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Periodic, Singular and Dark Solitons of a Generalized Geophysical KdV Equation by Using the Tanh-Coth Method

Surapol Naowarat [1], Sayed Saifullah [2], Shabir Ahmad [2,*] and Manuel De la Sen [3]

[1] Department of Mathematics, Faculty of Science and Technology, Suratthani Rajabhat University, Surat Thani 84100, Thailand
[2] Department of Mathematics, University of Malakand, Chakdara 18800, Pakistan
[3] Department of Electricity and Electronics, Institute of Research and Development of Processes, Faculty of Science and Technology, Campus of Leioa (Bizkaia), University of the Basque Country, 48940 Leioa, Spain
* Correspondence: shabir.maths@uom.edu.pk

Abstract: KdV equations have a lot of applications of in fluid mechanics. The exact solutions of the KdV equations play a vital role in the wave dynamics of fluids. In this paper, some new exact solutions of a generalized geophysical KdV equation are computed with the aid of tanh-coth method. To implement the tanh-coth procedure, we first convert the PDEs to ODEs with the help of wave transformation. Then, using a system of algebraic equations, we obtain several soliton solutions. To verify and clearly illustrate the exact solutions, several graphic presentations are developed by giving the parameter values, which are then thoroughly discussed in the relevant components.

Keywords: KdV equation; tanh-coth method; soliton solutions; wave transformation

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1. Introduction

In many areas of engineering, science, and mathematical physics, nonlinear partial differential equations (PDEs) are important and helpful [1,2]. There are a variety of exciting nonlinear models that have emerged in contemporary science, and several mathematicians and scientists have developed significant mathematical techniques for obtaining exact solutions. KdV equations have symmetries in many ways. Adler investigated solutions of the KdV equation governed by a stationary equation for symmetries from the non-commutative subalgebra, namely, for a linear combination of the master-symmetry and the scaling symmetry [3]. Some other studies related to symmetries in PDEs are listed in [4,5]. Nonlinear patterns have recently piqued the interest of a large number of researchers due to their diverse characteristics. The analysis of numerous fields of physical sciences and engineering, such as optical communication, telecoms, electrodynamics, etc., is placing an increasing emphasis on the building of soliton structures [6,7].

The interaction between interior waves and oceanic has often been a popular area of research [8]. The generating method of wave solutions affects the spread of surface and inner gravitational waves. These waves are common in the environment, oceans, and rivers, and understanding their process could help with ocean engineering. The wave and soliton dynamics of oceans are mainly described by KdV equations. The literature on KdV equation is very long, so we describe briefly some recent work on KdV equation. The basic nonlinear KdV equation [9] is given by:

$$\frac{\partial}{\partial t}\mathcal{W} + \frac{\partial^3}{\partial t^3}\mathcal{W} + 6\Psi\frac{\partial}{\partial t}\mathcal{W} = 0, \tag{1}$$

where $\mathcal{W}$ is the function of two variables x and t. The most well-known fifth order KdV equation [9] is expressed as:

$$\frac{\partial}{\partial t}\mathcal{W} + a\mathcal{W}^2\frac{\partial}{\partial x}\mathcal{W} + b\frac{\partial}{\partial x}\mathcal{W}\frac{\partial^2}{\partial x^2}\mathcal{W} + c\mathcal{W}\frac{\partial^3}{\partial x^3} + \frac{\partial^5}{\partial x^5}\mathcal{W} = 0, \tag{2}$$

where a, b and c denote real parameters. Numerous applications of the fifth-order KdV equation are found in quantum physics and quantum optics. For more details and the family of KdV equation, readers are referred to [9]. Geyer et al. generalized Equation (1) by adding the effects of the Earth's rotation on the fluid, the so-called Coriolis effect [10], to study one-layer oceanic flows in the equatorial region. Geyer et al. defined the geophysical KdV as:

$$\frac{\partial}{\partial t}\mathcal{W} - \omega_0\frac{\partial}{\partial x}\mathcal{W} + \frac{3}{2}\mathcal{W}\frac{\partial}{\partial x}\mathcal{W} + \frac{1}{6}\frac{\partial^3}{\partial x^3}\mathcal{W} = 0. \tag{3}$$

Recently, Alquran et al. proposed a generalized geophysical KdV equation [11] as:

$$\mathcal{W}_t + \varrho\mathcal{W}_x + \Omega\mathcal{W}\mathcal{W}_x + \eta\mathcal{W}_{xxx} = 0. \tag{4}$$

In the considered equation, the ϱ is a perturbation parameter, which is used for Coriolis effect, Ω represents non linearity and η is the dispersion factor, and $\mathcal{W}$ denotes the free surface advancement. The generalized geophysical KdV equation occurs in the disciplines of mechanics, acoustics, and medical engineering and depicts the scientific explanation of sound transmission in fluid.

In recent years, scientists have tried improve the existing methods or develop new analytical and numerical methods to solve nonlinear PDEs. Here, we are interested in analytical methods for solutions of nonlinear PDEs. We briefly discuss some analytical techniques that have been used by researchers. For instance, the homogeneous balance technique has been used to study periodic solitons of the BBM equation [12]. The authors in [13] used the simple equation technique to study the exact solution of nonlinear PDEs. Khaliq et al. utilized the novel G-expansion method to study the travelling waves solution of nonlinear Boussinesq equation [14]. Wazwaz used teh Tanh technique to analyze the solitonic solution of nonlinear parabolic PDEs [15]. Saifullah et al. investigated the considered perturbed KdV equation using the Hirota bilinear method [16]. Sun et al. investigated N-solitons and the collision properties of a generalized three-component Hirota–Satsuma coupled KdV equation using the Hirota bilinear method [17]. Chen et al. studied the exact solutions and interaction behaviour of the (3+1)-dimensional Hirota-Satsuma-Ito-like equation via Bäcklund transformation [18]. Zhang et al. analyzed localized solutions of (5+1)-dimensional evolution equations via the Hirota bilinear method [19]. Some special cases of the considered perturbed KdV were studied using different analytical methods [20,21].

In this work, we used the tanh-coth technique to investigate the considered PDE. There are several studies on extended tanh methods. The readers may refer to the papers [22,23]. The advantage of the proposed approach is that it transform PDE to ODE using travelling transformation. Then, ODE is converted to a system of algebraic equations. We solved this system to obtain new soliton solutions for the considered method.

2. The Tanh-Coth Method

Here, we present a brief description of the suggested method, which was presented by Wazwaz [9]. A wave variable $\zeta = x - \varpi t$ is used to convert any PDE

$$\mathbb{V}(\mathcal{W}, \mathcal{W}_t, \mathcal{W}_x, \mathcal{W}_{xx}, \mathcal{W}_{xxx}, \mathcal{W}_{xxxx}, \ldots) = 0, \tag{5}$$

to an ODE

$$\mathbb{G}(\mathcal{W}, \mathcal{W}', \mathcal{W}'', \mathcal{W}''', \mathcal{W}'''', \ldots) = 0, \tag{6}$$

where $\mathcal{W}' = \frac{d}{d\zeta}\mathcal{W}$, etc. We integrated the ODE (6) and considered all constants of integration to be equal to zero. Tanh was used as a new variable in the tanh approach, and tanh itself served as a representation for all of its derivatives. In the suggested tanh-coth approach, a new independent variable was introduced as

$$\mathcal{Z} = \tanh(\Upsilon\zeta),\ \zeta = x - \varpi t, \tag{7}$$

where Υ represents wave number, which leads to:

$$\begin{aligned}
\frac{d}{d\zeta} &= \Upsilon(1-\mathcal{Z}^2)\frac{d}{d\mathcal{Z}},\\
\frac{d^2}{d\zeta^2} &= -2\Upsilon^2\mathcal{Z}(1-\mathcal{Z}^2)\frac{d}{dZ} + \Upsilon^2(1-\mathcal{Z}^2)^2\frac{d^2}{d\mathcal{Z}^2},\\
\frac{d^3}{d\zeta^3} &= 2\Upsilon^3(1-\mathcal{Z}^2)(3\mathcal{Z}^2-1)\frac{d}{d\mathcal{Z}} - 6\Upsilon^3\mathcal{Z}(1-\mathcal{Z}^2)^2\frac{d^2}{d\Upsilon^2} + \Upsilon^3(1-\mathcal{Z}^2)^3\frac{d^3}{d\mathcal{Z}^3}.
\end{aligned} \tag{8}$$

The tanh-coth method posses the following finite expansion

$$\mathcal{W}(\Upsilon\zeta) = \mathcal{S}(\mathcal{Z}) = \sum_{m=0}^{K} \mathrm{r}_m\mathcal{Z}^m + \sum_{m=1}^{K} \mathrm{s}_m\mathcal{Z}^{-m}, \tag{9}$$

and

$$\mathcal{Z}' = \Upsilon(1-\mathcal{Z}^2), \tag{10}$$

where K is positive integer, mostly the value of K, which is determined by the homogeneous balance method. Equation (9) uses the classical tanh technique for $s_m = 0, 1 \leq m \leq K$. By substituting Equation (9) into the reduced ODE, one can obtain the algebraic equation in different powers of $\mathcal{Z}$. We found that for the value of K from Equations (8) and (9), the largest exponent for the function $\mathcal{W}$ and its derivatives are

$$\mathcal{W} \to K, \quad \mathcal{W}^q \to qK, \tag{11}$$

$$\mathcal{W}' \to K+1, \quad \mathcal{W}'' \to K+2, \quad \mathcal{W}^n \to K+n. \tag{12}$$

In order to find the value of the parameter K, it is usually necessary to balance the highest-order linear terms in the reduced equation with the terms with high-order derivatives using of the strategy presented above. Next, in the simplified equation, when the coefficients must vanish, take the coefficients of every power of $\mathcal{Z}$. In doing so, an algebraic system of equations can be obtained with the parameters r_m, s_m, ζ, and ϖ. Solving the algebraic system and determining these parameters, one can achieve an analytical solution $\mathcal{W}(x,t)$ in a closed form. The obtained analytic solutions may be the soliton solution in terms of $sech^2$, or kinks in the terms of tanh. Moreover, this procedure can also provide the periodic solutions.

3. Novel Solutions of Perturbed KdV Equation with Tanh-Coth Method

In this part of the paper, we will apply the proposed method to obtain some novel analytical solution for the perturbed KdV Equation (4). For this, consider the following wave transformation:

$$\zeta = x - \varpi t. \tag{13}$$

Substituting Equation (13) into Equation (4), we obtain

$$-\varpi\mathcal{W}'(\zeta) + \eta\mathcal{W}'''(\zeta) - \Omega\mathcal{W}(\zeta)\mathcal{W}'(\zeta) + \varrho\mathcal{W}'(\zeta) = 0. \tag{14}$$

Now, integrating Equation (14) with respect to ζ, we obtain

$$-\varpi\mathcal{W}(\zeta) + \eta\mathcal{W}''(\zeta) - \frac{1}{2}\Omega\mathcal{W}(\zeta)^2 + \varrho\mathcal{W}(\zeta) = 0. \tag{15}$$

To implement the proposed method we find the value of K. Using the homogeneity principle, we get $K = 2$. Therefore, the expansion Equation (9) for the considered equation is

$$\mathcal{W}(\Upsilon\zeta) = \mathcal{S}(\mathcal{Z}) = \sum_{m=0}^{2} \mathrm{r}_m \mathcal{Z}^m + \sum_{m=1}^{2} \mathrm{s}_m \mathcal{Z}^{-m}. \tag{16}$$

Inserting Equation (16) into Equation (15), we reach:

$$\begin{aligned} &-\varpi\Big(\mathrm{r}_0 + \mathrm{r}_1\tanh(\Upsilon\zeta) + \mathrm{r}_2\tanh^2(\Upsilon\zeta) + \mathrm{s}_1\coth(\Upsilon\zeta) + \mathrm{s}_2\coth^2(\Upsilon\zeta)\Big) - \frac{1}{2}\Omega \\ &\Big(\mathrm{r}_0 + \mathrm{r}_1\tanh(\Upsilon\zeta) + \mathrm{r}_2\tanh^2(\Upsilon\zeta) + \mathrm{s}_1\coth(\Upsilon\zeta) + \mathrm{s}_2\coth^2(\Upsilon\zeta)\Big)^2 + \varrho \\ &\Big(\mathrm{r}_0 + \mathrm{r}_1\tanh(\Upsilon\zeta) + \mathrm{r}_2\tanh^2(\Upsilon\zeta) + \mathrm{s}_1\coth(\Upsilon\zeta) + \mathrm{s}_2\coth^2(\Upsilon\zeta)\Big) + \eta \\ &\Big(-2\mathrm{r}_1\Upsilon^2\tanh(\Upsilon\zeta)\mathrm{sech}^2(\Upsilon\zeta) + 2\mathrm{r}_2\Upsilon^2\mathrm{sech}^4(\Upsilon\zeta) - 4\mathrm{r}_2\Upsilon^2\tanh^2(\Upsilon\zeta)\mathrm{sech}^2(\Upsilon\zeta) \\ &+2\mathrm{s}_1\Upsilon^2\coth(\Upsilon\zeta)\mathrm{csch}^2(\Upsilon\zeta) + 2\mathrm{s}_2\Upsilon^2\mathrm{csch}^4(\Upsilon\zeta) + 4\mathrm{s}_2\Upsilon^2\coth^2(\Upsilon\zeta)\mathrm{csch}^2(\Upsilon\zeta)\Big) = 0. \end{aligned} \tag{17}$$

Using the trigonometric identities and equating the coefficients of different powers of tanh and coth to zero, we obtain:

$$\begin{cases} 3\mathrm{r}_0{}^2\Omega - 10\mathrm{r}_0\mathrm{r}_2\Omega - 10\mathrm{r}_0\mathrm{s}_2\Omega + 6\mathrm{r}_0 c - 6\mathrm{r}_0\varrho - 5\mathrm{r}_1{}^2\Omega + 6\mathrm{r}_1\mathrm{s}_1\Omega + 35\mathrm{r}_2{}^2\Omega - 512\mathrm{r}_2\eta \\ \mu^2 + 6\mathrm{r}_2\mathrm{s}_2\Omega - 10\mathrm{r}_2 c + 10\mathrm{r}_2\varrho - 5\mathrm{s}_1{}^2\Omega + 35\mathrm{s}_2{}^2\Omega - 512\eta\mathrm{s}_2\mu^2 - 10\mathrm{s}_2 c + 10\mathrm{s}_2\varrho = 0, \\ 8\mathrm{r}_0\mathrm{r}_2\Omega - 8\mathrm{r}_0\mathrm{s}_2\Omega + 4\mathrm{r}_1{}^2\Omega - 56\mathrm{r}_2{}^2\Omega + 736\mathrm{r}_2\eta\mu^2 + 8\mathrm{r}_2 c - 8\mathrm{r}_2\varrho - 4\mathrm{s}_1{}^2\Omega + 56\mathrm{s}_2{}^2\Omega \\ -736\eta\mathrm{s}_2\mu^2 - 8\mathrm{s}_2 c + 8\mathrm{s}_2\varrho = 0, \\ -4\mathrm{r}_0{}^2\Omega + 8\mathrm{r}_0\mathrm{r}_2\Omega + 8\mathrm{r}_0\mathrm{s}_2\Omega - 8\mathrm{r}_0 c + 8\mathrm{r}_0\varrho + 4\mathrm{r}_1{}^2\Omega - 8\mathrm{r}_1\mathrm{s}_1\Omega + 28\mathrm{r}_2{}^2\Omega - 256\mathrm{r}_2\eta\mu^2 \\ -8\mathrm{r}_2\mathrm{s}_2\Omega + 8\mathrm{r}_2 c - 8\mathrm{r}_2\varrho + 4\mathrm{s}_1{}^2\Omega + 28\mathrm{s}_2{}^2\Omega - 256\eta\mathrm{s}_2\mu^2 + 8\mathrm{s}_2 c - 8\mathrm{s}_2\varrho = 0, \\ -8\mathrm{r}_0\mathrm{r}_2\Omega + 8\mathrm{r}_0\mathrm{s}_2\Omega - 4\mathrm{r}_1{}^2\Omega - 8\mathrm{r}_2{}^2\Omega + 32\mathrm{r}_2\eta\mu^2 - 8\mathrm{r}_2 c + 8\mathrm{r}_2\varrho + 4\mathrm{s}_1{}^2\Omega + 8\mathrm{s}_2{}^2\Omega \\ -32\eta\mathrm{s}_2\mu^2 + 8\mathrm{s}_2 c - 8\mathrm{s}_2\varrho = 0, \\ \mathrm{r}_0{}^2\Omega + 2\mathrm{r}_0\mathrm{r}_2\Omega + 2\mathrm{r}_0\mathrm{s}_2\Omega + 2\mathrm{r}_0 c - 2\mathrm{r}_0\varrho + \mathrm{r}_1{}^2\Omega + 2\mathrm{r}_1\mathrm{s}_1\Omega + \mathrm{r}_2{}^2\Omega + 2\mathrm{r}_2\mathrm{s}_2\Omega + 2\mathrm{r}_2 c \\ -2\mathrm{r}_2\varrho + \mathrm{s}_1{}^2\Omega + \mathrm{s}_2{}^2\Omega + 2\mathrm{s}_2 c - 2\mathrm{s}_2\varrho = 0, \\ 12\mathrm{r}_0\mathrm{r}_1\Omega - 12\mathrm{r}_0\mathrm{s}_1\Omega - 28\mathrm{r}_1\mathrm{r}_2\Omega + 80\mathrm{r}_1\eta\mu^2 - 12\mathrm{r}_1\mathrm{s}_2\Omega + 12\mathrm{r}_1 c - 12\mathrm{r}_1\varrho + 12\mathrm{r}_2\mathrm{s}_1\Omega \\ -80\eta\mathrm{s}_1\mu^2 + 28\mathrm{s}_1\mathrm{s}_2\Omega - 12\mathrm{s}_1 c + 12\mathrm{s}_1\varrho = 0, \\ -4\mathrm{r}_0\mathrm{r}_1\Omega - 4\mathrm{r}_0\mathrm{s}_1\Omega + 28\mathrm{r}_1\mathrm{r}_2\Omega - 64\mathrm{r}_1\eta\mu^2 - 4\mathrm{r}_1\mathrm{s}_2\Omega - 4\mathrm{r}_1 c + 4\mathrm{r}_1\varrho - 4\mathrm{r}_2\mathrm{s}_1\Omega \\ -64\eta\mathrm{s}_1\mu^2 + 28\mathrm{s}_1\mathrm{s}_2\Omega - 4\mathrm{s}_1 c + 4\mathrm{s}_1\varrho = 0, \\ -4\mathrm{r}_0\mathrm{r}_1\Omega + 4\mathrm{r}_0\mathrm{s}_1\Omega - 12\mathrm{r}_1\mathrm{r}_2\Omega + 16\mathrm{r}_1\eta\mu^2 + 4\mathrm{r}_1\mathrm{s}_2\Omega - 4\mathrm{r}_1 c + 4\mathrm{r}_1\varrho - 4\mathrm{r}_2\mathrm{s}_1\Omega \\ -16\eta\mathrm{s}_1\mu^2 + 12\mathrm{s}_1\mathrm{s}_2\Omega + 4\mathrm{s}_1 c - 4\mathrm{s}_1\varrho = 0, \\ 2\mathrm{r}_0\mathrm{r}_1\Omega + 2\mathrm{r}_0\mathrm{s}_1\Omega + 2\mathrm{r}_1\mathrm{r}_2\Omega + 2\mathrm{r}_1\mathrm{s}_2\Omega + 2\mathrm{r}_1 c - 2\mathrm{r}_1\varrho + 2\mathrm{r}_2\mathrm{s}_1\Omega + 2\mathrm{s}_1\mathrm{s}_2\Omega \\ +2\mathrm{s}_1 c - 2\mathrm{s}_1\varrho = 0. \end{cases} \tag{18}$$

Solving system (18) using Mathematica software, we obtain the following sets for non-trivial solutions:

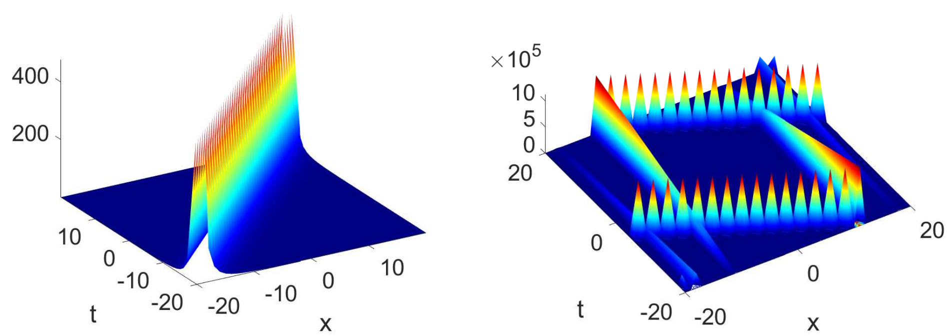
1. $\mathrm{r}_0 = -\frac{3(\varpi-\varrho)}{\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = 0; \mathrm{s}_1 = 0; \mathrm{s}_2 = \frac{3(\varpi-\varrho)}{\Omega}; \mu = \frac{\sqrt{\varpi-\varrho}}{2\sqrt{\eta}}.$
2. $\mathrm{r}_0 = -\frac{3(\varpi-\varrho)}{\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = \frac{3(\varpi-\varrho)}{\Omega}; \mathrm{s}_1 = 0; \mathrm{s}_2 = 0; \mu = -\frac{\sqrt{\varpi-\varrho}}{2\sqrt{\eta}}.$
3. $\mathrm{r}_0 = \frac{\varrho-\varpi}{2\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = -\frac{3(\varpi-\varrho)}{4\Omega}; \mathrm{s}_1 = 0; \mathrm{s}_2 = -\frac{3(\varpi-\varrho)}{4\Omega}; \mu = \frac{i\sqrt{\varpi-\varrho}}{4\sqrt{\eta}}.$
4. $\mathrm{r}_0 = \frac{\varpi-\varrho}{\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = 0; \mathrm{s}_1 = 0; \mathrm{s}_2 = -\frac{3(\varpi-\varrho)}{\Omega}; \mu = -\frac{i\sqrt{\varpi-\varrho}}{2\sqrt{\eta}}.$
5. $\mathrm{r}_0 = \frac{\varpi-\varrho}{\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = -\frac{3(\varpi-\varrho)}{\Omega}; \mathrm{s}_1 = 0; \mathrm{s}_2 = 0; \mu = \frac{i\sqrt{\varpi-\varrho}}{2\sqrt{\eta}}.$
6. $\mathrm{r}_0 = -\frac{3(\varpi-\varrho)}{2\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = \frac{3(\varpi-\varrho)}{4\Omega}; \mathrm{s}_1 = 0; \mathrm{s}_2 = \frac{3(\varpi-\varrho)}{4\Omega}; \mu = \frac{\sqrt{\varpi-\varrho}}{4\sqrt{\eta}}.$
7. $\mathrm{r}_0 = -\frac{3(\varpi-\varrho)}{\Omega}; \mathrm{r}_1 = 0; \mathrm{r}_2 = 0; \mathrm{s}_1 = 0; \mathrm{s}_2 = \frac{3(\varpi-\varrho)}{\Omega}; \mu = \frac{\sqrt{\varpi-\varrho}}{2\sqrt{\eta}}.$

The corresponding solution to each case is given below:

1. $\mathcal{W}_1(x,t) = \frac{3(\varpi-\varrho)}{\Omega} + \frac{3(\varpi-\varrho)\coth^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{2\sqrt{\eta}}\right)}{\Omega}.$
2. $\mathcal{W}_2(x,t) = \frac{3(\varpi-\varrho)\tanh^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{2\sqrt{\eta}}\right)}{\Omega} - \frac{3(\varpi-\varrho)}{\Omega}.$
3. $\mathcal{W}_3(x,t) = \frac{\varrho-\varpi}{2\Omega} + \frac{3(\varpi-\varrho)\tan^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{4\sqrt{\eta}}\right)}{4\Omega} + \frac{3(\varpi-\varrho)\cot^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{4\sqrt{\eta}}\right)}{4\Omega}.$
4. $\mathcal{W}_4(x,t) = \frac{\varpi-\varrho}{\Omega} + \frac{3(\varpi-\varrho)\cot^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{2\sqrt{\eta}}\right)}{\Omega}.$
5. $\mathcal{W}_5(x,t) = \frac{\varpi-\varrho}{\Omega} + \frac{3(\varpi-\varrho)\tan^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{2\sqrt{\eta}}\right)}{\Omega}.$
6. $\mathcal{W}_6(x,t) = -\frac{3(\varpi-\varrho)}{2\Omega} + \frac{3(\varpi-\varrho)\tanh^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{4\sqrt{\eta}}\right)}{4\Omega} + \frac{3(\varpi-\varrho)\coth^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{4\sqrt{\eta}}\right)}{4\Omega}.$
7. $\mathcal{W}_7(x,t) = \frac{3(\varpi-\varrho)\coth^2\left(\frac{\sqrt{\varpi-\varrho}(x-\varpi t)}{2\sqrt{\eta}}\right)}{\Omega} - \frac{3(\varpi-\varrho)}{\Omega}.$

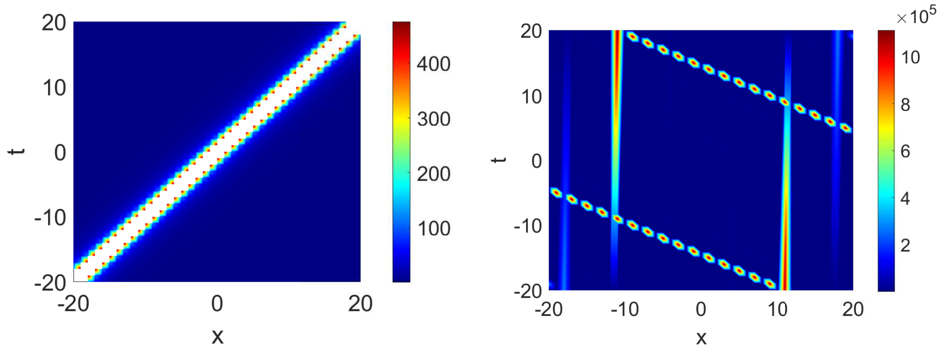
4. Simulation and Discussion

Here, we present the graphical simulations of the different exact solutions of the perturbed KdV Equation (4). We consider two different sets of values to see the affects of the parameter ϖ on the dynamics and structure of the wave solutions. We consider $\varpi - \varrho > 0$ for the (a,c) sub-plots, while $\varpi - \varrho < 0$ for (b,d) sub-plots in all of the simulated figures. Figure 1 depicts the dynamics of the exact solution $\mathcal{W}_1(x,t)$. In Figure 1, we see the singular soliton solution of the perturbed KdV when $\varpi = 1$; while decreasing the value of ϖ to -2, the hybrid type multi-solitons can be observed. The exact solution $\mathcal{W}_2(x,t)$ is visualized in Figure 2, where the dark soliton solution is observed, whose intensity is less than the plane; upon changing ϖ from a positive to negative value, the multi-soliton solution is obtained, which is demonstrated in Figure 1b. The exact solution $\mathcal{W}_3(x,t)$ is presented graphically in Figure 3. There is a singular quasi-periodic singular soliton solution for ϖ = 1, which is presented in Figure 3a, while varying ϖ to -1 gives the singular soliton solution of $\mathcal{W}_3(x,t)$. Similarly, Figure 4, presents the simulation of $\mathcal{W}_4(x,t)$, which shows the singular quasi-periodic soliton and the singular soliton solution. Furthermore, the exact solution $\mathcal{W}_5(x,t)$ is depicted in Figure 5, where the quasi-periodic soliton solution is observed with $\varpi = 1$, while the dark soliton solution can be seen with $\varpi = -1$. Finally, Figures 6 and 7 show the physical behavior of the exact solutions $\mathcal{W}_6(x,t)$ and $\mathcal{W}_7(x,t)$, respectively. The singular soliton solutions are obtained with $\varpi = 2$ and the singular quasi-periodic soliton solutions are observed with $\varpi = -1$.



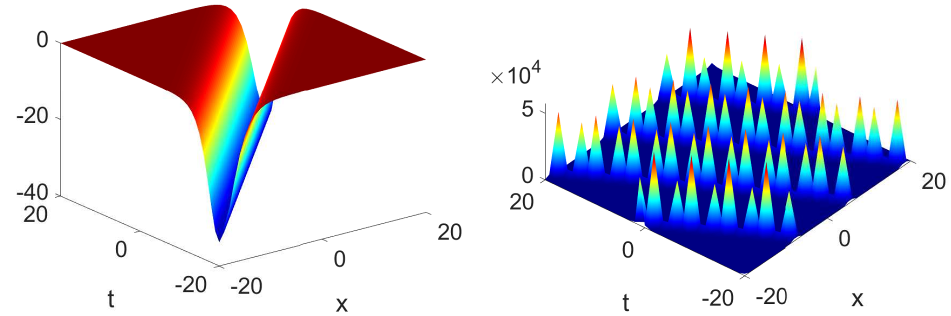
(a) $\varpi = 1,\ \varrho = 0.5$ **(b)** $\varpi = -2,\ \varrho = 1$

Figure 1. *Cont.*

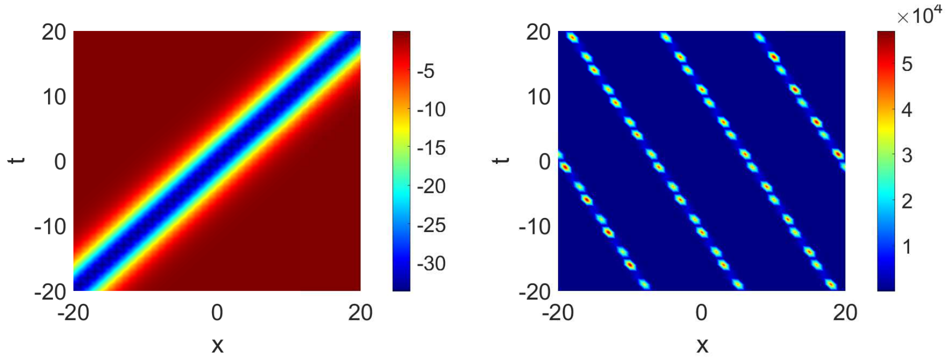


(**c**) $\varpi = 1,\ \varrho = 0.5$ (**d**) $\varpi = -2,\ \varrho = 1$

Figure 1. Simulation of exact solution $\mathcal{W}_1(x,t)$ with parameters $\Omega = 0.1$ and $\eta = 4$.

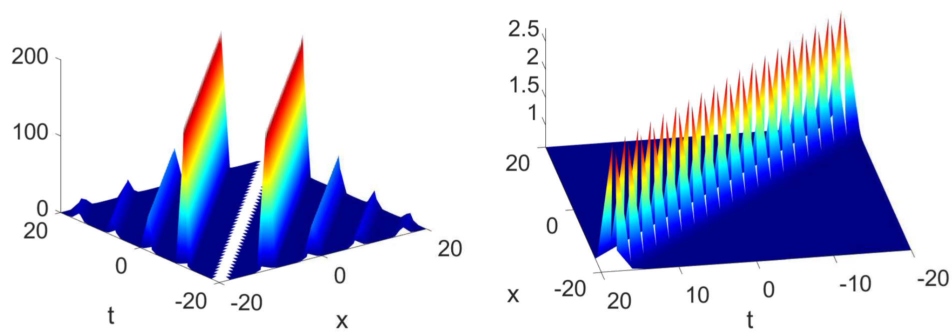


(**a**) $\varpi = 1,\ \varrho = 0.1$ (**b**) $\varpi = -0.6,\ \varrho = 0.1$



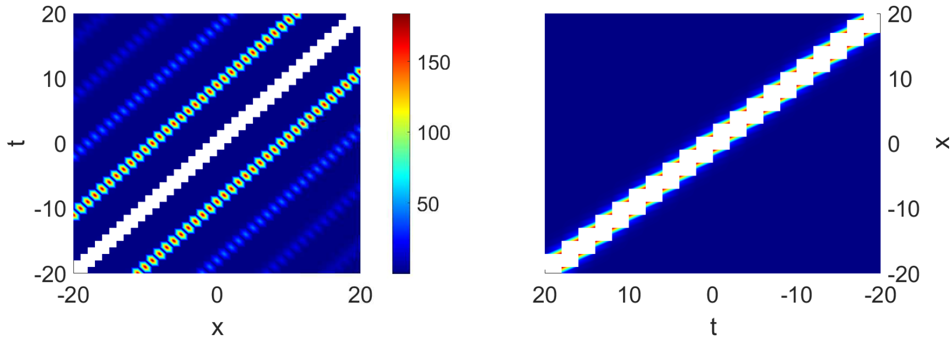
(**c**) $\varpi = 1,\ \varrho = 0.1$ (**d**) $\varpi = -0.6,\ \varrho = 0.1$

Figure 2. Simulation of exact solution $\mathcal{W}_2(x,t)$ with parameters $\Omega = 3$ and $\eta = 0.08$.



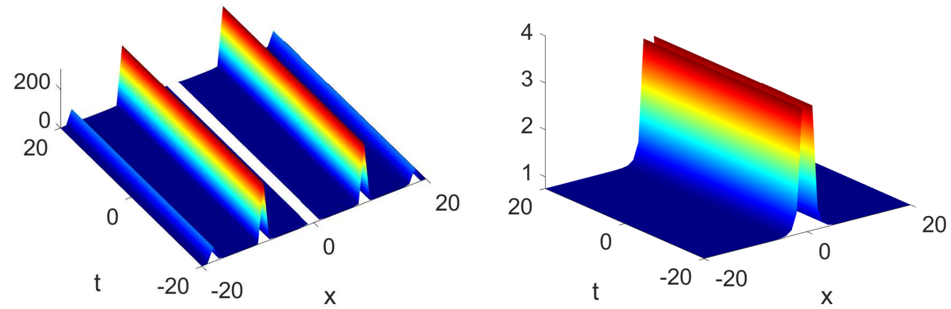
(**a**) $\varpi = 1,\ \varrho = 0.5$ (**b**) $\varpi = -1,\ \varrho = 0.5$

Figure 3. *Cont.*

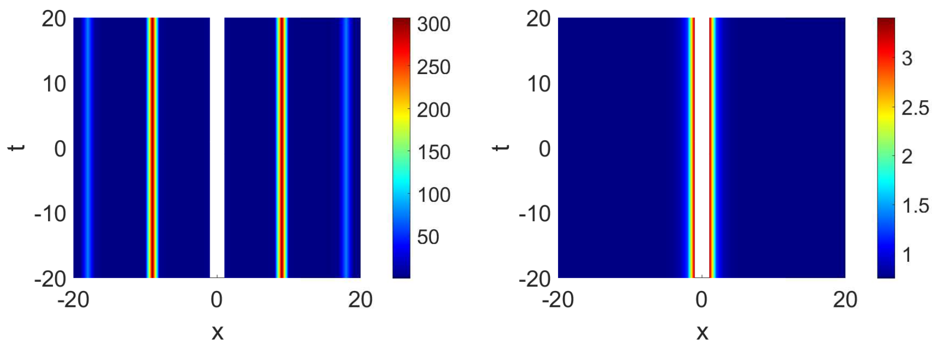


(**c**) $\varpi = 1, \ \varrho = 0.5$ (**d**) $\varpi = -1, \ \varrho = 0.5$

Figure 3. Simulation of exact solution $\mathcal{W}_3(x,t)$ with parameters $\Omega = 5$ and $\eta = 1$.

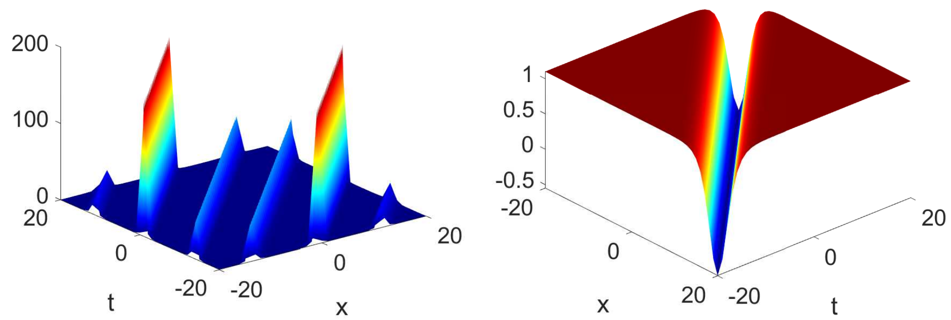


(**a**) $\varpi = 1, \ \varrho = 0.5$ (**b**) $\varpi = -1, \ \varrho = 0.5$



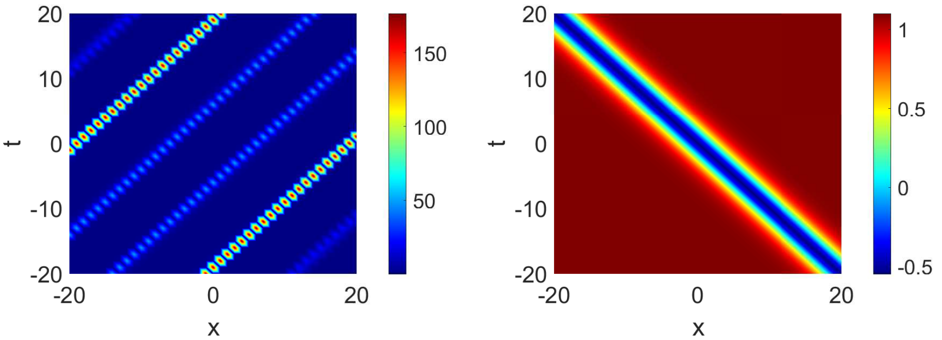
(**c**) $\varpi = 1, \ \varrho = 0.5$ (**d**) $\varpi = -1, \ \varrho = 0.5$

Figure 4. Simulation of exact solution $\mathcal{W}_4(x,t)$ with parameters $\Omega = 3$ and $\eta = 1$.



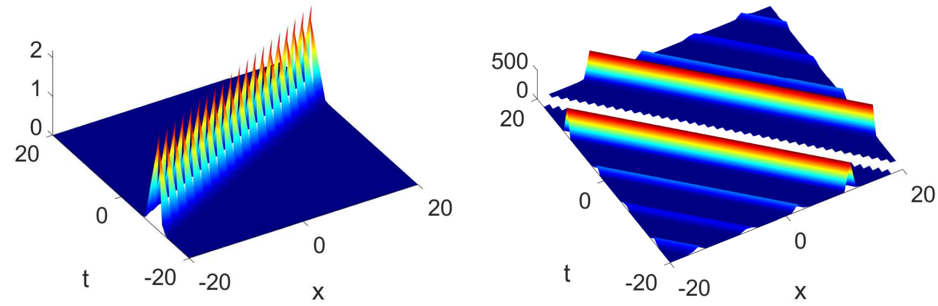
(**a**) $\varpi = 1, \ \varrho = 0.5$ (**b**) $\varpi = -1, \ \varrho = 0.1$

Figure 5. *Cont.*

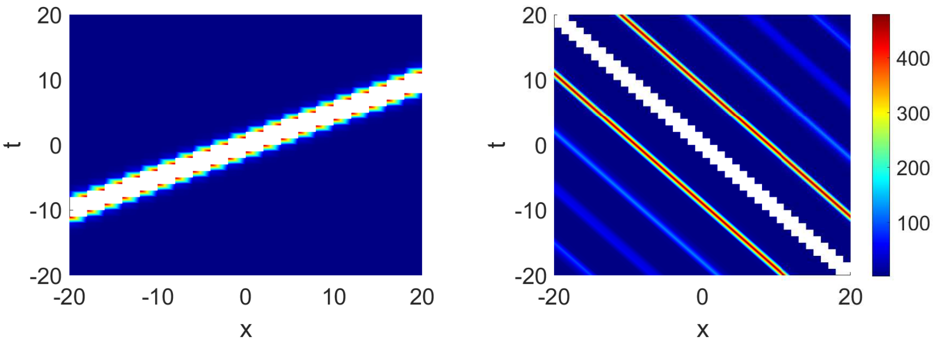


(**c**) $\varpi = 1,\ \varrho = 0.5$ (**d**) $\varpi = -1,\ \varrho = 0.1$

Figure 5. Simulation of exact solution $\mathcal{W}_5(x,t)$ with parameters $\Omega = 5$ and $\eta = 2$.

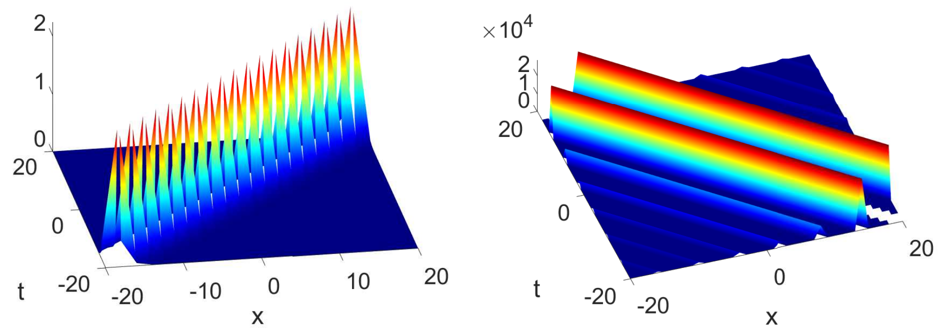


(**a**) $\varpi = 2,\ \varrho = 0.01$ (**b**) $\varpi = -1,\ \varrho = 0.01$



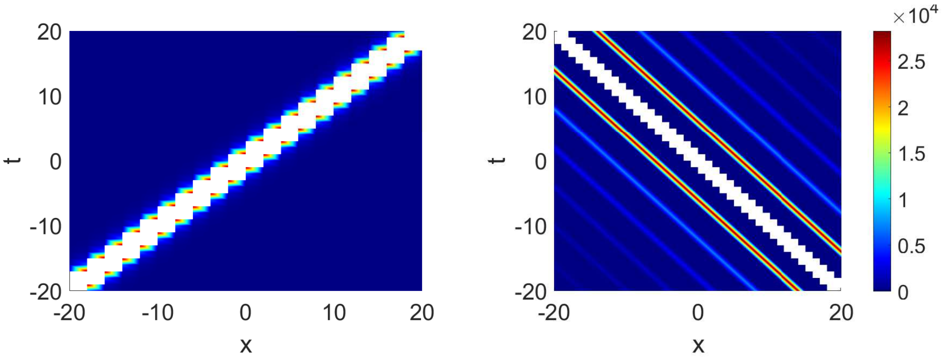
(**c**) $\varpi = 2,\ \varrho = 0.01$ (**d**) $\varpi = -1,\ \varrho = 0.01$

Figure 6. Simulation of exact solution $\mathcal{W}_6(x,t)$ with parameters $\Omega = 2$ and $\eta = 2$.



(**a**) $\varpi = 1,\ \varrho = 0.1$ (**b**) $\varpi = -1,\ \varrho = 0.1$

Figure 7. *Cont.*



(**c**) $\varpi = 1,\ \varrho = 0.1$ (**d**) $\varpi = -1,\ \varrho = 0.1$

Figure 7. Simulation of exact solution $\mathcal{W}_7(x,t)$ with parameters $\Omega = 5$ and $\eta = 1$.

5. Conclusions

In this paper, we computed a new travelling waves solution for the considered generalized geophysical KdV equation. The extended tanh approach, i.e., the tanh-coth method, was utilized to solve the considered PDE. To implement the technique, we used wave transformation, which transforms the given PDE to ODE. Then, solving ODE via a suitable substitution, we obtained a new exact solution in the form of tanh and coth functions. All the solutions were displayed via 3D and surface plots. We observed different types of soliton structures for the obtained results. The parameter ϖ has a great impact on the shape of the soliton solutions. In all figures, we noticed the transition in soliton solutions from one shape to another.

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