Electronic version of an article published as Journal of Algebra and Its Applications Vol. 16, No. 06, 1750119 (2017). https://doi.org/10.1142/S0219498817501195. © 2017 World Scientific Publishing Company. [https://www.worldscientific.com/worldscinet/jaa]

TRANSPORTING COHOMOLOGY IN LAZARD CORRESPONDENCE

OIHANA GARAIALDE OCAÑA AND JON GONZÁLEZ-SÁNCHEZ

Abstract. Lazard correspondence provides an isomorphism of categories between finitely generated nilpotent pro-p groups of nilpotency class smaller than p and finitely generated nilpotent $\mathbb{Z}_p$-Lie algebras of nilpotency class smaller than p. Denote by H^i_{Gr} and H^i_{Lie} the group cohomology functors and the Lie cohomology functors, respectively. The aim of this paper is to show that for $i = 0$, 1 and 2, and for a given category of modules the cohomology functors $H^i_{Gr} \circ \mathbf{exp}$ and H^i_{Lie} are naturally equivalent. A similar result is proved for $i = 3$ with the relative cohomology groups.

1. Introduction

Throughout the text let p denote a fixed prime and $\mathbb{Z}_p$ the p-adic integers. In his work of 1954 M. Lazard established an isomorphism of categories between finitely generated nilpotent pro-p groups of nilpotency class smaller than p and finitely generated nilpotent $\mathbb{Z}_p$-Lie algebras of nilpotency class smaller than p (see [11]). This well-known result of Lazard has had strong influence in the study of finite p-groups. For example, classifying "small" finite p-groups is equivalent to classifying "small" nilpotent $\mathbb{Z}_p$-Lie algebras which turned out to be fundamental in the classification of "small" finite p-groups (see [12]). This isomorphism of categories is given by the exponential and logarithm functors. In the context of complex representations, the orbit method of A. Kirillov can be applied to nilpotent pro-p groups of nilpotency class smaller than p. This method establishes a bijection between the complex irreducible representations of the group and the orbits of the coadjoint action of the group in the dual space of its Lie algebra (see [9] and [6]).

If G is a finite p-group and K is a field of characteristic p, there is only one irreducible representation over K, the trivial one. Therefore, irreducible representations over K do not give information about the group. However, the homology and cohomology groups provide more information about the group and about the modules. For example, if G is a finite p-group and $\mathbb{F}_p$ denotes the trivial module, $H^1(G, \mathbb{F}_p)$ gives the minimal number of generators of G and $H^2(G, \mathbb{F}_p)$ the minimal number of relations.

In a recent paper of B. Eick, M. Horn and S. Zandi it has been proved that if G is finite p-group of nilpotency class smaller than $p - 1$ and L is its Lie algebra then, the Schur multiplier of G and L are isomorphic (see [4]). Note that the

Date: July 6, 2014.

The authors are supported by the Spanish Government, grant MTM2011-28229-C02, and by the Basque Government, grant IT753-13. First author is supported by Basque Government grant PRE_2013_1_653. The second author is supported by the Ramón y Cajal program from the spanish government RYC-2011-08885.

Schur multipliers of G and L are given by the cohomology groups $H^2(G, C_{p^\infty})$ and $H^2(L, C_{p^\infty})$, respectively.

The aim of this text is to study cohomology groups of small dimensions in the context of the Lazard correspondence. In order to do it we will introduce certain categories of modules over groups $\mathbf{Tpl}_{Gr}^{c,d}$ (see Section 3) and certain categories of modules over Lie algebras $\mathbf{Tpl}_{Lie}^{c,d}$ (see Section 4). In these categories c denotes the nilpotency class of the group or of the Lie algebra while d denotes the length action of the group or of the Lie algebra over the corresponding module. In our first result we give an isomorphism of categories using truncated exponential and logarithm functors.

Theorem A. *Let c and d be smaller than p. Then,*

$$\mathbf{Exp} : \mathbf{Tpl}_{Lie}^{c,d} \to \mathbf{Tpl}_{Gr}^{c,d} \quad \textit{and} \quad \mathbf{Log} : \mathbf{Tpl}_{Gr}^{c,d} \to \mathbf{Tpl}_{Lie}^{c,d}$$

are isomorphisms of categories one inverse of the other.

We use this result to establish the main result of this manuscript.

Theorem B. *Denote by $H^i_{Lie} : \mathbf{Tpl}_{Lie}^{c,d} \longrightarrow \boldsymbol{Ab}$ and $H^i_{Gr} : \mathbf{Tpl}_{Gr}^{c,d} \longrightarrow \boldsymbol{Ab}$ the natural cohomology functors. Then*

(1) If $c, d < p$, then H^0_{Lie} and $H^0_{Gr} \circ \mathbf{Exp}$ are naturally equivalent.
(2) If $c < p$ and $d < p-1$, then H^1_{Lie} and $H^1_{Gr} \circ \mathbf{Exp}$ are naturally equivalent.
(3) If $c + d < p$, then H^2_{Lie} and $H^2_{Gr} \circ \mathbf{Exp}$ are naturally equivalent.

In degree three cohomology we give a similar result but for relative cohomology (see Proposition 15).

Theorem B.(3) provides a natural proof of the result of B. Eick, M. Horn and S. Zandi on the Schur multipliers of the groups and the Lie algebras.

Theorem C. *Let $\boldsymbol{Gr}_{p-2}$ be the category of a finite p-group of nilpotency class smaller than $p-1$ and $\boldsymbol{Lie}_{p-2}$ the category of finite and nilpotent $\mathbb{Z}_p$-Lie algebras of nilpotency class smaller than $p-1$. Denote by*

$$\begin{aligned} \mathcal{M}_{Gr} &: \boldsymbol{Gr}_{p-2} \longrightarrow \boldsymbol{Ab} \quad \textit{and} \\ \mathcal{M}_{Lie} &: \boldsymbol{Lie}_{p-2} \longrightarrow \boldsymbol{Ab} \end{aligned}$$

the group and Lie algebra Schur Multiplier functors respectively. Then $\mathcal{M}_{Gr} \circ \boldsymbol{exp}$ and $\mathcal{M}_{Lie}$ are naturally equivalent. In particular, for $L \in \boldsymbol{Lie}_{p-2}$ one has $\mathcal{M}_{Gr}(\boldsymbol{exp}(L)) \cong \mathcal{M}_{Lie}(L)$.

The paper is divided as follows. In the second section we state the Lazard corresponde for pro-p groups and Lie rings. In the third and fourth sections we introduce the categories of modules $\mathbf{Tpl}_{Gr}^{c,d}$ and $\mathbf{Tpl}_{Lie}^{c,d}$ and then we recall the definition of the cohomology groups for small dimensions. In the fifth section we define the **Exp** and **Log** functors between the categories of modules and we prove Theorem A. In the next section we prove Theorem B and finally, in the last section, we prove Theorem C.

2. Lazard correspondence for finite nilpotent pro-p groups and Lie rings

Let X be a set and $A(X)$ denote the $\mathbb{Z}_p$-algebra of non-commuting polynomials over X. In fact, $A(X)$ is the free associative $\mathbb{Z}_p$-algebra over X. Consider $A_{(p)}(X)$

the ideal generated by the monomials of degree p. Then $A(X)_p = A(X)/A_{(p)}(X)$ is the free associative $\mathbb{Z}_p$-algebra of nilpotency class $p-1$ over X. The algebra $A(X)_p$ with the Lie bracket given by $[a,b] = ab - ba$ with $a, b \in A(X)_p$ is a $\mathbb{Z}_p$-Lie algebra and the $\mathbb{Z}_p$-Lie subalgebra $L(X)_p$ generated by X is the free nilpotent $\mathbb{Z}_p$-Lie algebra of nilpotency class $p-1$. In $A(X)_p$ we can define the exponential and logarithm functions

$$\begin{aligned} exp(a) &= \sum_{k=0}^{p-1} \frac{1}{k!} a^k \quad \text{and} \\ log(1+a) &= \sum_{k=1}^{p-1} (-1)^{k+1} \frac{a^k}{k}, \end{aligned}$$

and the Baker-Campbell-Hausdorff formula

$$H(a,b) = log(exp(a) \cdot exp(b)).$$

By Lazard [11] we know that $L(X)_p$ together with the multiplication coming from the Baker-Campbell-Hausdorff formula is the free pro-p group of nilpotency class $p-1$ over X which will be denoted by $F(X)_p$. Furthermore, one can invert the Baker-Campbell-Hausdorff formula to get the Lie algebra structure from the group structure. Denote by

$$\begin{aligned} h_1(a,b) &= exp(log(a) + log(b)) \quad \text{and} \\ h_2(a,b) &= exp([log(a), log(b)]). \end{aligned}$$

Then $L(X)_p$ is isomorphic to $(F(X)_p, h_1, h_2)$ as $\mathbb{Z}_p$-Lie algebras. Moreover this isomorphism can be extended to finitely generated nilpotent $\mathbb{Z}_p$-Lie algebras of nilpotency class smaller than p and finitely generated nilpotent pro-p groups of nilpotency class smaller than p as it is stated in the next theorem.

Theorem 1 (Lazard Correspondence)**.** *Let Gr_{p-1} denote the category of finitely generated nilpotent pro-p groups of nilpotency class smaller than p and Lie_{p-1} the category of finitely generated nilpotent $\mathbb{Z}_p$-Lie algebras of nilpotency class less than p. Then there exist isomorphisms of categories one inverse of the other*

$$\begin{aligned} \boldsymbol{exp} &: Lie_{p-1} \longrightarrow Gr_{p-1} \\ \boldsymbol{log} &: Gr_{p-1} \longrightarrow Lie_{p-1}, \end{aligned}$$

such that for $G \in Gr_{p-1}$ and $L \in Lie_{p-1}$ the following statements hold:

(a) $\boldsymbol{exp}(L) = (L, H)$,
(b) $\boldsymbol{log}(G) = (G, h_1, h_2)$,
(c) *K is a subgroup of G if and only if $log(K)$ is a sub-Lie algebra of $log(G)$,*
(d) *K is a normal subgroup of G if and only if $log(K)$ is an ideal in $log(G)$,*
(e) *Nilpotency class of G = nilpotency class of $log(G)$,*
(f) $\mathrm{End}(G) = \mathrm{End}(log(G))$. *In particular,* $\mathrm{Aut}(G) = \mathrm{Aut}(log(G))$.

Proof. Compare [11] and [7] in Section 10. □

3. Group, modules and cohomology of low dimension

For a finite pro-p group G, we recall that a G-module M is a $\mathbb{Z}_p$-module together with a homomorphism $\phi : G \to \mathrm{Aut}(M)$. We define the category $\mathbf{Tpl}_{Gr}$ that takes as objects the triples (G, M, ϕ) where G is a finite pro-p group, M is a $\mathbb{Z}_p$-module and $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(M))$, that is, M is a $\mathbb{Z}_p[G]$-module. Given two objects (G_1, M_1, ϕ_1) and (G_2, M_2, ϕ_2) in $\mathbf{Tpl}_{Gr}$, a morphism from (G_1, M_1, ϕ_1) to (G_2, M_2, ϕ_2) is defined by a pair (α, β) where $\alpha \in \mathrm{Hom}(G_2, G_1)$, $\beta \in \mathrm{Hom}(M_1, M_2)$ and for all $m_1 \in M_1$ and $g_2 \in G_2$ the following holds:

$$\beta((\phi_1 \circ \alpha(g_2))(m_1)) = (\phi_2(g_2))(\beta(m_1)).$$

3.1. **Defining H^0 in the category $\mathbf{Tpl}_{Gr}$.** For a triple $(G, M, \phi) \in \mathbf{Tpl}_{Gr}$ the degree zero cohomology group is defined by

$$H^0(G, M, \phi) = H^0(G, M) = \{m \in M \mid m \cdot \phi(g) = m, \ \forall g \in G\}.$$

Furthermore, if one has (α, β) a morphism between two triples (G_1, M_1, ϕ_1) and (G_2, M_2, ϕ_2), then

$$\beta : H^0(G_1, M_1) \subseteq M_1 \longrightarrow H^0(G_2, M_2) \subseteq M_2$$

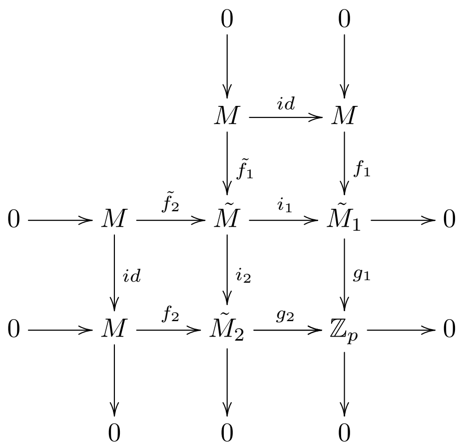
is a homomorphism of abelian groups. In fact, $H^0(\cdot)$ defines a covariant functor between the categories $\mathbf{Tpl}_{Gr}$ and the category of abelian groups $\mathbf{Ab}$.

3.2. **Defining H^1 in the category $\mathbf{Tpl}_{Gr}$.** For a triple $(G, M, \phi) \in \mathbf{Tpl}_{Gr}$, each equivalence class in $H^1(G, M, \phi)$ is in correspondence with the equivalent exact sequence of G-modules $0 \to M \to \tilde{M} \to \mathbb{Z}_p \to 0$. That is,

$$H^1(G, M, \phi) = \frac{\{\text{Equivalent extension of } \ \mathbb{Z}_p[G]\text{-modules}\}}{\{\text{Equivalent split extensions of } \ \mathbb{Z}_p[G]\text{-modules}\}}.$$

In order to give the additive structure over $H^1(G, M)$ one defines the sum of any such two extensions $0 \to M \to \tilde{M}_1 \to \mathbb{Z}_p \to 0$ and $0 \to M \to \tilde{M}_2 \to \mathbb{Z}_p \to 0$ as follows:

(1)



where $\tilde{M} \subseteq \tilde{M}_1 \oplus \tilde{M}_2$ is the pull-back of the arrows $\tilde{M}_1 \to \mathbb{Z}_p$ and $\tilde{M}_2 \to \mathbb{Z}_p$. Notice that $\tilde{f}_1$ and $\tilde{f}_2$ are defined by $\tilde{f}_1(m) = (f_1(m), 0)$ and $\tilde{f}_2(m) = (0, f_2(m))$. Then, the following exact sequence is the sum of the above extensions:

$$0 \to \frac{M \oplus M}{\Delta^-(M)} \to \tilde{M} \to \mathbb{Z}_p \to 0$$

where $\Delta^{-}(M)$ denotes the anti-diagonal map of M and $\frac{M\oplus M}{\Delta^{-}(M)}$ is isomorphic to M.

Furthermore, given (G_1, M_1, ϕ_1) and (G_2, M_2, ϕ_2) two objects in $\mathbf{Tpl}_{Gr}$ and a morphism $(\alpha, \beta) \in \mathrm{Mor}((G_1, M_1, \phi_1), (G_2, M_2, \phi_2))$, there is an induced homomorphism in cohomology.

Indeed, let $0 \to M_1 \to \tilde{M_1} \to \mathbb{Z}_p \to 0$ be an element in $H^1(G_1, M_1, \phi_1)$. On the one hand, it is easy to see that this short exact sequence of G_1-modules can be seen as a short exact sequence of G_2-modules by considering the action of G_2 over such modules given by the homomorphism $\phi_1 \circ \alpha$. On the other hand, given $\beta : M_1 \to M_2$ consider the following diagram constructed by taking the push-out of the arrows $M_1 \to \tilde{M_1}$ and $M_1 \to M_2$:

(2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & \tilde{M_1} & \xrightarrow{j_1} & \mathbb{Z}_p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\beta}} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M_2 & \longrightarrow & \tilde{M_2} & \xrightarrow{\theta} & \mathbb{Z}_p & \longrightarrow & 0. \end{array}$$

The homomorphism $\theta : M_2 \to \mathbb{Z}_p$ is given by $\theta(a, b) = j_1(a)$. In this way, the second row is an element in $H^1(G_2, M_2, \phi_2)$.

In fact one has that $H^1(\cdot)$ is a covariant functor between $\mathbf{Tpl}_{Gr}$ and $\mathbf{Ab}$.

3.3. **Defining H^2 in the category $\mathbf{Tpl}_{Gr}$.** Each equivalence class of $H^2(G, M)$ classifies the equivalent extensions (see [2, Chapt IV]) of G by M which consider the extensions of the form $1 \to M \to \tilde{G} \to G \to 1$, that is,

$$H^2(G, M) = \frac{\{\text{Equivalent extensions of G by M}\}}{\{\text{Equivalent split extensions of G by M}\}}.$$

By extensions of G by M, we mean the above extensions that give rise to the given action of G on M. The additive structure over $H^2(G, M)$ is given by the Baer sum [1]. Namely, for two extensions of G by M,

$$1 \longrightarrow M \xrightarrow{f_1} \tilde{G_1} \xrightarrow{g_1} G \longrightarrow 1$$

and

$$1 \longrightarrow M \xrightarrow{f_2} \tilde{G_2} \xrightarrow{g_2} G \longrightarrow 1$$

the Baer sum is defined as follows. Consider the next diagram

(3)
$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M & \xrightarrow{id} & M & & \\
 & & & & \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle f_1} & & \\
1 & \longrightarrow & M & \xrightarrow{\tilde{f}_2} & \tilde{H} & \xrightarrow{i_1} & \tilde{G}_1 & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle g_1} & & \\
1 & \longrightarrow & M & \xrightarrow{f_2} & \tilde{G}_2 & \xrightarrow{g_2} & G & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

where $\tilde{H} \subset \tilde{G}_1 \times \tilde{G}_2$ is the pull-back of the arrows $\tilde{G}_1 \to G$ and $\tilde{G}_2 \to G$. Now take

$$\tilde{G} = \frac{\tilde{H}}{\{(f_1(m), 1) - (1, f_2(m)) \mid m \in M\}}.$$

Then, the Baer sum is the following extension:

$$1 \longrightarrow M \xrightarrow{\tilde{f}} \tilde{G} \xrightarrow{\tilde{g}} G \longrightarrow 1$$

where $\tilde{f}(m) = (f_1(m), 1) = (1, f_2(m))$ and $\tilde{g}(a, b) = g_1(a) = g_2(b)$.

Consider (G_1, M_1, ϕ_1) and (G_2, M_2, ϕ_2) two objects in $\mathbf{Tpl}_{Gr}$ and $(\alpha, \beta) \in \mathrm{Mor}((G_1, M_1, \phi_1), (G_2, M_2, \phi_2))$. The induced homomorphism

$$H^2(G_1, M_1, \phi_1) \to H^2(G_2, M_2, \phi_2)$$

is defined as follows: take a class in $H^2(G_1, M_1, \phi_1)$, that is, an extension,

$$1 \to M_1 \to \tilde{G}_1 \to G_1 \to 1,$$

and consider the following diagram:

(4)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_1 & \xrightarrow{i_1} & \tilde{G}_1 & \xrightarrow{\pi_1} & G_1 & \longrightarrow & 1\\
 & & & \searrow{\scriptstyle \tilde{i}_1} & \uparrow{\scriptstyle \tilde{\alpha}} & & \uparrow{\scriptstyle \alpha} & & \\
 & & & & \tilde{H} & \xrightarrow{\tilde{\pi}_1} & G_2 & \longrightarrow & 1
\end{array}$$

where

$$\tilde{H} = \{(\tilde{g_1}, g_2) \in \tilde{G}_1 \times G_2 | \pi_1(\tilde{\alpha}(\tilde{g_1}, g_2)) = \alpha(\tilde{\pi}_1(\tilde{g_1}, g_2))\}$$

is the pull-back of the arrows $\tilde{G}_1 \to G_1$ and $G_2 \to G_1$ and $\tilde{i}_1 : M_1 \to \tilde{H}$ is defined by $\tilde{i}_1(m_1) = (i_1(m_1), 1)$. It can be shown that in fact, $1 \to M_1 \to \tilde{H} \to G_2 \to 1$ is an exact sequence.

Similarly, construct the following diagram using the homomorphism $\beta : M_1 \to M_2$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M_1 & \xrightarrow{\tilde{i_1}} & \tilde{H} & \xrightarrow{\tilde{\pi}_1} & G_2 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\beta}} & \nearrow{\scriptstyle \pi_2} & & & \\ 1 & \longrightarrow & M_2 & \xrightarrow{i_1} & \tilde{G_2} & & & & \end{array} \tag{5}$$

where

$$\tilde{G_2} = \frac{M_2 \oplus \tilde{H}}{\{(\beta(m_1), -\tilde{i}_1(m_1))|m_1 \in M_1\}}$$

is the push-out of the arrows $M_1 \to M_2$ and $M_1 \to \tilde{H}$ and $\pi_2((m_2, \tilde{h})) = \tilde{\pi}_1(\tilde{h})$. It follows that

$$1 \to M_2 \to \tilde{G_2} \to G_2 \to 1$$

is a short exact sequence. This construction defines a homomorphism between $H^2(G_1, M_1, \phi_1)$ and $H^2(G_2, M_2, \phi_2)$. In fact this defines a covariant functor $H^2(\cdot)$ between $\mathbf{Tpl}_{Gr}$ and $\mathbf{Ab}$.

3.4. **Defining H^3 in the category $\mathbf{Tpl}_{Gr}$.** It is well-known that each class in the cohomology group $H^3(G; M)$ is in correspondence with a short exact sequence of the following form [2],

$$0 \longrightarrow M \longrightarrow H \longrightarrow \tilde{H} \longrightarrow G \longrightarrow 1$$

where $0 \to M \to H_1 \to \tilde{H}_1 \to G \to 1$ and $0 \to M \to H_2 \to \tilde{H}_2 \to G \to 1$ are equivalent if there exist f_1 and f_2 that make the following diagram commute:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & H_1 & \longrightarrow & \tilde{H}_1 & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M & \longrightarrow & H_2 & \longrightarrow & \tilde{H}_2 & \longrightarrow & G & \longrightarrow & 1 \end{array} \tag{6}$$

Constructing this short exact sequence is equivalent to saying that there is a crossed module $f : H \to \tilde{H}$, that is, f is a homomorphism of groups together with an action of $\tilde{H}$ over H denoted by $\eta : \tilde{H} \to \mathrm{Aut}(H)$ such that for $h_2, h_3 \in H$ and $h_1 \in \tilde{H}$

(i) $f(h_2.h_1) = h_1.f(h_2).h_1^{-1} =^{h_1} f(h_2)$
(ii) $f(h_2).h_3 = h_2.h_3.h_2^{-1} =^{h_2} h_3$.

Notice that in such a short exact sequence we are only able to control two out of four terms, namely, the nilpotency class of G and the action length of M (see 3.5). The challenge is to keep the rest of the groups in our category of triples so that we can apply the correspondence of Lazard.

Our first approach, however, will be the following. Fix a surjective homomorphism of groups $\alpha : G_1 \to G_2$ and a G_2-module M. Then, we consider all the crossed modules $\mu : G \to G_1$ that have M as the kernel of μ and α as the cokernel. We say that two crossed moduless $\mu : G \to G_1$ and $\mu' : G' \to G_1$ are equivalent if there exists an isomorphism $f : G \to G'$ such that it is compatible with the actions of G_1 over G and G', $\mu' \circ f = \mu$ and $f_M = id_M$. That is, the following commutes:

(7)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & G & \xrightarrow{\mu} & G_1 & \longrightarrow & G_2 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M & \longrightarrow & G' & \xrightarrow{\mu'} & G_1 & \longrightarrow & G_2 & \longrightarrow & 1. \end{array}$$

Denote by $\mathrm{CMG}(G_2, G_1; M)$ the group of equivalence classes of all the crossed modules $\mu : G \to G_1$ with kernel M and cokernel $\alpha : G_1 \to G_2$. Then, there is a one to one correspondence between $\mathrm{CMG}(G_2, G_1, M)$ and the relative cohomology group $H^3(G_2, G_1; M)$. Observe that in this case, we control three out of four terms in the short exact sequence and thus, it will be easier to establish the necessary conditions to keep the reminding group in the category of modules so that we can apply the Lazard correspondence as mentioned before.

Remark 2. *The relative cohomology comes from the cochain complex $C^*(G_2, G_1; M)$ that fits in the following short exact sequence:*

$$0 \to C^*(G_2; M) \to C^*(G_1; M) \to C^*(G_2, G_1; M) \to 0.$$

In order to give group structure to these crossed modules one needs to define an addition of such short exact sequences. As in the previous section, this is given by the Baer sum. Given two short exact sequences

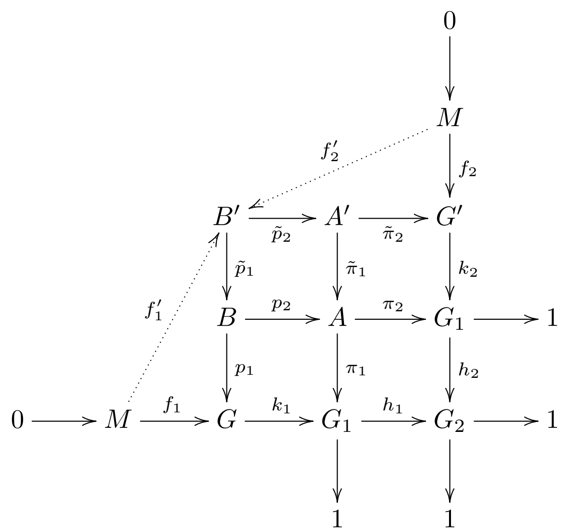
$$0 \to M \to G \to G_1 \to G_2 \to 1$$

and

$$0 \to M \to G' \to G_1 \to G_2 \to 1$$

the Baer sum is described as follows:

(8)



where A is the pull-back of h_1 and h_2; A' is the pull-back of π_2 and k_2; B is the pull-back of π_1 and k_1 and B' is the pull-back of $\tilde{\pi}_1$ and p_2 containing 6-tuples $(g, g_1, g_2, g_3, g_4, g') \in G \times G_1 \times G_1 \times G_1 \times G_1 \times G'$ such that $p_2(g, g_1, g_2) = (g_1, g_2) = (g_3, g_4) = \tilde{\pi}_1(g_3, g_4, g')$.

Take

$$\tilde{B} = \frac{B'}{\{(f_1(m),1,1,1,1,1)-(1,1,1,1,1,f_2(m))|m \in M\}}$$

and then the Baer sum is defined naturally by the next short exact sequence:

$$0 \longrightarrow M \xrightarrow{\tilde{f}} \tilde{B} \xrightarrow{\tilde{k}_1} G_1 \xrightarrow{\tilde{h}_1} G_2 \longrightarrow 0$$

where $\tilde{f}(m) = (f_1(m),1,1,1,1,1) = (1,1,1,1,1,f_2(m))$, $\tilde{k}_1(g_1,g_2,g_3,g_4,g_5,g_6) = g_1$ and $\tilde{h_1} = h_1$.

3.5. **The subcategory $\mathbf{Tpl}_{Gr}^{c,d}$.** Let (G,M,ϕ) be a triple in $\mathbf{Tpl}_{Gr}$. Let c denote the nilpotency class of G and let d denote the length of the action as follows:

Let $M_G = \frac{M}{[M,G]}$ where

$$[M,G] = \mathrm{Span}\{m - gm \mid g \in G, m \in M\}$$

and $[M, {}_iG] = [[M, {}_{i-1}G], G]$ for all $i \in \mathbb{N}$. Notice that

$$M \supseteq [M,G] \supseteq [M, {}_2G] \supseteq \cdots \supseteq [M, {}_iG] \supseteq \cdots$$

Then, the smallest $d \in \mathbb{N}$ for which $[M, {}_dG] = 1$ holds is called the length of the action of G on M.

We say that a triple (G,M,ϕ) is contained in the subcategory $\mathbf{Tpl}_{Gr}^{c,d}$ of $\mathbf{Tpl}_{Gr}$ if the nilpotency class of G is at most c and the length of the action of G on M is at most d.

4. Lie algebras, modules and cohomology of low dimension

Given a Lie ring L, we recall that an L-module M is a $\mathbb{Z}_p$-module together with a homomorphism of Lie algebras $\phi : L \to \mathrm{End}(M)$. We define the category $\mathbf{Tpl}_{Lie}$ which takes as objects the triples (L,M,ψ) where L is a Lie ring, M is a $\mathbb{Z}_p$-module and $\psi \in \mathrm{Hom}(L, \mathrm{End}(M))$. For any two objects (L_1,M_1,ψ_1) and (L_2,M_2,ψ_2) in $\mathbf{Tpl}_{Lie}$, a morphism from (L_1,M_1,ψ_1) to (L_2,M_2,ψ_2) is given by a pair (α,β) where $\alpha \in \mathrm{Hom}(L_2,L_1)$, $\beta \in \mathrm{Hom}(M_1,M_2)$ and for all $a_2 \in L_2$ and $m_1 \in M$, the following holds:

$$\beta((\psi_1 \circ \alpha(a_2))(m_1)) = (\psi_2(a_2))(\beta(m_1)).$$

4.1. **Definition of H^0 in the category $\mathbf{Tpl}_{Lie}$.** As in the previous section, given a triple (L,M,ψ) in $\mathrm{Obj}(\mathbf{Tpl}_{Lie})$, one can define the cohomology group $H^0(L,M,\psi)$ as the module of invariant elements of M under the L-action. That is,

$$H^0(L,M,\psi) = H^0(L,M) = M^L = \{m \in M \mid \psi(a) \cdot m = 0, \ \forall a \in L\}$$

Furthermore, if one has (α,β) a morphism between two triples (L_1,M_1,ψ_1) and (L_2,M_2,ψ_2), then

$$\beta : H^0(L_1,M_1) \subseteq M_1 \longrightarrow H^0(L_2,M_2) \subseteq M_2$$

is a homomorphism of abelian groups. In fact, $H^0(\cdot)$ defines a covariant functor between the category $\mathbf{Tpl}_{Lie}$ and the category of abelian groups $\mathbf{Ab}$.

4.2. **Defining H^1 in the category $\mathbf{Tpl}_{Lie}$.** Given a triple $(L, M, \psi) \in \mathbf{Tpl}_{Lie}$, each equivalence class in $H^1(L, M, \psi)$ is in correspondence with the equivalent exact sequence of L-modules $0 \to M \to \tilde{M} \to \mathbb{Z}_p \to 0$. That is,

$$H^1(L, M, \psi) = \frac{\{\text{Equivalent extensions of } \ L\text{-modules}\}}{\{\text{Equivalent split extensions of } \ L\text{-modules}\}}.$$

The additive structure of $H^1(L, M, \psi)$ is defined as in the diagram (1). In fact one has that $H^1(\cdot)$ is a covariant functor between $\mathbf{Tpl}_{Lie}$ and $\mathbf{Ab}$.

4.3. **Definition of H^2 in the category $\mathbf{Tpl}_{Lie}$.** Each equivalence class of $H^2(L, M)$ classifies the equivalent extensions of L by M by considering the extensions of the form $0 \to M \to \tilde{L} \to L \to 0$ [3].That is,

$$H^2(L, M) = \frac{\{\text{Equivalent extensions of L by M}\}}{\{\text{Equivalent split extensions of L by M}\}}.$$

As for the extensions of groups, the sum of extensions of Lie algebras is given by the Baer sum in diagram (3). In fact, $H^2(\cdot)$ is a covariant functor between $\mathbf{Tpl}_{Lie}$ and $\mathbf{Ab}$.

4.4. **Defining H^3 in the category $\mathbf{Tpl}_{Lie}$.** .

As in Section 3.4, each class in the cohomology group $H^3(L; M)$ is in correspondence with a short exact sequence of the following form

$$0 \to M \to \mathbf{g} \to \tilde{\mathbf{g}} \to L \to 0$$

under the equivalence class defined as in diagram (6). Constructing this short exact sequence is equivalent to saying that there is a crossed module $f : \mathbf{g} \to \tilde{\mathbf{g}}$, that is, f is a homomorphism of Lie algebras together with an action of $\tilde{\mathbf{g}}$ over $\mathbf{g}$ denoted by $\eta : \tilde{\mathbf{g}} \to \mathrm{Der}(\mathbf{g})$ such that for $g_1, g_2 \in \mathbf{g}$ and $\tilde{g} \in \tilde{\mathbf{g}}$

(i) $f(\eta(\tilde{g}).g_1) = [\tilde{g}, f(g_1)]$

(ii) $\eta(f(g_1)).g_2 = [g_1, g_2]$.

Notice that in such a short exact sequence we are only able to control two out of four terms, namely, the nilpotency class of L and the length action on M. The challenge is to keep the rest of the Lie algebras in our category of triples so that we can apply the correspondence of Lazard.

Our first approach, however, will be as in the case for the groups. Fix a surjective homomorphism of Lie algebras $\alpha : L_1 \to L_2$ and a L_2-module M. Then, we consider all the crossed modules $\mu : L \to L_1$ that have M as the kernel of μ and α as the cokernel. We say that two crossed homomorphisms $\mu : L \to L_1$ and $\mu' : L' \to L_1$ are equivalent if there exists an isomorphism $f : L \to L'$ such that it is compatible with the actions of L_1 over L and L', $\mu' \circ f = \mu$ and $f_L = id_L$. That is, the following diagram commutes:

(9)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & L & \xrightarrow{\mu} & L_1 & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & M & \longrightarrow & L' & \xrightarrow{\mu'} & L_1 & \longrightarrow & L_2 & \longrightarrow & 0.
\end{array}$$

Denote by $\mathrm{CML}(L_2, L_1; M)$ the group of the equivalence classes of all the crossed modules $\mu : L \to L_1$ with kernel M and cokernel $\alpha : L_1 \to L_2$. Then, there is a one to one correspondence between $\mathrm{CML}(L_2, L_1, M)$ and the relative cohomology group $H^3(L_2, L_1; M)$ as it is proven in the Appendix A of [10]. In this way, we control three terms out of four terms in the short exact sequence as before.

Remark 3. *As in the case of the cohomology of groups, the relative cohomology goup $H^*(L_2, L_1; M)$ comes from the cochain complex $C^*(L_2, L_1; M)$ that fits in the following exact sequence:*

$$0 \to C^*(L_2, M) \to C^*(L_1, M) \to C^*(L_2, L_1; M) \to 0.$$

The Baer sum of such two extensions is defined as in diagram (8).

4.5. **The subcategory $\mathbf{Tpl}^{c,d}_{Lie}$.** Let (L, M, ψ) be a triple in $\mathbf{Tpl}_{Lie}$. Let c denote the nilpotency class of L and let d denote the length action defined as follows:

Let $M_L = \frac{M}{[M,L]}$ where

$$[M, L] = \mathrm{Span}\{am \mid a \in L, m \in M\}$$

and $[M, {}_iL] = [[M, {}_{i-1}L], L]$. Notice that

$$M \supseteq [M, L] \supseteq [M, {}_2L] \supseteq \cdots \supseteq [M, {}_iL] \supseteq \cdots$$

Then, the smallest $d \in \mathbb{N}$ for which $[M, {}_dL] = 0$ is called the length action of L on M.

We say that a triple (L, M, ψ) is contained in the subcategory $\mathbf{Tpl}^{c,d}_{Lie}$ of $\mathbf{Tpl}_{Lie}$ if the nilpotency class of L is at most c and the length action of L on M is at most d.

5. Exp and Log for triples

We define the exponential and logarithm maps for the objects $(G, M, \phi) \in \mathbf{Tpl}^{c,d}_{Gr}$ and $(L, M, \psi) \in \mathbf{Tpl}^{c,d}_{Lie}$ for $c, d < p$ as follows:

$$\begin{aligned} \mathbf{Exp} :\ \mathbf{Tpl}^{c,d}_{Lie} &\longrightarrow \mathbf{Tpl}^{c,d}_{Gr} \\ (L, M, \psi) &\to (\mathbf{exp}(L), M, \exp(\psi)) \end{aligned}$$

and

$$\begin{aligned} \mathbf{Log} :\ \mathbf{Tpl}^{c,d}_{Gr} &\longrightarrow \mathbf{Tpl}^{c,d}_{Lie} \\ (G, M, \phi) &\to (\mathbf{log}(G), M, \log(\phi)) \end{aligned}$$

where $\exp(\psi) = \sum_{k=0}^{p-1} \frac{\psi^k}{k!}$ and $\log(\phi) = \sum_{k=1}^{p-1} (-1)^{k+1} \frac{(\phi - \mathrm{id})^k}{k}$.

Similarly, one can also define **Exp** and **Log** for morphisms $(\alpha, \gamma) \in \mathrm{Mor}(\mathbf{Tpl}^{c,d}_{Gr})$ and $(\beta, \gamma) \in \mathrm{Mor}(\mathbf{Tpl}^{c,d}_{Lie})$ by $\mathbf{Exp}(\beta, \gamma) = (\mathbf{exp}(\beta), \gamma)$ and $\mathbf{Log}(\alpha, \gamma) = (\mathbf{log}(\alpha), \gamma)$. In the following lemmas we will show that these maps are in fact well-defined.

Lemma 4. *Let $(G, M, \phi) \in \mathbf{Tpl}^{c,d}_{Gr}$ and $c, d < p$, then $(\mathbf{log}(G), M, log(\phi)) \in \mathbf{Tpl}^{c,d}_{Lie}$. Moreover, the following statements hold:*

1. *$(\phi - id)(g)^d = 0$ for all $g \in G$.*
2. *$[M, {}_i\boldsymbol{log}(G)] \subseteq [M, {}_iG]$ for all $i \geq 1$.*

Proof. We will start proving (1). Take $(G, M, \phi) \in \mathbf{Tpl}^{c,d}_{Gr}$. For all $g \in G$ and $m_1, m_2 \in M$, we have

$$\begin{aligned}(\phi(g) - \mathrm{id})(m_1 + m_2) &= \phi(g)(m_1 + m_2) - (m_1 + m_2)\\ &= \phi(g)(m_1) - m_1 + \phi(g)(m_2) - m_2\\ &= (\phi(g) - \mathrm{id})(m_1) + (\phi(g) - \mathrm{id})(m_2).\end{aligned}$$

Therefore $(\phi(g) - id) \in \mathrm{End}(M)$. Furthermore, for any G-invariant submodule U of M one has by definition $(\phi(g)-id)(U) \subseteq [U, G]$. In particular $(\phi(g)-id)^d(M) \subseteq [M,_d G] = 0$ and therefore $(\phi(g) - id)^d = 0$.

We will continue by proving that $(\mathbf{log}(G), M, \log(\phi)) \in \mathbf{Tpl}_{Lie}$. We would like to see that the map $\tilde{\phi} : G \to \mathrm{Aut}(M)$ given by $\tilde{\phi}(g) = \log(\phi(g))$ is also a homomorphism of Lie algebras so that we conclude that M is also a $\mathbf{log}(G)$-module. We claim that for $g_1, g_2 \in G$, we have $\tilde{\phi}(g_1 + g_2) = \tilde{\phi}(g_1) + \tilde{\phi}(g_2)$ and $\tilde{\phi}([g_1, g_2]) = [\tilde{\phi}(g_1), \tilde{\phi}(g_2)]$. Indeed

$$\begin{aligned}\tilde{\phi}(g_1 + g_2) = \log(\phi(g_1 + g_2)) &= \log(\phi(h_1(g_1, g_2))) = \log(h_1(\phi(g_1), \phi(g_2)))\\ &= \log(\exp(\log(\phi(g_1)) + \log(\phi(g_2)))) = \log(\phi(g_1)) + \log(\phi(g_2))\\ &= \tilde{\phi}(g_1) + \tilde{\phi}(g_2)\end{aligned}$$

and

$$\begin{aligned}\tilde{\phi}([g_1, g_2]) = \log(\phi([g_1, g_2])) &= \log(\phi(h_2(g_1, g_2))) = \log(h_2(\phi(g_1), \phi(g_2)))\\ &= \log(\exp([\log(\phi(g_1)), \log(\phi(g_2))])) = [\tilde{\phi}(g_1), \tilde{\phi}(g_2)]\end{aligned}$$

where $h_1(\cdot, \cdot)$ and $h_2(\cdot, \cdot)$ are the inverse Backer-Campbell- Hausdorff formulae. Therefore, $\tilde{\phi}$ is a homomorphism of Lie algebras and thus, M is a $\mathbf{log}(G)$-module and $(\mathbf{log}(G), M, \log(\phi)) \in \mathbf{Tpl}_{Lie}$.

For proving (2) notice that for a G-invariant submodule U of M we have that U is also a $\mathbf{log}(G)$-module. Furthermore, the action of $\mathbf{log}(G)$ in $U/[U, G]$ is trivial. Therefore $[U, \mathbf{log}(G)] \subseteq [U, G]$. An induction on i now shows that $[M,_i \mathbf{log}(G)] \subseteq [M,_i G]$.

In particular, we have $(\mathbf{log}(G), M, \log(\phi)) \in \mathbf{Tpl}^{c,d}_{Lie}$. □

Lemma 5. *Let $(L, M, \psi) \in \mathbf{Tpl}^{c,d}_{Lie}$ and $c, d < p$, then $(\mathbf{exp}(L), M, exp(\psi)) \in \mathbf{Tpl}^{c,d}_{Gr}$. Moreover, the following conditions hold:*

(1) $\psi(a)^d = 0$ for all $a \in L$.
(2) $[M,_i \boldsymbol{exp}(L)] \subseteq [M,_i L]$ for all $i \geq 1$.

Proof. We will start proving (1). Take $(L, M, \psi) \in \mathbf{Tpl}^{c,d}_{Lie}$. We have that $\psi(a) \in \mathrm{End}(M)$. Furthermore, for any L-invariant submodule U of M one has $\psi(a)(U) \subseteq [U, L]$. In particular $(\psi(a))^d(M) \subseteq [M,_d L] = 0$ and therefore $(\psi(a))^d = 0$.

We will continue by proving that $(\mathbf{exp}(L), M, \exp(\psi)) \in \mathbf{Tpl}_{Lie}$. We would like to see that the map $\tilde{\psi} : L \to \mathrm{End}(M)$ given by $\tilde{\psi}(a) = \exp(\psi(a))$ is also a homomorphism of Lie algebras so that we conclude that M is also a $\mathbf{exp}(L)$-module. We claim that for $a, b \in L$, $\tilde{\psi}(ab) = \tilde{\psi}(a)\tilde{\psi}(b)$ hold. Indeed

$$\tilde{\psi}(ab) = \exp(\psi(ab)) = \exp(\psi(H(a,b))) = \exp(H(\psi(a),\psi(b))) =$$
$$= \exp(\log(\exp(\psi(a))\exp(\psi(b)))) = \exp(\psi(a))\exp(\psi(b)) = \tilde{\psi}(a)\tilde{\psi}(b).$$

where $H(\cdot,\cdot)$ denotes the Backer-Campbell-Hausdorff formula. Therefore, $\tilde{\psi}$ is a homomorphism of groups and thus, M is a $\mathbf{exp}(L)$-module.

For proving (2) notice that for an L-invariant submodule U of M we have that U is also an $\mathbf{exp}(L)$-module. Furthermore, the action of $\mathbf{exp}(L)$ in $U/[U,L]$ is trivial. Therefore $[U,\mathbf{exp}(L)] \subseteq [U,L]$. An induction on i now shows that $[M,_i \mathbf{exp}(L)] \subseteq [M,_i L]$.

In particular, we have $(\mathbf{exp}(L), M, \exp(\psi)) \in \mathbf{Tpl}^{c,d}_{Gr}$. □

Theorem 6. *Let c and d be smaller than p. Then,*

$$\mathbf{Exp} : \mathbf{Tpl}^{c,d}_{Lie} \to \mathbf{Tpl}^{c,d}_{Gr} \quad \text{and} \quad \mathbf{Log} : \mathbf{Tpl}^{c,d}_{Gr} \to \mathbf{Tpl}^{c,d}_{Lie}$$

are well-defined functors. Moreover, these functors are isomorphisms of categories one inverse of the other.

Proof. By previous lemmas **Exp** and **Log** are well-defined functors. Moreover,

$$\begin{aligned}(\mathbf{Log} \circ \mathbf{Exp})(L,M,\psi) &= \mathbf{Log}(\mathbf{exp}(L), M, \exp(\psi)) \\ &= (\mathbf{log}(\mathbf{exp}(L)), M, \log(\exp(\psi))) \\ &= (L, M, \psi),\end{aligned}$$

and

$$\begin{aligned}(\mathbf{Exp} \circ \mathbf{Log})(G,M,\phi) &= \mathbf{Exp}(\mathbf{log}(G), M, \log(\phi)) \\ &= (\mathbf{exp}(\mathbf{log}(G)), M, \exp(\log(\phi))) \\ &= (G, M, \phi),\end{aligned}$$

which completes the proof of the theorem. □

Corollary 7. *Let $c, d < p$ and $(L, M, \psi) \in \mathbf{Tpl}^{c,d}_{Lie}$. Then, $[M,_i\mathbf{exp}(L)] = [M,_iL]$ for all $i \geq 1$.*

Proof. This is just a direct consequence of the Lemma 5 and Lemma 6. □

6. Transporting cohomology

In this section we would like to show the relation between the cohomology groups of a finite pro-p group over a module M and the cohomology groups of the corresponding Lie algebra over the same module M for dimensions 0, 1, 2 and 3. In fact, we will show that after taking the **Exp** and **Log** functors defined in the previous section such cohomology functors are naturally equivalent.

6.1. **H^0 and H_0 for triples.** We will show that both H^0 and H_0 are the same for a Lie algebra and its corresponding finite pro-p group.

Theorem 8. *Let $c, d < p$ and let*

$$H^0_{Lie} : \mathbf{Tpl}^{c,d}_{Lie} \longrightarrow \mathbf{Ab}$$

and

$$H^0_{Gr} : \mathbf{Tpl}^{c,d}_{Gr} \longrightarrow \mathbf{Ab},$$

be the cohomology functors. Then

$$H^0_{Lie} = H^0_{Gr} \circ \mathbf{Exp}.$$

Proof. Let (L, M, ψ) be in $\mathbf{Tpl}^{c,d}_{Lie}$ and put $(G, M, \phi) = \mathbf{Exp}(L, M, \psi) \in \mathbf{Tpl}^{c,d}_{Gr}$. Recall that

$$H^0_{Lie}(\mathbf{Log}(G, M, \phi)) = \{m \in M \mid log(\phi)(a) \cdot m = 0, \quad \text{for all} \quad a \in \mathbf{log}(G)\}.$$

Similarly,

$$H^0_{Gr}(\mathbf{Exp}(L, M, \psi)) = \{m \in M \mid exp(\phi)(g) \cdot m = m, \quad \text{for all} \quad g \in \mathbf{exp}(L)\}.$$

We want to see that $H^0_{Lie}(L, M, \psi) = H^0_{Gr}(G, M, \phi)$.

Subclaim 1 : $H^0_{Gr}(G, M, \phi) \subset H^0_{Lie}(\mathbf{Log}(G, M, \phi))$.

Subproof. Let $m \in H^0_{Gr}(G, M, \phi)$, that is, $\phi(g) \cdot m = m$ for all $g \in G$. Then,

$$\log(\phi)(g) \cdot m = \sum_{k=1}^{p-1} (1)^{k+1} \frac{(\phi - id)(g)^k}{k} \cdot m = 0$$

for all $g \in G$. Hence, $m \in H^0_{Lie}(\mathbf{Log}(G, M, \phi))$. □

Subclaim 2 : $H^0_{Lie}(L, M, \psi) \subset H^0_{Gr}(\mathbf{Exp}(L, M, \psi))$.

Subproof. Let $m \in H^0_{Lie}(L, M, \psi)$, that is, $\psi(a) \cdot m = 0$ for all $a \in L$. Then,

$$\exp(\psi)(a) \cdot m = \sum_{k=0}^{p-1} \frac{\psi(a)^k}{k!} \cdot m = m$$

for all $a \in L$. Hence, $m \in H^0_{Gr}(\mathbf{Exp}(L, M, \psi))$. □

Since **Exp** and **Log** are isomorphisms of categories, one inverse of the other, and by the first subclaim, we have

$$H^0_{Gr}(\mathbf{Exp}(L, M, \psi)) \subset H^0_{Lie}(\mathbf{Log}(\mathbf{Exp}(L, M, \psi))) = H^0_{Lie}(L, M, \psi).$$

Now the equality of the functors is clear by Subclaim 2. □

Theorem 9. *Let $c, d < p$ and let*

$$H_0^{Lie} : \mathbf{Tpl}^{c,d}_{Lie} \longrightarrow \mathbf{Ab}$$

and

$$H_0^{Gr} : \mathbf{Tpl}^{c,d}_{Gr} \longrightarrow \mathbf{Ab},$$

be the homology functors. Then

$$H_0^{Lie} = H_0^{Gr} \circ \mathbf{Exp}.$$

Proof. Let (L, M, ψ) be in $\mathbf{Tpl}_{Lie}^{c,d}$ and put $(G, M, \phi) = \mathbf{Exp}(L, M, \psi)$. Since $\mathbf{Exp}$ is an isomorphism of categories between $\mathbf{Tpl}_{Lie}^{c,d}$ and $\mathbf{Tpl}_{Gr}^{c,d}$, it is enough to prove that $H_0^{Lie}(L, M, \psi) = H_0^{Gr}(\mathbf{Exp}(L, M, \psi))$.

By definition $H_0^{Gr} = M_G$ and $H_0^{Lie} = M_L$ where $M_G = \frac{M}{[M,G]}$ and $M_L = \frac{M}{[M,L]}$. Then, as $[M, L] = [M, G]$ by Corollary 8, the equality $H_0^{Lie}(L, M, \psi) = H_0^{Gr}(\mathbf{Exp}(L, M, \psi))$ holds. □

6.2. **Exp and Log for exact sequences of modules and H^1.** We will show that the classes in H^1 remain unchanged after applying the **Exp** and **Log** functors.

Proposition 10. *Let $c, d < p$ and $(L, M_1, \psi_1), (L, M_2, \psi_2), (L, M_3, \psi_3) \in \mathbf{Tpl}_{Lie}^{c,d}$. Then,*

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

is an exact sequence of L-modules if and only if

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

is an exact sequence of $\mathbf{exp}(L)$*-modules.*

Proof. Notice that the action of L over the M_i modules commutes with α and β and thus, by theorem 7 the same will happen with the action of $\mathbf{exp}(L)$ over such modules. This fact proves the proposition. □

Theorem 11. *Let $c < p$, $d < p - 1$ and let*

$$H_{Lie}^1 : \mathbf{Tpl}_{Lie}^{c,d} \longrightarrow \mathbf{Ab}$$

and

$$H_{Gr}^1 : \mathbf{Tpl}_{Gr}^{c,d} \longrightarrow \mathbf{Ab},$$

be the cohomology functors. Then H_{Lie}^1 and $H_{Gr}^1 \circ \mathbf{Exp}$ are naturally equivalent and the natural transformation that provides the equivalence of functors is the one given by Proposition 11.

Proof. Take $(L, M, \psi) \in \mathbf{Tpl}_{Lie}^{c,d}$, and let $0 \to M \to \tilde{M} \to \mathbb{Z}_p \to 0$ be an element in $H_{Lie}^1(L, M, \psi)$. This means that $\frac{\tilde{M}}{M} \cong \mathbf{Z}_p$ and thus, $[\tilde{M}, L] \leq M$. Then, $[\tilde{M},_{d+1} L] = 0$. That is, $(L, \tilde{M}, \tilde{\psi}) \in \mathbf{Tpl}_{Lie}^{c,d+1}$ and Lazard correspondence holds as $d < p - 1$. We have a similar argument if we start with a triple in $\mathbf{Tpl}_{Gr}^{c,d}$.

Notice that the Baer sum of an extension of G-modules is the Baer sum of the corresponding L-module extensions which follows from the diagram (1). This shows that $H_{Lie}^1(L, M, \psi) \cong (H_{Gr}^1 \circ \mathbf{Exp})(L, M, \psi)$.

Finally, given a morphism $(\alpha, \beta) \in \mathrm{Mor}(\mathbf{Tpl}_{Lie}^{c,d})$, the morphism $H_{Lie}^1(\alpha, \beta)$ is transformed into the morphism $H_{Gr}^1 \circ \mathbf{Exp}(\alpha, \beta)$ by the diagram (2). □

6.3. **Exp and Log for group extensions and H^2.** Let $(G, M, \phi) \in \mathbf{Tpl}_{Gr}^{c,d}$ and consider

$$1 \to M \to \tilde{G} \to G \to 1.$$

Then the nilpotency class of $\tilde{G}$ is at most $c + d$. Similarly if $(L, M, \psi) \in \mathbf{Tpl}_{Lie}^{c,d}$ and

$$0 \to M \to \tilde{L} \to L \to 0$$

is an extension of Lie algebras, the nilpotency class of $\tilde{L}$ is at most $c + d$. The following result is now clear from Lazard correspondence.

Proposition 12. *Let $c + d < p$ and $(L, M, \psi) \in \mathbf{Tpl}_{Lie}^{c,d}$. Then,*

$$0 \to M \to \tilde{L} \to L \to 0$$

is an extension of the Lie algebra L if and only if

$$1 \to M \to \mathbf{exp}(\tilde{L}) \to \mathbf{exp}(L) \to 1$$

is an extension of the group $\mathbf{exp}(L)$.

Now, we are ready to show the next result.

Theorem 13. *Let $c + d < p$ and let*

$$H^2_{Lie} : \mathbf{Tpl}_{Lie}^{c,d} \longrightarrow \mathbf{Ab}$$

and

$$H^2_{Gr} : \mathbf{Tpl}_{Gr}^{c,d} \longrightarrow \mathbf{Ab},$$

be the cohomology functors. Then H^2_{Lie} and $H^2_{Gr} \circ \mathbf{Exp}$ are naturally equivalent and the natural transformation that provides the equivalence of functors is given by Proposition 13.

Proof. Notice that by Proposition 13, extensions and split extensions are preserved by the **exp-log** functors. By diagram (3), the Baer sum is also preserved. Finally one can see that by diagrams (4) and (5) the morphisms are transformed. □

6.4. **Exp and Log for group extensions and H^3.** Let (G, M, ϕ) and (G_2, M, ϕ_2) be in $\mathbf{Tpl}_{Gr}^{c,d}$ with nilpotency classes $\tilde{c} < c$ and $c_2 < c$ respectively and length action d. Consider $0 \to M \to G_1 \to G_2 \to G \to 1$. Then, the nilpotency class of G_1 is at most $c_2 + d$ and thus, $(G_1, M, \phi_1) \in \mathbf{Tpl}_{Gr}^{c_2+d,d}$.

Similarly, if $(L, M, \psi), (L_2, M, \psi_2) \in \mathbf{Tpl}_{Lie}^{c,d}$ with nilpotency classes $\tilde{c} < c$ and $c_2 < c$, respectively, length action d and if we consider $1 \to M \to L_1 \to L_2 \to L \to 1$, then $(L_1, M, \psi_1) \in \mathbf{Tpl}_{Lie}^{c_2+d,d}$.

Proposition 14. *Let $c + d < p$, $(G, M, \phi), (G_2, M, \phi_2) \in \mathbf{Tpl}_{Gr}^{c,d}$. Then,*

$$0 \to M \to G_1 \to G_2 \to G \to 1$$

is a crossed module for $\nu : G_2 \to G$ and a G-module M if and only if

$$0 \to M \to \boldsymbol{log}(G_1) \to \boldsymbol{log}(G_2) \to \boldsymbol{log}(G) \to 0$$

is a crossed module for $\boldsymbol{log}(\nu) : \boldsymbol{log}(G_2) \to \boldsymbol{log}(G)$ and the $\boldsymbol{log}(G)$-module M. In particular

$$H^3_{RGr}(G, G_2; M) \cong H^3_{RLie}(\boldsymbol{log}(G), \boldsymbol{log}(G_2); M).$$

Proof. Let

$$0 \longrightarrow M \xrightarrow{f} G_1 \xrightarrow{k} G_2 \xrightarrow{h} G \longrightarrow 1$$

be a crossed module with (G, M, ϕ) and (G_2, M, ψ) in $\mathbf{Tpl}^{c,d}_{Gr}$.

By definition, G_2 acts on G_1 and we can consider M to be contained in G_1. Then, the length action of G_1 over M is at most d as

$$[M, {}_iG_1] \leq [M, {}_iG_2] \text{ for all } i \geq 1.$$

It is enough to show that the nilpotency class of G_1 is at most p. Indeed, $\frac{G_1}{\mathrm{Ker}(k)} \simeq \mathrm{Im}(k) \subset G_2$ and thus $\gamma_{c+1}(G_1) \subset \mathrm{Ker}(k) = M$. Hence $\gamma_{c+d+1}(G_1) \leq [M, {}_dG_2] = 1$. By hypothesis, $c + d < p$ and thus, the Lazard correspondence holds, that is,

$$0 \to M \to \mathbf{log}(G_1) \to \mathbf{log}(G_2) \to \mathbf{log}(G) \to 0$$

is a crossed module for $\mathbf{log}(\nu) : \mathbf{log}(G_2) \to \mathbf{log}(G)$.

Similarly, one can give the same arguments starting with an extension of Lie algebras. The last statement of the proposition follows from diagram (8). □

7. Applications

Let G be a finite p-group and L be a Lie algebra. We have the following results based on Theorem 14.

Theorem 15. *Let $\boldsymbol{Gr}_{p-2}$ be the category of a finite p-group of nilpotency class smaller than $p-1$ and $\boldsymbol{Lie}_{p-2}$ the category of finite and nilpotent $\mathbb{Z}_p$-Lie algebras of nilpotency class smaller than $p-1$. Denote by*

$$\begin{aligned} \mathcal{M}_{Gr} &: \boldsymbol{Gr}_{p-2} \longrightarrow \boldsymbol{Ab} \quad \text{and} \\ \mathcal{M}_{Lie} &: \boldsymbol{Lie}_{p-2} \longrightarrow \boldsymbol{Ab} \end{aligned}$$

the group and Lie algebra Schur Multiplier functors, respectively. Then $\mathcal{M}_{Gr} \circ \boldsymbol{exp}$ and $\mathcal{M}_{Lie}$ are naturally equivalent. In particular, for $L \in \boldsymbol{Lie}_{p-2}$ one has $\mathcal{M}_{Gr}(\boldsymbol{exp}(L)) \cong \mathcal{M}_{Lie}(L)$.

Proof. Let G be a finite p-group of nilpotency class smaller than $p-1$ and write $L = \mathbf{log}(G)$. Observe that on the one hand $H^2_{Gr}(G, C_{p^\infty}) = \varinjlim H^2_{Gr}(G, C_{p^i})$ and $H^2_{Lie}(L, C_{p^\infty}) = \varinjlim H^2_{Lie}(L, C_{p^i})$ as $C_{p^\infty} = \varinjlim(C_p \hookrightarrow C_{p^2} \hookrightarrow C_{p^3} \ldots)$. On the other hand, by Theorem 14 there exists an isomorphism $\tau_i : H^2_{Gr}(G, C_{p^i}) \longrightarrow H^2_{Lie}(L, C_{p^i})$ which is given by the natural transformation between the functors H^2_{Gr} and H^2_{Lie} . The isomorphism $\tau_\infty = \varinjlim \tau_i : H^2(G, C_{p^\infty}) \longrightarrow H^2(L, C_{p^\infty})$ defines a natural equivalence between $\mathcal{M}_{Gr}$ and $\mathcal{M}_{Lie}$. □

The exact sequences of cohomology groups give another way of computing the cohomology of some groups. For example, the inflation-restriction-transgression exact sequence comes from studying spectral sequences. We want to show that this exact sequence commutes with the functors **exp** and **log**.

Consider $\mathbb{F}_p$ with trivial group action and denote the cohomology groups by $H^*_{Gr}(G)$ and $H^*_{Lie}(L)$ for short. Let G be a finite group with nilpotency class c and $N \trianglelefteq G$. We use the next characterization of $H^1_{Gr}(\cdot)$ in [5] to simplify the computations. Define $H^1_{Gr}(G) = \mathrm{Hom}(G, \mathbb{F}_p)$ and $H^1_{Gr}(N)^{\frac{G}{N}} = \mathrm{Hom}(\frac{N}{N^p[G,N]}, \mathbb{F}_p)$. Analogously, for a Lie algebra L and a Lie ideal I one has that $H^1_{Lie}(L) = \mathrm{Hom}(L, \mathbb{F}_p)$ and $H^1_{Lie}(I)^{\frac{L}{I}} = \mathrm{Hom}(\frac{I}{pI+[L,I]}, \mathbb{F}_p)$.

Lemma 16. *Let G be a finite p-group of nilpotency class smaller than p and N a normal subgroup of G. Let $L = \boldsymbol{log}(G)$ and $I = \boldsymbol{log}(N)$ and denote by $\mathbf{tr}_{Gr} : H^1_{Gr}(N)^{\frac{G}{N}} \to H^2_{Gr}(\frac{G}{N})$ and by $\mathbf{tr}_{Lie} : H^1_{Lie}(I)^{\frac{L}{I}} \to H^2_{Lie}(\frac{L}{I})$ the transgression maps. Then the following diagram is commutative:*

$$\begin{array}{ccc} H^1_{Gr}(N)^{\frac{G}{N}} & \xrightarrow{tr_{Gr}} & H^2_{Gr}(\frac{G}{N}) \\ \downarrow{\scriptstyle \tau^1_N} & & \downarrow{\scriptstyle \tau^2_{\frac{G}{N}}} \\ H^1_{Lie}(I)^{\frac{L}{I}} & \xrightarrow{tr_{Lie}} & H^2_{Lie}(\frac{L}{I}), \end{array} \tag{10}$$

where τ^1_N and $\tau^2_{\frac{G}{N}}$ are the isomorphism given by the natural equivalence of functors.

Proof. The transgression map tr_{Gr} is defined as follows: let $f \in H^1(N)^{\frac{G}{N}} = \mathrm{Hom}(\frac{N}{N^p[G,N]}, \mathbb{F}_p)$ and consider $M \leq G$ such that $\mathrm{Ker}(f) = \frac{M}{N^p[G,N]}$. Notice that $\frac{N}{M} \cong \mathbb{F}_p$ Then, the image of f is the next exact sequence in $H^2(\frac{G}{N})$:

$$1 \longrightarrow \frac{M}{N} \longrightarrow \frac{G}{M} \longrightarrow \frac{G}{N} \longrightarrow 1.$$

If $c < p-1$, it is clear that the **log** of this exact sequence

$$0 \to \frac{\mathbf{log}(N)}{\mathbf{log}(M)} \to \frac{\mathbf{log}(G)}{\mathbf{log}(M)} \to \frac{\mathbf{log}(G)}{\mathbf{log}(N)} \to 0$$

is the image of $f \in H^1(\mathbf{log}(N))^{\mathbf{log}(\frac{G}{N})}$ by the transgression map $\mathbf{tr}_{Lie}$. □

A consequence of this lemma is the next result.

Proposition 17. *Suppose that $c < p-1$. Then the following diagram commutes:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1_{Gr}(\frac{G}{N}) & \longrightarrow & H^1_{Gr}(G) & \longrightarrow & H^1_{Gr}(N)^{\frac{G}{N}} & \xrightarrow{tr_{Gr}} & H^2_{Gr}(\frac{G}{N}) & \longrightarrow & H^2_{Gr}(G) \\ & & \downarrow{\scriptstyle \tau^1_{\frac{G}{N}}} & & \downarrow{\scriptstyle \tau^1_G} & & \downarrow{\scriptstyle \tau^1_N} & & \downarrow{\scriptstyle \tau^2_{\frac{G}{N}}} & & \downarrow{\scriptstyle \tau^2_G} \\ 0 & \longrightarrow & H^1_{Lie}(\frac{L}{I}) & \longrightarrow & H^1_{Lie}(L) & \longrightarrow & H^1_{Lie}(I)^{\frac{L}{I}} & \xrightarrow{tr_{Lie}} & H^2_{Lie}(\frac{L}{I}) & \longrightarrow & H^2_{Lie}(L) \end{array}$$

where $L = \boldsymbol{log}(G)$ and $I = \boldsymbol{log}(N)$ and τ^i are the isomorphisms given by the natural equivalence of functors.

Proof. This is is a direct consequence of Theorem 12, Theorem 14 and Lemma 17. □

References

[1] Baer, R. *Erweiterungen von Gruppen und ihren Isomorphismen*, Math. Z.38 (1934), 375416.
[2] Brown, Kenneth S. *Cohomology of groups.* Springer-Verlag, N.Y. 1982.
[3] Chevalley, C. and Eilenberg, S. *Cohomology Theory of Lie Groups and Lie Algebras.* Transactions of the American Mathematical Society (Providence, R.I.: American Mathematical Society) 1948.
[4] Eick, B., Horn, M. and Zandi, S. *Schur multipliers and the Lazard correspondence.* Arch. Math. (Basel) **99** (2012), no. 3, 217–226.
[5] Evens, L. *The cohomology of groups.* Oxford Mathematical Monographs, Claredon Pres, New York, 1991.
[6] González-Sánchez, J. *Kirillov's orbit method for p-groups and pro-p groups.* Comm. Algebra **37** no. 12 (2009), 4476–4488.

[7] Khukhro, E.I. *p-Automorphisms of Finite p-Groups.* Cambridge Univ. Press, Cambridge (1998).
[8] Jacobson, N. *Lie algebras.* Interscience, New York / London (1962).
[9] Kazhdan, D. *Proof of Springer's hypothesis.* Israel J. Math. **28** (1977), 272–286.
[10] Kassel, C. and Loday, J.-L. *Extensions centrales d'algèbres de Lie.* Ann. Inst.Fourier Grenoble 32,4 (1982) 119–142
[11] Lazard, M. *Sur les groupes nilpotents et les anneaux de Lie.* Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 101–190.
[12] O'Brien, E.A. and Vaughan-Lee, M.R. *The groups with order p^7 for odd prime p.* J. Algebra **292** (2005), 243–258.

E-mail address: oihana.garayalde@ehu.es

Departamento de Matemáticas, Facultad de Ciencia y Tecnologa, UPV/EHU University of the Basque Country

E-mail address: jon.gonzalez@ehu.es

Jon González-Sánchez, Departamento de Matemáticas, Facultad de Ciencia y Tecnologa, UPV/EHU University of the Basque Country