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A BEHAVIOURAL FOUNDATION FOR MODELS OF EVOLUTIONARY DRIFT

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A Behavioural Foundation for Models of Evolutionary Drift

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ABSTRACT

Binmore and Samuelson (1999) have shown that perturbations (drift) are crucial to study the stability properties of Nash equilibria. We contribute to this literature by providing a behavioural foundation for models of evolutionary drift. In particular, this article introduces a microeconomic model of drift based on the similarity theory developed by Tversky (1977), Kahneman and Tversky (1979) and Rubinstein (1988),(1998). An innovation with respect to those works is that we deal with similarity relations that are derived from the perception that each agent has about how well he is playing the game. In addition, the similarity relations are adapted to a dynamic setting.

We obtain different models of drift depending on how we model the agent´s assessment of his behaviour in the game. The examples of the ultimatum game and the chain-store game are used to show the conditions for each model to stabilize elements in the component of Nash equilibria that are not subgame-perfect. It is also shown how some models approximate the laboratory data about those games while others match the data.

NOTE

The present paper is an extension of Uriarte (2003). That work introduced a model of drift which here is named as “drift based on socially induced similarity relations”. Hence, the reader will note that Section 3 in both works are substantially the same and that Proposition 4 of the previous work is numbered as Proposition 1 in the present one. Here we propose three additional models of drift and show the derived results. A shorter version of this work is forthcoming in the *Journal of Economic Behavior and Organization.*

A Behavioural Foundation for Models of Evolutionary Drift

1. Introduction

It is common place to observe that the Nash equilibrium selected by a theory depends on the manner in which perturbations are handled (see, for example, Selten's (1975) perfect equilibrium and Myerson's (1978) proper equilibrium and, more recently, McKelvey and Palfrey's (1995) quantal response equilibrium). Binmore et al.(1995) and Binmore and Samuelson(1999),-B&S henceforth- also emphasize the importance of perturbations, but they place these in the selection or learning process that takes the players to equilibrium, rather than perturbing the game itself. This paper follows the approach taken by the latter authors.

B&S studied the stability properties of Nash equilibria and dealt, most notably, with the issue of equilibrium selection, particularly in the Ultimatum Game (or the Chain-Store Game). This game is chosen because it is often used to justify subgame-perfect equilibria, even though the choices observed in the laboratory experiments on this game are other than the subgame-perfect (too many experiments have been carried out on the Ultimatum Game to quote them here; see, for instance, Güth et al. (2001) and the references quoted there). B&S show that states near equilibria that are not subgame-perfect can be stabilized by drift. Although their result is not particularly close to the experimental data, B&S have shown that evolutionary drift (which accounts for the perturbations that affect the selection or learning process through which equilibrium might be achieved) is a relevant element in game theory.

The difficulty to match the experimental data of the Ultimatum Game is that player's behaviour in this game seems to be led by fairness considerations. When the perception of a game is influenced by individual values or by social norms and conventions it seems that different strategies in that game are, a priory, valued differently and therefore the tendencies to abandon them might differ from one individual to another. If this is true then, how do we model this behaviour?. B&S argue that drift could be a key tool to deal with this issue. Drift may capture some of the real world imperfections left aside by the learning process embedded in the selection dynamic model, thus adding more realism to the equilibrating process. But the model of drift proposed by B&S lacks psychological foundations and so little insight is given into what perturbations one should expect.

It is in modelling such perturbations realistically that the present paper is concerned. The approach we take is based on the similarity theory developed first by Tversky (1977) in psychology, and later applied to choice theory by Kahneman and Tversky (1979) and Rubinstein (1988), (1998) to explain observed choice behavior (such as the one leading to the Allais Paradox). To do so, the present work develops a similarity theory valid for a dynamic setting to model the behaviour of *the perturbing agents* (to whom we call the $\sum$ agents).

The most relevant assumption in the article is that each $\sum$ agent is endowed with a threshold function which measures the vagueness or ambiguity felt by the agent about how well is playing the game. The level of vagueness felt might be sensitive to the proportions of agents in his population playing the same strategy he is currently using. The threshold function plays the role of a primitive concept in modelling the $\sum$ agents' choice behaviour. In essence, what a $\sum$ agent does is the following: he uses the threshold function to build a pair of similarity relations, one defined on the space of all possible expected payoffs to his current strategy and the other on the space of proportions of agents in his population playing that strategy. Then, with this pair of similarity relations, the $\sum$ agent builds a preference relation on the (product) space of payoffs-strategy proportions that tells him, at each period, how satisfied is with his current strategy. This preference relation and the agent's aspiration set, which is assumed to be the preferred set, change over time. Dissatisfied $\sum$ agents will switch strategy and, as rule of thumb, choose every strategy with the same probability. Hence, drift ensures that small fractions of all strategies, including those that are not currently played in the population, will be continuously injected and perturb the selection dynamic system.

Depending on the type of threshold function we deal with, we may build two classes of similarity relations. If the vagueness felt by the agent decreases when he observes that the proportion of agents playing his current strategy increases then, we would obtain the "*Socially Induced Similarity Relations*" (which are related to those introduced by Uriarte (1999) for the space of simple lotteries). But when the level of vagueness felt is not sensitive to what the others are doing then, we would obtain the "*AINU Similarity Relations*" (which are related to those introduced by Aizpurua, Ichiisi, Nieto and Uriarte (1993)) or, in short, AINU (1993)). Both classes of similarity relations are assumed to satisfy the axioms introduced by Rubinstein (1988).

From the "*Socially Induced Similarity Relations*" we derive three models of drift. In one model the $\sum$ agents are endowed with extremely sensitive threshold functions (*the playing modes*). This model seems to capture well the influence of social norms and convention on the agents' behaviour. But, on the negative side, the model presents, in some cases, a certain degree of "adhocery". We avoid this issue, with two additional models of drift that use the data about expected payoffs and strategy proportions to determine endogenously the threshold function of each agent. Finally, from the "*AINU Similarity Relations*" we obtain a model of drift that is sensitive only to payoffs: perturbations decrease when payoffs increase (as, for instance, in B&S and, in a different setting, McKelvey and Palfrey (1995)). Each model has different capabilities to stabilize Nash equilibria that are not subgame-perfect. As a consequence, some models match the experimental data about the Ultimatum Game, while others approach the data.

The present article is organized as follows. Section 2 introduces the notation and explains (and justifies) the methodology used in the paper. Section 3 presents a detailed account of how the "Socially Induced Similarity Relations" are obtained and how the model of drift is built. This is the main section of the paper. Its extension is well justified as it facilitates the understanding of Sections 4-5. Section 4 presents the "AINU Similarity Relations" and the drift model based on them. Section 5 presents the results obtained with the four models of evolutionary drift. In that section it is used the example of the ultimatum minigame (or the chain-store game) to show the sufficient conditions for each model to create stationary states with different stability properties in the component of Nash equilibria that are not subgame-perfect. Section 6 relates the results with those obtained, in particular, with the QRE model of McKelvey and Palfrey (1998) and with the B&S model. Section 7 presents numerical calculations made for the Full Ultimatum Game of Roth et al. (1991) and Roth and Erev (1995) showing that some models approximate the experimental results while others match the results. Section 8 concludes the article.

2. Notation

Let G be a noncooperative finite game in normal form, with $K = \{1, 2, ..., n\}$ as the set of players. We assume that there are n large player-populations. Randomly drawn members of the n player-populations, one from each population, are repeatedly matched to play the game. For each player $k \in K$, let $S_k = \{1, 2, ..., m_k\}$ be his finite set of pure strategies, for some integer $m_k \geqslant 2$. Throughout the paper, we shall refer to agent ki, a member of player-population $k \in K$ playing strategy $i \in S_k$. Thus, f_{ki} will denote the proportion of agents in player population $k \in K$ who play strategy $i \in S_k$ at time t, with f_k being the vector collecting such proportions in population k and $f = (f_{1,...,}f_n)$ the population state at time t. Hence, $f \in \Delta = \times_{k=1}^{n} \Delta_k$, where Δ_k is the simplex of mixed strategies for player $k \in K$. $F_{ki} = [0, 1]$ is the space of proportions of agents in player-population k playing strategy i. Let $\pi_{ki}(f)$ denote the (expected) payoff to agent ki given the population state f at time t. The term $\overline{\pi}_k(f) = \sum_{i=1}^{m_k} f_{ki}\pi_{ki}(f)$ denotes the average expected payoff to player population $k \in K$. More specifications about payoffs are given in section 3.2, below.

Methodology: Evolutionary Analysis and Laboratory Data

As B&S, we start with a system of continuous, deterministic differential equations that describe how the proportions of the player populations attached to each pure strategy evolve over time. This system is represented by a selection dynamic model which one can find in biological models of natural selection. Then, for a better approximation to the underlying stochastic strategy-adjustment process we add the drift term to the selection dynamics.

The relevance of evolutionary analysis to experimental data is emphasized by Binmore, Gale and Samuelson (1995), Binmore and Samuelson (1999), Samuelson (1998) and Binmore et al. (2002) in relation to the influence of social norms in the laboratory behaviour. In particular, the Ultimatum Game is relevant in this respect because the theoretical prediction of subgame-perfection is contradicted by experimental data. The influence of norms triggered by the framing of the Ultimatum Game explains the short run behaviour which locate the initial conditions of the laboratory learning in the basin of attraction of equilibria that are not subgame-perfect. In the medium run, it can be seen that the perturbed RD goes fast to equilibria that are not subgame perfect, lingering there until reaches a complete stop. But long run analysis by means of the replicator dynamics (RD) could be relevant too to the long run laboratory learning. Borgers and Sarin (1997) have shown that the RD may serve as long-run approximations to simple learning rules related to that used by Roth and Erev (1995) - the reinforcement learning rule- to explain experimental data.

3. Drift based on Socially Induced Similarity Relations.

It is natural to observe different behaviors when different agents face the same decision problem. For this reason, diversity of tastes and values are central to economic analysis. Thus, following the economist's approach, to the society that evolves according to the agents whose behaviour leads to the selection dynamic model, the so called $SD-$ agents, (Binmore et al. (1995), Borgers and Sarin(1997), Schlag (1998) and Cabrales (2000) show how different individual behaviours may lead to the most popular selection dynamic model, the Replicator Dynamics) we shall add a new type of agents, the so-called $\sum$ agents. Thus, the perturbed selection dynamic will have large player-populations and inside each population we assume there are two types of agents, the $SD-$ agents and the $\sum$ agents. Both population types are assumed to be equally large. We proceed now to describe the features of the $\sum$ agent ki.

3.1 The Threshold Function

We introduce first a function which plays the role of a primitive concept in the model.

It seems natural to assume that the participant in a continuously repeated interaction builds experience-based conjectures about how good or bad is playing the underlying game and that he may relate that evaluation, for instance, to the proportion of individuals who are playing exactly like him. We will assume that the $\sum -$ agent ki has, at each stage, information about those proportions and thinks as follows: *"the higher is the proportion of agents in my player population who are currently using the same strategy as mine, the less ambiguity (or insecurity or uncertainty or vagueness) I should feel about how well I am playing the game"*. Thus, we are relating the idea of "how well I am playing the game" with society (i.e. the rest of agents in my player population) and, therefore, with the social experience, conventions and norms that might exist therein. We assume that it is a subjective measure. It will become clear that we are not proposing a herding model. For some agents the measure depends exactly on playing according to what the rest of the society is doing. But for other agents, it depends less on what the others are doing and more on playing according to certain moral judgements or social norms or, simply, on what the experience is telling him (this would be the alert mode of playing described in Remark 1a, below). Formally,

Assumption 1

Each $\sum -$agent ki is endowed with a differentiable function d_{ki}, called threshold function, in the set

$$D = \left\{ \begin{array}{c} d_{ki} : F_{ki} \to [0,1] : \text{with } d_{ki}(0) = 1,\ d_{ki}(1) = 0 \\ \text{and } \widehat{f}_{ki} > \widetilde{f}_{ki} \Rightarrow d_{ki}(\widehat{f}_{ki}) < d_{ki}(\widetilde{f}_{ki}) \end{array} \right\} ......(1)$$

Given a proportion $f_{ki} \in F_{ki}$ and any $d_{ki} \in D$, $d_{ki}(f_{ki})$ measures the ambiguity (about how well is playing the game) felt by the $\sum$ agent ki when the proportion of agents in player population k playing strategy $i \in S_k$ at time t is f_{ki}. The ambiguity gradually decreases when he observes that more and more agents from his population come to play the same strategy as his. For a different use of the strategy proportions information, see Young (1993a), (1993b) and (1996).

Remark 1: **The Playing Modes**.

For any d_{ki}, $\widehat{d}_{ki}$ in D, if for all $f_{ki} \in (0,1)$, $d_{ki}(f_{ki}) < \widehat{d}_{ki}(f_{ki})$, then we say that d_{ki} is sharper than $\widehat{d}_{ki}$. Two important cases should be considered: for all $f_{ki} \in (0,1)$, the extremely sharp threshold function, $\overline{d} \in D$, for which $\overline{d}(f_{ki})$ takes values which are "very close" to 0 (i.e., $\overline{d}(f_{ki}) \cong 0$) and the extremely unsharp threshold function, $\underline{d} \in D$, for which $\underline{d}(f_{ki})$ takes values which are "very close" to $1 (i.e., \underline{d}(f_{ki}) \cong 1$). When $d_{ki} = \overline{d}$, we would say that the $\sum$ agent ki is in the *alert mode* of play and when $d_{ki} = \underline{d}$ then, we say the agent is in the *absent mode* of play.

3.2. Vagueness Modelled by Socially Induced Similarity Relations.

We assume that the level of vagueness felt by the $\sum$ agent ki, develops intervals (in both the payoff and strategy frequency spaces) inside which events are not distinguishable. To model these intervals we use the similarity theory axiomatized by Rubinstein(1988),(1998). The main innovation with respect Rubinstein's similarities is that those developed in this section are built with the help of the threshold functions in D and therefore it can be said that they are socially induced.

In essence, a similarity relation serves to capture the capacity of an individual to discriminate between events. Correlated similarity relations, a concept introduced by Aizpurúa et al.(1993), describe how that discrimination capacity changes depending on the values of some relevant parameter. For instance, we shall define here correlated similarity relations on F_{ki} to capture the idea that the efforts dedicated to discriminate on F_{ki} increase if the payoffs at stake increase. Similarly, we define correlated similarity relations on the set of all expected payoffs to pure strategy $i \in S_k$ to formalize the idea that discrimination between different payoffs increases when the proportion f_{ki} increases (i.e. a finer discrimination is obtained if the accumulated experience is increased and this is assumed to occur when more agents from population k come to play strategy i).

Without loss of generality, we may assume that the payoffs to the $\sum agents$ are strictly positive and do not exceed 1 (we might imagine that to put in practice their similarity based decision procedure, the $\sum agents$ would perform suitable positive affine transformations of the payoff functions of the underlying game G combined with local shifts of such functions; these operations, -assumed to be the same for all $\sum agents$ -, would leave invariant the best reply correspondences and hence, the set of Nash equilibria of G. Thus, the game G', obtained by means of those transformations, and G are equivalent). Then, $\pi_{ki}(f)$ denotes the expected payoff to $SD\ agent\ ki$ and $p_{ki}(f)$ the expected payoff to $\sum$ agent ki, with $p_{ki}(f) \in \Pi_{ki} = (0, 1]$.

Let $(p_{ki}(f), f_{ki})$ be the vector of expected payoff-proportion of agents of player population k attached to strategy $i \in S_k$ at time t. The d_{ki} function serves for two purposes:

(a) To define on Π_{ki} correlated similarities of the difference-type: each $\sum -$ agent ki calculates expected payoffs correctly but, due to the ambiguity about his play, develops a similarity interval for $p_{ki}(f)$: given f_{ki}, the interval is $[p_{ki}(f) - d_{ki}(f_{ki}), p_{ki}(f) + d_{ki}(f_{ki})]$. Thus, given f_{ki}, $d_{ki}(f_{ki})$ defines the threshold level on Π_{ki}. By Assumption 1, if f_{ki} increases, the threshold decreases and so the similarity interval of $p_{ki}(f)$ shrinks.

(b) To build the λ_{ki} function which will be used to define on F_{ki} correlated similarity relations of the ratio-type (see Rubinstein (1988)). Assuming that each $\sum$ agent ki is endowed with a $d_{ki} \in D$, the function λ_{ki} is defined as follows: given some $f_{ki} \in (0, 1)$, for all $p_{ki}(f) > d_{ki}(f_{ki})$

$$\lambda_{ki}(p_{ki}(f)) = \frac{p_{ki}(f)}{p_{ki}(f) - d_{ki}(f_{ki})} > 1 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Thus, given d_{ki} and a proportion $f_{ki} \in (0, 1)$, the domain of λ_{ki} will be the payoffs satisfying $p_{ki}(f) > d_{ki}(f_{ki})$. Hence, there is a family of λ_{ki} functions parameterized by $f_{ki} \in (0, 1)$. If $p_{ki}(f) \leqq d_{ki}(f_{ki})$ then, λ_{ki} is not defined and we would have the degenerate similarity relation, i.e. a relation for which the similarity interval of any point in Π_{ki} is the whole space Π_{ki} (see Rubinstein (1988)).

The λ_{ki} function is a threshold function that measures the discrimination capacity on F_{ki}.Hence, given $p_{ki}(f)$, the similarity interval for f_{ki} is $[f_{ki}/\lambda_{ki}(p_{ki}(f))$ $, f_{ki}.\lambda_{ki}(p_{ki}(f))]$. Note that if the payoffs at stake $p_{ki}(f)$ increases then[1], the similarity interval of f_{ki} shrinks because $\frac{\partial \lambda_{ki}}{\partial p_{ki}(f)} < 0$.

3.3. Satisficing $\sum$ Preferences and Endogenous Aspiration Sets.

For any vector $(p_{ki}(f), f_{ki}) \in \Pi_{ki} \times F_{ki}$, attached to strategy $i \in S_k$ at time t, the pair of similarity relations are used to define a preference relation that would determine the upper and lower contour sets relative to that vector as well as its indifference set (the procedure is described in Appendix 1). Figure 1 depicts the resulting (non-complete and non-transitive) $\sum$ preference relation, $\succsim_{ki}$defined on $\Pi_{ki} \times F_{ki}$, when the $\sum$ agent ki is outside the two playing modes mentioned in Remark 1. We assume that the preferred set, denoted by $U =$

$U\alpha \cup U\beta \cup U\delta$, represents the $\sum -$agent $ki's$ aspiration set at time t. By definition, as $(p_{ki}(f), f_{ki})$ changes the corresponding aspiration set, obviously, changes.

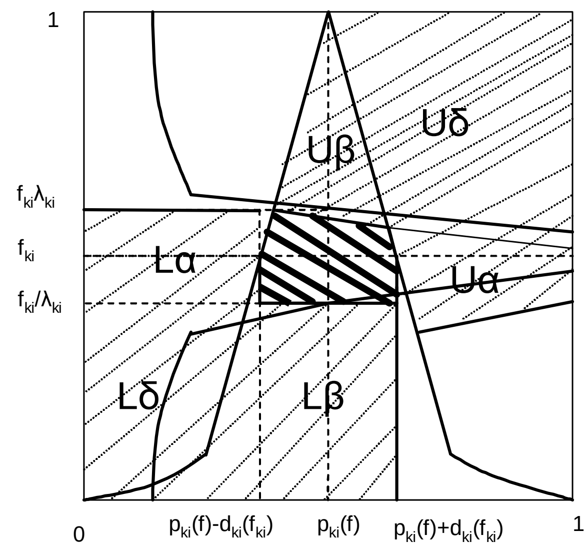


Figure 1. The $\sum$ preference $\succsim_{ki}$. Given the vector $(p_{ki}(f), f_{ki})$, its upper and lower contour sets, obtained by a procedure based on a pair of socially induced similarity relations, are denoted by U and L, repectively. The indifference set is the darkest area

We assume that a $\sum$ agent ki is a $\sum$ preference satisficer, in the sense that he chooses a strategy just to minimize the distance from $(p_{ki}(f), f_{ki})$ to his aspiration set U. We should note that:

(i) given f_{ki} and d_{ki}, since $\frac{\partial \lambda_{ki}}{\partial p_{ki}(f)} < 0$, the similarity interval of f_{ki}, $[f_{ki}/\lambda_{ki}(p_{ki}(f))$ $, f_{ki}.\lambda_{ki}(p_{ki}(f))]$, shrinks if $p_{ki}(f)$ increases. The horizontal wedge-shaped form of Figure 1 shows this property.

(ii) an increase in f_{ki} (say, to $\overline{f}_{ki}$) has more subtle implications: given $p_{ki}(f)$ and d_{ki}, since $d_{ki}(\overline{f}_{ki}) < d_{ki}(f_{ki})$, the similarity interval of p_{ki}, $[p_{ki}(f) - d_{ki}(\overline{f}_{ki}), p_{ki}(f) + d_{ki}(\overline{f}_{ki})]$, will shrink. This property is shown by the vertical triangle-like form of Figure 1. Furthermore, the change in f_{ki} creates a new function, $\overline{\lambda_{ki}}$, such that $\overline{\lambda}_{ki}(p_{ki}(f)) = \frac{p_{ki}(f)}{p_{ki}(f) - d_{ki}(\overline{f}_{ki})} < \lambda_{ki}(p_{ki}(f))$ for every $p_{ki}(f) > d_{ki}(f_{ki})$.

Therefore, both in (i) and (ii) we get a thinner indifference set $\sim_{ki} [(p_{ki}(f), f_{ki})]$ - the dark area of Figure 1- and a smaller value of λ_{ki}. A thinner indifference set implies a smaller distance from $(p_{ki}(f), f_{ki})$ to the aspiration set and hence, the agent will feel more satisfied with his current strategy. Since in this event we would have a smaller value of λ_{ki}, the function λ_{ki} could be thought of as an indicator of the degree of satisfaction of $\sum -$ agent ki. The smaller the value of λ_{ki}, the happier would feel the agent with his current strategy.

3.4. The Dynamic of Drift

We take the ratio $\frac{1}{\lambda_{ki}}$ as the probability that $\sum$ agent ki will retain his current strategy $i \in S_k$ in the next period ; $1 - \frac{1}{\lambda_{ki}}$ will then be the probability of switching to a different strategy in S_k.

The $\sum -$ agent ki only has information about the strategy he is currently using (i.e. the payoffs and frequencies attached to strategy i) . The next assumption describes his behaviour when he feels dissatisfied with his current strategy.

Assumption 3

When a $\sum$ agent ki is dissatisfied with his current strategy, he will choose the $m_k - 1$ available strategies $j \in S_k$, $j \neq i$, with the same probability $\frac{1}{m_k-1}(1 - \frac{1}{\lambda_{ki}})$.

Thus, the $\sum$ agents follow the rule "try every other action if you feel dissatisfied with your current strategy". Given Assumption 3, the $\sum$ agents perturb the SD system by injecting strategies that are not currently played. We assume that when a $\sum$ agent switches strategy, he learns, by imitation or education, to measure how well is playing with the newly adopted threshold function. Inside the $\sum$ agents of population k, strategy $i \in S_k = \{1, 2, ..., m_k\}$ will be played by those dissatisfied $\sum$ agents kj, $j \neq i$, coming from the rest of $m_k - 1$ strategies (the inflow): $\frac{1}{m_k-1} \sum_{j \neq i}^{m_k} f_{kj}(1 - \frac{1}{\lambda_{kj}})$. The outflow is the proportion of dissatisfied $\sum$ agents ki who abandon the strategy i: $f_{ki}(1 - \frac{1}{\lambda_{ki}})$. We shall assume that the drift term added to the $ki-th$ selection dynamics equation (see equation (5) below) is just the difference between these two flows. Hence, those who retain their current strategy are not included in the drift term. Therefore, we would have

$$\frac{1}{m_k - 1} \sum_{j \neq i}^{m_k} f_{kj}(1 - \frac{1}{\lambda_{kj}}) - f_{ki}(1 - \frac{1}{\lambda_{ki}})$$

If we simplify the notation by denoting $\theta_{ki}(f) = \frac{1}{m_k-1} \sum_{j \neq i}^{m_k} f_{kj}(1 - \frac{1}{\lambda_{kj}}) + f_{ki}\frac{1}{\lambda_{ki}}$ then, the drift term takes the following form

$$[\theta_{ki}(f) - f_{ki}]....................(3)$$

It is worth mentioning that $\theta_{ki}(f)$ in equation (3) is not exogenously given as in B&S and the noise models of Hopkins (2002). B&S interpret $\theta_{ki}(f)$ as mistake probabilities which might reflect rules of thumb and favour the assumption that each $\theta_{ki}(f)$ is fixed and positive. For instance, the specification of $\theta_{ki}(f)$ could reflect the rule of calculations based on a uniform distribution over strategies. The assumption implies that drift will point inward the state space and this could become problematic because it prevents exact convergence on the equilibria observed in the laboratory. In the present model, $\theta_{ki}(f) = \frac{1}{m_k-1}\sum_{j\neq i}^{m_k} f_{kj}(1-\frac{1}{\lambda_{kj}}) + f_{ki}\frac{1}{\lambda_{ki}}$ is endogenous and so drift is not necessarily inward-pointing, therefore the problem of approximate convergence and quantitative matching with laboratory data can be avoided.

3.5. The Connection between Threshold Function, Socially Induced Similarity Relations and Drift

We have two limiting values of drift. If the $\sum$ agent ki is almost completely certain that is playing well (*alert mode*) then, no matter what the values of $p_{ki}(f)$ and f_{ki} are, the similarity intervals and the indifference sets will have almost an empty interior and so $(p_{ki}(f), f_{ki})$ is "near" the agent's aspiration set. In other words, the agent is very satisfied with his current strategy and λ_{ki} will have very small values. Hence, the probability $1-\frac{1}{\lambda_{ki}}$ of switching, which is the source of drift, will be negligible. If the $\sum$ agent ki were in the *absent mode* then, the above implications will be reversed, his indifference sets would cover almost the entire space $\Pi_{ki}\times F_{ki}$ and the agent could be said to be "highly dissatisfied" with his current strategy.

Payoffs and strategy proportions will play an active role to determine the value of λ_{ki} and therefore the level of drift when the agent is outside the two playing modes (Figure 1 depicts this case). In this case, increases in f_{ki} and $p_{ki}(f)$ imply a decrease in drift[2].

4. Drift based on AINU's Similarity Relations.

Now we shall deal with similarity relations that are not socially induced. To this end, we use the correlated similarity relations introduced by Aizpurua, Ichiishi, Nieto and Uriarte (1993), - AINU-, for the space of simple lotteries. It is assumed that each $\sum$ agent feels a constant level of vagueness about how well is playing. That is, we assume that the $\sum$ agents of each player role $k\in K$ is endowed with a threshold function g_k, such that for every $i\in S_k$

$$g_k : F_{ki} \to [0,1]\,,\ \text{with } g_k(f_{ki}) = \epsilon_k, \text{ for all } f_{ki}\in[0,1]\,, \text{ being } \epsilon_k\in[0,1)$$

Hence, the threshold function that measures the ambiguity felt by the $\sum$ agent ki is not sensitive to the proportions f_{ki} (thus, the agent is not influenced by the social norms and conventions that those proportions might convey)). Assuming that each $\sum$ agent ki is endowed with the constant function g_k, a new function ψ_{ki} is defined as follows: for all $p_{ki}(f) > \epsilon_k$

$$\psi_{ki}(p_{ki}(f)) = \frac{p_{ki}(f)}{p_{ki}(f)-\epsilon_k} > 1 \qquad (4)$$

Notice that the λ_{ki} function defined in (2) was parameterized by f_{ki}, so that there is one λ_{ki} for each $f_{ki}\in(0,1)$. This is not the case for the ψ_{ki} function, because ϵ_k is constant. With the aid of g_k and ψ_{ki}, we can define similarity relations on the payoff and the strategy proportion spaces respectively. On Π_{ki} we define similarity relations of the difference type, for which the similarity intervals are $[p_{ki}(f) - \epsilon_k, p_{ki}(f) + \epsilon_k]$. On F_{ki}, we define correlated similarities of the ratio type. With a choice procedure (described in Appendix I), we can build a preference relation on $\Pi_{ki} \times F_{ki}$ depicted in Figure 2. Note that for a given payoff $p_{ki}(f)$, the size of the indifference set and, therefore, the drift level introduced by the $\sum$ agent ki depends on ϵ_k. Since $\partial\psi_{ki}/\partial p_{ki}(f) < 0$ then, if the expected payoff increases, the size of the indifference set shrinks and the level of drift decreases.

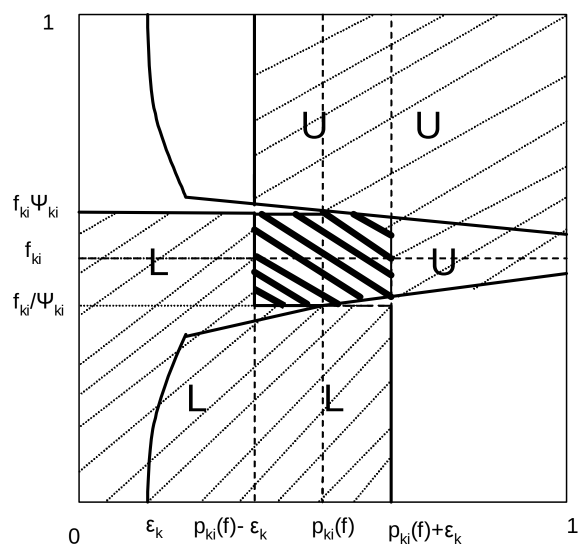


Figure 2. The AINU $\sum$ preference $\succsim_{ki}$. Given the parameters ϵ_I, ϵ_{II} and the vector $(p_{ki}(f), f_{ki})$, the upper and lower contour sets, obtained by a procedure based on a pair of AINU similarity relations, are denoted by U and L, repectively. The indifference set is the darkest area

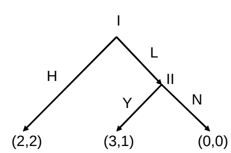
We proceed as in the previous case to build the drift term and we get

$$\begin{aligned} & \frac{1}{m_k - 1} \sum_{j \neq i}^{m_k} f_{kj}(1 - \frac{1}{\psi_{kj}}) - f_{ki}(1 - \frac{1}{\psi_{ki}}) \\ = \quad & \epsilon_k [\frac{1}{m_k - 1} \sum_{j \neq i}^{m_k} \frac{f_{kj}}{p_{kj}(f)} - \frac{f_{ki}}{p_{ki}(f)}] \end{aligned}$$

5. Results

We present now the results which may be obtained with the two models of drift described in the previous sections. The results refer to the drift specifications needed to stabilize Nash equilibria, in particular, those that are not subgame-perfect.

To this end, we shall consider first a simplified version of the Ultimatum Game (UG) (see Binmore et al. (1995)) whose strategic form is described in Figure 3. Player I is the population of Proposers who has two available strategies: to propose a high division (H) or a low division (L) of a cake of size 4. Player population II, the responders, may choose, when they are offered the low division, between accepting it (Y) or rejecting it (N). The Chain-Store Game has the same structure as the Ultimatum Minigame but is of a different economic nature. The empirical results of these two games suggest that different interpretations of (formally) the same game,- a game of entry-deterrence or a bargaining game-, may have different observed behaviours.



	(y) Y	N
(x) H	2,2	2 ,2
L	3,1	0,0

Figure 3. The extensive and strategic forms of the Ultimatum Minigame.

5.1. Drift produced by Socially Induced Similarity Relations with $\sum$ agents in Playing Modes.

When the perception of a game is influenced by individual values or by social norms and conventions then, different strategies in that game are a priory valued differently (i.e. before the game and during the play), and therefore the tendencies to abandon them might differ from one individual to another. To capture this situation, we assume that each $\sum$ *agent* plays his current strategy in a given playing mode. Note that given a game, the existence of a playing mode attached to a strategy must be deduced, in our opinion, from the data obtained in the laboratory about that game, as well as from the knowledge of society's modal tastes and values. In other words, the specification of the playing modes must be empirically determined.

For concreteness, the selection dynamics of this section is assumed to be the standard Replicator Dynamics (RD). The resulting perturbed deterministic

RD, derived by the joint behaviour of the $SD-ki$ agents (whose behaviour leads to the RD) and the $\sum -ki$ agents, will therefore be

$$\overset{\bullet}{f}_{ki} = f_{ki}\left(\pi_{ki}(f) - \overline{\pi}_k(f)\right) + [\theta_{ki}(f) - f_{ki}]\text{.................}(5)$$

Note that for each player-population k

$$\sum_{i=1}^{m_k} \overset{\bullet}{f}_{ki} = \sum_{i=1}^{m_k} f_{ki}\left(\pi_{ki}(f) - \overline{\pi}_k(f)\right) + \sum_{i=1}^{m_k} \ [\theta_{ki}(f) - \ f_{ki}] = 0$$

Example 1: **The Ultimatum Minigame (UM)**

Let the probabilities of playing H and Y be denoted as x and y, respectively. This game has a unique Subgame-perfect equilibrium $(x, y) = (0, 1)$ and a component of Nash equilibria, denoted NC, the segment joining (1, 0) and (1, 2/3). Let λ_H , λ_L, λ_Y, λ_N, d_H, d_L, d_Y and d_N denote the threshold functions of the $\sum$ agents playing strategy High, Low, Yes and No, respectively. Then, the perturbed system (5) for the UM is the following (time index suppressed)

$$\begin{aligned}
\dot{x} &= x(1-x)(2-3y) - x(1-\frac{1}{\lambda_H}) + (1-x)(1-\frac{1}{\lambda_L}) \\
&= x(1-x)(2-3y) - xd_H(x)/p_H(y). + (1-x)d_L(1-x)/p_L(y)\text{...............}(6) \\
\dot{y} &= y(1-y)(1-x) - y(1-\frac{1}{\lambda_Y}) + (1-y)(1-\frac{1}{\lambda_N}) \\
&= y(1-y)(1-x) - yd_Y(y)/p_Y(x) + (1-y)d_N(1-y)/p_N(x)\text{................}(7)
\end{aligned}$$

The experimental findings about the UG are very robust (see also the findings of Güth et al.(2001) though) and show that people share a common notion about what is a fair, reasonable or acceptable offer and that their play is largely guided by those notions. How is this result interpreted in terms of our model?. Let us suppose that someone's behaviour is guided by the following norm: "Be magnanimous and learn to say no to injustice". Then, our model would capture this (pregame) attitude by saying that this person would very likely play $High$ (in the role of proposer) and No (in the role of responder) in the *alert mode*. The robustness of the experimental findings about the UM seems to suggest that a high percentage of people are inequity averse and have an a priory idea of what is the right way of playing the UG. This would mean, in terms of the present model, that they would probably play the fair strategies in the *alert mode*. But, we may think as well that some agents -mainly proposers- might initially experiment with strategies that are not fair, just to see how the opponent reacts. We conjecture that those agents will play those strategies knowing, in advance, that sooner or later must abandon them. In other words, they will play in the *absent mode*.

Case I of Proposition 1 shows that, even if initially there is a very small percentage of $\sum$ agents playing H and N in the *alert mode* and the rest of agents in both populations are in the *absent mode*, $(1, 0) \in NC$ will be the only asymptotically stable outcome. In other words, if the initial play for the perturbed system (6)-(7) is arbitrarily near, say, the subgame-perfect equilibrium, $(0, 1)$, where there is only a small percentage of highly fairness-motivated $\sum$ agents in both player populations, the theorem shows that both the SD agents (now the replicator dynamic agents) and the $\sum$ agents learn to coordinate in the non perfect equilibrium where all proposers choose H and all responders N.

Example 2: The Chain-Store Game (CH-S)

The UM and CH-S games describe different economic situations and therefore the drift terms need not be the same. We shall assume, for simplicity, that both games have the payoffs of Figure 3. Player I is now the potential entrant and player II the incumbent (Monopolist). Thus, change in Figure 3, H and L for NE (Not Enter) and E(Enter), respectively; Y (Yield) and F(Fight) are now the strategies for player II. Hence, the CH-S game would correspond to the (only) Weak Monopolist Game of Jung et al.(1994) in which the incumbent would prefer to share the market if entry occurred. We may conjecture two different situations modelled by two different specifications of drift. For instance, let us consider the case when potential entrant $\sum$ agents playing NE are in the *alert* mode, those playing E are in the *absent mode* (that is being action E riskier, agents playing E have a high uncertainty about how well are playing) and all $\sum$ incumbent agents, i.e. those playing Y and F, are in the *absent* mode or almost in that mode. Thus, $\sum$ incumbents think to know well the trade, overestimate their power and do not care about their play. Let x denote the proportion of potential entrants playing NE and y the proportion of incumbents playing Y. Then, in Proposition 1, **Case II**, below, we get (1, 1/2) as a global asymptotic attractor.

The next situation would approach the case of experienced players with sufficient time and learning with no experimenter-induced strong monopolist of Jung et al.(1994). The appropriate specification of drift for this situation could be when both potential entrants playing E and incumbents playing Y are in the *alert mode*, while the rest of agents in both populations are in the *absent mode*. Then, in **Case III** of Proposition 1, we show that the subgame-perfect equilibrium is a global asymptotic attractor (and elements of NC are not local attactors). The next result is a full stability study of these two games.

Proposition 1

Case I. Suppose in the UM Game that the $\sum$ H agents (i.e. $\sum$ proposers offering H) are in the *alert mode* (so $d_H = \overline{d}$) and the $\sum$ L agents are in the *absent mode* ($d_L = \underline{d}$). Then, if responders playing Y are in the *absent mode* ($d_Y = \underline{d}$) and *those* playing N are in the *alert mode* ($d_N = \overline{d}$), the only asymptotic attractor is the equal-split Nash equilibrium $(1, 0)$.

Case II. Suppose in the CH-S Game that the $\sum$ NE agents are in the *alert mode* ,($d_{NE} = \overline{d}$) and the $\sum$ E agents are in the *absent mode* ($d_E = \underline{d}$). Then, if both $\sum$ agents Y and F are almost in the *absent mode* and $d_Y(.) = d_F(.)$, the only asymptotic attractor is $(1, 1/2)$.

Case III. Suppose in the CH-S Game that the $\sum$ NE agents are in the absent mode (so $d_{NE} = \underline{d}$) and the $\sum$ E agents are in the alert mode ($d_E = \overline{d}$). Then if $d_Y = \overline{d}$ and $d_F = \underline{d}$, the only asymptotic attractor is the subgame-perfect equilibrium $(0, 1)$.

Proof : In Appendix II. ■

Remark 2.1

It is easy to see that with the playing modes model of drift it can be stabilized Nash equilibria in components with empty interior (see, for instance, the game shown in p.382 of B&S(1999)). Hence, contrary to the B&S model, predictions could be based on arguments that do not depend on the size of the Nash component (for the notion of "size of a component", see B&S (1999), Section 6, p.378).

Remark 2.2

The assumption of Case I that the profile of strategies that are a Nash equilibrium are played in the *alert mode* and those outside the profile are played in the *absent mode* can be weaken. If we only assume that the strategies that are a Nash equilibrium are played in the *alert mode,* then we would get two asymptotic attractors $(1, 0)$ and the subgame-perfect equilibrium $(0, 1)$.

5.2. Drift Produced when the $\sum$ agents are outside the Playing Modes and the Use of Laboratory Data

The previous model could be, in some cases, said that uses ad hoc assumptions about how the playing modes are assigned to strategies. To avoid this issue, we propose now two methods (of course, they are not the only ones) to determine the d_{ki} functions in D by using laboratory data. We could say that in this way the $\sum$ agents are endowed with an endogenously determined threshold function d_{ki}, as the result of a process of interactive learning.

Method I: We should note first that in the set D, for any $r_{ki} \in [0, 1]$, $d_{ki} = r_{ki}\overline{d} + (1 - r_{ki})\underline{d} \in D$. Then, we can use the laboratory data about f_{ki} at each round to give values to r_{ki} and estimate the d_{ki} of each $\sum$ agent ki. Thus, for instance in the UM Game, d_H and d_L could be defined as: $d_H(x) = x\overline{d}(x) + (1 - x)\underline{d}(x)$ and $d_L(1 - x) = (1 - x)\overline{d}(1 - x) + x\underline{d}(1 - x)$; d_Y and d_N are defined in a similar manner. An alternative definition would be, $d_{ki}(f_{ki}) = f_{ki}\overline{d}(s) + (1 - f_{ki})\underline{d}(s)$, for some fixed $s \in (0, 1)$, but this would not imply any change on Proposition 2 below.

Method II: A more interesting approach seems to be when we use both f_{ki} and the data about the payoffs $p_{ki}(f)$ of each round to determine the d_{ki} functions (in the lab we would use the realized payoffs). We define the threshold functions as $d_{ki} = [f_{ki}\overline{d} + (1 - f_{ki})\underline{d}]^{\beta(p_{ki}(f))}$ and assume that the degree of alertness increases with payoffs. That is, $\partial\beta(p_{ki}(f))/\partial p_{ki}(f) > 0$, so that when $p_{ki}(f)$ approaches 1, $d_{ki}(f_{ki})$ approaches $\overline{d}$ and the $\sum$ agent ki would be near the *alert mode* and when $p_{ki}(f)$ tends to 0 the agent would be near the *absent mode*.

The equation system for the UM and CH-S games is now

$$\dot{x} = x(1-x)(2-3y) - xd_H(x)/p_H(y) + (1-x)d_L(1-x)/p_L(y) \text{.............(8)}$$
$$\dot{y} = y(1-y)(1-x) - yd_Y(y)/p_Y(x) + (1-y)d_N(1-y)/p_N(x) \text{.............(9)}$$

The next result shows that the perturbed system resulting from Method I has the same properties as the unperturbed replicator dynamics .

Proposition 2

If each $\sum$ agent ki is endowed with a threshold function defined as $d_{ki}(f_{ki}) = f_{ki}\overline{d} + (1-f_{ki})\underline{d}$ then, the subgame-perfect equilibrium $(0,1)$ is the only asymptotic attractor of the system (8)-(9) and each element of the interior of NC is a local attractor.

Proof: In Appendix II.

Proposition 3

If each $\sum$ agent ki is endowed with a threshold function defined as $d_{ki} = [f_{ki}\overline{d} + (1-f_{ki})\underline{d}]^{\beta(p_{ki}(f))}$ with $\partial\beta(p_{ki}(f))/\partial p_{ki}(f) > 0$ then,

(a) if d_H, d_L, d_Y and d_N are convex functions, the asymptotic attractors are $(0,1)$and $(1,0)$ and

(b) if d_H and d_L are convex and d_Y and d_N are concave functions, the asymptotic attractors are $(0,1)$ and $(1,1/2)$.

Proof: In Appendix II.

5.3. Drift based on AINU's type of similarity relations.

Let ϵ_I, $\epsilon_{II} \in [0,1)$ denote the constant levels of vagueness or uncertainty about how well are playing felt by proposers and responders, respectively. Then, the system would be

$$\dot{x} = x(1-x)(2-3y) + \epsilon_I[-x/p_H(y) + (1-x)/p_L(y)] \text{.............(10)}$$
$$\dot{y} = y(1-y)(1-x) + \epsilon_{II}[-y/p_Y(x) + (1-y)/p_N(x)] \text{.............(11)}$$

We have the following result, described by Figure 4.

Proposition 4

For values of ϵ_I/ϵ_{II} greater than 0, the system $(10) - (11)$ has a unique asymptotic attractor located in the vicinity of the subgame-perfect equilibrium $(x,y) = (0,1)$. As the ratio ϵ_I/ϵ_{II} approaches 0, there is an additional asymptotic attractor, which in the limit, when $\epsilon_I/\epsilon_{II} = 0$, is the element $(x,y) = (1,1/2)$.

Proof: In Appendix II.

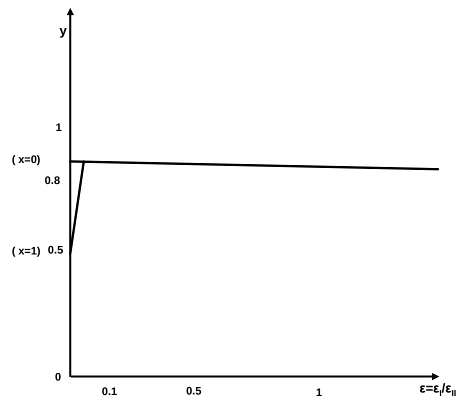


Figure 4. The ϵ−correspondence shows the asymptotic attractors for each value of the ratio ϵ_I/ϵ_{II}. The figure assumes $\epsilon_{II} = 0.1$ and ϵ_I taking values from 0 onwards.

6. Relation with the Literature

One might find some resemblances between the theory of this paper and that of the quantal response equilibrium (QRE). In the latter, players make correct estimates of expected payoffs but have an additive payoff disturbance (or error). Here, the $\sum$ agents too make correct estimates of the expected payoffs, but since they have doubts about how well they are playing the game, the estimated value is not distinguished from those on a similarity interval. In the QRE, experience implies a decrease in the errors. The similarity theory used in the present paper assumes as well that an increase in the number of $\sum$ agents ki is equivalent to an increase in experience and therefore the size of the similarity interval of expected payoff $p_{ki}(f)$ decreases and that if the payoffs at stake increase perturbations decreases. We find more coincidences between the QRE and the model of *drift based on AINU's type of similarity relations* which imply the above system (10)-(11). Recall that $\epsilon_k, (k = I, II)$, is a parameter that measures the vagueness felt by player k about how well is playing; this parameter determines the size of the similarity interval of the expected payoffs to each strategy i available to k: $[p_{ki}(f) - \epsilon_k, p_{ki}(f) + \epsilon_k]$. This is a "noisy interval" for both the QRE theory and the present paper´s theory. Hence, the ratio $\epsilon = \epsilon_I/\epsilon_{II}$ would be the relative noise, between proposers and responders, in the payoff space. In the system (10)-(11) we also see that the value of ϵ_k, $(k = I, II)$, determines the degree of influence of drift upon the replicator dynamic equation for each pure strategy of player k. In other words, ϵ_k is also a parameter measuring the sensitivity of player $k\prime s$ replicator equations to the noise produced by each $\sum$agent ki. This feature has some vague resemblance with the parameter λ of the logit QRE which measures the sensitivity of the response function to the level of error.

Needless to say that the main difference between the two models is the dynamic approach taken here. But, if we concentrate on the results, we find, under some conditions, coincidences between the logit equilibria and the limit points of some of the models of drift presented above. McKelvey and Palfrey (1998) study the quantal response equilibrium of two extensive versions of the chainstore paradox game, -the extensive (i.e. sequential) version and, what they call, the "strategy" version-, having both the same normal form, with the payoffs of the above Figure 3. The purpose is to test the invariance property. When studying the features of the Agent QRE correspondence (for the probabilities of each action) as a function of λ they find that there is the AQRE that converges to the subgame-perfect equilibrium and, as opposed to the extensive version, in the strategy version, for a large λ, there is an additional QRE that converges on the imperfect equilibrium $x = 1$, $y = 1/2$ (recall that x is the probability of NE and y is the probability of Y). The present paper assumes, as usual, that the invariance property is satisfied. Let us turn our attention to Proposition 4 and to the $\epsilon-$ correspondence of Figure 4 which summarizes this result. The $\epsilon-$ correspondence shows, for each ϵ, the asymptotic attractors of the system (10)-(11). For high values of ϵ, there is only one limit point which is near (x, y) $= (0, 1)$, depending on the values of ϵ_I and ϵ_{II}. But, as ϵ goes to 0, the system converges in the limit, (i.e., when $\epsilon_I = 0$), to $x = 1$, $y = 1/2$, as well as to (an approximation of) the subgame-perfect equilibrium. When $\epsilon_I = \epsilon_{II} = 0$ the system (10)-(11) is reduced to the replicator dynamic equations and hence the limit point will be $(0, 1)$.

Let us look now to the model of *drift produced by Socially Induced Similarity Relations with* $\sum$ *agents in Playing Modes*. In Case II of Proposition I, $x = 1$, $y = 1/2$ appears again as an asymptotic attractor (of the system (6)-(7)). How is this result explained?. McKelvey and Palfrey (1998) find empirical support to their result in Schotter et al. (1994) and rationalize it in terms of plausibility, in the sense that it is more likely to observe that player 2 perceives the suboptimality of F only when player 1 has chosen E (see p.19 of MacKelvey and Palfrey (1998)). We instead explain the result in terms of relative drift. As we said above, in Case II we assume that incumbents overestimate their market power and play without taking too much care, while potential entrants take a lot of care and pay much more attention to their decisions. In other words, they are more alert (and hence less noisy) than incumbents. Therefore, under the assumptions of Case II, perturbations in the entrant population introduce NE much more frequently than E, while in the incumbent population the $\sum$ agents are assumed to equally noisy and so perturbations introduce both Y and F with the same frequency. Hence, as the frequency of NE increases, drift approaches the component NC, where Y and F get equal payoff. Then, the state of NC which will be stabilized depends on the relative sharpness of d_Y and d_F and, since in Case II it is assumed that they are equally sharp, the stabilized state will be $(1, 1/2)$. We have seen in Proposition 4 that responders should be noisier than proposers to stabilize $x = 1$, $y = 1/2$. This happen when the relative noise, measured by $\epsilon = \epsilon_I/\epsilon_{II}$, is 0.

Therefore, we can conclude that the equilibria obtained in the Chain-Store Game with both the *model of drift based on AINU's type of similarity relations* and the *model of drift with* $\sum$ *agents in Playing Modes* might coincide, under some conditions, with those of the logit equilibria.

It is also worth noting that Binmore et al. (1995), assuming endogenous drift and uniform mistake probabilities for both populations, -i.e. $\theta_{ki}(f) = 1/2$ for each k and each i-, show, in their Proposition 3, that the asymptotically stable outcomes are the subgame-perfect equilibrium, $(0, 1)$ and (due to the assumption of fixed and uniform mistakes (which implies *inward pointing drift*) an approximation to $(1, 1/2)$. A key element in Binmore et al.'s (1995) Proposition 3 is that responders should drift more than proposers and this can happen only if drift is sufficiently sensitive to payoffs. Similarly, Proposition 4 shows that to stabilize $(1, 1/2)$, the ratio of noise $\epsilon = \epsilon_I/\epsilon_{II}$ should be 0.

Finally, Case I of Proposition 1 shows the conditions to stabilize the state $(1, 0)$ and be obtained as the only asymptotically stable outcome. This case is related to the work of Abbink et al.(2001). We may say, in their words, that we are dealing with fairness motivated agents, loyal to the strategy that would implement the equal split equilibrium. Thus, in our model $\sum$ responders playing No would be "programmed" to punish unfair offers. The result of Case I predicts learning in both populations, whereas in Abbink et al.(2001), there is only evidence for first movers learning. Remark 2.2 shows a less stringent assumption to have $(1, 0)$ as an outcome of the system.

7. Example 3: The Full Ultimatum Game.

In Roth et al. (1991) and Roth and Erev (1995) it is reported an experiment with the Ultimatum Game[3] carried out in four countries: Israel, Japan, USA and Slovenia. It is observed that the norms that are commonly used in real-life bargaining situations prompt individuals to initially allocate a significant share of the surplus to the responders. As argued by Roth and Erev (1995), players' initial propensities can have a long-term influence in the players learning. In the evolutionary framework, it can be said that the initial propensities in the Ultimatum Game tend to be located in the basin of attraction of the observed equilibria: that is, in the basin of $\{5\}$ for Slovenia and USA and $\{6\}$ for Israel and Japan. Hence, both the initial propensities and the final outcomes are the data that should be predicted by the theory. We report here a computer simulation[4] of this game using the four models of drift presented in section 5.

First, a few words about the B&S model. In Binmore et al. (1995) it is said that the range of potential equilibria obtained with the B&S model does not include, in general, the "fair outcome" in the Full Ultimatum Game (we shall see below that this is not the case with the models developed in this paper). It was reported there (see p.68) that even increasing the mistake probabilities attached to the "fair" offer has little effect on the results of the calculations. Furthermore, even though the observed initial propensities are in the neighborhood of the final outcomes of this game, simulations show that in the B&S model the solution trajectories starting from those initial propensities can lead, in many cases, towards the subgame-perfect equilibrium or equilibria close to it. Hence, we may say that the predictions derived with the B&S model are somehow inconsistent with the experimental results[5].

A brief account of the numerical calculations is the following.

(i) Drift with $\sum$ agents in Playing Modes.

We mentioned above that the degree of sharpness of d_{ki} could be used to model how sensitive is agent ki to society's norms and conventions which are encoded in the strategy frequencies f_{ki}. Let $\{A_{ki}\}$ denote that the $\sum$ agents in population k play strategy i in the *alert mode* and that the rest of strategies $j \neq i$ in population k are played by $\sum$ agents in the *absent mode*. Hence, we shall assume that in Japan and Israel proposers and responders play strategy 6 in the *alert mode* and the rest of strategies are played in the *absent mode*; that is $\{A_{I6}, A_{II6}\}$ denotes the set of threshold functions of those two countries, and $\{A_{I5}, A_{II5}\}$ those of Slovenia and USA (where $k = I$ stands for proposers and $k = II$ for responders).

The computations with the Full Ultimatum Game show that the observed laboratory equilibria, $\{5\}$ in USA and Slovenia and $\{6\}$ in Israel and Japan, appear as global asymptotic attractors[6]. This means that, contrary to what happens to the reinforcement learning model, in the present model of perturbed learning, the experimental outcomes can be obtained independently of the initial play. Hence, to simulate the path to the observed outcomes in the Ultimatum Game, the model does not need to take the observed initial play (or the initial propensities of Roth and Erev (1995)) as given. This model of drift predicts that the observed laboratory data, i.e. both the initial propensities and the equilibria, are due to a rather high sensitivity of individuals to fairness-oriented norms, leading them to use their analytical and perceptual resources at their highest level, i.e. in the *alert mode*.

We conclude that this model of drift does account well for the final data, but, on the negative side, it should be said that it is certainly *ad hoc*. We might accept that the equal split strategy 5 be played by both players in the *alert mode,* but how do we justify the same for playing 6?.

(ii) Drift with $\sum$ agents outside the two Playing Modes and Laboratory Data

To avoid the degree of "*adhocery*" that the previous model might have, Table I and Table II (in Appendix III) show the modal equilibrium demands (i.e. what player I demands for himself) for the models of drift in which the threshold functions are endogenously formed as in Proposition 2, $d_{ki}(f_{ki}) = f_{ki}\overline{d} + (1 - f_{ki})\underline{d}$, and Proposition 3, $d_{ki} = [f_{ki}\overline{d} + (1 - f_{ki})\underline{d}]^{\beta(p_{ki}(f))}$. In both tables, the entry in row i and column j is the modal equilibrium demand when the initial conditions are that all proposers start playing i and all responders start playing j (we follow Binmore et al.(1995) p.63-64 to decide when the system has converged to a point). Table I shows that the modal demand of 6 is independent of the initial demands made by proposers and the initial maximal acceptable demands made by responders. The outcome of 6 is quite robust as it appears in many of the cells of Table II. Furthermore, we should note that 6 is also the outcome when the system, for these two models of drift, is initialized with each of the strategies being played with probability 1/9. Hence, the laboratory result of Japan and Israel could be matched without any *ad hoc* model of drift. A different case is the outcome 5. Table II shows that it appears when all responders start playing 5 and all proposers start playing any demand from 1 to 5. But as we said above, there is a difference between 5 and 6 and maybe it is not too *ad hoc* to think that fairness motivated agents in both populations will play the equal split strategy 5 in the *alert mode.*

(iii) Drift based on AINU's similarity relations.

Table III summarizes calculations made for various levels of fixed vagueness ϵ_I and ϵ_{II}. The system is initialized with each strategy being played with probability 1/9. As we decrease the vagueness (and hence, the perturbation) of responders, ϵ_{II}, the outcome changes from 6 to 7; that is, the modal demand tends to (but still is far from) the subgame-perfect equilibrium, 9. This is in contrast with Binmore et al.(1995) where, when responders' (fixed) noise is sufficiently small relative to that of proposers, the subgame-perfect equilibrium appears. When $\epsilon_I = \epsilon_{II} = 0$, then the system coincides with the Replicator Dynamics and the outcome is 7.

8. Conclusions

What we have done is to complete a (biologically based) selection dynamic model by adding different models of drift originated by agents whose behaviour is based on decision procedures compatible with similarity relations. We have shown that the addition of this type behaviour, studied by authors like Kahneman and Tversky (1979), has positive implications. With a threshold function that measures the ambiguity felt by each perturbing agent about how well is playing, we build three models of drift based on Socially Induced Similarity Relations. The *playing modes model of drift* combined with the Replicator Dynamics (RD) seems to fit well to explain the influence of norms and conventions, such as fairness, but it could be said that the model makes a certain use of ad hoc assumptions. To avoid this issue we endogenize the threshold functions by using the data about payoffs and strategy proportions and obtain two additional models. We show that both of them are capable to stabilize equilibria that are not subgame-perfect. If similarity relations are not socially induced, we get the *AINU model of drift*. This last model shows resemblances with the QRE model of McKelvey and Palfrey (1995) and stabilizes the same equilibria as McKelvey and Palfrey (1998) in the Chain-Store Game.

We deduce that the failure of Binmore and Samuelson's (1995) model to match the observed data could be the sensitivity of drift to just payoffs and to being inward pointing (see section 6 above). In a different terrain, the results mitigate somehow the Cheung and Friedman's (1998) disappointing tests with the (unperturbed) RD[7].

There are things to be done in future works. One, in particular, is that it should be tested experimentally in which sense the knowledge of how many subjects are playing like me influences on my decisions[8]. This may help to understand the building of new conventions or the robustness (i.e. survival) of "old" individual and social values to evolutionary pressures. Another work left for future research is to use the two classes of similarity relations mentioned here to build, instead of drift, different models of selection dynamics.

9. Notes

1. Given f_{ki} and d_{ki}

$$\frac{\partial \lambda_{ki}}{\partial p_{ki}(f)} = \frac{-d_{ki}(f_{ki})}{(p_{ki}(f) - d_{ki}(f_{ki}))^2} < 0$$

This is the shrinking property of the correlated similarity relation defined by λ_{ki} on F_{ki} (see Uriarte (1999)). It means that if the expected payoff increases, $\sum$ agent ki's perception increases.

2. We can compare the present model of drift with that of Binmore and Samuelson 's (1999). The property $\frac{\partial \lambda_{ki}}{\partial p_{ki}(f)} < 0$ has some similarity with B & S's assumption of a decreasing and Lipschitz continuous drift function on expected payoff differences. This property would also be satisfied had we defined the functions λ_{ki} as

$$\lambda_{ki}(\Lambda_k(f) = \frac{\Lambda_k(f)}{\Lambda_k(f) - d_{ki}(f_{ki})}$$

where $\Lambda_k(f)(> d_{ki}(f_{ki}))$ is the difference between the maximum and the minimum of the expected payoffs attached to player $k's$ strategies given the current strategy frequencies in the opposing populations. Hence, perception increases with B & S 's measure of potential cost of making a mistake, $\Lambda_k(f)$.In the present model, drift is not the payoff sensitive under the two playing modes; that is, when $d_{ki} = \overline{d}$, the derivative $\frac{\partial \lambda_{ki}}{\partial p_{ki}(f)}$is almost zero and if $d_{ki} = \underline{d}$, λ_{ki} will be defined only when $p_{ki}(t) = 1$ and so we cannot take derivatives. Hence, as d_{ki} moves away from $\overline{d}$ and $\underline{d}$, λ_{ki} becomes relatively more sensitive to expected payoffs. As a consequence, we would approach a model of drift such as the one proposed by B & S.

The θ_{ki} of the present paper can be said to be $\sum -agent\ ki's$ "adaptive mistake probability". Mistake probabilities are fixed in the B & S model and agents may avoid the error by increasing their cognitive efforts when the potential cost of making the mistake increases. Instead, the approach taken here seems to be more natural, as agents can learn from their "mistakes", by adjusting them. Endogenous θ_{ki} 's implies that drift is not inward-pointing.

3. The experiment consisted of dividing an amount of money, 10 tokens, and the interpretation of the Ultimatum Game is that Player I is proposing to Player II what he is demanding for himself; the second player's strategies are maximal acceptable demands.

4. To run the simulations, we shall use the subclass of threshold functions in the set D, $d_{ki}(f_{ki}) = (1 - f_{ki})^{n_{ki}}$, with $n_{ki} \in (0, \infty)$ and $i \in S_k$. This subclass is large enough for the purpose of computations. When $n_{ki} \to 0$, the degree of sharpness diminishes. For the simulations, we consider that the $\sum$ agent ki is playing strategy i in the *alert mode* when $d_{ki}(f_{ki}) = (1 - f_{ki})^{10^8}$; he would be in the *absent mode* when $0 < n_{ki} \leq 1$, say $d_{ki}(f_{ki}) = (1 - f_{ki})^{10^{-8}}$. As in Binmore et al. (1995), we approach equation (4) by means of the equation $f_{ki}(t + \tau) - f_{ki}(t) = \tau f_{ki}(t)(\pi_{ki}(f)(t) - \overline{\pi}_{ki}(f)(t)) + \tau(\theta_{ki}(f)(t) - f_{ki}(t))$, where the step size $\tau = 0.01$. When $d_{ki} = [f_{ki}\overline{d} + (1 - f_{ki})\underline{d}]^{\beta(p_{ki}(f))}$, we assume that $\beta(p_{ki}(f)) = p_{ki}(f)/1 - p_{ki}(f)$. We shall consider, like Binmore et al.(1995), that the system has converged

on a point when the first 15 decimals are unchanging. Multiplying the number of iterations of the above discrete equation by τ we would get an approximation to how much learning -number of times to asses strategies- has been needed to reach temporary (medium run) and definitive (long run) stability.

5. Of course, we do not think that the issue is reduced to a mere quantitative matching of the theoretical results with the laboratory data. What is relevant here is the motivation of the perturbations that push the system to converge to the observed equilibrium in a given country as being something closely related to, say, some cultural characteristic that distinguishes the country that is being examined. The model developed in Binmore et al.(1995) requires a mistake probability of 0.95 attached to the equilibrium demand reached in each country and the remaining probabilities being equal to one another. Under this specification of drift and for some values of their α and β parameters, only starting from those observed initial propensities or from a neighborhood of them, the model may match the observed equilibria. The remaining problems are the motivations for this specification of drift that allows the quantitative matching and the point made in the above note 3. Another issue is that, in the Binmore et al. (1995) model of drift, agents' mistakes depend on their capability to compute the potential cost - measured in expected payoffs- of making a mistake.

6. There are other combinations leading to the same result; for instance, when agents playing strategies in the vicinity of 5 and 6 are in the alert mode too, we may simulate the path to the observed equilibria. What happens is that the basin of attraction of 5 and 6 will shrink.

7. In the Matching Pennies Game, -just to mention the behaviour of the *playing modes model of drift* with respect to an interior Nash equilibrium-, when all the $\sum -$ agents in both player populations are in the absent mode, the perturbed system converges to the Nash equilibrium $(1/2, 1/2)$.

8. After each round, every subject should be given information about the proportion f_{ki} of people in his population who have used the same strategy as his current one.

10. Appendix I:

10.1. Socially Induced Correlated Similarity Relations.

Given a pair of vectors, $(\overline{p}_{ki}(\overline{f}), \overline{f}_{ki})$ and $(p_{ki}(f), f_{ki})$ in $\Pi_{ki} \times F_{ki}$, with $\overline{f}_{ki}$, $f_{ki} \in (0,1)$, we define similarity relations on Π_{ki} and F_{ki} in the following way. To simplify notation, we write $p_{ki}(f)$ and $\overline{p}_{ki}(\overline{f})$ as p_{ki} and $\overline{p}_{ki}$, respectively.

(i) On the space Π_{ki} , we define **correlated** similarity relations of the difference type as follows: given $\overline{f}_{ki}$,we say that $\overline{p}_{ki}$ is similar to p_{ki}, (formally written as $\overline{p}S\Pi[\overline{f}_{ki}]p_{ki}$), if and only if $|\overline{p}_{ki} - p_{ki}| \leqq d_{ki}(\overline{f}_{ki})$, where $|.|$ stands for absolute value. Note that $d_{ki}(\overline{f}_{ki})$, the uncertainty or ambiguity level felt by $\sum$ agent ki given the proportion $\overline{f}_{ki}$, becomes the threshold level in the definition of this type of similarity relation. The vertical "cone-shaped" part of Figure 1 is defined by the set of similars to $p_{ki}(f)$ given any $f_{ki} \in (0,1)$. When $f_{ki} = 0$, the whole set Π_{ki} is similar to p_{ki} and when $f_{ki} = 1$ only p_{ki} is similar to itself.

(ii) On F_{ki} , we define **correlated** similarity relations of the ratio-type as follows: given $\overline{p}_{ki}$ and $\overline{f}_{ki}$, we say that $\overline{f}_{ki}$ is similar to f_{ki},(formally written as, $\overline{f}_{ki} SF[\overline{p}_{ki}, \overline{f}_{ki}] f_{ki}$), if and only if $1/\lambda_{ki} \leqq \overline{f}_{ki}/f_{ki} \leqq \lambda_{ki}$. The λ_{ki} function (recall that λ_{ki} is defined for a given strategy proportion f_{ki}) defines a threshold on F_{ki}. The horizontal "cone-shaped" part of Figure 1 is defined by the set of similars to a given f_{ki}, as payoffs go from $p_{ki} > d_{ki}(f_{ki})$ to 1.

10.2. The $\sum$ Preference on $\Pi_{ki} \times F_{ki}$

We shall assume that each $\sum$ agent ki compares pairs of alternatives in $\Pi_{ki} \times F_{ki}$ with the aid of the above pair of correlated similarity relations, $S\Pi$ and SF, to decide which of the two is preferred. Thus, the agent may define his $\sum$ procedural preference $\succsim_{ki}$on $\Pi_{ki} \times F_{ki}$ and know his aspiration set U at each t (which we identify with the upper contour set of the vector (p_{ki}, f_{ki}) at t). That is, given a pair of vectors $(\overline{p}_{ki}, \overline{f}_{ki})$ and (p_{ki}, f_{ki}) in $\Pi_{ki} \times F_{ki}$, the vector $(\overline{p}_{ki}, \overline{f}_{ki})$ will be declared to be preferred to (p_{ki}, f_{ki}), i.e. $(\overline{p}_{ki}, \overline{f}_{ki}) \succ_{ki} (p_{ki}, f_{ki})$, whenever $\sum$ agent ki perceives that one of the following three conditions is met. Note that since $(\overline{p}_{ki}, \overline{f}_{ki})$ is to be preferred, the conditional similarity relation $S\Pi$ on Π_{ki} given $\overline{f}_{ki}$ and the conditional similarity relation SF on F_{ki} given $\overline{p}_{ki}$ and $\overline{f}_{ki}$ are to be used.

Condition α : $\overline{p}_{ki} > p_{ki}$, and *no* $\overline{p}_{ki} S\Pi[\overline{f}_{ki}] p_{ki}$; while $\overline{f}_{ki} SF[\overline{p}_{ki}, \overline{f}_{ki}] f_{ki}$.

In words, $\overline{p}_{ki}$ is bigger than p_{ki} and, given $\overline{f}_{ki}$, $\overline{p}_{ki}$ is perceived to be *not similar* to p_{ki} ; while , $\overline{f}_{ki}$ is perceived to be *similar* to f_{ki}. U_α in Figure 1 is the area implied by this condition.

Condition β : $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki} SF[\overline{p}_{ki}, \overline{f}_{ki}] f_{ki}$;while $\overline{p}_{ki} S\Pi[\overline{f}_{ki}] p_{ki}$.

In words, $\overline{f}_{ki}$ is bigger than f_{ki} and, given $\overline{p}_{ki}$ and $\overline{f}_{ki}$, $\overline{f}_{ki}$ is perceived to be *not similar* to f_{ki}; while, given $\overline{f}_{ki}$, $\overline{p}_{ki}$ is perceived to be *similar* to p_{ki}.U_β in Figure 1 is the area implied by this condition.

Condition δ : $\overline{p}_{ki} > p_{ki}$ and *no* $\overline{p}_{ki} S\Pi_{ki}[\overline{f}_{ki}] p_{ki}$; $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki} SF[\overline{p}_{ki}, \overline{f}_{ki}] f_{ki}$.

That is, vector $(\overline{p}_{ki}, \overline{f}_{ki})$ is strictly bigger than (p_{ki}, f_{ki}) and no similarity is perceived in both instances. U_δ in Figure 1 is the area implied by this condition.

Whenever both expected payoffs and strategy proportions are perceived to be *similar*, then the two vectors will be declared *indifferent*; i.e. when $\overline{p}_{ki} S\Pi[\overline{f}_{ki}] p_{ki}$, $p_{ki} S\Pi[f_{ki}] \overline{p}_{ki}$, $\overline{f}_{ki} SF[\overline{p}_{ki}, \overline{f}_{ki}] f_{ki}$ and $f_{ki} SF[p_{ki}, f_{ki}] \overline{f}_{ki}$, then $(\overline{p}_{ki}, \overline{f}_{ki}) \sim_{ki} (p_{ki}, f_{ki})$. When none of these four situations takes place, then the two vectors would be non-comparable (see Figure 1).

10.2. AINU Similarity Relations.

(i) On the space Π_{ki} , we define correlated similarity relations of the difference type as follows: we say that $\overline{p}_{ki}$ is similar to p_{ki} (formally written as

$\overline{p}_{ki} S\Pi p_{ki}$), if and only if $|\overline{p}_{ki} - p_{ki}| \leqq \epsilon_k$. The vertical cilinder-shaped part of Figure 2 is defined by the set of similars to p_{ki} given ϵ_k.

(ii) On F_{ki} , we define correlated similarity relations of the ratio-type as follows: given $\overline{p}_{ki}$, we say that $\overline{f}_{ki}$ is similar to f_{ki},(formally written as, $\overline{f}_{ki} SF[\overline{p}_{ki}] f_{ki}$), if and only if $1/\psi_{ki} \leqq \overline{f}_{ki}/f_{ki} \leqq \psi_{ki}$. The horizontal cone-shaped part of Figure 2 is defined by the set of similars to f_{ki}, as payoffs go from $p_{ki} > \epsilon_k$ to 1.

The preference relation is defined by means of a procedure similar to the previous case.

11. Appendix II:

Proof of Proposition 1:

Case I: Rewrite (6)-(7) as follows (we shall use the same notation for the Ultimatum Minigame and for the Chain-Store Game)

$$\dot{x} = (1-x)[x(2-3y) + \underline{d}(1-x)/p_L(y)] - x\overline{d}(x)/p_H(y) \ldots\ldots\ldots\ldots\ldots\ldots (6')$$
$$\dot{y} = y[(1-y)(1-x) - \underline{d}(y)/p_Y(x)] + (1-y)\overline{d}(1-y)/p_N(x) \ldots\ldots\ldots\ldots (7')$$

Writing $\dot{x} = \dot{y} = 0$ in (6)-(7) yields $(0,0)$, $(1,0)$, $(0,1)$ (since $d_Y(y) = 0$ when $y = 1$) and $(1,1)$, as the possible stationary points of the system. Recall that the payoffs to the $\sum$ agents take values in $(0,1]$ and are obtained from those of the original Ultimatum Minigame by making use of the invariance properties of the set of Nash equilibria (for instance, we may add 1 to each payoff of the Ultimatum Minigame - see Figure 2- and then divide each by 4).

By Remark 2 we know that, since agents playing H and N are in the *alert mode* and those playing L and Y are in the *absent mode* then, the probabilities $\overline{d}(x)/p_H(y)$ and $\overline{d}(1-y)/p_N(x)$ are almost 0 and can be ignored, while both $\underline{d}(1-x)/p_L(y)$and $\underline{d}(y)/p_Y(x)$ are nearly 1 (due to the assumptions that $\underline{d}(.)$ is almost 1 and that, by (2), $p_{ki}(f) > \underline{d}(f_{ki})$). As a consequence, in the interior of the state space, i.e. for values of $x \in (0,1)$ and $y \in (0,1)$, $\dot{x} > 0$ and $\dot{y} < 0$. When $x = 1$, all responders earn the same and so the system (6)-(7) is reduced to

$$\dot{y} = -y\underline{d}(y) + (1-y)\overline{d}(1-y)$$

and so $\dot{y} < 0$ for all $y \in (0,1)$. When $y = 1$, it can be seen that $\dot{x} > 0$ for all $x \in (0,1)$. When $x = 0$, $\dot{y} < 0$ for all $y \in (0,1)$ and when $y = 0$, $\dot{x} > 0$ for all $x \in (0,1)$. Therefore, $(1,0)$ is a global asymptotic attractor.

Case II: writing $\dot{x} = \dot{y} = 0$ in the system (6)-(7) yields (0,0), (1,0), (0, 1), (1, 1) and (1, 1/2) as the possible stationary points. As in the previous case, $\dot{x} > 0$

in the interior of the state space. When $x = 1$, $\overset{\bullet}{y} = -yd_Y(y) + (1-y)d_N(1-y)$ and since we have assumed that $d_Y(.) = d_N(.)$ then, $\overset{\bullet}{y} \lesseqgtr 0$ if $y \gtreqless 1/2$;thus, $\overset{\bullet}{y} = 0$ when $y = 1/2$. The rest of the behaviour in the boundary is the same as in Case I, therefore, $(1, 1/2)$ is a global asymptotic attractor.

Case III. After the study of the previous cases, It can be easily verified that, given the assumed playing modes, $\overset{\bullet}{x} < 0$ and $\overset{\bullet}{y} > 0$ in the interior of the state space; in the boundary the behaviour is such that $(0, 1)$ is a global asymptotic attractor. ■

Proof of Proposition 2: Note first that $d_{ki}(f_{ki}) = f_{ki}\overline{d}(f_{ki}) + (1 - f_{ki})\underline{d}(f_{ki}) = (1 - f_{ki})$ because for all $f_{ki} \in (0, 1)$, $\overline{d}(f_{ki})$ is nearly zero and $\underline{d}(f_{ki})$ is nearly one ; hence, we may think of $d_{ki}(f_{ki})$ as if it were the linear threshold function of set D. Thus, the (8)-(9) system can be rewritten as

$$\dot{x} = x(1-x)(2-3y) + x(1-x)[-1/p_H(y) + 1/p_L(y)] \text{.............}(8')$$
$$\dot{y} = y(1-y)(1-x) + y(1-y)[-1/p_Y(x) + 1/p_N(x)] \text{..............}(9')$$

We shall see that the trajectory of the above system coincides with that of the replicator dynamics. If $x \in (0, 1)$ and $y < 2/3$ then, $\dot{x} > 0$ because $x(1-x)(2-3y) > 0$ and $p_H(y) > p_L(y)$. On the other hand, for all values of $x \in (0, 1)$ and $y \in (0, 1)$, $\dot{y} > 0$ because $y(1-y)(1-x)$ and $p_Y(x) > p_N(x)$. When $x = 1$, $\dot{x} = 0$ and $p_Y(x) = p_N(x) = p_{II}(x)$, therefore $\dot{y} = y(1-y)[-1/p_{II}(x) + 1/p_{II}(x)] = 0$ for all $y \in [0, 1]$.Now it is an easy task to show that the interior points of NC are local attractors: take any point $\omega \neq (1, 0), (1, 2/3)$ in NC and any neighborhood B of ω; then, there will be another neighborhood $\tilde{O} \subseteq B$ with $\omega \in \tilde{O}$, such that if the initial point is in $\tilde{O}$ it is clear that the trajectory will remain in B.

If $x \in (0, 1)$ and $y > 2/3$ then, $\dot{x} < 0$ because $p_H(y) < p_L(y)$; since $\dot{y} > 0$, then, it is easy to see that the system $(8')-(9')$ converges to the subgame-perfect equilibrium $(0, 1)$ and so it is an asymptotic attractor.

Proof of Proposition 3: Note first that $d_{ki} = [f_{ki}\overline{d} + (1 - f_{ki})\underline{d}]^{\beta(p_{ki}(f))} = (1 - f_{ki})^{\beta(p_{ki}(f))}$, so the (8)-(9) system can be rewritten as

$$\overset{\bullet}{x} = x(1-x)(2-3y) - x(1-x)^{\beta(p_H(y))}/p_H(y) + (1-x)x^{\beta(p_L(y))}/p_L(y) \text{.............}(8")$$
$$\overset{\bullet}{y} = y(1-y)(1-x) - y(1-y)^{\beta(p_Y(x))}/p_Y(x) + (1-y)y^{\beta(p_N(x))}/p_N(x) \text{..............}(9")$$

Writing $\overset{\bullet}{x} = \overset{\bullet}{y} = 0$ in (8")-(9") yields $(0, 0)$, $(0, 1), (1, 0)$, $(1, 1/2)$ and $(1, 1)$ as the possible stationary points of the system. When $x = 1$, d_Y and d_N are the same function because $p_Y(x) = p_N(x) = p_{II}(x)$. Hence, the system (8")-(9") is reduced to $\dot{y} = \frac{1}{p_{II}(x)}[-y(1-y)^{\beta(p_{II}(x))} + (1-y)y^{\beta(p_{II}(x))}]$.Thus, $\dot{y} = 0$ when $y = 0, 1/2$ and 1. Suppose $0 \leq y < 2/3$ and let B be any neighborhood of $(1,0)$ and $\tilde{O}$ a neighborhood with $(1,0) \in \tilde{O} \subseteq B$. Then, $x(1-x)(2-3y) > 0$; on the other hand, $p_H(y) > p_L(y)$ and if both $d_H = (1-x)^{\beta(p_H(y))}$and $d_L = x^{\beta(p_L(y))}$ are convex, the drift part will also be positive, hence $\dot{x} > 0$. The component $y(1-y)(1-x)$ of $\dot{y}$ is positive and, as we approach $x = 1$, tends to 0. Then, if, in the drift part of $\dot{y}$ we assume that both $d_Y = (1-y)^{\beta(p_Y(x))}$ and $d_N = y^{\beta(p_N(x))}$ are convex functions, any trajectory starting in $\tilde{O}$ will converge to $(1,0)$ and so $(1,0)$ will be an asymptotic attractor. If d_Y and d_N were concave functions, those trajectories will diverge from $(1,0)$. With a similar procedure, it can be shown that $(1,1/2)$ is an asymptotic attractor if d_Y and d_N are concave functions. It is rather simple to show that each of these two asymptotic attractors is accompanied by another one, the subgame-perfect equilibrium, $(0,1)$. The rest points (0, 0) and (1, 1) are easily excluded as limit points.

Proof of Proposition 4. Suppose first that $\epsilon_I = 0$ and $\epsilon_{II} > 0$. Then, the rest points of the system are $(x,y) = (1,1/2)$ and $(0,\widetilde{y})$, where, for a given value of ϵ_{II}, $\widetilde{y}$ is the positive root of the quadratic equation $y(1-y)+\epsilon_{II}[-y/p_Y(x)+(1-y)/p_N(x)] = 0$.

We show first that $(1,1/2)$ is an asymptotic attractor. Let $x \in (0,1)$ and $y < 2/3$, then $\dot{x} > 0$.On the other hand, as x approaches 1, $p_Y(x)$ and $p_N(x)$ tend to be equal. Hence, the replicator dynamic term $y(1-y)(1-x)$ of $\dot{y}$ tends to 0 and, for values of $y \in (1/2, 2/3)$, the drift term $\epsilon_{II}[-y/p_Y(x)+(1-y)/p_N(x)]$ tends to take negative values, making $\dot{y} < 0$. Using the same reasoning, for values of $y < 1/2$, the drift term will be positive and so $\dot{y} > 0$. Therefore, for any neighborhood B of $(1,1/2)$, we can find another neighborhood $\tilde{O} \subseteq B$ with $(1,1/2) \in \tilde{O}$, such that all trajectories starting in $\tilde{O}$ will converge to $(1,1/2)$.

If $x \in (0,1)$ and $y > 2/3$, then, $\dot{x} < 0$ and $\dot{y} > 0$ because, for a value of ϵ_{II} sufficiently small, the replicator dynamic term, $y(1-y)(1-x)$, is greater than 0 and dominates the drift part. Then, it is easy to see that $(x,y) = (0,\widetilde{y})$ must be an asymptotic attractor (where $\widetilde{y}$ is in the vicinity of 1, depending on the value of the parameter ϵ_{II}: as ϵ_{II} decreases in value,$\widetilde{y}$ approaches 1).

Suppose now that both ϵ_I and $\epsilon_{II} > 0$ then, the exact limit points of $(10) - (11)$ depend on the values of ϵ_I and ϵ_{II}. The element of the Nash component $(1,1/2)$ cannot be now an asymptotic attractor because if we set $x = 1$, $\dot{x} = -\epsilon_I/p_H(y)$. Hence the drift part of (10) pushes the system away from $(1,1/2)$, with increasing power as ϵ_I increases in value relative to ϵ_{II}. The only asymptotic attractor in this case would be in the vicinity of the subgame-perfect equilibrium $(x,y) = (0,1)$, its exact location depends on the values of ϵ_I and ϵ_{II}.

12. Appendix III:

Table I

	1	2	3	4	5	6	7	8	9
1	6	6	6	6	6	6	6	6	6
2	6	6	6	6	6	6	6	6	6
3	6	6	6	6	6	6	6	6	6
4	6	6	6	6	6	6	6	6	6
5	6	6	6	6	6	6	6	6	6
6	6	6	6	6	6	6	6	6	6
7	6	6	6	6	6	6	6	6	6
8	6	6	6	6	6	6	6	6	6
9	6	6	6	6	6	6	6	6	6

Table II

	1	2	3	4	5	6	7	8	9
1	6	6	6	4	5	6	6	6	6
2	6	6	6	4	5	6	6	6	6
3	6	6	6	4	5	6	6	6	6
4	6	6	6	4	5	6	6	6	6
5	6	6	6	6	5	6	6	6	6
6	6	6	6	6	6	6	6	6	6
7	6	6	6	6	6	6	7	7	7
8	6	6	6	6	6	6	6	6	6
9	6	6	6	6	6	6	6	6	6

Table III

	$\epsilon_{II} = 0.1$	$\epsilon_{II} = 0.01$	$\epsilon_{II} = 0.001$	$\epsilon_{II} = 0$
$\epsilon_I = 0.1$	6	7	7	7
$\epsilon_I = 0.01$	6	7	7	7
$\epsilon_I = 0.001$	6	7	7	7
$\epsilon_I = 0$	6	7	7	7

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