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Some New Hermite–Hadamard Type Inequalities Pertaining to Generalized Multiplicative Fractional Integrals

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Abstract: There is significant interaction between the class of symmetric functions and other types of functions. The multiplicative convex function class, which is intimately related to the idea of symmetry, is one of them. In this paper, we obtain some new generalized multiplicative fractional Hermite–Hadamard type inequalities for multiplicative convex functions and for their product. Additionally, we derive a number of inequalities for multiplicative convex functions related to generalized multiplicative fractional integrals utilising a novel identity as an auxiliary result. We provide some examples for the appropriate selections of multiplicative convex functions and their graphical representations to verify the authenticity of our main results.

Keywords: generalized multiplicative fractional integrals; Hermite–Hadamard type inequalities; multiplicative convex functions; Hölder's inequality; power mean inequality; numerical analysis

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

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1. Introduction and Preliminaries

One effective technique that is primarily utilised to address several challenges in both pure and applied research is the convexity of functions. There are several important properties of symmetric convex sets. The Hermite–Hadamard inequality (H-H), one of the most significant mathematical inequality linked to convex function, is often utilised in many other branches of computer mathematics and is as follows:

Theorem 1 ((H-H) [1])**.** *Suppose that $\mathscr{Q} : \mathrm{I} \subseteq \mathrm{R} \to \mathrm{R}$ is a convex function and $\mathtt{r}, \mathtt{w} \in \mathrm{I}$ with $\mathtt{r} \neq \mathtt{w}$, then*

$$\mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right) \leq \frac{1}{\mathtt{w}-\mathtt{r}} \int_{\mathtt{r}}^{\mathtt{w}} \mathscr{Q}(\mathtt{y})\mathtt{dy} \leq \frac{\mathscr{Q}(\mathtt{r}) + \mathscr{Q}(\mathtt{w})}{2}.$$

Due to the significance of this inequality, over the past ten years, numerous types of convexity, such as harmonically convex, exponentially convex, coordinated convex functions, etc., have been explored in the literature along with versions of the H-H inequality. The H-H inequality was also refined for an improved power mean inequality [2] and R-convex and R-concave functions [3]. In addition, numerous authors have helped this subject advance by using the discovered kernels and identities; see [4–15] and the references therein.

For the simplicity of notations, let us denote, respectively, $\mathrm{Q} = [0, \infty)$, where $\mathrm{Q}^{\circ} = (0, \infty)$, and $\mathrm{L}[\mathtt{r}, \mathtt{w}]$ the set of all Lebesgues integrable functions on $[\mathtt{r}, \mathtt{w}]$.

In the study of inequalities, the multiplicative convex function is crucial, and it is summed up as follows:

Definition 1 (Refs. [16,17]). *A function* $\mathscr{Q} : \mathrm{I} \subseteq \mathrm{R} \to \mathrm{Q}$ *is called multiplicative convex or* log-*convex, if* $\log \mathscr{Q}$ *is convex or equivalently, for all* $\mathtt{r}, \mathtt{w} \in \mathrm{I}$ *and* $\varrho \in [0,1]$, *i.e.,*

$$\mathscr{Q}(\varrho\mathtt{r} + (1-\varrho)\mathtt{w}) \leq [\mathscr{Q}(\mathtt{r})]^{\varrho}[\mathscr{Q}(\mathtt{w})]^{1-\varrho}.$$

As stated by Definition 1, it is easily revealed that

$$\mathscr{Q}(\varrho\mathtt{r} + (1-\varrho)\mathtt{w}) \leq [\mathscr{Q}(\mathtt{r})]^{\varrho}[\mathscr{Q}(\mathtt{w})]^{1-\varrho} \leq \varrho\mathscr{Q}(\mathtt{r}) + (1-\varrho)\mathscr{Q}(\mathtt{w}),$$

which indicates that every multiplicative convex function is a convex function, but the converse is not true.

Researchers have looked into a range of multiplicative-convex-function-related inequalities and characteristics. For instance, the H-H type of integral inequalities for multiplicative convex functions was studied by Bai and Qi [18]. Many H-H type weighted inequalities that are pertinent to multiplicative convex functions constructed on real intervals were established by Dragomir [19]. A few H-H type integral inequalities were presented by Set and Ardiç [20] concerning multiplicative convex functions and p-functions. Many multiplicative convex functions' features were provided by Zhang and Jiang [21]. Kadakal [22] established new versions of the H-H inequality for subadditive functions. Furthermore, interested readers can see [23–27] and the references therein for recent developments based on multiplicative convex functions.

The term "*integral operators" is used to refer to a class of multiplicative operators developed by Bashirov et al. [28], represented as $\int_{\mathtt{r}}^{\mathtt{w}} (\mathscr{Q}(\mathrm{y}))^{\mathtt{dy}}$. The term "ordinary integral" is represented by the symbol $\int_{\mathtt{r}}^{\mathtt{w}} \mathscr{Q}(\mathrm{y})\mathtt{dy}$. In retrospect, if the function $\mathscr{Q}$ is positive and Riemann integrable, defined on $[\mathtt{r}, \mathtt{w}]$, then it is multiplicatively integrable and moreover,

$$\int_{\mathtt{r}}^{\mathtt{w}} (\mathscr{Q}(\mathrm{y}))^{\mathtt{dy}} = \exp\left\{ \int_{\mathtt{r}}^{\mathtt{w}} \ln \mathscr{Q}(\mathrm{y})\, \mathtt{dy} \right\}. \tag{1}$$

Compared to the calculus of Newton and Leibnitz, multiplicative calculus has a limited range of applications. In fact, it exclusively includes positive functions. Nonetheless, the requirement of inventing and implementing multiplicative calculus mirrors the relevance of the polar coordinates when rectangular systems already exist. Moreover, we believe that multiplicative calculus is a helpful mathematical tool for economics, finance, and other scientific fields since multiplicative derivatives allow for several interpretations of situations where the logarithmic scale emerges. In reality, a lot of scientific tables use logarithmic scales. For instance, logarithmic scales are used to describe earthquake magnitude, chemical acidities, and sound signal levels.

Definition 2 (Ref. [28]). *Considering the function* $\mathscr{Q} : \mathrm{R} \to \mathrm{Q}$. *The multiplicative derivative of the function* $\mathscr{Q}$ *is given by*

$$\frac{\mathrm{d}^*\mathscr{Q}}{\mathrm{dy}}(\mathrm{y}) = \mathscr{Q}^*(\mathrm{y}) = \lim_{\mathrm{h}\to 0}\left(\frac{\mathscr{Q}(\mathrm{y}+\mathrm{h})}{\mathscr{Q}(\mathrm{y})}\right)^{\frac{1}{\mathrm{h}}}.$$

The link between the function $\mathscr{Q}^*$ and the ordinary derivative $\mathscr{Q}$ is as follows, if the function $\mathscr{Q}$ is differentiable and positive at y:

$$\mathscr{Q}^*(\mathrm{y}) = \exp\{[\ln \mathscr{Q}(\mathrm{y})]'\} = \exp\left\{\frac{\mathscr{Q}'(\mathrm{y})}{\mathscr{Q}(\mathrm{y})}\right\}.$$

These *differentiable characteristics are true.

Definition 3 (Ref. [28]). *Suppose that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are *differentiable functions. If the constant* c *is arbitrary, then the functions $c\mathscr{Q}_1, \mathscr{Q}_1\mathscr{Q}_2, \mathscr{Q}_1+\mathscr{Q}_2, \mathscr{Q}_1/\mathscr{Q}_2$, and $\mathscr{Q}_1^{\mathscr{Q}_2}$ are *differentiable, and the following identities are true:*

1. $(c\mathscr{Q}_1)^*(y) = \mathscr{Q}_1^*(y)$;
2. $(\mathscr{Q}_1\mathscr{Q}_2)^*(y) = \mathscr{Q}_1^*(y)\mathscr{Q}_2^*(y)$;
3. $(\mathscr{Q}_1+\mathscr{Q}_2)^*(y) = \mathscr{Q}_1^*(y)^{\frac{\mathscr{Q}_1(y)}{\mathscr{Q}_1(y)+\mathscr{Q}_2(y)}}\mathscr{Q}_2^*(y)^{\frac{\mathscr{Q}_2(y)}{\mathscr{Q}_1(y)+\mathscr{Q}_2(y)}}$;
4. $\left(\frac{\mathscr{Q}_1}{\mathscr{Q}_2}\right)^*(y) = \frac{\mathscr{Q}_1^*(y)}{\mathscr{Q}_2^*(y)}$;
5. $\left(\mathscr{Q}_1^{\mathscr{Q}_2}\right)^*(y) = \mathscr{Q}_1^*(y)^{\mathscr{Q}_2(y)}\mathscr{Q}_1(y)^{\mathscr{Q}_2'(y)}$.

Bashirov et al. [28] showed the following characteristics of the multiplicative integral operators:

Theorem 2 ((Multiplicative Integration by Parts) [28]). *Let the functions $\mathscr{Q}_1 : [r,w] \to \mathbb{R}$ be *differentiable and $\mathscr{Q}_2 : [r,w] \to \mathbb{R}$ be differentiable. Then, the function $\mathscr{Q}_1^{\mathscr{Q}_2}$ is *integrable and the following equality holds true:*

$$\int_r^w \left(\mathscr{Q}_1^*(y)^{\mathscr{Q}_2(y)}\right)^{dy} = \frac{\mathscr{Q}_1(w)^{\mathscr{Q}_2(w)}}{\mathscr{Q}_1(r)^{\mathscr{Q}_2(r)}} \cdot \frac{1}{\int_r^w \left(\mathscr{Q}_1(y)^{\mathscr{Q}_2'(y)}\right)^{dy}}.$$

Theorem 3 (Ref. [28]). *Let the functions $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be positive and Riemann integrable defined on $[r,w]$. Then, the function $\mathscr{Q}_1\mathscr{Q}_2$ is *integrable on $[r,w]$ and the following identities hold true:*

1. $\int_r^w ((\mathscr{Q}_1(y))^p)^{dy} = \int_r^w ((\mathscr{Q}_1(y))^{dy})^p, \quad p \in \mathbb{R}$;
2. $\int_r^w (\mathscr{Q}_1(y)\mathscr{Q}_2(y))^{dy} = \int_r^w (\mathscr{Q}_1(y))^{dy} \cdot \int_r^w (\mathscr{Q}_2(y))^{dy}$;
3. $\int_r^w \left(\frac{\mathscr{Q}_1(y)}{\mathscr{Q}_2(y)}\right)^{dy} = \frac{\int_r^w (\mathscr{Q}_1(y))^{dy}}{\int_r^w (\mathscr{Q}_2(y))^{dy}}$;
4. $\int_r^w (\mathscr{Q}_1(y))^{dy} = \int_r^v (\mathscr{Q}_1(y))^{dy} \cdot \int_v^w (\mathscr{Q}_1(y))^{dy}, \quad r \le v \le w$;
5. $\int_r^r (\mathscr{Q}_1(y))^{dy} = 1$ *and* $\int_r^w (\mathscr{Q}_1(y))^{dy} = \left(\int_w^r (\mathscr{Q}_1(y))^{dy}\right)^{-1}$.

The H-H inequalities for multiplicative convex functions, which are visually appealing geometric-mean-type inequalities, are shown below.

Theorem 4 (Ref. [29]). *Let the function $\mathscr{Q}$ be positive and multiplicative convex on $[r,w]$. Then,*

$$\mathscr{Q}\left(\frac{r+w}{2}\right) \le \left(\int_r^w (\mathscr{Q}(y))^{dy}\right)^{\frac{1}{w-r}} \le \sqrt{\mathscr{Q}(r)\mathscr{Q}(w)},$$

holds true.

According to Abdeljawad and Grossman [30], the multiplicative Riemann–Liouville fractional integrals are an exciting refinement of the Riemann–Liouville fractional integrals (R-L).

Multiplicative fractional calculus is the combination of multiplicative calculus and fractional calculus. Basic forms of fractional integrals and derivatives are defined and described to understand what exponential varying quantities look like with an arbitrary order reflected by delay effects. This makes the multiplicative fractional integrals more interesting to work with.

Definition 4 (Ref. [30]). *The multiplicative left-sided Riemann–Liouville fractional integral ${}_rI_*^\alpha\mathscr{Q}(x)$ of order $\alpha \in \mathbb{C}$, where $\mathrm{Re}(\alpha) > 0$, is defined as*

$${}_rI_*^\alpha\mathscr{Q}(x) = \exp\{J_{r^+}^\alpha \ln \mathscr{Q}(x)\},$$

and the multiplicative right-sided one ${}_{*}\mathrm{I}^{\alpha}_{\mathtt{w}}\mathscr{Q}(\mathrm{x})$ *is defined as*

$$ {}_{*}\mathrm{I}^{\alpha}_{\mathtt{w}}\mathscr{Q}(\mathrm{x}) = \exp\{\mathrm{J}^{\alpha}_{\mathtt{w}^-} \ln \mathscr{Q}(\mathrm{x})\}. $$

Here, we denote $\mathrm{J}^{\alpha}_{\mathtt{r}^+}\mathscr{Q}(\mathrm{x})$ as the left-sided R-L integral and $\mathrm{J}^{\alpha}_{\mathtt{w}^-}\mathscr{Q}(\mathrm{x})$ as the right-sided R-L integral, which are defined as

$$ \mathrm{J}^{\alpha}_{\mathtt{r}^+}\mathscr{Q}(\mathrm{x}) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{r}}^{\mathrm{x}} (\mathrm{x}-\varrho)^{\alpha-1}\mathscr{Q}(\varrho)d\varrho, \quad (\mathrm{x} > \mathtt{r}) $$

and

$$ \mathrm{J}^{\alpha}_{\mathtt{w}^-}\mathscr{Q}(\mathrm{x}) = \frac{1}{\Gamma(\alpha)}\int_{\mathrm{x}}^{\mathtt{w}} (\varrho-\mathrm{x})^{\alpha-1}\mathscr{Q}(\varrho)d\varrho, \quad (\mathrm{x} < \mathtt{w}), $$

respectively, for all $\alpha > 0$.

Moreover, Sarikaya et al. [31], using R-L fractional integrals, proved the following impressive inequality.

Theorem 5 (Ref. [31])**.** *Assume that the function* $\mathscr{Q} : [\mathtt{r}, \mathtt{w}] \to \mathrm{R}$ *is positive together with* $0 \le \mathtt{r} < \mathtt{w}$ *and* $\mathscr{Q} \in \mathrm{L}[\mathtt{r}, \mathtt{w}]$*. If the function* $\mathscr{Q}$ *defined on* $[\mathtt{r}, \mathtt{w}]$ *is convex, then the following inequalities with* $\alpha > 0$ *for fractional integrals hold true:*

$$ \mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(\mathtt{w}-\mathtt{r})^{\alpha}}\left[\mathrm{J}^{\alpha}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w}) + \mathrm{J}^{\alpha}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})\right] \le \frac{\mathscr{Q}(\mathtt{r}) + \mathscr{Q}(\mathtt{w})}{2}. $$

The importance of using fractional calculus as a tool for integrating and differentiating real or complex number orders has been demonstrated. It has developed swiftly as a result of being used to simulate a variety of issues, particularly when tackling stochastic difficulties, the dynamics of complex systems, and decision-making in structural engineering. For further information, please see [32,33]. Fractional calculus has a lot of applications with reference to fluid dynamics, for example see the articles [34–36], and engineering [37,38]. For recent developments on fluid dynamics, one can refer to [39–45] and the references therein. Researchers have been paying close attention to this subject recently. Many studies have been conducted on the H-H type inequalities involving various types of fractional integral operators. For example, [46] for R-L fractional integrals, Pshtiwan et al. [47] for tempered fractional integrals, Chen et al. [48] for Katugampola fractional integrals, Sahoo et al. [49] for Caputo–Fabrizio fractional integrals, Botmart et al. [50] for fractional integrals with exponential kernels, and Peng et al. [51] for new bounds estimates on the multiplicative fractional integral. Due to its potential applications in several sectors of the pure and applied sciences, fractional integral inequalities have captured the attention of many mathematicians and researchers. The primary function of fractional operators is to build connections between continuous and discrete situations.

The goal of this work was to define the left- and right-sided generalised multiplicative fractional integral operators, as well as several H-H inequalities, including the generalized multiplicative fractional integrals via the multiplicative convexity property. The research in the publications described above served as an inspiration for this one. Our findings using these novel generalizations may be used to assess a wide range of mathematical issues with applications in the real world. We expect that the innovative approaches presented in this study will motivate researchers working in the field of analysis, numerical analysis, and mathematical inequalities. Future research in this area is fascinating. Our theories might inspire a good number of additional studies.

Our paper is organised as follows: For multiplicative convex functions and their product, we find some new generalised multiplicative fractional H-H type inequalities in Section 2. We derive many inequalities for multiplicative convex functions relevant to generalized multiplicative fractional integrals in Section 3, utilising a new identity as an auxiliary result. In Section 4, we provide several examples for appropriate selections of multiplicative convex functions and their graphical representations to verify the veracity of our main results. In Section 5, the conclusions and recommendations for further study are provided.

2. Main Results

Now, we are in position to introduce the following definition regarding generalized multiplicative fractional integral operators.

Definition 5. *Let $\mathscr{Q} \in \mathrm{L}[\mathtt{r}, \mathtt{w}]$ and $\phi : \mathrm{Q} \to \mathrm{Q}$. The left- and right-sided generalized multiplicative fractional integrals $\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathrm{x})$ and $\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathrm{x})$ with respect to the function ϕ are defined as follows:*

$$\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathrm{x}) := \exp\left\{ \int_{\mathtt{r}}^{\mathrm{x}} \frac{\phi(\mathrm{x}-\varrho)}{\mathrm{x}-\varrho} \cdot \ln \mathscr{Q}(\varrho) d\varrho \right\} \quad (\mathrm{x} > \mathtt{r})$$

and

$$\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathrm{x}) := \exp\left\{ \int_{\mathrm{x}}^{\mathtt{w}} \frac{\phi(\varrho-\mathrm{x})}{\varrho-\mathrm{x}} \cdot \ln \mathscr{Q}(\varrho) d\varrho \right\} \quad (\mathrm{x} < \mathtt{w}), \tag{2}$$

where the function ϕ was constructed by Sarikaya et al. in [52].

Remark 1.

- *Taking $\phi(\varrho) = \varrho$ in Definition 5, we have the notion of multiplicative integral operator given by (1).*
- *Choosing $\phi(\varrho) = \frac{\varrho^{\alpha}}{\Gamma(\alpha)}$ in Definition 5, we get the notion of multiplicative Riemann–Liouville fractional integral operators given by Definition 4.*

For other suitable choices of function ϕ, for example $\varrho(\mathrm{x}-\varrho)^{\alpha-1}$, $\frac{\varrho}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}\varrho\right)$, etc., in Definition 5, we can deduce some new interesting multiplicative fractional integral operators.

The generalized multiplicative fractional H-H type inequalities for multiplicative convex functions are given as follows:

Theorem 6. *Let $\mathscr{Q} : \mathrm{Q} \to \mathrm{Q}$ be a multiplicative convex function with $\mathtt{r}, \mathtt{w} \in \mathrm{Q}^{\circ}$ and $\mathtt{r} < \mathtt{w}$, then*

$$\left[\mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]^{\Omega} \leq \sqrt{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w}) \cdot \mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})} \leq [\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{\Omega}{2}}, \tag{3}$$

where

$$\Omega := \int_0^1 \frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho} d\varrho.$$

Proof. Since $\mathscr{Q}$ is a multiplicative convex function on Q, we have

$$\begin{aligned} \mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right) &= \mathscr{Q}\left(\frac{\varrho\mathtt{r}+(1-\varrho)\mathtt{w}+(1-\varrho)\mathtt{r}+\varrho\mathtt{w}}{2}\right) \\ &\leq [\mathscr{Q}(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})]^{\frac{1}{2}} \cdot [\mathscr{Q}((1-\varrho)\mathtt{r}+\varrho\mathtt{w})]^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\ln \mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right) \leq \frac{1}{2}[\ln \mathscr{Q}(\varrho\mathtt{r}+(1-\varrho)\mathtt{w}) + \ln \mathscr{Q}((1-\varrho)\mathtt{r}+\varrho\mathtt{w})]. \tag{4}$$

Multiplying both sides of (4) by $\frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho}$, integrating the resultant inequality with respect to ϱ over $[0,1]$, and changing the variables of integration, we get

$$\ln \mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\int_0^1 \frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho}d\varrho = \Omega\cdot\left[\ln \mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]$$
$$\leq \frac{1}{2}\left[\int_0^1 \frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho}\ln \mathscr{Q}(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})d\varrho + \int_0^1 \frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho}\ln \mathscr{Q}((1-\varrho)\mathtt{r}+\varrho\mathtt{w})d\varrho\right]$$
$$= \frac{1}{2}\left[\int_{\mathtt{r}}^{\mathtt{w}} \frac{\phi(\varrho-\mathtt{r})}{\varrho-\mathtt{r}}\ln \mathscr{Q}(\varrho)d\varrho + \int_{\mathtt{r}}^{\mathtt{w}} \frac{\phi(\mathtt{w}-\varrho)}{\mathtt{w}-\varrho}\ln \mathscr{Q}(\varrho)d\varrho\right].$$

It readily yields that

$$\left[\mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]^{\Omega} \leq \sqrt{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})\cdot \mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})},$$

which shows that the left-side inequality of (3) is proved. For the proof of the right-side inequality in (3), we first note that $\mathscr{Q}$ is a is multiplicative convex function on Q; then, for $\varrho \in [0,1]$, it yields

$$\mathscr{Q}(\varrho\mathtt{r}+(1-\varrho)\mathtt{w}) \leq [\mathscr{Q}(\mathtt{r})]^{\varrho}\cdot[\mathscr{Q}(\mathtt{w})]^{1-\varrho} \tag{5}$$

and

$$\mathscr{Q}((1-\varrho)\mathtt{r}+\varrho\mathtt{w}) \leq [\mathscr{Q}(\mathtt{r})]^{1-\varrho}\cdot[\mathscr{Q}(\mathtt{w})]^{\varrho}. \tag{6}$$

By taking the logarithmic function on both sides of inequalities (5) and (6), and adding them, we obtain

$$\frac{1}{2}[\ln \mathscr{Q}(\varrho\mathtt{r}+(1-\varrho)\mathtt{w}) + \ln \mathscr{Q}((1-\varrho)\mathtt{r}+\varrho\mathtt{w})] \leq \frac{\ln \mathscr{Q}(\mathtt{r})+\ln \mathscr{Q}(\mathtt{w})}{2}. \tag{7}$$

Multiplying both sides of (7) by $\frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho}$, integrating the resultant inequality with respect to ϱ over $[0,1]$, and using the change of variables, we have the right-side inequality in (3). The proof of Theorem 6 is completed. □

Corollary 1. *Taking $\phi(\varrho)=\varrho$ in Theorem 6, we have*

$$\left[\mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]^{\mathtt{w}-\mathtt{r}} \leq \int_{\mathtt{r}}^{\mathtt{w}}(\mathscr{Q}(\mathtt{y}))^{\mathtt{dy}} \leq [\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{\mathtt{w}-\mathtt{r}}{2}}, \tag{8}$$

which is Theorem 4 established by Ali et al. ([29], Theorem 5).

Corollary 2. *Choosing $\phi(\varrho)=\frac{\varrho^{\alpha}}{\Gamma(\alpha)}$ in Theorem 6, we get*

$$\left[\mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]^{\frac{(\mathtt{w}-\mathtt{r})^{\alpha}}{\Gamma(\alpha+1)}} \leq \exp\left\{\frac{\mathrm{J}^{\alpha}_{\mathtt{r}^+}\ln \mathscr{Q}(\mathtt{w})+\mathrm{J}^{\alpha}_{\mathtt{w}^-}\ln \mathscr{Q}(\mathtt{r})}{2}\right\} \leq [\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{(\mathtt{w}-\mathtt{r})^{\alpha}}{2\Gamma(\alpha+1)}}. \tag{9}$$

Corollary 3. *Let $\mathscr{Q}_1,\mathscr{Q}_2:\mathrm{Q}\to\mathrm{Q}$ be two multiplicative convex functions; then, from Theorem 6, we obtain*

$$\left[\mathscr{Q}_1\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\cdot\mathscr{Q}_2\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]^{\Omega}$$
$$\leq \sqrt{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}_1(\mathtt{w})\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}_2(\mathtt{w})\cdot\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}_1(\mathtt{r})\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}_2(\mathtt{r})}$$
$$\leq [\mathscr{Q}_1(\mathtt{r})\mathscr{Q}_2(\mathtt{r})\cdot\mathscr{Q}_1(\mathtt{w})\mathscr{Q}_2(\mathtt{w})]^{\frac{\Omega}{2}}. \tag{10}$$

Proof. As the functions $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are positive multiplicative convex, the product function $\mathscr{Q}_1\mathscr{Q}_2$ is positive and multiplicative convex.

Indeed,

$$\mathscr{Q}_1(\varrho r + (1-\varrho)w) \leq [\mathscr{Q}_1(r)]^{\varrho} \cdot [\mathscr{Q}_1(w)]^{1-\varrho},$$

and

$$\mathscr{Q}_2(\varrho r + (1-\varrho)w) \leq [\mathscr{Q}_2(r)]^{\varrho} \cdot [\mathscr{Q}_2(w)]^{1-\varrho}.$$

Multiplying the above two inequalities, we get

$$\mathscr{Q}_1(\varrho r + (1-\varrho)w) \cdot \mathscr{Q}_2(\varrho r + (1-\varrho)w) \leq [\mathscr{Q}_1(r)\mathscr{Q}_2(r)]^{\varrho} \cdot [\mathscr{Q}_1(w)\mathscr{Q}_2(w)]^{1-\varrho}.$$

If we apply Theorem 6 to the function $\mathscr{Q}_1\mathscr{Q}_2$, then we deduce the required double inequality (10). □

Corollary 4. *Taking $\phi(\varrho) = \varrho$ in Corollary 3, we have*

$$\left[\mathscr{Q}_1\left(\frac{r+w}{2}\right) \cdot \mathscr{Q}_2\left(\frac{r+w}{2}\right)\right]^{w-r} \leq \int_r^w (\mathscr{Q}_1(y)\mathscr{Q}_2(y))^{dy} \leq [\mathscr{Q}_1(r)\mathscr{Q}_2(r) \cdot \mathscr{Q}_1(w)\mathscr{Q}_2(w)]^{\frac{w-r}{2}}. \tag{11}$$

Corollary 5. *Choosing $\phi(\varrho) = \frac{\varrho^{\alpha}}{\Gamma(\alpha)}$ in Corollary 3, we get*

$$\begin{aligned} &\left[\mathscr{Q}_1\left(\frac{r+w}{2}\right) \cdot \mathscr{Q}_2\left(\frac{r+w}{2}\right)\right]^{\frac{(w-r)^{\alpha}}{\Gamma(\alpha+1)}} \leq \exp\left\{\frac{J^{\alpha}_{r^+} \ln \mathscr{Q}_1(w)\mathscr{Q}_2(w) + J^{\alpha}_{w^-} \ln \mathscr{Q}_1(r)\mathscr{Q}_2(r)}{2}\right\} \\ &\leq [\mathscr{Q}_1(r)\mathscr{Q}_2(r) \cdot \mathscr{Q}_1(w)\mathscr{Q}_2(w)]^{\frac{(w-r)^{\alpha}}{2\Gamma(\alpha+1)}}. \end{aligned} \tag{12}$$

3. Further Results

Lemma 1. *Let $\mathscr{Q} : \mathrm{Q} \to \mathrm{Q}$ be a *differentiable function on Q° with $r, w \in \mathrm{Q}^{\circ}$ and $r < w$. If the integrable function $\mathscr{Q}^*$ is defined on $[r, w]$, then the following generalized multiplicative fractional integral equality holds true:*

$$\frac{T^{\phi}_{r^+}\mathscr{Q}(w)}{T^{\phi}_{w^-}\mathscr{Q}(r)} = [\mathscr{Q}(r)\mathscr{Q}(w)]^{\Omega(1)} \cdot \int_0^1 \left[\mathscr{Q}^*(\varrho r + (1-\varrho)w)^{\Omega(1-\varrho)-\Omega(\varrho)}\right]^{d\varrho}, \tag{13}$$

where

$$\Omega(\varrho) := \int_0^{\varrho} \frac{\phi(x(w-r))}{x} dx.$$

Proof. Let us denote

$$I := \int_0^1 \left[\mathscr{Q}^*(\varrho r + (1-\varrho)w)^{\Omega(1-\varrho)-\Omega(\varrho)}\right]^{d\varrho}.$$

Employing the multiplicative integration by parts and changing the variables of integration, we find that

$$\begin{aligned} I &= \frac{1}{[\mathscr{Q}(r)\mathscr{Q}(w)]^{\Omega(1)}} \cdot \frac{1}{\int_0^1 [\mathscr{Q}(\varrho r + (1-\varrho)w)^{-\Omega'(1-\varrho)-\Omega'(\varrho)}]^{d\varrho}} \\ &= \frac{1}{[\mathscr{Q}(r)\mathscr{Q}(w)]^{\Omega(1)}} \cdot \frac{1}{\int_0^1 \left[\mathscr{Q}(\varrho r + (1-\varrho)w)^{\frac{\phi((1-\varrho)(w-r))}{1-\varrho} - \frac{\phi(\varrho(w-r))}{\varrho}}\right]^{d\varrho}} \end{aligned}$$

$$= \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{1}{\exp\left\{ \int_0^1 \left[\frac{\phi((1-\varrho)(\mathtt{w}-\mathtt{r}))}{1-\varrho} - \frac{\phi(\varrho(\mathtt{w}-\mathtt{r}))}{\varrho} \right] \ln \mathscr{Q}(\varrho\mathtt{r}+(1-\varrho)\mathtt{w}) d\varrho \right\}}$$
$$= \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{1}{\exp\left\{ \int_{\mathtt{r}}^{\mathtt{w}} \frac{\phi(\varrho-\mathtt{r})}{\varrho-\mathtt{r}} \ln \mathscr{Q}(\varrho) d\varrho - \int_{\mathtt{r}}^{\mathtt{w}} \frac{\phi(\mathtt{w}-\varrho)}{\mathtt{w}-\varrho} \ln \mathscr{Q}(\varrho) d\varrho \right\}}$$
$$= \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})}{\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})}.$$

Multiplying the above equality by the factor $[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}$, we obtain the desired result (13). □

Remark 2.

(a) *Taking $\phi(\varrho) = \varrho$ in Lemma 1, we have*

$$[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\mathtt{w}-\mathtt{r}} \cdot \int_0^1 \left[\mathscr{Q}^*(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})^{(\mathtt{w}-\mathtt{r})(1-2\varrho)} \right]^{d\varrho} = 1. \tag{14}$$

(b) *Choosing $\phi(\varrho) = \frac{\varrho^\alpha}{\Gamma(\alpha)}$ in Lemma 1, we get*

$$\frac{{}_{\mathtt{r}}\mathrm{I}^{\alpha}_{*}\mathscr{Q}(\mathtt{w})}{{}_{*}\mathrm{I}^{\alpha}_{\mathtt{w}}\mathscr{Q}(\mathtt{r})} = [\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{(\mathtt{w}-\mathtt{r})^\alpha}{\Gamma(\alpha+1)}} \cdot \int_0^1 \left[\mathscr{Q}^*(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})^{\frac{(\mathtt{w}-\mathtt{r})^\alpha}{\Gamma(\alpha+1)}((1-\varrho)^\alpha-\varrho^\alpha)} \right]^{d\varrho}. \tag{15}$$

Theorem 7. *Let $\mathscr{Q} : \mathrm{Q} \to \mathrm{Q}$ be a *differentiable function on Q° with $\mathtt{r}, \mathtt{w} \in \mathrm{Q}^\circ$ and $\mathtt{r} < \mathtt{w}$. If the integrable function $|\mathscr{Q}^*|^{\mathrm{q}}$ defined on $[\mathtt{r}, \mathtt{w}]$ is multiplicative convex with $\mathrm{p} > 1$ and $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$, then*

$$\left| \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})}{\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})} \right| \le \exp\left\{ \mathrm{C}^{\frac{1}{\mathrm{p}}}(\Omega, \mathrm{p}) \cdot \left(\frac{\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + \ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}}{2} \right)^{\frac{1}{\mathrm{q}}} \right\}, \tag{16}$$

where

$$\mathrm{C}(\Omega, \mathrm{p}) := \int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)|^{\mathrm{p}} d\varrho.$$

Proof. By using Lemma 1, Hölder's inequality, the multiplicative convexity of $|\mathscr{Q}^*|^{\mathrm{q}}$ on Q, changing the variables of integration, and the properties of the modulus, we have

$$\begin{aligned}
&\left| \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})}{\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})} \right| \\
&= \left| \int_0^1 \left[\mathscr{Q}^*(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})^{\Omega(1-\varrho)-\Omega(\varrho)} \right]^{d\varrho} \right| \\
&\le \exp\left\{ \int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)| |\ln \mathscr{Q}^*(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})| d\varrho \right\} \\
&\le \exp\left\{ \left(\int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)|^{\mathrm{p}} d\varrho \right)^{\frac{1}{\mathrm{p}}} \cdot \left(\int_0^1 |\ln \mathscr{Q}^*(\varrho\mathtt{r}+(1-\varrho)\mathtt{w})|^{\mathrm{q}} d\varrho \right)^{\frac{1}{\mathrm{q}}} \right\} \\
&\le \exp\left\{ \mathrm{C}^{\frac{1}{\mathrm{p}}}(\Omega, \mathrm{p}) \cdot \left(\int_0^1 [\varrho \ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + (1-\varrho)\ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}] d\varrho \right)^{\frac{1}{\mathrm{q}}} \right\} \\
&= \exp\left\{ \mathrm{C}^{\frac{1}{\mathrm{p}}}(\Omega, \mathrm{p}) \cdot \left(\frac{\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + \ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}}{2} \right)^{\frac{1}{\mathrm{q}}} \right\}.
\end{aligned}$$

This concludes the proof. □

Corollary 6. *Taking $\phi(\varrho) = \varrho$ in Theorem 7 and using Remark 2(a), we have*

$$\left|\frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\mathtt{w}-\mathtt{r}}}\right| \leq \exp\left\{\frac{(\mathtt{w}-\mathtt{r})}{(\mathrm{p}+1)^{\frac{1}{\mathrm{p}}}} \cdot \left(\frac{\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + \ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}}{2}\right)^{\frac{1}{\mathrm{q}}}\right\}. \tag{17}$$

Corollary 7. *Choosing $\phi(\varrho) = \frac{\varrho^\alpha}{\Gamma(\alpha)}$ in Theorem 7 and using Remark 2(b), we get*

$$\begin{aligned}&\left|\frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{(\mathtt{w}-\mathtt{r})^\alpha}{\Gamma(\alpha+1)}}} \cdot \frac{{}_{\mathtt{r}}\mathrm{I}^{\alpha}_{*}\mathscr{Q}(\mathtt{w})}{{}_{*}\mathrm{I}^{\alpha}_{\mathtt{w}}\mathscr{Q}(\mathtt{r})}\right| \\ &\leq \exp\left\{\frac{(\mathtt{w}-\mathtt{r})^\alpha}{\Gamma(\alpha+1)}\mathrm{F}^{\frac{1}{\mathrm{p}}}(\mathrm{p},\alpha) \cdot \left(\frac{\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + \ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}}{2}\right)^{\frac{1}{\mathrm{q}}}\right\},\end{aligned} \tag{18}$$

where

$$\mathrm{F}(\mathrm{p},\alpha) := \int_0^1 |(1-\varrho)^\alpha - \varrho^\alpha|^{\mathrm{p}} d\varrho.$$

Theorem 8. *Let $\mathscr{Q} : \mathrm{Q} \to \mathrm{Q}$ be a *differentiable function on Q° with $\mathtt{r}, \mathtt{w} \in \mathrm{Q}^\circ$ and $\mathtt{r} < \mathtt{w}$. If the integrable function $|\mathscr{Q}^*|^{\mathrm{q}}$ defined on $[\mathtt{r}, \mathtt{w}]$ is multiplicative convex with $\mathrm{q} \geq 1$, then*

$$\begin{aligned}&\left|\frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})}{\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})}\right| \\ &\leq \exp\left\{\mathrm{C}^{1-\frac{1}{\mathrm{q}}}(\Omega) \cdot [\mathrm{D}(\Omega)\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + (\mathrm{C}(\Omega) - \mathrm{D}(\Omega))\ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}]^{\frac{1}{\mathrm{q}}}\right\},\end{aligned} \tag{19}$$

where

$$\mathrm{C}(\Omega) := \int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)| d\varrho, \quad \mathrm{D}(\Omega) := \int_0^1 \varrho|\Omega(1-\varrho) - \Omega(\varrho)| d\varrho.$$

Proof. By using Lemma 1, the power mean inequality, the multiplicative convexity of $|\mathscr{Q}^*|^{\mathrm{q}}$ on Q, changing the variables of integration, and the properties of the modulus, we have

$$\begin{aligned}&\left|\frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})}{\mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})}\right| \\ &= \left|\int_0^1 \left[\mathscr{Q}^*(\varrho\mathtt{r} + (1-\varrho)\mathtt{w})^{\Omega(1-\varrho)-\Omega(\varrho)}\right]^{d\varrho}\right| \\ &\leq \exp\left\{\int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)||\ln\mathscr{Q}^*(\varrho\mathtt{r} + (1-\varrho)\mathtt{w})|d\varrho\right\} \\ &\leq \exp\left\{\left(\int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)|d\varrho\right)^{1-\frac{1}{\mathrm{q}}} \cdot \left(\int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)||\ln\mathscr{Q}^*(\varrho\mathtt{r} + (1-\varrho)\mathtt{w})|^{\mathrm{q}}d\varrho\right)^{\frac{1}{\mathrm{q}}}\right\} \\ &\leq \exp\left\{\mathrm{C}^{1-\frac{1}{\mathrm{q}}}(\Omega) \cdot \left(\int_0^1 |\Omega(1-\varrho) - \Omega(\varrho)|\left[\varrho\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + (1-\varrho)\ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}\right]d\varrho\right)^{\frac{1}{\mathrm{q}}}\right\} \\ &= \exp\left\{\mathrm{C}^{1-\frac{1}{\mathrm{q}}}(\Omega) \cdot [\mathrm{D}(\Omega)\ln|\mathscr{Q}^*(\mathtt{r})|^{\mathrm{q}} + (\mathrm{C}(\Omega) - \mathrm{D}(\Omega))\ln|\mathscr{Q}^*(\mathtt{w})|^{\mathrm{q}}]^{\frac{1}{\mathrm{q}}}\right\}.\end{aligned}$$

This concludes the proof. □

Corollary 8. *Taking* $q = 1$ *in Theorem 8, we have*

$$\left| \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\Omega(1)}} \cdot \frac{T^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w})}{T^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})} \right| \le \exp\{D(\Omega)\ln|\mathscr{Q}^*(\mathtt{r})| + (C(\Omega) - D(\Omega))\ln|\mathscr{Q}^*(\mathtt{w})|\}. \tag{20}$$

Corollary 9. *Choosing* $\phi(\varrho) = \varrho$ *in Theorem 8 and using Remark 2(a), we get*

$$\left| \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\mathtt{w}-\mathtt{r}}} \right| \le \exp\left\{ \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (\mathtt{w}-\mathtt{r}) \cdot [\ln|\mathscr{Q}^*(\mathtt{r})|^q + \ln|\mathscr{Q}^*(\mathtt{w})|^q]^{\frac{1}{q}} \right\}. \tag{21}$$

Corollary 10. *Letting* $\phi(\varrho) = \frac{\varrho^\alpha}{\Gamma(\alpha)}$ *in Theorem 8 and using Remark 2(b), we obtain*

$$\left| \frac{1}{[\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{(\mathtt{w}-\mathtt{r})^\alpha}{\Gamma(\alpha+1)}}} \cdot \frac{{}_{\mathtt{r}}I^{\alpha}_{*}\mathscr{Q}(\mathtt{w})}{{}_{*}I^{\alpha}_{\mathtt{w}}\mathscr{Q}(\mathtt{r})} \right| \le \exp\left\{ C^{1-\frac{1}{q}}(\alpha)\frac{(\mathtt{w}-\mathtt{r})^\alpha}{\Gamma(\alpha+1)} \cdot [D(\alpha)\ln|\mathscr{Q}^*(\mathtt{r})|^q + (C(\alpha)-D(\alpha))\ln|\mathscr{Q}^*(\mathtt{w})|^q]^{\frac{1}{q}} \right\}, \tag{22}$$

where

$$C(\alpha) := \int_0^1 |(1-\varrho)^\alpha - \varrho^\alpha| d\varrho, \qquad D(\alpha) := \int_0^1 \varrho|(1-\varrho)^\alpha - \varrho^\alpha| d\varrho.$$

Remark 3. *For suitable choices of the function* ϕ*, for example* $\varrho(x-\varrho)^{\alpha-1}$, $\frac{\varrho}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}\varrho\right)$*, etc., in Theorems 6–8, we can deduce some new interesting integral inequalities. We omit their proofs and the details are left to the interested reader.*

4. Numerical and Graphical Computations

Before starting this section, it is easy to prove that the functions $\mathscr{Q}_1(x) = \sqrt{\frac{1}{x}}$ and $|\mathscr{Q}_2{}^*(x)| = \exp\{\frac{2}{x}\}$ for all $x > 0$ are multiplicative convex. Using this fact and Theorems 6–8 above, we have the following numerical results and their graphical representations to verify the correctness of the derived findings given in Table 1, Table 2 and Table 3 for Figure 1, Figure 2 and Figure 3, respectively.

Table 1. Numerical validation of Theorem 6 for $\mathtt{r} = 1$, $\mathtt{w} = 2$, $\mathscr{Q}_1(x) = \sqrt{\frac{1}{x}}$, *and* $\phi(\varrho) = \frac{\varrho^\alpha}{\Gamma(\alpha)}$.

α	Values of the Left Term	Values of the Middle Term	Values of the Right Term
0.1	0.808069	0.828942	0.833478
0.2	0.801869	0.819867	0.828009
0.3	0.797796	0.81346	0.824413
0.4	0.795726	0.809497	0.822584
0.5	0.795513	0.807751	0.822396
0.6	0.796999	0.807991	0.823709
0.7	0.800014	0.809991	0.826372
0.8	0.804385	0.813529	0.83023
0.9	0.809936	0.818388	0.835123

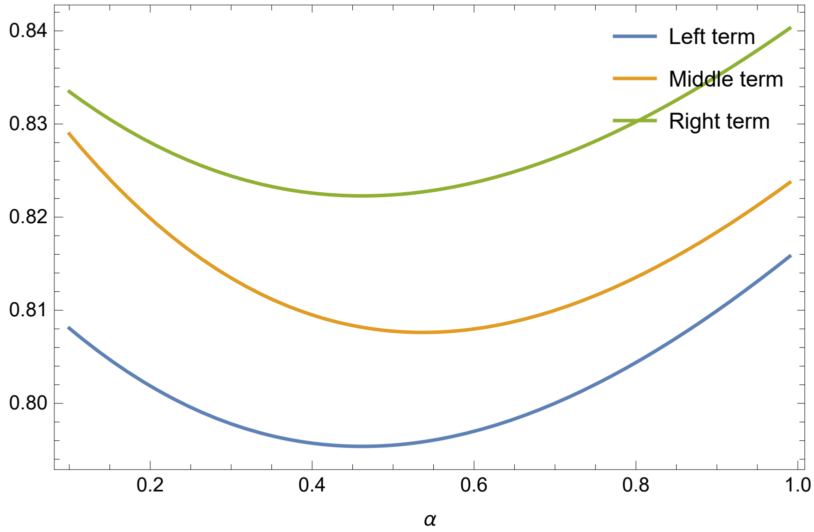

Figure 1. Graphical behaviour of Theorem 6 for $\mathtt{r} = 1$, $\mathtt{w} = 2$, $\mathscr{Q}_1(\mathrm{x}) = \sqrt{\frac{1}{\mathrm{x}}}$, $\phi(\varrho) = \frac{\varrho^{\alpha}}{\Gamma(\alpha)}$, and $0 < \alpha < 1$.

Table 2. Numerical validation of Theorem 7 for $\mathtt{r} = 1$, $\mathtt{w} = 2$, $|\mathscr{Q}_2{}^*(\mathrm{x})| = \exp\{\frac{2}{\mathrm{x}}\}$, *and* $\phi(\varrho) = \frac{\varrho^{\alpha}}{\Gamma(\alpha)}$.

α	Values of the Left Term	Values of the Right Term
0.1	0.765017	1.32413
0.2	0.601802	1.65435
0.3	0.487657	1.96873
0.4	0.407373	2.24682
0.5	0.350791	2.47288
0.6	0.311124	2.63777
0.7	0.283816	2.73922
0.8	0.2658	2.78087
0.9	0.254999	2.77053

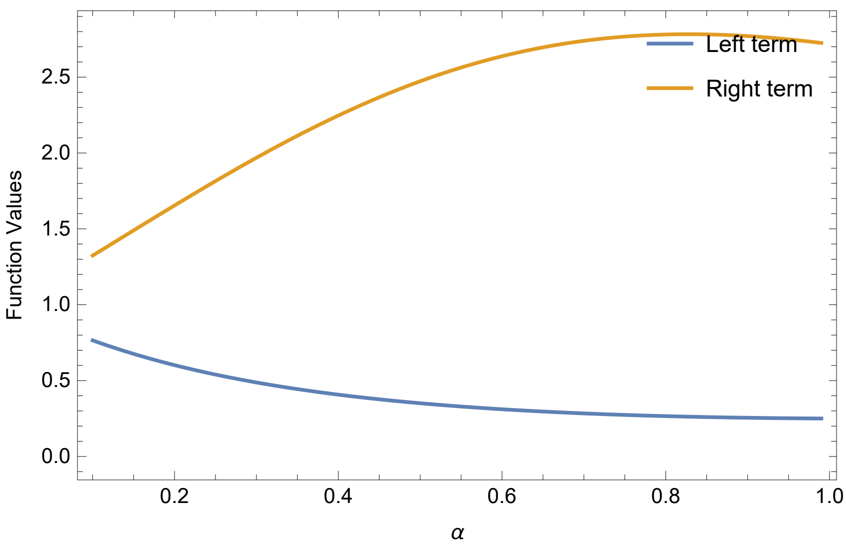

Figure 2. Graphical behaviour of Theorem 7 for $\mathtt{r} = 1$, $\mathtt{w} = 2$, $|\mathscr{Q}_2{}^*(\mathrm{x})| = \exp\{\frac{2}{\mathrm{x}}\}$, $\phi(\varrho) = \frac{\varrho^{\alpha}}{\Gamma(\alpha)}$, and $0 < \alpha < 1$.

Table 3. Numerical validation of Theorem 8 for $\mathtt{r}=1$, $\mathtt{w}=2$, $|\mathscr{Q}_2{}^*(\mathrm{x})| = \exp\{\frac{2}{\mathrm{x}}\}$, *and* $\phi(\varrho) = \frac{\varrho^\alpha}{\Gamma(\alpha)}$.

α	Values of the Left Term	Values of the Right Term
0.1	0.765017	1.24817
0.2	0.601802	1.50229
0.3	0.487657	1.7462
0.4	0.407373	1.96457
0.5	0.350791	2.14523
0.6	0.311124	2.28061
0.7	0.283816	2.36816
0.8	0.2658	2.40973
0.9	0.254999	2.41046

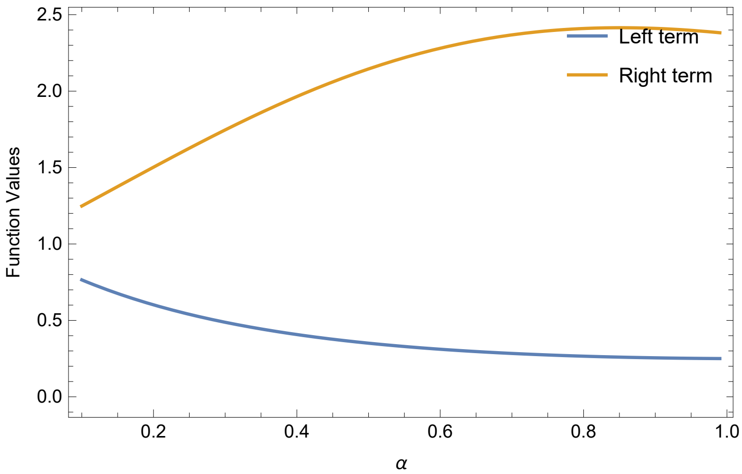

Figure 3. Graphical behaviour of Theorem 8 for $\mathtt{r}=1$, $\mathtt{w}=2$, $|\mathscr{Q}_2{}^*(\mathrm{x})| = \exp\{\frac{2}{\mathrm{x}}\}$, $\phi(\varrho) = \frac{\varrho^\alpha}{\Gamma(\alpha)}$, and $0 < \alpha < 1$.

From the above figure, it is clear that for $0 < \alpha < 1$, the inequalities of Theorem 6 are satisfied. The green line represents the right-side inequality $\left([\mathscr{Q}(\mathtt{r})\mathscr{Q}(\mathtt{w})]^{\frac{\Omega}{2}}\right)$, the yellow line represents the middle inequality $\left(\sqrt{\mathrm{T}^{\phi}_{\mathtt{r}^+}\mathscr{Q}(\mathtt{w}) \cdot \mathrm{T}^{\phi}_{\mathtt{w}^-}\mathscr{Q}(\mathtt{r})}\right)$, and the blue line represents the left-side inequality $\left(\left[\mathscr{Q}\left(\frac{\mathtt{r}+\mathtt{w}}{2}\right)\right]^{\Omega}\right)$. Thus, it shows that the inequality holds true for different values of α.

5. Conclusions

First, a generalized multiplicative fractional integral operator was introduced. Then, for multiplicative convex functions and their products, some new generalized multiplicative fractional H-H inequalities were constructed in this study. Additionally, we derived a number of inequalities for multiplicative convex functions related to generalized multiplicative fractional integrals utilizing a novel identity as an auxiliary result. To validate the accuracy of our main results, we gave some examples for suitable choices of multiplicative convex functions and their graphical representations. Some future aspects of this concept could be defining interval-valued multiplicative convex functions and establishing new kinds of interval-valued inequalities. A novel approach on this concept could be linking the multiplicative convex functions to coordinates i.e. establishing new generalized multiplicative coordinated convex functions and related inequalities. We believe that these novel definitions can be treated also using quantum calculus.

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