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Article

Stability and Controllability Study for Mixed Integral Fractional Delay Dynamic Systems Endowed with Impulsive Effects on Time Scales

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Abstract: In this article, we investigate a novel class of mixed integral fractional delay dynamic systems with impulsive effects on time scales. Also, fixed-point techniques are applied to study the existence and uniqueness of a solution to the considered systems. Furthermore, sufficient conditions for Ulam–Hyers stability and controllability of the considered systems are established. It turns out that controllability is a very relevant property in dynamic systems and also in differential equations since, if controllability holds, then the solution of a system of differential equations also holds. Finally, an illustrative example of the obtained results is provided.

Keywords: controllability; generalized Ulam-Hyers stability; time scale; impulsive effect; fixed point technique; Leray–Schauder theorem

MSC: 47H10; 26A33; 34A08; 34B27

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1. Background and Building Systems

Fractional differential equations (FDEs) and partial differential equations have been used to model some natural phenomena and physical problems as initial and boundary value problems. Numerous studies on the concept of FDEs have been conducted during the last few decades. Due to a chance meeting between Leibniz and L'Hospital in 1695, the concept of FDEs was initially put forth. The development of mathematical models for a wide range of issues emerging in mathematical networks, control theory, aerodynamics, bioengineering, physics, blood flows, signal processing, engineering, etc., is now significantly facilitated by FDEs [1,2].

From several investigations, we can deduce that FDEs have a far higher prominence than integer-order derivatives. As a result, fractional calculus attracted a lot of interest and attention from experts and scholars. Additionally, it improved the sketch of the hereditary characteristics of diverse materials and processes. As a result, numerous monographs, and research articles have been published in this area [3–10].

For many types of functional equations, the theory of stability analysis, including Mittag-Leffler function (MLF), exponential, Lyapunov, and finite time stability, has recently been addressed. In 1940, Ulam and Hyers developed the most significant and intriguing type of stability, known as Ulam-Hyers (UH) stability [11]. The stability of homomorphisms across groups was a topic Ulam touched on during his lecture at Wisconsin University. Hyers [12] responded favorably to Ulam's dilemma in 1941 on the basis of the idea that groups can be thought of as Banach spaces (BS), and this stability came to be known as UH stability. See [13–16] for more details.

There are many phenomena in the real world that are exposed to short-term external influences as they evolve. These external effects durations are incredibly short compared to the overall period of the phenomenon being witnessed. Therefore, it is reasonable to assume that these outside forces are genuine impulses. Now to investigate these abrupt changes, impulsive differential equations play a key role in modeling physical real-world problems. Such type of impulsive differential equations has argued concern to different applications, including biological systems such as heartbeat, blood flow, and impulse rate. Also, it has many applications in electrodynamics, population dynamics, viscoelastic, radio physics, metallurgy, mathematical economy, electric technology, theoretical physics, pharmacokinetics, control theory, and chemical engineering technology [17–23].

The core idea in mathematical control theory, controllability, was introduced by Kalman in 1960. In general, controllability refers to the ability to use a proper control function to steer the state of a control dynamical equation from an arbitrary initial state to the desired terminal state. The controllability results have been studied by many authors [24,25]. Additionally, the study of controllability findings on temporal scales is a recent field with limited evidence [26,27]. The existence, controllability, and Ulam type stability of the impulsive fractional dynamical system with respect to a mixed structure have specifically been investigated in a few studies.

Based on the above contributions, in this manuscript, we consider a new class of dynamic systems with mixed integral fractional delays and impulsive effects on time scales. Also, fixed-point approaches are used to demonstrate the existence and originality of solutions to the systems under discussion. Moreover, Ulam-Hyers stability is also proven for the aforementioned mixed impulsive systems, as well as a generalized version of it. Furthermore, the controllability of the aforementioned systems was then examined. Finally, an illustrative example is used to explore the results from the earlier parts.

Now, we investigate the following mixed integral fractional dynamic systems on the time scale τ, which are generalizations of the results of [28]:

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Theta\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Theta\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right), \\ \vartheta\in\tau' = \tau\backslash\{\vartheta_1,\vartheta_2,\vartheta_1\ldots,\vartheta_n\},\ \rho\in(0,1),\ \nu(\vartheta_0)=\nu_0,\ \xi(\vartheta_0)=\xi_0, \\ \nu(\vartheta_r^+)-\nu(\vartheta_r^-) = O_r(\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-)),\ r=1,2,\ldots,n, \\ \xi(\vartheta_r^+)-\xi(\vartheta_r^-) = O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-)),\ r=1,2,\ldots,n, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Xi\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(s,s,\nu(s),\xi(s))\Delta s\right), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Xi\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right), \\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\ldots,n,\ \rho\in(0,1),\ \nu(\vartheta_0)=\nu_0,\ \xi(\vartheta_0)=\xi_0, \\ \nu(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\ldots,n, \\ \xi(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\ldots,n. \end{cases} \tag{2}$$

Furthermore, we study the controllability of the following systems:

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Theta\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s, \int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right) + H\ell(\vartheta), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Theta\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s, \int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right) + H\ell(\vartheta), \\ \vartheta \in \tau' = \tau\backslash\{\vartheta_1,\vartheta_2,\vartheta_1...,\vartheta_n\},\ \rho\in(0,1),\ \nu(\vartheta_0)=\nu_0,\ \xi(\vartheta_0)=\xi_0, \\ \nu(\vartheta_r^+) - \nu(\vartheta_r^-) = O_r(\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-)),\ r=1,2,\dots,n, \\ \xi(\vartheta_r^+) - \xi(\vartheta_r^-) = O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-)),\ r=1,2,\dots,n, \end{cases} \tag{3}$$

and

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Xi\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s, \int\limits_{\vartheta_0}^{\vartheta_l} Z_2(s,s,\nu(s),\xi(s))\Delta s\right) + H\ell(\vartheta), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Xi\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s, \int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right) + H\ell(\vartheta), \\ \vartheta \in (s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\dots,n,\ \rho\in(0,1),\ \nu(\vartheta_0)=\nu_0,\ \xi(\vartheta_0)=\xi_0, \\ \nu(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\dots,n, \\ \xi(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\dots,n, \end{cases} \tag{4}$$

where ${}^{c,\tau}D^{\rho}$ denotes the Caputo derivative (CD) [1] of fractional order ρ on time scales τ, $B(\vartheta)$ represents a regressive square matrix, which is piecewise continuous, $H : \tau \to \tau$ is a bounded linear operator. By assuming $\mathbb{R}$ as the real number, $\nu, \xi \in L^2(I,\mathbb{R})$ are control maps, $\tau^0 = [\vartheta_0, \vartheta_l]_\tau$, the pre-fixed numbers are $\vartheta_0 = s_0 < \vartheta_1 < s_1 < \vartheta_2 < \cdots < \vartheta_n < s_n < \vartheta_{n+1} = \vartheta_l$ and the mappings $Z : \tau^0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $Z_{1,2} : \tau^0 \times \tau^0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\Theta : \tau^0 \times (\mathbb{R}^n)^4 \to \mathbb{R}^n$, (where $(\mathbb{R}^n)^4 = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$), $\varphi_i : (\vartheta_i, s_i] \cap \tau \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1,2,\dots,n$, $\Xi : (s_i, \vartheta_{i+1}] \cap \tau \times (\mathbb{R}^n)^4 \to \mathbb{R}^n$, $i = 1,2,\dots,n$, $O_r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $D_r : \tau^0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. We also provide the right limit and the left limit of the $\nu(\vartheta)$ and $\xi(\vartheta)$ at ϑ_r as $\nu(\vartheta_r^+) = \lim_{\epsilon\to 0^+} \nu(\vartheta_r + \epsilon)$, $\nu(\vartheta_r^-) = \lim_{\epsilon\to 0^-} \nu(\vartheta_r - \epsilon)$, $\xi(\vartheta_r^+) = \lim_{\epsilon\to 0^+} \xi(\vartheta_r + \epsilon)$, $\xi(\vartheta_r^-) = \lim_{\epsilon\to 0^-} \xi(\vartheta_r - \epsilon)$, respectively.

2. Definitions and Auxiliary Lemmas

In this part, we give some definitions, fundamental ideas, lemmas and hypotheses for this study.

Assume that $C(I,\mathbb{R})$ is the space of all continuous mapping from the interval I onto $\mathbb{R}$. Clearly, $C(I,\mathbb{R})$ is a BS equipped with the norm $\|\nu\|_C = \sup_{\vartheta\in\tau}|\nu(\vartheta)|$. The product space (PS) $Q = C_4(I,\mathbb{R})$ (where $C_4(I,\mathbb{R}) = C(I,\mathbb{R}) \times C(I,\mathbb{R}) \times C(I,\mathbb{R}) \times C(I,\mathbb{R})$) is also BS under the norm

$$\|(\nu_1,\nu_2,\nu_3,\nu_4)\|_C = \|\nu_1\|_C + \|\nu_2\|_C + \|\nu_3\|_C + \|\nu_4\|_C.$$

Also, we define a BS $C^1(I,\mathbb{R}) = \{\nu \in C(I,\mathbb{R}) : \nu^\Delta \in C(I,\mathbb{R})\}$ with the norm $\|\nu\|_{C^1} = \max\{\|\nu\|_C, \|\nu^\Delta\|_{C^1}\}$. Moreover, $Q^1 = C^1_4(I,\mathbb{R})$ is a PS with

$$\|(\nu_1,\nu_2,\nu_3,\nu_4)\|_{C^1} = \|\nu_1\|_{C^1} + \|\nu_2\|_{C^1} + \|\nu_3\|_{C^1} + \|\nu_4\|_{C^1}.$$

A time scale τ is a closed, nonempty subset of $\mathbb{R}$. An interval on a time scale is defined as $[a,b]_\tau = \{\vartheta \in \tau : a \leq \vartheta \leq b \in C(I,\mathbb{R})\}$. The same is true for $(a,b)_\tau$ and $[a,b)_\tau$.

The operators for forward and backward jumps $\rho : \tau \to \tau$ and $\upsilon : \tau \to \tau$ are described as

$$\rho(\vartheta) = \inf\{s \in \tau : s > \vartheta\} \text{ and } \upsilon(\vartheta) = \sup\{s \in \tau : s < \vartheta\},$$

respectively. To determine the existing distance between two consecutive points, the operator $\Omega : \tau \to [0,\infty)$, which is defined as $\Omega(\vartheta) = \rho(\vartheta) - \vartheta$ is applied. Along these lines, the derived version τ^r of τ is

$$\tau^r = \begin{cases} \tau \backslash (\upsilon, \sup \tau], & \text{if } \sup \tau < \infty, \\ \tau, & \text{if } \sup \tau = \infty. \end{cases}$$

For all $\vartheta \in \tau^r$, the regressive (respectively positively regressive) function $\beth : \tau \to \mathbb{R}$ is described as $1 + \Omega(\vartheta)\beth(\vartheta) \neq 0$ (respectively $1 + \Omega(\vartheta)\beth(\vartheta) > 0$).

Definition 1 ([29]). *The delta derivative $f^\Delta(\vartheta)$ of a mapping $f : \tau \to \mathbb{R}$ at a point $\vartheta \in \tau^r$ is a number (assuming it exists) if for $\varepsilon > 0$, a neighborhood V of ϑ exists if and only if*

$$\Big| (f(\rho(\vartheta)) - f(\varrho)) - f^\Delta(\vartheta)(\rho(\vartheta) - \varrho) \Big| \leq \varepsilon |\rho(\vartheta) - \varrho|, \ \forall \varrho \in V.$$

The following results are very important in the sequel.

Theorem 1 ([29]). *Assume that $a, b \in \tau$ and $h \in C_{rd}(\tau,\mathbb{R})$. Then*

(i) $\tau = \mathbb{R}$ *leads to*

$$\int_a^b f(\vartheta)\Delta\vartheta = \int_a^b f(\vartheta)d\vartheta.$$

(ii) *If only isolated points make up $[a,b)$, then*

$$\int_a^b f(\vartheta)\Delta\vartheta = \begin{cases} \sum_{\vartheta \in [a,b)} \varkappa(\vartheta) f(\vartheta), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{\vartheta \in [a,b)} \varkappa(\vartheta) f(\vartheta), & \text{if } a > b. \end{cases}$$

(iii) *If $\tau = y\mathbb{Z} = \{yr : r \in \mathbb{Z}\}$, $y > 0$, we obtain*

$$\int_a^b f(\vartheta)\Delta\vartheta = \begin{cases} \sum\limits_{r=\frac{a}{y}}^{\frac{b}{y}-1} y f(yr), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum\limits_{r=\frac{a}{y}}^{\frac{b}{y}-1} y f(yr), & \text{if } a > b. \end{cases}$$

Theorem 2 ([30]). *Let the function $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing, and $a, b \in \tau$, then*

$$\int_a^b f(\vartheta)\Delta\vartheta = \int_a^b f(\vartheta)d\vartheta.$$

Definition 2 ([30]). *The delta fractional integral for the integrable mapping $\psi : [a,b]_\tau \to \mathbb{R}$ is described as*

$${}^\Delta I_{c^+}^\rho \psi(\vartheta) = \frac{1}{\Gamma(\rho)} \int\limits_c^\vartheta (\vartheta - s)^{\rho-1} \psi(s) \Delta s.$$

Definition 3 ([30]). *For the mapping $f : \tau \to \mathbb{R}$, the fractional CD on the time scale is*

$$ {}^{c,\tau}D^{\rho}_{c^+}f(\vartheta) = \frac{1}{\Gamma(n-\rho)}\int\limits_{c}^{\vartheta}(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta s, $$

where $n = [\rho] + 1$ and f^{Δ_n} is the n-th derivative of f.

- If $\tau = \cup_{i=0}^{\infty}[2i, 2i+1]$, we have

$$ \begin{aligned} {}^{c,\tau}D^{\rho}_{c^+}f(\vartheta) &= \frac{1}{\Gamma(n-\rho)}\int\limits_{c}^{\vartheta}(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta\, s \\ &= \frac{1}{\Gamma(n-\rho)}\left(\sum_{r=0}^{i-1}\int\limits_{2r}^{2r+1}(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta s + \int\limits_{2i}^{2i+1}(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta s\right), \end{aligned} $$

  for $\vartheta \in [2i, 2i+1]$, $i = 0, 1, \dots$.
- If $\tau = y\mathbb{Z}$, $y > 0$, we get

$$ \begin{aligned} {}^{c,\tau}D^{\rho}_{c^+}f(\vartheta) &= \frac{1}{\Gamma(n-\rho)}\int\limits_{c}^{\vartheta}(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta\, s \\ &= \frac{1}{\Gamma(n-\rho)}\sum_{r=0}^{\frac{\vartheta}{y}-1} y(\vartheta-ry)^{n-\rho-1}f^{\Delta^n}(y),\ \vartheta \in \tau. \end{aligned} $$

- If $\tau = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \mathbb{Z}$, then

$$ \begin{aligned} {}^{c,\tau}D^{\rho}_{c^+}f(\vartheta) &= \frac{1}{\Gamma(n-\rho)}\int\limits_{c}^{\vartheta}(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta\, s \\ &= \frac{1}{\Gamma(n-\rho)}\sum_{\vartheta\in\tau}\varkappa(s)(\vartheta-s)^{n-\rho-1}f^{\Delta^n}(s)\Delta s. \end{aligned} $$

Take into account the MLF as

$$ O_{\rho,\alpha}(\vartheta) = \sum_{r=0}^{\infty}\frac{\vartheta^r}{\Gamma(\rho r+\alpha)} \text{ for } \rho, \alpha > 0. $$

If $\alpha = 1$, we get

$$ O_{\rho,1}(\varsigma\vartheta^{\rho}) = O_{\rho}(\varsigma\vartheta^{\rho}) = \sum_{r=0}^{\infty}\frac{\varsigma^r\vartheta^{\rho r}}{\Gamma(\rho r+1)},\ \varsigma, \vartheta \in \mathbb{C}, $$

has the fascinating property ${}^{c}D^{\rho}_{0^+}O_{\rho}(\varsigma\vartheta^{\rho}) = \varsigma O_{\rho}(\vartheta^{\rho})$.

Remark 1. *Motivated by the results of [31], the solution of the problem (1) takes the form*

$$\nu(\vartheta)=\begin{cases}\Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\ \vartheta\in(\vartheta_0,\vartheta_1],\\ \Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\\ +\sum_{t=1}^{i}O_t\big(\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big)+D_t\big(\vartheta_t^-,\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big),\ \vartheta\in(\vartheta_i,\vartheta_{i+1}],\ i=1,2,\ldots,n,\end{cases}$$

and

$$\xi(\vartheta)=\begin{cases}\Lambda_\rho(B\vartheta^\rho)\xi_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\ \vartheta\in(\vartheta_0,\vartheta_1],\\ \Lambda_\rho(B\vartheta^\rho)\xi_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\\ +\sum_{t=0}^{i}O_t\big(\xi(\vartheta_t^-),\nu(\vartheta_t^-)\big)+D_t\big(\vartheta_t^-,\xi(\vartheta_t^-),\nu(\vartheta_t^-)\big),\ \vartheta\in(\vartheta_i,\vartheta_{i+1}],\ i=1,2,\ldots,n,\end{cases}$$

where $\Lambda_\rho(B\vartheta^\rho)$ represents the matrix for the aforementioned MLF, which is given by

$$\Lambda_\rho(B\vartheta^\rho)=\sum_{t=0}^{i}\frac{B^t\vartheta^{\rho t}}{\Gamma(\rho t+1)}.$$

Now, in order to reach our results, we take into account the following assertions:

(1) For the mappings $\Theta,\Xi:\tau^0\times(\mathbb{R}^n)^4\to\mathbb{R}^n$, there are positive constants T_{Θ_i},T_{Ξ_i}, $i=1,2,3,4$ so that

$$\begin{aligned}|\Theta(\vartheta,\sigma_1,\sigma_2,\sigma_3,\sigma_4)-\Theta(\vartheta,\omega_1,\omega_2,\omega_3,\omega_4)| &\leq \sum_{i=1}^{4}T_{\Theta_i}|\sigma_i-\omega_i|,\text{ for each }\vartheta\in I,\ \sigma_i,\omega_i\in\mathbb{R},\\ |\Xi(\vartheta,\sigma_1,\sigma_2,\sigma_3,\sigma_4)-\Xi(\vartheta,\omega_1,\omega_2,\omega_3,\omega_4)| &\leq \sum_{i=1}^{4}T_{\Xi_i}|\sigma_i-\omega_i|,\text{ for each }\vartheta\in I,\ \sigma_i,\omega_i\in\mathbb{R}.\end{aligned}$$

(2) The mappings $\Theta, \Xi : \tau^0 \times (\mathbb{R}^n)^4 \to \mathbb{R}^n$ are continuous and there are positive constants c_i, h_i, $i = 1,2,3,4$ so that

$$\begin{aligned} |\Theta(\vartheta, \sigma_1, \sigma_2, \sigma_3, \sigma_4)| &\le c_0 + c_1|\sigma_1| + c_2|\sigma_2| + c_3|\sigma_3| + c_4|\sigma_4|, \text{ for each } \vartheta \in I,\ \sigma_i \in \mathbb{R}, \\ |\Xi(\vartheta, \sigma_1, \sigma_2, \sigma_3, \sigma_4)| &\le h_0 + h_1|\sigma_1| + h_2|\sigma_2| + h_3|\sigma_3| + h_4|\sigma_4|, \text{ for each } \vartheta \in I,\ \sigma_i \in \mathbb{R}. \end{aligned}$$

(3) There is a bounded invertible operator $({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}$ for the linear operator $\left({}^{\rho}\Re^{A}_{\vartheta_0}\right) : L^2(I, \mathbb{R}) \to \mathbb{R}$, which is described as

$$ {}^{\rho}\Re^{A}_{\vartheta_0} L = \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta s, \tag{5}$$

these operators admit values in $L^2(I,\mathbb{R}) \setminus \ker({}^{\rho}\Re^{A}_{\vartheta_0})$.

(4) There exists a positive constant $Y^{\rho}_{\Re}$ so that $\left\|({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}\right\| \le Y^{\rho}_{\Re}$.

(5) The operator $H : \tau \to \tau$ is continuous and there is $Y_{\Re}$ so that $\|H\| \le Y_{\Re}$.

Equation (5) can be computed for various τ using Theorem 1 as follows:

- If $\tau = y\mathbb{Z}$, $y > 0$, we have

$$ {}^{\rho}\Re^{A}_{\vartheta_0} L = \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta s = \sum_{r=0}^{\frac{\vartheta}{y}-1} y(A - rsy)^{\rho-1} HL(sy).$$

- If $\tau = \cup_{i=0}^{\infty}[2i, 2i+1]$ and $\vartheta \in [4,5]$, we obtain

$$\begin{aligned} &{}^{\rho}\Re^{A}_{\vartheta_0} L \\ = &\int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta\, s \\ = &\int_{0}^{1} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta s + \int_{2}^{3} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta\, s \\ &+ \int_{4}^{A} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta s. \end{aligned}$$

- When $\tau = \{q^m : q > 1,\ m \in \mathbb{Z}\} \cup \mathbb{Z}$. Then

$$ {}^{\rho}\Re^{A}_{\vartheta_0} L = \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big) HL(s)\Delta s = \sum_{\vartheta \in [0,A]} \varkappa(A-\vartheta)^{\rho-1} HL(\vartheta).$$

For simplicity, we consider the following:

$$\begin{aligned} &u_1 = \sup_{\vartheta\in\tau}\big\|\Lambda_{\rho}(B\vartheta^{\rho})\nu_0\big\|, \quad u_2 = \sup_{\vartheta\in\tau}\big\|\Lambda_{\rho,\rho}\big(B(\vartheta-s)^{\rho}\big)\big\|, \quad u_3 = \sup_{\vartheta\in\tau}\left\|(\vartheta-s)^{\rho-1}\right\|, \\ &u_4 = \sup_{\vartheta\in\tau}\big\|\Lambda_{\rho}(B\vartheta^{\rho})\xi_0\big\|, \quad E_1 = \textstyle\sum_{j=1}^{i}(T_O+T_D)\delta + u_1, \quad E_2 = \sum_{j=1}^{i}(T_O+T_D), \\ &E_3 = \tfrac{1}{\Gamma(\rho)} u_3 T_{\varphi}\delta(s_i - \vartheta_i) + u_1, \quad E_4 = \tfrac{1}{\Gamma(\rho)} u_3 T_{\varphi}\delta(s_i - \vartheta_i), \quad E_5 = \textstyle\sum_{j=1}^{i}(T_O+T_D)\delta + u_4, \\ &E_6 = \tfrac{1}{\Gamma(\rho)} u_3 T_{\varphi}\delta(s_i - \vartheta_i) + u_4, \end{aligned}$$

$$\begin{array}{rcl} J_1 & = & u_3\big(u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_4}T_{Z_2}\big)(s_l - s_0) + Y^*_{\Theta}\big)(\vartheta_l - \vartheta_0) \\ J_2 & = & u_3\big(u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_4}T_{Z_2}\big)(s_l - s_0)\big)(\vartheta_l - \vartheta_0) \\ J^*_1 & = & u_3\big(u_2T_Z + u_2T_{\Xi_1} + u_2T_{\Xi_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_4}T_{Z_2}\big)(s_l - s_0) + Y^*_{\Xi}\big)(\vartheta_l - \vartheta_0), \\ J^*_2 & = & u_3\big(u_2T_Z + u_2T_{\Xi_1} + u_2T_{\Xi_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_4}T_{Z_2}\big)(s_l - s_0)\big)(\vartheta_l - \vartheta_0) \\ J_3 & = & E_2 + J_2,\ J^*_3 = E_4 + J^*_2. \end{array}$$

3. The Existence and Uniqueness Study

In this part, existence and uniqueness criteria are studied.

Theorem 3. *Under the assertion* (1)*, the mixed impulsive system (MIS) (1) possesses a unique solution (US) provided that*

$$\max_{1\le i\le 3}\{J_i\} < 0.5. \tag{6}$$

Proof. Assume that $\mho \subseteq Q$ and $\mho = \{(K_1, K_2, K_3, K_4) \in Q : \|K_1, K_2, K_3, K_4\| \le \delta_2\}$, where $\delta_2 = \max\{\delta, \delta_1\}$ and $\delta, \delta_1 \in (0, 1)$ so that

$$\delta > \max\{E_1, E_2, E_3\} \text{ and } \delta > \max\{E_5, E_2, E_3\},$$

and the rest constants are presented in the sequel. Define an operator $\Im_\rho : \mho \times \mho \to \mho$ by

$$\Im_\rho(\nu,\xi)(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\ \vartheta \in (\vartheta_0, \vartheta_1], \\ \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\ + \sum_{t=0}^{i} O_t\big(\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big) + D_t\big(\vartheta_t^-,\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big),\ \vartheta \in (\vartheta_i, \vartheta_{i+1}],\ i = 1, 2, \ldots, n, \end{cases} \tag{7}$$

Suppose that

$$\begin{array}{rcl} \|Z(\vartheta,\nu,\xi)\| & \le & \|Z(\vartheta,\nu,\xi) - Z(\vartheta,0,0)\| + \|Z(\vartheta,0,0)\| \le T_Z(\|\nu\| + \|\xi\|) + Y_Z, \\ \big\|O\big(\nu(\vartheta^-),\xi(\vartheta^-)\big)\big\| & \le & T_O(\|\nu\| + \|\xi\|), \\ D_t\big(\vartheta^-,\nu(\vartheta^-),\xi(\vartheta^-)\big) & \le & T_D(\|\nu\| + \|\xi\|), \end{array}$$

and

$$\begin{array}{rcl} \|\Theta(\vartheta,K_1,K_2,K_3,K_4)\| & \le & \|\Theta(\vartheta,K_1,K_2,K_3,K_4) - \Theta(\vartheta,0,0,0,0)\| - \|\Theta(\vartheta,0,0,0,0)\| \\ & \le & T_{\Theta_1}\|K_1\| + T_{\Theta_2}\|K_2\| + T_{\Theta_3}\|K_3\| + T_{\Theta_4}\|K_4\| + Y_\Theta, \end{array}$$

where $Y_Z = \sup_{\vartheta\in\tau}\|Z(\vartheta,0,0)\|$, $Y_\Theta = \sup_{\vartheta\in\tau}\|\Theta(\vartheta,0,0,0,0)\|$, and $Y^*_\Theta = Y_Z + Y_\Theta$. Additionally, $K_1 = \nu(s)$, $K_2 = \xi(s)$,

$$K_3 = \int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p \text{ and } K_4 = \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p.$$

Now, we show that $\Im_\rho : \mho \times \mho \to \mho$ is a self-mapping. For $\vartheta \in (\vartheta_i, \vartheta_{i+1}], i = 1, 2, \ldots, n$, we can obtain

$$\begin{aligned}
\|\Im_\rho(\nu,\xi)(\vartheta)\| \leq\ & \sum_{j=1}^{i}\left\|O_j\left(\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)\right\| + \sum_{j=1}^{i}\left\|D_j\left(\vartheta_j^-,\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)\right\| + \|\Lambda_\rho(B\vartheta^\rho)\nu_0\| \\
& + \int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\
& + \Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\Bigg\|\Delta s \\
\leq\ & \sum_{j=1}^{i}T_O\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\| + \sum_{j=1}^{i}T_D\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\| + \|\Lambda_\rho(B\vartheta^\rho)\nu_0\| \\
& + \int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\|(\|Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\| \\
& + \left\|\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\right\|\Delta s,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\Im_\rho(\nu,\xi)(\vartheta)\| \leq\ & \sum_{j=1}^{i}T_O\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\| + \sum_{j=1}^{i}T_D\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\| + \|\Lambda_\rho(B\vartheta^\rho)\nu_0\| \\
& + \int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\|\big(T_Z(\|\nu\|+\|\xi\|)+T_{\Theta_1}\|\nu(s)\|+T_{\Theta_2}\|\xi(s)\| \\
& + T_{\Theta_3}\left\|\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p\right\| + T_{\Theta_4}\left\|\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right\| + Y_\Theta^*\Bigg)\Delta s \\
\leq\ & \sum_{j=1}^{i}T_O\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\| + \sum_{j=1}^{i}T_D\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\| + \|\Lambda_\rho(B\vartheta^\rho)\nu_0\| \\
& + \int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\|\big(T_Z(\|\nu\|+\|\xi\|)+T_{\Theta_1}\|\nu(s)\|+T_{\Theta_2}\|\xi(s)\| \\
& + T_{\Theta_3}T_{Z_1}\int_{s_0}^{s_l}\|\nu(p)+\xi(p)\|\Delta p + T_{\Theta_4}T_{Z_2}\int_{s_0}^{s_l}\|\nu(p)+\xi(p)\|\Delta p + Y_\Theta^*\Bigg)\Delta s,
\end{aligned}$$

hence

$$
\begin{aligned}
\left\|\Im_\rho(\nu,\xi)(\vartheta)\right\| \leq\ & (T_O+T_D)\sum_{j=1}^{i}\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\|+\left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
& +\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\big(T_Z(\|\nu\|+\|\xi\|)+T_{\Theta_1}\|\nu(s)\|+ \\
& +T_{\Theta_2}\|\xi(s)\|+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)\|\nu(p)+\xi(p)\|(s_l-s_0)+Y_\Theta^*\big)\Delta\, s \\
\leq\ & (T_O+T_D)\sum_{j=1}^{i}\sup_{\vartheta\in\tau}\left\|\nu(\vartheta_j^-)+\xi(\vartheta_j^-)\right\|+\sup_{\vartheta\in\tau}\left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
& +\int_{\vartheta_0}^{\vartheta_l}\sup_{\vartheta\in\tau}\left\|(\vartheta-s)^{\rho-1}\right\|\sup_{\vartheta\in\tau}\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\Bigg(T_Z\left(\sup_{\vartheta\in\tau}\|\nu\|+\sup_{\vartheta\in\tau}\|\xi\|\right)+T_{\Theta_1}\sup_{\vartheta\in\tau}\|\nu(s)\| \\
& +T_{\Theta_2}\sup_{\vartheta\in\tau}\|\xi(s)\|+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)\sup_{\vartheta\in\tau}\|\nu(p)+\xi(p)\|(s_l-s_0)+Y_\Theta^*\Bigg)\Delta s,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\left\|\Im_\rho(\nu,\xi)(\vartheta)\right\| \leq\ & (T_O+T_D)\sum_{j=1}^{i}\|\nu+\xi\|_\infty+u_1+\int_{\vartheta_0}^{\vartheta_l}u_2u_3\big(T_Z\|\nu+\xi\|_\infty+T_{\Theta_1}\|\nu\|_\infty \\
& +T_{\Theta_2}\|\xi\|_\infty+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)\|\nu(p)+\xi(p)\|_\infty(s_l-s_0)+Y_\Theta^*\big)\Delta\, s \\
\leq\ & \sum_{j=1}^{i}(T_O+T_D)\delta+u_1+u_3\big(\delta u_2T_Z+\delta u_2T_{\Theta_1}+\delta u_2T_{\Theta_2} \\
& +\delta u_2\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)+\delta Y_\Theta^*\big)\times\int_{\vartheta_0}^{\vartheta_l}\Delta\, s \\
\leq\ & \sum_{j=1}^{i}(T_O+T_D)\delta+u_1+\delta u_3\big(u_2T_Z+u_2T_{\Theta_1}+u_2T_{\Theta_2} \\
& +u_2\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)+Y_\Theta^*\big)(\vartheta_l-\vartheta_0).
\end{aligned}
$$

So

$$\left\|\Im_\rho(\nu,\xi)(\vartheta)\right\|\leq E_1+\delta J_1\leq\delta+\delta J_1=\delta_1.$$

Similarly, one can prove that

$$\left\|\Im_\rho(\xi,\nu)(\vartheta)\right\|\leq E_5+\delta J_1\leq\delta+\delta J_1=\delta_1,$$

where $\delta_1=\delta+\delta J_1$. This implies that

$$\left\|\Im_\rho(\nu,\xi)(\vartheta)\right\|\leq\delta_2 \text{ and } \left\|\Im_\rho(\xi,\nu)(\vartheta)\right\|\leq\delta_2. \tag{8}$$

It follows from (8) that $\Im(\mho\times\mho)\subseteq\mho$. Again for $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\ldots,n$, with $\nu_0=\widetilde{\nu}_0$ and $\xi_0=\widetilde{\xi}_0$, one gets

$$
\begin{aligned}
&\left\|\Im_\rho(\nu,\xi)(\vartheta)-\Im_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\
\leq\ & \sum_{j=1}^{i}\left\|O_j\left(\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)-O_j\left(\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\right)\right\|+\sum_{j=1}^{i}\left\|D_j\left(\vartheta_j^-,\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)-D_j\left(\vartheta_j^-,\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\right)\right\| \\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^\rho\right)\right)\right\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\
&+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg) \\
&-\left(Z\left(\vartheta,\widetilde{\nu}(\vartheta),\widetilde{\xi}(\vartheta)\right)+\Theta\left(s,\widetilde{\nu}(s),\widetilde{\xi}(s),\int_{s_0}^{s_l}Z_1\left(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\right)\Delta p,\int_{s_0}^{s_l}Z_2\left(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\right)\Delta p\right)\right)\Bigg\|\Delta s
\end{aligned}
$$

which yields that

$$
\begin{aligned}
&\left\|\Im_\rho(\nu,\xi)(\vartheta)-\Im_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\
\leq\ & \sum_{j=1}^{i}T_O\left(\left\|\nu(\vartheta_j^-)-\widetilde{\nu}(\vartheta_j^-)\right\|+\left\|\xi(\vartheta_j^-)-\widetilde{\xi}(\vartheta_j^-)\right\|\right)+\sum_{j=1}^{i}T_D\left(\left\|\nu(\vartheta_j^-)-\widetilde{\nu}(\vartheta_j^-)\right\|+\left\|\xi(\vartheta_j^-)-\widetilde{\xi}(\vartheta_j^-)\right\|\right) \\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^\rho\right)\right)\right\|\Big[T_Z\left(\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\right) \\
&+T_{\Theta_1}\|\nu(s)-\widetilde{\nu}(s)\|+T_{\Theta_2}\left\|\xi(s)-\widetilde{\xi}(s)\right\|+T_{\Theta_3}\int_{s_0}^{s_l}\left\|Z_1(s,p,\nu(p),\xi(p))-Z_1\left(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\right)\right\|\Delta p \\
&+T_{\Theta_4}\int_{s_0}^{s_l}\left\|Z_2(s,p,\nu(p),\xi(p))-Z_2\left(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\right)\right\|\Delta p\Big]\Delta s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\|\Im_\rho(\nu,\xi)(\vartheta)-\Im_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\
\leq\ & \sum_{j=1}^{i}(T_O+T_D)\left(\left\|\nu(\vartheta_j^-)-\widetilde{\nu}(\vartheta_j^-)\right\|+\left\|\xi(\vartheta_j^-)-\widetilde{\xi}(\vartheta_j^-)\right\|\right) \\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^\rho\right)\right)\right\|\Big[T_Z\left(\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\right)+T_{\Theta_1}\|\nu(s)-\widetilde{\nu}(s)\| \\
&+T_{\Theta_2}\left\|\xi(s)-\widetilde{\xi}(s)\right\|+T_{\Theta_3}T_{Z_1}\int_{s_0}^{s_l}\left(\|\nu(p)-\widetilde{\nu}(p)\|+\left\|\xi(p)-\widetilde{\xi}(p)\right\|\right)\Delta p \\
&+T_{\Theta_3}T_{Z_2}\int_{s_0}^{s_l}\left(\|\nu(p)-\widetilde{\nu}(p)\|+\left\|\xi(p)-\widetilde{\xi}(p)\right\|\right)\Delta p\Big]\Delta s
\end{aligned}
$$

$$\leq \sum_{j=1}^{i}(T_O+T_D)\left(\sup_{\vartheta\in\tau}\left\|\nu(\vartheta_j^-)-\widetilde{\nu}(\vartheta_j^-)\right\|+\sup_{\vartheta\in\tau}\left\|\xi(\vartheta_j^-)-\widetilde{\xi}(\vartheta_j^-)\right\|\right)$$
$$+\int_{\vartheta_0}^{\vartheta_l}\sup_{\vartheta\in\tau}\left\|(\vartheta-s)^{\rho-1}\right\|\sup_{\vartheta\in\tau}\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^{\rho}\right)\right)\right\|$$
$$\times\Bigg[\begin{array}{c} T_Z\left(\sup_{\vartheta\in\tau}\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\sup_{\vartheta\in\tau}\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\right) \\ +T_{\Theta_1}\sup_{\vartheta\in\tau}\|\nu(s)-\widetilde{\nu}(s)\| \end{array}$$
$$+T_{\Theta_2}\sup_{\vartheta\in\tau}\left\|\xi(s)-\widetilde{\xi}(s)\right\|+T_{\Theta_3}T_{Z_1}\left(\sup_{\vartheta\in\tau}\|\nu(p)-\widetilde{\nu}(p)\|+\sup_{\vartheta\in\tau}\left\|\xi(p)-\widetilde{\xi}(p)\right\|\right)(s_l-s_0)$$
$$+T_{\Theta_3}T_{Z_2}\left(\sup_{\vartheta\in\tau}\|\nu(p)-\widetilde{\nu}(p)\|+\sup_{\vartheta\in\tau}\left\|\xi(p)-\widetilde{\xi}(p)\right\|\right)(s_l-s_0)\Bigg]\Delta s.$$

Using the fact $\|\nu-\widetilde{\nu}\|_\infty \leq \|\nu-\widetilde{\nu}\|_\infty + \left\|\xi-\widetilde{\xi}\right\|_\infty$ and $\left\|\xi-\widetilde{\xi}\right\|_\infty \leq \left\|\xi-\widetilde{\xi}\right\|_\infty + \|\nu-\widetilde{\nu}\|_\infty$, one can write

$$\left\|\Im_\rho(\nu,\xi)(\vartheta)-\Im_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\|$$
$$\leq \sum_{j=1}^{i}(T_O+T_D)\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)+u_3u_2\Big[T_Z\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)+T_{\Theta_1}\|\nu-\widetilde{\nu}\|_\infty$$
$$+T_{\Theta_2}\left\|\xi-\widetilde{\xi}\right\|_\infty+T_{\Theta_3}T_{Z_1}\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)(s_l-s_0)$$
$$+T_{\Theta_3}T_{Z_2}\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)(s_l-s_0)\Big]\times\int_{\vartheta_0}^{\vartheta_l}\Delta s,$$
$$\leq \sum_{j=1}^{i}(T_O+T_D)+u_3\left[u_2T_Z+u_2T_{\Theta_1}+u_2T_{\Theta_2}+u_2T_{\Theta_3}T_{Z_1}(s_l-s_0)+u_2T_{\Theta_3}T_{Z_2}(s_l-s_0)\right](\vartheta_l-\vartheta_0)$$
$$\times\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)$$
$$= (E_2+J_2)\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)=J_3\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right). \tag{9}$$

Analogously, we obtain that

$$\left\|\Im_\rho(\nu,\xi)(\vartheta)-\Im_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\|\leq J_3\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right). \tag{10}$$

From (6), (9) and (10), we conclude that the operator $\Im_\rho$ is strictly contractive. As a result, using the Banach principle, the MIS (1) has a US. □

We now have the following conclusion for the MIS (2):

Theorem 4. *Via the assertion* (1), *the MIS* (2) *admits a US provided that*

$$\max_{1\leq i\leq 3}\{J_i^*\}<0.5. \tag{11}$$

Proof. Assume that $\mho \subseteq Q$ and $\mho = \{(K_1,K_2,K_3,K_4)\in Q : \|K_1,K_2,K_3,K_4\| \leq \delta_2\}$, where $\delta_2 = \max\{\delta,\delta_1\}$ and $\delta,\delta_1\in(0,1)$ so that

$$\delta > \max\{E_1,E_2,E_3\} \text{ or } \delta > \max\{E_5,E_2,E_3\}.$$

Describe $\wp_\rho : \mho\times\mho\to\mho$ as

$$\wp_\rho(\nu,\xi)(\vartheta)=\begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad +\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\ \vartheta\in(\vartheta_i,s_i]\cap\tau, \\ \Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad +\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\ \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\dots,n. \end{cases}$$

Also, let

$$\begin{aligned} \|Z(\vartheta,\nu,\xi)\| &\le \|Z(\vartheta,\nu,\xi)-Z(\vartheta,0,0)\|+\|Z(\vartheta,0,0)\|\le T_Z(\|\nu\|+\|\xi\|)+Y_Z, \\ \|\varphi(s,\nu(s),\xi(s))\| &\le T_\varphi(\|\nu\|+\|\xi\|), \end{aligned}$$

and

$$\begin{aligned} \|\Xi(\vartheta,K_1,K_2,K_3,K_4)\| &\le \|\Xi(\vartheta,K_1,K_2,K_3,K_4)-\Xi(\vartheta,0,0,0,0)\|-\|\Xi(\vartheta,0,0,0,0)\| \\ &\le T_{\Xi_1}\|K_1\|+T_{\Xi_2}\|K_2\|+T_{\Xi_3}\|K_3\|+T_{\Xi_4}\|K_4\|+Y_\Xi, \end{aligned}$$

where $Y_Z=\sup_{\vartheta\in\tau}\|Z(\vartheta,0,0)\|$, $Y_\Xi=\sup_{\vartheta\in\tau}\|\Xi(\vartheta,0,0,0,0)\|$, and $Y^*_\Xi=Y_Z+Y_\Xi$. Additionally, $K_1=\nu(s)$, $K_2=\xi(s)$,

$$K_3=\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p \text{ and } K_4=\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p.$$

Now, we claim that $\wp_\rho:\mho\times\mho\to\mho$ is a self-mapping. For $\vartheta\in(\vartheta_i,\vartheta_{i+1}]\cap\tau$, $i=1,2,\dots,n$, we have

$$\begin{aligned} \|\wp_\rho(\nu,\xi)(\vartheta)\| \le{}& \left\|\frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s\right\|+\|\Lambda_\rho(B\vartheta^\rho)\nu_0\| \\ &+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ &+\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\Bigg\|\Delta s \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\rho)} \int_{\vartheta_i}^{s_i} \left\|(\vartheta - s)^{\rho-1}\right\| \|\varphi_i(s,\nu(s),\xi(s))\| \Delta s + \left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
&+ \int_{\vartheta_0}^{\vartheta_l} \left\|(\vartheta - s)^{\rho-1}\right\| \left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta - s)^\rho\right)\right)\right\| (\|Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\| \\
&+ \left\|\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\right\| \Bigg) \Delta s,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left\|\wp_\rho(\nu,\xi)(\vartheta)\right\| &\leq \frac{1}{\Gamma(\rho)} \int_{\vartheta_i}^{s_i} \left\|(\vartheta - s)^{\rho-1}\right\| T_\varphi(\|\nu(s)\| + \|\xi(s)\|)\Delta s + \left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
&+ \int_{\vartheta_0}^{\vartheta_l} \left\|(\vartheta - s)^{\rho-1}\right\| \left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta - s)^\rho\right)\right)\right\| \big(T_Z(\|\nu\| + \|\xi\|) + T_{\Xi_1}\|\nu(s)\| + T_{\Xi_2}\|\xi(s)\| \\
&+ T_{\Xi_3}\left\|\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p\right\| + T_{\Xi_4}\left\|\int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right\| + Y^*_\Xi \Bigg) \Delta s \\
&\leq \frac{1}{\Gamma(\rho)} \left\|(\vartheta - s)^{\rho-1}\right\| T_\varphi\|\nu + \xi\|(s_i - \vartheta_i) + \left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
&+ \int_{\vartheta_0}^{\vartheta_l} \left\|(\vartheta - s)^{\rho-1}\right\| \left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta - s)^\rho\right)\right)\right\| \big(T_Z\|\nu + \xi\| + T_{\Xi_1}\|\nu(s)\| + T_{\Xi_2}\|\xi(s)\| \\
&+ T_{\Theta_3}T_{Z_1}\int_{s_0}^{s_l} \|\nu(p) + \xi(p)\|\Delta p + T_{\Theta_4}T_{Z_2}\int_{s_0}^{s_l} \|\nu(p) + \xi(p)\|\Delta p + Y^*_\Xi \Bigg) \Delta s,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\left\|\wp_\rho(\nu,\xi)(\vartheta)\right\| \\
\leq\ & \frac{1}{\Gamma(\rho)} \left\|(\vartheta - s)^{\rho-1}\right\| T_\varphi\|\nu(s) + \xi(s)\|(s_i - \vartheta_i) + \left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
&+ \int_{\vartheta_0}^{\vartheta_l} \left\|(\vartheta - s)^{\rho-1}\right\| \left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta - s)^\rho\right)\right)\right\| \big(T_Z\|\nu + \xi\| + T_{\Xi_1}\|\nu(s)\| + T_{\Xi_2}\|\xi(s)\| \\
&T_{\Theta_3}T_{Z_1}\|\nu(p) + \xi(p)\|(s_l - s_0) + T_{\Theta_4}T_{Z_2}\|\nu(p) + \xi(p)\|(s_l - s_0) + Y^*_\Xi\big)\Delta s \\
\leq\ & \frac{1}{\Gamma(\rho)} \sup_{\vartheta\in\tau}\left\|(\vartheta - s)^{\rho-1}\right\| T_\varphi \sup_{\vartheta\in\tau}\|\nu(s) + \xi(s)\|(s_i - \vartheta_i) + \sup_{\vartheta\in\tau}\left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\
&+ \int_{\vartheta_0}^{\vartheta_l} \sup_{\vartheta\in\tau}\left\|(\vartheta - s)^{\rho-1}\right\| \sup_{\vartheta\in\tau}\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta - s)^\rho\right)\right)\right\| \Bigg(T_Z \sup_{\vartheta\in\tau}\|\nu + \xi\| + T_{\Xi_1}\sup_{\vartheta\in\tau}\|\nu(s)\| + T_{\Xi_2}\sup_{\vartheta\in\tau}\|\xi(s)\| \\
&T_{\Theta_3}T_{Z_1}\sup_{\vartheta\in\tau}\|\nu(p) + \xi(p)\|(s_l - s_0) + T_{\Theta_4}T_{Z_2}\sup_{\vartheta\in\tau}\|\nu(p) + \xi(p)\|(s_l - s_0) + Y^*_\Xi \Bigg)\Delta s,
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\left\|\wp_\rho(\nu,\xi)(\vartheta)\right\| \\
\leq\ & \frac{1}{\Gamma(\rho)} u_3 T_\varphi \|\nu+\xi\|_\infty (s_i-\vartheta_i) + u_1 + \int_{\vartheta_0}^{\vartheta_l} u_2 u_3 \big(T_Z \|\nu+\xi\|_\infty + T_{\Xi_1}\|\nu\|_\infty + T_{\Xi_2}\|\xi\|_\infty \\
& T_{\Theta_3} T_{Z_1} \|\nu+\xi\|_\infty (s_l - s_0) + T_{\Theta_4} T_{Z_2} \|\nu+\xi\|_\infty (s_l-s_0) + Y^*_\Xi\big)\Delta\, s \\
\leq\ & \frac{1}{\Gamma(\rho)} u_3 T_\varphi \delta (s_i-\vartheta_i) + u_1 + u_3\big(u_2 T_Z \delta + u_2 T_{\Xi_1}\delta + u_2 T_{\Xi_2}\delta \\
& + u_2 T_{\Theta_3} T_{Z_1}(s_l-s_0) + T_{\Theta_4} T_{Z_2}\delta(s_l-s_0) + Y^*_\Xi\big) \int_{\vartheta_0}^{\vartheta_l} \Delta\, s \\
\leq\ & \frac{1}{\Gamma(\rho)} u_3 T_\varphi \delta (s_i-\vartheta_i) + u_1 + \delta u_3\big(u_2 T_Z + u_2 T_{\Xi_1} + u_2 T_{\Xi_2} \\
& + u_2\big(T_{\Theta_3} T_{Z_1} + T_{\Theta_4} T_{Z_2}\big)(s_l-s_0) + Y^*_\Xi\big)(\vartheta_l - \vartheta_0).
\end{aligned}$$

Thus

$$\left\|\wp_\rho(\nu,\xi)(\vartheta)\right\| \leq E_3 + \delta J_1^* = \delta_1.$$

Similarly, one can write

$$\left\|\wp_\rho(\xi,\nu)(\vartheta)\right\| \leq E_6 + \delta J_1^* = \delta_1.$$

where $\delta_1 = \delta + \delta J_1^*$. Hence

$$\left\|\wp_\rho(\nu,\xi)(\vartheta)\right\| \leq \delta_2 \text{ and } \left\|\wp_\rho(\xi,\nu)(\vartheta)\right\| \leq \delta_2. \tag{12}$$

Therefore, from (12), we get $\wp(\mho\times\mho) \subseteq \mho$. Also, for $\vartheta \in (s_i, \vartheta_{i+1}] \cap \tau$, $i = 1,2,\ldots,n$, with $\nu_0 = \widetilde{\nu}_0$ and $\xi_0 = \widetilde{\xi}_0$, we get

$$\begin{aligned}
&\left\|\wp_\rho(\nu,\xi)(\vartheta) - \wp_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\
\leq\ & \left\|\frac{1}{\Gamma(\rho)} \int_{\vartheta_i}^{s_i} (\vartheta-s)^{\rho-1}[\varphi_i(s,\nu(s),\xi(s)) - \varphi_i(s,\nu(s),\xi(s))]\Delta s\right\| \\
& + \int_{\vartheta_0}^{\vartheta_l} \left\|(\vartheta-s)^{\rho-1}\right\| \left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\| \|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\
& + \Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg) \\
& - \left(Z\Big(\vartheta,\widetilde{\nu}(\vartheta),\widetilde{\xi}(\vartheta)\Big) + \Xi\left(s,\widetilde{\nu}(s),\widetilde{\xi}(s),\int_{s_0}^{s_l} Z_1\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p, \int_{s_0}^{s_l} Z_2\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p\right)\right)\Bigg\|\Delta s
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\varphi_i(s,\nu(s),\xi(s))-\varphi_i(s,\widetilde{\nu}(s),\widetilde{\xi}(s))\right\|\Delta s\\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^{\rho}\right)\right)\right\|\Big[T_Z\Big(\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\Big)\\
&+T_{\Xi_1}\|\nu(s)-\widetilde{\nu}(s)\|+T_{\Xi_2}\left\|\xi(s)-\widetilde{\xi}(s)\right\|+T_{\Xi_3}\int_{s_0}^{s_l}\left\|Z_1(s,p,\nu(p),\xi(p))-Z_1\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\right\|\Delta p\\
&+T_{\Xi_4}\int_{s_0}^{s_l}\left\|Z_2(s,p,\nu(p),\xi(p))-Z_2\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\right\|\Delta p\Big]\Delta s,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\left\|\wp_\rho(\nu,\xi)(\vartheta)-\wp_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\|\\
\leq\ & \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}\left\|(\vartheta-s)^{\rho-1}\right\|L_\varphi\Big[\|\nu(s)-\widetilde{\nu}(s)\|+\left\|\xi(s)-\widetilde{\xi}(s)\right\|\Big]\Delta s\\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^{\rho}\right)\right)\right\|\Big[T_Z\Big(\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\Big)+T_{\Xi_1}\|\nu(s)-\widetilde{\nu}(s)\|\\
&+T_{\Xi_2}\left\|\xi(s)-\widetilde{\xi}(s)\right\|+T_{\Xi_3}T_{Z_1}\int_{s_0}^{s_l}\Big(\|\nu(p)-\widetilde{\nu}(p)\|+\left\|\xi(p)-\widetilde{\xi}(p)\right\|\Big)\Delta p\\
&+T_{\Xi_3}T_{Z_2}\int_{s_0}^{s_l}\Big(\|\nu(p)-\widetilde{\nu}(p)\|+\left\|\xi(p)-\widetilde{\xi}(p)\right\|\Big)\Delta p\Big]\Delta s\\
\leq\ & \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}\sup_{\vartheta\in\tau}\left\|(\vartheta-s)^{\rho-1}\right\|L_\varphi\sup_{\vartheta\in\tau}\Big[\|\nu(s)-\widetilde{\nu}(s)\|+\left\|\xi(s)-\widetilde{\xi}(s)\right\|\Big]\Delta s\\
&+\int_{\vartheta_0}^{\vartheta_l}\sup_{\vartheta\in\tau}\left\|(\vartheta-s)^{\rho-1}\right\|\sup_{\vartheta\in\tau}\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^{\rho}\right)\right)\right\|\left[\begin{array}{c}T_Z\Big(\sup_{\vartheta\in\tau}\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\sup_{\vartheta\in\tau}\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\Big)\\+T_{\Xi_1}\sup_{\vartheta\in\tau}\|\nu(s)-\widetilde{\nu}(s)\|\end{array}\right.\\
&+T_{\Xi_2}\sup_{\vartheta\in\tau}\left\|\xi(s)-\widetilde{\xi}(s)\right\|+T_{\Xi_3}T_{Z_1}\left(\sup_{\vartheta\in\tau}\|\nu(p)-\widetilde{\nu}(p)\|+\sup_{\vartheta\in\tau}\left\|\xi(p)-\widetilde{\xi}(p)\right\|\right)(s_l-s_0)\\
&\left.+T_{\Xi_3}T_{Z_2}\left(\sup_{\vartheta\in\tau}\|\nu(p)-\widetilde{\nu}(p)\|+\sup_{\vartheta\in\tau}\left\|\xi(p)-\widetilde{\xi}(p)\right\|\right)(s_l-s_0)\right]\Delta s,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\left\|\wp_\rho(\nu,\xi)(\vartheta)-\wp_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\|\\
\leq\ & \frac{1}{\Gamma(\rho)}u_3L_\varphi\Big[\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\Big](s_i-\vartheta_i)+u_3u_2\left[\begin{array}{c}T_Z\Big(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\Big)\\+T_{\Xi_1}\|\nu-\widetilde{\nu}\|_\infty\end{array}\right.\\
&+T_{\Xi_2}\left\|\xi-\widetilde{\xi}\right\|_\infty+T_{\Xi_3}T_{Z_1}\Big(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\Big)(s_l-s_0)\\
&\left.+T_{\Xi_3}T_{Z_2}\Big(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\Big)(s_l-s_0)\right]\times\int_{\vartheta_0}^{\vartheta_l}\Delta s,
\end{aligned}
$$

$$\leq \frac{1}{\Gamma(\rho)}u_3 L_\varphi(s_i-\vartheta_i)+u_3\left[\begin{array}{c} u_2 T_Z+u_2 T_{\Theta_1}+u_2 T_{\Theta_2} \\ +u_2 T_{\Theta_3}T_{Z_1}(s_l-s_0)+u_2 T_{\Theta_3}T_{Z_2}(s_l-s_0)\end{array}\right](\vartheta_l-\vartheta_0)$$
$$\times\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)$$
$$= (E_4+J_2^*)\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right)=J_3^*\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right). \tag{13}$$

Analogously, we can write

$$\left\|\Im_\rho(\nu,\xi)(\vartheta)-\Im_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\|\leq J_3^*\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right). \tag{14}$$

It follows from (11), (13) and (14) that the impulsive problem (2) has a US based on the Banach principle. □

The existence of at least one solution is next examined for both the MISs (1) and (2) using the Leray-Schauder alternative fixed point (FP) method and the weaker stipulation (2).

Theorem 5. *Assume that the assertion* (2) *holds. If there is* $\beth>0$ *so that*

$$u_1+u_4+2J_3\beth<\beth, \tag{15}$$

then the MIS (1) has at least on solution.

Proof. In the beginning, we demonstrate that the operator $\Im_\rho$ defined by (7) is completely continuous (CC). We can see that it is a continuous operator due to the continuity of the mappings O, D, Z, and Θ. Moreover, suppose that $\mho_1\subseteq Q$ and by using the fact O, D, Z, and Θ are bounded operators. Then there are positive constants T_1, T_2, Y_1 and Y_2 so that

$$\begin{aligned}\sum_{j=1}^{i}O_j(\nu(\vartheta),\xi(\vartheta)),\sum_{j=1}^{i}O_j(\xi(\vartheta),\nu(\vartheta)) &\leq T_1,\ \sum_{j=1}^{i}D_j(\vartheta,\nu(\vartheta),\xi(\vartheta)),\sum_{j=1}^{i}D_j(\vartheta,\xi(\vartheta),\nu(\vartheta))\leq T_2,\\ Z(\vartheta,\nu(\vartheta),\xi(\vartheta)),Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) &\leq Y_1 \text{ and } \|\Theta(\vartheta,K_1,K_2,K_3,K_4)\|\leq Y_2,\end{aligned}$$

where $K_1=\nu(s), K_2=\xi(s), K_3=\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p$ and $K_4=\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p$. Consider $T_1+T_2+u_1=\daleth$, $T_1+T_2+u_4=\daleth^*$, $Y_1+Y_2=Y$, $\left\|(\vartheta-s)^{\rho-1}\right\|\leq\daleth_1$, $\daleth+\daleth_1 u_2 Y(\vartheta_l-\vartheta_0)\leq\aleph$, and $\daleth^*+\daleth_1 u_2 Y(\vartheta_l-\vartheta_0)\leq\aleph$.

For any $\nu,\xi\in\mho_1$ and $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\ldots,n$, we get

$$\begin{aligned}\left\|\Im_\rho(\nu,\xi)(\vartheta)\right\| &\leq \sum_{j=1}^{i}\left\|O_j\big(\nu(\vartheta_j),\xi(\vartheta_j)\big)\right\|+\sum_{j=1}^{i}\left\|D_j\big(\vartheta_j,\nu(\vartheta_j),\xi(\vartheta_j)\big)\right\|+\left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\|\\ &\quad+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\Big\|(Z(s,\nu(s),\xi(s))\\ &\quad\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\bigg)\bigg\|\Delta s\\ &\leq T_1+T_2+u_1+\daleth_1 u_2(Y_1+Y_2)\int_{\vartheta_0}^{\vartheta_l}\Delta s\\ &= \daleth+\daleth_1 u_2 Y(\vartheta_l-\vartheta_0).\end{aligned} \tag{16}$$

Similarly

$$\left\|\Im_\rho(\xi,\nu)(\vartheta)\right\| \le T_1 + T_2 + u_4 + \daleth_1 u_2(Y_1+Y_2)\int_{\vartheta_0}^{\vartheta_l}\Delta s = \daleth^* + \daleth_1 u_2 Y(\vartheta_l - \vartheta_0). \tag{17}$$

From (16) and (17), we have

$$\left\|\Im_\rho(\nu,\xi)(\vartheta)\right\| \le \aleph \text{ and } \left\|\Im_\rho(\xi,\nu)(\vartheta)\right\| \le \aleph.$$

Therefore, $\Im$ is uniformly bounded.

Now, we establish that $\Im$ is CC. For this regard, we discuss the following options:

Option 1. Consider that time scales τ are made up of distinct points, with each point on τ being isolated. Based on Theorem 1, $\Im_\rho$ takes the form

$$\Im_\rho(\nu,\xi)(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \sum_{\vartheta\in\tau}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \sum_{\vartheta\in\tau}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \ \vartheta\in(\vartheta_0,\vartheta_1], \\ \Lambda_\rho(B\vartheta^\rho)\nu_0 + \sum_{\vartheta\in\tau}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \sum_{\vartheta\in\tau}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\ +\sum_{t=1}^{i}O_t\big(\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big) + D_t\big(\vartheta_t^-,\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big), \ \vartheta\in(\vartheta_i,\vartheta_{i+1}], \ i=1,2,\ldots,n. \end{cases} \tag{18}$$

It is obvious that (18) is a collection of summing operators on a discrete finite set. Furthermore, it is implied from the continuity of oj, O_t, D_t, Z and Θ that Λ_ρ is CC.

Option 2. Suppose that τ is continuous and $\vartheta_{l_1},\vartheta_{l_2}\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\ldots,n$ so that $\vartheta_{l_1} < \vartheta_{l_2}$, then

$$\begin{aligned} &\left\|\Im_\rho(\nu,\xi)(\vartheta_{l_2}) - \Im_\rho(\nu,\xi)(\vartheta_{l_1})\right\| \\ \le &\left\|\sum_{j=1}^{i}O_j\Big(\nu(\vartheta_{l_{2j}}^-),\xi(\vartheta_{l_{2j}}^-)\Big) - O_j\Big(\nu(\vartheta_{l_{1j}}^-),\xi(\vartheta_{l_{1j}}^-)\Big)\right. \\ &+\sum_{j=1}^{i}D_j\Big(\vartheta_{l_{2j}}^-,\nu(\vartheta_{l_{2j}}^-),\xi(\vartheta_{l_{2j}}^-)\Big) - D_j\Big(\vartheta_{l_{1j}}^-,\nu(\vartheta_{l_{1j}}^-),\xi(\vartheta_{l_{1j}}^-)\Big) \\ &+\left[\left(\int_{\vartheta_0}^{\vartheta_{l_2}}(\vartheta_{l_2}-s)^{\rho-1}\Lambda_{\rho,\rho}\Big(B\Big((\vartheta_{l_2}-s)^\rho\Big)\Big)(Z(s,\nu(s),\xi(s))\right.\right. \\ &\left.+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\Delta s\right) \\ &-\left(\int_{\vartheta_0}^{\vartheta_{l_1}}(\vartheta_{l_1}-s)^{\rho-1}\Lambda_{\rho,\rho}\Big(B\Big((\vartheta_{l_1}-s)^\rho\Big)\Big)(Z(s,\nu(s),\xi(s))\right. \\ &\left.\left.\left.+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\Delta s\right]\right\|, \end{aligned}$$

it follows that

$$\begin{aligned}
&\left\|\Im_\rho(\nu,\xi)(\vartheta_{l_2}) - \Im_\rho(\nu,\xi)(\vartheta_{l_1})\right\| \\
\leq\ & \sum_{j=1}^{i}\left\|O_j\left(\nu(\vartheta^-_{l_{2j}}),\xi(\vartheta^-_{l_{2j}})\right) - O_j\left(\nu(\vartheta^-_{l_{1j}}),\xi(\vartheta^-_{l_{1j}})\right)\right\| \\
&+\sum_{j=1}^{i}\left\|D_j\left(\vartheta^-_{l_{2j}},\nu(\vartheta^-_{l_{2j}}),\xi(\vartheta^-_{l_{2j}})\right) - D_j\left(\vartheta^-_{l_{1j}},\nu(\vartheta^-_{l_{1j}}),\xi(\vartheta^-_{l_{1j}})\right)\right\| \\
&+\left\|\int_{\vartheta_0}^{\vartheta_{l_2}}\left[(\vartheta_{l_1}-s)^{\rho-1}\Lambda_{\rho,\rho}\left(B\left((\vartheta_{l_2}-s)^\rho\right)\right) - (\vartheta_{l_1}-s)^{\rho-1}\Lambda_{\rho,\rho}\left(B\left((\vartheta_{l_1}-s)^\rho\right)\right)\right]\times Z(s,\nu(s),\xi(s))\right\| \\
&+\left\|\int_{\vartheta_0}^{\vartheta_{l_2}}\left[(\vartheta_{l_1}-s)^{\rho-1}\Lambda_{\rho,\rho}\left(B\left((\vartheta_{l_2}-s)^\rho\right)\right) - (\vartheta_{l_1}-s)^{\rho-1}\Lambda_{\rho,\rho}\left(B\left((\vartheta_{l_1}-s)^\rho\right)\right)\right]\right. \\
&\quad\left.\times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Delta s\right\| \\
&\quad+\left\|\int_{\vartheta_{l_1}}^{\vartheta_{l_2}}\left[(\vartheta_{l_2}-s)^{\rho-1}\Lambda_{\rho,\rho}\left(B\left((\vartheta_{l_2}-s)^\rho\right)\right)\right]Z(s,\nu(s),\xi(s))\right\| \\
&\quad+\left\|\int_{\vartheta_{l_1}}^{\vartheta_{l_2}}\left[(\vartheta_{l_2}-s)^{\rho-1}\Lambda_{\rho,\rho}\left(B\left((\vartheta_{l_2}-s)^\rho\right)\right)\right]Z(s,\nu(s),\xi(s))\right. \\
&\quad\left.\times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Delta s\right\|.
\end{aligned}$$

It is easy to see that the above inequality tends to 0 as $\vartheta_{l_1}\to\vartheta_{l_2}$. The operator $\Im_\rho$ is hence equicontinuous. Finally, we come to the conclusion that $\Im_\rho$ is CC using the Arzela-Ascoli theorem.

Option 3. Suppose that τ contains isolated points along with dense ones, that is τ is continuous and discrete. In light of Theorem 1, we can represent $\Im_\rho$ as the summation operator, which is CC (analyzed in option 1). Also, one may establish that $\Im_\rho$ is a CC operator for the dense points (analyzed in option 2). Therefore, for isolated and dense points, $\Im_\rho$ can be expressed as the sum of two operators. Since the two operators are CC, we can infer that their sum is also CC. As a result, the operator $\Im_\rho$ is CC. Thus, by adding together the three aforementioned scenarios, we determine that $\Im_\rho$ is a CC operator.

Ultimately, let $\alpha\in[0,1]$ and there are ν and ξ so that $\nu(\vartheta)=\Im_\rho(\nu,\xi)(\vartheta)$ and $\xi(\vartheta)=\Im_\rho(\xi,\nu)(\vartheta)$. Hence for $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\ldots,n$, one can write

$$
\begin{aligned}
& \|\xi(\vartheta)\| \\
= \; & \left\|\alpha\left(\Im_\rho(\xi,\nu)(\vartheta_l)\right)\right\| \\
& \Bigg\|\alpha\Bigg[\sum_{j=1}^{i} O_j\Big(\nu(\vartheta_{l_j}^-),\xi(\vartheta_{l_j}^-)\Big)+\sum_{j=1}^{i} D_j\Big(\vartheta_{l_j}^-,\nu(\vartheta_{l_j}^-),\xi(\vartheta_{l_j}^-)\Big)+\Lambda_\rho(B\vartheta^\rho)\nu_0 \\
& +\int_{\vartheta_0}^{\vartheta_l}(\vartheta_l-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta_l-s)^\rho\big)\big)\big(Z(s,\nu(s),\xi(s)) \\
& +\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\Delta s\Bigg]\Bigg\|
\end{aligned}
$$

$$
\begin{aligned}
\leq \; & \sum_{j=1}^{i} L_O\|\nu+\xi\|_\infty+\sum_{j=1}^{i} L_D\|\nu+\xi\|_\infty+u_1+\left\|(\vartheta_l-s)^{\rho-1}\right\|u_2\big(L_Z\|\nu+\xi\|_\infty+L_{\Theta_1}\|\nu\|_\infty \\
& +L_{\Theta_2}\|\xi\|_\infty+L_{\Theta_3}L_{Z_1}\|\nu+\xi\|_\infty(s_l-s_0)+L_{\Theta_4}L_{Z_2}\|\nu+\xi\|_\infty(s_l-s_0)\big)(\vartheta_l-\vartheta_0) \\
= \; & u_1+\left[\begin{array}{c}\sum_{j=1}^{i} L_O+\sum_{j=1}^{i} L_D \\ +u_3\big(u_2L_Z+u_2L_{\Theta_1}+u_2L_{\Theta_2}+u_2\big(L_{\Theta_3}L_{Z_1}+L_{\Theta_4}L_{Z_2}\big)(s_l-s_0)\big)(\vartheta_l-\vartheta_0)\end{array}\right] \\
& \times\|\nu+\xi\|_\infty,
\end{aligned}
$$

which yields that

$$
\|\xi\|_\infty \leq u_1+[E_2+J_2]\|\nu+\xi\|_\infty = u_1+J_3\|\nu+\xi\|_\infty. \tag{19}
$$

Similarly, we have

$$
\|\nu\|_\infty \leq u_4+[E_2+J_2]\|\nu+\xi\|_\infty = u_4+J_3\|\nu+\xi\|_\infty. \tag{20}
$$

Combining (19) and (20), we obtain that

$$
\|\xi+\nu\|_\infty \leq \|\xi\|_\infty+\|\nu\|_\infty = u_1+u_4+2J_3\|\nu+\xi\|_\infty.
$$

Hence

$$
\frac{\|\xi+\nu\|_\infty}{u_1+u_4+2J_3\|\nu+\xi\|_\infty} \leq 1.
$$

From (15), we have $\daleth>0$ so that $\|\xi+\nu\|_\infty\neq\daleth$. Let us consider the set

$$
\Phi=\{\xi,\nu\in\tau,\ \|\xi+\nu\|_\infty<\daleth\}.
$$

Consequently, the operator $\Im_\rho:\Phi\times\Phi\to\tau$ is both continuously and CC. Therefore from the choice of Φ there are no $\xi,\nu\in\mho_1(\Phi)$ provided that $\xi=\alpha\big(\Im_\rho(\xi,\nu)(\vartheta)\big)$ and $\nu=\alpha\big(\Im_\rho(\nu,\xi)(\vartheta)\big)$, $\alpha\in[0,1]$. Due to the nonlinear Leray-Schauder alternative, $\Im$ has a coupled FP, which is a solution to the MIS (1). □

Theorem 6. *Assume that the assertion* (2) *is true. If there is* $\daleth^*>0$ *so that*

$$
u_1+u_4+2J_3^*\daleth^*<\daleth^*,
$$

then the MIS (2) has at least on solution.

Proof. It is comparable to the earlier defense of $\wp_\rho$ in Theorem 5. □

4. Stability Study

We begin this part with the inequalities below:

$$\begin{cases} \|^{c,\tau}D^{\rho}\nu(\vartheta) - B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ -\Theta\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right)\Bigg\| \le \varepsilon,\ \vartheta\in\tau' \\ \|^{c,\tau}D^{\rho}\xi(\vartheta) - B(\vartheta)\xi(\vartheta) - Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ -\Theta\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right)\Bigg\| \le \varepsilon,\ \vartheta\in\tau' \\ \|\nu(\vartheta_r^+) - \nu(\vartheta_r^-) - O_r(\nu(\vartheta_r^-),\xi(\vartheta_r^-)) - D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-))\| \le \varepsilon,\ r=1,2,\dots,n, \\ \|\xi(\vartheta_r^+) - \xi(\vartheta_r^-) - O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-)) - D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-))\| \le \varepsilon,\ r=1,2,\dots,n, \end{cases} \tag{21}$$

and

$$\begin{cases} \|^{c,\tau}D^{\rho}\nu(\vartheta) - B(\vartheta)\nu(\vartheta) - Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ -\Xi\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(s,s,\nu(s),\xi(s))\Delta s\right)\Bigg\| \le \varepsilon, \\ \|^{c,\tau}D^{\rho}\xi(\vartheta) - B(\vartheta)\xi(\vartheta) - Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ -\Xi\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right)\Bigg\| \le \varepsilon, \\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\dots n, \\ \left\|\nu(\vartheta) - \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\right\| \le \varepsilon, \\ \left\|\xi(\vartheta) - \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s\right\| \le \varepsilon,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\dots,n. \end{cases} \tag{22}$$

for $\varepsilon > 0$.

Definition 4. *We say that the MIS (1) is UH stable on τ, if for every $\xi,\nu\in\mho_1(\tau,\mathbb{R}^n)$ satisfying (21), there are $\widehat{\xi},\widehat{\nu}\in Q^1(\tau,\mathbb{R}^n)$ as a solution of (1) so that*

$$\left\|\xi(s)-\widehat{\xi}(s)\right\| + \|\nu(s)-\widehat{\nu}(s)\| \le F\varepsilon,\ \textit{for}\ F>0,\ s\in\tau.$$

Definition 5. *We say that the MIS (2) is UH stable on τ, if for every $\xi,\nu\in Q^1(\tau,\mathbb{R}^n)$ fulfilling (22), there are $\widehat{\xi},\widehat{\nu}\in Q^1(\tau,\mathbb{R}^n)$ as a solution of (2) so that*

$$\left\|\xi(\vartheta)-\widehat{\xi}(\vartheta)\right\| + \|\nu(\vartheta)-\widehat{\nu}(\vartheta)\| \le F^*\varepsilon,\ \textit{for}\ F^*>0,\ s\in\tau.$$

Remark 2. *The solutions $\xi,\nu\in Q^1(\tau,\mathbb{R}^n)$ fulfill (21), if there is $l\in Q(\tau,\mathbb{R}^n)$ with the sequence l_r so that $\|l_r\|\le\varepsilon$ and*

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Theta\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right) + l(\vartheta), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Theta\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right) + l(\vartheta), \\ \vartheta\in\tau',\ \nu(\vartheta_0)=\nu_0,\ \xi(\vartheta_0)=\xi_0, \\ \nu(\vartheta_r^+) - \nu(\vartheta_r^-) = O_r(\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + l_r,\ r=1,2,\dots,n, \\ \xi(\vartheta_r^+) - \xi(\vartheta_r^-) = O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + l_r,\ r=1,2,\dots,n, \end{cases}$$

Lemma 1. *Every functions $\xi,\nu \in Q^1(\tau,\mathbb{R}^n)$ that fulfill (21) also satisfy the inequalities shown below:*

$$\begin{cases} \Big\|\nu(\vartheta) - \Lambda_\rho(B\vartheta^\rho)\nu_0 - \sum_{r=1}^{n}\left(O_r(\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-))\right) \\ - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\nu(s),\xi(s))\right)\Delta s\Big\| \le \delta\varepsilon, \\ \Big\|\xi(\vartheta) - \Lambda_\rho(B\vartheta^\rho)\xi_0 - \sum_{r=1}^{n}\left(O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-))\right) \\ - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\xi(s),\nu(s))\Delta s - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\ \times\Theta\left(s,\xi(s),\nu(s),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\xi(s),\nu(s))\right)\Delta s\Big\| \le \delta\varepsilon, \end{cases}$$

for $\vartheta \in (\vartheta_r,\vartheta_{r+1}] \subset \tau$, where $\big\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\big\| \le \frac{1}{2}u_2$ and $\delta = (n + \frac{1}{2}u_3u_2(\vartheta_l - \vartheta_0))$.

Proof. When $\xi,\nu \in Q^1(\tau,\mathbb{R}^n)$ justify (21), then by Remark 2, we have

$$\begin{cases} {}^{c,\tau}D^\rho\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Theta\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right) + l(\vartheta), \\ {}^{c,\tau}D^\rho\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Theta\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right) + l(\vartheta), \\ \vartheta \in \tau',\ \nu(\vartheta_0) = \nu_0,\ \xi(\vartheta_0) = \xi_0, \\ \nu(\vartheta_r^+) - \nu(\vartheta_r^-) = O_r(\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-)) + l_r,\ r = 1,2,\ldots,n, \\ \xi(\vartheta_r^+) - \xi(\vartheta_r^-) = O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-)) + l_r,\ r = 1,2,\ldots,n, \end{cases}$$

yields

$$\begin{aligned} \nu(\vartheta) &= \Lambda_\rho(B\vartheta^\rho)\nu_0 + \sum_{r=1}^{n}\left(O_r\big(\nu(\vartheta_r^-),\xi(\vartheta_r^-)\big) + D_r\big(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-)\big)\right) + \sum_{t=1}^{n} l_t \\ &+ \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\ &\times\Theta\left(s,\nu(s),\xi(s),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\nu(s),\xi(s))\right)\Delta\, s \\ &+ \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)f(s)\Delta s \end{aligned}$$

and

$$
\begin{aligned}
\xi(\vartheta) \quad = \quad & \Lambda_\rho(B\vartheta^\rho)\xi_0 + \sum_{r=1}^{n}\big(O_r\big(\xi(\vartheta_r^-),\nu(\vartheta_r^-)\big) + D_r\big(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-)\big)\big) + \sum_{t=1}^{n} l_t \\
& + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\xi(s),\nu(s))\Delta s + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\
& \times\Theta\left(s,\xi(s),\nu(s),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\xi(s),\nu(s))\right)\Delta\, s \\
& \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)f(s)\Delta s.
\end{aligned}
$$

So

$$
\begin{aligned}
& \Bigg\|\nu(\vartheta) - \Lambda_\rho(B\vartheta^\rho)\nu_0 - \sum_{r=1}^{n}\big(O_r\big(\nu(\vartheta_r^-),\xi(\vartheta_r^-)\big) + D_r\big(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-)\big)\big) \\
& - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\
& \times\Theta\left(s,\nu(s),\xi(s),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\nu(s),\xi(s))\right)\Delta s\Bigg\| \\
\leq \quad & \int_{\vartheta_0}^{\vartheta_l}\Big\|(\vartheta-s)^{\rho-1}\Big\|\Big\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Big\|\|f(s)\|\Delta s + \sum_{t=1}^{n}\|l_t\| \leq \delta\varepsilon.
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
& \Bigg\|\xi(\vartheta) - \Lambda_\rho(B\vartheta^\rho)\xi_0 - \sum_{r=1}^{n}\big(O_r\big(\xi(\vartheta_r^-),\nu(\vartheta_r^-)\big) + D_r\big(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-)\big)\big) \\
& - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\xi(s),\nu(s))\Delta s - \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\
& \times\Theta\left(s,\xi(s),\nu(s),\int_{\vartheta_0}^{\vartheta_l}Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int_{\vartheta_0}^{\vartheta_l}Z_2(\vartheta,s,\xi(s),\nu(s))\right)\Delta s\Bigg\| \\
\leq \quad & \int_{\vartheta_0}^{\vartheta_l}\Big\|(\vartheta-s)^{\rho-1}\Big\|\Big\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Big\|\|f(s)\|\Delta s + \sum_{t=1}^{n} l_t \leq \delta\varepsilon.
\end{aligned}
$$

Thus, we get the desired result. □

Remark 3. *The mappings $\xi, \nu \in Q^1(\tau, \mathbb{R}^n)$ verify (22), if there is a map $l \in Q(\tau, \mathbb{R}^n)$ and bounded sequences $\{l_r, r = 1,2,\ldots,n\}$ provided that $\|l_r(\vartheta)\| \le \varepsilon$, $\vartheta \in \tau$, $r = 1,2,\ldots,n$ so that*

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Xi\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right) + l(\vartheta), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Xi\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right) + l(\vartheta), \\ \vartheta \in (s_i,\vartheta_{i+1}]\cap\tau,\ i = 1,2,\ldots n, \\ \nu(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s + l_i, \\ \xi(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s + l_i,\ \vartheta \in (\vartheta_i,s_i]\cap\tau,\ i = 1,2,\ldots,n. \end{cases}$$

Lemma 2. *Every mappings $\xi, \nu \in Q^1(\tau, \mathbb{R}^n)$ that fulfill (22) also fulfill the inequalities shown below:*

$$\begin{cases} \Bigg\|\nu(\vartheta) - \Lambda_{\rho}(B\vartheta^{\rho})\nu_0 - \int\limits_{s_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)Z(s,\nu(s),\xi(s)) \\ -\int\limits_{s_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B(\vartheta-s)^{\rho}\big)\Xi\left(s,\nu(s),\xi(s),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\right)\Delta s \\ -\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\Bigg\| \le (u_3u_2(\vartheta-\vartheta_i)+n)\varepsilon, \\ \Bigg\|\xi(\vartheta) - \Lambda_{\rho}(B\vartheta^{\rho})\xi_0 - \int\limits_{s_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)Z(s,\xi(s),\nu(s)) \\ -\int\limits_{s_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B(\vartheta-s)^{\rho}\big)\Xi\left(s,\xi(s),\nu(s),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\right)\Delta s \\ -\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\Bigg\| \le (u_3u_2(\vartheta-\vartheta_i)+n)\varepsilon, \\ \vartheta \in (s_i,\vartheta_{i+1}]\cap\tau,\ i = 1,2,\ldots n, \end{cases}$$

and

$$\begin{cases} \left\|\xi(s) - \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\right\| \le n\varepsilon, \\ \left\|\nu(s) - \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s\right\| \le n\varepsilon, \end{cases} \quad \vartheta \in (\vartheta_i,s_i]\cap\tau,\ i = 1,2,\ldots,n,$$

in which $\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)\| \le \frac{1}{2}u_2$.

Proof. If $\xi,\nu \in Q^1(\tau,\mathbb{R}^n)$ justify (21), by virtue of Remark 3, we get

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta) = B(\vartheta)\nu(\vartheta) + Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\ +\Xi\left(\vartheta,\nu(\vartheta),\xi(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\Delta s\right) + l(\vartheta), \\ {}^{c,\tau}D^{\rho}\xi(\vartheta) = B(\vartheta)\xi(\vartheta) + Z(\vartheta,\xi(\vartheta),\nu(\vartheta)) \\ +\Xi\left(\vartheta,\xi(\vartheta),\nu(\vartheta),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\Delta s\right) + l(\vartheta), \\ \nu(\vartheta_0)=\nu_0,\ \xi(\vartheta_0)=\xi_0,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\ldots,n, \\ \nu(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s + l_i,\ \vartheta,s\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\ldots,n, \\ \xi(\vartheta) = \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s + l_i,\ \vartheta,s\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\ldots,n. \end{cases} \tag{23}$$

Obviously, Equation (23) leads to

$$\nu(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int\limits_{\vartheta_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s + \int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\ \times\Xi\left(s,\nu(s),\xi(s),\int\limits_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p,\int\limits_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Delta s \\ +\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\ldots,n, \\ \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s + l_i,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\ldots,n, \end{cases}$$

and

$$\xi(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int\limits_{\vartheta_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\xi(s),\nu(s))\Delta s + \int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) \\ \times\Xi\left(s,\xi(s),\nu(s),\int\limits_{s_0}^{s_l} Z_1(s,p,\xi(p),\nu(p))\Delta p,\int\limits_{s_0}^{s_l} Z_2(s,p,\xi(p),\nu(p))\Delta p\right)\Delta s \\ +\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\ldots,n, \\ \frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s + l_i,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\ldots,n. \end{cases}$$

For $\vartheta\in(s_i,\vartheta_{i+1}]\cap\tau$, $i=1,2,\ldots,n$, we have

$$\begin{aligned} &\Bigg\|\nu(\vartheta) - \Lambda_\rho(B\vartheta^\rho)\nu_0 - \int\limits_{s_i}^{\vartheta}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(s,\nu(s),\xi(s)) \\ &-\int\limits_{s_i}^{\vartheta}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B(\vartheta-s)^\rho\big)\Xi\left(s,\nu(s),\xi(s),\int\limits_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\nu(s),\xi(s))\Delta s,\int\limits_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\nu(s),\xi(s))\right)\Delta\, s \\ &-\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\Bigg\| \\ &\le \int\limits_{s_i}^{\vartheta}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\|f(s)\|\Delta s + \sum_{t=1}^{n}\|l_t\| \le \left(\frac{1}{2}u_3u_2(\vartheta-\vartheta_i)+n\right)\varepsilon, \end{aligned}$$

and

$$\Bigg\| \xi(\vartheta) - \Lambda_\rho(B\vartheta^\rho)\xi_0 - \int_{s_i}^{\vartheta} \Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big) Z(s,\xi(s),\nu(s))$$
$$- \int_{s_i}^{\vartheta}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B(\vartheta-s)^\rho\big)\Xi\left(s,\nu(s),\xi(s),\int_{\vartheta_0}^{\vartheta_l} Z_1(\vartheta,s,\xi(s),\nu(s))\Delta s, \int_{\vartheta_0}^{\vartheta_l} Z_2(\vartheta,s,\xi(s),\nu(s))\right)\Delta\, s$$
$$-\frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\Bigg\|$$
$$\leq \int_{s_i}^{\vartheta}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\|f(s)\|\Delta s + \sum_{t=1}^{n}\|l_t\| \leq \left(\tfrac{1}{2}u_3u_2(\vartheta-\vartheta_i)+n\right)\varepsilon,$$

with a similar approach, we obtain

$$\begin{cases} \left\|\xi(s) - \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\right\| \leq n\varepsilon, \\ \left\|\nu(s) - \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{\vartheta}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s\right\| \leq n\varepsilon, \end{cases} \quad \vartheta \in (\vartheta_i, s_i]\cap\tau,\ i=1,2,\ldots,n.$$

Consequently, the debate is over. □

Currently, we present a sufficient stipulation for the UH stability of MISs (1) and (2).

Theorem 7. *If both inequality (6) and assumption* (1) *are true, the MIS (1) is UH stable.*

Proof. Assume that ξ and ν are a solution of the MIS (1) and $\widetilde{\xi}$ and $\widetilde{\nu}$ are a solution of (21). Using Theorem 3, one has

$$\nu(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\ \vartheta\in(\vartheta_0,\vartheta_1], \\ \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\ +\sum_{t=0}^{i} O_t\big(\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big) + D_t\big(\vartheta_t^-,\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big),\ \vartheta\in(\vartheta_i,\vartheta_{i+1}],\ i=1,2,\ldots,n, \end{cases}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF. In a manner similar to Theorem 3, we obtain

$$
\begin{aligned}
&\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\| \\
\leq\ & \sum_{j=1}^{i}\left\|O_j\left(\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)-O_j\left(\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\right)\right\| \\
&+\sum_{j=1}^{i}\left\|D_j\left(\vartheta_j^-,\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)-D_j\left(\vartheta_j^-,\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\right)\right\| \\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}(B((\vartheta-s)^\rho))\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\
&+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg) \\
&-\left(Z\left(\vartheta,\widetilde{\nu}(\vartheta),\widetilde{\xi}(\vartheta)\right)\right. \\
&+\Theta\left(s,\widetilde{\nu}(s),\widetilde{\xi}(s),\int_{s_0}^{s_l}Z_1\left(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\right)\Delta p,\int_{s_0}^{s_l}Z_2\left(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\right)\Delta p\right)\Bigg)\Bigg\|\Delta s \\
\leq\ & \frac{1}{2}u_3u_2(\vartheta_l-\vartheta_i)\varepsilon+J_3\left(\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\right).
\end{aligned}
\tag{24}
$$

Similarly, we can obtain

$$
\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\leq\frac{1}{2}u_3u_2(\vartheta_l-\vartheta_i)\varepsilon+J_3\left(\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|+\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|\right). \tag{25}
$$

From (24) and (25), we get

$$
\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\leq u_3u_2(\vartheta_l-\vartheta_i)\frac{\varepsilon}{1-2J_3}=\nabla\varepsilon,
$$

where $\nabla=\frac{u_3u_2(\vartheta_l-\vartheta_i)}{1-2J_3}$. It follows that the MIS (1) is UH stable. In addition, if $\overline{\nabla}(\varepsilon)=\overline{\nabla}(0)=0$, once this happens, our considered system (1) becomes generalized UH stable. □

Theorem 8. *If both inequality (11) and assumption* (1) *are true, the MIS (2) is UH stable.*

Proof. Assume that ξ and ν are a solution of the MIS (1) and $\widetilde{\xi}$ and $\widetilde{\nu}$ are a solution of (22). From Theorem 4, we have

$$
\nu(\vartheta)=\begin{cases}
\Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}(B((\vartheta-s)^\rho))Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\
\quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}(B((\vartheta-s)^\rho))\Delta s \\
\times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\ \vartheta\in(\vartheta_i,s_i]\cap\tau, \\
\Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}(B((\vartheta-s)^\rho))Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\
\quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}(B((\vartheta-s)^\rho))\Delta s \\
\times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\
+\frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\dots,n,
\end{cases}
$$

and

$$\xi(\vartheta)=\begin{cases}\Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\\ \Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \quad\times\Xi\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\\ \quad+\frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\ldots,n,\end{cases}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF. In a manner similar to Theorem 4, we get

$$\begin{aligned}&\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|\\ \le\ &\frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\varphi_i(s,\nu(s),\xi(s))-\varphi_i(s,\widetilde{\nu}(s),\widetilde{\xi}(s))\right\|\Delta s\\ &+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\right\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\\ &+\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\\ &-\left(Z\Big(\vartheta,\widetilde{\nu}(\vartheta),\widetilde{\xi}(\vartheta)\Big)+\Xi\left(s,\widetilde{\nu}(s),\widetilde{\xi}(s),\int_{s_0}^{s_l}Z_1\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p,\int_{s_0}^{s_l}Z_2\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p\right)\right)\Bigg\|\Delta s\\ \le\ &\frac{1}{2}u_3u_2(\vartheta_l-\vartheta_i)\varepsilon+J_3^*\left(\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\right).\end{aligned}\tag{26}$$

Analogously, we can obtain that

$$\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\le\frac{1}{2}u_3u_2(\vartheta_l-\vartheta_i)\varepsilon+J_3^*\left(\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|+\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|\right).\tag{27}$$

Adding (26) and (27), we find that

$$\|\nu(\vartheta)-\widetilde{\nu}(\vartheta)\|+\left\|\xi(\vartheta)-\widetilde{\xi}(\vartheta)\right\|\le(u_3u_2(\vartheta_l-\vartheta_i))\frac{\varepsilon}{1-2J_3^*}=\Delta\varepsilon,$$

where $\Delta=\frac{u_3u_2(\vartheta_l-\vartheta_i)}{1-2J_3^*}$. Therefore, the MIS (2) is UH stable. Further, if $\Delta^*=\Delta^*(0)=0$, once this happens, our supposed system (2) becomes generalized UH stable. □

5. Controllability Study

The third part of the article analyzes the controllability of given impulsive systems. We start by providing some definitions in this line.

Definition 6. *The MIS (3) has a solution $(\nu, \xi) \in \tau \times \tau$ if $\nu(0) = \nu_0$, $\xi(0) = \xi_0$ and the pair (ξ, ν) is a solution to the integral equations below:*

$$\nu(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) Z(\vartheta, \nu(\vartheta), \xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s, \nu(s), \xi(s), \int_{s_0}^{s_l} Z_1(s, p, \nu(p), \xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s, p, \nu(p), \xi(p))\Delta p\right), \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) H\ell(\vartheta)\Delta s, \ \vartheta \in (\vartheta_0, \vartheta_1], \\ \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) Z(\vartheta, \nu(\vartheta), \xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s, \nu(s), \xi(s), \int_{s_0}^{s_l} Z_1(s, p, \nu(p), \xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s, p, \nu(p), \xi(p))\Delta p\right), \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) H\ell(\vartheta)\Delta s \\ + \sum_{t=1}^{i} O_t\big(\nu(\vartheta_t^-), \xi(\vartheta_t^-)\big) + D_t\big(\vartheta_t^-, \nu(\vartheta_t^-), \xi(\vartheta_t^-)\big), \ \vartheta \in (\vartheta_i, \vartheta_{i+1}], \ i = 1, 2, \ldots, n, \end{cases} \tag{28}$$

and

$$\xi(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\xi_0 + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) Z(\vartheta, \xi(\vartheta), \nu(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s, \xi(s), \nu(s), \int_{s_0}^{s_l} Z_1(s, p, \xi(p), \nu(p))\Delta p, \int_{s_0}^{s_l} Z_2(s, p, \xi(p), \nu(p))\Delta p\right), \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) H\ell(\vartheta)\Delta s, \ \vartheta \in (\vartheta_0, \vartheta_1], \\ \Lambda_\rho(B\vartheta^\rho)\xi_0 + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) Z(\vartheta, \xi(\vartheta), \nu(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\ \times\Theta\left(s, \xi(s), \nu(s), \int_{s_0}^{s_l} Z_1(s, p, \xi(p), \nu(p))\Delta p, \int_{s_0}^{s_l} Z_2(s, p, \xi(p), \nu(p))\Delta p\right), \\ \quad + \int_{\vartheta_0}^{\vartheta_l} (\vartheta - s)^{\rho-1} \Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big) H\ell(\vartheta)\Delta s \\ + \sum_{t=1}^{i} O_t\big(\xi(\vartheta_t^-), \nu(\vartheta_t^-)\big) + D_t\big(\vartheta_t^-, \xi(\vartheta_t^-), \nu(\vartheta_t^-)\big), \ \vartheta \in (\vartheta_i, \vartheta_{i+1}], \ i = 1, 2, \ldots, n, \end{cases} \tag{29}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF.

Definition 7. *We say that the MIS (3) is controllable on τ, if for each $\nu_0, \xi_0, \nu_A, \xi_A \in \tau$ where $\vartheta_{i+1} = A$ is any arbitrary point, there is an rd-continuous function $\ell \in L^2(I, \mathbb{R})$ so that the relevant solution of (3) fulfills $\nu(0) = \nu_0$, $\xi(0) = \xi_0$ and $\ell(A) = \ell_A$.*

For convenience, we choose to use:

$$\begin{aligned}
J_4 &= u_3\big(u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_4}T_{Z_2}\big)(s_l - s_0) + Y^*_\Theta\big) \\
&\quad \times(\vartheta_l - \vartheta_0)\Big(1 + T_H T^\rho_{\Re}(\vartheta_l - \vartheta_0)\Big), \\
J_5 &= u_3\big(u_2T_Z + u_2T_{\Xi_1} + u_2T_{\Xi_2} + u_2\big(T_{\Xi_3}T_{Z_1} + T_{\Xi_4}T_{Z_2}\big)(s_l - s_0)\big)(\vartheta_l - \vartheta_0), \\
J_6 &= \Big(1 + T_H T^\rho_{\Re}(\vartheta_l - \vartheta_0)\Big) \times \left(\sum_{j=1}^{i}(T_O + T_D)\right. \\
&\quad \left. + \big(u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_4}T_{Z_2}\big)(s_l - s_0)\big)(\vartheta_l - \vartheta_0)\right), \\
J_7 &= \Big(1 + T_H T^\rho_{\Re}(\vartheta - \vartheta_l)\Big) \times \left(\frac{1}{\Gamma(\rho)}u_3T_\varphi(s_i - \vartheta_i)\right. \\
&\quad \left. + \big(u_2T_Z + u_2T_{\Xi_1} + u_2T_{\Xi_2} + u_2\big(T_{\Xi_3}T_{Z_1} + T_{\Xi_4}T_{Z_2}\big)(s_l - s_0)\big)(\vartheta_l - \vartheta_0)\right),
\end{aligned}$$

$$\begin{aligned}
E_7 &= \Big(1 + T_H T^\rho_{\Re}\Big)\left(\sum_{j=1}^{i}(T_O + T_D)\delta + u_1\right) + T_H T^\rho_H\|\nu_A\|, \\
E_8 &= \Big(1 + T_H T^\rho_{\Re}\Big)\left(\sum_{j=1}^{i}(T_O + T_D)\delta + u_4\right) + T_H T^\rho_H\|\xi_A\|, \\
E_9 &= \Big(1 + T_H T^\rho_{\Re}\Big)\left(\frac{1}{\Gamma(\rho)}u_3T_\varphi\delta^{''}(s_i - \vartheta_i) + u_1 + Y^*_\Theta\right) + T_H T^\rho_H\|\nu_A\|, \\
E_{10} &= \Big(1 + T_H T^\rho_{\Re}\Big)\left(\frac{1}{\Gamma(\rho)}u_3T_\varphi\delta^{''}(s_i - \vartheta_i) + u_4 + Y^*_\Theta\right) + T_H T^\rho_H\|\xi_A\|.
\end{aligned}$$

Lemma 3. *Assume that the hypotheses* (1), $(3)-(5)$ *are true and* $\nu_A, \xi_A \in \tau$ *where* $\vartheta_{i+1} = A$ *is any chosen point. Then the pair* $(\nu, \xi) \in \tau \times \tau$*, which is defined in (28) and (29) is a solution to the MIS (3) with the control function*

$$\ell(\vartheta) = \begin{cases}
({}^\rho\Re^A_{\vartheta_0})^{-1}\Bigg[\nu_T - \Lambda_\rho(B\vartheta^\rho)\nu_0 - \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)Z(\vartheta, \nu(\vartheta), \xi(\vartheta))\Delta s \\
\quad - \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\
\quad \times\Theta\left(s, \nu(s), \xi(s), \int_{s_0}^{s_l} Z_1(s, p, \nu(p), \xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s, p, \nu(p), \xi(p))\Delta p\right)\Bigg], \ \vartheta \in (\vartheta_0, \vartheta_1], \\
({}^\rho\Re^A_{\vartheta_0})^{-1}\Bigg[\nu_T - \Lambda_\rho(B\vartheta^\rho)\nu_0 - \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)Z(\vartheta, \nu(\vartheta), \xi(\vartheta))\Delta s \\
\quad - \int_{\vartheta_0}^{\vartheta_l}(\vartheta - s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta - s)^\rho\big)\big)\Delta s \\
\quad \times\Theta\left(s, \nu(s), \xi(s), \int_{s_0}^{s_l} Z_1(s, p, \nu(p), \xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s, p, \nu(p), \xi(p))\Delta p\right), \\
-\sum_{t=1}^{i}\big(O_t\big(\nu(\vartheta_t^-), \xi(\vartheta_t^-)\big) - D_t\big(\vartheta_t^-, \nu(\vartheta_t^-), \xi(\vartheta_t^-)\big)\big)\Bigg], \ \vartheta \in (\vartheta_i, \vartheta_{i+1}], \ i = 1, 2, \ldots, n,
\end{cases} \tag{30}$$

or

$$\ell(\vartheta)=\begin{cases}({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}\Big[\xi_T-\Lambda_\rho(B\vartheta^\rho)\xi_0-\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad-\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right)\Big],\ \vartheta\in(\vartheta_0,\vartheta_1],\\ ({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}\Big[\xi_T-\Lambda_\rho(B\vartheta^\rho)\xi_0-\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad-\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\\ -\sum_{t=1}^{i}\big(O_t\big(\xi(\vartheta_t^-),\nu(\vartheta_t^-)\big)+D_t\big(\vartheta_t^-,\xi(\vartheta_t^-),\nu(\vartheta_t^-)\big)\big)\Big],\ \vartheta\in(\vartheta_i,\vartheta_{i+1}],\ i=1,2,\dots,n,\end{cases}\tag{31}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF, $\nu(A)=\nu_A$ and $\xi(A)=\xi_A$. Further, the control function $\ell(\vartheta)$ estimated by $\|\ell(\vartheta)\|\le\Pi_\ell$ or $\|\ell(\vartheta)\|\le\widehat{\Pi}_\ell$, where Π_ℓ and $\widehat{\Pi}_\ell$ are defined by

$$\begin{aligned}\Pi_\ell &= \Pi^\rho_\Re\Bigg[\|\nu_A\|+\big\|\Lambda_\rho(B\vartheta^\rho)\nu_0\big\|+\left(\sum_{j=1}^{i}(T_O+T_D)\right)\|\nu+\xi\|_\infty\\ &\quad+\big(u_3u_2\big(T_Z+T_{\Theta_1}+T_{\Theta_2}+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)\big)\|\nu+\xi\|_\infty+Y^*_\Theta\big)(\vartheta_l-\vartheta_0)\Bigg],\end{aligned}$$

and

$$\begin{aligned}\widehat{\Pi}_\ell &= \Pi^\rho_\Re\Bigg[\|\xi_A\|+\big\|\Lambda_\rho(B\vartheta^\rho)\xi_0\big\|+\left(\sum_{j=1}^{i}(T_O+T_D)\right)\|\nu+\xi\|_\infty\\ &\quad+\big(u_3u_2\big(T_Z+T_{\Theta_1}+T_{\Theta_2}+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)\big)\|\nu+\xi\|_\infty+Y^*_\Theta\big)(\vartheta_l-\vartheta_0)\Bigg],\end{aligned}$$

for $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\dots,n$.

Proof. For $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\dots,n$, let $(\nu,\xi)\in\tau\times\tau$, which is defined in (28) and (29)) is a solution to the MIS (3), then for $\vartheta\in A$, and using (30), we get

$$\begin{aligned}\nu(A) &= \Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s\\ &\quad+\int_{\vartheta_0}^{\vartheta_l}(A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)\\ &\quad\times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right)\Delta s\\ &\quad+\sum_{t=1}^{i}\big(O_t\big(\xi(\vartheta_t^-),\nu(\vartheta_t^-)\big)+D_t\big(\vartheta_t^-,\xi(\vartheta_t^-),\nu(\vartheta_t^-)\big)\big)\\ &\quad+\int_{\vartheta_0}^{\vartheta_l}(A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)H\ell(A)\Delta s\end{aligned}$$

$$
\begin{aligned}
= \quad & \Lambda_\rho(BA^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta\, s \\
& + \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big) \\
& \times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l} Z_1(s,p,\xi(p),\nu(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\xi(p),\nu(p))\Delta p\right)\Delta\, s \\
& + \sum_{t=1}^{i}\big(O_t\big(\xi(A_t^-),\nu(A_t^-)\big) + D_t\big(A_t^-,\xi(A_t^-),\nu(A_t^-)\big)\big) + \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)H \\
& \quad \times({}^\rho\Re^A_{\vartheta_0})^{-1}\Bigg[\nu_T - \Lambda_\rho(BA^\rho)\nu_0 - \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s \\
& \quad - \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big) \\
& \quad \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Delta\, s \\
& \quad - \sum_{t=1}^{i}\big(O_t\big(\nu(A_t^-),\xi(A_t^-)\big) + D_t\big(A_t^-,\nu(A_t^-),\xi(A_t^-)\big)\big)\Bigg]\Delta s,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\nu(A) \quad = \quad & \Lambda_\rho(BA^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta\, s \\
& + \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big) \\
& \times\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l} Z_1(s,p,\xi(p),\nu(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\xi(p),\nu(p))\Delta p\right)\Delta\, s \\
& + \sum_{t=1}^{i}\big(O_t\big(\xi(A_t^-),\nu(A_t^-)\big) - D_t\big(A_t^-,\xi(A_t^-),\nu(A_t^-)\big)\big) + \\
& \times({}^\rho\Re^A_{\vartheta_0})({}^\rho\Re^A_{\vartheta_0})^{-1}\Bigg[\nu_T - \Lambda_\rho(BA^\rho)\nu_0 - \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big)Z(s,\nu(s),\xi(s))\Delta s \\
& - \int_{\vartheta_0}^{\vartheta_l} (A-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((A-s)^\rho\big)\big) \\
& \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p, \int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Delta\, s \\
& - \sum_{t=1}^{i}\big(O_t\big(\nu(A_t^-),\xi(A_t^-)\big) + D_t\big(A_t^-,\nu(A_t^-),\xi(A_t^-)\big)\big)\Bigg]\Delta\, s \\
= \quad & \nu_A.
\end{aligned}
$$

In the same way, by using (31), we can prove that $\xi(A) = \xi_A$. Also, for $\vartheta \in (\vartheta_i, \vartheta_{i+1}]$, $i = 1, 2, \ldots, n$, the estimation

$$\begin{aligned}\|\ell(\vartheta)\| &= \Pi^{\rho}_{\Re}\Big[\|\nu_A\| - \big\|\Lambda_\rho(B\vartheta^\rho)\nu_0\big\| \\ &\quad - \int_{\vartheta_0}^{\vartheta_l}\Big\|(\vartheta-s)^{\rho-1}\Big\|\big\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\big\|\|Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\|\Delta\, s \\ &\quad - \int_{\vartheta_0}^{\vartheta_l}\Big\|(\vartheta-s)^{\rho-1}\Big\|\big\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\big\| \\ &\quad \times\left\|\Theta\left(s,\xi(s),\nu(s),\int_{s_0}^{s_l} Z_1(s,p,\xi(p),\nu(p))\Delta p,\int_{s_0}^{s_l} Z_2(s,p,\xi(p),\nu(p))\Delta p\right)\right\|\Delta\, s \\ &\quad - \sum_{t=1}^{i}\Big(\big\|O_t\big(\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big)\big\| + \big\|D_t\big(\vartheta_t^-,\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big)\big\|\Big)\Big],\end{aligned}$$

leads to

$$\begin{aligned}\|\ell(\vartheta)\| &= \Pi^{\rho}_{\Re}\Bigg[\|\nu_A\| + \big\|\Lambda_\rho(B\vartheta^\rho)\nu_0\big\| + \left(\sum_{j=1}^{i}(T_O+T_D)\right)\|\nu+\xi\|_\infty \\ &\quad +\big(u_3u_2\big(T_Z+T_{\Theta_1}+T_{\Theta_2}+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)\big)\|\nu+\xi\|_\infty + \Upsilon^*_\Theta\big)(\vartheta_l-\vartheta_0)\Bigg] = \Pi_\ell\end{aligned}$$

Similarly,

$$\begin{aligned}\|\ell(\vartheta)\| &= \Pi^{\rho}_{\Re}\Bigg[\|\xi_A\| + \big\|\Lambda_\rho(B\vartheta^\rho)\xi_0\big\| + \left(\sum_{j=1}^{i}(T_O+T_D)\right)\|\nu+\xi\|_\infty \\ &\quad +\big(u_3u_2\big(T_Z+T_{\Theta_1}+T_{\Theta_2}+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)\big)\|\nu+\xi\|_\infty + \Upsilon^*_\Theta\big)(\vartheta_l-\vartheta_0)\Bigg] = \widehat{\Pi}_\ell.\end{aligned}$$

This finishes the proof. □

Definition 8. *The MIS (4) has a solution $(\nu, \xi) \in \tau \times \tau$ if $\nu(0) = \nu_0$, $\xi(0) = \xi_0$ and the pair (ξ, ν) is a solution to the following integral equations:*

$$\nu(\vartheta) = \begin{cases} \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\ + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)H\ell(\vartheta)\Delta s,\ \vartheta \in (\vartheta_i, s_{i+1}] \cap \tau,\ i = 1,2,\ldots,n, \\ \Lambda_\rho(B\vartheta^\rho)\nu_0 + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s \\ \times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l} Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l} Z_2(s,p,\nu(p),\xi(p))\Delta p\right), \\ \quad + \int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)H\ell(\vartheta)\Delta s \\ + \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s,\ \vartheta \in (s_i, \vartheta_{i+1}] \cap \tau,\ i = 1,2,\ldots,n, \end{cases} \tag{32}$$

and

$$\xi(\vartheta)=\begin{cases}\Lambda_\rho(B\vartheta^\rho)\xi_0+\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad+\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\xi(s),\nu(s),\int\limits_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int\limits_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\\ +\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)H\ell(\vartheta)\Delta s,\ \vartheta\in(\vartheta_i,s_{i+1}]\cap\tau,\ i=1,2,\ldots,n,\\ \Lambda_\rho(B\vartheta^\rho)\xi_0+\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad+\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\xi(s),\nu(s),\int\limits_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int\limits_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\right),\\ \quad+\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)H\ell(\vartheta)\Delta s\\ +\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\ldots,n,\end{cases}\tag{33}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF.

Definition 9. *We say that the MIS (4) is controllable on τ, if for each $\nu_0,\xi_0,\nu_A,\xi_A\in\tau$ where $\vartheta_{i+1}=A$, there is an rd-continuous function $\ell\in L^2(I,\mathbb{R})$ so that the corresponding solution of (4) justifies $\nu(0)=\nu_0$, $\xi(0)=\xi_0$ and $\ell(A)=\ell_A$.*

Lemma 4. *Suppose that the hypotheses (1), (3), (4), (5) hold and $\nu_A,\xi_A\in\tau$ where $\vartheta_{i+1}=A$ is any chosen point. Then the pair $(\nu,\xi)\in\tau\times\tau$, which is defined in (32) and (33) is a solution to the MIS (3) on $(s_i,\vartheta_{i+1}]\cap\tau$, $i=1,2,\ldots,n$, with the control function*

$$\ell(\vartheta)=\begin{cases}({}^\rho\Re_{\vartheta_0}^A)^{-1}\Bigg[\nu_T-\Lambda_\rho(B\vartheta^\rho)\nu_0-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\nu(s),\xi(s),\int\limits_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int\limits_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg],\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\\ ({}^\rho\Re_{\vartheta_0}^A)^{-1}\Bigg[\nu_T-\Lambda_\rho(B\vartheta^\rho)\nu_0-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\nu(s),\xi(s),\int\limits_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int\limits_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\\ -\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\nu(s),\xi(s))\Delta s\Bigg],\ \vartheta\in(s_i,\vartheta_{i+1}],\ i=1,2,\ldots,n,\end{cases}$$

or

$$\ell(\vartheta)=\begin{cases}({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}\Bigg[\xi_T-\Lambda_\rho(B\vartheta^\rho)\xi_0-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\Bigg(s,\xi(s),\nu(s),\int\limits_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int\limits_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\Bigg)\Bigg],\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\\ ({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}\Bigg[\xi_T-\Lambda_\rho(B\vartheta^\rho)\xi_0-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\xi(\vartheta),\nu(\vartheta))\Delta s\\ \quad-\int\limits_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\Bigg(s,\xi(s),\nu(s),\int\limits_{s_0}^{s_l}Z_1(s,p,\xi(p),\nu(p))\Delta p,\int\limits_{s_0}^{s_l}Z_2(s,p,\xi(p),\nu(p))\Delta p\Bigg),\\ -\frac{1}{\Gamma(\rho)}\int\limits_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\Bigg],\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\dots,n,,\end{cases}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF, $\nu(A)=\nu_A$, and $\xi(A)=\xi_A$. Further, the control function $\ell(\vartheta)$ estimated by $\|\ell(\vartheta)\|\le\beth_\ell$ or $\|\ell(\vartheta)\|\le\widetilde{\beth}_\ell$, where $\beth_\ell$ and $\widetilde{\beth}_\ell$ are defined by

$$\begin{aligned}\beth_\ell&=\Pi^\rho_\Re\Bigg[\|\nu_A\|+\big\|\Lambda_\rho(B\vartheta^\rho)\nu_0\big\|+\frac{1}{\Gamma(\rho)}(s_i-\vartheta_i)u_1T_\varphi\|\nu+\xi\|_\infty\\&+\big(u_3u_2\big(T_Z+T_{\Theta_1}+T_{\Theta_2}+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)\big)\|\nu+\xi\|_\infty+Y^*_\Theta\big)(\vartheta_l-\vartheta_0)\Big],\end{aligned}$$

and

$$\begin{aligned}\widetilde{\beth}_\ell&=\Pi^\rho_\Re\Bigg[\|\xi_A\|+\big\|\Lambda_\rho(B\vartheta^\rho)\xi_0\big\|+\frac{1}{\Gamma(\rho)}(s_i-\vartheta_i)u_4T_\varphi\|\nu+\xi\|_\infty\\&+\big(u_3u_2\big(T_Z+T_{\Theta_1}+T_{\Theta_2}+\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)\big)\|\nu+\xi\|_\infty+Y^*_\Theta\big)(\vartheta_l-\vartheta_0)\Big],\end{aligned}$$

for $\vartheta\in(s_i,\vartheta_{i+1}]\cap\tau$, $i=1,2,\dots,n$.

Proof. Similar to Lemma 3, the proof is given. □

Theorem 9. *If the assertions* (1), (3), (4), (5) *are true, then the MIS (3) is controllable on τ provided that*

$$\max\{J_4,J_5,J_6\}<1. \tag{34}$$

Proof. Assume that $\mho''\subseteq Q$ where $\mho''=\{(K_1,K_2,K_3,K_4)\in Q:\|K_1,K_2,K_3,K_4\|\le\delta_2''\}$, where $\delta_2''=\max\{\delta'',\delta_1''\}$ and $\delta'',\delta_1''\in(0,1)$, and also $\delta_2''\ge E_7$. Define an operator $\Im''_\rho:\mho''\times\mho''\to\mho''$ by

$$\Im''_{\rho}(\nu,\xi)(\vartheta)=\begin{cases}\Lambda_{\rho}(B\vartheta^{\rho})\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)\Delta s\\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)H\ell(\vartheta)\Delta s,\ \vartheta\in(\vartheta_0,\vartheta_1],\\ \Lambda_{\rho}(B\vartheta^{\rho})\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)\Delta s\\ \times\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)H\ell(\vartheta)\Delta s\\ +\sum_{t=1}^{i}\big(O_t\big(\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big)+D_t\big(\vartheta_t^-,\nu(\vartheta_t^-),\xi(\vartheta_t^-)\big)\big),\ \vartheta\in(\vartheta_i,\vartheta_{i+1}],\ i=1,2,\dots,n,\end{cases}$$

where $\Lambda_{\rho}(B\vartheta^{\rho})$ is the matrix form of the MLF. Now, we show that $\Im''_{\rho}:\mho''\times\mho''\to\mho''$ is a self-mapping. For $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\dots,n$, we have

$$\begin{aligned}\left\|\Im''_{\rho}(\nu,\xi)(\vartheta)\right\| \leq\ & \sum_{j=1}^{i}\left\|O_j\left(\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)\right\|+\sum_{j=1}^{i}\left\|D_j\left(\vartheta_j^-,\nu(\vartheta_j^-),\xi(\vartheta_j^-)\right)\right\|+\|\Lambda_{\rho}(B\vartheta^{\rho})\nu_0\|\\ &+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)\right\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\\ &+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg)\Bigg\|\Delta s\\ &+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^{\rho}\big)\big)\right\|\|H\|\|\ell(s)\|\Delta s\\ \leq\ & \sum_{j=1}^{i}(T_O+T_D)\delta''+u_1+\delta''u_3\big(u_2T_Z+u_2T_{\Theta_1}+u_2T_{\Theta_2}\\ &+u_2\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)+Y^*_{\Theta}\big)(\vartheta_l-\vartheta_0)\\ &+T_HT^{\rho}_{\Re}(\vartheta_l-\vartheta_0)\left[\sum_{j=1}^{i}(T_O+T_D)\delta''+\|\nu\|_A+u_1+\delta''u_3\big(u_2T_Z+u_2T_{\Theta_1}+u_2T_{\Theta_2}\right.\\ &\left.+u_2\big(T_{\Theta_3}T_{Z_1}+T_{\Theta_4}T_{Z_2}\big)(s_l-s_0)+Y^*_{\Theta}\big)(\vartheta_l-\vartheta_0)\right]\end{aligned}$$

which implies that

$$\left\|\Im''_{\rho}(\nu,\xi)(\vartheta)\right\|\leq E_7+\delta''J_6\leq\delta''+\delta''J_6=\delta''_1$$

Similarly, if $\delta''_2\geq E_8$, one can show that

$$\left\|\Im_{\rho}(\xi,\nu)(\vartheta)\right\|\leq E_8+\delta''J_1\leq\delta''+\delta''J_1=\delta''_1,$$

Hence

$$\left\|\Im''_{\rho}(\nu,\xi)(\vartheta)\right\| \leq \delta''_2 \text{ and } \left\|\Im''_{\rho}(\xi,\nu)(\vartheta)\right\| \leq \delta''_2. \tag{35}$$

It follows from (35) that $\Im''(\mho''\times\mho'') \subseteq \mho''$. Also, for $\vartheta \in (\vartheta_i, \vartheta_{i+1}]$, $i = 1,2,\ldots,n$, with $\nu_0 = \widetilde{\nu}_0$ and $\xi_0 = \widetilde{\xi}_0$, one has

$$\begin{aligned}
&\left\|\Im''_{\rho}(\nu,\xi)(\vartheta) - \Im''_{\rho}(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\
\leq\ & \sum_{j=1}^{i}\left\|O_j\Big(\nu(\vartheta_j^-),\xi(\vartheta_j^-)\Big) - O_j\Big(\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\Big)\right\| + \sum_{j=1}^{i}\left\|D_j\Big(\vartheta_j^-,\nu(\vartheta_j^-),\xi(\vartheta_j^-)\Big) - D_j\Big(\vartheta_j^-,\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\Big)\right\| \\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}(B((\vartheta-s)^{\rho}))\|\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\
&+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg) \\
&-\left(Z\Big(\vartheta,\widetilde{\nu}(\vartheta),\widetilde{\xi}(\vartheta)\Big)+\Theta\left(s,\widetilde{\nu}(s),\widetilde{\xi}(s),\int_{s_0}^{s_l}Z_1\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p,\int_{s_0}^{s_l}Z_2\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p\right)\right)\Bigg\|\Delta s \\
&-\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}(B((\vartheta-s)^{\rho}))\|\|H\|\left\|({}^{\rho}\Re^{A}_{\vartheta_0})^{-1}\right\|(\vartheta-\vartheta_l) \\
&\times\Bigg[\sum_{j=1}^{i}\left\|O_j\Big(\nu(\vartheta_j^-),\xi(\vartheta_j^-)\Big) - O_j\Big(\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\Big)\right\| + \sum_{j=1}^{i}\left\|D_j\Big(\vartheta_j^-,\nu(\vartheta_j^-),\xi(\vartheta_j^-)\Big) - D_j\Big(\vartheta_j^-,\widetilde{\nu}(\vartheta_j^-),\widetilde{\xi}(\vartheta_j^-)\Big)\right\| \\
&+\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\|\Lambda_{\rho,\rho}(B((\vartheta-s)^{\rho}))\|\Delta s\|(Z(\vartheta,\nu(\vartheta),\xi(\vartheta)) \\
&+\Theta\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\Bigg) \\
&-\left(Z\Big(\vartheta,\widetilde{\nu}(\vartheta),\widetilde{\xi}(\vartheta)\Big)+\Theta\left(s,\widetilde{\nu}(s),\widetilde{\xi}(s),\int_{s_0}^{s_l}Z_1\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p,\int_{s_0}^{s_l}Z_2\Big(s,p,\widetilde{\nu}(p),\widetilde{\xi}(p)\Big)\Delta p\right)\right)\Bigg\|\Bigg]\Delta s,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left\|\Im''_{\rho}(\nu,\xi)(\vartheta) - \Im''_{\rho}(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\
\leq\ & \left(\sum_{j=1}^{i}(T_O+T_D) + \left[u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_3}T_{Z_2}\big)(s_l-s_0)\right](\vartheta_l-\vartheta_0)\right) \\
&\times\left(\|\nu-\widetilde{\nu}\|_\infty + \left\|\xi-\widetilde{\xi}\right\|_\infty\right) \\
&+T_HT^{\rho}_{\Re}(\vartheta_l-\vartheta_0)\left[\sum_{j=1}^{i}(T_O+T_D) + \left[u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_3}T_{Z_2}\big)(s_l-s_0)\right](\vartheta_l-\vartheta_0)\right] \\
&\times\left(\|\nu-\widetilde{\nu}\|_\infty + \left\|\xi-\widetilde{\xi}\right\|_\infty\right) \\
\leq\ & (1+T_HT^{\rho}_{\Re}(\vartheta_l-\vartheta_0)) \\
&\times\left[\sum_{j=1}^{i}(T_O+T_D) + \left[u_2T_Z + u_2T_{\Theta_1} + u_2T_{\Theta_2} + u_2\big(T_{\Theta_3}T_{Z_1} + T_{\Theta_3}T_{Z_2}\big)(s_l-s_0)\right](\vartheta_l-\vartheta_0)\right] \\
&\times\left(\|\nu-\widetilde{\nu}\|_\infty + \left\|\xi-\widetilde{\xi}\right\|_\infty\right).
\end{aligned}$$

Hence

$$\left\|\Im''_\rho(\nu,\xi)(\vartheta)-\Im''_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \leq J_6\Big(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\Big).$$

Analogously, we can write

$$\left\|\Im''_\rho(\nu,\xi)(\vartheta)-\Im''_\rho(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \leq J_6\Big(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\Big).$$

Therefore, the operator $\Im''_\rho$ is strictly contractive. As a result, using the Banach FP theorem, $\Im''_\rho$ has only one FP, that is the MIS (3) has a US. Moreover, we deduce from Lemma 3 that $\nu(\vartheta)$ and $\xi(\vartheta)$ satisfy the conditions $\nu(A)=\nu_A$ and $\xi(A)=\xi_A$. As a result, the controllability exists for the MIS (3). □

Theorem 10. *If the assertions (1), (3), (4) and (5) hold, then the MIS (4) is controllable on τ provided that*

$$\max\{J_i\}<1,\ i=5,7.$$

Proof. Assume that $\mho''\subseteq Q$ where $\mho''=\{(K_1,K_2,K_3,K_4)\in Q:\|K_1,K_2,K_3,K_4\|\leq\delta''_2\}$, where $\delta''_2=\max\{\delta'',\delta''_1\}$ and $\delta'',\delta''_1\in(0,1)$, and also $\delta''_2\geq E_9$. Define an operator $\Im^{**}_\rho:\mho''\times\mho''\to\mho''$ by

$$\Im^{**}_\rho(\nu,\xi)(\vartheta)=\begin{cases}\Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\\ +\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)H\ell(\vartheta)\Delta s,\ \vartheta\in(\vartheta_i,s_i]\cap\tau,\ i=1,2,\dots,n,\\ \Lambda_\rho(B\vartheta^\rho)\nu_0+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\Delta s\\ \quad+\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)\Delta s\\ \times\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right),\\ \quad+\frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}(\vartheta-s)^{\rho-1}\varphi_i(s,\xi(s),\nu(s))\Delta s\\ +\int_{\vartheta_0}^{\vartheta_l}(\vartheta-s)^{\rho-1}\Lambda_{\rho,\rho}\big(B\big((\vartheta-s)^\rho\big)\big)H\ell(s)\Delta s,\ \vartheta\in(s_i,\vartheta_{i+1}]\cap\tau,\ i=1,2,\dots,n,\end{cases}$$

where $\Lambda_\rho(B\vartheta^\rho)$ is the matrix form of the MLF. Now, we illustrate that $\Im^{**}_\rho:\mho''\times\mho''\to\mho''$ is a self-mapping. For $\vartheta\in(s_i,\vartheta_{i+1}]\cap\tau$, $i=1,2,\dots,n$, we get

$$\begin{aligned}\left\|\Im_\rho^{**}(\nu,\xi)(\vartheta)\right\| \leq\; & \frac{1}{\Gamma(\rho)}\int_{\vartheta_i}^{s_i}\left\|(\vartheta-s)^{\rho-1}\right\|\|\varphi_i(s,\nu(s),\xi(s))\|\Delta s+\left\|\Lambda_\rho(B\vartheta^\rho)\nu_0\right\| \\ & +\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^\rho\right)\right)\right\|(\|Z(\vartheta,\nu(\vartheta),\xi(\vartheta))\| \\ & +\left\|\Xi\left(s,\nu(s),\xi(s),\int_{s_0}^{s_l}Z_1(s,p,\nu(p),\xi(p))\Delta p,\int_{s_0}^{s_l}Z_2(s,p,\nu(p),\xi(p))\Delta p\right)\right\|\right)\Delta s \\ & +\int_{\vartheta_0}^{\vartheta_l}\left\|(\vartheta-s)^{\rho-1}\right\|\left\|\Lambda_{\rho,\rho}\left(B\left((\vartheta-s)^\rho\right)\right)\right\|\|H\|\|\ell(s)\|\Delta s \\ \leq\; & \frac{1}{\Gamma(\rho)}u_3T_\varphi\delta(s_i-\vartheta_i)+u_1+u_3\big(u_2T_Z\delta''+u_2T_{\Xi_1}\delta''+u_2T_{\Xi_2}\delta'' \\ & +u_2\delta''T_{\Theta_3}T_{Z_1}(s_l-s_0)+T_{\Theta_4}T_{Z_2}\delta''(s_l-s_0)+Y_\Xi^*\big)(\vartheta_l-\vartheta_0) \\ & +T_HT_{\Re}^\rho\Big[\frac{1}{\Gamma(\rho)}u_3T_\varphi\delta(s_i-\vartheta_i)+u_1+u_3\big(u_2T_Z\delta''+u_2T_{\Xi_1}\delta''+u_2T_{\Xi_2}\delta'' \\ & +u_2\delta''T_{\Theta_3}T_{Z_1}(s_l-s_0)+T_{\Theta_4}T_{Z_2}\delta''(s_l-s_0)+Y_\Xi^*\big)(\vartheta_l-\vartheta_0)\Big] \\ \leq\; & E_9+\delta''J_5\leq\delta''+\delta''J_5=\delta_1'',\end{aligned}$$

also, if $E_{10}\leq\delta''$, we have

$$\left\|\Im_\rho^{**}(\xi,\nu,)(\vartheta)\right\|\leq E_{10}+\delta''J_5\leq\delta''+\delta''J_5=\delta_1''.$$

Hence

$$\left\|\Im_\rho^{**}(\nu,\xi)(\vartheta)\right\|\leq\delta_2''\text{ and }\left\|\Im_\rho^{**}(\xi,\nu,)(\vartheta)\right\|\leq\delta_2''. \tag{36}$$

It follows from (36) that $\Im^{**}(\mho''\times\mho'')\subseteq\mho''$. Also, for $\vartheta\in(\vartheta_i,\vartheta_{i+1}]$, $i=1,2,\ldots,n$, with $\nu_0=\widetilde{\nu}_0$ and $\xi_0=\widetilde{\xi}_0$, one has

$$\begin{aligned}&\left\|\Im_\rho^{**}(\nu,\xi)(\vartheta)-\Im_\rho^{**}(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\| \\ \leq\; & \left(\frac{1}{\Gamma(\rho)}u_3T_\varphi(s_i-\vartheta_i)+u_3\big[u_2T_Z+u_2T_{\Xi_1}+u_2T_{\Xi_2}+u_2\big(T_{\Xi_3}T_{Z_1}+T_{\Xi_3}T_{Z_2}\big)(s_l-s_0)\big](\vartheta_l-\vartheta_0)\right) \\ & \times\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right) \\ & +T_HT_{\Re}^\rho(\vartheta-\vartheta_l)\left[\frac{1}{\Gamma(\rho)}u_3T_\varphi(s_i-\vartheta_i)+u_3\left[\begin{array}{c}u_2T_Z+u_2T_{\Xi_1}+u_2T_{\Xi_2}\\+u_2\big(T_{\Xi_3}T_{Z_1}+T_{\Xi_3}T_{Z_2}\big)(s_l-s_0)\end{array}\right](\vartheta_l-\vartheta_0)\right] \\ & \times\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right) \\ \leq\; & (1+T_HT_{\Re}^\rho(\vartheta-\vartheta_l)) \\ & \times\left[\frac{1}{\Gamma(\rho)}u_3T_\varphi(s_i-\vartheta_i)+u_3\big[u_2T_Z+u_2T_{\Xi_1}+u_2T_{\Xi_2}+u_2\big(T_{\Xi_3}T_{Z_1}+T_{\Xi_3}T_{Z_2}\big)(s_l-s_0)\big](\vartheta_l-\vartheta_0)\right] \\ & \times\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right).\end{aligned}$$

Hence

$$\left\|\Im_\rho^{**}(\nu,\xi)(\vartheta)-\Im_\rho^{**}(\widetilde{\nu},\widetilde{\xi})(\vartheta)\right\|\leq J_7\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right).$$

Similarly, we can obtain

$$\left\|\Im^{**}_{\rho}(\xi,\nu)(\vartheta)-\Im^{**}_{\rho}(\widetilde{\xi},\widetilde{\nu})(\vartheta)\right\|\leq J_7\left(\|\nu-\widetilde{\nu}\|_\infty+\left\|\xi-\widetilde{\xi}\right\|_\infty\right).$$

Therefore, the operator $\Im^{**}_{\rho}$ is strictly contractive. Thus, from the Banach FP theorem, $\Im^{**}_{\rho}$ has a unique FP, which is a US to the MIS (4). In addition, we deduce from Lemma 4 that $\nu(\vartheta)$ and $\xi(\vartheta)$ satisfy the stipulations $\nu(A)=\nu_A$ and $\xi(A)=\xi_A$. This proves that, the MIS (4) is controllable. □

6. Supportive Example

This part is devoted to analyzing the findings from the earlier steps.

Example 1. *Consider the MIS below:*

$$\begin{cases} {}^{c,\tau}D^{\rho}\nu(\vartheta)=\frac{3}{\vartheta-1}\nu(\vartheta)+\frac{3}{\vartheta-1.3}\phi(\vartheta)+e_u(\vartheta,\psi(\nu(\vartheta)),\psi(\xi(\vartheta)))+\int_0^{\vartheta}\Lambda_{\rho,\gamma}(\nu,\xi)\Delta s+\phi(\vartheta),\\ {}^{c,\tau}D^{\rho}\xi(\vartheta)=\frac{3}{\vartheta-1}\xi(\vartheta)+\frac{3}{\vartheta-1.3}\phi(\vartheta)+e_u(\vartheta,\psi(\xi(\vartheta)),\psi(\nu(\vartheta)))+\int_0^{\vartheta}\Lambda_{\rho,\gamma}(\xi,\nu)\Delta s+\phi(\vartheta),\\ \quad\nu(0)=1,\ \xi(0)=1,\ \vartheta=[0,4]_\tau\backslash\{1,1.3\},\\ \nu(\vartheta_r)=O_r(\nu(\vartheta_r),\xi(\vartheta_r))+D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-),\phi(\vartheta_r^-)),\ r=1,2,\\ \xi(\vartheta_r)=O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-))+D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-),\phi(\vartheta_r^-)),\ r=1,2. \end{cases} \tag{37}$$

It is equivalent to

$$\begin{cases} \left|{}^{c,\tau}D^{\rho}\nu(\vartheta)-\frac{3}{\vartheta-1}\nu(\vartheta)-\frac{3}{\vartheta-1.3}\phi(\vartheta)-e_u(\vartheta,\psi(\nu(\vartheta)),\psi(\xi(\vartheta)))-\int_0^{\vartheta}\Lambda_{\rho,\gamma}(\nu,\xi)\Delta s+\phi(\vartheta)\right|\leq 1,\\ \left|{}^{c,\tau}D^{\rho}\xi(\vartheta)-\frac{3}{\vartheta-1}\xi(\vartheta)+\frac{3}{\vartheta-1.3}\phi(\vartheta)+e_u(\vartheta,\psi(\xi(\vartheta)),\psi(\nu(\vartheta)))+\int_0^{\vartheta}\Lambda_{\rho,\gamma}(\xi,\nu)\Delta s+\phi(\vartheta)\right|\leq 1,\\ \quad\vartheta=[0,4]_\tau\backslash\{1,1.3\},\\ |\nu(\vartheta_r)-O_r(\nu(\vartheta_r),\xi(\vartheta_r))-D_r(\vartheta_r^-,\nu(\vartheta_r^-),\xi(\vartheta_r^-),\phi(\vartheta_r^-))|\leq 1,\ r=1,2,\\ |\xi(\vartheta_r)-O_r(\xi(\vartheta_r^-),\nu(\vartheta_r^-))-D_r(\vartheta_r^-,\xi(\vartheta_r^-),\nu(\vartheta_r^-),\phi(\vartheta_r^-))|\leq 1,\ r=1,2. \end{cases} \tag{38}$$

We put $\tau^r=[0,4]_\tau\backslash\{1,1.3\}$, $\vartheta_1=1$, $\vartheta_2=1.3$, $K_1(\vartheta)=\frac{3}{\vartheta-1}$, $K_2(\vartheta)=\frac{3}{\vartheta-1.3}$, $\Lambda_{\rho,\gamma}(\nu,\xi)=\sum_{r=0}^{\infty}\frac{\nu^r\xi^r}{\Gamma(r\rho+\gamma)}$, *for* $\rho,\gamma>0$. *Also, we take*

$$Z\big(\vartheta,\nu(\vartheta),\xi(\vartheta),S_{\nu,\xi}(\vartheta),\phi(\vartheta)\big)=e_u(\vartheta,\psi(\nu(\vartheta)),\psi(\xi(\vartheta)))+\int_0^t\Lambda_{\rho,\gamma}(\nu,\xi)\Delta s+\phi(\vartheta),$$

where $\phi(\vartheta)$ *is a control map for* $\vartheta\in\tau^r$ *and* $S_{\nu,\xi}(\vartheta)=\int_0^t\Lambda_{\rho,\gamma}(\nu,\xi)\Delta s$ *and put* $\varepsilon=1$. *Assume that* $\widehat{\nu},\widehat{\xi}\in Q^1([0,3],\mathbb{R})$ *satisfy (38), then there is* $\varphi\in Q^1([0,3]_{\tau^r},\mathbb{R})$ *with* $\varphi_0\in\mathbb{R}$ *so that* $|\varphi(\vartheta,\nu,\xi)|\leq 1$, *for all* $\vartheta\in\tau^r$ *and* $|\varphi_0|\leq 1$. *So, (38) can be written as*

$$\begin{cases} {}^{c,\tau}D^{\rho}\widehat{\nu}(\vartheta)=\frac{3}{\vartheta-1}\widehat{\nu}(\vartheta)+\frac{3}{\vartheta-1.3}\phi(\vartheta)+e_u\left(\vartheta,\psi(\widehat{\nu}(\vartheta)),\psi(\widehat{\xi}(\vartheta))\right)+\int_0^{\vartheta}\Lambda_{\rho,\gamma}\left(\widehat{\nu},\widehat{\xi}\right)\Delta s+\phi(\vartheta)+\varphi(\vartheta,\widehat{\nu},\widehat{\xi}),\ \vartheta\in\tau^r,\\ {}^{c,\tau}D^{\rho}\widehat{\xi}(\vartheta)=\frac{3}{\vartheta-1}\widehat{\xi}(\vartheta)+\frac{3}{\vartheta-1.3}\phi(\vartheta)+e_u\left(\vartheta,\psi(\widehat{\xi}(\vartheta)),\psi(\widehat{\nu}(\vartheta))\right)+\int_0^{\vartheta}\Lambda_{\rho,\gamma}\left(\widehat{\xi},\widehat{\nu}\right)\Delta s+\phi(\vartheta)+\varphi(\vartheta,\widehat{\xi},\widehat{\nu}),\ \vartheta\in\tau^r,\\ \widehat{\nu}(\vartheta_r)=O_r\left(\widehat{\nu}(\vartheta_r),\widehat{\xi}(\vartheta_r)\right)+D_r\left(\vartheta_r^-,\widehat{\nu}(\vartheta_r^-),\widehat{\xi}(\vartheta_r^-),\phi(\vartheta_r^-)\right)+\varphi_0\left(\vartheta,\widehat{\nu},\widehat{\xi}\right),\ r=1,2,\\ \widehat{\xi}(\vartheta_r)=O_r\left(\widehat{\xi}(\vartheta_r^-),\widehat{\nu}(\vartheta_r^-)\right)+D_r\left(\vartheta_r^-,\widehat{\xi}(\vartheta_r^-),\widehat{\nu}(\vartheta_r^-),\phi(\vartheta_r^-)\right)+\varphi_0\left(\vartheta,\widehat{\xi},\widehat{\nu}\right),\ r=1,2. \end{cases}$$

So, (37) has the solution

$$\begin{aligned} \nu(\vartheta)&=O_1(\nu(\vartheta_1),\xi(\vartheta_1))+O_2(\nu(\vartheta_2),\xi(\vartheta_2))+D_1(\vartheta_1^-,\nu(\vartheta_1^-),\xi(\vartheta_1^-),\phi(\vartheta_1^-))+D_2(\vartheta_2^-,\nu(\vartheta_2^-),\xi(\vartheta_2^-),\phi(\vartheta_2^-))\\ &\quad+\int_0^t e_u(\vartheta,\psi(\vartheta))\left[e_u(\vartheta,\psi(\nu(\vartheta)),\psi(\xi(\vartheta)))+\int_0^s\Lambda_{\rho,\gamma}(\nu,\xi)\Delta p+\phi(s)\right]\Delta s,\\ \xi(\vartheta)&=O_1(\xi(\vartheta_1),\nu(\vartheta_1))+O_2(\xi(\vartheta_2),\nu(\vartheta_2))+D_1(\vartheta_1^-,\xi(\vartheta_1^-),\nu(\vartheta_1^-),\phi(\vartheta_1^-))+D_2(\vartheta_2^-,\xi(\vartheta_2^-),\nu(\vartheta_2^-),\phi(\vartheta_2^-))\\ &\quad+\int_0^t e_u(\vartheta,\psi(\vartheta))\left[e_u(\vartheta,\psi(\xi(\vartheta)),\psi(\nu(\vartheta)))+\int_0^s\Lambda_{\rho,\gamma}(\xi,\nu)\Delta p+\phi(s)\right]\Delta s. \end{aligned}$$

Therefore, the MIS (37) has just one solution on $Q^1([0,3]_\tau,\mathbb{R})$, *according to our obtained results, and the system (37) is UH stable on* τ^r.

7. Conclusions and Future Works

Over the past year, impulsive fractional differential equations have advanced quickly. Particularly for those issues that are subject to abrupt changes and discontinuous jumps and cannot be described by integer-order differential equations, it plays a significant role in providing a natural framework for modeling various real processes arising in phenomena discussed in various scientific fields. Numerous mathematicians have recently demonstrated an interest in the qualitative theory of impulsive differential equations. In a connected fractional dynamic system with initial boundary and impulsive conditions on time scales, we have successfully demonstrated its existence, uniqueness, UH stability and controllability criterion. For the existence of at least one solution, the Leray-Schauder alternative type FP theorem was applied. The unique solution was looked into while using the Banach contraction theorem. In addition, an example was given to demonstrate the results drawn from the analysis. For future projects, the main aim of the authors is that these qualitative specifications can be checked and established on some real-world impulsive systems arising in mathematical models of brain.

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Abbreviations

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism
FDEs	Fractional differential equations
UH	Ulam-Hyers
BS	Banach spaces
CD	Caputo derivative
PS	Product space
MLF	Mittag-Leffler function
MIS	Mixed impulsive system
US	Unique solution
FP	Fixed point
CC	Com pletely continuous
rd-continuous	Right dense continuous

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