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**A MODEL OF EVOLUTIONARY
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A Model of Evolutionary Drift

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ABSTRACT

Drift appears to be crucial to study the stability properties of Nash equilibria in a component specifying different out-of-equilibrium behaviour. We propose a new microeconomic model of drift to be added to the learning process by which agents find their way to equilibrium. A key feature of the model is the sensitivity of the noisy agent to the proportion of agents in his player population playing the same strategy as his current one. We show that, 1. Perturbed Payo[®]-Positive and Payo[®] Monotone selection dynamics are capable of stabilizing pure non strict Nash equilibria in either singleton or nonsingleton component of equilibria; 2. The model is relevant to understand the role of drift in the behaviour observed in the laboratory for the Ultimatum Game and for predicting outcomes that can be experimentally tested. Hence, the selection dynamics model perturbed with the proposed drift may be seen as well as a new learning tool to understand observed behaviour.

Key words: similarity relations, drift, Nash equilibrium, replicator dynamics, learning.

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1. Introduction

It is common place to observe that the equilibrium selected by a theory depends on the manner in which perturbations are handled (see, for example, Selten's (1975) perfect equilibrium and Myerson's (1978) proper equilibrium). Binmore et al. (1995) and Binmore and Samuelson (1999) also emphasize the importance of perturbations, but they place these in the dynamic process that takes the players to equilibrium rather than perturbing the game itself (like Selten or Myerson). It is in modelling such perturbations realistically that the present paper is concerned.

The important work of Binmore and Samuelson (1999), B & S, from now on, studies the limiting behaviour of perturbed selection dynamic systems, but little insight is given into what perturbations one should expect. The implicit message of B & S is the need of an explicit microeconomic model of observed human choice behaviour that originates the drift that perturbs a selection dynamics.

We tackle this issue by relating drift with the choice procedure used by the so-called P agents. The procedure derives explicitly from the similarity theory developed first by Tversky (1977) in psychology and later applied to choice theory by Kahneman and Tversky (1979) and Rubinstein (1988) to explain observed behavior, such as the one leading to the Allais Paradox. The present paper extends the similarity based choice theory developed in Aizpurua et al. (1993) and Uriarte (1999), by building a model valid for a dynamic setting.

We use the same methodology as B & S to study the stability properties of Nash equilibria specifying different out-of-equilibrium path appearing in connected components of stationary states. Thus, we work with a continuous, deterministic selection dynamic model -derived from a biological model of natural selection. Then, for a better approximation to an underlying stochastic strategy-adjustment process that governs the players' behaviour, the selection model will be completed by adding perturbations that incorporates some of the real-life imperfections or "anomalies" of the human choice behaviour which are excluded by the model. These appear in the model as drift. For B & S, the ultimate goal of including drift is to obtain a more realistic model of how society evolves and coordinates on some equilibria and not in others.

In the present paper drift is not derived from the behaviour of agents who misread the game. We assume a society composed of agents who use different strategy choice procedures. Thus, a different type of agents, the P procedural agents, is added to the population of agents whose behaviour leads to the selection dynamic model (the SD -agents). Drift arises from the strategy-adjustment process governing the behaviour of the P agents. In that process, the P agents inject continuously strategies that are not currently played by the population. Thus, the learning process of the P agents will perturb that of the SD agents. Hence, we shall deal with perturbed selection dynamic systems containing large player pop-

ulations and each population being composed of two equally large subpopulations of agent types, the SD_i agent type and \bar{P} -agent type.

The most important innovation introduced in this paper is a feature observed in real life decision-making procedures that is excluded by selection dynamic models. In those models, payoffs are the only indicator considered by the agents when taking decisions on the space of available strategies. But it is well known that, in real choice situations, people often imitate successful actions, whereby "success" is measured by the fraction of the population taking that action. In our model, it is the \bar{P} -agent who captures that way of learning. It is true that both payoffs and strategy frequencies are indicators of how well one is doing in a game, but when social norms and conventions are introduced into the picture, their influence on individuals is that different strategies are a priori perceived differently and therefore the tendencies to abandon them might differ. That is, in some games, playing some strategy might be viewed as more ethical (or, just more according to the custom) than playing another strategy and therefore, people will be less inclined to abandon the former than the latter. For this reason, when the choices of an individual are led by a set of norms, it seems natural to assume that the influence of payoffs on the strategy choices of a \bar{P}_i agent is less important than the influence of the fraction of agents in the same population playing the same strategy.

The analytic results of the present paper are obtained under such extreme sensitivities of the \bar{P}_i agents to the strategy frequencies that the possible influence of payoffs on their strategy choices is eliminated. Our model allows drift to be governed by non-payoff factors only and payoffs would influence effectively the motion of the perturbed system through the selection dynamic model. This model of drift shows that there exists a trade-off between the stability properties of the component of non strict Nash equilibria and the sensitivity of drift to either expected payoffs (as in B & S) or to strategy frequencies. We show that the more sensitive is drift to strategy frequencies the stronger are the stability properties of the components. It is worth mentioning that contrary to B & S and to the noise models of Hopkins (2002), drift in the present model is non inward-pointing. The reason for this is that if the purpose of the perturbed model is to match observed behaviour then, inward-pointing drift is problematic.

With respect to B & S's results, extensions are obtained in two directions. First, we show that, with a suitable choice of the parameters that specify drift, both perturbed payoff-positive and payoff-monotone selection models may stabilize pure non strict Nash equilibria in either singleton or non singleton component of equilibria. Thus, contrary to the model of B & S, stability in components of Nash equilibria specifying out of equilibrium behaviour is achieved independently of the size of the component. Second, as a model of perturbed learning, the model seems to be a better description of the interactive learning that actually takes

place in laboratory environments. It is known that the range of equilibria that can be obtained with the B & S model does not include those that are observed in the experiments of Roth and Erev (1995) with the Full Ultimatum Game (see Binmore et al. (1995)). In the present model, the equilibria are consistent with the experimental data and, moreover, it is predicted that they are independent of the observed initial propensities. To our knowledge, the independence of the outcomes from the initial play in the Ultimatum Game has never been tested. Hence, we propose an experiment to address this question and test the prediction of the model. From the experiment we would know how realistic are the assumptions built in the theory of the agents and the role of drift in the equilibrium selection.

The paper is organized as follows. The notation is presented in Section 2. In Section 3, we present our model of evolutionary drift. In section 4, we show how individual similarity compatible adaptive choices approximate, in the aggregate, selection dynamic models such as the adjusted Replicator Dynamics and Weakly Payo-Positive dynamics. In section 5 the main analytic results are presented. In Section 6 we study the perturbed learning implicit in the model in the light of the observed equilibria of different games.

2. Notation

Let G be a noncooperative finite game in normal form, with $K = \{1, 2, \dots, n\}$ as the set of players. We assume that there are n large player-populations. Randomly drawn members of the n player-populations, one from each population, are repeatedly matched to play the game. For each player $k \in K$, let $S_k = \{1, 2, \dots, m_k\}$ be his finite set of pure strategies, for some integer $m_k > 2$. Throughout the paper, we shall refer to agent ki ; a member of player-population $k \in K$ playing strategy $i \in S_k$. Thus, f_{ki} will denote the proportion of agents in player population $k \in K$ who play strategy $i \in S_k$ at time t , with f_k being the vector collecting such proportions in population k and $f = (f_1, \dots, f_n)$ the population state at time t . Hence, $f \in \Phi = \prod_{k=1}^n \Phi_k$; where Φ_k is the simplex of mixed strategies for player $k \in K$. $F_{ki} = [0, 1]$ is the space of proportions of agents in player-population k playing strategy i . Let $u_{ki}(f)$ denote the (expected) payo to agent ki given the population state f at time t and U_{ki} the space of expected payos $u_{ki}(f)$; $k \in K$; $i \in S_k$ and $f \in \Phi$ (more specifications about payos are given in section 3.2, below). The term $\bar{u}_k(f) = \sum_{i=1}^{m_k} f_{ki} u_{ki}(f)$ denotes the average expected payo to player population $k \in K$.

Methodology

We start with a system of continuous, deterministic differential equations that describe how the proportions of the player populations attached to each pure strategy evolve over time. For B & S, this system is represented by a selection dynamic model which one can find in biological models of natural selection. Obviously, no selection function can take into account the many factors that affect the actual learning process that guide agents' decisions. Therefore, B & S assume that a better approximation to the underlying stochastic strategy-adjustment process could be obtained by adding a drift term to the selection dynamics. In the next section we shall extend the analysis of B & S by adding behavioral features initially excluded in the selection dynamics and providing microeconomic foundations for the drift term.

3. Drift.

It is natural to observe different behaviours when different agents face the same decision problem. For this reason, diversity of tastes and values are central to economic analysis. Thus, following the economist's approach, to the society that evolves according to those agents whose behaviour leads to the selection dynamic model (the SD_i agents), we shall add a new type of agents, the so-called β -procedural agents. We shall assume below that the source of perturbations to the selection dynamic is the strategy-adjustment process followed by the agents. Thus, the perturbed selection dynamic (studied in section 5) will have large player-populations and inside each population we assume there are two types of agents, the SD_i agents and the β agents. Both population types are assumed to be equally large. We proceed now to describe the features of the β agent k_i .

It seems natural to assume that the participant in a repeated interaction builds experience-based conjectures about how good or bad is playing the underlying game and that he may relate that evaluation to, among other factors, the proportion of individuals who are playing exactly like him. We will assume that in a given interaction that takes place many times the β agent k_i has, at each stage, information about those proportions and thinks as follows: "the higher is the proportion of agents in my player population who are currently using the same strategy as mine, the less ambiguity (or insecurity or uncertainty or vagueness) I should feel about how well I am playing the game". Formally,

Assumption 1

Each β agent k_i is endowed with a differentiable function d_{k_i} in the set

$$D = \left\{ d_{k_i} : F_{k_i} \subset [0; 1] : d_{k_i}(0) = 1; d_{k_i}(1) = 0; \text{ and } \begin{matrix} \frac{1}{2} & & \frac{3}{4} \\ d_{k_i}(p_{k_i}) > d_{k_i}(e_{k_i}) & & d_{k_i}(p_{k_i}) < d_{k_i}(e_{k_i}) \end{matrix} \right\}$$

Given a proportion $f_{ki} \in F_{ki}$ and any $d_{ki} \in D$; $d_{ki}(f_{ki})$ measures the ambiguity (about how well is playing the game at time t) felt by the agent ki when the proportion of agents in player population k playing strategy $i \in S_k$ at time t is f_{ki} . The ambiguity gradually decreases when he observes that more and more agents from his population come to play the same strategy as his: We call the d_{ki} function agent ki 's threshold function. For a different use of the strategy proportions information, see Young (1993a), (1993b) and (1996).

Remark 1: The Playing Modes.

For any d_{ki}, \bar{d}_{ki} in D ; if for all $f_{ki} \in (0; 1)$; $d_{ki}(f_{ki}) < \bar{d}_{ki}(f_{ki})$; then we say that d_{ki} is sharper than \bar{d}_{ki} : Two important cases should be considered: for all $f_{ki} \in (0; 1)$, the extremely sharp threshold function, $\bar{d} \in D$; for which $\bar{d}(f_{ki})$ takes values which are "very close" to 0 (i.e., $\bar{d}(f_{ki}) \cong 0$) and the extremely unsharp threshold function, $\underline{d} \in D$; for which $\underline{d}(f_{ki})$ takes values which are "very close" to 1 (i.e.; $\underline{d}(f_{ki}) \cong 1$). Clearly, for any $r_{ki} \in [0; 1]$, $d_{ki} = r_{ki}\bar{d} + (1 - r_{ki})\underline{d}$ is in D : Hence, r_{ki} may be used as a measure for the degree of sharpness of agent ki 's threshold function d_{ki} . If $r_{ki} = 1$; $d_{ki}(f_{ki}) = \bar{d}(f_{ki})$ and we say that agent ki is in the alert mode of play. When $r_{ki} = 0$; $d_{ki}(f_{ki}) = \underline{d}(f_{ki})$; and we say that agent ki is in the absent mode of play.

3.1. Vagueness Modelled by Means of Similarity Relations.

We assume that the level of vagueness felt by the agent ki ; about how well is playing, develops intervals (in both the payoff and strategy frequency spaces) inside which events are not distinguishable. To model these intervals, we use the similarity theory introduced by Tversky(1977) and later adapted to decision theory by Rubinstein(1988),(1998).

In essence, a similarity relation serves to capture the capacity of an individual to discriminate between events. Correlated similarity relations, a concept introduced by Aizpurua et al.(1993), describe how that discrimination capacity changes depending on the values of some relevant parameter. For instance, the correlated similarity relations defined on F_{ki} could capture the idea that the efforts dedicated to discriminate on F_{ki} increase if the payoffs at stake, $v_{ki}(f)$; increase. Correlated similarity relations on f_{ki} would formalize the idea that discrimination on f_{ki} increases when the proportion of agents f_{ki} increases (i.e. a finer discrimination is obtained if experience is increased and this is assumed to occur when more agents from population k come to play strategy i).

We shall see that the d_{ki} function serves to define on f_{ki} (correlated) similarities of the difference type and the s_{ki} function (defined below) is used to define

on F_{ki} a similarity of the ratio type.

3.2. The Role of Payoffs.

Assuming that each agent ki is endowed with a $d_{ki} \in D$ and given $f_{ki} \in (0; 1)$, the function ρ_{ki} is defined as follows: for all $\frac{1}{4}_{ki}(f) > d_{ki}(f_{ki}) > 0$

$$\rho_{ki}(\frac{1}{4}_{ki}(f)) = \frac{\frac{1}{4}_{ki}(f)}{\frac{1}{4}_{ki}(f) - d_{ki}(f_{ki})} > 1(2)$$

In this definition of ρ_{ki} we assume that all payoffs are strictly positive and do not exceed 1. Hence, $\frac{1}{4}_{ki} = (0; 1]$; for all $k \in K$ and $i \in S_k$: If $\frac{1}{4}_{ki}(f) \leq d_{ki}(f_{ki})$ then, ρ_{ki} is not defined and we would then have the degenerate similarity relation, i.e. a relation for which the similarity interval of any point in $\frac{1}{4}_{ki}$ is the whole space $\frac{1}{4}_{ki}$ (see Rubinstein (1988)): (Nevertheless, this situation cannot be easily modelled. When doing the computations, we can avoid the problems caused by the degenerate case, -i.e. when $\rho_{ki} < 0$ -, to the dynamics (4) and (6) below, by adding a constant $c > 1$ to both numerator and denominator of (2)):

But, in fact, payoffs play some role in the determination of ρ_{ki} functions only for values of r_{ki} different from 0 and 1, that is, when agents are not in one of the two playing modes. Under the definition (2) of ρ_{ki} ; if $r_{ki} = 1$ then, for any $f_{ki} \in (0; 1)$; $\rho_{ki}(\frac{1}{4}_{ki}(f))$ is nearly 1 for any $\frac{1}{4}_{ki}(f) > \bar{d}(f_{ki})$: When $r_{ki} = 0$ then, $\rho_{ki}(\frac{1}{4}_{ki}(f))$ is a very large number for any $\frac{1}{4}_{ki}(f) > \underline{d}(f_{ki})$. In the latter case, $\rho_{ki}(\cdot)$ would only be defined for $\frac{1}{4}_{ki}(f)$ equal to 1. Therefore, both in the alert and the absent modes, it is the extreme sensitivity of the threshold function $d_{ki}(\cdot)$ to f_{ki} that determines the value of the function ρ_{ki} . In other words, expected payoffs would not matter in these two extreme cases and then, the ρ_{ki} functions would be defined as in (3) below.

Assumption 2

When the agent ki is in a given playing mode, then, for all $\frac{1}{4}_{ki}(f) \in \frac{1}{4}_{ki}$ and $f_{ki} \in (0; 1)$; the ρ_{ki} functions are defined as

$$\rho_{ki}(\frac{1}{4}_{ki}(f)) = \frac{1}{1 - d_{ki}(f_{ki})} > 1(3)$$

where $d_{ki}(\cdot) = \bar{d}(\cdot)$ if agent ki is playing in the alert mode or $d_{ki}(\cdot) = \underline{d}(\cdot)$ if he is playing in the absent mode.

Now all payoffs and the space \mathcal{I}_{ki} are less restricted than in the previous case. In this paper, we assume that the ρ_{ki} functions are defined as in (3): Some words of warning should be said here. Even though the analytic results (of section 5) are obtained with ρ_{ki} defined as in (3), we think it is worth mentioning the definition of ρ_{ki} given in (2), because there are real life (or laboratory) situations in which agents are not exactly in a given playing mode, while in other cases one might think that they are (see the examples presented in section 6). Further, (2) allows us to establish a link with the B & S model of drift (explained in note 1 below). We call ρ_{ki} the i agent ki 's perception function. These functions play a key role in the dynamics of drift presented in section 3.4.

3.3. Satisfying Preferences

The pair of similarity relations defined by d_{ki} and ρ_{ki} , respectively, facilitate the decision-making to the agent ki by means of the following procedure (the paper would be clearer and less cryptic if the specific features of the similarity theory used here are not hidden; hence, we show in Appendix I how the similarity relations are defined and how a preference on $\mathcal{I}_{ki} \in F_{ki}$ could be defined from them): given a pair of vectors $(\bar{v}_{ki}(f); \bar{f}_{ki})$ and $(v_{ki}(f); f_{ki})$ in $\mathcal{I}_{ki} \in F_{ki}$; the agent ki proceeds by checking first whether " $\bar{v}_{ki}(f)$ is similar to $v_{ki}(f)$ " and " \bar{f}_{ki} is similar to f_{ki} ": If, say, only the first statement is true and $\bar{f}_{ki} > f_{ki}$, then $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki})$ is declared to be preferred to $(v_{ki}(f); f_{ki})$: In this manner, the agent ki may build a (non-complete and non-transitive) preference relation, $\%_{ki}$, on $\mathcal{I}_{ki} \in F_{ki}$. Thus, for any vector $(v_{ki}(f); f_{ki}) \in \mathcal{I}_{ki} \in F_{ki}$; attached to strategy $i \in S_k$ at time t ; the corresponding upper, lower and indifference sets can be defined (see Figure 1; to understand how this figure is built, we have assumed that $d_{ki} \in \mathcal{d}$; \mathcal{d} and ρ_{ki} is defined as in (3)). The preferred set would represent i agent ki 's aspiration set at time t . By definition, as $(v_{ki}(f); f_{ki})$ changes the corresponding aspiration set, obviously, changes.

[Place here Figure 1]

We assume that a P agent ki is a P preference satisfier, in the sense that he chooses a strategy just to minimize the distance from $(v_{ki}(f); f_{ki})$ to his aspiration set. But this amounts to minimize the uncertainty, $d_{ki}(f_{ki})$; felt about how well is playing the game. That is, $d_{ki}(f_{ki})$ determines both the size of the similarity interval on \mathcal{I}_{ki} and, through ρ_{ki} ; the size of the interval on F_{ki} . And both intervals determine the thickness of the indifference set (see Appendix I and Figure 1). The smaller is the value of ρ_{ki} ; the thinner is the indifference set $S_{ki} [(v_{ki}(f); f_{ki})]$; the dark area of Figure 1- , and therefore, the smaller is the distance to the aspiration: Hence, the function ρ_{ki} could be thought of as an

indicator of the degree of satisfaction of P_i agent k_i with the strategy he is currently using. The smaller the value of s_{ki} ; the happier would feel the agent with his current strategy. Then, no matter what the values of $u_{ki}(f)$ and f_{ki} are, the indifference set of a P_i agent k_i in the alert mode would have almost an empty interior and so $(u_{ki}(f); f_{ki})$ is "near" the agent's aspiration set. Then, it could be said that he is "very satisfied" with his current strategy. In the other extreme, the indifference set of someone in the absent mode would cover almost the entire space $u_{ki} \in F_{ki}$ and the agent could be said to be "highly dissatisfied" with his current strategy.

3.4. The Dynamic of Drift

We take the ratio $\frac{1}{s_{ki}}$ as the probability that P_i agent k_i will retain his current strategy $i \in S_k$ in the next period; $1 - \frac{1}{s_{ki}}$ will then be the probability of switching to a different strategy in S_k .

Assumption 3: "The Mistake".

When a P_i agent k_i is dissatisfied with his current strategy, he will choose the $m_{ki} - 1$ available strategies $j \in S_k; j \neq i$; with the same probability $\frac{1}{m_{ki} - 1} (1 - \frac{1}{s_{ki}})$.

Thus, the P_i agents follow the rule "try every other action if you feel dissatisfied with your current strategy". We call it "mistake" just to keep using the same words of B&S; but, of course, we do not think it is a mistake to behave that way. Assumption 3 is introduced with the only purpose of perturbing the SD system by injecting strategies that are not currently played. Since we are completing a selection model that excludes some features of human choice behaviour, we think that there are some arguments that may justify the use of this assumption. In a selection dynamic model, extinct strategies remain extinct forever. This assumption should be used with care in non biological contexts. In real choice situations, there are agents who, given the uncertainties they face, give a chance not only to technologies that have survived the evolutionary pressures (i.e. the market pressures) but to all the technologies they know. They may even update "recipes" used in the past. This might explain, for instance, the existence of agricultural products grown in mass production farms and those produced in a smaller scale in "bio-farms" (where no pesticides and hormones are used) and industrial products obtained either by means of robotized technologies or with "hand made" technologies.

We assume that when an agent switches strategy instantly learns, by imitation or education, the playing mode of the newly adopted strategy. Inside the agents of population k , strategy $i \in S_k = \{1; 2; \dots; m_k\}$ will be played "mistake-

only" by those dissatisfied \mathbf{P} agents $k_j, j \in i$; coming from the rest of $m_k - 1$ strategies (the in-flow), $\frac{1}{m_k - 1} \sum_{j \in i}^{m_k} f_{kj} (1 - \frac{1}{s_{kj}})$. The out-flow is the proportion of dissatisfied \mathbf{P} agents k_i who abandon the strategy and "mistakenly" choose with equal probability the $m_k - 1$ strategies different to i , $f_{ki} (1 - \frac{1}{s_{ki}})$: We shall assume that the drift term or the "mistaken" dynamics added to the k_i th selection dynamics equation is just the difference between these two flows originated by the "mistake" in which incur dissatisfied \mathbf{P} agents. Hence, those who retain their current strategy are not included in the drift term.

$$\begin{aligned} & \frac{1}{m_k - 1} \sum_{j \in i}^{m_k} f_{kj} (1 - \frac{1}{s_{kj}}) - f_{ki} (1 - \frac{1}{s_{ki}}) \\ = & \frac{1}{m_k - 1} \sum_{j \in i}^{m_k} f_{kj} (1 - \frac{1}{s_{kj}}) + f_{ki} \frac{1}{s_{ki}} - f_{ki} \\ = & [\mu_{ki}(f) - f_{ki}] \end{aligned}$$

where $\mu_{ki}(f) = \frac{1}{m_k - 1} \sum_{j \in i}^{m_k} f_{kj} (1 - \frac{1}{s_{kj}}) + f_{ki} \frac{1}{s_{ki}}$. If each \mathbf{P} agent k_i ; in every $k \in K = \{1; 2; \dots; ng\}$; is playing each strategy $i \in S_k$ under some playing mode then, by (3), the drift term is not sensitive to expected payoffs and will take the following form¹

$$[\mu_{ki}(f) - f_{ki}] = \frac{1}{m_k - 1} \sum_{j \in i}^{m_k} f_{kj} d_{kj}(f_{kj}) - f_{ki} d_{ki}(f_{ki}) \quad (4)$$

Remark 2:

The limiting values of drift are modelled as follows. When a \mathbf{P} agent k_i is in the alert mode, the probability of switching, $1 - \frac{1}{s_{ki}}$, will always be very small and so the drift he will introduce will be negligible. We may say that he is, either by experience or due to some other reasons (such as education, the ethical values held by the agent and the social norms he honors), "loyal" to his current strategy $i \in S_k$. In the absent mode an agent feels a high degree of uncertainty about the righteousness of his play and therefore will feel very dissatisfied with his current strategy. Hence, the probability $1 - \frac{1}{s_{ki}}$, will be large because s_{ki} is very large; therefore the drift he will introduce will be very high:

4. From Similarity Compatible Choices to Selection Dynamic Models.

So far, we have used similarity theory to build a model of drift. Now we show how one can build a selection dynamic model (SD) from agents whose choices are based on a pair of similarity relations. Assume that there are k large player populations. Inside each population, there is only one type of agents, the $SD_{i_{ki}}$ agents, who are described below (and specifically named as the \odot agent ki). Members of the k populations chosen at random - one member from each player population - are repeatedly matched to play the noncooperative finite game G : We assume that every $SD_{i_{ki}}$ agent has always a constant level of uncertainty about how well is playing the game. Hence,

Assumption 4

For all $k \in K$ and $i \in S_k$, the function $d_{ki} : F_{ki} \rightarrow [0; 1]$ is defined as:
 $d_{ki}(f_{ki}) = z_k$; for all $f_{ki} \in F_{ki} = [0; 1]$; where $z_k \in (0; 1)$:

As before, we assume that the level of uncertainty z_k felt by agent $SD_{i_{ki}}$ generates similarity intervals in both i_{ki} and F_{ki} , which are described by means of similarity relations. Assumption 4 will allow us to introduce a similarity relation of the difference type on i_{ki} . The next function, - called perception function, - defined for all $\mathcal{V}_{ki} > z_k > 0$ as

$$\hat{A}_{ki}(\mathcal{V}_{ki}(f)) = \frac{\mathcal{V}_{ki}(f)}{\mathcal{V}_{ki}(f) - z_k} > 1$$

We assume now that all payoffs are strictly positive and do not exceed 1. Since z_k is fixed, we might restrict \mathcal{V}_{ki} to the set $i_{ki} = (z_k; 1]$: The function \hat{A}_{ki} serves to build ratio-type correlated similarity relations on F_{ki} (both similarity relations are built very much like those described in Appendix 1 for the \odot agents. For specific details that distinguish this case from the similarities used to build drift, see Aizpurua et al.(1993)). The similarities will then be used to build a preference - now a \odot procedural preference, - on $i_{ki} \in F_{ki}$: As in the behaviour of the preference holders, at each point in time, a \odot agent ki 's degree of satisfaction with strategy i depends on the distance from the vector $(\mathcal{V}_{ki}(f); f_{ki})$; attached to strategy $i \in S_k$ at time t ; and its corresponding preferred (or aspiration) set. That distance depends on the thickness of the indifference set, $\gg_{ki} [(\mathcal{V}_{ki}(f); f_{ki})]$; which in turn depends on both z_k and $\hat{A}_{ki}(\mathcal{V}_{ki}(f))$: Assuming differentiability of \hat{A}_{ki} with respect $\mathcal{V}_{ki}(t)$; we have that $\frac{\partial \hat{A}_{ki}(\mathcal{V}_{ki}(f))}{\partial \mathcal{V}_{ki}(f)} < 0$: Thus, the higher is $\mathcal{V}_{ki}(f)$ the

smaller is $\hat{A}_{ki}(\mathcal{Y}_{ki}(f))$ and the thinner is the indifference set. Hence, we assume that at each period of time, every agent ki chooses a strategy in S_k to reduce the distance from $(\mathcal{Y}_{ki}(f); f_{ki})$ to its corresponding aspiration (or preferred) set.

Let us consider now the ratio (for notational simplicity we write $\hat{A}_{ki}(f)$ instead of $\hat{A}_{ki}(\mathcal{Y}_{ki}(f))$);

$$\frac{\hat{A}_{ki}(f) - 1}{\sum_{i=1}^{m_k} \hat{A}_{ki}(f)} = \frac{\hat{A}_k(f) - 1}{\hat{A}_k(f)}$$

where $\sum_{i=1}^{m_k} \hat{A}_{ki}(f) = \hat{A}_k(f)$; is the total perception in player population k at t . As $\mathcal{Y}_{ki}(f)$ increases (decreases) the ratio $\frac{\hat{A}_{ki}(f) - 1}{\hat{A}_k(f)}$ decreases (increases). Hence, we may interpret $\frac{\hat{A}_{ki}(f) - 1}{\hat{A}_k(f)}$ ($1 - \frac{\hat{A}_{ki}(f) - 1}{\hat{A}_k(f)}$) as a measure of the proportion of ki strategists who feel dissatisfied (satisfied) with strategy i . Assume that time is divided in discrete periods of length ζ and that $(\zeta) 1 - \zeta$ is the probability that each agent (does not) retain(s) his current strategy. Then $\zeta \frac{(\hat{A}_k(f(t)) - 1)}{\hat{A}_k(f(t))} f_{ki}(t)$ denotes the proportion of ki strategists who will choose a new strategy at time t (the outflow). If a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\zeta \prod_{j=1}^{m_k} \frac{(\hat{A}_k(f(t)) - 1)}{\hat{A}_k(f(t))} f_{kj} f_{ki} = \zeta \frac{(\hat{A}_k(f(t)) - 1)}{\hat{A}_k(f(t))} f_{ki}(t)$ denotes the inflow of agents who choose strategy i (the inflow). (where $\hat{A}_k(f(t)) = \sum_{j=1}^{m_k} \hat{A}_{kj}(f(t)) f_{kj}(t)$ is the average perception in player population k at time t). Therefore

$$f_{ki}(t + \zeta) = f_{ki}(t) - \zeta \frac{[\hat{A}_{ki}(f(t)) - 1]}{\hat{A}_k(f(t))} f_{ki}(t) + \zeta \frac{(\hat{A}_k(f(t)) - 1)}{\hat{A}_k(f(t))} f_{ki}(t):$$

As $\zeta \rightarrow 0$; in the limit we have

$$\dot{f}_{ki} = f_{ki} \frac{(\hat{A}_k(f) - 1) \hat{A}_{ki}(f)}{\hat{A}_k(f)} \quad (5)$$

Proposition 1

(a) If for all player position $k \in K = \{1; 2; \dots; n\}$; the strategy set S_k consists of two elements, i.e. if $m_k = 2$ then, equation (5) is just the standard Replicator Dynamics (RD) multiplied by a positive function.

(b) If $m_k > 2$; then, from (5) we obtain a selection dynamics that approximates the RD, but preserves only the positive sign of the RD, i.e. is a Weakly Payo[®]-Positive selection dynamics (WP-PD).

Proof: In the Appendix II. \forall

5. Analysis.

It seems that the previous result tells us that we should study selection dynamic models, ranging from the Payo[®]-Positive ones to the wider class of Weakly Payo[®]-Positive dynamics. To this end, let, as before, $K = \{1, 2, \dots, n\}$ be the set of players and for each player $k \in K$, let $S_k = \{1, 2, \dots, m_k\}$ be his finite set of pure strategies, for some integer $m_k > 2$. We start with the RD; results will not be changed if we consider the standard RD. The resulting perturbed deterministic RD is (note that the expected payo[®]s of the selection dynamic model are not normalized)

$$\dot{f}_{ki} = f_{ki} \left(\frac{1}{m_k} \sum_{j \in S_k} g_{kj}(f) - \bar{g}_k(f) \right) + [\mu_{ki}(f) - f_{ki}] \quad (6)$$

As it was mentioned above, (6) is derived by the joint behaviour of the S_k agents and the i agents. Notice that for each player-population $k \in K$; $\sum_{i=1}^{m_k} [\mu_{ki}(f) - f_{ki}] = 0$ and so $\sum_{i=1}^{m_k} f_{ki} \left(\frac{1}{m_k} \sum_{j \in S_k} g_{kj}(f) - \bar{g}_k(f) \right) + \sum_{i=1}^{m_k} [\mu_{ki}(f) - f_{ki}] = 0$.

Let f^* be a pure strategy equilibrium of the general game G in normal form. We say that f^* is a Nash equilibrium in the alert mode when for every $i \in S_k$ and every player population $k \in \{1, 2, \dots, n\}$; the agents ki 's threshold function $d_{ki} = \bar{d}$; while for the rest of agents kj 's the functions $d_{kj} = \underline{d}$, $j \in S_k$. Thus, a pure-strategy Nash equilibrium f^* is in alert mode when all strategies in that equilibrium profile are played by agents in the alert mode while all strategies not in f^* are played by agents in the absent mode.

Proposition 2

Let f^* be pure but not a strict Nash equilibrium in the alert mode. Then, f^* is stabilized in the Replicator Dynamics perturbed by the drift term (4).

Proof: In Appendix II. \forall .

Let the growth-rate function g that assigns to each population state f ; player population k and pure strategy i available to k the growth rate $g_{ki}(f)$ of the population share f_{ki} . Let g be a Payo[®] Positive growth-rate function (i.e. for all $f \in \mathbb{C}$; player population $k \in K$ and pure strategies $i \in S_k$; $\text{sgn}[g_{ki}(f)] = \text{sgn}[f_{ki} - \bar{f}_{ki}]$).

$\text{sgn}[\mu_{ki}(f) - \mu_k(f)]$). The RD is an example of such g : By adding the drift term (4) we get the perturbed equation

$$\dot{f}_{ki} = g_{ki}(f)f_{ki} + [\mu_{ki}(f) - f_{ki}](7)$$

The next corollary is immediate.

Corollary 1

Proposition 2 is valid for any Payo[®] Positive g : ¥

Consider now the class of Payo[®] Monotonic growth-rate functions (i.e. for all $f \in \mathbb{C}$; player population $k \in K$ and pure strategies $i, j \in S_k$; $\mu_{ki}(f) > \mu_{kj}(f)$, $g_{ki}(f) > g_{kj}(f)$).

Proposition 3

Let f^* be pure but not a strict Nash equilibrium in the alert mode. Then, f^* is stabilized in any Payo[®] Monotonic selection dynamics perturbed by the drift term (4).

Proof: In Appendix II. ¥

Now, what about the class of Weakly Payo[®] Positive Dynamics?. Here we face the problem that when in a player population, say $k \in K$; everyone earns the same payo[®], apart from regularity, weak payo[®] positivity imposes no restrictions on g_k . Then, inside population k ; we might have two opposed general drifts: one caused by the selection dynamics itself, driving the population toward some pure strategy (see Weibull(1995), Example 5.7) and the other caused by the drift term, sending the population toward another pure strategy, and both strategies belonging to different Nash equilibria. The limiting result will depend on the relative power of both drifts. Hence, nothing general can be established in this class of selection dynamics.

6. Perturbed Learning (or Learning to Be Imperfect) and Predictions

The specification of the drift term $[\mu_{ki}(f) - f_{ki}]$ will change from game to game. The changes will be captured by the playing modes attached to each strategy in each population. Given a game, in particular, a game with multiple Nash equilibria, the choice of a specification of drift must be guided, in our opinion, by

the data obtained in the laboratory, as well as from the knowledge of society's modal tastes and values. For concreteness, the selection dynamics of this section is assumed to be the standard RD.

6.1. Example 1(Outcomes do not depend on the size of the Nash Component)

Let us consider a game having a component with empty interior, such as the one presented by B & S, reproduced here in Figure 2. In this game, (L;U) is an equilibrium in weakly dominated strategies (a non strict-path equilibrium in B & S words) that cannot be stabilized with the model of drift of B & S because there is no possibility for inward- pointing drift to be compatible with this equilibrium. If we assume that agents playing L and U are in the alert mode and those playing R and D are in the absent mode then, it can be verified that (L;U) is a local asymptotic attractor for the perturbed system (6) (the formal proof is similar to that of Proposition 2). Figure 2 is obtained from a computer simulation²: Figure 2a corresponds to the phase diagram of the unperturbed replicator dynamics and Figure 2b depicts the phase diagram of drift. Figure 2c depicts the perturbed replicator dynamics with two asymptotic attractors, the subgame-perfect equilibrium (R;D) and (L;U). The basin of attraction of (L;U) will increase in size relative to that of (R;D) if the sharpness of d_L and d_U were higher relative to those of d_R and d_D :

[Place here Figure 2]

Similarly, in the version of the Dalek Game studied by B & S (see Figure 3), the non subgame-perfect equilibrium (T;R) will appear as a local asymptotic attractor whenever these strategies are played in the alert mode and the strategies of the subgame-perfect equilibrium (M;L) are played in the absent mode (note that these playing modes might make sense when the game is interpreted as how to divide 12 units of surplus). Thus, contrary to the B&S model, predictions with our model are based on arguments that do not depend on the size of the component.

We have established:

Corollary 2

Proposition 2 shows that the stability power of the present model of drift does not depend on the size of the Nash Component. ¥

[Place here Figure 3]

In the next two examples we are going to present the drift specifications needed to stabilize every Nash equilibrium, whether subgame-perfect or not, in two games of different economic nature, but equal structure: the Ultimatum Minigame and the Chain-Store Game. The sections below could be called "learning to be imperfect", as they are, in a sense, a different answer to the issues dealt by Binmore et al. (1995).

| | | | |
|-------|---|------------|-----|
| | | (y) Y N | |
| | | 2,2 2,2 | |
| (x) H | L | 3,1 | 0,0 |

Figure 4. The Ultimatum Minigame or the Chain-Store Game.

6.2. Example 3: The Ultimatum Minigame (UM)

Let us suppose a simplified version of the Ultimatum Game (see Binmore et al. (1995)). Figure 4 describes the game in strategic form; the amount to be divided is 4 and x and y are the probabilities of playing H and Y, respectively. The game has a unique Subgame-perfect equilibrium $(0, 1)$ and a component of Nash equilibria, denoted NC, the segment joining $(1, 0)$ and $(1, 2/3)$. Let $d_H, d_L, d_Y, d_N; d_H, d_L; d_Y$ and d_N denote the perception and threshold functions of agents playing strategy High, Low, Yes and No, respectively.

The experimental findings about the Ultimatum Game are very robust (see also the findings of Güth et al. (2001) though) and show that people share a common notion about what is a fair, reasonable or acceptable offer and that their play is largely guided by those notions. Hence, for the UM, there is only one specification of drift that is compatible with the experimental findings; that is, the drift derived when we assume that β_i proposers playing H and β_j responders playing N are in the alert mode while the rest of β_i agents in both populations are in the absent mode. In other words, agents in both populations must be loyal to the strategies that allow a coordination in the equal-split equilibrium and unloyal to those strategies that do not allow that outcome. Proposition 4, Case I, below, shows that this particular specification of the drift term, and no other, stabilizes the point $x = 1$ and $y = 0$ of the Nash component. Further, even if initially there is a very small percentage of β_i agents playing H and N in their respective populations in the alert mode and the rest of agents in both populations are in the absent mode, the theorem shows that $(1; 0)$ will be the outcome. In Binmore et al. (1995), outcomes are locally asymptotic and obtained as approximations to an element in the component NC:

The perturbed system (6) with drift term (4) for the UM is the following (time index suppressed)

$$\begin{aligned}\dot{x} &= x(1-x)(2-3y) - xd_H(x) + (1-x)d_L(1-x) \quad (8) \\ \dot{y} &= y(1-y)(1-x) - yd_V(y) + (1-y)d_N(1-y) \quad (9)\end{aligned}$$

6.3. Example 4: The Chain-Store Game (CH-S)

Selten's Chain-Store Game has the same structure as the UM game. The two games describe different economic situations and therefore the drift term need not be the same. We shall assume, for simplicity, that both the UM and CH-S games have the payoffs of Figure 4. Hence, the UM game would correspond to the (only) Weak Monopolist Game of Jung et al.(1994) in which the incumbent (Monopolist) would prefer to share the market if entry occurred. We may conjecture two different situations modelled by two different specifications of drift. For instance, let us consider the case when potential entrant agents playing NE (Not Enter) are in the alert mode, those playing E (Enter) are in the absent mode and all incumbent agents, i.e. those playing Y (Yield) and F (Fight), are in the absent mode or almost in that mode (change in Figure 2, H and L for NE and E, respectively; F substitutes N). Thus, incumbents think to know well the trade, overestimate their power and do not worry; while entrants calculate well their moves. Let x denote the proportion of potential entrants playing NE and y the proportion of incumbents playing Y: Then, in Proposition 4, Case II, below, we get $(1, 1/2)$ as a global asymptotic attractor. Note that this result is related to Proposition 3 of Binmore et al. (1995), where, assuming endogenous drift and uniform mistake probabilities for both populations, they found that the asymptotic attractors are $(0,1)$, the subgame-perfect equilibrium, and $(1, 1/2)$.

The next situation would approach the case of experienced players with sufficient time and learning with no experimenter-induced strong monopolist of Jung et al.(1994). The appropriate specification of drift for this situation could be when both potential entrants playing E and incumbents playing Y are in the alert mode, while the rest of agents in both populations are in the absent mode. Then, in Case III of Proposition 4, we show that the subgame-perfect equilibrium is a global asymptotic attractor (and elements of NC are not local attractors). The next result is a full stability study of these two games.

Proposition 4

Case I. Suppose in the UM Game that the P_H agents (i.e. P proposers offering H) are in the alert mode (so $d_H = \bar{d}$) and the P_L agents are in the absent mode ($d_L = \underline{d}$). Then, if responders playing Y are in the absent mode ($d_Y = \underline{d}$) and those playing N are in the alert mode ($d_N = \bar{d}$), the only asymptotic attractor is the equal-split Nash equilibrium $(1; 0)$.

Case II. Suppose in the CH-S Game that the P_{NE} agents are in the alert mode ($d_{NE} = \bar{d}$) and the P_E agents are in the absent mode ($d_E = \underline{d}$). Then, if both agents Y and F are almost in the absent mode and $d_Y(\cdot) = d_F(\cdot)$, the only asymptotic attractor is $(1; 1=2)$.

Case III. Suppose in the CH-S Game that the P_{NE} agents are in the absent mode (so $d_{NE} = \underline{d}$) and the P_E agents are in the alert mode ($d_E = \bar{d}$). Then if $d_Y = \bar{d}$ and $d_F = \underline{d}$, the only asymptotic attractor is the subgame-perfect equilibrium $(0; 1)$:

Proof : In Appendix II. \forall

Corollary 3

Any Nash equilibrium in the interior of NC can be stabilized.

Proof. This can be accomplished by a combination of some strategies in the alert mode and others with threshold functions displaying different levels of low sensitivity to the corresponding strategy frequency, as in Case II of Proposition 4. \forall

"Be magnanimous and learn to say no". If someone's behaviour is guided by this norm, he would play High (in the role of proposer) and No (in the role of responder) in the alert mode. Proposition 2, Case I, may serve to relate the present work with that of Abbink et al.(2001). We may say, in their words, that this is a fairness motivated person, loyal to the strategy that would implement the equal split equilibrium, so that, responders playing No are "programmed" to punish unfair offers. Suppose the initial play is near the subgame perfect equilibrium, $(0; 1)$; where there is only a small percentage of highly fairness-motivated agents in both player populations. The theorem shows that both proposers and responders learn to coordinate in the non perfect equilibrium $(1; 0)$ where they all play H and N; respectively, in the alert mode. Hence, the theorem predicts learning in both noisy player populations, whereas in Abbink et al.(2001), there is only evidence for first movers learning.

Numerical Example: the drift terms needed to stabilize equilibria in the Nash Component NC of the UM and CH-S games.

As in Example 1, consider the subclass $d_{ki}(f_{ki}) = (1 - f_{ki})^{n_{ki}}$, $n_{ki} \in (0; 1)$ of threshold functions in D . In this set, the degree of sharpness of $d_{ki}(f_{ki})$ increases with the value of n_{ki} : Suppose we want to stabilize the points $(1; 0:6)$ and $(1; 0:3)$; both in the component NC of the UM or Ch-S Game. If we set $x = 1$ in the system (5)-(6) and make $\dot{y} = 0$; we get, (in the following, we use the notation for the Ultimatum Minigame; for the Chain-Store Game, we would write d_F instead of d_N)

$$\frac{d_Y(y)}{d_N(1 - y)} = \frac{1 - y}{y}$$

When $y = 0:6$; we know from Proposition 4 that above $(1; 1=2)$; we must have $1 > r_Y > r_N$ (but r_Y cannot be very high). Substituting y by its value and taking logarithms in the above expression, we get, among many others, the following three pairs of values for n_Y and n_N - determining the sharpness or sensitivity of d_Y and d_N to strategy frequencies y and $1 - y$; respectively- compatible with the drift needed to stabilize $(1; 0:6) \in NC$: $(n_Y = 0.72126, n_N = 0.5)$, $(0.554, 0.2)$ and $(0.4481, 0.01)$, respectively. If $y = 0:3$; Proposition 4 says that below $(1; 1=2)$; $r_Y < r_N < 1$; we get the following three pairs values for n_Y and n_N compatible with the drift that stabilizes $(1; 0:3) \in NC$: $(0.6624, 0.9)$, $(0.3249, 0.8)$ and $(0.1561, 0.75)$, respectively.

In Figure 5 it is depicted the graph of the function that relates r_Y with r_N . Above the diagonal D , it is indicated the pairs of $(r_Y; r_N)$ needed to stabilize the equilibria in the subset $[(1,0), (1,1/2)]$ of the component $NC = [(1, 0), (1, 2/3)]$. When $(r_Y; r_N) = (0; 1)$; Proposition 4 shows that $(H; N)$ is a global asymptotic attractor and when the graph cuts D at low levels of $r_Y = r_N$; then, $(1; 1=2)$ is the global asymptotic attractor. The graph below and near D are the pairs $(r_Y; r_N)$ needed to stabilize the rest of the equilibria in NC : When $(r_Y; r_N) = (1; 0)$; then, under the assumptions of Case III, we would obtain the subgame-perfect equilibrium $(L; Y)$ as a global attractor. ¥

[Place here Figure 5]

6.4. Example 5: The Full Ultimatum Game.

To test the role of drift in the selection of outcomes observed in the laboratory we propose an experiment for the Full Ultimatum Game. We have seen that the model allows to establish a clear relation between drift and the modal tastes and values in a society or, in a smaller scale, between drift and the observed behaviour

in specific environments. This relation serves to obtain the information needed to specify the drift term (4) by determining the playing mode of each strategy. Once this is accomplished, the model may make predictions or at least replicate the learning that takes place, if any, in the laboratory.

In Roth et al. (1991) and Roth and Erev (1995) it is reported an experiment with the Ultimatum Game³ carried out in four countries: Israel, Japan, USA and Slovenia. It is observed that the norms that are commonly used in real-life bargaining situations prompt individuals to initially allocate a significant share of the surplus to the responders.

We shall refer in this section to a computer simulation for this game carried out with system (6). A more complete study of this sort can be found in Uriarte (2000), where it is shown under which conditions the model converges to the observed equilibria in each of the four countries⁴. As argued by Roth and Erev (1995), players' initial propensities can have a long-term influence in the players learning. It can be said too that the initial propensities in the Ultimatum Game tend to be located in the basin of attraction of the observed equilibria: that is, in the basin of f_{5g} for Slovenia and USA and f_{6g} for Israel and Japan. Hence, both the initial propensities and the final outcomes are the data that should be predicted by the theory. Let us refer first to the B & S theory.

In Binmore et al. (1995) it is said that the range of potential equilibria obtained with the B & S model does not include, in general, the "fair outcome" in the Full Ultimatum Game. It was reported there (see p.68) that increasing the mistake probabilities attached to the "fair" offer has little effect on the results of the calculations. Furthermore, even though the observed initial propensities are in the neighborhood of the final outcomes of this game, simulations show that in the B & S model the solution trajectories starting from those initial propensities can lead, in many cases, towards the subgame-perfect equilibrium or equilibria close to it. Hence, we may say that the predictions derived with the B & S model are somehow inconsistent with the experimental results⁵.

We mentioned above that the degree of sharpness of d_{ki} could be used to model how sensitive is agent ki to society's norms and conventions which are encoded in the strategy frequencies f_{ki} . Thus, d_{ki} would be a kind of agent ki 's "initial cultural endowment", shaped by his/her life experience as a member of a given society and capturing part of the cultural background that would bring to the laboratory. From Proposition 2, we know how to introduce the drift needed to stabilize equilibria that are not subgame-perfect. So, let P_{ki} denote that the agents ki playing strategy i are in the alert mode and that the rest of strategies $j \neq i$ for population k are played by agents in the absent mode. For the simulations, we shall suppose that the "initial cultural endowment" in Japan and Israel is given by $f_{P_{16}}; P_{116g}$ and in Slovenia and USA $f_{P_{15}}; P_{115g}$; where $k = 1$ stands for

proposers and $k = 11$ for responders⁶:

Under these conditions, the computations with the Full Ultimatum Game show that the observed laboratory equilibria, $f5g$ in USA and Slovenia and $f6g$ in Israel and Japan, appear as global asymptotic attractors⁵. Something similar happened in Proposition 4 with the Ultimatum Minigame. This means that, contrary to what happens to the reinforcement learning model, in the present model of perturbed learning the experimental outcomes can be obtained independently of the initial play. Hence, to simulate the path to the observed outcomes in the ultimatum game, the model does not need to take the observed initial play (the initial propensities of Roth and Erev (1995)) as given. The present model of drift predicts that the observed laboratory data, i.e. both the initial propensities and the equilibria, are due to a rather high sensitivity of individuals to fairness-oriented norms. Society's inequality-averse modal values will motivate a behaviour in which the "fair" strategies will be played by the agents using their analytical and perceptual resources at their highest level, i.e. in the alert mode.

To our knowledge, the independence of the experimental outcomes in the Ultimatum Game from the initial propensities has not been tested yet. Therefore we propose the following experiment to address this question and have more information about the role of drift in the outcome. A group of subjects is anonymously matched in a way so that their play is conditioned to coordinate on equilibria close to the subgame-perfect equilibrium; another group is conditioned to play equilibria with high payoffs for responders. This could be done, for instance, by anonymously matching individual proposers against dummy agents (or computers) programmed to play like "soft" responders who would accept any positive amount. Responders, on the other hand, would be matched against dummy or computerized "soft" proposers. After few rounds, and due to the softness of the opponents, the initial propensities of the real subjects, that are usually seen in this game, would have disappeared; the fairness oriented ethical preferences would have been temporarily "forgotten" by both proposers and responders and their egoistic preferences will be maximized. Without any interruption of the game, the dummy or computerized proposers and responders are eliminated and the subjects are matched against each other⁷. Our prediction is that if we let them play enough rounds, the behaviour oriented to get reasonable settlements for both parties will reappear generating a strong drift which will influence the learning process and push the system to the outcomes observed in the mentioned experiments.

7. Conclusions

Clearly, what a regular selection dynamic model, -payo° -monotonic, payo° -positive or weakly payo° -positive-, excludes is a type of agent whose strategy

adjustment process depends on non-payoffs variables, such as the proportion of agents in his player population playing the same strategy as his current one. We have shown that the inclusion of a choice behaviour sensitive to those proportions, under the form of drift, significantly changes the stability power of a payoff led dynamics.

In fact, what we have done is to complete a (biologically based) selection model by adding new type of agents whose behaviour could be influenced only by social norms and conventions. The results are shown to match those observed in the laboratory and mitigate somehow the Cheung and Friedman 's (1998) disappointing tests with the (unperturbed) RD⁸. We deduce that the failure of Binmore and Samuelson's (1995) drift model to match the observed data is due to their dependence on payoffs and to being inward pointing.

There are things to be done in future works. First, we should endogenize the playing modes, whether alert or absent, through a transition from one to the other which could be either continuous or not. In other words, we should relate the playing modes with experience. Second, we should carry out experiments to test the predictions mentioned in the last section and, in particular, to know in which sense the knowledge of how many subjects are playing like me influences my decisions.

8. Notes

1. We can compare the present model of drift with that of Binmore and Samuelson 's (1999). Suppose that the agent k_i is neither in the alert nor in the absent mode and that ρ_{k_i} is defined as in (2). Given d_{k_i} , the drift introduced by this agent depends on the value of ρ_{k_i} at $(\rho_{k_i}(f); f_{k_i})$; then, given f_{k_i} and d_{k_i}

$$\frac{\partial \rho_{k_i}}{\partial \rho_{k_i}(f)} = \frac{-d_{k_i}(f_{k_i})}{(\rho_{k_i}(f) - d_{k_i}(f_{k_i}))^2} < 0$$

This means that if the expected payoff at stake increases, perception increases and, as a consequence, the drift introduced by agent k_i will be reduced. This is the shrinking property of the correlated similarity relation defined by ρ_{k_i} on F_{k_i} (see Uriarte (1999)): This property has some similarity with B & S's assumption of a decreasing and Lipschitz continuous drift function on expected payoff differences. This property would also be satisfied had we defined the functions ρ_{k_i} as

$$\rho_{k_i}(\alpha_k(f)) = \frac{\alpha_k(f)}{\alpha_k(f) - d_{k_i}(f_{k_i})}$$

where $\alpha_k(f) (> d_{k_i}(f_{k_i}))$ is the difference between the maximum and the minimum of the expected payoffs attached to player k 's strategies given the current strategy

frequencies in the opposing populations. Hence, perception increases with B & S's measure of potential cost of making a mistake, $\alpha_k(f)$: As noted above, the role of payoffs in the determination of the level of drift depends on the parameter r_{ki} . If $r_{ki} = 1$, $d_{ki} = \bar{d}$ and the derivative $\frac{\partial \mu_{ki}}{\partial \alpha_{ki}(f)}$ is almost zero; if $r_{ki} = 0$, $d_{ki} = \underline{d}$ and μ_{ki} will be defined only when $\alpha_{ki}(t) = 1$ and so we cannot take derivatives. Hence, As $r_{ki} \neq 0$, d_{ki} becomes less sensitive to α_{ki} and so μ_{ki} will be more sensitive to expected payoffs. Thus, we would approach a model of drift which would be more sensitive to expected payoffs $\alpha_{ki}(f)$ (or to potential costs $\alpha_k(f)$) and less sensitive to the strategy proportion f_{ki} . In other words, we would approach the model of drift proposed by B & S.

The μ_{ki} of the present paper can be said to be "agent ki 's adaptive mistake probability": Mistake probabilities are fixed in the B & S model and agents may avoid the error by increasing their cognitive efforts when the potential cost of making the mistake increases. Instead, the approach taken here seems to be more natural, as agents can learn from their "mistakes", by adjusting them. Endogenous μ_{ki} 's means that drift is not inward-pointing.

2. To run the simulations, we shall use the subclass of threshold functions in the set D ; $d_{ki}(f_{ki}) = (1 - f_{ki})^{n_{ki}}$; with $n_{ki} \in (0; 1)$ and $i \in S_k$. This subclass is large enough for the purpose of computations. When $n_{ki} \rightarrow 0$, the degree of sharpness diminishes. For the simulations, we consider that the agent ki is playing strategy i in the alert mode when $d_{ki}(f_{ki}) = (1 - f_{ki})^{500}$; he would be in the absent mode when $0 < n_{ki} < 1$; say $d_{ki}(f_{ki}) = (1 - f_{ki})^{10^{i-8}}$. As in Binmore et al. (1995), we approach equation (6) by means of the equation $f_{ki}(t + \zeta) - f_{ki}(t) = \zeta f_{ki}(t)(\alpha_{ki}(f)(t) - \mu_{ki}(f)(t)) + \zeta(\mu_{ki}(f)(t) - f_{ki}(t))$, where the step size $\zeta = 0.01$. We shall consider, like Binmore et al.(1995), that the system has converged on a point when the first 15 decimals are unchanging.

3. The experiment consisted of dividing an amount of money and the interpretation of the Ultimatum Game is that Player I is proposing to Player II what he is demanding for himself; the second player's strategies are maximal acceptable demands.

4. Of course, we do not think that the issue is reduced to a mere quantitative matching of the theoretical results with the laboratory data. What is relevant here is the motivation of the perturbations that push the system to converge to the observed equilibrium in a given country as being something closely related to, say, some cultural characteristic that distinguishes the country that is being examined.

5. The model developed in Binmore et al.(1995) requires a mistake probability of 0.95 attached to the equilibrium demand reached in each country and the remaining probabilities being equal to one another. Under this specification of drift and for some values of their α and μ parameters, only starting from those observed initial propensities or from a neighborhood of them, the model may match the observed equilibria. The

remaining problems are the motivations for this specification of drift that allows the quantitative matching and the point made in the above note 3. A more basic issue is that in the Binmore et al. (1995) model of drift agents' mistakes depend on their capability to compute the potential cost - measured in expected payoffs - of making a mistake.

6. There are other combinations leading to the same result, but as we put in the alert mode more and more strategies, in particular, those in the vicinity of 5 and 6, we may simulate the path to the observed equilibria, but their basin will shrink.

7. After each round, every subject should be given information about the proportion f_{ki} of people in his population who have used the same strategy as his current one.

8. In the Matching Pennies Game, -just to mention the behaviour of the model with respect to an interior Nash equilibrium-, when all the π_j agents in both player populations are in the absent mode, the perturbed system (6) converges to the Nash equilibrium ($\pi_1=2; \pi_2=2$):

9. Appendix I:

9.1. Similarity relations.

Given a pair of vectors, $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki})$ and $(v_{ki}(f); f_{ki})$; in $\mathbb{I}_{ki} \in F_{ki}$; with $\bar{f}_{ki}, f_{ki} \in (0; 1)$; we define similarity relations on \mathbb{I}_{ki} and F_{ki} as follows.

(i) On the space \mathbb{I}_{ki} we define correlated similarity relations of the difference type: given $\bar{v}_{ki}(\bar{f}); \bar{f}_{ki}$ is said to be similar to $v_{ki}(f); f_{ki}$; (formally written as $\bar{v}_{ki}(\bar{f}) S_{\mathbb{I}_{ki}}(\bar{f}_{ki}) v_{ki}(f)$), if and only if $|\bar{v}_{ki}(\bar{f}) - v_{ki}(f)| \leq d_{ki}(\bar{f}_{ki})$, where $|j|$ stands for absolute value. Note that $d_{ki}(\bar{f}_{ki})$; the uncertainty or ambiguity level felt by agent ki , becomes the threshold level in the definition of this similarity relation.

(ii) on F_{ki} we define (Rubinstein-like) ratio-type similarity relations: \bar{f}_{ki} is similar to f_{ki} ; (formally, $\bar{f}_{ki} S_{F_{ki}} f_{ki}$), if and only if $1 - \epsilon_{ki} \leq \bar{f}_{ki} = f_{ki} \leq 1 + \epsilon_{ki}$ (where ϵ_{ki} is defined as in (3)).

9.2. The \mathbb{P}_{ki} Procedural Preference on $\mathbb{I}_{ki} \in F_{ki}$

We shall assume that each agent ki compares pairs of alternatives in $\mathbb{I}_{ki} \in F_{ki}$ with the aid of the above pair of similarities, $S_{\mathbb{I}_{ki}}(\bar{f}_{ki})$ and $S_{F_{ki}}$; and may know which of the two is preferred. Thus, he may define his procedural preference $\%_{ki}$ on $\mathbb{I}_{ki} \in F_{ki}$ and know his aspiration set at each t : That is, given a pair of vectors $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki})$ and $(v_{ki}(f); f_{ki})$ in $\mathbb{I}_{ki} \in F_{ki}$, the vector $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki})$ will be declared to be preferred to $(v_{ki}(f); f_{ki})$, i.e. $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki}) \hat{A}_{ki} (v_{ki}(f); f_{ki})$; whenever agent ki perceives that one of the following three conditions is met.

Condition \textcircled{R} : $\bar{v}_{ki}(\bar{f}) > v_{ki}(f)$, and no $\bar{v}_{ki}(\bar{f}) S_{\mathbb{I}_{ki}}(\bar{f}_{ki}) v_{ki}(f); \bar{f}_{ki} S_{F_{ki}} f_{ki}$:

In words, $\bar{v}_{ki}(\bar{f})$ is bigger than $v_{ki}(f)$ and, given \bar{f}_{ki} , $\bar{v}_{ki}(\bar{f})$ is perceived to be not similar to $v_{ki}(f)$; while, \bar{f}_{ki} is perceived to be similar to f_{ki} .

Condition $\bar{\cdot}$: $\bar{f}_{ki} > f_{ki}$ and no $\bar{f}_{ki} S_{ki} f_{ki}$; $\bar{v}_{ki}(\bar{f}) S_{ki}(\bar{f}_{ki}) v_{ki}(f)$:

In words, \bar{f}_{ki} is bigger than f_{ki} and \bar{f}_{ki} is perceived to be not similar to f_{ki} ; and, given \bar{f}_{ki} , $\bar{v}_{ki}(\bar{f})$ is perceived to be similar to $v_{ki}(f)$:

Condition \pm : $\bar{v}_{ki}(\bar{f}) > v_{ki}(f)$ and no $\bar{v}_{ki}(\bar{f}) S_{ki}(\bar{f}_{ki}) v_{ki}(f)$; $\bar{f}_{ki} > f_{ki}$ and no $\bar{f}_{ki} S_{ki} f_{ki}$: Vector $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki})$ is strictly bigger than $(v_{ki}(f); f_{ki})$ and no similarity is perceived in both instances.

Whenever both expected payoffs and strategy proportions are perceived to be similar, then the two vectors will be declared indifferent; i.e. when $v_{ki}(f) S_{ki}(\bar{f}_{ki}) \bar{v}_{ki}(\bar{f})$; $\bar{v}_{ki}(\bar{f}) S_{ki}(f_{ki}) v_{ki}(f)$; and $\bar{f}_{ki} S_{ki} f_{ki}$, then $(\bar{v}_{ki}(\bar{f}); \bar{f}_{ki}) \gg_{ki} (v_{ki}(f); f_{ki})$. When none of these four situations takes place, then the two vectors would be non-comparable.

Appendix II:

Proof of Proposition 1. (a) Let $S_k = \{f_1, f_2\}$ be player population k 's strategy set. Without loss of generality, let us refer to the dynamics of strategy 1: Then, by equation (5), we have

$$\begin{aligned}
 \bar{f}_{k1} &= f_{k1} \frac{\bar{A}_k(f) - A_{k1}(f)}{A_k(f)} \\
 &= f_{k1} \frac{\frac{v_{k1}(f)}{v_{k1}(f) - 2_k} f_{k1} + \frac{v_{k2}(f)}{v_{k2}(f) - 2_k} f_{k2} - \frac{v_{k1}(f)}{v_{k1}(f) - 2_k}}{\frac{v_{k1}(f)}{v_{k1}(f) - 2_k} + \frac{v_{k2}(f)}{v_{k2}(f) - 2_k}} \\
 &= f_{k1} (1 - f_{k1}) \frac{v_{k2}(f)(v_{k1}(f) - 2_k) - v_{k1}(f)(v_{k2}(f) - 2_k)}{v_{k1}(f)(v_{k2}(f) - 2_k) + v_{k2}(f)(v_{k1}(f) - 2_k)} \\
 &= \frac{f_{k1}(1 - f_{k1}) 2_k}{v_{k1}(f)(v_{k2}(f) - 2_k) + v_{k2}(f)(v_{k1}(f) - 2_k)} (v_{k1}(f) - v_{k2}(f)) \\
 &= \frac{2_k}{v_{k1}(f)(v_{k2}(f) - 2_k) + v_{k2}(f)(v_{k1}(f) - 2_k)} f_{k1} (v_{k1}(f) - \bar{v}_k(f))
 \end{aligned}$$

Hence,

$$\dot{f}_{ki} = \frac{z_k}{D(f)} f_{ki} [\mathcal{V}_{ki}(f) - \mathcal{T}_k(f)] \quad (10)$$

Since $\mathcal{V}_{ki}(f) > z_k > 0$; $i = 1; 2$; then, $D(f) = 2\mathcal{V}_{k1}(f)\mathcal{V}_{k2}(f) - z_k > 0$ and the dynamics is well defined. By equation (10), growth rates $\frac{\dot{f}_{ki}}{f_{ki}}$ equal payoff differences $[\mathcal{V}_{ki}(f) - \mathcal{T}_k(f)]$ multiplied by a (Lipschitz) continuous, positive function $\frac{z_k}{D(f)}$: This concludes the proof of (a).

(b) Consider now the case of three strategies $m_k = 3$. Without loss of generality, let the strategy $i \in S_k = \{1; 2; 3\}$ be 3. Then, rearranging (6) yields the following:

$$\begin{aligned} \dot{f}_{k3} &= f_{k3} \frac{\dot{\bar{A}}_k(f) - \dot{A}_{k3}(f)}{\bar{A}_k} \\ &= \frac{f_{k3}}{D(f)} [z_k^2 (f_{k1}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k1}(f)) + f_{k2}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k2}(f))) + z_k (f_{k1}(\mathcal{V}_{k3}(f) \\ &\quad - \mathcal{V}_{k1}(f))\mathcal{V}_{k2}(f) + f_{k2}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k2}(f))\mathcal{V}_{k1}(f))]; \end{aligned}$$

Thus,

$$\dot{f}_{k3} = \frac{z_k}{D(f)} f_{k3} [f_{k1}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k1}(f))(\mathcal{V}_{k2}(f) - z_k) + f_{k2}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k2}(f))(\mathcal{V}_{k1}(f) - z_k)] \quad (11)$$

where $D(f) = \mathcal{V}_{k1}(f)(\mathcal{V}_{k2}(f) - z_k)(\mathcal{V}_{k3}(f) - z_k) + \mathcal{V}_{k2}(f)(\mathcal{V}_{k1}(f) - z_k)(\mathcal{V}_{k3}(f) - z_k) + \mathcal{V}_{k3}(f)(\mathcal{V}_{k1}(f) - z_k)(\mathcal{V}_{k2}(f) - z_k) =$

$$3\mathcal{V}_{k1}(f)\mathcal{V}_{k2}(f)\mathcal{V}_{k3}(f) + z_k^2 [\mathcal{V}_{k1}(f)\mathcal{V}_{k2}(f) + \mathcal{V}_{k1}(f)\mathcal{V}_{k3}(f) + \mathcal{V}_{k2}(f)\mathcal{V}_{k3}(f)];$$

As before,

$D(f) > 0$; because to build function \bar{A}_{ki} ; we assumed $\mathcal{V}_{ki}(f) > z_k$: Thus, $\frac{z_k f_{k3}}{D(f)} > 0$ in (11). Since $\mathcal{V}_{k3}(f) - \mathcal{T}_k(f) = f_{k1}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k1}(f)) + f_{k2}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k2}(f))$; it is easy to see that, when $\mathcal{V}_{k3}(f) - \mathcal{T}_k(f) > 0$; then

$$f_{k1}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k1}(f))(\mathcal{V}_{k2}(f) - z_k) + f_{k2}(\mathcal{V}_{k3}(f) - \mathcal{V}_{k2}(f))(\mathcal{V}_{k1}(f) - z_k) > 0$$

and therefore $\dot{f}_{k3} > 0$: In other words, when $f_{ki} > 0$ in the RD then, $\dot{f}_{ki} > 0$ in equation (11).

When $\mathcal{V}_{k3}(f) - \mathcal{T}_k(f) < 0$ and so $\dot{f}_{ki} < 0$ in the RD, it can be seen that there are cases in which the negative sign is not preserved by (10). Similarly, when $\mathcal{V}_{k3}(f) - \mathcal{T}_k(f) = 0$, equation (11) is not always 0.

(c) Consider now the general case with $i \in S_k = \{1; 2; \dots; n_k\}$. Then, rearranging (6) yields the following:

$$\begin{aligned} \dot{f}_{ki} &= f_{ki} \frac{\overline{A}_k(f) - A_{ki}(f)}{A_k(f)} \\ &= \frac{1}{D(f)} f_{ki} \left[\sum_{j=1}^{k-1} f_{kj} (\mu_{ki}(f) - \mu_{kj}(f)) + \sum_{h \in j} (\mu_{kh}(f) - \mu_k) \right] \end{aligned}$$

where $D(f) = \sum_{j=1}^{k-1} \sum_{h \in j} (\mu_{kh}(f) - \mu_k) > 0$ if $\mu_{kh}(f) > \mu_k$; for all $h \in S_k$: As before, since $\mu_{k3}(f) - \mu_k(f) = f_{k1}(\mu_{k3}(f) - \mu_{k1}(f)) + f_{k2}(\mu_{k3}(f) - \mu_{k2}(f))$; it is easy to see that, when $\mu_{ki}(f) - \mu_k(f) > 0$, then, $\sum_{j=1}^{k-1} f_{kj} (\mu_{ki}(f) - \mu_{kj}(f)) + \sum_{h \in j} (\mu_{kh}(f) - \mu_k) > 0$ and therefore $\dot{f}_{ki} > 0$.

Proof of Proposition 2: Let i^* be a strategy in the equilibrium profile $f^* \in \Phi$: Let $i^* \in S_k = \{1; 2; \dots; m_k\}$, $k \in K = \{1; 2; \dots; n\}$: Since we are assuming that f^* is a pure Nash equilibrium played in the alert mode then, (6) could be written as:

$$\begin{aligned} \dot{f}_{ki^*} &= f_{ki^*} (\mu_{ki^*}(f) - \mu_k(f)) + \frac{1}{m_k - 1} \sum_{j \in i^*}^{k-1} f_{kj} (\mu_{ki^*}(f) - \mu_{kj}(f)) \\ &= \sum_{j \in i^*}^{k-1} f_{kj} (f_{ki^*} (\mu_{ki^*}(f) - \mu_{kj}(f)) + \frac{1}{m_k - 1} (\mu_{kj}(f) - \mu_{ki^*}(f))) \end{aligned} \quad (12)$$

We must show that $\dot{f}_{ki^*} > 0$ for any f in some neighborhood of f^* : Let $\bar{\epsilon}$ be an open set assumed to contain only the vertex f^* of Φ . Note first that, for all $f_{ki^*} \in (0; 1)$ and all $f_{kj} \in (0; 1)$, the drift term affecting equation \dot{f}_{ki^*} is always positive and greater than any of the drift terms affecting the rest of the equations \dot{f}_{kj} , $j \in i^*$; in population k . That is, $[\mu_{ki^*}(f) - \mu_{ki^*}] > 0$; $[\mu_{kj}(f) - \mu_{kj}] \leq 0$ and

$$\frac{1}{m_k - 1} \sum_{j \in i^*}^{k-1} f_{kj} (\mu_{kj}(f) - \mu_{ki^*}(f)) > \frac{1}{m_k - 1} \left(\sum_{l \in j; l \neq i^*}^{k-1} f_{kl} (\mu_{kl}(f) - \mu_{ki^*}(f)) + f_{kj} (\mu_{kj}(f) - \mu_{ki^*}(f)) \right)$$

This inequality holds for all i^a in f^a and $k \in K$: This is so because $\bar{d}(f_{ki^a})$ is nearly 0 and $\underline{d}(f_{kj})$ is nearly 1, for all f_{ki^a} and f_{kj} in $(0; 1)$; respectively.

Since f^a is pure but not a strict Nash equilibrium, player k may have an alternative pure best reply say, $l \in S_k$: Then, for any f in the edge of Φ joining the vertices $f^a = (i^a; f_{i^a k}^a)$ and $f^l = (l; f_{i^a k}^a)$; payoffs to k are constant: $\mathcal{V}_{ki^a}(f) = \mathcal{V}_{kl}(f) = \mathcal{V}_k(f)$: Hence, for any payoff-positive g ; $g_{ki^a}(f) = g_{kl}(f) = 0$. But then, the above inequality in the drift terms generate, inside population k ; a movement

toward vertex f^a : $\dot{f}_{ki^a} = \frac{1}{m_{ki^a} - 1} \sum_{j \in i^a}^{m_k} f_{kj} \underline{d}(f_{kj}) - f_{ki^a} \bar{d}(f_{ki^a}) > 0$:

Suppose now that $\mathcal{V}_{ki^a}(f) - \mathcal{V}_k(f) < 0$; f not in that edge. We show now that in (12) the expression inside brackets is positive. Since f^a is a Nash equilibrium, \mathcal{V}_k is continuous and S_k is finite, we may choose a neighborhood $-\epsilon - \epsilon \setminus \Phi$ of f^a so that, for all $f \in -\epsilon$, $k \in K$, $j \in i^a$, $j \in S_k$; $\frac{1}{m_{ki^a} - 1} > \max_j |\mathcal{V}_{ki^a}(f) - \mathcal{V}_k(f)|$, (where $|j|$ stands for absolute value). By assumption, $\underline{d}(f_{kj})$ is close to 1 for all $f_{kj} \in (0; 1)$ and therefore, on $-\epsilon$; $\underline{d}(f_{kj}) > f_{ki^a}$. Hence, on $-\epsilon$; for all $k \in K$, i^a in f^a and $f \in f^a \setminus -\epsilon$, we are sure that $f_{ki^a}(\mathcal{V}_{ki^a}(f) - \mathcal{V}_k(f)) + \frac{1}{m_{ki^a} - 1} \underline{d}(f_{kj}) > 0$. Then, since the negative term, $-f_{ki^a} \bar{d}(f_{ki^a})$; has a negligible absolute value

$$\sum_{j \in i^a}^{m_k} f_{kj} - f_{ki^a} (\mathcal{V}_{ki^a}(f) - \mathcal{V}_k(f)) + \frac{1}{m_{ki^a} - 1} \underline{d}(f_{kj}) > -f_{ki^a} \bar{d}(f_{ki^a})$$

Therefore, if the system of perturbed equations \dot{f}_{ki^a} starts in any $f^0 \in -\epsilon$; $f^0 \in f^a$; then, for all $k \in K$ and i^a in f^a ;

$$\dot{f}_{ki^a} = \sum_{j \in i^a}^{m_k} f_{kj} - f_{ki^a} (\mathcal{V}_{ki^a}(f) - \mathcal{V}_k(f)) + \frac{1}{m_{ki^a} - 1} \underline{d}(f_{kj}) - f_{ki^a} \bar{d}(f_{ki^a}) > 0$$

Hence; $\lim_{t \rightarrow \infty} f^0(t) = f^a$. Once f^a is reached then, for each $k \in K$ and each i^a in f^a ; $\dot{f}_{ki^a} = 0$; because $f_{ki^a} = 1$ and so $\bar{d}(f_{ki^a}) = 0$. Thus, we can conclude that the Nash equilibrium f^a is asymptotically stable. \forall

Proof of Proposition 3. Let i^a be a strategy in $f^a \in \Phi$: Let $i^a; j \in S_k$, $i^a \in j$, $k \in K = \{1; 2; \dots; n\}$: Write equation (7) for i^a and j :

$$\dot{f}_{ki^a} = f_{ki^a} g_{ki^a}(f) + [\mu_{ki^a}(f) - f_{ki^a}] = f_{ki^a} (g_{ki^a}(f) - \bar{d}(f_{ki^a})) + \frac{1}{m_{ki^a} - 1} \sum_{j \in i^a}^{m_k} f_{kj} \underline{d}(f_{kj})$$

$$\dot{f}_{kj} = f_{kj} g_{kj}(f) + [\mu_{kj}(f) - f_{kj}] = f_{kj} (g_{kj}(f) - \underline{d}(f_{kj})) + \frac{1}{m_{ki^a} - 1} \left(\sum_{l \in j; i^a}^{m_k} f_{kl} \underline{d}(f_{kl}) + f_{ki^a} \bar{d}(f_{ki^a}) \right)$$

Let \bar{f} be an open set assumed to contain only the vertex f^* of Φ : As in Proposition 2, if $i \in S_k$ is an alternative pure best reply for player k ; then $g_{ki}(f) = g_{ki^*}(f)$ along the edge of Φ joining the two vertices. Hence, there is a drift, inside population k ; toward pure strategy i^* :

Now, let $\mu_{ki^*}(f) < \mu_{kj}(f)$; for f not in that edge: Then, by payoff monotonicity, $g_{ki^*}(f) < g_{kj}(f)$ and, since f^* is a Nash equilibrium (and therefore, a stationary state for a payoff monotonic g); $g_{ki^*}(f^*) = g_{kj}(f^*) = 0$. As f approaches f^* ; those payoff (and growth-rate) differences will decrease continuously.

Notice first that, in some neighborhood of f^* ; the monotonic relationship, $\mu_{ki^*}(f) < \mu_{kj}(f)$, $g_{ki^*}(f) < g_{kj}(f)$; breaks down. This is due to the drift term of f_{kj} that makes, necessarily, $(g_{kj}(f) - d(f_{kj})) < 0$, $f \notin f^*$ in that neighborhood, where $d(f_{kj})$ is nearly 1 for all $f_{kj} \in (0; 1)$; while $\bar{d}(f_{ki^*})$ is nearly 0 for all $f_{ki^*} \in (0; 1)$: The differences between $f_{kj}(g_{kj}(f) - d(f_{kj}))$ and $f_{ki^*}(g_{ki^*}(f) - \bar{d}(f_{ki^*}))$ tend to 0 as we approach f^* :

On the other hand, if we take into account what is left of the drift terms, we have

$$\frac{1}{m_k - 1} \sum_{j \in i^*} f_{kj} d(f_{kj}) > \frac{1}{m_k - 1} \left(\sum_{l \in j; i^*} f_{kl} d(f_{kl}) + f_{ki^*} \bar{d}(f_{ki^*}) \right) > 0$$

for all f_{ki^*} and f_{kj} in $(0; 1)$; $j = 1; \dots; m_k$: This is true because the number of $d(\cdot)$ functions that the expression on the left has is $(m_k - 1)$, while the one on the right has $(m_k - 2)$ plus the negligible term $f_{ki^*} \bar{d}(f_{ki^*})$; where f_{ki^*} is the proportion that is assumed to be increasing, as we approach f^* : In other words, there exists a neighborhood $\bar{f} \subset \bar{f}^*$ of f^* in which, for all $f \in \bar{f} \subset \bar{f}^* \setminus \Phi$; i^* in \bar{f}^* , $j \in i^* \in S_k$ and $k \in K$, unambiguously

$$f_{ki^*}^2 = f_{ki^*} g_{ki^*}(f) + [\mu_{ki^*}(f) - f_{ki^*}] > 0$$

Therefore we can say that f^* is asymptotically stable. \forall

Proof of Proposition 4:

Case I: rewriting the perturbed system (8)-(9), we get

$$\begin{aligned} \dot{x} &= (1 - x) [d(1 - x) + x(2 - 3y)] - x\bar{d}(x) \\ \dot{y} &= y[(1 - y)(1 - x) - d(y)] + (1 - y)\bar{d}(1 - y) \end{aligned}$$

Writing $\dot{x} = \dot{y} = 0$ in (8)-(9) yields (0;0), (1;0); (0;1) (since $d_Y(y) = 0$ when $y = 1$) and (1,1), as the possible stationary points of the system. In the interior of the state space, i.e. for values of $x \in (0;1)$ and $y \in (0;1)$, we can see in the above equations that (noting that $\bar{d}(\cdot)$ is almost 0 and $\underline{d}(\cdot)$ is almost 1), $\dot{x} > 0$ and $\dot{y} < 0$: When $x = 1$; the system (8)-(9) is reduced to

$$\dot{y} = y \underline{d}(y) + (1 - y) \bar{d}(1 - y)$$

and so $\dot{y} < 0$ for all $y \in (0;1)$: When $y = 1$; it can be seen that $\dot{x} > 0$ for all $x \in (0;1)$: When $x = 0$; $\dot{y} < 0$ for all $y \in (0;1)$ and when $y = 0$; $\dot{x} > 0$ for all $x \in (0;1)$: Therefore, (1;0) is a global asymptotic attractor.

Case II: writing $\dot{x} = \dot{y} = 0$ in the system (8)-(9) yields (0,0), (1,0), (0, 1), (1, 1) and (1, 1/2) as the possible stationary points. As in the previous case, $\dot{x} > 0$ in the interior of the state space. When $x = 1$; $\dot{y} = y \underline{d}_Y(y) + (1 - y) \underline{d}_N(1 - y)$ and since we have assumed that $d_Y(\cdot) = d_N(\cdot)$ then, $\dot{y} > 0$ if $y < 1/2$; thus, $\dot{y} = 0$ when $y = 1/2$: The rest of the behaviour in the boundary is the same as in Case I, therefore, (1; 1/2) is a global asymptotic attractor.

Case III. After the study of the previous cases, It can be easily verified that, given the assumed playing modes, $\dot{x} < 0$ and $\dot{y} > 0$ in the interior of the state space; in the boundary the behaviour is such that (0; 1) is a global asymptotic attractor. ¥

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Figure 1. The P_i procedural preference \succ_{ki} : Given the vector $(\frac{1}{4}_{ki}(f); f_{ki})$; its upper and lower contour sets, -obtained by means of a procedure specified by Conditions \textcircled{R} ; $\textcircled{+}$ and $\textcircled{\pm}$, described in Appendix 1-, are $U(\textcircled{R}) [U(\textcircled{+}) [U(\textcircled{\pm})$ and $L(\textcircled{R}) [L(\textcircled{+}) [L(\textcircled{\pm})$; respectively. The dark area is the indifference set $\succ_{ki} [(\frac{1}{4}_{ki}(f); f_{ki})]$:

Figure 5. The relationship established by Proposition 4 between the degree of sharpness r_Y and r_N of the threshold functions d_Y and d_N , respectively in the Ultimatum Minigame.