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Hyers Stability in Generalized Intuitionistic *P*-Pseudo Fuzzy 2-Normed Spaces

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Abstract: In this article, we defined the generalized intuitionistic *P*-pseudo fuzzy 2-normed spaces and investigated the Hyers stability of m-mappings in this space. The m-mappings are interesting functional equations; these functional equations are additive for $m = 1$, quadratic for $m = 2$, cubic for $m = 3$, and quartic for $m = 4$. We have investigated the stability of four types of functional equations in generalized intuitionistic *P*-pseudo fuzzy 2-normed spaces by the fixed point method.

Keywords: intuitionistic fuzzy 2-normed space; Hyers–Ulam–Rassias stability; fuzzy mathematics

MSC: 39B82; 39B52; 46S40; 47H10

1. Introduction

Functional equations generalize the subject of a modern branch of mathematics. The first articles in the field of functional equations were published by J. D'Alembert during 1747–1750. The apparent simplicity and harmonic nature have caused the subject of functional equations to be studied by many mathematicians. In the fall of 1940, Ulam [1] presented several unsolved problems in his famous speech at the University of Wisconsin. This lecture became the starting point for the theory of stability of functional equations. The question raised by Ulam was as follows: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D? If the problem admits a solution, we say that equation D is stable.

Ulam's problem was solved by Hyers [2] for additive mappings in 1941, and Hyers's results were generalized by Rassias [3] for linear mappings by various control functions. The results of Rassias had a great impact on the issue of the stability of functional equations. Today this type of stability is called the Hyers–Ulam–Rassias stability.

Mathematicians have proposed and proved many other theorems in the field of stability by changing the type of functional equation, control function, and space in the above theorem. In some of the articles in this field, the control function ϵ has been replaced by another function, and the stability theorem has been re-examined with new conditions. Similarly, by changing the type of functional equation in the above theorem from additive to quadratic, cubic, Jensen, etc., or replacing the functional equation with a differential or integral equation, the conditions of the stability theorem have been investigated and proven. We refer readers to [4–13] references for consideration of the stability of various functional equations in different spaces.

L. Zadeh [14] proposed the concept of fuzzy sets in 1965. The fuzzy metric space was introduced by Kromosil and Michalek [15]. This space is a generalization of the probabilistic metric space. In 1986, Atanasos [16] founded the concept of intuitionistic fuzzy sets by developing fuzzy sets. The idea of intuitionistic fuzzy normed space by Saadati and Park [17] was introduced in 2006.

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In 2012, Gordji et al. [4] introduced the following functional equation

$$f(ru+v)+f(ru-v) = r^{m-2}[f(u+v)+f(u-v)] \\ +2\left(r^2-1\right)\left[r^{m-2}f(u)+\frac{(m-2)(1-(m-2)^2)}{6}f(v)\right] \quad (1)$$

for every fixed integer r with $r \neq 0, \pm 1$. It is easily proven that $f(u) = cu^m (u \in \mathbb{R}, \mathrm{m} = 1, 2, 3, 4)$ satisfies the functional Equation (1). More precisely, if $m = 1$, the functional Equation (1) is additive, if $m = 2$, then it is quadratic, if $m = 3, 4$, then it is the cubic and quartic functional equation, respectively. We call a solution of the functional Equation (1) m-mapping.

Theorem 1 ([18]). *If (Δ, d) is a complete generalized metric space and $\mathcal{Z} : \Delta \to \Delta$ is a strictly contractive mapping with Lipschitz constant $\kappa < 1$, then for each element $u \in \Delta$, either*

$$d(\mathcal{Z}^n u, \mathcal{Z}^{n+1} u) = +\infty,$$

for every non-negative integer n, or there exists a $n_0 \in \mathbb{Z}^+$ such that
(1) $d(\mathcal{Z}^n u, \mathcal{Z}^{n+1} u) < +\infty$ for every $n \geq n_0$;
(2) The sequence $\{\mathcal{Z}^n u\} \to v^*$, where v^* is a fixed point of $\mathcal{Z}$;
(3) v^* is the unique fixed point of $\mathcal{Z}$ in the set $V = \{v \in \Delta \mid d(\mathcal{Z}^{n_0} u, v) < +\infty\}$.
(4) $d(v, v^*) \leq \frac{1}{1-\kappa} d(v, \mathcal{Z} v)$ for every $v \in V$.

Let Δ be a linear space over the field $\mathcal{F}$ and $\bigstar$ be a continuous t-norm and $\blacklozenge$ be a continuous t-conorm, in the following; we define the concepts of fuzzy and anti-fuzzy 2-norm.

Definition 1 ([19]). *A fuzzy subset μ of $\Delta \times \Delta \times \mathbb{R}$ is said to be a fuzzy 2-norm on Δ if and only if for $u, v, w \in \Delta$, $p, q \in \mathbb{R}$, and $\alpha \in \mathcal{F}$ the following items hold.*
(FT1) $\mu(u, v, p) = 0$ if $p \leq 0$.
(FT2) $\mu(u, v, p) = 1$ if and only if u, v are linearly dependent for all $p > 0$.
(FT3) $\mu(u, v, p)$ is invariant under any permutation of u, v.
(FT4) $\mu(u, \alpha v, p) = \mu\left(u, v, \frac{p}{|\alpha|}\right)$, for all $p > 0$ and $\alpha \neq 0$.
(FT5) $\mu(u + w, v, p + q) \geq \mu(u, v, p) \bigstar \mu(w, v, q)$ for all $p, q > 0$.
(FT6) $\mu(u, v, .)$ is a non-decreasing function on $\mathbb{R}$ and

$$\lim_{p \to \infty} \mu(u, v, p) = 1.$$

In this case, the (Δ, μ) is said to be a fuzzy 2-normed space.

Example 1 ([19]). *Let $(\Delta, \|., .\|)$ be a 2-normed linear space. Define*

$$\mu(u, v, p) = \begin{cases} \frac{p}{p + \|u, v\|} & p > 0 \\ 0 & p \leq 0 \end{cases}$$

where $u, v \in \Delta$ and $p \in \mathbb{R}$. Then (Δ, μ) is a fuzzy 2-normed linear space.

Definition 2 ([20]). *A fuzzy subset ν of $\Delta \times \Delta \times \mathbb{R}$ is said to be an anti fuzzy 2-norm on Δ if and only if for all $u, v, w \in \Delta$, $p, q \in \mathbb{R}$ and $\alpha \in \mathcal{F}$, the following items hold.*
(FN1) $\nu(u, v, p) = 1$, for every $p \leq 0$.
(FN2) $\nu(u, v, p) = 0$ if and only if u, v are linearly dependent for all $p > 0$.
(FN3) $\nu(u, v, p)$ is invariant under any permutation of u, v.
(FN4) $\nu(u, \alpha v, p) = \nu\left(u, v, \frac{p}{|\alpha|}\right)$ for every $p > 0, \alpha \neq 0$.

(FN5) $\nu(u, v+w, p+q) \leq \nu(u, v, p) \blacklozenge \nu(u, w, q)$ *for all* $p, q > 0$.
(FN6) $\nu(u, v, .)$ *is a non-increasing function and*

$$\lim_{p\to\infty} \nu(u, v, p) = 0.$$

In this case, the (Δ, ν) *is said to be an anti-fuzzy 2-normed linear space.*

Example 2 ([20])**.** *Let* $(\Delta, \|.,.\|)$ *be a 2-normed linear space. Define*

$$\nu(u, v, p) = \begin{cases} \frac{\|u,v\|}{p+\|u,v\|} & p > 0 \\ 1 & p \leq 0 \end{cases}$$

where $u, v \in \Delta$ *and* $p \in \mathbb{R}$*. Then* (Δ, ν) *is an anti-fuzzy 2-normed linear space.*

Lemma 1 ([20])**.** *We define the set* Υ^* *and operation* $\leq_{\Upsilon^*}$ *by*

$$\Upsilon^* = \left\{(\sigma_1, \sigma_2) : (\sigma_1, \sigma_2) \in [0,1]^2 \text{ and } \sigma_1 + \sigma_2 \leq 1\right\}$$
$$(\sigma_1, \sigma_2) \leq_{\Upsilon^*} (\pi_1, \pi_2) \Longleftrightarrow \sigma_1 \leq \pi_1, \sigma_2 \geq \pi_2$$

for all $(\sigma_1, \sigma_2), (\pi_1, \pi_2) \in \Upsilon^*$*. Then* $(\Upsilon^*, \leq_{\Upsilon^*})$ *is a complete lattice.*

Definition 3 ([20])**.** *A continuous t-norm* τ *on* $\Upsilon = [0,1]^2$ *is said to be continuous t-representable if there is a continuous t-norm* $\bigstar$ *and a continuous t-conorm* $\blacklozenge$ *on* $[0,1]$ *such that, for every* $\sigma = (\sigma_1, \sigma_2), \pi = (\pi_1, \pi_2) \in \Upsilon$

$$\tau(\sigma, \pi) = (\sigma_1 \bigstar \pi_1, \sigma_2 \blacklozenge \pi_2)$$

2. Main Results

2.1. Generalized Intuitionistic P-Pseudo Fuzzy 2-Normed Space

In this section, we introduce generalized intuitionistic P-pseudo fuzzy 2-normed space, and then we investigate the stability of functional equations in this space.

Definition 4 ([8])**.** *Let* Δ *be a linear space over the field* $\mathcal{F}$*,* μ *and* ν *be a fuzzy 2-norm and anti fuzzy 2-norm, respectively, such that* $\nu(u, v, p) + \mu(u, v, p) \leq 1$*,* τ *is a continuous* $t-$*representable, and*

$$\rho_{\mu,\nu} : \Delta \times \Delta \times \mathbb{R} \to \Upsilon^*$$
$$\rho_{\mu,\nu}(u, v, p) = (\mu(u, v, p), \nu(u, v, p))$$

is a function satisfying the following condition for all $u, v, w \in \Delta$*,* $p, q \in \mathbb{R}$ *and* $\alpha \in \mathcal{F}$

(P1) $\rho_{\mu,\nu}(u, v, p) = (0, 1) = 0_{\Upsilon^*}$ *for all* $p \leq 0$.
(P2) $\rho_{\mu,\nu}(u, v, p) = (1, 0) = 1_{\Upsilon^*}$ *if and only if* u, v *are linearly dependent for all* $p > 0$.
(P3) $\rho_{\mu,\nu}(\alpha u, v, p) = \rho_{\mu,\nu}\left(u, v, \frac{p}{|\alpha|}\right)$ *for all* $p > 0$ *and* $\alpha \neq 0$.
(P4) $\rho_{\mu,\nu}(u, v, p)$ *is invariant under any permutation of* u, v.
(P5) $\rho_{\mu,\nu}(u+w, v, p+q) \geq_{\Upsilon^*} \tau(\rho_{\mu,\nu}(u, v, p), \rho_{\mu,\nu}(w, v, q))$ *for all* $p, q > 0$.
(P6) $\rho_{\mu,\nu}(u, v, .)$ *is continuous and*

$$\lim_{p\to 0} \rho_{\mu,\nu}(u, v, p) = 0_{\Upsilon^*} \quad \text{and} \quad \lim_{p\to\infty} \rho_{\mu,\nu}(u, v, p) = 1_{\Upsilon^*}$$

Then $\rho_{\mu,\nu}$ *is said to be an intuitionistic fuzzy 2-norm on a linear space* Δ*, and the 3-tuple* $(\Delta, \rho_{\mu,\nu}, \tau)$ *is called to be an intuitionistic fuzzy 2-normed space (for short IF2NS).*

Example 3. *Let* $(\Delta, \|.,.\|)$ *be a 2-normed space,*

$$\tau(r,s) = (r_1 s_1, \min(r_2 + s_2, 1))$$

be a continuous t-representable for all $r = (r_1, r_2), s = (s_1, s_2) \in \Upsilon^*$ *and* μ *be a fuzzy 2-norm and* ν *be an anti fuzzy 2-norm. We put*

$$\rho_{\mu,\nu}(u,v,p) = \left(\frac{p}{p + m\|u,v\|}, \frac{\|u,v\|}{p + m\|u,v\|} \right)$$

for all $p \in \mathbb{R}^+$ *in which* $m > 1$*. Then* $(\Delta, \rho_{\mu,\nu}, \tau)$ *is an IF2NS.*

Definition 5 ([8])**.** *A sequence* $\{u_n\}$ *in* $(\Delta, \rho_{\mu,\nu}, \tau)$ *is said to be convergent to a point* $u \in \Delta$*, if*

$$\lim_{n\to\infty} \rho_{\mu,\nu}(u_n - u, v, p) = 1_{\Upsilon^*} \quad (v \in \Delta)$$

for all $p > 0$*.*

Definition 6. *In Definition (4), we replace condition* $(P5)$ *with the following condition; in this case,* $(\Delta, \rho_{\mu,\nu}, \tau)$ *is called to be an intuitionistic pseudo fuzzy 2-normed space.*

$$(P5')\ \rho_{\mu,\nu}(u + w, v, K(p+q)) \geq_{\Upsilon^*} \tau(\rho_{\mu,\nu}(u,v,p), \rho_{\mu,\nu}(w,v,q))$$

for constant $K \geq 1$*.*

Definition 7. *The intuitionistic pseudo fuzzy 2-normed space* $(\Delta, \rho_{\mu,\nu}, \tau)$ *is called generalized intuitionistic* P*-pseudo fuzzy 2-normed space, if for all* $u, v \in \Delta$*,* $p, q > 0$ *and* $0 < P \leq 1$*, the following inequality holds.*

$$\rho_{\mu,\nu}(u + w, v, \sqrt[P]{p+q}) \geq_{\Upsilon^*} \tau\Big(\rho_{\mu,\nu}(u, v, \sqrt[P]{t}), \rho_{\mu,\nu}(w, v, \sqrt[P]{s})\Big)$$

Example 4. *Let* $(\Delta, \|.,.\|)$ *be a 2-normed space with conditions of Example (3); we define*

$$\rho_{\mu,\nu}(u,v,p) = \left(\frac{p}{p + m\|u,v\|}, \frac{\|u,v\|}{p + m\|u,v\|} \right),$$

then $(\Delta, \rho_{\mu,\nu}, \tau)$ *is a generalized intuitionistic* P*-pseudo fuzzy 2-normed space.*

It follows from $(P2)$ and $(P5')$ that in a generalized intuitionistic P-pseudo fuzzy 2-normed space $(\Delta, \rho_{\mu,\nu}, \tau)$ for all $q > p > 0$ and $u, v \in \Delta$, we have

$$\rho_{\mu,\nu}(u,v,q) = \rho_{\mu,\nu}(u + 0, v, \sqrt[P]{p^P + (q^P - p^P)}) \geq_{\Upsilon^*} \tau\{\rho_{\mu,\nu}(u,v,p), \rho_{\mu,\nu}(0, v, \sqrt[P]{q^P - p^P})\} = \rho_{\mu,\nu}(u,v,p). \tag{2}$$

Therefore, $\rho_{\mu,\nu}(u,v,.)$ is a non-decreasing function on $\mathbb{R}^+$ for all $u, v \in \Delta$. Next, we present the following concepts of convergence and Cauchy sequences in a generalized intuitionistic P-pseudo fuzzy 2-normed space $(\Delta, \rho_{\mu,\nu}, \tau)$.

Definition 8. *A sequence* $\{u_n\}$ *in* Δ *is said to be convergent if there exists* $u \in \Delta$ *such that*

$$\lim_{n\to\infty} \rho_{\mu,\nu}(u_n - u, v, p) = 1_{\Upsilon^*} \quad (v \in \Delta)$$

for all $p > 0$*. In this case, we write*

$$u_n \xrightarrow{\rho_{\mu,\nu}} u \ \text{ or } \ u := \rho_{\mu,\nu} - \lim_{n\to\infty} u_n.$$

Definition 9. *A sequence $\{u_n\}$ in Δ is called to be Cauchy sequence, if for each $0 < \epsilon < 1$ and $p > 0$ there exists $n_0 \in \mathbb{N}$, such that*

$$\rho_{\mu,\nu}(u_n - u_m, v, p) \geq_{\Upsilon^*} (1-\epsilon, \epsilon) \ \ (n, m \geq n_0) \ \ (v \in \Delta)$$

If any Cauchy sequence is convergent, then generalized intuitionistic P-pseudo fuzzy 2-normed space $(\Delta, \rho_{\mu,\nu}, \tau)$ is said to be complete and the complete generalized intuitionistic P-pseudo fuzzy 2-normed space is said to be a Banach generalized intuitionistic P-pseudo fuzzy 2-normed space.

2.2. Stability of m-Mapping in Generalized Intuitionistic P-Pseudo Fuzzy 2-Normed Space

In this section, using the fixed point theorem, we investigate the stability of m-mapping in a generalized intuitionistic P-pseudo fuzzy 2-normed space. We suppose that $0 < P \leq 1$ and $Q = \frac{1}{P}$, Δ is a real vector space, $(\Theta, \rho_{\mu,\nu}, \tau)$ and is a Banach generalized intuitionistic P-pseudo fuzzy 2-normed space and $(\chi, \rho'_{\mu,\nu}, \tau')$ is generalized intuitionistic P-pseudo fuzzy 2-normed space. Furthermore, let $f : \Delta \to \Theta$ be a mapping. We define

$$\begin{aligned} D_m f(u,v) := f(ru+v) + f(ru-v) - r^{m-2}[f(u+v) + f(u-v)] \\ -2\left(r^2-1\right)\left[r^{m-2}f(u) + \frac{(m-2)(1-(m-2)^2)}{6}f(v)\right] \end{aligned} \tag{3}$$

for all $u, v \in \Delta$, fixed integer number $r \neq 0, \pm 1$ and $0 < m < 5$.

Theorem 2. *Let $\varphi_m, \psi_m : \Delta \times \Delta \to \chi$ be two functions such that for all $u, v \in \Delta$ and $p > 0$, the following relations are satisfied,*

$$\rho'_{\mu,\nu}(\varphi_m(ru, rv), \psi_m(ru, rv), p) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,v), \psi_m(u,v), \tfrac{p}{\alpha}) \tag{4}$$

moreover,

$$\lim_{n\to\infty}\left(\varphi_m(r^n u, r^n u), \psi_m(r^n u, r^n u), \tfrac{r^{mn}p}{2\alpha^n}\right) = 1, \tag{5}$$

where $\alpha > 0$ and $\alpha^2 < r^m$. Let $\xi : \Delta \to \Theta$ be a function so that

$$\xi(ru) = \tfrac{1}{\alpha}\xi(u), \tag{6}$$

for all $u \in \Delta$ and, $f : \Delta \to \Theta$ be a mapping such that,

$$\begin{aligned} \rho_{\mu,\nu}(D_m f(u,v), \xi(u), p+q) \geq_{\Upsilon^*} \tau'\{\rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), \\ \rho'_{\mu,\nu}(\varphi_m(v,v), \psi_m(v,v), q)\}. \end{aligned} \tag{7}$$

Then there exists a unique m-mapping $F : \Delta \to \Theta$ such as that satisfied in (1), and

$$\rho_{\mu,\nu}(f(u) - F(u), \xi(u), p) \geq \rho'_{\mu,\nu}\left(\varphi_m(u,u), \psi_m(u,u), (r^{mP} - \alpha^{2P})^Q\right) \tag{8}$$

Proof. Putting $v = 0$ and $p = q$ in (7), we have

$$\begin{aligned} \rho_{\mu,\nu}(2f(ru) - 2r^m f(u), \xi(u), 2p) \geq_{\Upsilon^*} \tau'\left[\rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), 1\right] \\ = \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), \end{aligned} \tag{9}$$

therefore,

$$\rho_{\mu,\nu}(f(ru) - r^m f(u), \xi(u), p) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p). \tag{10}$$

Now, we define the set $\mathcal{S}$ and the function d on it as follows

$$\mathcal{S} := \{g : \ g : \Delta \to \Theta, \ \ g(0) = 0\}$$

$$d(g,h) := \inf\Big\{\delta \in \mathbb{R}^+ \mid \rho_{\mu,\nu}\big(g(u) - h(u), \xi(u), \delta^Q t\big) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), \ \forall u \in \Delta \,, \ \forall p > 0\Big\}, \tag{11}$$

where $\inf \emptyset = +\infty$. The following shows that $(\mathcal{S}, d)$ is a complete generalized metric space.

(1) It is obvious that d has a symmetry property, i.e., $d(g,h) = d(h,g)$.

(2) Using Definition (11), we have

$$d(g,g) := \inf\Big\{\delta \in \mathbb{R}^+ \mid \underbrace{\rho_{\mu,\nu}\big(g(u) - g(u), \xi(u), \delta^Q p\big)}_{=1_{\Upsilon^*}} \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), \ \forall u \in \Delta \,, \ \forall p > 0\Big\} \tag{12}$$

The right side of the above definition is satisfied for every $\delta \in \mathbb{R}^+$, then $d(g,g) = 0$.

(3) Let $d(g,h) = 0$, using the definition of d, the following inequality holds, for every constant u and $p > 0$.

$$\rho_{\mu,\nu}\big(g(u) - h(u), \xi(u), p\big) \geq_{\Upsilon^*} \rho'_{\mu,\nu}\Big(\varphi_m(u,u), \psi_m(u,u), \frac{p}{\delta^Q}\Big).$$

As $\delta \to 0$, by $(P6)$, we have

$$\rho_{\mu,\nu}\big(g(u) - h(u), \xi(u), p\big) \geq_{\Upsilon^*} 1 \Rightarrow g(u) = h(u),$$

for all $u \in \Delta$ and $p > 0$.

(4) Triangular inequality: Let $g, h, j \in \mathcal{S}$ such that $d(g,h) \leq \eta_1$ and $d(j,h) \leq \eta_2$. Using (7), we have

$$\begin{aligned}\rho_{\mu,\nu}\big(g(u) - h(u), \xi(u), \eta_1^Q p\big) &\geq_{\Upsilon^*} \rho_{\mu,\nu}\big(g(u) - h(u), \xi(u), \delta^Q p\big) \\ &\geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p)\end{aligned} \tag{13}$$

and

$$\begin{aligned}\rho_{\mu,\nu}\big(h(u) - j(u), \xi(u), \eta_1^Q p\big) &\geq_{\Upsilon^*} \rho_{\mu,\nu}\big(h(u) - j(u), \xi(u), \delta^Q p\big) \\ &\geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p).\end{aligned} \tag{14}$$

Therefore, for all $u \in \Delta$ and $p > 0$, we obtain

$$\begin{aligned}\rho_{\mu,\nu}\big(g(u) - j(u), \xi(u), (\eta_1 + \eta_2)^Q p\big) &= \rho_{\mu,\nu}\big(g(u) - j(u), \xi(u), \sqrt[P]{(\eta_1 + \eta_2)p^P}\big) \\ &\geq_{L^*} \tau\Big(\rho_{\mu,\nu}\big(g(u) - j(u), \xi(u), \sqrt[P]{\eta_1 p^P}\big), \rho_{\mu,\nu}\big(g(u) - j(u), \xi(u), \sqrt[P]{\eta_2 p^P}\big)\Big) \\ &= \tau\Big(\rho_{\mu,\nu}\big(g(u) - j(u), \xi(u), \sqrt[P]{\eta_1} p\big), \rho_{\mu,\nu}\big(g(u) - j(u), \xi(u), \sqrt[P]{\eta_2} p\big)\Big) \\ &\geq \rho'_{\mu,\nu}\Big(\varphi_m(u,u), \psi_m(u,u), p\Big),\end{aligned}$$

by (11), we have

$$d(g,j) \leq \eta_1 + \eta_2 \Rightarrow d(g,j) \leq d(g,h) + d(h,j), \tag{15}$$

that is, the property of triangular inequality holds, then d is a generalized metric on $\mathcal{S}$. Next, we show that $(\mathcal{S}, d)$ is a complete generalized metric space. For this, we prove that every Cauchy sequence $\{g_n\}$ in $\mathcal{S}$ is convergent to $g \in \mathcal{S}$. Let $u \in \Delta$ be fixed and $\varepsilon > 0$, $\epsilon \in (0,1)$ and $p > 0$ be given, such that

$$\rho'_{\mu,\nu}\Big(\varphi_m(u,u), \psi_m(u,u), p\Big) > 1 - \epsilon.$$

Since $\{g_n\}$ is a Cauchy sequence in $\mathcal{S}$ for $\delta^{Q} < \frac{\varepsilon}{p}$ there exists $n_0 \in \mathbb{N}$ such that

$$d(g_n, g_m) < \tfrac{\varepsilon}{p} \ \ \forall n, m \geq n_0,$$

therefore, we have

$$\begin{aligned}\rho_{\mu,\nu}\Big(g_n(u) - g_m(u), \xi(u), \varepsilon\Big) &\geq_{\Upsilon^*} \rho_{\mu,\nu}\Big(g_n(u) - g_m(u), \xi(u), \delta^{Q} p\Big) \\ &\geq_{\Upsilon^*} \rho'_{\mu,\nu}\Big(\varphi_m(u,u), \psi_m(u,u), p\Big) > 1 - \epsilon. \end{aligned} \tag{16}$$

Hence, the sequence $\{g_n(u)\}$ is a Cauchy sequence in Θ since Θ is a Banach space, so $\{g_n(u)\}$ is a convergent sequence. It means that there exists $g : \Delta \to \Theta$ such that

$$\lim_{n\to\infty} g_n(u) = g(u).$$

It is enough to show that $g \in \mathcal{S}$. Assume that $\alpha, \beta > 0$ be given. There is $n_0 \in \mathbb{N}$ such that the following inequality holds for all $n \geq n_0$ and $m > 0$.

$$\rho_{\mu,\nu}\Big(g_n(u) - g_{n+m}(u), \xi(u), \alpha^{Q} p\Big) \geq_{\Upsilon^*} \rho'_{\mu,\nu}\Big(\varphi_m(u,u), \psi_m(u,u), p\Big).$$

Fix $n \geq n_0$ and $p > 0$, we have

$$\begin{aligned}&\rho_{\mu,\nu}(g_n(u) - g_{n+m}(u), \xi(u), (\alpha+\beta)^{Q} p) \\ &\quad = \rho_{\mu,\nu}(g_n(u) - g_{n+m}(u), \xi(u), \sqrt[P]{(\alpha+\beta) p^{P}}) \\ &\quad \geq_{\Upsilon^*} \tau\Big[\rho_{\mu,\nu}(g_n(u) - g_{n+m}(u), \xi(u), \alpha^{Q} p), \rho_{\mu,\nu}(g_{n+m}(u) - g(u), \xi(u), \beta^{Q} p)\Big] \\ &\quad \geq_{\Upsilon^*} \tau\Big[\rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), \rho_{\mu,\nu}(g_{n+m}(u) - g(u), \xi(u), \beta^{Q} p)\Big].\end{aligned}$$

By passing $m \to \infty$, so

$$\begin{aligned}\rho_{\mu,\nu}(g_n(u) - g_{n+m}(u), \xi(u), (\alpha+\beta)^{Q} p) &\geq_{\Upsilon^*} \tau\Big[\rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), 1\Big] \\ &= \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p).\end{aligned} \tag{17}$$

By (11), we can deduce that $g \in \mathcal{S}$. Hence, $(\mathcal{S}, d)$ is a complete generalized metric space. Next, we define the mapping $\mathcal{Z} : \mathcal{S} \to \mathcal{S}$ by

$$\mathcal{Z}g(u) := \frac{1}{r^m} g(ru), \quad \forall g \in \mathcal{S}, u \in \Delta.$$

Assume that $g, h \in \mathcal{S}$, such that $d(g,h) < \delta$, where $\delta \in (0,\infty)$ is an arbitrary constant. Then, by (11) we obtain

$$\rho_{\mu,\nu}(g(u) - h(u), \xi(u), \delta^Q p) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p), \quad \forall u \in \Delta,\ p > 0. \tag{18}$$

Replacing ru by u in (18), we have

$$\rho_{\mu,\nu}(g(ru) - h(ru), \xi(ru), \delta^Q p) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(ru,ru), \psi_m(ru,ru), p). \tag{19}$$

Therefore, using $(P3)$ and Definition (6), we have

$$\rho_{\mu,\nu}\left(\frac{1}{r^m}g(ru) - \frac{1}{r^m}h(ru), \frac{1}{\alpha}\xi(u), \frac{\delta^Q p}{r^m}\right) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(ru,ru), \psi_m(ru,ru), p). \tag{20}$$

It means that

$$\begin{aligned}\rho_{\mu,\nu}\left(\mathcal{Z}g(u) - \mathcal{Z}h(u), \xi(u), \frac{\alpha}{r^m}\delta^Q p\right) &\geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(ru,ru), \psi_m(ru,ru), p)\\ &\geq_{\Upsilon^*} \rho'_{\mu,\nu}\left(\varphi_m(u,u), \psi_m(u,u), \frac{p}{\alpha}\right).\end{aligned} \tag{21}$$

Hence,

$$\rho_{\mu,\nu}\left(\mathcal{Z}g(u) - \mathcal{Z}h(u), \xi(u), \frac{\alpha^2}{r^m}\delta^Q p\right) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,u), \psi_m(u,u), p). \tag{22}$$

Therefore, by (11), we have

$$d(\mathcal{Z}g, \mathcal{Z}h) \leq \left(\frac{\alpha^2}{r^m}\right)^P \delta.$$

It means that $\mathcal{Z}$ is a strictly contractive self-mapping on $\mathcal{S}$ with the Lipschitz constant $L = \left(\frac{\alpha^2}{r^m}\right)^P < 1$.

Moreover, by (10), we obtain

$$d(f, \mathcal{Z}f) \leq \left(\frac{1}{r^m}\right)^P.$$

It follows from (1) that the $\{\mathcal{Z}^n f\}$ converges to a fixed point F of $\mathcal{Z}$. Therefore,

$$F : \Delta \to \Theta \tag{23}$$

$$F(u) := \rho_{\mu,\nu} - \lim_{n\to\infty} \mathcal{Z}^n f(u) = \lim_{n\to\infty} \frac{1}{r^{mn}} f(r^n u),$$

for all $u \in \Delta$ and $p > 0$. Furthermore,

$$F(ru) = r^m F(u). \tag{24}$$

Also, F is the unique fixed point of $\mathcal{Z}$ in the set $S^* = \{g \in S \,:\, d(f,g) < \infty\}$. Hence, there exists a $\delta \in \mathbb{R}^+$ such that

$$\rho_{\mu,\nu}\left(g(u) - f(u), \xi(u), \delta^Q p\right) \geq_{\Upsilon^*} \rho'_{\mu,\nu}\left(\varphi_m(u,u), \psi_m(u,u), \frac{p}{\alpha}\right) \tag{25}$$

for all $u \in \Delta$ and $p > 0$. Furthermore,

$$d(f, F) \leq \frac{1}{1-L} d(f, \mathcal{Z}f) \leq \frac{1}{r^{mP} - \alpha^{2P}}.$$

It means that (8) holds. It is enough to show that F satisfies (1). Putting $p = q$, $v := r^n v$ and $u := r^n u$ in (7), we obtain

$$\begin{aligned}\rho_{\mu,\nu}(D_m f(r^n u, r^n v), \xi(r^n u), 2p) \geq_{\Upsilon^*} \\ \tau'\Big\{\rho'_{\mu,\nu}(\varphi_m(r^n u, r^n u), \psi_m(r^n u, r^n u), p), \\ \rho'_{\mu,\nu}(\varphi_m(r^n v, r^n v), \psi_m(r^n v, r^n v), p)\Big\}.\end{aligned} \tag{26}$$

According to $(P3)$, we have

$$\begin{aligned}\rho_{\mu,\nu}\left(\frac{1}{r^{mn}} D_m f(r^n u, r^n v), \xi(u), p\right) \geq_{L^*} \\ \tau'\Big\{\rho'_{\mu,\nu}\left(\varphi_m(r^n u, r^n u), \psi_m(r^n u, r^n u), \frac{r^{mn} p}{2\alpha^n}\right), \\ \rho'_{\mu,\nu}\left(\varphi_m(r^n v, r^n v), \psi_m(r^n v, r^n v), \frac{r^{mn} p}{2\alpha^n}\right)\Big\}.\end{aligned} \tag{27}$$

By letting $n \to \infty$ and using (5) and (23), we have

$$\rho_{\mu,\nu}(D_m F(u,v), \xi(u), p) \geq_{\Upsilon^*} 1 \overset{(P2)}{\longrightarrow} D_m F(u,v) = 0, \ \forall u, v \in \Delta, p > 0.$$

Thus, F satisfies (3) and as a result, F is an $m-$mapping. □

Corollary 1. *Let α be a real positive number with $\alpha > r^m$, such that the mappings φ_m, ψ_m: $\Delta \times \Delta \to \chi$ satisfy in the following inequality, for all $u, v \in \Delta$ and $p > 0$.*

$$\rho'_{\mu,\nu}\left(\varphi_m\left(\frac{u}{r}, \frac{v}{r}\right), \psi_m\left(\frac{u}{r}, \frac{v}{r}\right), p\right) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi_m(u,v), \psi_m(u,v), \alpha p). \tag{28}$$

Furthermore, suppose that $\xi : \Delta \to \Theta$ is a function that satisfies

$$\xi(ru) = \frac{1}{\alpha}\xi(u), \tag{29}$$

for all $u \in \Delta$. If $f : \Delta \to \Theta$ is a mapping satisfying $f(0) = 0$ and the inequality (7), then there exists a unique $m-$mapping $F : \Delta \to \Theta$ satisfying (3) such that

$$\rho_{\mu,\nu}(f(u) - F(u), \xi(u), p) \geq \rho'_{\mu,\nu}\left(\varphi_m(u,u), \psi_m(u,u), (\alpha^P - r^{mP})^Q\right) \tag{30}$$

for all $u \in \Delta$ and $p > 0$.

Proof. It is similar to the proof of the above theorem. □

Corollary 2. *Let φ, ψ be functions from $\Delta \times \Delta$ to χ such that for all $u, v \in \Delta$ and $p > 0$, the following inequality is held.*

$$\rho'_{\mu,\nu}(\varphi(u,v), \psi(u,v), p) \geq_{\Upsilon^*} \rho'_{\mu,\nu}(\varphi(2u, 2v), \psi_m(2u, 2v), 5p). \tag{31}$$

Furthermore, assume that $\xi : \Delta \to \Theta$ is a function satisfying

$$\xi(u) = 5\xi(2u), \tag{32}$$

for all $u \in \Delta$. *If* $f : \Delta \to \Theta$ *is a mapping satisfying* $f(0) = 0$ *and the inequality*

$$\begin{aligned}\rho_{\mu,\nu}\big(f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x),\ \xi(u),\ p+q\big)\\ \geq_{\Upsilon^*} \tau'\{\rho'_{\mu,\nu}(\varphi(u,u),\psi(u,u),p),\rho'_{\mu,\nu}(\varphi(v,v),\psi(v,v),q)\}.\end{aligned} \quad (33)$$

Then there exists a unique quadratic mapping $F : \Delta \to \Theta$ *such that*

$$\rho_{\mu,\nu}(f(u)-F(u),\xi(u),p) \geq \rho'_{\mu,\nu}\Big(\varphi_m(u,u),\psi_m(u,u),(5^P-4^P)^Q\Big) \quad (34)$$

for all $u \in \Delta$ *and* $p > 0$.

Proof. Putting $m = r = 2$ and $\alpha = 5$ in the above theorem, we can easily show the stability of quadratic functional equations in generalized intuitionistic P-pseudo fuzzy 2-normed space. □

3. Conclusions

In this paper, we defined the generalized intuitionistic P-pseudo fuzzy 2-normed space and investigated its features. Furthermore, we defined the convergent and Cauchy sequences in this space; then, we investigated the stability of m-mapping in this space by the fixed point method. By changing m and choosing the appropriate r, α from Theorem 2.1, we can prove the stability of the additive, cubic and quartic functional equation.

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