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INFORMATION, STABILITY AND DYNAMICS IN NETWORKS UNDER INSTITUTIONAL CONSTRAINTS (I)

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Information, stability and dynamics in networks under institutional constraints (I)[*]

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Abstract

In this paper we study the effects of institutional constraints on stability, efficiency and network formation. More precisely, an exogenous "societal cover" consisting of a collection of possibly overlapping subsets that covers the whole set of players and such that no set in this collection is contained in another specifies the social organization in different groups or "societies". It is assumed that a player may initiate links only with players that belong to at least one society that s/he also belongs to, thus restricting the feasible strategies and networks. In this way only the players in the possibly empty "societal core", i.e., those that belong to all societies, may initiate links with all individuals. In this setting the part of the current network within each connected component of the cover is assumed to be common knowledge to all players in that component. Based on this two-ingredient model, network and societal cover, we examine the impact of societal constraints on stable/efficient architectures and on dynamics.

Journal of Economic Literature Classification Numbers: C72

Key words: Network, Non-cooperative game, Dynamics.

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1 Introduction

In recent years the study of the economics of networks has attracted considerable attention from researchers and become one of the hottest topics of economic research[1]. The economics of networks is, in Goyal's words, "*an ambitious research program which combines aspects of markets (e.g., prices and competition) along with explicit patterns of connections between individual entities to explain economic phenomena*" (Goyal, 2007, p. 6).

Several seminal papers study the stability and efficiency of networks providing the basic models. In the simplest model links are formed unilaterally (Goyal (1993), Bala (1996)). In this setting Bala and Goyal (2000a) study Nash stability and provide a dynamic model. A model where links are formed on the basis of bilateral agreements is studied by Jackson and Wolinsky (1996), who introduce the notion of pairwise stability. In these seminal papers it is assumed that there is homogeneity across players and also that the current network is common knowledge to all node-players. Galeotti et al. (2006) consider heterogeneous players, while Bloch and Dutta (2009) consider endogenous link strength. The common knowledge assumption may be unrealistic in many cases, and indeed is dropped by McBride (2006), who studies the effects of limited perception, namely, assuming that each node-player perceives the current network only up to a certain distance from the node.

In the seminal models networks provide a means for the flow of information or other benefits through the links, but the current network is assumed to be common knowledge to all players, who may unrestrictedly initiate links with any other players. In some cases this may be an unrealistic assumption, and in general the larger the network is the more unrealistic it will be. It seems more realistic to assume that because they belong to the same group (family, club, professional association, department, etc.) individuals may have a clear idea of the connections within such smaller groups and initiate links only within the groups they belong to. Moreover, an individual may belong to more than one of these groups, sharing common knowledge of the links connecting members of each group. In a way this is an unorthodox approach if, as put by Goyal, "*the theoretical research on network effects (..) is motivated by the idea that,* within the same group [in italics]*, individuals will have different connections and that this difference in connections will have a bearing on their behavior.*" (Goyal, 2007, p. 7). Nevertheless, this is the approach adopted here, and it is worth remarking that the orthodox single-group assumption is in fact a particular case of the more general setting adopted here. In particular, this allows Bala and Goyal's (2000a) "two-way flow" basic model, on which we concentrate in this paper, to be integrated into a wider model which sheds new light on various conclusions of their model, showing which prevail and up to which point, and which do not in this wider setting.

Based on this idea, in this paper we focus on the effects of institutional and/or in-

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[1]Some recent books surveying this literature are Goyal (2007), Jackson (2008) and Vega-Redondo (2007).

formational constraints on stability, efficiency and network formation. More precisely, an exogenous "societal cover" specifies social organization in different groups or "societies". A societal cover is a collection of possibly overlapping subsets of the set of players or "societies" that covers the whole set (i.e., each player belongs to at least one set in this collection) such that no set in this collection is contained in another. It is assumed that a player may initiate links only with players that belong to one or more of the societies that s/he also belongs to, thus restricting her/his feasible strategies, and as a consequence the feasible networks. Note that in this scenario only the players in the possibly empty "societal core", i.e., those that belong to all societies, may have direct access to all individuals. It is also assumed that only the part of the current network within each "component" (in a sense to be specified later) *of the societal cover* is common knowledge to all players in that "component". Note also that this model collapses to Bala and Goyal's (2000a) unrestricted setting for the particular case of the trivial societal cover consisting of a single society including all players.

Based on this two-ingredient model -network and societal cover- we examine the impact of the societal constraints, which are interpreted in general terms as institutional constraints, on stable/efficient architectures and on dynamics.

For any given societal cover we constrain our attention to the admissible networks (i.e., those consistent with the cover) and first extend Bala and Goyal's (2000a) notion of a Nash network as those admissible networks where no player has an incentive to change her/his strategy, i.e., her/his choice of admissible links. We then extend their characterization of Nash networks as those among the admissible networks which are minimally connected. In this way the set of such Nash networks is a subset of the set of Bala and Goyal's unrestricted Nash networks. Then Bala and Goyal's (2000a) notion of strict Nash network is also naturally extended to this setting. Now a strict Nash network is a network consistent with the societal cover where no player may initiate and/or delete any admissible link(s) without loss. By contrast with Nash networks, things turn out to be much more complicated with *strict* Nash networks. In Bala and Goyal's setting the center-sponsored star is the only (non empty) architecture of strict Nash networks, while in our setting the center-sponsored architecture is feasible only when the societal core, i.e. the set of players belonging to all societies, is not empty. Moreover, even when the center-sponsored star architecture is feasible it is not the only possible architecture of strict Nash networks. A variety of architectures of strict Nash networks appear for any non single-society cover, and the more complex the societal cover the greater this variety is. Nevertheless, some patterns are common to these architectures. Moreover, a full characterization of all strict Nash networks for a societal cover is provided by means of a condition that encapsulates synthetically the essence of the architecture of these networks, embodying an implicit form of hierarchical principle. The main features of their architectures, where stars continue to play a prominent role, are studied. Particular attention is paid to the role of players who belong to more than one society, by means of whom different but overlapping societies can be connected. It turns out that strict Nash networks incorporate a clear hierarchical structure: They

are either oriented trees (also called "arborescences" in graph theory) or a sort of "grafted" oriented trees or arborescences. The latter are possible only when there are "hinge-players", i.e., players who are the *unique* common member of two societies.

Finally we extend Bala and Goyal's dynamic model, where starting from any initial network each player with some positive probability plays a best response or randomizes across them when there is more than one, otherwise the player exhibits inertia, i.e., keeps his/her links unchanged. In this way a Markov chain on the state space of all networks is defined. In Bala and Goyal's setting, the absorbing states are precisely the strict Nash networks and they prove that starting from any network the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. While when adapted to our setting the best response dynamic model *does not* necessarily lead to strict Nash networks. The reason is that in our more complex setting this dynamic process may lead to the formation of stable "incomplete" strict Nash networks that cannot be part of the same "general" strict Nash network. Thus the same logic that in their setting leads to the absorbing strict Nash networks, may lead in ours to stable "incomplete" strict Nash incompatible networks that block the converging process. Thus, in a way, institutional constraints may hinder the way towards strict Nash networks. Nevertheless, best response dynamics lead to something very close to a strict Nash network that we call "*quasi strict Nash networks*", if not to a strict Nash network. These constitute absorbing sets of minimally connected networks, closed with respect to best response dynamics, and fully connected by those dynamics in the sense that any network in one of these sets is reachable from any other in the same set by best response dynamics. Thus, with probability 1, best response dynamics would lead either to a strict Nash network (whenever the set of quasi strict Nash networks reached is a singleton) or one of these sets of quasi strict Nash networks where the best response dynamics would oscillate for ever. Nevertheless this is not a serious drawback because stability is reached *in terms of payoffs* as it is proved they all quasi strict Nash networks within each of these sets yield the same payoffs to all players.

The rest of the paper is organized as follows. In section 2 the basic model is specified along with the necessary notation and terminology. Section 3 studies stability and efficiency under institutional constraints. In section 4 Bala and Goyal's dynamic model is extended to this setting. Finally, section 5 summarizes the main conclusions and points out some lines of further research.

2   The model

Let $N = \{1, 2, .., n\}$ denote the set of *nodes* or *players*. Players may initiate or delete *links* with other players. By $g_{ij} \in \{0, 1\}$ we denote the existence ($g_{ij} = 1$) or not ($g_{ij} = 0$) of a link connecting i and j initiated by i. Vector $g_i = (g_{ij})_{j \in N \setminus i} \in \{0, 1\}^{N \setminus i}$

specifies[2] the set of links initiated by i and will be referred to as an (unrestricted) *strategy* of player i. $G_i := \{0,1\}^{N \setminus i}$ denotes the set of i's (unrestricted) strategies and $G_N = G_1 \times G_2 \times .. \times G_n$ the set of (unrestricted) strategy profiles. An unrestricted strategy profile g univocally determines a directed *network*[3] that we identify with g

$$g = \{(i,j) \in N \times N : g_{ij} = 1\}.$$

Given an network g, and $M \subseteq N$ we denote by $g \mid_M$ the network that results by restricting g to M, more precisely

$$g \mid_M := \{(i,j) \in M \times M : g_{ij} = 1\}.$$

We now consider the following situation. An exogenous "societal cover" specifies a set of possibly overlapping "societies" that represent a social constraint in the following sense: Each player in N can initiate links with any other player as long as they belong to the same society. Formally, we have the following

Definition 1 *A "societal cover" of N is a collection of subsets of N (called "societies"), $\mathcal{K} \subseteq 2^N$, such that: (i)* $\bigcup_{A \in \mathcal{K}} A = N$*, and (ii) for all $A, B \in \mathcal{K}$ ($A \neq B$), $A \not\subseteq B$.*

Condition *(i)* ensures that every player belongs to at least one society; while condition *(ii)* precludes superfluous societies: if $A \subseteq B$, A would be superfluous given the interpretation of societies.

The following notation and terminology is useful. We denote by $\mathcal{K}_i$ the set of societies that i belongs to, and by $N(\mathcal{K}_i)$ the set of nodes that i may directly access, that is:

$$\mathcal{K}_i := \{A \in \mathcal{K} : i \in A\}$$

and

$$N(\mathcal{K}_i) := \bigcup_{A \in \mathcal{K}_i} A.$$

Two nodes i, j have *identical affiliation* if they belong to the same societies, i.e., $\mathcal{K}_i = \mathcal{K}_j$. Two nodes i, j have the *same reach* if $N(\mathcal{K}_i) = N(\mathcal{K}_j)$. Note that *identical affiliation* implies the *same reach*, but the converse is not true.

Example: If $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$$\mathcal{K} := \{\{1,2,3,4,5,6\}, \{4,5,6,7,8,9\}, \{1,2,4,5,7,8\}, \{2,3,5,6,8,9\}\},$$

then 2 and 4 have the *same reach*: $N(\mathcal{K}_2) = N(\mathcal{K}_4) = N$, but *different affiliations* as $\mathcal{K}_2 \neq \mathcal{K}_4$:

$$\mathcal{K}_2 = \{\{1,2,3,4,5,6\}, \{1,2,4,5,7,8\}, \{2,3,5,6,8,9\}\}$$

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[2]We always drop the brackets "{..}" in expresions such as $N \setminus \{i\}$.

[3]In graph theory this is called a "digraph" without loops, i.e., edges connecting a node with itself (see, for instance, Tutte (1984)).

and
$$\mathcal{K}_4 = \{\{1,2,3,4,5,6\}, \{1,2,4,5,7,8\}, \{4,5,6,7,8,9\}\}.$$

The following terminology is used. A *component* $\mathcal{C}$ of a societal cover $\mathcal{K}$ is a subset $\mathcal{C} \subseteq \mathcal{K}$ such that *(i)* for all $A, B \in \mathcal{C}$ there exist $A_1, .., A_k \in \mathcal{K}$ s.t. $A_1 = A$ and $B = A_k$, and $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, .., k-1$, and *(ii)* for all $B \in \mathcal{K} \backslash \mathcal{C}$, $B \cap (\cup_{A \in \mathcal{C}} A) = \varnothing$. The subset $\cup_{A \in \mathcal{C}} A$ of N covered by a component $\mathcal{C}$ is denoted by $N(\mathcal{C})$. For each i, $\mathcal{C}_i(\mathcal{K})$ denotes the component of $\mathcal{K}$ that contains $\mathcal{K}_i$. A societal cover is *connected* if it has a unique component. The *societal core* of a societal cover is the set of nodes that belong to all societies

$$core(\mathcal{K}) := \bigcap_{A \in \mathcal{K}} A.$$

This set may be empty. Note that only the players in the societal core may have direct access to all individuals in N.

Let $\mathcal{K}$ be a societal cover of N, if $\mathcal{K}' \subseteq \mathcal{K}$ we say that $\mathcal{K}'$ is a *subcover* of $\mathcal{K}$ if $\mathcal{K}'$ is a societal cover of $N(\mathcal{K}') := \cup_{A \in \mathcal{K}'} A$ s.t. for all $A \in \mathcal{K}$, $A \subseteq N(\mathcal{K}')$ implies $A \in \mathcal{K}'$. In particular, a component of a a societal cover $\mathcal{K}$ is a (connected) subcover of $\mathcal{K}$.

The following definition constrains the structure of a network so as to be consistent with a given societal cover of N by ruling out links connecting individuals who are not members of at least one society in common.

Definition 2 *A network g is consistent with a societal cover $\mathcal{K}$ (or is a $\mathcal{K}$-network) if for every link $g_{ij} = 1$ there exists some $A \in \mathcal{K}$ s.t. $i, j \in A$ (i.e., $\mathcal{K}_i \cap \mathcal{K}_j \neq \varnothing$).*

A vector $g_i = (g_{ij})_{j \in N(\mathcal{K}_i) \backslash i} \in \{0,1\}^{N(\mathcal{K}_i) \backslash i}$ specifies a set of $\mathcal{K}$-*feasible* links initiated by i and is referred to as a $\mathcal{K}$-*admissible strategy* of player i, as we assume i's capacity to choose which links to initiate *in $N(\mathcal{K}_i)$*. $G_i(\mathcal{K}) := \{0,1\}^{N(\mathcal{K}_i) \backslash i}$ denotes the set of i's $\mathcal{K}$-admissible strategies and $G_{\mathcal{K}} = G_1(\mathcal{K}) \times G_2(\mathcal{K}) \times .. \times G_n(\mathcal{K})$ the set of $\mathcal{K}$-admissible strategy profiles. A $\mathcal{K}$-admissible strategy profile g univocally determines a $\mathcal{K}$-network that we identify with g.

Observe that this setting is *not* narrower than Bala and Goyal's standard one. It is in fact more general as the standard (i.e., unrestricted) notions of network, strategy and strategy profile correspond to the particular case of the simplest societal cover $\mathcal{K} = \{N\}$, where a single society includes all players and all links are feasible.

Given a network g, we denote $\bar{g}_{ij} := max\{g_{ij}, g_{ji}\}$. In this way a nondirected network $\bar{g}$ is defined[4]. $\bar{g}$ represents the communication provided by network g, which is independent of who initiated the existing links according to the assumptions of the model. We denote by $N(g)$ the set of non isolated nodes, that is,

$$N(g) := \{i \in N : \bar{g}_{ij} \neq 0 \text{ for some } j \in N\}.$$

We say that there is a *path* from i to j in g if there exist players $j_1, .., j_k$, s.t. $i = j_1$, $j = j_k$, and for all $l = 1, .., k-1$, $\bar{g}_{j_l j_{l+1}} = 1$, and we say that such a path is *i-oriented*

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[4]In graph theory terms, $\bar{g}$ is the "underlying graph" of digraph g (see, e.g., Tutte, 1984).

if for all $l = 1, .., k - 1$, $g_{j_l j_{l+1}} = 1$. A path (directed or not) is $\mathcal{K}$-feasible if all its links are $\mathcal{K}$-feasible. The set of players with whom i initiated a link is denoted by $N^d(i; g)$, and the set of players connected with i by a path (union $\{i\}$) by $N(i; g)$, and their cardinalities by $\mu_i^d(g) := \#N^d(i; g)$ and $\mu_i(g) := \#N(i; g)$. Note that if g is a $\mathcal{K}$-network then $N^d(i; g) \subseteq N(\mathcal{K}_i)$ and $N(i; g) \subseteq \cup_{A \in \mathcal{C}_i(\mathcal{K})} A$. We say that a network g is an *arborescence* or an *oriented tree* if there is a node i_0 such that for any other node there is a unique i_0-*oriented* path connecting it with the *node root* i_0.

It is assumed that each node contains valuable information and a link allows that information to flow in both directions without decay independently of who initiates it, so that each node receives the information from all nodes with which it is connected by a path. Let $v_{ij} > 0$ be the payoff that player i derives from connecting directly (by a link) or indirectly (by a path) with player j, and $c_{ij} > 0$ the cost for player i of initiating a link with j. Thus the payoff of player i in g is

$$\Pi_i(g) = \sum_{j \in N(i;g)} v_{ij} - \sum_{j \in N^d(i;g)} c_{ij}.$$

If we assume costs and benefits to be homogeneous across players (i.e., $v_{ij} = v$ and $c_{ij} = c$, for all i, j) and $v > c$, connections with *new* nodes are always profitable and[5]

$$\Pi_i(g) = v\mu_i(g) - c\mu_i^d(g).$$

A $\mathcal{K}$-network is *efficient* if it maximizes the aggregate payoff under the constraint of $\mathcal{K}$-feasible payoffs, that is, those that can be obtained by means of $\mathcal{K}$-networks. As a term of comparison we sometimes consider standard unrestricted efficiency to which we refer as efficient networks.

A *component* of a network g is a set $C(g) \subseteq N$ such that any two players in $C(g)$ are connected by a path, and no player in $N \setminus C(g)$ is connected by a path with a player in $C(g)$. We say g is *connected* if N is the unique component of g. A network is *minimal* if for all i, j s.t. $g_{ij} = 1$, the number of components of g is smaller than the number of components of $g - g_{ij}$, where $g - g_{ij}$ is the network that results by replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g (similarly, when $g_{ij} = 0$ we write $g + 1_{ij}$ to represent the network that results by replacing $g_{ij} = 0$ by $g_{ij} = 1$ in g).

Remark: Note the relationship between the notions of connected component of a societal cover $\mathcal{K}$ of N and connected component of a $\mathcal{K}$-network: a connected component of a $\mathcal{K}$-network is always covered by a connected component of the societal cover $\mathcal{K}$.

We denote by g_{-i} the network where all links initiated by i are deleted, and by (g_{-i}, g_i') the strategy profile and network that results by replacing g_i by g_i' in g. In particular, $(g_{-i}, g_i) = g$.

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[5] Although the results presented here can easily be extended with some slight modifications to the case where payoffs are, as in Bala and Goyal (2000a), given by a function $\Phi(\mu_i(g), \mu_i^d(g))$, where $\Phi(x, y)$ is strictly increasing in x and strictly decreasing in y, we prefer this simpler assumption about payoffs so as to make the statements of the basic results simpler.

We next discuss some notions of stability of networks consistent with a given societal cover $\mathcal{K}$.

3 Stability and efficiency

The following definitions are natural extensions of the notions of Nash stability and strict Nash stability due to Bala and Goyal (2000a) for a network in a scenario where: (i) a societal cover $\mathcal{K}$ allows only for links connecting individuals belonging to the same society, and (ii) all players in a same component $\mathcal{C}$ of $\mathcal{K}$, i.e., in $N(\mathcal{C})$, have common knowledge of the part of the current network connecting individuals of $N(\mathcal{C})$. The common knowledge assumption restricted to players in the same component of the cover can be justified by assuming that information about the current network propagates between overlapping societies. Note that this scenario yields the unconstrained and common-knowledge environment of Bala and Goyal (2000a) for the particular case of the trivial societal cover: $\mathcal{K} = \{N\}$.

Definition 3 *A Nash $\mathcal{K}$-network is a $\mathcal{K}$-network g that is stable under $\mathcal{K}$-admissible strategies, that is, for all $i \in N$:*

$$\Pi_i(g) \geq \Pi_i(g_{-i}, g_i') \quad \text{for all } g_i' \in G_i(\mathcal{K}). \tag{1}$$

When (1) holds we say that g_i is a *best (admissible) response* of i to g_{-i}. Thus, in a Nash $\mathcal{K}$-network every player is playing a best $\mathcal{K}$-admissible response to those played by the others. Note that for $\mathcal{K} = \{N\}$ a Nash $\mathcal{K}$-network is a Nash network in the standard setting.

The stability notion can be refined in the strict sense by extending Bala and Goyal's strict Nash networks.

Definition 4 *A strict Nash $\mathcal{K}$-network is a Nash $\mathcal{K}$-network such that for all $i \in N$:*

$$\Pi_i(g) > \Pi_i(g_{-i}, g_i') \quad \text{for all } g_i' \in G_i(\mathcal{K}) \ (g_i' \neq g_i). \tag{2}$$

Thus (2) means that in a strict Nash $\mathcal{K}$-network every player is playing her/his *unique* best (admissible) response to those played by the others. Also note that for $\mathcal{K} = \{N\}$ a strict Nash $\mathcal{K}$-network is a Nash network in the standard setting.

Given the constraints on information, strategies and feasible networks that a societal cover imposes, the set of players $N(\mathcal{C})$ in each component $\mathcal{C}$ of the cover, where subcover $\mathcal{C}$ prescribes what links are feasible, form an entirely "separate world": No link with $N \backslash N(\mathcal{C})$ is possible and no information about it reaches $N(\mathcal{C})$. In particular we have the following straightforward result.

Proposition 1 *A $\mathcal{K}$-network g is a Nash (strict Nash) $\mathcal{K}$-network if and only if $g \mid_{N(\mathcal{C})}$ is a Nash (strict Nash) $\mathcal{C}$-network for each component $\mathcal{C}$ of $\mathcal{K}$.*

Remark: Note also that although societies consisting of a single individual are included in the model, such trivial societies are of no interest in this setting. Moreover, the only connected societal cover $\mathcal{K}$ that contains a society A s.t. $\#A = 1$ is $\mathcal{K} = \{A\}$.

Therefore, in view of Proposition 1 and the preceding remark, in what follows *we constrain our attention to connected societal covers and we always assume that all societies have at least two individuals,* unless otherwise specified. The following proposition extends Bala and Goyal's result to this setting.

Proposition 2 *Given a connected societal cover $\mathcal{K}$ of N, a $\mathcal{K}$-network g is a Nash $\mathcal{K}$-network if and only if it is minimally connected.*

Proof. Let $\mathcal{K}$ be a connected societal cover of N, and g a $\mathcal{K}$-network. Assume g is not connected. Then there exist two nodes $i, j \in N$ not connected by a path in g. As cover $\mathcal{K}$ is connected, there exists a finite sequence of nodes $x_1, .., x_m$, such that $x_1 = i$, $x_m = j$ and for each $k = 1, .., m-1$, there is some $A \in \mathcal{K}$ s.t. $x_k, x_{k+1} \in A$. Then for at least two consecutive nodes among these m nodes, say x_k and x_{k+1}, there is no path in g connecting them. But then it is profitable for either of these two nodes to initiate a link with the other. Thus g must be connected. If it were not minimal there would be some redundant link that could be eliminated and that would benefit the player that did so, and consequently g is not a Nash $\mathcal{K}$-network.

Reciprocally, assume that g is minimally connected. Let i be any player and g_i' be any strategy $g_i' \in G_i(\mathcal{K})$ ($g_i' \neq g_i$). We show that $\Pi_i(g) \geq \Pi_i(g_{-i}, g_i')$. A new strategy $g_i' \neq g_i$ means deleting some links and initiating new ones. If g is minimally connected, then each deletion means disconnecting i with a set of nodes, and if there is more than one deletion any two of these sets of nodes disconnected from i must also be disconnected from each other (otherwise a deleted link would be redundant). Thus the number of links initiated should be at least equal to the number deleted, otherwise the payoff would decrease. But then i's payoff for (g_{-i}, g_i') cannot be greater than for g. Therefore if g is minimally connected no player has an incentive to make any $\mathcal{K}$-admissible change. ∎

In Bala and Goyal (2000a) the following result is established (in our terminology and under the assumptions about costs and benefits made here[6]): A network is efficient if and only if it is minimally connected, and Nash networks are those minimally connected. In view of this, we have the following

Corollary 1 *When the societal cover $\mathcal{K}$ is connected the following conditions are equivalent for a network g:*
(i) g is a Nash $\mathcal{K}$-network.
(ii) g is a $\mathcal{K}$-consistent Nash network.
(iii) g is an efficient $\mathcal{K}$-network.

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[6]In fact, given their weaker assumptions on the payoffs (see footnote 5), the empty network may also be Nash stable in their setting.
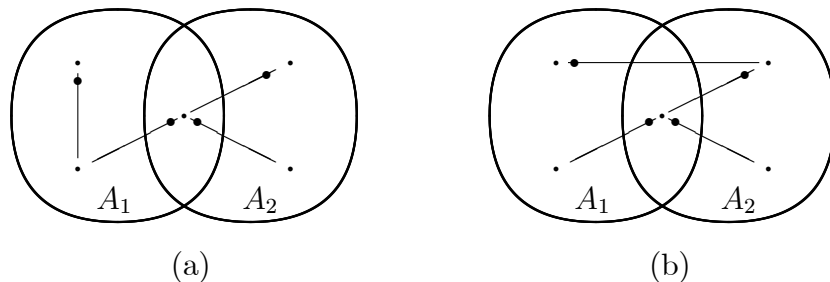
Figure 1: Nash networks and Nash $\mathcal{K}$-networks.

Therefore, for any given set of nodes N and any societal cover $\mathcal{K}$, the set of Nash $\mathcal{K}$-networks is a subset of the set of standard unrestricted Nash networks. In Figure 1 two minimally connected networks are represented[7]: (a) is a Nash $\mathcal{K}$-network, while (b) is not a Nash $\mathcal{K}$-network because one link connects two nodes that do not belong to the same society.

We now focus on strict Nash $\mathcal{K}$-networks. "Stars" of different types play an important role in network stability in different contexts (see, Bala and Goyal (2000a, 2006), Jackson and Wolinsky (1996), Bloch and Dutta (2009)), and, as we show below, they are also important in connection with strict Nash $\mathcal{K}$-networks. In this context the following variant of the notion of center-sponsored star proves useful.

Definition 5 *A set of players $M \subseteq N$ ($\#M \geq 2$) is said to be connected by a center-sponsored star s in a network g if $g \mid_M = s$ and there is a node $i \in M$ s.t. $N^d(i; g) = M \backslash i$ and $g_{jk} = 0$ for all $j \in M \backslash i$ and all $k \in M \backslash j$.*

Note that according to this definition (i) a center-sponsored star does *not* necessarily connect all players in N; (ii) its center i can be linked from other nodes different from those in the star; and (iii) the nodes in the periphery, i.e., those j in M s.t. $g_{ij} = 1$ can be connected with other nodes that do not belong to the star.

Re-stated in terms of the current setting, notation and terminology, and adapted to it, Bala and Goyal (2000a) establish the following result: *The only strict Nash networks are those consisting of a single center-sponsored star that connects all players*[8].

As we show below, the societal cover diversifies the stable/efficient networks as strict Nash $\mathcal{K}$-networks are not necessarily center-sponsored stars. A variety of constellations of linked stars emerges as possible strict Nash $\mathcal{K}$-networks depending on the structure of the societal cover; moreover, in general, several architectures appear as strict Nash for a given societal cover. Our next goal is to identify and characterize these networks.

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[7]As in all figures, nodes are represented by dots (without labels unless conveninet for the purpose of the illustration), links by segments between them, and a filled circle indicates the node that initiated it.

[8]Given their weaker assumptions on the payoffs (see footnote 5), the empty network may also be strict Nash in their setting.

In the characterization of strict Nash $\mathcal{K}$-networks the following binary relation on N associated with a network g plays an important role. Let $\stackrel{g}{\rightarrow}$ be the transitive closure of the binary relation L_g defined by

$$i \ L_g \ j \quad \Leftrightarrow (i = j \ \ or \ \ g_{ij} = 1)$$

That is to say, $i \stackrel{g}{\rightarrow} j$ if $i = j$ or there exists an *i-oriented* path from i to j. This relation is obviously transitive, but in general, for an arbitrary network g, is not complete, antisymmetric or acyclic[9]. But if g is minimally connected, then $\stackrel{g}{\rightarrow}$ is certainly antisymmetric and acyclic (otherwise at least one link would be redundant). Thus, in view of Proposition 2, we have the following

Lemma 1 *For any Nash $\mathcal{K}$-network g, the binary relation $\stackrel{g}{\rightarrow}$ is a partial order on N.*

For any Nash $\mathcal{K}$-network g, we use the following terminology. We say that i is a *predecessor* of j (or that j is a *successor* of i) in g if $i \neq j$ and $i \stackrel{g}{\rightarrow} j$. We say that a node is *terminal* in g if it has no successors, and we say that a node is *maximal* in g if it has no predecessors.

The following theorem characterizes strict Nash $\mathcal{K}$-networks by means of a condition that captures synthetically the essence of these networks, embodying an implicit form of hierarchical principle in their architecture: In such networks every player initiates links with every node within his/her reach unless it is connected (directly or indirectly) with any of his/her predecessors. Formally, we have the following result.

Theorem 1 *A network g is a strict Nash $\mathcal{K}$-network if and only if g is a minimally connected $\mathcal{K}$-network such that for each node i, and all $j \neq i$ within i's reach (i.e., all $j \in N(\mathcal{K}_i)/i$), $g_{ij} = 1$ unless j is a predecessor of i or there exists a k predecessor of i such that $j \in N(k; g)$.*

Proof. Necessity ($\Rightarrow$): Obviously, a $\mathcal{K}$-network g that is a strict Nash $\mathcal{K}$-network is also a Nash $\mathcal{K}$-network, and by Proposition 2, necessarily minimally connected, and by Lemma 1, $\stackrel{g}{\rightarrow}$ is a partial order. Now let i be a node in g and assume $g_{ij} = 0$, for some $j \in N(\mathcal{K}_i)/i$ that is not a predecessor of i and for which there is no k predecessor of i such that $j \in N(k; g)$. As g is minimally connected, there must be a path connecting i and j, that then does not contain any predecessor of i. In particular, on that path the first link must be a link initiated by i. But then i can delete that link and initiate a link with j without altering i's payoff, and consequently g is not a strict Nash $\mathcal{K}$-network.

Sufficiency ($\Leftarrow$): Assume that g is a minimally connected $\mathcal{K}$-network. By Proposition 2, g is a Nash $\mathcal{K}$-network. Let i be any node and any $g_i' \in G_i(\mathcal{K})$ s.t. $g_i' \neq g_i$. We show that $\Pi_i(g) > \Pi_i(g_{-i}, g_i')$ if the condition in the theorem holds. Reasoning

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[9]A binary relation R on a set X is *antisymmetric* if, for all $x, y \in X$, xRy and yRx, implies $a = b$; and R is said to be *acyclic* if there is no finite chain $x_1, x_2, .., x_n$ in X s.t. $x_k R_{k+1}$ for $k = 1, .., n-1$, and $x_1 R x_n$, unless $x_k = x_{k+1}$ for $k = 1, .., n-1$.

as in Proposition 2, as g is minimally connected, $g'_i \neq g_i$ involves deleting some links and initiating an at least equal number of new links for (g_{-i}, g'_i) to be also minimally connected, otherwise i's payoffs would be smaller in (g_{-i}, g'_i), but in fact the number of links deleted and that of those newly initiated by i should be the same for the same reason. Let link ii' be one of the former (i.e., $g_{ii'} = 1$ and $g'_{ii'} = 0$) and let ij be one of the latter (i.e., $g_{ij} = 0$ and $g'_{ij} = 1$). If the condition in the theorem holds, either j is a predecessor of i in g or there exists a k predecessor of i in g such that $j \in N(k; g)$. But this implies a cycle in (g_{-i}, g'_i). The reason is this: Evidently adding link $g'_{ij} = 1$ to g means a cycle in $g - g_{ii'} + 1_{ij}$, but it must be proved that this cycle is contained in (g_{-i}, g'_i). This is so because no link in the path in g connecting i and j can have been initiated by i (this would imply a cycle in g, which is assumed to be minimally connected). Therefore, no matter what other links in g_i are deleted in g'_i, the cycle is entirely contained in (g_{-i}, g'_i). The same can be said about all new links in g'_i w.r.t. g_i, all new links are redundant in (g_{-i}, g'_i). Therefore necessarily $\Pi_i(g) > \Pi_i(g_{-i}, g'_i)$. ∎

This characterization allows in particular for a constructive proof of existence of strict Nash $\mathcal{K}$-networks for any societal cover $\mathcal{K}$: Start at any node i_0 and initiate links with all nodes in $N(\mathcal{K}_{i_0})$, then extend the network by initiating new links from those nodes, always respecting the characterizing condition. In fact we have the following result:

Proposition 3 *For any connected societal cover $\mathcal{K}$ and any node $i_0 \in N$ there exists an oriented tree g rooted at i_0 that is a strict Nash $\mathcal{K}$-network.*

Proof. Iterate the following procedure:

- Step 0: Initially let i_0 be any player in N, and g^0 the $\mathcal{K}$-network that results by i_0 initiating links with all players in $N(\mathcal{K}_{i_0})$.

- Step from k to $k+1$: If g^k is the current $\mathcal{K}$-network resulting form step k, take any terminal node, say i_{k+1}, in g^k, for which the set of nodes in $N(\mathcal{K}_{i_{k+1}})/i_{k+1}$ which are not predecessors of i_{k+1} in g^k and there is no l predecessor of i_{k+1} such that $i_{k+1} \in N(l; g^k)$ is *not empty*, and let i_{k+1} initiate links with all those players. If no such node exists, stop; otherwise, let g^{k+1} be the $\mathcal{K}$-network that results by adding to g^k all these links initiated by i_{k+1}.

It is clear that if $\mathcal{K}$ is connected this iterated process must stop in a finite number of steps and the resulting network will be an oriented tree rooted in i_0 that forms a strict Nash $\mathcal{K}$-network connecting all players in N. If $\mathcal{K}$ were not connected the same iterated procedure could be applied within each component of the cover and by Proposition 2 a strict Nash $\mathcal{K}$-network would result. ∎

As a corollary of Theorem 1, the following propositions establish some prominent features of the architecture of strict Nash $\mathcal{K}$-networks that help to form a clearer idea about these networks, which we later illustrate with some examples. The first shows the role of stars in strict Nash $\mathcal{K}$-networks.

Proposition 4 *In a strict Nash $\mathcal{K}$-network g:*

(i) There must be an $i \in N$ who is the center of a center-sponsored star that links with all players in $N(\mathcal{K}_i)$, that is, s.t. $N^d(i;g) = N(\mathcal{K}_i)/i$, and no other player in N initiated a link with i.

(ii) For each society $A \in \mathcal{K}$, either no link connects two nodes of that society or all or some of the members of that society are connected by center-sponsored stars and no other link exists connecting a pair of nodes in A.

Proof. *(i)* By Lemma 1, given that g is minimally connected, $\xrightarrow{g}$ is a partial order and necessarily exists at least one maximal element, i.e., with no predecessor. Let i_0 be a maximal element. As i_0 is maximal, by Theorem 1, necessarily $N^d(i_0;g) \cup \{i_0\} = N(\mathcal{K}_{i_0})$.

(ii) Let A be a society in the cover $\mathcal{K}$. Assume that for some $i, j \in A$, $g_{ij} = 1$. It is enough to show that the only other link that may exist connecting any $k \in A \backslash \{i, j\}$ with i or j is a link initiated by i. Assume that $g_{kj} = 1$. Then k can delete the link with j and initiate one with i and have the same payoff. Assume that $g_{jk} = 1$. Then i can delete the link with j and initiate one with k and have the same payoff. Finally, assume that $g_{ki} = 1$. Then k can delete the link with i and initiate one with j and have the same payoff. Thus the only remaining possibility of a link connecting any $k \in A \backslash \{i, j\}$ with i or j is a link $g_{ik} = 1$. $\blacksquare$

As an immediate corollary of part *(i)*, we have the following conclusion that yields Bala and Goyal's result as a particular case.

Corollary 2 *There exists a center-sponsored star that is a strict Nash $\mathcal{K}$-network if and only if the societal core is not empty and the center belongs to it.*

Observe the similarity of the proof of part *(ii)* with Bala and Goyal's proof of their result, and its differences: Minimal connectedness and strict "Nash-ness" do *not* entail that *all* nodes are connected by a *single* star. Now the possibility of other center-sponsored stars within a society is left open, and even the possibility of some nodes being left outside these stars (but linked through nodes belonging to societies other than A).

Thus we have in short that in a strict Nash $\mathcal{K}$-network g within each society either no pair of nodes is connected by a link or *some* center-sponsored stars connect *some* of the nodes in that society. But there is at least one center-sponsored star whose center connects *all* nodes of *all* societies to which the center belongs. The question now is: How do these stars interconnect in g? Evidently through overlapping societies. The following proposition answers this question more precisely by establishing the possible connections through overlapping societies: Stars "hand in hand", i.e., interconnected through a free-rider player, are possible only if a *single* player belongs to both societies. Otherwise, if more than one player belongs to both societies, a player interconnecting them necessarily initiates link(s) with players of one or both societies.

Proposition 5 *Let A, B be two overlapping societies in a societal cover $\mathcal{K}$ with $i \in A \cap B$, and g a strict Nash $\mathcal{K}$-network. If for some $j \in A \backslash (A \cap B)$ and $k \in B \backslash (A \cap B)$ it is $\bar{g}_{ij} = \bar{g}_{ik} = 1$, then $g_{ji} = g_{ki} = 1$ is possible only if $A \cap B = \{i\}$.*

Proof. Assume that $i \in A \cap B$ and for some $j \in A \setminus (A \cap B)$ and some $k \in B \setminus (A \cap B)$, $g_{ji} = g_{ki} = 1$. If $\{i\} \subsetneq A \cap B$ take $i' \in A \cap B$, $i \neq i'$. If i and i' were linked then j (or k) could delete the link with i and initiate a link with i' without loss. Thus we should have $\bar{g}_{ii'} = 0$. As g is minimally connected either there exists a path connecting i' and j and not containing k, or there exists a path connecting i' and k and not containing j. In the first case k can delete the link with i and initiate a link with i', and in the second j can delete the link with i and initiate a link with i'. In both cases this is without loss for the player deleting the link, therefore proving that g is not a strict Nash $\mathcal{K}$-network. ∎

The examples in Figure 2 illustrate the characterization and its corollaries and convey the logic of strict Nash $\mathcal{K}$-networks. Of course, the characterizing condition holds in all cases, as the reader may check. Examples (a) and (b) represent societal covers with a nonempty core where a center-sponsored star is *one* of the possible architectures of strict Nash $\mathcal{K}$-networks: (d) and (c) represent other strict Nash $\mathcal{K}$-networks for the same covers. In examples (a), (b) and (d) a single center-sponsored star covers (partially) each society, while *two* center-sponsored stars cover society A_3 in (c) and society A_5 in (e), and in both cases no other link exists between pairs of individuals. In all cases a maximal node exists (represented by a white circle "○"), but *there may exist more than one,* as in examples (e), (f) and (g), which illustrate Proposition 5: Stars connecting "hand in hand" by means of a "free rider" node are possible when a single player belongs to both societies. We have in fact the following conclusion: When no pair of societies in the societal cover $\mathcal{K}$ share a single player a strict Nash $\mathcal{K}$-network is an oriented tree, as is proved by the following

Theorem 2 *Let $\mathcal{K}$ be a connected societal cover of N, then if for all $A, B \in \mathcal{K}$, $\#(A \cap B) \neq 1$, then a strict Nash $\mathcal{K}$-network is a $\mathcal{K}$-network which necessarily forms an arborescence or oriented tree.*

Proof. There is a unique path connecting any maximal node with each node. Assume that there are two maximal nodes i_0 and i_1. Then there is a path connecting i_0 and i_1, but then there must exist three nodes on that path i, j and k such that $g_{ij} = g_{kj} = 1$. Now if the intersection of any two societies in $\mathcal{K}$ is either empty or contains *more* than a single player, by Proposition 5, this is impossible. Therefore there can be only one maximal node connected with any other node by a unique path and consequently g is an oriented tree. ∎

But note that, as examples (f) and (g) in Figure 2 show, when there are two or more societies to which a single player belongs it is possible "to start" at different nodes at the same time, i.e., several maximal nodes may exist. In such cases an oriented tree does not result. In this case two or more "grafted" oriented trees may emerge, so that any node is connected by an oriented tree with at least one but possibly more maximal nodes.

Finally, in the spirit of the "community detection" problem (see, e.g., Jackson, 2009), we address an issue reciprocal to that considered so far: Given a network g, can
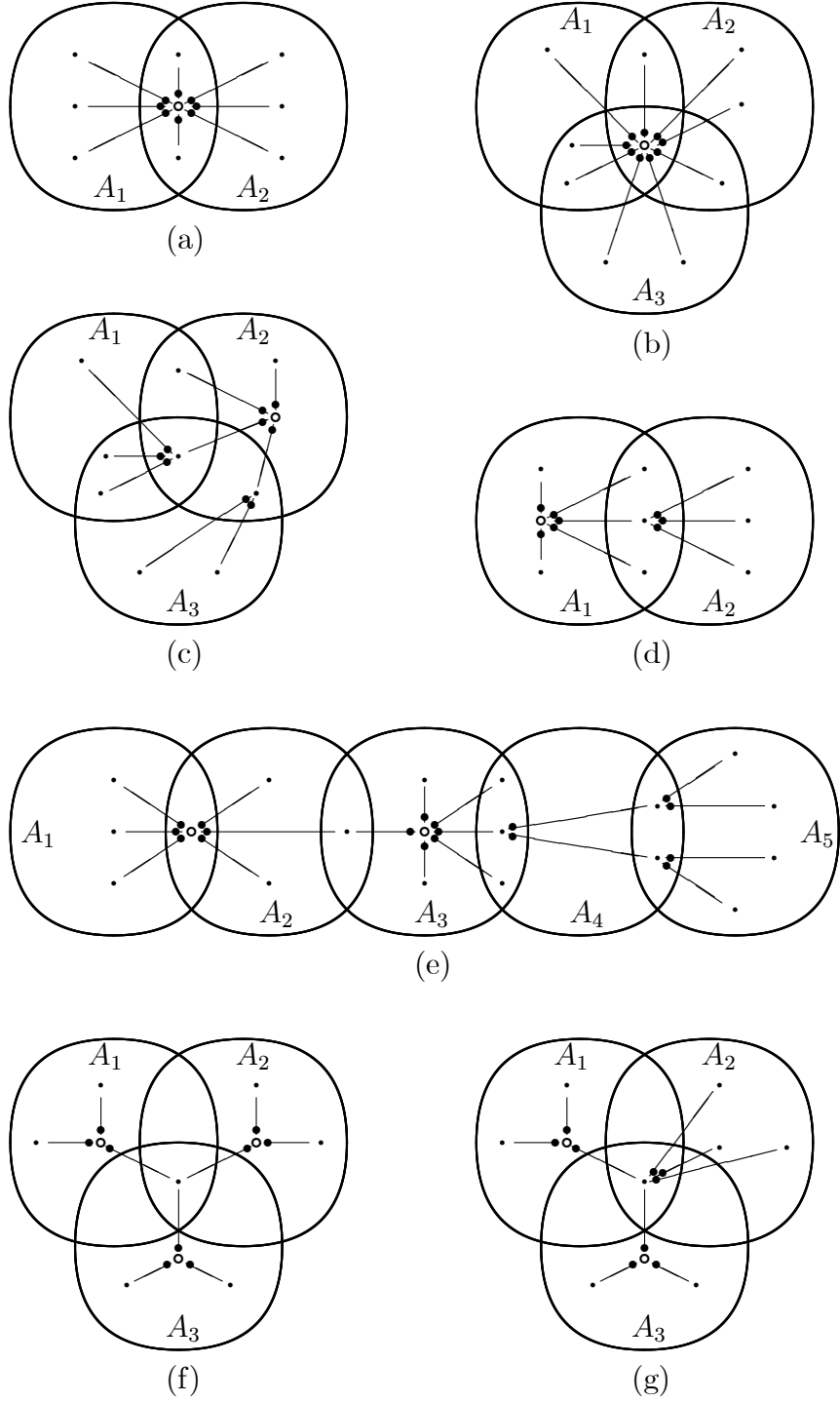
Figure 2: Strict Nash $\mathcal{K}$-networks.

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it be interpreted as a strict Nash $\mathcal{K}$-network for any particular societal cover $\mathcal{K}$? Given the multiplicity of strict Nash $\mathcal{K}$-networks for a societal cover $\mathcal{K}$, it is easy to see that this question admits many answers: In general, an oriented tree (or several grafted trees) can be seen as a strict Nash $\mathcal{K}$-network for different societal covers. Restricting attention to oriented trees, the following associated covers are worth noting. Let g be an oriented tree rooted at i_0. The *generational cover*, consisting of a minimal number of societies, each consisting of all nodes at the same distance from the root that are not terminal along with their "offspring"; the *family cover* where each node forms a society with its offspring; and the trivial *binary cover* where any two directly linked nodes form a society. For all the three societal covers the oriented tree g is a strict Nash $\mathcal{K}$-network, and for the latter two it is the only one with maximal node i_0.

4   Dynamics

We now apply Bala and Goyal's (2000a) dynamic model adapted to this setting. Namely, starting from any initial $\mathcal{K}$-network g each player i with some positive probability responds with a $\mathcal{K}$-admissible best response[10] to g_{-i} or randomizes across them when there is more than one, otherwise player i exhibits *inertia*, i.e., keeps his/her links unchanged. In this way a Markov chain on the state space of all $\mathcal{K}$-networks is defined. In Bala and Goyal's setting, i.e., for $\mathcal{K} = \{N\}$, the absorbing states are precisely the strict Nash networks and they prove that starting from any network the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. The following example shows that this may not be the case for the same dynamic model in the context of $\mathcal{K}$-networks.

Example: In Figure 3 (a) players in A_1 have no best response but keep their strategies, while player 1 is indifferent between initiating a link with 2 or 3 or 4, and consequently the best response dynamic process would oscillate forever. Similarly, in Figure 3 (b) all players in A_1 and players in A_3 keep their strategies, while player 1 is indifferent between initiating a link with 2 or 3, and consequently the best response dynamic process would oscillate forever among these two networks. Note that in both examples the set of $\mathcal{K}$-networks among which the best response dynamics oscillates are minimally connected and yield the same payoffs to all players.

The example shows an interesting difference with respect to Bala and Goyal's setting. It is not difficult to show that Bala and Goyal's best response dynamics lead to a Nash $\mathcal{K}$-network (i.e., $\mathcal{K}$-minimally connected)[11]. The problem appears when one tries to extend the rest of their proof in search of a *strict* Nash $\mathcal{K}$-network. The reason is that in their single-society setting, starting from any network g that is not

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[10]Note that if g is a Nash $\mathcal{K}$-network any strategy g_i' of player i such that $\Pi_i(g) = \Pi_i(g_{-i}, g_i')$, is a best response to g_{-i}.

[11]The proof is similar to that of Lemma 4.1 in Bala and Goyal (2000a), now just taking into account and respecting $\mathcal{K}$-feasibilty.
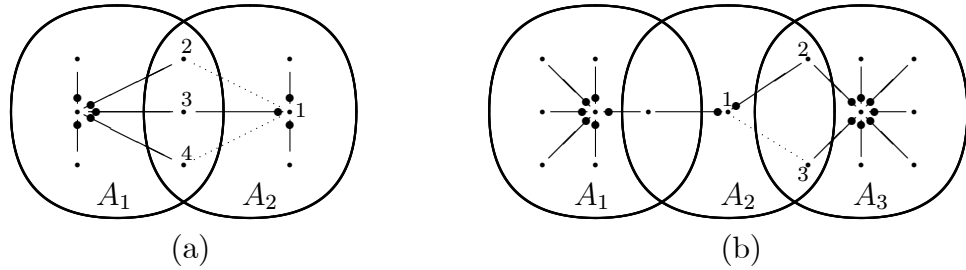
Figure 3: Dynamic deadlock towards a strict Nash $\mathcal{K}$-network.

strict Nash one can show that with a positive probability a strict Nash network may be reached by a finite sequence of best responses. In our more complex setting, $\mathcal{K}$-admissible best response dynamics lead to the formation of architectures consisting of interconnected center-sponsored stars, but it might happen that these interconnected center-sponsored stars that appear in different parts of the $\mathcal{K}$-network as the dynamic process runs cannot be part of the same strict Nash $\mathcal{K}$-network, thus blocking the possibility of transition to an absorbing state. Note that in Bala and Goyal's setting with a single-society cover this last situation never occurs. Then, the same logic that in their setting leads to the absorbing strict Nash networks, in ours may lead to the formation of interconnected center-sponsored stars incompatible in any strict Nash $\mathcal{K}$-network that block the converging process. Nevertheless, we have a similar result if we replace strict Nash $\mathcal{K}$-networks, not longer absorbing states, by the following sets:

Definition 6 *Let $\mathcal{K}$ be a connected societal cover of N; a quasi strict Nash $\mathcal{K}$-set is a set Q of minimally connected $\mathcal{K}$-networks that is closed under best response dynamics, and verifies full reachability by best response dynamics; and a quasi strict Nash $\mathcal{K}$-network is a network that belongs to a quasi strict Nash $\mathcal{K}$-set.*

By "full reachability" under best response dynamics we mean that any network in one of these sets is reachable from any other in the same set by best response dynamics.

Example (a) in Figure 3 shows a three-element quasi strict Nash $\mathcal{K}$-set, and example (b) a two-element quasi strict Nash $\mathcal{K}$-set. Note that a strict Nash $\mathcal{K}$-network is just a singleton quasi strict Nash $\mathcal{K}$-set. We then have the following result:

Theorem 3 *Starting from any $\mathcal{K}$-network g, best response dynamics reach a quasi strict Nash $\mathcal{K}$-set with probability 1.*

Proof. Let BR be the binary relation in the set $G_{\mathcal{K}}$ of all $\mathcal{K}$-networks defined by $gBRg'$ if g' is one of the possible results of a best response move from g, and let BR^* be the transitive closure of this relation, i.e., gBR^*g' if it is possible to reach g' from g by a finite sequence of best response moves. We now denote by H^* its associated equivalence relation, that is,

$$gH^*g' \Leftrightarrow gBR^*g' \ \& \ g'BR^*g.$$

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This relation establishes a partition of the set of all $\mathcal{K}$-networks into equivalence classes, so that in each class any network is reachable by a sequence of best response moves from any other in the same class. And note that a binary relation $\overline{BR}$ can be defined on the quotient set $G_\mathcal{K}/H^*$ (i.e., the set of equivalence classes):

$$[g]\overline{BR}[g'] \Leftrightarrow gBR^*g'.$$

Note that $\overline{BR}$ is a well defined binary relation that is reflexive, antisymmetric and transitive, i.e., $\overline{BR}$ is a partial order in the quotient set. As the quotient set is finite, minimal classes must exist, i.e., classes $[g]$ s.t. $[g]\overline{BR}[g']$ implies $[g] = [g']$. Now note that the Markov chain on $\mathcal{K}$-networks defined by the best response dynamics can be extended to a Markov chain on $G_\mathcal{K}/H^*$: For any two different related classes, i.e., $[g], [g']$ s.t. $[g]\overline{BR}[g']$ there is a positive probability of reaching the latter from the former. Thus, there is a positive probability of reaching a minimal class where the process will stay forever. Now a minimal class is just a set of $\mathcal{K}$-networks closed w.r.t. best response dynamics and such that any network in this set is reachable by a sequence of best response moves from any other in this set. ∎

Thus, the absorbing sets of the best response dynamic process are the quasi strict Nash $\mathcal{K}$-sets, that have been introduced in Definition 6 *in best response dynamics terms*, but it seems desirable a characterization of these sets, or of the quasi strict Nash $\mathcal{K}$-networks that form them, in terms independent of dynamics. With this purpose the following definition is convenient.

Definition 7 *Let g be a minimally connected $\mathcal{K}$-network, and let $s \subseteq g$ be a center-sponsored star with center i, then we say that s is $\mathcal{K}$-irreversible if for any other center-sponsored star $s' \subseteq g$ with center $j \neq i$ such that $N(\mathcal{K}_i) \cap N(\mathcal{K}_j) \neq \varnothing$, if $s'' = s \cup s'$ all nodes in $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_j))$ are linked by i or all by j.*

That is, a center-sponsored star s, that is part of a minimally connected $\mathcal{K}$-network is $\mathcal{K}$-irreversible if for any other center-sponsored star s' whose center's reach intersects with that of s, all nodes in $s \cup s'$ within the reach of both centers are linked from either the center of s or the center of s'. If this happens, spoke nodes have no best response and there is no possibility of miscoordination between the centers. Then, the only feasible best responses of the nodes (if at all any does exist) in s consists of the center replacing some spoke nodes with the same number of other nodes within its reach. In that case *the center of the star cannot change*, nor the number of nodes that form the star, and we say that the star is $\mathcal{K}$-irreversible. Observe that in Figure 3, both examples consist of minimally connected $\mathcal{K}$-networks formed by *interconnected $\mathcal{K}$-irreversible center-soponsored stars*. We have in fact the following characterization.

Theorem 4 *A $\mathcal{K}$-network g is a quasi strict Nash $\mathcal{K}$-network if and only if it is a minimally connected $\mathcal{K}$-network consisting of interconnected $\mathcal{K}$-irreversible center-sponsored stars.*

Proof. Necessity ($\Rightarrow$): By definition, a quasi strict Nash $\mathcal{K}$-network g is a minimally connected $\mathcal{K}$-network. Now let g be a $\mathcal{K}$-network not consisting of interconnected center-sponsored stars. Then best response dynamics lead with probability 1 to a $\mathcal{K}$-network formed by interconnected center-sponsored stars[12]. Now let g be a $\mathcal{K}$-network consisting of interconnected center-sponsored stars, and let $s, s' \subseteq g$ be two center-sponsored stars with centers i and j such that $N(\mathcal{K}_i) \cap N(\mathcal{K}_j) \neq \varnothing$ and not all nodes in $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_j))$ are linked by i neither by j, where $s'' = s \cup s'$. In that case both i and j have a best response among the $\mathcal{K}$-feasible moves in s'' and they can therefore miscoordinate. Then there is a positive probability of transition to a $\mathcal{K}$-network where either i or j initiate links with all nodes in $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_j))$. Applying this to every such pairs s and s' in sequence, the process leads to a $\mathcal{K}$-network consisting of interconnected center-sponsored stars that are $\mathcal{K}$-irreversible, reaching thus an equivalence class different from $[g]$, and consequently g is not a quasi strict Nash $\mathcal{K}$-network.

Sufficiency ($\Leftarrow$): Assume that g is a minimally connected $\mathcal{K}$-network consisting of interconnected $\mathcal{K}$-irreversible center-sponsored stars. Since for every center-sponsored stars $s, s' \subseteq g$ with centers i and j such that $N(\mathcal{K}_i) \cap N(\mathcal{K}_j) \neq \varnothing$, all the nodes in $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_j))$, where $s'' = s \cup s'$, are linked either from i or j, the only $\mathcal{K}$-feasible best response (if any does exist) that nodes in s'' may have are those where just only one of the centers, say i, deletes some links with nodes in $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_j))$ and replaces each of them by a link with another node in $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_j))$ or in some $N(s'') \cap (N(\mathcal{K}_i) \cap N(\mathcal{K}_k))$, where k is the center of a center-sponsored star $t \subseteq g$, while all other nodes in s'' have no best response, not even center j. Since this happens for every such pairs s and s', then miscoordination cannot occur and g is a quasi strict Nash $\mathcal{K}$-network. ∎

As a corollary, we have the following result that shows that when an absorbing quasi strict Nash $\mathcal{K}$-set is reached, in spite of the possibly perpetual oscillation, stability is essentially reached given that all networks in the same quasi strict Nash $\mathcal{K}$-set *yield the same payoffs to all players*.

Corollary 3 *If Q is a quasi strict Nash $\mathcal{K}$-set, for all $g, g' \in Q$ and all $i \in N$, $\Pi_i(g) = \Pi_i(g')$.*

Proof. Let Q be a quasi strict Nash $\mathcal{K}$-set and $g \in Q$. As g is a minimally connected $\mathcal{K}$-network that consists of interconnected $\mathcal{K}$-irreversible center-sponsored stars, whose centers are fixed, any $\mathcal{K}$-admissible best response move (if any does exist) can only involve some center(s) changing some links keeping the network minimally connected. Therefore the payoffs must remain unchanged for all players. ∎

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[12]The proof is similar to that of Theorem 4.1 in Bala and Goyal (2000a), just respecting $\mathcal{K}$-feasibilty.

5 Concluding remarks

We have studied the impact of institutional constraints as modeled by a societal cover in Bala and Goyal's (2000a) benchmark two-way flow model, by extending their model in a natural way. The notion of societal cover seems suitable for capturing in a formal and tractable way many factual constraints that are to be observed in real world situations to which we refer generically as "institutional". Such constraints emerge due to social (cultural, economic, geographic, etc.) reasons and cannot be ignored in many contexts. In this paper we characterize and study in some detail the structure of stable and efficient networks under these constraints by extending Bala and Goyal's approach and results. In a nutshell, the conclusions are these: Center-sponsored star (when feasible) is no longer the only stable (in the strict Nash sense) architecture, but center-sponsored stars continue to be the basic building blocks of stable networks. Moreover, the architecture of such stable networks embodies a formal hierarchical principle that yields oriented trees[13] or "grafted" oriented trees adapted to the constraints imposed by the cover. Finally, simple best response dynamics "work" basically well in this more complicated setting: They may fail to reach a strict Nash network if incompatible irreversible center-sponsored stars form, but a stable configuration of payoffs associated with an absorbing quasi strict Nash set is sure to be reached.

The interesting results obtained with this approach suggest several lines of further research. In fact, this paper is the first step of a research project to explore the effects of institutional constraints as modeled here. In a second paper, continuation of this one, we address the effects of further restricting information, assuming that individuals in each society have common knowledge only of the part of the current network that connects individuals in that society. Other lines of research that can be suggested are the following: Given the multiplicity of strict Nash networks in the setting considered, it may be interesting to study possible selection/refinement among them, perhaps combined with the introduction of decay or non full reliability[14]. Another line of work is the study of the effects of heterogeneity in this setting. It may also be worth trying an extension of the one-way flow model of Bala and Goyal (2000a) similar to the one achieved here for the two-way flow model. Finally, it could be interesting to see the impact of institutional constraints as modeled here on Jackson and Wolinsky's (1996) model based on pairwise stability.

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