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ADMISSIBLE HIERARCHIC SETS

by

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Admissible Hierarchic Sets*

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Abstract

In this paper we present a solution concept for abstract systems called the admissible hierarchic set. The solution we propose is a refinement of the hierarchic solution, a generalization of the von Neumann and Morgenstern solution. For finite abstract systems we show that the admissible hierarchic sets and the von Neumann and Morgenstern stable sets are the only outcomes of a coalition formation procedure (Wilson, 1972 and Roth, 1984). For coalitional games we prove that the core is either a vN&M stable set or an admissible hierarchic set.

JEL classification: C70.

Keywords: Abstract system, coalitional game, von Neumann and Morgenstern stable set, core, hierarchic solution, subsolution.

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1 Introduction

The idea of this paper was suggested by Wilson (1972) and Roth (1984) works that describe a dynamic procedure, hereafter called the *W-R* procedure, of coalition formation. Roth uses an ecological metaphor to explain the dynamics of a process of formation of habitats for microorganisms which arrive at the seashore in successive tides and may or may not become dwellers of tidepools. In this process, populations of microorganisms win and lose members and may stabilize over time. The *W-R* procedure is characterized by three features: (i) there are infinite successive arrivals of microorganisms, (ii) weak dwellers may be eliminated by the entrance of strong microorganisms and (iii) one dweller is enough to block the entrance of a newcomer.

It may be argued that the last two features, though not the first one, are observed in a number of decision problems. For instance, the admission of countries to an international association or the admission of individuals to an "exclusive" club may depend critically on the members already in the "institution". Regarding the first feature it seems reasonable to suppose that each agent is only allowed to apply for admission once. Consequently, we extend the *W-R* procedure to the case where agents have only one opportunity to become members of an institution. The reformulation of the *W-R* procedure that results from doing this provides sequence-sensitive outcomes determined by the dominance relationship between agents. The extension proposed is of interest in its own right and may well represent agents' behavior in some socioeconomic institutions, and each outcome of the reformulated procedure could be considered as a "standard of behavior" of a social organization.

The main objective of this paper is to determine the outcomes provided by the reformulated *W-R* procedure. In the search for this result we consider that the domination relationships that exist between agents are represented by an abstract system (An abstract system is pair $(X, \mathcal{R})$ where X is an arbitrary set X and $\mathcal{R}$ an irreflexive binary relation defined on it.). Then we consider von Neumann and Morgenstern (*vN&M*) stable sets and two of their most significant generalizations: subsolutions and hierarchic sets,[1] and introduce a refinement of the latter generalization that we call *admissible hierarchic sets*. Let us explain

[1] Arce (1994) uses some stability criteria for social norms that include the vN&M stable sets and the hierarchic solution, and compares the way they solve the 3-person prisoner 's dilemma.

the idea of stability behind this new notion.

Given an arbitrary set X and an irreflexive binary relation $\mathcal{R}$ defined on it, if $S \subseteq X$, satisfies the internal stability condition[2] and it is not a *vN&M* stable set, then there will be some element in $X \backslash S$ which is undominated by any element of S. Hence, the external stability condition will not be satisfied. This observation suggests the division of X into disjoint sets: S, $\mathcal{D}(S)$ (the set of elements dominated by some element of S) and $\mathcal{P}(S)$ (the set of elements undominated by some element of S from which all elements of S have been excluded). Kahn and Mookherjee (1992) call these three sets the *good*, the *ugly* and the *bad* respectively. Therefore, the non emptiness of $\mathcal{P}(S)$, prevents S from being a *vN&M* stable set and consequently any proposal of generalization of a *vN&M* stable set may be understood as a "neutralization" of $\mathcal{P}(S)$.

From this perspective, a *subsolution* (Roth, 1976) proposes the following counteraction for $\mathcal{P}(S)$: every element of the set $\mathcal{P}(S)$ is dominated by some other element of this set, and no element of S is dominated by any element of $\mathcal{P}(S)$. But perhaps the most immediate counteraction for $\mathcal{P}(S)$ is simply to require that any element outside S be indirectly dominated by some element of S (*i.e.*, S satisfies the generalized external stability condition). Kosheleva and Kreinovich (1990) call this generalization the *hierarchic solution*, while Vasil'ev (2001) calls it the *weak NM solution*. However, as we shall see, some hierarchic sets may not be efficient in their role of counteraction for $\mathcal{P}(S)$. Consequently we propose a refinement of the solution concept given by Kosheleva and Kreinovich and we introduce the notion of *effective domination system* for the counteraction of $\mathcal{P}(S)$, so that if the hierarchic set S has no effective domination system for $\mathcal{P}(S)$ then S should be discarded as an admissible solution.

The concept of the admissible hierarchic solution proves very convincing since we find that admissible hierarchic sets and *vN&M* stable sets are the *only* outcomes of the reformulated *W-R* procedure mentioned above. This result is obtained for any abstract system $(X, \mathcal{R})$ where X is a finite set and $\mathcal{R}$ is an irreflexive and asymmetric binary relation. In addition, we show that if an abstract system has no *vN&M* stable set then there will be admissible hierarchic sets. Other results concern the relationship between admissible hierarchic sets

[2] Delver and Monsuur (2001) introduce a generalization of vN&M stable sets called socially stable sets which does not require the fulfillment of the internal stability condition.

and some other solution concepts such as the core, subsolutions and generalized stable sets (van Deemen, 1991).

Furthermore, for abstract systems associated to coalitional games we show, by making use of the binary relation introduced by Sengupta and Sengupta (1996) that the core is either a *vN&M* stable set or an admissible hierarchic set.

The paper is organized as follows: Section 2 contains the preliminaries. Section 3 contains the definition of admissible hierarchic sets. In Section 4 we analyze admissible hierarchic sets for finite sets and an asymmetric binary relation. Then the *W-R* procedure is reformulated and the main result of the paper is proved. In Section 5 we analyze the core for balanced coalitional games.

2 Preliminaries

Let X be an arbitrary set and let $\mathcal{R}$ be a binary relation defined on X. The binary relation $\mathcal{R}$ is irreflexive. $\mathcal{R}$ is read "dominates". Hence, if for x, $y \in X$, $x\mathcal{R}y$ then we say that x dominates y. The pair $(X, \mathcal{R})$ is called an *abstract system.*

For each $x \in X$, let $\mathcal{D}(x) = \{y \in X : x\mathcal{R}y\}$. Given a nonempty $S \subseteq X$, let $\mathcal{D}(S) = \bigcup_{x \in S} \mathcal{D}(x)$, $\mathcal{U}(S) = X \backslash \mathcal{D}(S)$ and $\mathcal{P}(S) = \mathcal{U}(S) \backslash S$.

A set $S \subseteq X$ is a *vN&M stable set* if : *i)* for all x, $y \in S$: not $x\mathcal{R}y$ and *ii)* for all $y \notin S$ there is a $x \in S$ such that $x\mathcal{R}y$. These are called the *internal* and *external stability conditions* respectively.

A set $C \subseteq X$ is the *core* if $C = \mathcal{U}(X)$. That is, the core is the set of elements undominated by any other element.

Let $\mathcal{R}^T$ be the *transitive closure* of $\mathcal{R}$ on X. Then $x\mathcal{R}^T y$ if there is a path from x to y in X, $x = x_0, x_1, ..., x_m = y$ such that $x_{i-1} R x_i$ for all $i \in \{1, ..., m\}$. For each $x \in X$, let $\mathcal{D}^T(x) = \{y \in X : x\mathcal{R}^T y\}$. Given a nonempty $S \subseteq X$, let $\mathcal{D}^T(S) = \bigcup_{x \in S} \mathcal{D}^T(x)$ and $\mathcal{U}^T(S) = X \backslash \mathcal{D}^T(S)$.

Let X be a finite set and $\mathcal{R}$ be an asymmetric binary relation. A set $S \subseteq X$ is a *generalized stable set (van Deemen (1990))* if: *i)* for all x, $y \in S$ $(x \neq y)$: not $x\mathcal{R}^T y$ and *ii)* for all $y \notin S$ there is a $x \in S$ such that $x\mathcal{R}^T y$. The first condition, called the *generalized internal stability condition*, says that no element of S is directly or indirectly dominated by any other element of S. The second condition is the *generalized external stability condition* which says that any element outside S is directly or indirectly dominated by some element of S.

A set $S \subseteq X$ *is a hierarchic set (Kosheleva and Kreinovich (1990)*[3]*)* if*: i)* for all x, $y \in S$: not $x\mathcal{R}y$ and *ii)* for all $y \notin S$ there is a $x \in S$ such that $x\mathcal{R}^T y$. This solution concept satisfies the internal stability condition and the generalized external stability condition.

A set $S \subseteq X$ is a *subsolution* (Roth (1976)) if: *i)* $S \subseteq \mathcal{U}(S)$ and *ii)* $S = \mathcal{U}^2(S)(= \mathcal{U}(\mathcal{U}(S))$. Thus, a subsolution satisfies the internal stability condition but not the external stability condition. Instead, it requires that no element of $\mathcal{P}(S)$ dominate any element of S and yet every element of $\mathcal{P}(S)$ must be dominated by another element of this set.

3 Admissible Hierarchic Sets for $(X, \mathcal{R})$

Let $(X, \mathcal{R})$ be an abstract system and S a hierarchic set for $(X, \mathcal{R})$ such that $\mathcal{P}(S) \neq \emptyset$. As S satisfies the generalized external stability condition, for each $z \in \mathcal{P}(S)$ there are $x \in S$ and $y \in \mathcal{D}(S)$ such that $x\mathcal{R}y$ and $y\mathcal{R}^T z$ in $(X \backslash S, \mathcal{R})$. That is, z is *indirectly dominated by x through y*. Consequently, we say that y is an *intermediary* of x.

Then, the condition of generalized external stability of S compels S and $\mathcal{D}(S)$ to play a crucial role in the domination of $\mathcal{P}(S)$. However, not all elements of S and $\mathcal{D}(S)$ are necessary to counteract $\mathcal{P}(S)$ and it seems desirable to identify the parts of these sets that perform this role. To that end we define a domination system for $\mathcal{P}(S)$ in the following way.

Let $A \subseteq S$ and $B \subseteq \mathcal{D}(S)$. We say that (A, B) is a *domination system for* $\mathcal{P}(S)$ if: *i)* for all $z \in \mathcal{P}(S)$ there are $x \in A$ and $y \in B$ such that $x\mathcal{R}y$ and $y\mathcal{R}^T z$ in $(X \backslash S, \mathcal{R})$, *ii)* for all $x \in A$ there are $y \in B$ and $z \in \mathcal{P}(S)$ such that $x\mathcal{R}y$ and $y\mathcal{R}^T z$ in $(X \backslash S, \mathcal{R})$ and *iii)* for all $y \in B$ there are $x \in A$ and $z \in \mathcal{P}(S)$ such that $x\mathcal{R}y$ and $y\mathcal{R}^T z$ in $(X \backslash S, \mathcal{R})$.

[3] Kosheleva and Kreinovich's definition of hierarchy stability for coalitional games does not use the transitive closure, but their definition is equivalent to the one used in this paper, see Arce (1994).

Condition i) says that $\mathcal{P}(S)$ is indirectly dominated by A through B. Conditions ii) and iii) say also that all the elements of A and B intervene in the counteraction for $\mathcal{P}(S)$. Given the role played by set B we say that it is a *set of intermediaries of* A.

To illustrate the notion of the domination system consider the following examples.

Example 1 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *and* $\mathcal{R} = \{(x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6), (x_3, x_2), (x_4, x_1)\}$. *The associated directed graph is shown in Figure 1. Observe that the set* $S = \{x_1, x_2\}$ *is a hierarchic set such that* $\mathcal{D}(S) = \{x_3, x_4\}$ *and* $\mathcal{P}(S) = \{x_5, x_6\}$. *Thus,* S *has a unique domination system* (A, B) *for* $\mathcal{P}(S)$ *where* $A = \{x_1, x_2\}$ *and* $B = \{x_3, x_4\}$.

Example 2 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ *and* $\mathcal{R} = \{(x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6), (x_3, x_2), (x_4, x_1), (x_7, x_8), (x_8, x_2), (x_8, x_9)\}$. *Now consider the set* $S = \{x_1, x_2, x_7\}$. *Then,* S *is a hierarchic set such that* $\mathcal{D}(S) = \{x_3, x_4, x_8\}$ *and* $\mathcal{P}(S) = \{x_5, x_6, x_9\}$. *Again, we have that* S *has a unique domination system* (A, B) *for* $\mathcal{P}(S)$, *where* $A = \{x_1, x_2, x_7\}$ *and* $B = \{x_3, x_4, x_8\}$ *(See Figure 2).*

Example 3 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ *and* $\mathcal{R} = \{(x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6), (x_3, x_2), (x_4, x_1), (x_1, x_7), (x_7, x_5), (x_8, x_9), (x_9, x_2)\}$. *The associated directed graph is shown in Figure 3. The set* $S = \{x_1, x_2, x_8\}$ *is a hierarchic set such that* $\mathcal{D}(S) = \{x_3, x_4, x_7, x_9\}$ *and* $\mathcal{P}(S) = \{x_5, x_6\}$. *In this case,* S *has three domination systems for* $\mathcal{P}(S)$*:* (A_1, B_1), (A_2, B_2) *and* (A_3, B_3) *where* $A_1 = A_2 = A_3 = \{x_1, x_2\}$, $B_1 = \{x_3, x_4\}$, $B_2 = \{x_4, x_7\}$ *and* $B_3 = \{x_3, x_4, x_7\}$.

Figure 1

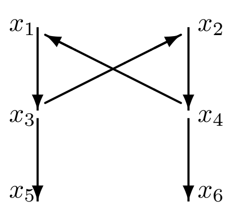


Figure 2

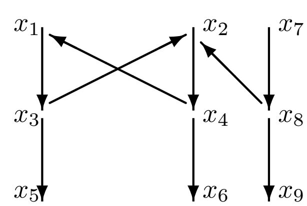
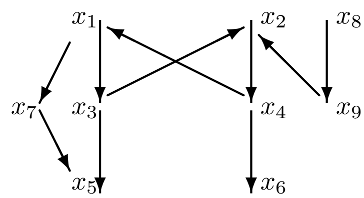


Figure 3



We now define when a domination system for $\mathcal{P}(S)$ is effective.

Let (A, B) be a domination system for $\mathcal{P}(S)$ and $\overline{B} \subseteq B$. Denote by $\overline{A}$ the set $\{x \in A : x\mathcal{R}y \text{ for some } y \in \overline{B}\}$ ($\overline{A}$ is the part of A that dominates indirectly through $\overline{B}$). We say that the domination system (A, B) is *effective* if $\overline{A} \nsubseteq \mathcal{D}(\overline{B})$ for all $\overline{B} \subseteq B$.

To illustrate the notion of the effective domination system consider the examples given above. In Example 1, the domination system (A, B) is not effective since A is dominated by the set of intermediaries B, that is, $A \subseteq \mathcal{D}(B)$ (It is entirely "trapped"). In Example 2, the domination system (A, B) is not effective either, since $\overline{A} \subseteq \mathcal{D}(\overline{B})$ for $\overline{B} = \{x_3, x_4\}$ (It is partially "trapped"). However, in Example 3, although the domination systems (A_1, B_1) and (A_3, B_3) are not effective, the domination system (A_2, B_2) is effective.

Finally, we define the concept of admissible hierarchic set as follows:

Definition 4 *A hierarchic set $S \subseteq X$ such that $\mathcal{P}(S) \neq \emptyset$ is admissible if it has an effective domination system for $\mathcal{P}(S)$.*

Notice that the hierarchic set S considered in Examples 1 and 2 is not admissible but the one given in Example 3 is admissible since it has an effective domination system.

4 Finite Abstract Systems

In this section, we assume that the set X is finite and $\mathcal{R}$ an irreflexive and asymmetric domination relation on X. In the first subsection we establish some relations between admissible hierarchic sets and other solution concepts. In the second subsection, considering a dynamic model of coalition formation, the *W-R* procedure, we define the Σ-solutions for the abstract system $(X, \mathcal{R})$ associated with that model. Then we prove the main result of the paper: admissible hierarchic sets and *vN&M* stable sets are the *only* outcomes of this dynamic procedure.

4.1 Admissible Hierarchic Sets and Other Solution Concepts for $(X, \mathcal{R})$

Proposition 5 *If S is a generalized stable set such that $\mathcal{P}(S) \neq \emptyset$ then it is an admissible hierarchic set.*

Proof. By the generalized external stability condition, for all $z \in \mathcal{P}(S)$ there are $x \in S$ and $y \in \mathcal{D}(S)$ such that $x\mathcal{R}y$ and $y\mathcal{R}^T z$ in $(X \backslash S,\ \mathcal{R})$. Consider (A, B) where $B = \{y \in \mathcal{D}(S) : y\mathcal{R}^T z$ in $(X \backslash S,\ \mathcal{R})$ for some $z \in \mathcal{P}(S)\}$ and $A = \{x \in S : x\mathcal{R}y$ for some $y \in B\}$. Then we have that (A, B) is a domination system for $\mathcal{P}(S)$. Moreover, as S satisfies the generalized internal stability condition, then $\overline{A} \nsubseteq D(\overline{B})$ for all $\overline{B} \subseteq B$. Hence, the domination system (A, B) is effective and therefore S is an admissible hierarchic set. ■

It is easy to show that not every subsolution is a hierarchic set. Restricted to subsolutions which are hierarchic sets, the following example shows that they are not admissible hierarchic sets in general.

Example 6 *Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ and $\mathcal{R} = \{(x_1, x_3),$ $(x_2, x_4), (x_3, x_5), (x_4, x_6), (x_3, x_2), (x_4, x_1), (x_5, x_7), (x_6, x_8), (x_7, x_9), (x_8, x_{10}),$ $(x_9, x_5), (x_{10}, x_6)\}$. The set $S = \{x_1, x_2\}$ is not an admissible hierarchic set, since it has a unique domination system (A, B) where $A = S$ and $B = \mathcal{D}(S)$ and $S \subseteq \mathcal{D}(\mathcal{D}(S))$. The dominion of S is $\mathcal{D}(S) = \{x_3, x_4\}$. Hence, $\mathcal{U}(S) = \{x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ and $\mathcal{U}^2(S) = \{x_1, x_2\}$. Consequently, $S \subseteq \mathcal{U}(S)$ and $S = \mathcal{U}^2(S)$, i.e. S is a subsolution but it is not admissible hierarchic set.*

The following result concerns the existence of an admissible hierarchic set.

Proposition 7 *Every finite abstract system $(X, \mathcal{R})$ where $\mathcal{R}$ is an irreflexive and asymmetric domination relation on X, has either a vN&M stable or an admissible hierarchic set.*

Proof. From Theorem 3 [10], there exists a nonempty $S \subseteq X$ such that S is a generalized stable set. If $\mathcal{P}(S) = \emptyset$ then S is a *vN&M* stable set. Otherwise, from Proposition 5, S is an admissible hierarchic set . ■

Denote by $\mathcal{A}$ the set of admissible hierarchic sets for $(X, \mathcal{R})$.

Proposition 8 *If $\mathcal{A} \neq \emptyset$ then the intersection of the admissible hierarchic sets is the core.*

Proof. The core is contained in any hierarchic set, hence it is contained in any admissible hierarchic set and therefore in the intersection between them. Conversely, if $\bigcap_{S \in \mathcal{A}} S = \emptyset$ then obviously $\bigcap_{S \in \mathcal{A}} S \subseteq C$. So, we may assume that $\bigcap_{S \in \mathcal{A}} S \neq \emptyset$. Let $x \in \bigcap_{S \in \mathcal{A}} S$. We prove that $x \in C$. Suppose that $x \notin C$. Then there is a $y \in X$ such that $y\mathcal{R}x$. Now consider $S \in \mathcal{A}$ and $G \subseteq S$ such that G is a generalized stable set. (This is possible given that if S is not minimal with respect to the generalized external stability of domination then it must contain a G with this property). As $\mathcal{P}(S) \subseteq \mathcal{P}(G)$ and $\mathcal{P}(S) \neq \emptyset$, then $\mathcal{P}(G) \neq \emptyset$, hence, by Proposition 5, $G \in \mathcal{A}$ and therefore $x \in G$. Now, as $y\mathcal{R}x$ we have $x\mathcal{R}^T y$. Let $G' = (G \backslash \{x\}) \cup \{y\}$. It is easy to check that G' is a generalized stable set such that $\mathcal{P}(G') \neq \emptyset$. Hence, $G' \in \mathcal{A}$, and therefore $x \in G'$ contradicting $x \notin G'$. ■

Proposition 9 *The core is an admissible hierarchic set if and only if it is a hierarchic set which is not vN&M stable.*

Proof. If the core C is an admissible hierarchic set then, by Definition 4, it is a hierarchic set which is not *vN&M* stable. Conversely, if the core is a hierarchic set which is not *vN&M* stable, then obviously it is a generalized stable set such that $\mathcal{P}(C) \neq \emptyset$. Hence, from Proposition 5, it is an admissible hierarchic set. ■

4.2 Σ-solutions for $(X, \mathcal{R})$

Let $X = \{x_1, ..., x_n\}$ be a set of agents. Consider that agents in X may form coalitions to participate in a common project. However, not all the agents may be equally suitable for the project. This information is described by an abstract system where the relationship between the agents in X may be represented by the domination graph. We assume that the formation of coalitions is dynamic in the sense that agents arrive sequentially at the "place" where the coalition forms.

Time is divided into n periods, and in each period only one agent arrives at the system and may form part of it.

An arrival order in X is a permutation $\sigma \in \Sigma_n$ such that $x_{\sigma(t)} \in X$ is the agent that arrives at the system at period t, for all $t \in \{1, ..., n\}$. The arrival order of $x \in X$ is denoted by $t(x)$.

Let $\sigma \in \Sigma_n$ be an arrival order. The set of *surviving* agents in the system at the end of period t is denoted by $S_\sigma(t)$.

The set $S_\sigma(t)$ evolves in the following way:

for $t = 1 : S_\sigma(1) = \{x_{\sigma(1)}\}$, and

for $t > 1 : S_\sigma(t) = \begin{cases} S_\sigma(t-1) & \text{if } x_{\sigma(t)} \in \mathcal{D}(S_\sigma(t-1)) \\ (S_\sigma(t-1) \cup \{x_{\sigma(t)}\}) \backslash \mathcal{D}(x_{\sigma(t)}) & \text{otherwise} \end{cases}$

That is, if the agent that arrives at the system in period t is dominated by some agent already there, then the system remains unchanged. Otherwise, the new agent occupies a place in the system and any agent already there that is dominated by the newcomer is eliminated from the system.

Definition 10 *A set $S \subseteq X$ is a $\Sigma-$solution if there is an arrival order $\sigma \in \Sigma_n$ such that $S_\sigma(n) = S$.*

We will prove that the $\Sigma - $ *solutions* are either *vN&M* stable sets or admissible stable sets (Theorems 13 and 16). To do this we need some prior lemmas.

Given an arrival order $\sigma \in \Sigma_n$ we define a binary relation on X, called elimination and denoted by E_σ as follows:

$xE_\sigma y$ if $x\mathcal{R}y$ and either $x \in S_\sigma(t(y)-1)$ or $y \in S_\sigma(t(x)-1)$ and $x \in S_\sigma(t(x))$.

That is, x eliminates y if x dominates y and x is in the system when y arrives or if y is in the system when x arrives and there is no z in the system such that $z\mathcal{R}x$.

Let E_σ^T be the *transitive closure* of E_σ on X. Then, if $xE_\sigma^T y$ there is a *path* from x to y in X, $x = x_0, x_1, ..., x_m = y$ such that $x_{i-1}E_\sigma x_i$ for all $i \in \{1, ..., m\}$.

Lemma 11 *If $\sigma \in \Sigma_n$ such that $S_\sigma(n) = S$ is verified:*

i) For all $x, y \in S$: not $xE_\sigma y$.

ii) For all $y \notin S$ there is a $x \in S$ such that $xE_\sigma^T y$.

iii) For all $x \in S$, $y \in \mathcal{D}(S)$ and $z \in \mathcal{P}(S)$ such that $xE_\sigma y$ and $yE_\sigma^T z$, then $t(y) < t(x)$.

Proof. Conditions i) and ii) are trivial. Condition iii) is also immediate, given that if there is an arrival order σ such that x elimates y and y eliminates z (directly or indirectly) then y has to arrive at the system earlier than x. ■

Lemma 12 *If S is a $\Sigma - solution$ then it is a hierarchic set.*

Proof. As S is a $\Sigma - solution$ then there exists $\sigma \in \Sigma_n$ such that $S_\sigma(n) = S$. Then, from Lemma 11 i), for all $x, y \in S$: not $xE_\sigma y$ hence not $x\mathcal{R}y$. Moreover, from Lemma 11 ii), for all $y \notin S$ there is a $x \in S$ such that $xE_\sigma^T y$, hence $x\mathcal{R}^T y$.
■

According to Lemma 12, $\Sigma - solutions$ are hierarchic sets. In the next theorem we prove that a non admissible hierarchic set is not a $\Sigma - solution$. We first give an intuitive idea of the proof by using Example 1.

> The hierarchic set $S = \{x_1, x_2\}$ is not admissible as shown in Example 1. The unique domination system for $\mathcal{P}(S)$ is (A, B) where $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$, which is not effective since $A \subseteq \mathcal{D}(B)$. Then, it is not possible to establish an arrival order σ in X such that $S_\sigma(6) = \{x_1, x_2\}$ since agents x_1 and x_2 are dominated by intermediary agents x_4 and x_3, respectively. Suppose that there is $\sigma \in \Sigma_6$ such that $S_\sigma(6) = \{x_1, x_2\}$. Then $E_\sigma = \{(x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6)\}$. If $t(x_1) < t(x_2)$, then agent x_1 is the first to arrive at the system. As $x_2 E_\sigma x_4$ and $x_4 E_\sigma x_6$, from Lemma 11 ii), agent x_4 arrives at the system before agent x_2. But then, as $x_4 \mathcal{R} x_1$, agent x_1 is eliminated by x_4 contradicting $x_1 \in S_\sigma(6)$. If $t(x_2) < t(x_1)$, arguing similarly, we conclude that agent x_2 is eliminated from the system by agent x_3.

Theorem 13 *If S is a $\Sigma - solution$ then it is either a vN&M stable set or an admissible hierarchic set.*

Proof. From Lemma 12, if S is a $\Sigma-$ *solution* then it is a hierarchic set. If $\mathcal{P}(S)=\emptyset$ then S is a *vN&M stable set.* Suppose that $\mathcal{P}(S)\neq\emptyset$. We prove that S is an admissible hierarchic set.

As S is a $\Sigma-$ *solution* there is a $\sigma\in\Sigma_n$ such that $S_\sigma(n)=S$. From Lemma 11 ii), for all $z\in\mathcal{P}(S)$ there is an $x\in S$ such that $xE_\sigma^T z$. As $z\notin\mathcal{D}(S)$ then there is a $y\in\mathcal{D}(S)$ such that $xE_\sigma y$ and $yE_\sigma^T z$. Let $A=\{x\in S: xE_\sigma^T z$ for some $z\in\mathcal{P}(S)\}$ and $B=\{y\in\mathcal{D}(S): xE_\sigma y$ and $yE_\sigma^T z$ for some $x\in A$ and $z\in\mathcal{P}(S)\}$. It is easy to check that (A,B) is a domination system for $P(S)$. Now we prove that (A,B) is effective *i.e.,* $\overline{A}\nsubseteq\mathcal{D}(\overline{B})$ for all $\overline{B}\subseteq B$ where $\overline{A}=\{x\in A: x\mathcal{R}y$ for some $y\in\overline{B}\}$.

Suppose that $\overline{A}\subseteq\mathcal{D}(\overline{B})$ for some $\overline{B}\subseteq B$. Let $\overline{x}\in\overline{A}$ such that $t(\overline{x})=\min\{t(x): x\in\overline{A}\}$, that is, $\overline{x}$ is the first agent of $\overline{A}$ that arrives at the system. Since $\overline{x}\in\mathcal{D}(\overline{B})$, then there is a $y\in\overline{B}$ such that $y\mathcal{R}\overline{x}$. Now as $y\in\overline{B}$ there exist $x'\in\overline{A}$ and $z\in\mathcal{P}(S)$ such that $x'E_\sigma y$ and $yE_\sigma^T z$. Then, by Lemma 11 iii), $t(y)<t(x')$. But, then $yE_\sigma\overline{x}$ since $y\mathcal{R}\overline{x}$, contradicting $\overline{x}\in S_\sigma(n)$. ■

In the following theorem we prove the converse of Theorem 13, that is, *vN&M* stable sets or admissible hierarchic sets are $\Sigma-$ *solutions*. First, we prove that *vN&M* stable sets are $\Sigma-$ *solutions*.

Lemma 14 *If S is a vN&M stable set then there is a $\sigma\in\Sigma_n$ such that $S_\sigma(n)=S$.*

Proof. Let $k=|S|$ and σ be any arrival order in X such that $x_{\sigma(t)}\in S$ if $1\leq t\leq k$ and $x_{\sigma(t)}\in X\backslash S$, otherwise. As S is a *vN&M* stable set, then for all x, $y\in S$: not xRy. Hence, not $xE_\sigma y$ and thus $S_\sigma(k)=S$. Moreover, for all $y\notin S$ there is a $x\in S$ such that $x\mathcal{R}y$. Therefore, $xE_\sigma y$ and $S_\sigma(t)=S$ if $t>k$. Thus $S_\sigma(n)=S$. ■

To prove that every admissible hierarchic set is a $\Sigma-$ *solution*, we first need a lemma. Also, first we will use Example 3 to give an intuitive idea of the proof.

Lemma 15 *Let S be an admissible hierarchic set and (A,B) a minimal effective domination system*[4] *of $\mathcal{P}(S)$. Then, there is an arrival order $\overline{\sigma}$ in A such that $\overline{x}_{\overline{\sigma}(1)}\notin\mathcal{D}(B)$ and $\overline{x}_{\overline{\sigma}(t)}\notin\mathcal{D}(B\backslash\bigcup_{k=1}^{t-1}\mathcal{D}(\overline{x}_{\overline{\sigma}(k)}))$ for all $t>1$.*

[4] The domination system (A,B) is *minimal* if there is no effective domination system (A',B') such that $A'\subset A$ or $B'\subset B$.

Proof. Since (A, B) is an effective domination system, then $\overline{A} \nsubseteq \mathcal{D}(\overline{B})$ for all $\overline{B} \subseteq B$, where $\overline{A} = \{x \in A : x\mathcal{R}y \text{ for some } y \in \overline{B}\}$. Then, using this condition we define $\overline{\sigma}$ inductively as follows:

for $t = 1$: as $A \nsubseteq \mathcal{D}(B)$ there is an $x \in A$ such that $x \notin \mathcal{D}(B)$. Set $\overline{x}_{\overline{\sigma}(1)} = x$.

For $t > 1$: let $\overline{B} = B \backslash \bigcup_{k=1}^{t-1} \mathcal{D}(\overline{x}_{\overline{\sigma}(k)})$. As $\overline{A} \nsubseteq \mathcal{D}(\overline{B})$ there is an $\overline{x} \in \overline{A}$ such that $\overline{x} \notin \mathcal{D}(\overline{B})$. Set $\overline{x}_{\overline{\sigma}(t)} = \overline{x}$.

Clearly, $\overline{x}_{\overline{\sigma}(t)} \neq \overline{x}_{\overline{\sigma}(t')}$ if $t \neq t'$. Moreover, given that (A, B) is a minimal effective domination system, we have $B \subseteq \bigcup_{k=1}^{l} \mathcal{D}(\overline{x}_{\overline{\sigma}(k)})$ if and only if $l = \mid A \mid$ (If $l < \mid A \mid$ then (A', B) where $A' = \{\overline{x}_{\overline{\sigma}(1)}, ..., \overline{x}_{\overline{\sigma}(l)}\}$ will be an effective domination system such that $A' \subset A$, contradicting that (A, B) is minimal). Hence, $\overline{B} = B \backslash \bigcup_{k=1}^{t-1} \mathcal{D}(\overline{x}_{\overline{\sigma}(k)}) \neq \emptyset$ for all $t = 1, ..., \mid A \mid$. Consequently, $\overline{\sigma}$ is an arrival order in A. ■

As we have seen, in Example 3, the set $S = \{x_1, x_2, x_8\}$ is an admissible hierarchic with a unique effective domination system (A, B) where $A = \{x_1, x_2\}$ and $B = \{x_4, x_7\}$. Now we give an arrival order σ in X such that $S_\sigma(9) = S$. Notice that x_2 is the agent of A that is not dominated by any intermediary agent in B, that is, $x_2 \notin \mathcal{D}(B)$. Hence, this agent is the first one to arrive at the system. Thus, $S_\sigma(9) = \{x_1, x_2, x_8\}$ can be obtained with the following arrival order:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 5 & 6 & 2 & 1 & 8 & 3 & 9 \end{pmatrix}$$

Theorem 16 *If S is either a vN&M stable set or an admissible hierarchic set then S is a $\Sigma -$ solution.*

Proof. We prove that $S_\sigma(n) = S$ for some $\sigma \in \Sigma_n$ using induction on n.

If $n = 2$ then S is obviously a *vN&M* stable set and the result follows from Lemma 14. Now, we prove the assertion for $n > 2$. If S is a *vN&M* stable set the result again follows from Lemma 14. Then we can assume that S is an admissible hierarchic set. Let (A, B) be an effective domination system for $\mathcal{P}(S)$. We can assume without loss of generality that (A, B) is minimal. As (A, B) is a domination system for $\mathcal{P}(S)$, for all $z \in \mathcal{P}(S)$ there is a $y \in B$ such that $y\mathcal{R}^T z$ in $(X \backslash S, \mathcal{R})$, that is, $\mathcal{P}(S) \subseteq \mathcal{D}^T(B)$ in $(X \backslash S, \mathcal{R})$. Set $X_1 = B \cup \mathcal{D}^T(B)$ in $(X \backslash S, \mathcal{R})$. Then B satisfies the generalized external stability condition for the abstract system $(X_1, \mathcal{R})$. By the minimality of the domination system, B satisfies the generalized internal stability condition in $(X_1, \mathcal{R})$ (Otherwise, we could consider another effective domination system (A, B') such that $B' \subset B$, contradicting that (A, B) is minimal). Then B is a generalized stable set for $(X_1, \mathcal{R})$. Hence, either B is a *vN&M* stable set, or, from Proposition 5, it is an admissible hierarchic set. Then, by inductive hypothesis, B is a $\Sigma-$solution for $(X_1, \mathcal{R})$, and hence, there is an arrival order σ' in X_1 such that $S_{\sigma'}(\mid X_1 \mid) = B$. Denote by $x'_{\sigma'(t)}$ the agent that arrives at the system $(X_1, \mathcal{R})$ in period t.

Let $X_2 = A$ and $\overline{\sigma}$ be the arrival order in A given in Lemma 15. Then $\overline{x}_{\overline{\sigma}(1)} \notin \mathcal{D}(B)$ and $\overline{x}_{\overline{\sigma}(t)} \notin \mathcal{D}(B \backslash \bigcup_{k=1}^{t-1} \mathcal{D}(\overline{x}_{\overline{\sigma}(k)}))$, for all $t > 1$.

Now consider $X_3 = S \backslash A$ and $X_4 = \mathcal{D}(S) \backslash X_1$. We have $X = \bigcup_{i=1}^{4} X_i$. Denote by $t_i =\mid X_1 \mid +...+ \mid X_i \mid$ for all $i = 1, ..., 4$. We give an arrival order $\sigma \in \Sigma_n$ in X such that $x_{\sigma(t)} = x'_{\sigma'(t)}$ if $1 \leq t \leq t_1$, $x_{\sigma(t)} = \overline{x}_{\overline{\sigma}(t-t_1)}$ if $t_1 < t \leq t_2$, $x_{\sigma(t)} \in X_3$ if $t_2 < t \leq t_3$ and $x_{\sigma(t)} \in X_4$ otherwise. Then, we have $S_\sigma(t_1) = B$, $S_\sigma(t_2) = A$ and $S_\sigma(t) = S$ if $t \geq t_3$. Hence, $S_\sigma(n) = S$. ■

We conclude this section with an observation about the frequency with which the *vN&M* stable sets occur. Recall that Definition 10 says that given an abstract system for every arrival order there is a $\Sigma-$*solution*. However, it is clear that every $\Sigma-$*solution* may be the result of several arrival orders.

Let S be a $\Sigma-$*solution*. Denote by $n(S)$ the cardinal of the set $\{\sigma \in \Sigma_n : S_\sigma(n) = S\}$. Let $\overline{n} = \max\{n(S) : S \text{ is a } \Sigma - solution\}$. The following example shows that the vN&M stable set V is not the $\Sigma-$*solution* such that $n(V) = \overline{n}$.

Example 17 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ *and* $\mathcal{R} = \{(x_1, x_2), (x_2, x_5), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_6, x_7), (x_7, x_5)\}$. *The abstract system* $(X, \mathcal{R})$ *has a unique vN&M stable set:* $V = \{x_1, x_3, x_5, x_6\}$. *Using a computer program it can be verified that the* $\Sigma-$*solution such that* $n(S) = \overline{n}$ *is the admissible hierarchic set* $S = \{x_1, x_3, x_6\}$.

5 Balanced Coalitional Games

In this section, we use the binary relation for coalitional games defined by Sengupta and Sengupta (1996) to prove that the core for balanced coalitional games is either a *vN&M* stable set or an admissible hierarchic set.

Let (N, v) be a coalitional game where $N = \{1, ..., n\}$ is the set of players and v the characteristic function of the game.

Given an allocation $x = (x^1, ..., x^n)$ and a nonempty $S \subseteq N$, x^S denote the proyection of x on S and $x(S) = \Sigma_{i \in S}\ x^i$. We write $x^S > y^S$ if $x^i \geq y^i$ for all $i \in S$ and $x^S \neq y^S$.

Denote by PI the set of feasible allocations and by I the set of imputations. The core of the game (N, v) is defined as $C(N, v) = \{x \in I : x(S) \geq v(S)$ for all $S \subset N\}$. Hereafter we write simply C.

Let $x, y \in PI$. We say that x *directly dominates* y *via coalition* S $(x \succ_D^S y)$ if $x^S > y^S$, $x(S) = v(S)$ and $x \in I$. Also, we say that x *directly dominates* y $(x \succ_D y)$ if there exists some coalition S such that $x \succ_D^S y$. We say that x *indirectly dominates* y $(x \succ_I y)$ if there exists a finite sequence of imputations $\{x_0, ..., x_m\}$, where $x_0 = x$ and $x_m = y$, and a finite sequence of coalitions $\{S_0, ..., S_{m-1}\}$ such that $x_{i-1} \succ_D^{S_{i-1}} x_i$ for all $i \in \{1, ..., m\}$.

Let $(X, \mathcal{R})$ be the abstract system associated to the game (N, v) where X is the set of feasible allocations and $\mathcal{R}$ the direct domination relation defined on PI. Hereafter we write $(PI, \succ_D)$. Then, the *transitive closure* of $\mathcal{R}$ on X is the indirect domination relation on PI.

Theorem 18 *The core of a balanced game is either a vN&M stable set or an admissible hierarchic set for* $(PI, \succ_D)$.

Proof. As C is nonempty, from Theorem [8], for all $y \in PI$ such that $y \notin C$ there exists $x \in C$ such that $x \succ_I y$. Thus, C satisfies the generalized external stability condition. Moreover, as no imputation in I can directly dominate any imputation in the core, then C satisfies the internal stability condition. Therefore, C is a hierarchic set for $(PI, \succ_D)$. If $\mathcal{P}(C) = \emptyset$ then C is a *vN&M* stable. Hence, we can assume that $\mathcal{P}(C) \neq \emptyset$. As C satisfies the generalized external stability condition, for all $z \in \mathcal{P}(C)$ there are $x \in C$ and $y \in \mathcal{D}(C)$ such that $x \succ_D y$ and $y \succ_I z$ in $(PI \backslash C, \succ_D)$. Consider (A, B) where $B = \{y \in \mathcal{D}(C) : y \succ_I z$ in $(PI \backslash C, \succ_D)$ for some $z \in \mathcal{P}(C)\}$ and $A = \{x \in C : x \succ_D y$ for some $y \in B\}$. Then, we have that (A, B) is a domination system for $\mathcal{P}(S)$. Also, obviously $\overline{A} \nsubseteq \mathcal{D}(\overline{B})$ for all $\overline{B} \subseteq B$ since $\overline{A} \subseteq C$, hence the domination system (A, B) is effective and C is an admissible hierarchic set. ■

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