An extension of the fuzzy unit interval to a tensor product with completely distributive first factor*

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To Professor Jerzy Albrycht for his 94th birthday

Abstract

The original Hutton interval $I(L)$ can algebraically be identified with the tensor product $I \otimes L$ of the real unit interval I and a complete lattice L. Due to this, the tensor product $M \otimes L$ with M a completely distributive lattice is considered as a generalization of the lattice $I(L)$. When appropriately endowed with an L-topology, the tensor product $M \otimes L$ becomes also an L-topological extension of $I(L)$. If M is $\lhd$-separable (= it has a countable join base free of supercompact elements), many of the L-topological features of $I(L)$ are retained. To wit, Urysohn lemma and Tietze–Urysohn extension theorem for $(M \otimes L)$-valued functions are then proved. The relationship of $M \otimes L$ to the L-fuzzy topological modification of M in the sense of D. Zhang and Y.-M. Liu [Math. Nachr. 168 (1994) 79–95] is discussed.

Keywords: Hutton fuzzy unit interval, The category Sup, Tensor product of complete lattices, Completely distributive lattice, Supercompact, $\lhd$-separability, Urysohn lemma, Tietze–Urysohn extension theorem, L-fuzzy modification of complete distributivity, Symmetric monoidal closed category

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1. Introduction

This paper continues our program initiated in [6, 7] of implementing tensor products of complete lattices into fuzzy set theory and, in particular, into many-valued topology. We are concerned with developing a new codomain for continuous functions in many-valued topology which would provide a common generalization of I-valued functions, $I(L)$-valued functions, $I(I(L))$-valued functions, and M-valued functions, where I is the real unit interval, L is a complete lattice, M is a completely distributive lattice, and $I(L)$ is the fuzzy unit interval of Hutton [11]. This has to do with second order fuzziness in the sense of Rodabaugh [25].

As observed in [7], $I(L)$ can on the algebraic level be viewed as a tensor product $I \otimes L$. It is therefore felt that a suitable candidate for the new codomain is the tensor product $M \otimes L$, for besides the order isomorphism $I(L) \cong I \otimes L$ we also have $M \otimes \mathbf{2} \cong M$ and $\mathbf{2} \otimes L \cong L$. The tensor product $M \otimes L$ is in this paper chosen – as in Shmuely [26] – to be the complete lattice of all join-reversing maps from M to L under pointwise order.

Extending the original Hutton's interval L-topology of $I(L)$ to $M \otimes L$ requires certain assumptions on the first factor M. We assume that M is a completely distributive lattice and that L is a complete lattice with an order-reversing involution. As in the case of $I(L)$, our tensor product $M \otimes L$ is appropriately endowed with three L-topologies: the upper, the lower, and the interval L-topology. The appropriateness of these L-topologies is confirmed by the fact that the upper, lower, and interval $\mathbf{2}$-topologies on $M \otimes \mathbf{2}$ coincide with the traditional upper, lower, and interval topologies on M, respectively.

Our investigations sometimes led to a few new insights into complete distributivity of lattices (including atomic Boolean algebras).

When proving Urysohn lemma and Tietze-Urysohn extension theorem for $(M \otimes L)$-valued functions, we choose M to be $\lhd$-separable — i.e. it has a countable join base which is free of supercompact elements. There are many examples of such lattices to choose from, for $\lhd$-separability is closed under tensor products and under countable Cartesian products.

There have already been made various attempts to generalize $I(L)$ (cf. [7]). In particular, Zhang and Liu [28] considered the set of all join-preserving maps from M to L, and called it the *L-fuzzy modification of M*, thereby not respecting the original antitone variant of $I(L)$. The relationship of $M \otimes L$ to the L-fuzzy topological modification of M is discussed.

Some deeper aspects of the tensor products are used to show that the recursive construction $I(L)$, $I(I(L)),\ldots$ terminates, thereby answering an open question of [16].

2. Preliminaries

We refer to the Compendium [5] for lattice-theoretic concepts not defined herein. Completeness of lattices M and L is assumed from the beginning. Members of L are denoted a, b, c, and members of M are denoted t, s, r, q, etc. The latter notation is because in this paper the real unit interval is a source example of M. No confusion will arise when using 0 and 1 to denote the universal lower and upper bounds of any complete lattice. In particular, the two point lattice $\{0,1\}$ is denoted **2**. Given a set X, the family L^X of all maps from X to L is a complete lattice under pointwise order:

$$f \le g \quad \text{in } L^X \qquad \text{iff} \qquad f(x) \le g(x) \quad \text{for all } x \in X.$$

2.1. Basics on tensor products of complete lattices

The material below is developed in great detail in the the forthcoming book [4] to which we refer for all the details and proofs (see also [6]).

A map λ of L^M is called *join-preserving* if

$$\lambda(\bigvee T) = \bigvee \lambda(T) \quad \text{for all } T \subseteq M.$$

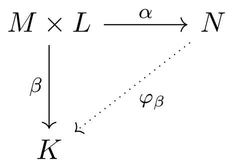
The category of all complete lattices and their join-preserving maps is denoted Sup.

The Cartesian product $M \times L$ is a complete lattice under componentwise order. Let K be a further complete lattice. A map $M \times L \xrightarrow{\beta} K$ is *separately join-preserving* (or a *bimorphism* in Sup) if

$$\beta(t, \bigvee A) = \bigvee_{a\in A} \beta(t,a) \qquad \text{and} \qquad \beta(\bigvee T, a) = \bigvee_{t\in T} \beta(t,a)$$

for all $t \in M$, $A \subseteq L$, $T \subseteq M$, and $a \in L$.

Definition 2.1. A *tensor product* of M and L in the category Sup is – by definition – a complete lattice N together with a separately join-preserving map $M \times L \xrightarrow{\alpha} N$ satisfying the following *universal property*: for every separately join-preserving map $M \times L \xrightarrow{\beta} K$ there exists a unique join-preserving map $N \xrightarrow{\varphi_\beta} K$ such that the following diagram is commutative:



In this context α is called the *universal bimorphism.*

As usually it follows from the universal property of the tensor product is unique up to an order isomorphism. The tensor product of M and L will be denoted $M \otimes L$. Similarly, the corresponding universal bimorphism α will be written as $M \times L \xrightarrow{\otimes} M \otimes L$.

We now proceed to describe a construction of a tensor product of M and L which suits our purposes best. It has for the first time been described by Shmuely [26]. To this end, define a map $\lambda \in L^M$ to be *join-reversing* if

$$\lambda(\bigvee T) = \bigwedge \lambda(T) \quad \text{for all} \quad T \subseteq M.$$

Let us keep in mind that such a λ is order-reversing and $\lambda(0) = 1$. The family of all join- reversing maps from M to L is a complete lattice under the pointwise order inherited from L^M (as arbitrary meets are pointwise meets). By some abuse of ideology and notation, already at this point we let

$$M \otimes L := \{\lambda \in L^M \mid \lambda \text{ is join-reversing}\}$$

(for historical reason we note that the above family in the context of fuzzy sets has already been considered in [9, 20]). Given $(t, a) \in M \times L$, define a map $M \xrightarrow{t \otimes a} L$ by

$$(t \otimes a)(s) = \begin{cases} 1 & \text{if} \quad s = 0, \\ a & \text{if} \quad 0 \neq s \leq t, \\ 0 & \text{if} \quad s \nleq t. \end{cases}$$

Then $t \otimes a$ is in $M \otimes L$ and the map $M \times L \xrightarrow{\otimes} M \otimes L$ defined by $(t, a) \mapsto t \otimes a$ is the universal bimorphism. The universal bounds 0 and 1 of $M \otimes L$ have the following form

$$0(t) = \begin{cases} 1 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad t \neq 0, \end{cases} \quad \text{and} \quad 1(t) = 1 \quad \text{for all} \quad t \in M$$

All this can be summarized as follows (we refer to [4] or [22] for categorical terminology):

Theorem 2.2. *Let M and L be complete lattices. Then*

$$M \otimes L = \{\lambda \in L^M \mid \lambda \text{ is join-reversing}\}$$

together with the bimorphism $M \times L \xrightarrow{\otimes} M \otimes L$ is the tensor product of M and L in the category Sup *and $\otimes$ makes* Sup *into a symmetric monoidal closed category.*

Elements of $M \otimes L$ are called *tensors* and $t \otimes a$ is called an *elementary tensor*. It is not hard to see that if λ is a tensor of $M \otimes L$, then $t \otimes a \leq \lambda$ iff $a \leq \lambda(t)$. From this immediately follows that each tensor λ has the following decomposition:

$$\lambda = \bigvee_{t \in M} t \otimes \lambda(t). \tag{2.2}$$

Remark 2.3 (Lattice embeddings of M and L to $M \otimes L$)**.** Both M and L completely embed into $M \otimes L$. Namely, $M \xrightarrow{e_M} M \otimes L$ is given by

$$e_M(t) = t \otimes 1,$$

and $L \xrightarrow{e_L} M \otimes L$ is given by

$$e_L(a) = 1 \otimes a$$

(this notation may cause problems if $M = L$ but we never consider such a case explicitly).

2.2. Classic L-topological terminology

Here we explain which sort of many-valued topologies are going to be used in this paper. Namely, a family $\mathcal{T} \subseteq L^X$ is an *L-valued topology* (cf. [10, Section 5.2]) or short an *L-topology* on X, members of $\mathcal{T}$ are *open*, and $(X, \mathcal{T})$ is an *L-valued topological space* or short an *L-topological space* if $\mathcal{T}$ is closed under finite meets and arbitrary joins formed in L^X. A map $(X, \mathcal{T}_X) \xrightarrow{f} (Y, \mathcal{T}_Y)$ is *continuous* if, given a V in $\mathcal{T}_Y$, the map $V \circ f$ belongs to $\mathcal{T}_X$. Obviously, L-topological spaces and continuous maps form a category $\mathsf{Top}(L)$ which is topological over Set. Finally, if we identify a subset U of X with its characteristic function 1_U, then the category of topological spaces is isomorphic to a coreflective and full subcategory of $\mathsf{Top}(L)$.

Given A in L^X, we let $\operatorname{Int} A = \bigvee\{U \in \mathcal{T} \mid U \leq A\}$. If L has an order-reversing involution $(\cdot)'$, then K of L^X is *closed* if K' is open where $K'(x) = K(x)'$ for each $x \in X$. Then $\overline{A} = \bigwedge\{K \in L^X \mid A \leq K \text{ and } K \text{ is closed}\}$.

Notation. A complete lattice L with an order-reversing involution $(\cdot)'$ is written as $(L, ')$ and is called a *complete De Morgan algebra*.

An L-topology $\mathcal{T}$ on X is *generated* by a *subbase* $\mathcal{S} \subseteq L^X$ if $\mathcal{T}$ is the intersection of all the L-topologies on X which contain $\mathcal{S}$. The *subbase characterization of continuity* states that for $\mathcal{W}$ a subbase of $\mathcal{T}_Y$, a map $(X, \mathcal{T}_X) \xrightarrow{f} (Y, \mathcal{T}_Y)$ is continuous if and only if $W \circ f \in \mathcal{T}_X$ for all $W \in \mathcal{W}$(see [13, p. 282] for historical remarks). The point here is that L is an *arbitrary complete lattice* and not a frame. A subset Z of X becomes a subspace of X with L-topology consisting of restrictions $U|_Z$ for all $U \in \mathcal{T}$. Hence the subspace L-topology on Z is the initial L-topology with respect to the set-inclusion of Z into X. The Cartesian product X^J is L-topologized by the subbase $\{U \circ \pi_j \mid U \text{ is open and } j \in J\}$ where π_j is the jth projection. A continuous injective map $(X, \mathcal{T}_X) \xrightarrow{f} (Y, \mathcal{T}_Y)$ is an *L-topological embedding* if the initial L-topology with respect to f and $\mathcal{T}_Y$ coincides with $\mathcal{T}_X$. Finally, let us assume that $(L, ')$ is a complete De Morgan algebra. Then an L-topological space $(X, \mathcal{T})$ is called *normal* if, whenever K is closed, U is open, and $K \leq U$, there exists an open V such that $K \leq V \leq \overline{V} \leq U$.

3. On complete distributivity and $\lhd$-separability

A *completely distributive* lattice is a complete lattice M in which for every family $\{T_j\}_{j \in J}$ of subsets of M the following holds:

$$\bigwedge_{j \in J} \bigvee T_j = \bigvee_{\Phi \in \prod_{j \in J} T_j} \bigwedge_{j \in J} \Phi(j). \tag{3.1}$$

Instead of using (3.1), we shall use Raney's [23] characterization of complete distributivity in terms of the totally below relation $\lhd$ (cf. [1] and [4, Subsection 2.1.2]). Namely, if M is a complete lattice and $s, t \in M$, then the symbol

$$s \lhd t$$

means that

$$t \leq \bigvee T \quad \text{with } T \subseteq M \qquad \text{implies} \qquad s \leq r \quad \text{for some } r \in T.$$

For all r, s, t and u in M we have the following properties:

(1) $s \lhd t$ implies $s \leq t$,
(2) $r \leq s \lhd t \leq u$ implies $r \lhd u$,

Moreover, M is completely distributive if and only if $\lhd$ is *approximating* — i.e.

$$t = \bigvee\{s \in M \mid s \lhd t\} \quad \text{for all } t \in M.$$

In this situation, this means that if M is completely distributive, the insertion property of $\lhd$ is satisfied:

(3) If $s \lhd t$, then there exists $q \in M$ such that $s \lhd q \lhd t$ holds.

(Cf. [5, p. 204] and [23] where $\lhd$ is denoted by ρ).

We shall freely make use of these three properties. In particular, the insertion property implies that for any subset T of a completely distributive lattice M we have:

$$s \lhd \bigvee T \qquad \text{iff} \qquad s \lhd t \quad \text{for some } t \in T. \tag{3.2}$$

Notation. For each $t \in M$ we write

$$\Downarrow t = \{s \in M \mid s \lhd t\} \quad \text{and} \quad \Uparrow t = \{s \in M \mid t \lhd s\}.$$

Note that $\Downarrow 0 = \varnothing$ and $\Uparrow 0 = M \setminus \{0\}$ (cf. [5, IV-2.29 (i)]). As always, we write $\downarrow t = \{s \in M \mid s \leq t\}$ and $\uparrow t = \{s \in M \mid t \leq s\}$.

Example 3.1. Let M be a completely distributive lattice and L be a complete lattice. Then the totally below relation of the tensor product $M \otimes L$ can be characterized on elementary tensors as follows (cf. [4, Lemma 2.1.21]). If $t, s \in M$ and $a, b \in L$ with $s \neq 0$ and $b \neq 0$, then

$$s \otimes b \lhd t \otimes a \qquad \text{iff} \qquad s \lhd t \text{ and } b \lhd a. \tag{3.3}$$

In [4], property (3.3) is responsible for the non-trivial "if" part of the following equivalence: *$M \otimes L$ is completely distributive iff M and L are completely distributive*. The "only if" part follows from the complete embeddings of M and L into $M \otimes L$ (cf. Remark 2.3). For historical reasons we note that the "if" part of the above equivalence has already been proved by Shmuely [26] by a direct use of the complete distributivity law (3.1).

Remark 3.2. This is a good place to mention that some lattice properties of $I \otimes L$ have been proved quite long before they were proved for $I(L)$. Examples include complete distributivity and continuity of $I(L)$ (cf. [19] and [15], respectively). As has already been mentioned, complete distributivity comes from Shmuely [26], while continuity comes from Bandelt [2].

A subset $Q \subseteq M$ is called a *join base* of a complete lattice M (in short: *base*) if each member of M is a join of a subset of Q. Equivalently, if $t = \bigvee\{q \in Q \mid q \leq t\}$ for all $t \in M$.

Remark 3.3. If Q and B are bases of M and L, respectively, then the subset

$$\{q \otimes b \mid q \in Q \text{ and } b \in B\}$$

is a base of $M \otimes L$.

The next fact gives a characterization of a base in the framework of completely distributive lattices.

Fact 3.4. *For a subset Q of a completely distributive lattice M the following assertions are equivalent*:

(1) *Q is a base for M.*
(2) *Given $s \lhd t$ in M, there is a $q \in Q$ such that $s \lhd q \lhd t$.*
(3) *$t = \bigvee\{q \in Q \mid q \lhd t\}$ for all $t \in M$.*

Proof. It is shown in [8, Fact 2.1] that (1) and (2) are equivalent. The implication (3) $\Longrightarrow$ (1) is obvious. To see that (2) implies (3), we use the approximation and insertion properties of $\lhd$:

$$t = \bigvee_{s \lhd t} s \leq \bigvee_{s \lhd t} \bigvee\{q \in Q \mid s \lhd q \lhd t\} \leq \bigvee\{q \in Q \mid q \lhd t\} \leq t. \qquad \square$$

The relation $s \lhd t$ allows for the possibility that s and t might be equal. Elements which fail to have this property will play a crucial role in Section 4.

3.1. An important corollary of complete distributivity

As a first step we present a further characterization of complete distributivity. It may be that this characterization is not new, but we have never seen it in print. For later purposes we begin with a very useful lemma.

Lemma 3.5. *Let M be a complete lattice. Then for every $t \in M$ there exists an element $s_t \in M$ such that $\Uparrow t = M \setminus \downarrow s_t$ holds.*

Proof. If $t \in M$ is given, then we define $s_t \in M$ as follows:

$$s_t = \bigvee\{s \in M \mid t \not\leq s\}. \tag{3.4}$$

Since $\Uparrow 0 = M \setminus \{0\}$ and $s_0 = 0$, the assertion is obvious in the case of $t = 0$. Hence it is sufficient to consider the case $t \neq 0$. If the inclusion $\Uparrow t \subseteq M \setminus \downarrow s_t$

fails to hold, then there exists some $r \in M$ with $t \lhd r \leq s_t$ and so there exists $s \in M$ such that $t \leq s$ and $t \not\leq s$ which is a contradiction. On the other hand, if we choose $r \in M$ such that t is not totally below r, then there exists a subset T of M such that the following relation holds:

$$r \leq \bigvee T \quad \text{and} \quad t \not\leq s \text{ for all } s \in T.$$

Hence the definition of s_t implies $r \leq \bigvee T \leq s_t$. Consequently $M \setminus \Uparrow t \subseteq \downarrow s_t$ follows. □

Comment 3.6. For historical reasons we point out that the equivalence $t \lhd r$ if and only if $r \not\leq s_t$ already appears in an equivalent formulation in [24, p. 422], where s_t is determined by (3.4). The statement of the previous lemma is closely related to [8, Proposition 5.2].

Proposition 3.7. *Let M be a complete lattice. Then M is completely distributive if and only if for every $t \in M$ the following property holds*:

$$M \setminus \downarrow t \subseteq \bigcup_{s \not\leq t} \Uparrow s. \tag{3.5}$$

Proof. Let us assume that M is completely distributive — i.e. the totally below relation $\lhd$ is approximating. Then for $r, t \in M$ with $r \not\leq t$ the following relation holds:

$$r = \bigvee_{s \lhd r} s = \Big(\bigvee_{s \lhd r,\, s \leq t} s\Big) \vee \Big(\bigvee_{s \lhd r,\, s \not\leq t} s\Big) \leq t \vee \Big(\bigvee_{s \lhd r,\, s \not\leq t} s\Big).$$

Since $r \not\leq t$, the last join is a non-empty join — i.e. there is an $s \in M$ with $s \lhd r$ and $s \not\leq t$. Thus $r \in \bigcup_{s \not\leq t} \Uparrow s$, and the relation (3.5) is verified.

Conversely, let us assume that (3.5) holds for all $t \in M$. Then for every $t \in M$ we define an element $\widehat{t}$ by

$$\widehat{t} = \bigvee \{r \in M \mid r \lhd t\} \leq t.$$

In order to show that $\lhd$ is approximating, it is sufficient to prove $t \leq \widehat{t}$. Let us assume the contrary $t \not\leq \widehat{t}$. Then in the case of $\widehat{t}$ we apply (3.5) and obtain:

$$t \in M \setminus \downarrow \widehat{t} \subseteq \bigcup_{s \not\leq \widehat{t}} \Uparrow s.$$

Hence there exists $s \in M$ such that $s \not\leq \widehat{t}$ and $s \lhd t$ — i.e. a contradiction to the definition of $\widehat{t}$. Hence $\lhd$ is approximating. □

Corollary 3.8. *In every completely distributive lattice M the following relation holds for all $t \in M$:*

$$M \setminus \downarrow\!\!\!\!\downarrow t = \bigcup_{s \not\leq t} \Uparrow s.$$

Proof. Since in any complete lattice M the relation $\bigcup_{s \not\leq t} \Uparrow s \subseteq M \setminus \downarrow\!\!\!\!\downarrow t$ is satisfied, the assertion follows immediately from Proposition 3.7. □

As an application of Proposition 3.7 we here present the non-trivial part of Tarski's theorem (see [3, p. 119, Theorem 17] and [18, Example (i)]).

Corollary 3.9. *Every completely distributive complete Boolean algebra is atomic.*

Proof. Let M be a complete Boolean algebra, t be an element of M and t' be its complement. Then

$$s_t = \begin{cases} 0 & \text{if} \quad t = 0, \\ t' & \text{if} \quad t \text{ is an atom}, \\ 1 & \text{otherwise}, \end{cases}$$

and

$$\Uparrow t = \begin{cases} M \setminus \{0\} & \text{if} \quad t = 0, \\ \uparrow t & \text{if} \quad t \text{ is an atom}, \\ \varnothing & \text{otherwise}. \end{cases}$$

If we assume that M is not atomic, then there exists $t \in M$ with $t \neq 0$ such that the element $\bar{t} = \bigvee \{s \in M \mid s \leq t, \; s \text{ an atom}\}$ satisfies the condition $t \not\leq \bar{t}$. Referring to the previous constructions it is easily seen that the relation $M \setminus \downarrow\!\!\!\!\downarrow \bar{t} \not\subseteq \bigcup_{s \not\leq \bar{t}} \Uparrow s$ holds. Hence Proposition 3.7 implies the non-complete distributivity of M. □

Having described these preliminary properties of the totally below relation we illustrate the situation by the following examples.

Examples 3.10. (1) Let M be a complete chain and t be an element of M. Referring to (3.4), we have $s_t = \bigvee((\downarrow t) \setminus \{t\})$, and so we conclude from Lemma 3.5 that $t \lhd t$ if and only if $\bigvee((\downarrow t) \setminus \{t\}) < t$ — i.e. if and only if t is *isolated from below*. In the particular case of the real unit interval I, $s_t = t$ for each $t \in I$, and so the totally below relation $\lhd$ coincides with the strictly less-than relation $<$.

(2) As a further illustration let us consider the cartesian product $M \times L$ of two complete lattices M and L endowed with the componentwise order, and (t, a) be an element of $M \times L$. Then

$$s_{(t,a)} = \begin{cases} (1, s_a) & \text{if} \quad t = 0,\ a > 0, \\ (s_t, 1) & \text{if} \quad t > 0,\ a = 0, \\ (1, 1) & \text{if} \quad t > 0,\ a > 0, \end{cases}$$

and

$$\Uparrow(t, a) = \begin{cases} M \times (\Uparrow a) & \text{if} \quad t = 0,\ a > 0, \\ (\Uparrow t) \times L & \text{if} \quad t > 0,\ a = 0, \\ \varnothing & \text{if} \quad t > 0,\ a > 0. \end{cases}$$

Hence

$$(t, a) \lhd (s, b) \qquad \text{iff} \qquad (t = 0 \text{ and } a \lhd b) \text{ or } (a = 0 \text{ and } t \lhd s).$$

Consequently $M \times L$ is completely distributive if and only if both M and L are completely distributive.

(3) Let M be the usual topology on $\mathbb{R}$ and let u be a non-empty open set in M. Then

$$s_u = \bigvee \{v \in M \mid u \not\leq v\} = \bigcup \{v \in M \mid u \not\subseteq v\} = \mathbb{R}$$

and so $\Uparrow u = \varnothing$. Hence if u is a non-empty open set in M different from $\mathbb{R}$, then $M \setminus \downarrow u \neq \bigcup_{v \not\leq u} \Uparrow v = \varnothing$ and M fails to be completely distributive.

(4) Let $\mathcal{H}$ be a Hilbert space with $2 \leq \dim(\mathcal{H})$. Then the complete lattice M of all closed linear subspaces of $\mathcal{H}$ is atomic. Referring to (3.4), for every atom u — i.e. for every 1-dimensional linear subspace — we define:

$$s_u = \bigvee \{v \in M \mid u \not\leq v\} = \text{top. closure}\big(\text{lin. hull}\big(\bigcup \{v \in M \mid u \not\subseteq v\}\big)\big).$$

Since s_u coincides with the given Hilbert space — i.e. $s_u = \mathcal{H}$, the proof of Lemma 3.5 shows that $\Uparrow u$ is empty. Hence for every non-trivial closed linear subspace w of $\mathcal{H}$ we have $\Uparrow w = \varnothing$. To sum up, we have shown that the totally below relation coincides with the trivial relation — i.e. $u \lhd v$ if and only if $u = 0$ and $v \neq 0$.

3.2. $\lhd$*-Separability*

Let M be a complete lattice. An element $t \in M$ is called *supercompact* (also known as: *completely join irreducible* or *completely coprime*) if

$$t \lhd t.$$

We recall that 0 is never supercompact.

Definition 3.11 ([8])**.** We say that a completely distributive lattice M is $\lhd$*-separable* if it has a countable base Q which is free of supercompact elements — i.e.

$$q \not\lhd q \quad \text{for every } q \in Q.$$

Proposition 3.12. (1) *Let M be a completely distributive lattice and L be a complete lattice. Then an elementary tensor $t \otimes a$ in $M \otimes L$ is supercompact if and only if t and a are supercompact.*

(2) *Let M and L be completely distributive lattices. If M is $\lhd$-separable and L has a countable base or L is $\lhd$-separable and M has a countable base, then the tensor product $M \otimes L$ is a $\lhd$-separable.*

Proof. (1) Since supercompact elements of complete lattices never coincide with the universal lower bound, the equivalence in (1) follows immediately from the characterization of the totally below relation $\lhd$ in Example 3.1.

(2) Since M and L are completely distributive, their tensor product $M \otimes L$ is also completely distributive (cf. Example 3.1). Further, if Q and B are countable bases of M and L respectively, then the subset

$$\{q \otimes b \mid q \in Q,\, b \in B\}$$

is a countable base of $M \otimes L$ (see Remark 3.3). On this background the assertion (1) implies immediately the assertion (2). □

Since the real unit interval I is a $\lhd$-separable completely distributive lattice in which the rationals of I form a countable base without supercompact elements (cf. Example 3.10 (1)), we also have the following

Corollary 3.13. *Let L be a completely distributive lattice with a countable base. Then $I(L)$, $I(I(L))$, and so on, are $\lhd$-separable completely distributive lattices.*

Corollary 3.14. *If M is a $\lhd$-separable completely distributive lattice, then the countable product $M^{\mathbb{N}}$ of M is again completely distributive and $\lhd$-separable.*

Proof. Let $\mathcal{P}(\mathbb{N})$ be the power set of the natural numbers. The tensor product

$$M \otimes \mathcal{P}(\mathbb{N})$$

is completely distributive and order isomorphic to $M^{\mathbb{N}}$ (cf. [12, p. 10] or [4, Example 2.1.9]). Hence the assertion follows from Proposition 3.12 (2). □

Remark 3.15. The Hilbert cube is a prominent example of Corollary 3.14. The proof of Corollary 3.14 provides an alternative argument (based on tensor products) for a special case of a statement of Proposition 3.5 in [8] which states that the Cartesian product of an *arbitrary countable family* of $\lhd$-separable completely distributive lattices is again $\lhd$-separable.

4. Three L-topologies on $M \otimes L$

Before defining some L-topologies on $M \otimes L$ we give an alternative description of members of $M \otimes L$ with M a completely distributive lattice.

Definition 4.1. Let M be completely distributive. A map $M \xrightarrow{\lambda} L$ is called *left-continuous* if

$$\lambda(t) = \bigwedge\{\lambda(s) \mid s \lhd t\} \quad \text{for all } t \in M.$$

Checking that λ is left-continuous may be a useful alternative to verifying that λ is a tensor, for the following holds.

Lemma 4.2. *Let M be completely distributive lattice and L be a complete lattice. Then a map $M \xrightarrow{\lambda} L$ is a tensor of $M \otimes L$ if and only if λ is left-continuous.*

Proof. Since the relation $\lhd$ is approximating on M, it follows that every tensor of $M \otimes L$ is left-continuous. On the other hand, if $M \xrightarrow{\lambda} L$ is left-continuous, then we apply (3.2) and obtain for $T \subseteq M$:

$$\lambda\big(\bigvee T\big) = \bigwedge\{\lambda(s) \mid s \lhd \bigvee T\} = \bigwedge\big(\bigcup_{t\in T}\{\lambda(s) \mid s \lhd t\}\big) = \bigwedge_{t\in T} \lambda(t).$$

Hence λ is a tensor of $M \otimes L$. □

Given an order-reversing map $M \xrightarrow{\lambda} L$, let

$$\lambda^+(t) = \bigvee_{t \lhd s} \lambda(s) \quad \text{for all } t \in M.$$

Clearly, λ^+ is order-reversing and $\lambda^+ \leq \lambda$. Further properties of λ^+ are presented in the following:

Lemma 4.3. *Let M be a completely distributive lattice with a base Q, and let L be a complete lattice. For each $\lambda \in M \otimes L$ and $t \in M$ the following hold, where q stands for a member of Q:*

(1) $\lambda^+(t) = \bigvee_{t \lhd q} \lambda^+(q)$.

(2) $\lambda^+(t) = \bigvee_{t \lhd q} \lambda(q)$.

(3) $\lambda(t) = \bigwedge_{q \lhd t} \lambda(q)$.

(4) $\lambda(t) = \bigwedge_{q \lhd t} \lambda^+(q)$.

Proof. Referring to Fact 3.4 (2) we infer from the definition of λ^+ that

$$\lambda^+(t) = \bigvee_{t \lhd s} \lambda(s) = \bigvee\{\lambda(s) \mid t \lhd q \lhd s \text{ for some } q \in Q\} = \bigvee_{t \lhd q} \lambda^+(q).$$

Hence λ^+ satisfies (1). Since $\lambda^+ \leq \lambda$, the property (1) implies (2). The property (3) follows from the properties that Q is a base of M and λ is join-reversing. With regard to (4) we argue as follows. By definition of λ^+ the relation $\lambda(t) \leq \bigwedge_{q \lhd t} \lambda^+(q)$ holds. The reverse inequality follows from (3) and $\lambda^+ \leq \lambda$. □

We are now prepared for an L-topologization of $M \otimes L$.

Definition 4.4. Let M be a completely distributive lattice and let $(L, ')$ be a complete De Morgan algebra. For every $t \in M$, consider the maps $M \otimes L \xrightarrow{R_t} L$ and $M \otimes L \xrightarrow{L_t} L$ determined by

$$R_t(\lambda) = \lambda^+(t) \quad \text{and} \quad L_t(\lambda) = \lambda(t)'.$$

Then we define three L-topologies on $M \otimes L$ as follows:

(a) the *upper* L-topology $\mathcal{R}_{M\otimes L}$ generated by $\{R_t \mid t \in M\}$,

(b) the *lower* L-topology $\mathcal{L}_{M\otimes L}$ generated by $\{L_t \mid t \in M\}$,

(c) the *interval* L-topology $\mathcal{I}_{M\otimes L}$ generated by $\{R_t, L_t \mid t \in M\}$.

Note that $R_0 = L_0$ is a constant map with value 0.

Remark 4.5. If I is the real unit interval, then the tensor product $I \otimes L$ coincides with Lowen's [21] simplification of the original Hutton's $I(L)$. For details see [7].

The following is a restatement of Lemma 4.3 in terms of the maps R_t and L_t.

Lemma 4.6. *Let M be a completely distributive lattice with a base Q, and let L be a complete lattice. For each $t \in M$ the following hold, where q stands for a member of Q:*

(1) $R_t = \bigvee_{t \lhd q} R_q$.

If $(L,')$ is a complete De Morgan algebra, then:

(2) $R_t = \bigvee_{t \lhd q} L'_q$.

(3) $L_t = \bigvee_{q \lhd t} L_q$.

(4) $L_t = \bigvee_{q \lhd t} R'_q$.

In the remaining of this section, we discuss L-topological embeddings of M into $M \otimes L$. We recall that every complete lattice M carries three intrinsic topologies: the upper topology $\nu(M)$ generated by all the sets $M \setminus \downarrow t$, the lower topology $\omega(M)$ generated by all the sets $M \setminus \uparrow t$, and the interval topology $\iota(M)$ generated by all the sets $M \setminus \downarrow t$ and $M \setminus \uparrow t$. The next proposition follows from Lemma 3.5 and Proposition 3.7:

Proposition 4.7. *Let M be a completely distributive lattice. Then the upper topology $\nu(M)$* is generated by the family $\{\Uparrow t \mid t \in M\}$.

Remark 4.8. Let M be an arbitrary completely distributive lattice and $L = \mathbf{2}$. Since $\mathbf{2}$ is the unit object of Sup (cf. Theorem 2.2), it follows immediately that the embedding $M \xrightarrow{e_M} M \otimes \mathbf{2}$ is an order isomorphism. Because of $\big(R_t \circ e_M(s)\big)(t) = (s \otimes 1)^+(t)$ the relation $R_t \circ e_M = 1_{\Uparrow t}$ holds. Hence we conclude from Proposition 4.7 that the upper $\mathbf{2}$-topology on $M \otimes \mathbf{2}$ coincides with the traditional upper topology $\nu(M)$ on M. Similarly, we have $L_t \circ e_M = 1_{M \setminus \uparrow t}$, so that the lower $\mathbf{2}$-topology on $M \otimes \mathbf{2}$ coincides with the traditional lower topology on $\omega(M)$.

We refer again to Lemma 3.5, Proposition 3.7 and Remark 4.8 and observe that in the case of a completely distributive lattice M and a complete De Morgan algebra $(L,')$ the embedding $M \otimes \mathbf{2} \cong M \xrightarrow{e_M} M \otimes L$ is L-topological in three senses. Therefore we record the following fact.

Fact 4.9. *Let M be a completely distributive lattice and let $(L,')$ be a complete De Morgan algebra. The map $M \xrightarrow{e_M} M \otimes L$ is an L-topological embedding of $(M, \nu(M))$, $(M, \omega(M))$, and $(M, \iota(M))$ into $(M \otimes L, \mathcal{R}_{M\otimes L})$, $(M \otimes L, \mathcal{L}_{M\otimes L})$, and $(M \otimes L, \mathcal{I}_{M\otimes L})$, respectively. In this context it is worthwhile to mention that $\mathcal{R}_{M\otimes L}$ and the first embedding is independent of the order-reversing involution* (cf. Comment 6.1).

We finish this section with a discussion which explains the role of complete distributivity in Definition 4.4.

Remark 4.10. It is evident that in Definition 4.4 the three L-topologies $\mathcal{R}_{M\otimes L}, \mathcal{L}_{M\otimes L}$ and $\mathcal{I}_{M\otimes L}$ do not require the complete distributivity of M. Therefore it is interesting that in the case of complete lattices M the **2**-topology $\mathcal{L}_{M\otimes \mathbf{2}}$ coincides with the lower topology $\omega(M)$, while the **2**-topology $\mathcal{R}_{M\otimes \mathbf{2}}$ may be strictly coarser than the upper topology $\nu(M)$ (cf. Lemma 3.5). For example, if $\mathcal{H}$ is a Hilbert space with $2 \leq \dim(\mathcal{H})$ and M is the complete lattice of all closed linear subspaces of $\mathcal{H}$, then we conclude from Example 3.10 (4)) that the **2**-topology $\mathcal{R}_{M\otimes \mathbf{2}}$ has the following form $\{\varnothing,\, M \setminus \{0\},\, M\}$ where 0 is the trivial linear subspace of $\mathcal{H}$.

5. Urysohn lemma and Tietze–Urysohn extension theorem for $(M \otimes L)$-valued functions

If M is a $\lhd$-separable completely distributive lattice and Q is a base that witnesses the $\lhd$-separability of M, then – by definition – the transitive relation $\lhd$ is irreflexive when restricted to $Q \times Q$. This way we have arrived at the following.

Lemma 5.1 ([8, Definition 6.1 + Lemma 6.3])**.** *Let K be an arbitrary complete lattice endowed with a relation $\Subset$ satisfying the following conditions for all elements $a, b, c \in K$:*

(1) $a \Subset b$ *implies* $a \leq b$,
(2) $a \leq b \Subset c \leq d$ *implies* $a \Subset d$,
(3) $a, b \Subset c$ *implies* $a \vee b \Subset c$,
(4) $a \Subset b, c$ *implies* $a \Subset b \wedge c$,
(5) $a \Subset b$ *implies* $a \Subset c \Subset b$ *for some* $c \in K$.

Let J be an arbitrary countable set endowed with a transitive and irreflexive relation $\prec$. Let $\{a_j \mid j \in J\}$ and $\{b_j \mid j \in J\}$ be families of K satisfying the following:

$$j \prec i \quad \textit{implies} \quad \begin{cases} a_i \leq a_j, \\ a_i \Subset b_j, \\ b_i \leq b_j. \end{cases}$$

Then there exists a family $\{c_j \mid j \in J\}$ such that

$$j \prec i \quad \textit{implies} \quad \begin{cases} a_i \Subset c_j, \\ c_i \Subset c_j, \\ c_i \Subset b_j. \end{cases}$$

Remark 5.2. Let $(L,')$ be a complete De Morgan algebra, let X be an L-topological space, and let $K = L^X$. Given $A, B \in L^X$, we let

$$A \Subset B \quad \text{iff} \quad \overline{A} \leq \operatorname{Int} B. \tag{5.1}$$

Then $\Subset$ satisfies (1)–(4) above, and $\Subset$ satisfies (5) iff X is *normal*.

In what follows, $M \otimes L$ is endowed with its interval L-topology.

Theorem 5.3 (Urysohn lemma for $(M \otimes L)$-valued functions)**.** *Let M be a $\lhd$-separable completely distributive lattice and let $(L,')$ be a complete De Morgan algebra. For X an L-topological space the following statements are equivalent*:

(1) *X is normal.*

(2) *If $K \in L^X$ is closed, $U \in L^X$ is open, and $K \leq U$, then there exists a continuous function $X \xrightarrow{f} M \otimes L$ such that*

$$K \leq L_1' \circ f \leq R_0 \circ f \leq U.$$

Proof. Let Q be a countable base of M consisting of non-supercompact elements. In what follows q, r and s stand for members of Q. To show (2) implies (1), we follow the standard Urysohn's technique based on a special case of Lemma 5.1 in which $J = Q$, in which $\lhd$ plays the role of $\prec$, and in which $a_j = K$ and $b_j = U$ for all j.

Conversely, let X be normal and $\Subset$ stand for the relation of (5.1). By Lemma 5.1, there is a family $\{F_q \mid q \in Q\}$ of elements of L^X such that $K \Subset F_r \Subset F_q \Subset U$ whenever $q \lhd r$. In particular

$$\overline{F_r} \leq \operatorname{Int} F_q \quad \text{if} \quad q \lhd r. \tag{5.2}$$

For each $x \in X$ we let

$$\lambda_x(t) = \bigwedge_{q \lhd t} F_q(x) \quad \text{for every } t \in M.$$

We check it is left-continuous. Indeed,

$$\lambda_x(t) = \bigwedge_{q \lhd t} F_q(x) = \bigwedge_{s \lhd t} \bigwedge_{q \lhd s} F_q(x) = \bigwedge_{s \lhd t} \lambda_x(s),$$

i.e. $\lambda_x \in M \otimes L$. Define $X \xrightarrow{f} M \otimes L$ by the formula $f(x) = \lambda_x$. Thus

$$f(x)(t) = \bigwedge_{q \lhd t} F_q(x). \tag{5.3}$$

We now show f is continuous by using the subbasic characterization of continuity — i.e. we are going to show that $L_t \circ f$ and $R_t \circ f$ are open for each $t \in M$. For each $t \in M$ we have

$$L_t \circ f = \bigvee_{q \lhd t} F'_q \tag{5.4}$$

and

$$R_t \circ f = \bigvee_{t \lhd q} F_q. \tag{5.5}$$

Clearly, (5.4) is a restatement of (5.3). To show (5.5), we use (2) of Lemma 4.6 and (5.3) to obtain

$$R_t \circ f = \bigvee_{t \lhd q} L'_q \circ f = \bigvee_{t \lhd q} \bigwedge_{r \lhd q} F_r \geq \bigvee_{t \lhd q} \bigvee_{q \lhd r} F_r = \bigvee_{t \lhd r} F_r.$$

For the reverse inequality notice that

$$R_t \circ f = \bigvee_{t \lhd q} \bigwedge_{r \lhd q} F_r \leq \bigvee_{t \lhd r} F_r,$$

so that (5.5) is verified. By (5.2), we obtain that

$$L_t \circ f = \bigvee_{q \lhd t} F'_q = \bigvee_{q \lhd t} \overline{F_q}'$$

and

$$R_t \circ f = \bigvee_{t \lhd q} F_q = \bigvee_{t \lhd q} \operatorname{Int} F_q.$$

are open.

Finally, since $K \leq F_q \leq U$ for all $q \in Q$, hence

$$K \leq \bigwedge_{q \lhd 1} F_q = L'_1 \circ f \leq R_0 \circ f = \bigvee_{0 \lhd q} F_q \leq U,$$

which completes the proof. □

Theorem 5.4 (Tietze–Urysohn extension theorem for $(M \otimes L)$-valued functions)**.** *Let M be a $\lhd$-separable completely distributive lattice and let $(L, ')$ be a complete De Morgan algebra. Let X be a normal L-topological space and let $Z \subseteq X$ be such that $1_Z \in L^X$ is closed. Then every continuous function $Z \xrightarrow{g} M \otimes L$ has a continuous extension to the whole X.*

Proof. Let Q be a countable base of M which is free of supercompact elements. In what follows p, q, r and s stand for members of Q. We follow the technique of Proof 2 of Theorem 4.10 of [14]. For every q there exist open V_q and W_q in X such that

$$L_q \circ g = W_q|_Z \quad \text{and} \quad R_q \circ g = V_q|_Z.$$

Let

$$K_q = W_q' \wedge 1_Z \quad \text{and} \quad U_q = V_q \vee 1_{X\setminus Z}.$$

Then K_q is closed, U_q is open for all $q \in Q$, and for each $x \in Z$ and $s \lhd r$ in Q we have

$$\begin{aligned} K_r(x) &= W_r'(x) = L_r(g(x))' = g(x)(r) \\ &\leq g(x)^+(s) = R_s(g(x)) = V_s(x) = U_s(x). \end{aligned}$$

Hence

$$K_r \leq U_s \quad \text{if} \quad s \lhd r.$$

Let $\Subset$ be the relation of (5.1): $A \Subset B$ iff $\overline{A} \subseteq \operatorname{Int} B$. Since X is normal, the families $\mathcal{K} = \{K_q \mid q \in Q\}$ and $\mathcal{U} = \{U_q \mid q \in Q\}$ satisfy the following:

$$s \lhd r \quad \text{implies} \quad \begin{cases} K_r \leq K_s, \\ K_r \Subset U_s, \\ U_r \leq U_s. \end{cases}$$

By Lemma 5.1, there exists a family $\mathcal{F} = \{F_q \mid q \in Q\}$ such that

$$s \lhd r \quad \text{implies} \quad \begin{cases} K_r \Subset F_s, \\ F_r \Subset F_s, \\ F_r \Subset U_s. \end{cases} \tag{5.6}$$

As in the proof of Theorem 5.3, define a function $X \xrightarrow{f} M \otimes L$ by the formula

$$f(x)(t) = \bigwedge_{q \lhd t} F_q(x).$$

Then f is continuous by the same argument as in the proof of Theorem 5.3. It remains to check that $f = g$ on Z. Let $x \in Z$. Clearly, $f(x)(0) = 1 =$

$g(x)(0)$. Let $t \neq 0$. We have $W'_q(x) = K_q(x)$, hence, by (3) of Lemma 4.6,

$$\begin{aligned} g(x)(t) &= \big(L'_t \circ g\big)(x) \\ &= \bigwedge_{q \lhd t} \big(L'_q \circ g\big)(x) \\ &= \bigwedge_{q \lhd t} K_q(x) \\ &\leq \bigwedge_{q \lhd t} F_q(x) = f(x)(t) \end{aligned}$$

where the inequality holds by (5.6). Likewise, since $U_q(x) = V_q(x)$, we have

$$\begin{aligned} f(x)(t) &= \bigwedge_{q \lhd t} F_q(x) \\ &\leq \bigwedge_{q \lhd t} U_q(x) \\ &= \bigwedge_{q \lhd t} (R_q \circ g)(x) \\ &= \big(L'_t \circ g\big)(t) = g(x)(t). \end{aligned}$$

We have shown that $g(x) = f(x)$ for all $x \in Z$. □

Remark 5.5. Theorems 5.3 and 5.4 provide common generalizations of results in three different situations. Because of Fact 4.9, if $M = [0, 1]$ and $L = \mathbf{2}$, then Theorems 5.3 and 5.4 become the Urysohn lemma and Tietze–Urysohn extension theorem for usual topological spaces, respectively. If $M = [0, 1]$ and $(L, ')$ is a complete De Morgan algebra, then these theorems reduce to the L-topological versions of the Urysohn lemma and Tietze–Urysohn extension theorem (cf. [11] and [14]). With $L = \mathbf{2}$ we arrive at [8, Theorem 6.5 (4) and (5)].

6. The relationship of $M \otimes L$ to the L-fuzzy topological modification of M

As has already been mentioned, as a generalization of $I(L)$, Zhang and Liu [28] considered the collection of all join-preserving maps from M to L and called it the *L-fuzzy modification* of M. Roughly speaking, in our paper, the relation $\lhd$ plays the role of the relation $<$ of I, while in [28] $<$ is replaced by $\not\geq$. Due to the fact that $I(L) = I \otimes L$, the Zhang-Liu's construction will be denoted here by $M[L]$ (and not by $M(L)$ as is in [28]). We observe that if L has an order-reversing involution $(\cdot)'$, then

$$\lambda \in M \otimes L \quad \text{iff} \quad \lambda' \in M[L]$$

and $\lambda \leq \mu$ in $M \otimes L$ iff $\mu' \leq \lambda'$ in $M[L]$. Hence the map $M \otimes L \xrightarrow{h} M[L]$ given by $h(\lambda) = \lambda'$ is an order-reversing bijection. For each $t \in M$ define two maps $M[L] \xrightarrow{\mathrm{R}_t} L$ and $M[L] \xrightarrow{\mathrm{L}_t} L$ by

$$\mathrm{R}_t(\mu) = \bigwedge_{s \not\leq t} \mu(s) \quad \text{and} \quad \mathrm{L}_t(\mu) = \mu(t)'.$$

In [28], two L-topologies have been introduced on $M[L]$. Here we shall discuss only the L-topology δ_L which is generated by the family $\{\mathrm{R}'_t, \mathrm{L}'_t \mid t \in M\}$.

We now proceed to show that $(M[L], \delta_L)$ and $(M \otimes L, \mathcal{I}_{M\otimes L})$ are homeomorphic. For this we need an alternative, but equivalent, description of the upper L-topology $\mathcal{R}_{M\otimes L}$ on $M \otimes L$.

Comment 6.1. In our paper, as in [11], the use of join-reversing maps shows that the upper L-topology is independent of the order-reversing involution, while in [28] all the subbasic elements depend on it.

Proposition 6.2. *Let M be a completely distributive lattice and let L be a complete lattice. For each $t \in M$ we define $M \otimes L \xrightarrow{\mathfrak{r}_t} L$ by*

$$\mathfrak{r}_t(\lambda) = \bigvee_{s \not\leq t} \lambda(s).$$

Then the family $\big\{\mathfrak{r}_t \mid t \in M\big\}$ is a subbase for the L-topology $\mathcal{R}_{M\otimes L}$.

Proof. Denote by $\mathfrak{R}$ the L-topology on $M \otimes L$ generated by $\{\mathfrak{r}_t \mid t \in M\}$. For every $t \in M$ and $\lambda \in M \otimes L$ we have

$$\mathfrak{r}_t(\lambda) = \bigvee_{s \not\leq t} \bigvee_{s \lhd r} \lambda(r) = \bigvee_{s \not\leq t} \lambda^+(s) = \bigvee_{s \not\leq t} R_s(\lambda),$$

where the first equality holds on account of Corollary 3.8. This shows the inclusion $\mathfrak{R} \subseteq \mathcal{R}_{M\otimes L}$. To show the reverse inclusion, fix $t \in M$. By Lemma 3.5, there exists an $s_t \in M$ such that $\Uparrow t = M \setminus \downarrow s_t$. Hence $R_t = \mathfrak{r}_{s_t}$. $\square$

Corollary 6.3. *Let M be completely distributive and let $(L, ')$ be a complete De Morgan algebra. Then $(M[L], \delta_L)$ and $(M \otimes L, \mathcal{I}_{M\otimes L})$ are homeomorphic.*

Proof. Let us consider the bijection $(M \otimes L, \mathcal{I}_{M\otimes L}) \xrightarrow{h} (M[L], \delta_L)$ given by $h(\lambda) = \lambda'$. Since by Proposition 6.2 the relations $\mathrm{L}'_t \circ h = L'_t \in \mathcal{L}_{M\otimes L}$ and $\mathrm{R}'_t \circ h = \mathrm{r}_t \in \mathcal{R}_{M\otimes L}$ hold, h and h^{-1} are continuous — i.e. h is a homeomorphism. $\square$

Remark 6.4 (Brouwer fixed point theorem)**.** Let M and $(L,')$ be completely distributive lattices. Further, let $\mathfrak{m}$ be a cardinal and let $(M\otimes L)^{\mathfrak{m}}$ be the L-topological product of $\mathfrak{m}$ copies of $M\otimes L$ with its interval L-topology. In [17] it is shown that $M[L]^{\mathfrak{m}}$ has the *fixed point property* — i.e. each continuous selfmap of $M[L]^{\mathfrak{m}}$ has a fixed point (when $M=I$ and $L=\mathbf{2}$ it becomes the Brouwer fixed point theorem for an arbitrary cube $I^{\mathfrak{m}}$). Since the L-topological spaces $M[L]$ and $M\otimes L$ are homeomorphic, we conclude that $(M\otimes L)^{\mathfrak{m}}$ has the fixed point property, too.

Appendix: Iterating the construction of $I(L)$

This section requires a good command of symmetric monoidal closed categories. All the material needed is elaborated in detail in [4].

In [16, Question 17], it is asked whether the recursive construction $I(L)$, $I(I(L))$, and so on, terminates. Now, if we know that $I(L)$ is the tensor product of I and L, we have a solution by a categorical argument applied to the monoidal closed category Sup. Let us first recall what is the tensor product of morphisms of Sup.

If $M \xrightarrow{\alpha} M_1$ and $L \xrightarrow{\beta} L_1$ are join-preserving maps, then the *tensor product* $\alpha\otimes\beta$ of α and β is the unique join-preserving map from $M\otimes L$ into $M_1\otimes L_1$ making the following diagram commutative:

$$\begin{array}{ccc} M\times L & \xrightarrow{\ \otimes\ } & M\otimes L \\ {\scriptstyle \alpha\times\beta}\downarrow & & \vdots\,{\scriptstyle \alpha\otimes\beta} \\ M_1\times L_1 & \xrightarrow{\ \otimes\ } & M_1\otimes L_1 \end{array}$$

In particular, $\alpha\otimes\beta$ coincides with the unique join preserving extension of the bimorphisms $(t,a)\longmapsto\alpha(t)\otimes\beta(a)$ from $M\times L$ to $M\otimes L$. Given a tensor λ of $M\otimes L$, the formula for $(\alpha\otimes\beta)(\lambda)$ is obtained by using (2.2) and the fact that $\alpha\otimes\beta$ is join-preserving.

Now, define

$$\mathbb{I}^n=\begin{cases} L & \text{if} \quad n=0,\\ I\otimes\mathbb{I}^{n-1} & \text{if} \quad n\geq 1,\end{cases}$$

and $\mathbb{I}^n \xrightarrow{f_{n+1,n}} \mathbb{I}^{n+1}$ by

$$f_{n+1,n}=\begin{cases} e_L & \text{if} \quad n=0,\\ \mathrm{id}_I\otimes f_{n,n-1} & \text{if} \quad n\geq 1,\end{cases}$$

where e_L is the embedding of L into $I \otimes L$ (see Remark 2.3). Then

$$\big(\mathbb{I}^n, f_{m,n}\big)_{n\geq 0}$$

is a direct system in Sup where $f_{m,n} = f_{m,m-1} \circ \cdots \circ f_{n+1,n}$ $(n < m)$. Since Sup is cocomplete, the direct limit $\mathbb{I}^\infty$ of the considered direct system exists. Further, we conclude from the symmetry and closedness of Sup that the endofunctor $I \otimes \_$ of Sup has a right adjoint functor. Hence $I \otimes \mathbb{I}^\infty$ and $\mathbb{I}^\infty$ are isomorphic — i.e. the sequence

$$I(L),\ I(I(L)),\ I(I(I(L))),\ \ldots$$

stops.

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