. Thus, $O_t = \left(B_t^1 - D_t\right)^+$ and $\hat{O}_t = \left(B_t^1 - \mu_t\right)^+$. Also, $O_{t+1} = \left(B_{t+1}^1 - D_{t+1}\right)^+$, with $W_t^1 = \min\left\{B_t^1, D_t\right\}$, $W_t^2 = \min\left\{B_t^2, D_t - W_t^1\right\}$, which yields $B_{t+1}^1 = \left(B_t^2 - \left(D_t - B_t^1\right)^+\right)^+$ and $O_{t+1} = \left(\left(B_t^2 - \left(D_t - B_t^1\right)^+\right)^+ - D_{t+1}\right)^+$. The derivation of $O_{t+2} = \left(B_{t+2}^1 - D_{t+2}\right)^+$ is more involved. Note that $B_{t+2}^1 = B_{t+1}^2 - W_{t+1}^2$, with $B_{t+1}^2 = \left(B_t^3 - \left(D_t - W_t^1 - W_t^2\right)\right)^+$ and $W_t^2 = \min\left\{B_t^2, \left(D_t - B_t^1\right)^+\right\}$. We have

$$\begin{aligned} B_{t+1}^2 &= \left(B_t^3 - \left(\left(D_t - B_t^1\right)^+ - B_t^2\right)^+\right)^+, \\ W_{t+1}^1 &= \min\left\{\left(B_t^2 - \left(D_t - B_t^1\right)^+\right)^+, D_{t+1}\right\}, \\ W_{t+1}^2 &= \min\left\{B_{t+1}^2, \left(D_{t+1} - \left(B_t^2 - \left(D_t - B_t^1\right)^+\right)^+\right)^+\right\}. \end{aligned}$$

Therefore, $B_{t+2}^1 = \left(B_t^3 - \left(D_{t+1} + \left(D_t - B_t^1\right)^+ - B_t^2\right)^+\right)^+$. Collecting all the terms above gives $O_{t+2} = (\left(B_t^3 - \left(D_{t+1} + \left(D_t - B_t^1\right)^+ - B_t^2\right)^+\right)^+ - D_{t+2})^+$. The value of $\hat{E}_t$ is obtained by summing up O_t, O_{t+1} and O_{t+2} and replacing D_t, D_{t+1} and D_{t+2} by μ_t, μ_{t+1} and μ_{t+2}, respectively.

The expressions are valid for general m. They are simple for Monday, Tuesday and Wednesday, but are rather complicated for Thursday and Friday. However, they get simpler when a concrete value of m is taken since some B_t^r are equal to 0. For instance, $m = 5$ gives $\hat{E}_t = (B_t^3 - \mu_{4,6})^+$ for Thursdays and $\hat{E}_t = (B_t^2 - \mu_{5,6})^+ + (B_t^3 - (\mu_{5,6} - B_t^2)^+ - \mu_7)^+$ for Fridays.

3. Approximations of performance measures under the EWA policy

We now derive analytical approximations of the main performance measures in the model. Table 1 summarises the random variables related to the inventory system. We use the same notation, with small instead of capital letters, for their expected values in the steady state. The model is weekly stationary, so these expected values depend on the day of the week, which means that the expected on-hand stock h_t, say, is the same for all $t + 7k$, $k \geq 0$. See Appendix A for a theoretical justification of the existence of the long-run distribution and its periodicity. In the rest of the Section we write the formulae for $t = 1, \ldots, 7$ only. On some occasions the subscripts in the formulae become negative or zero: for instance when day t is a Monday ($t = 1$) and we write $D_{t-3,t-1}$; in those cases the value t in the formula must be understood as $t + 7$.

In this section we keep m general as long as we can. When the approximations need a specific value for m, we take $m = 5$.

3.1. Approximation of the order quantities

Note that the order quantity of day t, Q_t, is $(S_t - B_t)^+$, where $S_t = SS_t + \mu_{t,t+L+R-1} + \hat{O}_t + \cdots + \hat{O}_{t+L+R-2}$, and $\hat{O}_{t+j}$ is the estimation of the outdated quantity on day $t + j$ in Section 2.2. Approximating $\hat{O}_t$ by o_t gives an approximation of the order-up-to quantities:

$$\begin{aligned} s_t &\sim \mu_{t,t+1} + k\sigma_{t,t+1} + k_1 + o_t, \quad t = 1, 2, 3 \\ s_4 &\sim \mu_{4,7} + k\sigma_{4,7} + k_2 + o_4 + o_5 + o_6, \\ s_5 &\sim \mu_{5,1} + k\sigma_{5,1} + k_2 + o_5 + o_6 + o_7. \end{aligned} \tag{3}$$

Note that, depending on the particular value of m, some of the o_t are 0. For instance, if $m = 5$, then $o_4 = o_5 = 0$ since there is no production on Saturdays or Sundays, so there is no outdating on Thursdays or Fridays.

We now make two assumptions. The first is $(B_t - D_t)^+ \sim B_t - D_t$, which is quite reasonable since the service level is high, so most days we have $B_t \geq D_t$ and, even if $B_t < D_t$, the difference $D_t - B_t = U_t$ (unsatisfied demand on day t) is small. In fact, this assumption is common when analysing inventory systems: for instance, Silver et al. (1998) assert (p. 253) that a usual assumption for the inventory management of items with random demand is "Unit shortage costs (explicit or implicit) are so high that a practical operating procedure will always result in the average level of backorders being negligibly small when compared with the average level of the on-hand stock". The second assumption is $S_t \geq B_t$ for every weekday t, as otherwise the stock at the beginning of the day is very large and $Q_t = 0$, which is infrequent in many inventory models, such as models for the production of platelet concentrates in blood banks, where the size orders are positive at every review point.

Now we approximate b_t. For $t = 2, 3, 4, 5$, we have

$$\begin{aligned} B_t &= (B_{t-1} - D_{t-1})^+ + Q_{t-1} - O_{t-1} = (B_{t-1} - D_{t-1})^+ \\ &\quad + (S_{t-1} - B_{t-1})^+ - O_{t-1} \sim S_{t-1} - D_{t-1} - O_{t-1}. \end{aligned}$$

By (3), $b_t \sim \mu_t + k\sigma_{t-1,t} + k_1$ for $t = 2, 3, 4$ and $b_5 = \mu_{5,7} + k\sigma_{4,7} + k_2 + o_5 + o_6$. For Saturday, $B_t = (B_{t-1} - D_{t-1} - O_{t-1})^+ \sim S_{t-2} - D_{t-2,t-1} - O_{t-2} - O_{t-1}$ which yields $b_6 \sim \mu_{6,7} + k\sigma_{4,7} + k_2 + o_6$. For Sunday, $B_t = (B_{t-1} - D_{t-1} - O_{t-1})^+ \sim S_{t-3} - D_{t-3,t-1} - O_{t-3} - O_{t-2} - O_{t-1}$ and $b_7 \sim \mu_7 + k\sigma_{4,7} + k_2$. Last, for Monday, $B_t \sim S_{t-3} - D_{t-3,t-1} - O_{t-3} - O_{t-2} - O_{t-1}$, so $b_1 \sim \mu_1 + k\sigma_{5,1} + k_2$.

Since $Q_t = (S_t - B_t)^+ \sim S_t - B_t$, the approximations for the expected order quantities are

$$
\begin{aligned}
q_1 &\sim \mu_2 + k(\sigma_{1,2} - \sigma_{5,1}) + k_1 - k_2 + o_1\\
q_2 &\sim \mu_3 + k(\sigma_{2,3} - \sigma_{1,2}) + o_2\\
q_3 &\sim \mu_4 + k(\sigma_{3,4} - \sigma_{2,3}) + o_3\\
q_4 &\sim \mu_{5,7} + k(\sigma_{4,7} - \sigma_{3,4}) + k_2 - k_1 + o_4 + o_5 + o_6\\
q_5 &\sim \mu_1 + k(\sigma_{5,1} - \sigma_{4,7}) + o_7
\end{aligned}
$$

Note that the formulae above depend on o_t, $t = 1, \dots, 7$, which are unknown. We give approximations for their values in Section 3.4.

3.2. Expected on-hand inventory

Since $H_t = (B_t - D_t)^+ - O_t$ and stockouts are assumed to be uncommon, we can approximate $H_t \sim B_t - D_t - O_t$ and the formulae in Section 3.1 give

$$
\begin{aligned}
h_1 &\sim k\sigma_{5,1} + k_2 - o_1\\
h_t &\sim k\sigma_{t-1,t} + k_1 - o_t, \qquad t = 2, 3, 4\\
h_5 &\sim \mu_{6,7} + k\sigma_{4,7} + k_2 + o_6\\
h_6 &\sim \mu_7 + k\sigma_{4,7} + k_2\\
h_7 &\sim k\sigma_{4,7} + k_2 - o_7
\end{aligned}
$$

3.3. A formula for v_t^i and the age distribution of the stock

In this section we derive a formula for $E[V_t^i]$, the expected number of units ordered on day t which are in stock at the end of day $t+i$ before outdated units are discarded. First note that $V_t^i = 0$ for $i = 1, \dots, m$ when $t = 6$ is a Saturday or $t = 7$ is a Sunday; also, $V_t^1 = V_t^2 = 0$ if $t = 5$ is a Friday, since units arrive on Monday. For the rest of the values V_t^i, recall that $Q_t = (S_t - B_t)^+$. Now, since the order placed on day t sets the inventory position to S_t and we use a FIFO issuing policy, V_t^i can be approximated by $(S_t - D_{t,t+i} - (O_t + \dots + O_{t+i-1}))^+$, with a limit of $(S_t - B_t)^+$. That is,

$$V_t^i \sim \min\left\{(S_t - D_{t,t+i} - (O_t + \dots + O_{t+i-1}))^+, (S_t - B_t)^+\right\}.$$

In order to compute $E[V_t^i]$, we condition on B_t. In what follows, we take S_t as if it was deterministic, which is not true because under the EWA policy it depends on the age distribution of the stock. Let $\tilde{D}_{t,t+i} = D_{t,t+i} + O_t + \dots + O_{t+i-1}$, for $t \geq 1$, $i = 1, \dots, m$. Note that if $B_t \geq S_t$, then $V_t^i = 0$ for every $i = 1, \dots, m$; if $B_t < S_t$, then

$$
\begin{aligned}
E[V_t^i | B_t] &\sim (S_t - B_t) P(\tilde{D}_{t,t+i} \leq B_t) + \int_{B_t}^{S_t} (S_t - x) f_{\tilde{D}_{t,t+i}}(x) dx\\
&= \int\int_{\{(x,y): B_t < y < S_t, 0 < x < y\}} f_{\tilde{D}_{t,t+i}}(x) dx dy\\
&= \int_{B_t}^{S_t} F_{\tilde{D}_{t,t+i}}(x) dx,
\end{aligned}
$$

where $f_{\tilde{D}_{t,t+i}}$ and $F_{\tilde{D}_{t,t+i}}$ are the PDF and CDF, respectively, of the demand in the interval $[t, t+i]$ plus the outdated units in the interval $[t, t+i-1]$. Thus,

$$
E[V_t^i | B_t] \sim \begin{cases} \int_{B_t}^{S_t} F_{\tilde{D}_{t,t+i}}(x) dx & \text{if } B_t < S_t,\\ 0 & \text{if } B_t \geq S_t. \end{cases}
$$

By the properties of conditional expectation,

$$
\begin{aligned}
E\left[V_t^i\right] &\sim \int_0^{S_t} \left(\int_y^{S_t} F_{\tilde{D}_{t,t+i}}(x) dx\right) f_{B_t}(y) dy\\
&= \int\int_{\{(x,y): 0 < y < x < S_t\}} F_{\tilde{D}_{t,t+i}}(x) f_{B_t}(y) dx dy\\
&= \int_0^{S_t} \left(\int_0^x f_{B_t}(y) dy\right) F_{\tilde{D}_{t,t+i}}(x) dx\\
&= \int_0^{S_t} F_{B_t}(x) F_{\tilde{D}_{t,t+i}}(x) dx. \qquad (4)
\end{aligned}
$$

To apply formula (4), we need approximations of $F_{\tilde{D}_{t,t+i}}(x)$ and $F_{B_t}(x)$. Since we approximate O_t by its expected value o_t, we have

$$
\begin{aligned}
F_{\tilde{D}_{t,t+i}}(x) &= P(D_{t,t+i} + O_t + \dots + O_{t+i-1} \leq x)\\
&\sim F_{t,t+i}(x - (o_t + \dots + o_{t+i-1})).
\end{aligned}
$$

The approximation of B_t depends on the day of the week. When day t is a Monday, using the approximations in Section 3.1,

$$
\begin{aligned}
F_{B_t}(x) &\sim P(S_{t-3} - D_{t-3,t-1} - O_{t-3} - O_{t-2} - O_{t-1} \leq x)\\
&\sim P(\mu_{5,1} + k\sigma_{5,1} + k_2 - D_{t-3,t-1} \leq x)\\
&= \overline{F}_{5,7}(\mu_{5,1} + k\sigma_{5,1} + k_2 - x),
\end{aligned}
$$

where $\overline{F}(t) = 1 - F(t)$.

When day t is Tuesday, Wednesday or Thursday,

$$
\begin{aligned}
F_{B_t}(x) &\sim P(S_{t-1} - D_{t-1} - O_{t-1} \leq x)\\
&\sim P(D_{t-1} \geq \mu_{t-1,t} + k\sigma_{t-1,t} + k_1 - x)\\
&= \overline{F}_{t-1}(\mu_{t-1,t} + k\sigma_{t-1,t} + k_1 - x).
\end{aligned}
$$

with $t = 2, 3, 4$, respectively.

Analogously, for Friday:

$$
\begin{aligned}
F_{B_t}(x) &= P(S_{t-1} - D_{t-1} - O_{t-1} \leq x)\\
&\sim \overline{F}_4(\mu_{4,7} + k\sigma_{4,7} + k_2 + o_5 + o_6 - x).
\end{aligned}
$$

The above expressions, together with (4), yield the required approximations of v_t^i, $t = 1, \dots, 5$, $i = 1, \dots, m$. For instance, the expected number of units that are ordered on Thursday and are in stock at the end of Sunday, v_4^3, can be approximated by

$$
\int_0^{\mu_{4,7} + k\sigma_{4,7} + k_2 + o_4 + o_5 + o_6} \overline{F}_3(\mu_{3,4} + k\sigma_{3,4} + k_1 - x) \times F_{4,7}(x - (o_4 + o_5 + o_6)) dx.
$$

These formulae are explicit; however, they depend on $o_1, \dots, o_7$, the expected number of outdated units each day. In the next section we show how to estimate these quantities. Once they have been estimated, their values can be plugged into the above formulae to compute the approximations of v_t^i, since the distributions $F_{t,s}$ are known.

The formulae above enable us to approximate the age distribution of the stock b_t^r. In fact, when day t is not a Monday, the number of units with r days of remaining lifetime at the beginning of day t, once the incoming order has arrived, B_t^r, is $V_{t+r-m-1}^{m-r}$, for $r = 1, \dots, m-1$, and $B_t^m = Q_{t-1}$. When day t is a Monday, $B_t^r = V_{t+r-m-1}^{m-r}$, for $r = 1, \dots, m-3$, $B_t^{m-2} = Q_{t-3}$ and $B_t^{m-1} = B_t^m = 0$. The approximation of b_t^r is obtained by substituting the values of Q and V by the corresponding approximations of q and v.

3.4. Expected outdating

In this section we set $m = 5$. Derivation of the formulae when m is 4 and 6 can be found in Appendix B. Other values of m can be worked out in a similar way.

Let $m = 5$. Recall first that $o_4 = o_5 = 0$ since there is no outdating on Thursdays or Fridays. Now, since $O_t = V^5_{t-5}$, we use formula (4) with $i = 5$, and get

$$o_1 \sim \int_0^{\mu_{3,4}+k\sigma_{3,4}+k_1+o_3} \overline{F}_2(\mu_{2,3}+k\sigma_{2,3}+k_1-x)F_{3,1}(x-o_3-o_6-o_7)dx,$$
$$o_2 \sim \int_0^{\mu_{4,7}+k\sigma_{4,7}+k_2+o_6} \overline{F}_3(\mu_{3,4}+k\sigma_{3,4}+k_1-x)F_{4,2}(x-o_6-o_7-o_1)dx,$$
$$o_3 \sim \int_0^{\mu_{5,1}+k\sigma_{5,1}+k_2+o_6+o_7} \overline{F}_4(\mu_{4,7}+k\sigma_{4,7}+k_2+o_6-x) \times F_{5,3}(x-o_6-o_7-o_1-o_2)dx,$$
$$o_6 \sim \int_0^{\mu_{1,2}+k\sigma_{1,2}+k_1+o_1} \overline{F}_{5,7}(\mu_{5,1}+k\sigma_{5,1}+k_2-x)F_{1,6}(x-o_1-o_2-o_3)dx,$$
$$o_7 \sim \int_0^{\mu_{2,3}+k\sigma_{2,3}+k_1+o_2} \overline{F}_1(\mu_{1,2}+k\sigma_{1,2}+k_1-x)F_{2,7}(x-o_2-o_3-o_6)dx. \quad (5)$$

The formulae above are cyclical, since o_s is needed to compute o_t. We solve this via an iterative procedure: we set all $o_t = 0$, compute the formulae in (5) to get an approximation of o_t and plug the new values into the formulae to get another approximation. This procedure is iterated until the changes in the o_t are smaller than a tolerance value. While we have not proved analytically that this procedure converges, in all our settings below a small number of iterations (less than 8) were needed for a tolerance of 10^{-3}.

3.5. Remaining lifetime of units issued and freshness

For a given day t, the number of units issued with a remaining lifetime of r days, W^r_t can be expressed as $W^r_t = V^{m-r}_{t+r-m-1} - V^{m+1-r}_{t+r-m-1}$, for $r = 1,\dots,m-1$ and $W^m_t = Q_{t-1} - V^1_{t-1}$. This formula has two exceptions related to the weekend: when t is a Monday and $r = m-2$, $W^{m-2}_t = Q_{t-3} - V^3_{t-3}$, and when t is a Saturday and $r = m$, $W^m_t = 0$. Thus, the values w^r_t can be approximated by substituting Q and V by their approximations in Sections 3.1 and 3.3, respectively. Also, freshness of units issued on day t, $t = 1,\dots,7$ can be approximated by

$$\frac{\sum_{r=1}^m r w^r_t}{\sum_{r=1}^m w^r_t}.$$

Note that extending the formula derived by Broekmeulen & van Donselaar (2019) for freshness in stationary models to the present situation is difficult, since the distribution of the demand, the values of the lead time L and the review interval R, and the remaining lifetime of units when they enter the inventory depend on the day of the week. Thus, there seems to be no easy way of using Little's formula in our context to find the freshness of units delivered each day of the week. Appendix C shows how that approach can be used to get a formula for the freshness of units without differentiating by the day of issue, although it still requires the approximations in Section 3.4.

3.6. Expected shortage

We are assuming that stockouts are rare, but they may still occur, so it is important to have an approximation of the expected shortage, u_t, where $U_t = (D_t - B_t)^+$. We begin when day t is a Tuesday, Wednesday or Thursday. Using the approximations for B_t, S_t and O_t in the previous sections, $U_t \sim (D_t - S_{t-1} + D_{t-1} + O_{t-1})^+$, so

$$u_t \sim E\left[(D_{t-1,t} - (\mu_{t-1,t}+k\sigma_{t-1,t}+k_1))^+\right]$$
$$= \int_{\mu_{t-1,t}+k\sigma_{t-1,t}+k_1}^{\infty} (x-(\mu_{t-1,t}+k\sigma_{t-1,t}+k_1))f_{t-1,t}(x)dx$$
$$= \int_{\mu_{t-1,t}+k\sigma_{t-1,t}+k_1}^{\infty} \overline{F}_{t-1,t}(x)dx,$$

for $t = 2, 3, 4$.

When t is a Friday, $U_t \sim (D_t - S_{t-1} + D_{t-1} + O_{t-1})^+$ and

$$u_5 \sim \int_{\mu_{4,7}+k\sigma_{4,7}+k_2+o_5+o_6}^{\infty} \overline{F}_{4,5}(x)dx.$$

When t is a Saturday, $U_t \sim (D_t - S_{t-2} + D_{t-2,t-1} + O_{t-2} + O_{t-1})^+$, so

$$u_6 \sim \int_{\mu_{4,7}+k\sigma_{4,7}+k_2+o_6}^{\infty} \overline{F}_{4,6}(x)dx.$$

For Sunday, $U_t \sim (D_t - S_{t-3} - D_{t-3,t-1} + O_{t-3} + O_{t-2} + O_{t-1})^+$ and

$$u_7 \sim \int_{\mu_{4,7}+k\sigma_{4,7}+k_2}^{\infty} \overline{F}_{4,7}(x)dx.$$

Lastly, when t is a Monday, $U_t \sim (D_t - S_{t-3} - D_{t-3,t-1} + O_{t-3} + O_{t-2} + O_{t-1})^+$, so

$$u_1 \sim \int_{\mu_{5,1}+k\sigma_{5,1}+k_2}^{\infty} \overline{F}_{5,1}(x)dx.$$

The fill rate of day t can be approximated by $100(1 - u_t/\mu_t)$.

3.7. Probability of on-hand inventory being lower than a threshold

The probability of the on-hand inventory at the end of day t (before outdated units are discarded) being less than a is $P[D_t > B_t - a]$, which can be approximated in a similar way to the previous section, obtaining:

$$P(D_t > B_t - a) = \begin{cases} \overline{F}_{5,1}(\mu_{5,1}+k\sigma_{5,1}+k_2-a) & \text{if day } t \text{ is a Monday,} \\ \overline{F}_{t-1,t}(\mu_{t-1,t}+k\sigma_{t-1,t}+k_1-a) & \text{if day } t \text{ is a Tuesday,} \\ & \text{Wednesday or Thursday,} \\ \overline{F}_{4,5}(\mu_{4,7}+k\sigma_{4,7}+k_2+o_5+o_6-a) & \text{if day } t \text{ is a Friday,} \\ \overline{F}_{4,6}(\mu_{4,7}+k\sigma_{4,7}+k_2+o_6-a) & \text{if day } t \text{ is a Saturday,} \\ \overline{F}_{4,7}(\mu_{4,7}+k\sigma_{4,7}+k_2-a) & \text{if day } t \text{ is a Sunday.} \end{cases}$$

In particular, taking $a = 0$ gives an approximation of the probability of a stockout, i.e. 1−service level.

4. Application to the CVTTH data

We assess the accuracy of our approximations by comparing them with the values obtained by Monte Carlo simulations in a real example. For the distributions of the daily demand we choose those fitted to the data of the CVTTH for 2012 (which were considered by Pérez Vaquero et al. (2016)), i.e. (discretised) normal distributions with means and standard deviations as shown in Table 2. The lifetime of platelet concentrates is $m = 5$.

The normality assumption was checked using the Shapiro test for normality. Significant p−values were found for Saturday and Sunday. Note however that the distributions of the demand on those days are never used on their own in our formulae; instead they are used at least together with Friday (for Saturday) and with Friday and Saturday (for Sunday). Therefore, what needs to be checked is not the normality of the demand on Saturdays and on Sundays but the normality on Friday + Saturday and Friday + Saturday + Sunday. Table 3 shows that the normality assumption is reasonable.

We remark that the hypothesis of normality is not necessary for the derivation of our formulae: they can be applied with any other distribution, including the empirical distribution of historical data if available.

Regarding independence of the daily demand, we performed the (Pearson) correlation test for each pair of consecutive days. Two of the pairs were found to be significant, namely Tue-Wed (p−value 0,045) and Thu-Fri (p−value 0,004). An analysis of the

Table 2
Means and standard deviation for daily demand of platelet concentrates in CVTTH in 2012.

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
μ_t	27,75	23,71	24,57	22,16	29,39	13,29	11,82
σ_t	6,85	5,65	7,86	6,90	7,81	4,89	4,38

Table 3
$p-$values of the Shapiro test of normality for demand of platelet concentrates in CVTTH in 2012.

Mon	Tue	Wed	Thu	Fri	Fri + Sat	Fri + Sat + Sun
0,05	0,32	0,95	0,22	0,50	0,59	0,41

Table 4
$p-$values of the Pearson correlation test for independence of demand on consecutive days in CVTTH in 2012.

Mon-Tue	Tue-Wed	Wed-Thu	Thu-Fri	Fr-Sat	Sat-Sun	Sun-Mon
0,10	0,06	0,38	0,11	0,67	0,07	0,79

scatterplots of these two pairs reveals the existence of influential points which correspond to very low demand (under 10 units) on public holidays over the year. Once these points are removed the $p-$values are greater than 0,05, so independence can be assumed: see Table 4.

We compare our approximations with the estimations obtained from simulations for different values of parameters k, k_1, k_2. Namely, we take all combinations where k ranges in $\{1.5, 2, 2.5, 3\}$ and (k_1, k_2) is (0,0) or (10,5). For the Monte Carlo simulations, we set 1000 runs and a run length of 520 weeks (10 years), where the first 52 weeks of each simulation run are taken as the warm-up period. The number of simulations and their length are chosen to give a small simulation error. This can be checked in Table D.8 in Appendix D which shows, for the case $k = 1.5, k_1 = k_2 = 0$, the standard deviation of the estimations using simulation together with their relative errors, measured as the ratio of the half-width of the 95% confidence interval over the sample mean. In almost every instance the relative error is smaller than 2%, with the only exceptions being quantities with a very small sample mean. Increasing the length of the simulation runs or their number (and thus increasing the simulation time) produces almost no changes in the sample means. The computation of our formulae in Section 3 for the setting $k = 1.5$, $k_1 = 0$, $k_2 = 0$ takes 4 ms. on an Intel(R) Core(TM) i5 (3.30 GHz), while simulation takes 1130 ms.

4.1. Results for the CVTTH data

Table 5 shows the results of our formulae and the simulation for $k = 1.5$, $k_1 = 0$, $k_2 = 0$. The tables for the rest of the cases are available in Section 1 of the Supplementary Material file. In the tables there are two rows for each quantity: the upper row (plain font) is the result of the application of the formulae in Section 3; the lower row (italic font) is the result obtained by averaging the results over the simulation runs. We explain Table 5 in detail, and the rest of the tables have the same structure.

The first part of the table gives the results for each day of the week. The first column shows the values of b_t; for instance b_1, the expected number of units in stock at the beginning of a Monday is 46,2 by our formula in Section 3.1 and 46,8 by the simulation. Below we write first the result of our formula and then, in brackets, the result of simulation, i.e. b_1 is 46,2 (46,8). The second column is q_t, the expected order size; thus, for instance, the expected order size on Mondays is 18,6 (18,3). The next column gives o_t, the expected quantity outdated on day t: so there are, on average, 0,17 (0,14) units outdated on Wednesdays. The next column is h_t, the

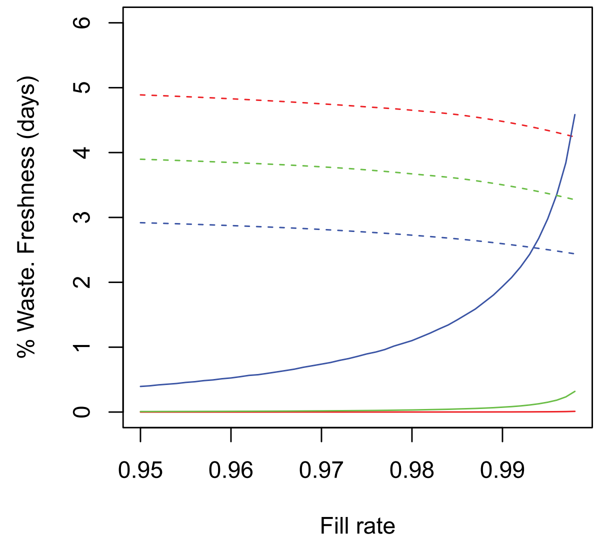

Fig. 1. Efficient Frontier relating waste with the fill rate (solid curve) for $m = 4$ (blue), $m = 5$ (green) and $m = 6$ (red). Dotted curves represent the freshness of units issued in each optimal configuration. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

expected number of units on-hand at the end of day t. Column u gives the value of u_t, the expected shortage on day t. The following two columns give the service level and the probability of the on-hand stock being below a threshold at the end of the day; we use 5 units as the threshold in all settings. Thus, for instance, the service level on Tuesdays is 94,0% (95,2%) and 14,7% (14,1%) of Tuesdays end with an on-hand stock lower than 5 units. The following columns $b(1), \ldots, b(5)$ are the values of b_t^r, the expected number of units with a remaining lifetime of r days at the beginning of day t; for instance, the expected number of units with 2 days of remaining lifetime at the beginning of a Monday is 18,8 (18,9). The following five columns $w(1), \ldots, w(5)$ are w_t^r, the number of units issued on day t with remaining lifetime equal to r days; for instance, the expected number of units issued on Monday with 2 days of remaining lifetime is 16,7 (16,7). The last column is freshness, the expected remaining lifetime of units issued on day t; for instance, units issued on Wednesday have an expected remaining lifetime of 4,14 (4,09).

The second part of the table, below the “Week” line, summarises the values over the whole week, computing sums, averages or percentages over the 7 days where appropriate. The first column gives the average of the b_t values, i.e. the average number of units on-hand at the beginning of a day: 43,9 (44,1). q is the total number of units ordered over the week: 152,9 (151,7), which correspond to 100,2% (99,4%) of the demand. The next column shows that 0,25 (0,22) units go out of date over the week, i.e. 0,16% (0,15%) of the units ordered. The average on-hand inventory at the end of the day is 22,0 (22,4) units. There are, on average, 1,426 (1,243) units not served over the whole week, i.e. 0,93% (0,81%) of the demand, so the fill rate is 99,07% (99,19%). The following two columns show that the service level is 95.6% (96,2%); i.e. 4,4% (3,8%) of the days have a stockout, and 9,6% (9,5%) of the days end with less than 5 units in stock. Columns $b(1), \ldots, b(5)$

Table 5
Performance measures for the CVTTH data in Section 4: approximations by formulae (plain font) and simulation (italic font). $k = 1.5$; $k_1 = 0$; $k_2 = 0$.

Day	b	q	o	h	u	s.l.	P(u.t.)	b(1)	b(2)	b(3)	b(4)	b(5)	w(1)	w(2)	w(3)	w(4)	w(5)	freshness
Monday	46,2	18,6	0,00	18,4	0,328	0,938	0,120	0,0	18,8	27,7	0,0	0,0	0,0	16,7	11,1	0,0	0,0	2,40
	46,8	*18,3*	*0,00*	*19,3*	*0,226*	*0,954*	*0,110*	*0,0*	*18,9*	*27,9*	*0,0*	*0,0*	*0,0*	*16,7*	*10,8*	*0,0*	*0,0*	*2,39*
Tuesday	37,0	25,9	0,08	13,2	0,229	0,940	0,147	2,2	16,6	0,0	0,0	18,6	2,1	13,6	0,0	0,0	8,0	2,93
	37,6	*25,4*	*0,08*	*14,0*	*0,197*	*0,952*	*0,141*	*2,2*	*17,1*	*0,0*	*0,0*	*18,3*	*2,1*	*13,9*	*0,0*	*0,0*	*7,5*	*2,87*
Wednesday	39,1	23,5	0,17	14,4	0,252	0,940	0,139	3,1	0,0	0,0	10,5	25,9	2,9	0,0	0,0	9,6	12,1	4,14
	39,5	*22,6*	*0,14*	*15,0*	*0,238*	*0,947*	*0,138*	*3,2*	*0,0*	*0,0*	*10,8*	*25,4*	*3,1*	*0,0*	*0,0*	*9,8*	*11,5*	*4,09*
Thursday	37,8	57,3	0,00	15,7	0,275	0,939	0,132	0,0	0,0	0,9	13,7	23,5	0,0	0,0	0,9	11,9	9,3	4,38
	37,5	*57,5*	*0,00*	*15,6*	*0,254*	*0,944*	*0,141*	*0,0*	*0,0*	*1,0*	*14,0*	*22,6*	*0,0*	*0,0*	*1,0*	*12,2*	*8,8*	*4,36*
Friday	73,0	27,7	0,00	43,6	0,000	1,000	0,000	0,0	0,0	1,8	14,2	57,3	0,0	0,0	1,8	13,2	14,7	4,43
	73,1	*27,9*	*0,00*	*43,7*	*0,000*	*1,000*	*0,000*	*0,0*	*0,0*	*1,8*	*13,8*	*57,5*	*0,0*	*0,0*	*1,8*	*12,9*	*14,6*	*4,44*
Saturday	43,6	0,0	0,00	30,3	0,013	0,996	0,011	0,0	0,0	1,0	42,6	0,0	0,0	0,0	0,8	12,4	0,0	3,93
	43,7	*0,0*	*0,00*	*30,4*	*0,011*	*0,997*	*0,011*	*0,0*	*0,0*	*0,9*	*42,8*	*0,0*	*0,0*	*0,0*	*0,8*	*12,5*	*0,0*	*3,94*
Sunday	30,3	0,0	0,00	18,5	0,329	0,938	0,120	0,0	0,1	30,2	0,0	0,0	0,0	0,1	11,4	0,0	0,0	2,99
	30,4	*0,0*	*0,00*	*18,9*	*0,317*	*0,942*	*0,124*	*0,0*	*0,1*	*30,3*	*0,0*	*0,0*	*0,0*	*0,1*	*11,4*	*0,0*	*0,0*	*2,99*
Week																		
Average	43,9			22,0		0,956	0,096	0,8	5,1	8,8	11,6	17,9						
	44,1			*22,4*		*0,962*	*0,095*	*0,8*	*5,2*	*8,8*	*11,6*	*17,7*						
Sum		152,9	0,25		1,426								5,0	30,4	25,9	47,2	44,2	3,62
		151,7	*0,22*		*1,243*								*5,2*	*30,7*	*25,8*	*47,4*	*42,4*	*3,60*
Percentage		100,2%	0,16%		0,93%								3,3%	19,9%	17,0%	30,9%	28,9%	
		99,4%	*0,15%*		*0,81%*								*3,4%*	*20,3%*	*17,0%*	*31,3%*	*28,0%*	

Table 6
Performance measures for the CVTTH data in Section 4: approximations by formulae (plain font) and simulation (italic font). Aggregated weekly results.

Safety stock	b	q	o	h	u	s.l.	P(u.t.)	b(1)	b(2)	b(3)	b(4)	b(5)	w(1)	w(2)	w(3)	w(4)	w(5)	freshness
k=1,5, k1=0, k2=0	43,9	100,2%	0,16%	22,0	0,93%	0,956	0,096	0,8	5,1	8,8	11,6	17,9	3,3%	19,9%	17,0%	30,9%	28,9%	3,62
	44,1	*99,4%*	*0,15%*	*22,4*	*0,81%*	*0,962*	*0,095*	*0,8*	*5,2*	*8,8*	*11,6*	*17,7*	*3,4%*	*20,3%*	*17,0%*	*31,3%*	*28,0%*	*3,60*
k=1,5, k1=10, k2=5	51,0	100,4%	0,36%	29,1	0,19%	0,990	0,024	1,3	6,3	10,7	14,8	17,9	5,5%	23,2%	20,0%	36,9%	14,4%	3,31
	51,0	*100,1%*	*0,31%*	*29,2*	*0,18%*	*0,991*	*0,025*	*1,3*	*6,4*	*10,7*	*14,8*	*17,8*	*5,6%*	*23,3%*	*20,1%*	*36,8%*	*14,2%*	*3,31*
k=2, k1=0, k2=0	49,4	100,4%	0,43%	27,5	0,26%	0,985	0,038	1,4	6,5	10,1	13,6	18,0	6,0%	23,4%	16,4%	34,1%	20,1%	3,39
	49,5	*100,1%*	*0,36%*	*27,6*	*0,25%*	*0,987*	*0,039*	*1,4*	*6,5*	*10,1*	*13,6*	*17,8*	*6,1%*	*23,5%*	*16,5%*	*34,4%*	*19,5%*	*3,38*
k=2, k1=10, k2=5	56,6	100,9%	0,86%	34,6	0,04%	0,997	0,007	2,2	7,8	12,4	16,2	18,0	9,1%	25,7%	21,2%	35,4%	8,6%	3,09
	56,5	*100,7%*	*0,72%*	*34,5*	*0,05%*	*0,998*	*0,008*	*2,2*	*7,8*	*12,5*	*16,1*	*17,9*	*9,2%*	*25,7%*	*21,4%*	*35,3%*	*8,3%*	*3,08*
k=2,5, k1=0, k2=0	55,0	101,0%	1,00%	33,0	0,06%	0,996	0,012	2,3	7,9	11,6	15,2	18,1	9,8%	25,4%	16,8%	35,1%	13,0%	3,16
	54,8	*100,7%*	*0,82%*	*32,8*	*0,06%*	*0,996*	*0,013*	*2,3*	*7,8*	*11,5*	*15,2*	*18,0*	*9,8%*	*25,4%*	*16,9%*	*35,3%*	*12,7%*	*3,16*
k=2,5, k1=10, k2=5	62,2	101,8%	1,80%	40,0	0,01%	0,999	0,002	3,4	9,2	14,2	17,2	18,3	13,6%	26,7%	22,8%	32,1%	4,7%	2,88
	61,8	*101,5%*	*1,47%*	*39,7*	*0,01%*	*1,000*	*0,002*	*3,3*	*9,1*	*14,2*	*17,1*	*18,1*	*13,6%*	*26,5%*	*23,5%*	*31,8%*	*4,5%*	*2,87*
k=3, k1=0, k2=0	60,6	102,1%	2,02%	38,4	0,01%	0,999	0,003	3,5	9,2	13,1	16,6	18,3	14,1%	25,8%	18,1%	34,1%	7,9%	2,96
	60,6	*101,7%*	*1,69%*	*38,4*	*0,01%*	*0,999*	*0,004*	*3,5*	*9,1*	*13,1*	*16,6*	*18,3*	*14,5%*	*25,6%*	*18,1%*	*34,0%*	*7,8%*	*2,95*
k=3, k1=10, k2=5	67,8	103,4%	3,28%	45,2	0,00%	1,000	0,000	4,7	10,5	15,9	18,1	18,6	18,3%	26,5%	24,5%	28,2%	2,5%	2,70
	67,7	*102,9%*	*2,78%*	*45,2*	*0,00%*	*1,000*	*0,000*	*4,7*	*10,4*	*15,9*	*18,1*	*18,6*	*18,7%*	*26,1%*	*25,5%*	*27,4%*	*2,4%*	*2,69*

Table 7
Mean absolute error between formulae and simulation in the settings of Section 4.

Day	b	q	o	h	u	s.l.	P(u.t.)	b(1)	b(2)	b(3)	b(4)	b(5)	w(1)	w(2)	w(3)	w(4)	w(5)	freshness
Monday	0,2	0,3	0,00	0,3	0,020	0,003	0,002	0,1	0,3	0,4	0,0	0,0	0,1	0,1	0,1	0,0	0,0	0,00
Tuesday	0,3	0,7	0,04	0,4	0,005	0,002	0,002	0,2	0,4	0,0	0,0	0,3	0,1	0,2	0,0	0,0	0,2	0,03
Wednesday	0,5	0,6	0,31	0,3	0,003	0,002	0,001	0,1	0,0	0,0	0,2	0,7	0,4	0,0	0,0	0,3	0,3	0,06
Thursday	0,5	0,6	0,00	0,5	0,005	0,001	0,002	0,0	0,0	0,3	0,5	0,6	0,0	0,0	0,3	0,3	0,3	0,03
Friday	0,3	0,4	0,00	0,3	0,000	0,000	0,000	0,0	0,1	0,2	0,4	0,6	0,0	0,1	0,2	0,2	0,1	0,01
Saturday	0,3	0,0	0,00	0,3	0,000	0,000	0,000	0,0	0,1	0,3	0,5	0,0	0,0	0,1	0,1	0,2	0,0	0,02
Sunday	0,3	0,0	0,00	0,3	0,006	0,001	0,003	0,0	0,2	0,4	0,0	0,0	0,0	0,1	0,1	0,0	0,0	0,01
Week																		
Average	0,1			0,1		0,001	0,001	0,0	0,1	0,0	0,0	0,1						
Sum		0,6	0,32		0,029								0,2	0,2	0,4	0,3	0,6	0,01
Percentage		0,4%	0,20%		0,02%								0,1%	0,2%	0,3%	0,3%	0,3%	

give the average number of units in stock with r days of remaining lifetime at the beginning of a day; thus, the average number of units with 4 days of remaining lifetime at the beginning of a day is 11,6 (11,6). The next five columns give the distribution of the remaining lifetime of units issued; for instance, 30,4 (30,7) units are issued with 2 days of remaining lifetime over the week, corresponding to 19,9% (20,3%) of the total units issued. The freshness of units issued is 3,62 (3,60).

Analysing the results for all the values of (k, k_1, k_2) (Table 5 and Tables 1–7 in the Supplementary Material file) close agreement is observed between the values given by the formulae in Section 3 and those obtained by simulation. For daily measures, there are no discrepancies greater than 1,5 units in any quantity and there are very few cases where the discrepancy is greater than 1 unit. One of the main novelties of the paper lies in the formulae for the age distribution of units issued (columns $w(1), \ldots, w(5)$). In all instances the discrepancies observed are lower than 1 unit for all ages and days. This results in a very accurate estimation of freshness, with no discrepancies greater than 0,1 days in any case, for every day. Note also that columns b and $b(1), \ldots, b(5)$ are computed using very different expressions (those in Sections 3.1 and 3.3, respectively); but the sum of columns $b(1), \ldots, b(5)$ is still very close to column b, which shows the internal consistency of our formulae.

A summary of the eight tables, with the aggregate (weekly) results, is given in Table 6. This table shows for each measure the value of either the Average or Percentage row, as in the last rows of Table 5. Table 6 also shows very close agreement between formulae and simulations: there are no discrepancies greater than 0,5 units or greater than 1% in the case of percentages (actually, most percentages show discrepancies lower than 0,5%).

Table 7 shows the mean absolute errors between formulae and simulations over the 8 settings; that is, for each quantity, the absolute values of the differences between the value given by the formula and the value obtained by simulation are added together and then divided by 8 (the number of settings). All the figures in the table are smaller than 1 unit (or 0,5% in the case of percentages) and the approximation of freshness in particular is very accurate, with a mean absolute error of 0,01 days.

4.2. Optimisation

We do not include costs in our model. This could be readily done since the cost function can be written in terms of the approximations given in Section 3. In this paper we do not look deeper into the use of the formulae for optimisation of the parameters in the model, but we show a direct application which consists on finding the safety factors to minimise outdating subject to the constraint of having a minimum fill rate. The optimisation can be done using the formulae in Section 3 via a grid search of the values k, k_1, k_2. An Efficient Frontier showing how the outdating depends on the target fill rate can then be computed. The Efficient Frontier is a tool proposed and analysed in Broekmeulen & van Donselaar (2019) which enables different settings to be compared. We use it here to compare three values of m (4, 5 and 6) in the model in Section 4 (normal distributed demand with mean and standard deviations in Table 2). See Fig. 1, which also includes the freshness of the delivered platelet concentrates for the optimal parameters for each fill rate. Observe that there is a small difference in waste between $m = 5$ and $m = 6$, and $m = 4$ gives much higher waste quantities. For freshness, there are three almost parallel curves with a shift slightly smaller than 1. The overall conclusion is that freshness is not greatly influenced by the fill rate and the differences in freshness for different values of m are basically the differences in the lifetimes of the units. On the other hand, waste is affected by the fill rate, especially when $m = 4$, with the differences between $m = 5$ and $m = 6$ being very small.

5. Simulation experiments

Section 4 shows that our formulae give accurate approximations in the case of the data analysed in Pérez Vaquero et al. (2016). Since these data can be seen as a specific environment, with $m = 5$, normally distributed demand, low demand uncertainty and similar average demand on non-weekend days, in this section we analyse other situations to determine the extent to which our approximations are applicable. There are many settings that can be considered, so we have chosen the following four "parameters" to vary: (1) Lifetime m: 4, 5 and 6; (2) demand distribution: normal distribution with CV=0,25, normal distribution with CV=0,5, exponential distribution and Poisson distribution; (3) mean demand on weekdays: flat demand (30,30,30,30,30) and peaked demand (22,32,42,32,22), with the mean demand on Saturday and Sunday set to 15 in both cases (note that the mean demand for the whole week is 180 units); (4) safety stock parameters (k, k_1, k_2): (1.5,0,0), (2,5,5) and (2.5,10,5). The choice of the normal and Poisson distributions is a natural one since they are, by far, the most widely used distributions for modelling demand in inventory systems, especially in blood banks. Setting CV at 0,25 is similar to the data in Pérez Vaquero et al. (2016); we also take a value of 0,5 to see the effect of greater variability in demand on the accuracy of the approximations. Note that the CV in the Poisson distribution is 1 over the square root of the mean, so the values obtained are 0,15 to 0,26 for that distribution in our scenarios. Haijema et al. (2007) take values of CV from 0,20 to 0,35 for their experiments, while Stanger et al. (2012) take values from 0,1 to 0,5. Other publications, such as van Donselaar & Broekmeulen (2011) and Haijema & Minner (2019), use the ratio variance/mean instead of CV as the parameter and take values in the range from 0,75 to 4; note that the variance/mean ratio is 1 in the Poisson distribution and ranges from 0,9 to 10,5 in the normal distributions that we consider. The exponential distribution is not commonly used for modelling demand, but it is considered, for instance, in Williams & Patuwo (1999); and the geometric distribution (which can be seen as its discretised version) appears in Cooper (2001) and van Donselaar & Broekmeulen (2012). We include it because it has a high variability, with a CV of 1 and variance/mean ratio ranging from 15 to 42 (the mean of the distribution).

For each of the $3 \times 4 \times 2 \times 3 = 72$ settings we compute our formulae and run the Monte Carlo simulations (using the same number and length of runs as in Section 4). The results are shown in Tables 8–13 of the Supplementary Material file. The tables are obtained by varying m and the mean demand on weekdays and each one contains 12 settings defined by the distribution of the demand and the values of (k, k_1, k_2). Their structure is the same as in Table 6.

Table 14 of the Supplementary Material file summarizes the results in the 72 settings. For each value of $m = 4, 5, 6$ and for each distribution, we compute the absolute and relative errors of four output variables (% outdating, on-hand stock, fill rate and freshness). The absolute error is defined as $|x_{sim} - x_{approx}|$ and the relative error as $|x_{sim} - x_{approx}|/x_{sim}$. We get 6 values of these variables for each pair $(m, distribution)$ and show their minimum, average and maximum.

The tables reveal that all parameters are relevant for the behaviour of the systems, but some of them seem to have little influence on the accuracy of our formulae. For instance, the system performs better when the demand on weekdays is equally distributed (flat demand), but the accuracy for flat and peaked demand is very similar, with the rest of the parameters being equal. Analogously, systems with longer unit lifetimes behave better, in the sense of having a smaller outdated quantities with the same service level, but the accuracy of our formulae is not clearly influenced by this parameter.

On the other hand the distribution and, especially, the variability of demand measured by its CV play an essential role in the closeness of our approximations to the simulated values. Very high accuracy is observed for the Poisson and normal distributions with CV$= 0,25$. For these distributions, the difference in outdating between our approximations and the simulated values is less than 1% of total demand in all instances, with a maximum difference of 1,6 units per week in the case of Poisson distribution and 1 unit per week in the normal distribution. Note that some relative errors in the column *% outdating* of Table 14 are large, due to a small value in the denominator of the formula. For instance, the Poisson distribution with $m = 4$ has a maximum relative error of $53,6$% corresponding to the setting with $k = 1,5$, $k_1 = k_2 = 0$ and flat demand. In this setting, our formula yields a mean number of outdating units per week equal to 0,77, while the value estimated by simulation is 0,50, so the difference is 0,27 units per week, which can be regarded as negligible when evaluating the performance of the model, regardless the large relative error. The approximations for freshness (differences lower than 0,05 days with relative errors smaller than 1,5%), unsatisfied demand (differences smaller than 0,4 units per week and relative errors of the fill rate smaller than 0,25%) and average on-hand stock (differences smaller than 0,8 units and relative errors smaller than 3%) are also very accurate.

For the normal distribution with CV$= 0,5$, the accuracy of the formulae for unsatisfied demand (differences between approximations and simulated values smaller than 1 unit per week in all settings, with a maximum relative error of the fill rate equal to 0,5%) and freshness (maximum difference of 0,08 days) can still be regarded as very good. For the number of outdated units, the formula behaves slightly worse than in the cases of the Poisson and normal distributions with CV=0,25, with a maximum difference of $2,40$% of total demand, which corresponds to 4,32 units per week.

The exponential distribution can be seen as an extreme example, as it has very high variance/mean ratios, ranging from 15 to 42 in our settings, far from those used in van Donselaar & Broekmeulen (2011) and Haijema & Minner (2019), where the maximum ratio considered is 4. In fact, the number of outdated units is very high: in the most favourable case ($m = 6$), outdating is 26% for a fill rate of $98,7$%; this means that an average of 2,5 units are not satisfied every week, while 47 units are outdated. When $m = 4, 5$ the situation is even more extreme; for instance, with $m = 5$, a fill rate of $98,7$% is associated with an outdating of $35,8$%. For this distribution the fill rate is poorly estimated by our formulae, since the approximations for unsatisfied demand are almost double the values obtained by simulation, which yields differences of up to 8 units per week between the formula and the simulation. Major differences are also observed in the estimation of freshness, with a maximum of 0,28 days and relative errors up to $10,9$%. Notably, the approximation for the number of outdated units is not so bad; actually, the differences between our approximations for outdating and the simulated values are smaller than 1% of the total demand in all but four of the settings, meaning a difference of less than 2 units per week, while the number of outdated units per week ranges from 22 to 90 across all the scenarios.

A common feature observed in the tables is that (leaving out the exponential distribution) our formulae can be seen as conservative, since there is a slight bias which presents the model as worse than the real situation. In fact, in almost all instances where there is a difference between the approximations and the simulations, our formulae give higher values for outdating and lower values for service levels and fill rates. Therefore, practitioners can use our formulae to get a good approximation of the actual performance measure and can expect the model to behave as predicted by the formulae or slightly better.

Given that we have observed no substantial differences in accuracy depending on the mean demand during the week (flat/peaked) or on the lifetime of the units, our advice on the use of the formulae depends essentially on the variability of demand. We recommend using them when demand has moderate variability such as the Poisson or others with a similar CV. The approximations are still good for distributions with a slightly higher variability, although the number of outdated units may be overestimated. On the other hand, we do not recommend it for systems where there is high uncertainty in demand, since it is very difficult in such cases to maintain a high service level, which leads to poor approximations.

6. Conclusions and future work

- We have analysed a complex inventory model for perishable items with fixed lifetimes, random demand, nonzero lead time and a weekly pattern, where stockouts must be kept to a minimum. No assumptions on the distribution of demand are made.
- We use a realistic model for production of platelet concentrates, with particular values of L, R, form of the safety stock, etc. This has the advantage of including a variety of situations (different days have different characteristics) which are present in many inventory systems. Therefore, users can easily adapt the formulae in this paper to their particular operating procedures.
- We derive analytical approximations of the main performance measures of the model, namely day by day approximations of expected on-hand inventory, size of stockouts, order size, number of outdated units, fill rate, probability of stockouts and of having on-hand inventory below a threshold, age distribution of units in stock, date of issue and freshness. To the best of our knowledge, no approximations of these quantities (except for the fill rate) in nonstationary models under the EWA policy have appeared in the literature.

- Monte Carlo simulations are used to assess the accuracy of the approximations. Platelet concentrate production at a real blood bank is analysed in detail. The results show a very close agreement between our formulae and the results obtained by simulation for that case. Then the comparison is extended to 72 scenarios depending on the lifetime of the units, the distribution of demand and the values of the safety stock. The results indicate that the approximations in this paper are reliable as long as the variability of daily demand is not very high (CV smaller than 0,5). More extensive simulations are needed to assess the usefulness of the approximations in more general settings.
- Our formulae are developed under the EWA replenishment policy. If the simpler base-stock policy is used, straightforward modifications can be made to obtain the corresponding formulae. In particular, the resulting approximations for the age distribution of the stock would be new, since they have appeared in the literature only for continuous review models where the distribution of demand is stationary.
- We use the approximations to find an Efficient Frontier which shows how the outdated quantity (and the freshness of units issued) depends on the target fill rate. We do not include costs in our model but this can be readily done since the cost function can be written in terms of the approximations given in Section 3. In future work we will look for the values of the parameters (k, k_1, k_2 in our model) which minimise the long-run expected cost subject to certain service restrictions (service level, freshness, etc.).
- We also seek to compute approximations of the performance measures for a LIFO issuing policy. Although the LIFO policy may not be very relevant for blood products, there are other systems where it is common, such as retail food distribution, where the customer can see the expiration dates of the products; see Section 9 of Silver et al. (1998). When dealing with food products, it will be desirable to consider also the effect of case pack sizes, as in Broekmeulen & van Donselaar (2019).

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Appendix A. Existence of the long-run expected values

In Section 3 we derive approximations for the long-run expectations of the random variables related to the model described in Table 1. The model has a week pattern, so the long-run expectations depend on the day of the week. For every random quantity X_t, the long-run expectation is defined as

$$x_t = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_{t+7k},$$

for $t = 1, \ldots, 7$.

The theory of discrete time Markov chains guarantees that the above limits exist. In fact, the system can be modelled as a homogeneous Markov chain $\mathbf{Z}_t = \left(Day_t, Z_t^1, \ldots, Z_t^m\right)$, where Day_t is the day of the week and Z_t^r is the number of units in stock at the beginning of day t with r days of remaining lifetime (except when t is a Saturday or a Sunday, where Z_t^m and Z_t^{m-1} are the number of units ordered on the previous Friday). Assuming an (arbitrarily large) daily production capacity M, the process $(\mathbf{Z}_t)$ is a discrete time irreducible periodic (period $=$ 7) Markov chain with finite space state E. This assures the existence of the almost sure limits of $\sum_{k=1}^{n} h(\mathbf{Z}_t + 7k)/n$, for any $h : E \to \mathbb{R}$. Therefore, all the expected values (interpreted as long-run averages) in Section 3 exist.

The theory of Markov chains enables the existence of the long-run expectations to be shown, but it is not useful for finding a formula for their values, since they are expressed as a function of the solution of a linear system where the coefficient matrix is the transition matrix of the chain. The dimension of the matrix is equal to the cardinal of E which, for $m \le 6$, is $(m-1)(M+1)^{m-2} + 2(M+1)^{m-1} + (6-m)(M+1)^m$. For instance, if the lifetime of units is $m = 5$ and the daily production capacity is $M = 100$, then the cardinal of E is greater than 10^{10}. Even if the solution could be obtained numerically, it would give no insight on the role of the different parameters in the model.

Appendix B. Derivation of the formulae in Section 3.4

For $m = 4$, $o_3 = o_4 = 0$. Using formula (4) with $i = 4$, we get:

$$o_1 \sim \int_0^{\mu_{4,7}+k\sigma_{4,7}+k_2+o_5+o_6} \overline{F}_3(\mu_{3,4} + k\sigma_{3,4} + k_1 - x) \times F_{4,1}(x - o_5 - o_6 - o_7)dx,$$

$$o_2 \sim \int_0^{\mu_{5,1}+k\sigma_{5,1}+k_2+o_5+o_6+o_7} \overline{F}_4(\mu_{4,7} + k\sigma_{4,7} + k_2 + o_5 + o_6 - x) \times F_{5,2}(x - o_5 - o_6 - o_7 - o_1)dx,$$

$$o_5 \sim \int_0^{\mu_{1,2}+k\sigma_{1,2}+k_1+o_1} \overline{F}_{5,7}(\mu_{5,1} + k\sigma_{5,1} + k_2 - x)F_{1,5}(x - o_1 - o_2)dx,$$

$$o_6 \sim \int_0^{\mu_{2,3}+k\sigma_{2,3}+k_1+o_2} \overline{F}_1(\mu_{1,2} + k\sigma_{1,2} + k_1 - x)F_{2,6}(x - o_2 - o_5)dx,$$

$$o_7 \sim \int_0^{\mu_{3,4}+k\sigma_{3,4}+k_1+o_3} \overline{F}_2(\mu_{2,3} + k\sigma_{2,3} + k_1 - x)F_{3,7}(x - o_5 - o_6)dx.$$

Analogously, for $m = 6$, we have $o_5 = o_6 = 0$. Using formula (4) with $i = 6$:

$$o_1 \sim \int_0^{\mu_{2,3}+k\sigma_{2,3}+k_1+o_2} \overline{F}_1(\mu_{1,2} + k\sigma_{1,2} + k_1 - x) \times F_{2,1}(x - o_2 - o_3 - o_4 - o_7)dx,$$

$$o_2 \sim \int_0^{\mu_{3,4}+k\sigma_{3,4}+k_1+o_3} \overline{F}_2(\mu_{2,3} + k\sigma_{2,3} + k_1 - x) \times F_{3,2}(x - o_3 - o_4 - o_7 - o_1)dx,$$

$$o_3 \sim \int_0^{\mu_{4,7}+k\sigma_{4,7}+k_2+o_4} \overline{F}_3(\mu_{3,4} + k\sigma_{3,4} + k_1 - x) \times F_{4,3}(x - o_4 - o_7 - o_1 - o_2)dx,$$

$$o_4 \sim \int_0^{\mu_{5,1}+k\sigma_{5,1}+k_2+o_7} \overline{F}_4(\mu_{4,7} + k\sigma_{4,7} + k_2 - x) \times F_{5,4}(x - o_7 - o_1 - o_2 - o_3)dx,$$

$$o_7 \sim \int_0^{\mu_{1,2}+k\sigma_{1,2}+k_1+o_1} \overline{F}_{5,7}(\mu_{5,1} + k\sigma_{5,1} + k_2 - x) \times F_{1,7}(x - o_1 - o_2 - o_3 - o_4)dx.$$

Appendix C. Use of Little's formula for freshness

In Broekmeulen & van Donselaar (2019) the authors use Little's formula $L = \lambda W$ from queuing theory to estimate freshness in a stationary model. Although Little's formula can be applied to non-stationary models under some circumstances, for instance when there is a fixed T such that the system is empty at times 0 and T (see Little, 2011), it seems that it cannot be used to estimate freshness of units issued on a certain day. However, if the goal is to find the mean freshness of all units issued (without drawing a distinction by the day they are issued) then, following Theorem 2.1 in Whitt (1991), and given that our system is weekly stationary, Little's formula can be applied using the approach in Broekmeulen & van Donselaar (2019). In our case, we consider the week as the time unit. A platelet concentrate enters the (queueing) system as soon as it is ordered and not when it arrives at the blood bank since in that case units ordered on Friday, which arrive three days later, would be different from units ordered on other days. The time in the system of a concentrate is 1 day if it is issued with m days of remaining lifetime, 2 days if it is issued with $m-1$ days of remaining lifetime and so on; the time in the system of an outdated unit is m days. Therefore W, the mean time in weeks that a unit spends in the system is

$$W = \frac{1}{7}\frac{n_m + 2n_{m-1} + \cdots + mn_1 + mn_0}{n_m + n_{m-1} + \cdots + n_1 + n_0},$$

where n_r is the mean number of units issued with r days of remaining lifetime for $r = 1, \ldots, m$ and n_0 is the mean number of outdated units. The mean number of units in the system is

$$L = \frac{\sum_{i=1}^{4} q_i + 3q_5 + \sum_{i=1}^{7} h_i}{7},$$

where the first two terms in the numerator correspond to the days that the units ordered spend in the queuing system before actually arriving at the blood bank and the last term is the number of units in the blood bank throughout the week. Lastly, λ, the mean number of units arriving during the week is $n_m + \cdots + n_1 + n_0$. Therefore, Little's formula yields $7L = n_m + 2n_{m-1} + \cdots + mn_1 + mn_0$ and we have

$$\frac{7L - mn_0}{\lambda - n_0} = p_m + 2p_{m-1} + \cdots + mp_1,$$

where $p_r = n_r/(n_m + \cdots + n_1)$ is the probability of a unit being issued with r days of remaining lifetime, $r = 1, \ldots, m$. Since freshness is equal to $m + 1 - (p_m + 2p_{m-1} + \cdots + mp_1)$ it follows that

$$freshness = m + 1 - \frac{7L - mn_0}{\lambda - n_0}. \tag{C.1}$$

For instance, in the case of Table 5 (simulated values), $m = 5$, $7L = 364,4$, $\lambda = 151,7$ and $n_0 = 0,22$, yielding a freshness value of 3,60 days, which coincides with the value in the table.

Formula (C.1) is exact but its application still requires approximations of main performance measures on specific days and not only their weekly aggregated values. In particular, it seems there is no way to circumvent our formulae in Section 3.4 to approximate freshness. Moreover, note that (C.1) provides only partial information on freshness as it gives the weekly average, while our formulae in Section 3.5 approximate freshness of units issued on each day of the week.

Appendix D. Accuracy of simulations

Results on the accuracy of simulations in the setting of Table 5 are shown in Table D1. For each day, the first row shows the sample standard deviation *SD* of each quantity across the 1000 simulation runs. The second row shows the ratio in percentage terms of the half-width of the 95% confidence interval for the expectation over the sample mean value, that is,

$$\frac{1,96 \times SD/\sqrt{1000}}{\overline{X}},$$

where *SD* is the value in the first row and $\overline{X}$ is the sample mean given in Table 5 (italic font). For values where the sample mean is smaller than 0,001 the ratio (second row) is not computed.

Table D1
Standard deviations and errors in estimations using Monte Carlo simulation in Table 5.

Day	Measure	b	q	o	h	u	s.l.	P(u.t.)	b(1)	b(2)	b(3)	b(4)	b(5)	w(1)	w(2)	w(3)	w(4)	w(5)
Monday	SD	0,455	0,440	0,000	0,547	0,062	0,010	0,015	0,015	0,520	0,287	0,000	0,000	0,015	0,410	0,460	0,000	0,000
	% error IC	0,06%	0,15%		0,18%	1,70%	1,34%	0,84%	6,89%	0,17%	0,06%			6,88%	0,15%	0,27%		
Tuesday	SD	0,305	0,301	0,040	0,382	0,051	0,010	0,016	0,232	0,422	0,000	0,000	0,440	0,216	0,343	0,000	0,000	0,392
	% error IC	0,05%	0,07%	3,17%	0,17%	1,61%	1,27%	0,71%	0,67%	0,15%			0,15%	0,65%	0,15%			0,32%
Wednesday	SD	0,256	0,255	0,052	0,405	0,060	0,010	0,016	0,284	0,000	0,000	0,316	0,301	0,266	0,000	0,000	0,298	0,422
	% error IC	0,04%	0,07%	2,30%	0,17%	1,57%	1,19%	0,70%	0,55%			0,18%	0,07%	0,54%			0,19%	0,23%
Thursday	SD	0,337	0,337	0,000	0,441	0,062	0,011	0,016	0,000	0,000	0,137	0,358	0,255	0,000	0,000	0,132	0,309	0,379
	% error IC	0,06%	0,04%		0,17%	1,52%	1,21%	0,70%			0,85%	0,16%	0,07%			0,85%	0,16%	0,27%
Friday	SD	0,286	0,286	0,000	0,454	0,000	0,000	0,000	0,000	0,018	0,193	0,351	0,337	0,000	0,018	0,190	0,341	0,484
	% error IC	0,02%	0,06%		0,06%					5,24%	0,66%	0,16%	0,04%		5,22%	0,66%	0,16%	0,20%
Saturday	SD	0,454	0,000	0,000	0,512	0,012	0,002	0,005	0,001	0,017	0,135	0,408	0,000	0,001	0,015	0,112	0,252	0,000
	% error IC	0,06%			0,10%	6,68%		2,71%		5,63%	0,95%	0,06%			5,52%	0,92%	0,12%	
Sunday	SD	0,512	0,000	0,001	0,521	0,074	0,011	0,015	0,005	0,047	0,501	0,000	0,000	0,004	0,040	0,210	0,000	0,000
	% error IC	0,10%			0,17%	1,45%	1,16%	0,76%		2,42%	0,10%				2,31%	0,11%		

Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.ejor.2022.03.018.

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