EIGENVALUE CURVES FOR GENERALIZED MIT BAG MODELS

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ABSTRACT. We study spectral properties of Dirac operators on bounded domains $\Omega \subset \mathbb{R}^3$ with boundary conditions of electrostatic and Lorentz scalar type and which depend on a parameter $\tau \in \mathbb{R}$; the case $\tau = 0$ corresponds to the MIT bag model. We show that the eigenvalues are parametrized as increasing functions of τ, and we exploit this monotonicity to study the limits as $\tau \to \pm\infty$. We prove that if Ω is not a ball then the first positive eigenvalue is greater than the one of a ball with the same volume for all τ large enough. Moreover, we show that the first positive eigenvalue converges to the mass of the particle as $\tau \downarrow -\infty$, and we also analyze its first order asymptotics.

CONTENTS

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1. Introduction

Dirac operators acting on domains $\Omega \subset \mathbb{R}^d$ are used in relativistic quantum mechanics to describe particles that are confined in a box. The so-called MIT bag model is a very remarkable example in dimension $d = 3$. It was introduced in the 1970s as a simplified model to study confinement of quarks in hadrons [29, 37, 38, 51]. The mathematical study of this and related three-dimensional models has gained attention in the recent years [6, 7, 8, 21, 25, 49, 64, 67]. In dimension $d = 2$, Dirac operators with special boundary conditions similar to the ones in the MIT bag model are used in the description of graphene [3, 28, 34, 43, 60, 66]. They have been also investigated in the past few years from the mathematical point of view [14, 15, 23, 32, 42, 55, 65, 70]. The present work focuses on the spectral study of a family $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ of Dirac operators acting on bounded domains $\Omega \subset \mathbb{R}^3$ with electrostatic plus Lorentz scalar type boundary conditions which depend on a parameter $\tau \in \mathbb{R}$; the particular case $\tau = 0$ corresponds to the MIT bag model.

Throughout this article we assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^2 boundary. The unit normal vector field at $\partial\Omega$ which points outwards of Ω is denoted by ν, and the surface measure on $\partial\Omega$, by σ. Given $\tau \in \mathbb{R}$, let $\mathcal{H}_\tau$ be the Dirac operator on Ω defined by

$$\begin{aligned} \mathrm{Dom}(\mathcal{H}_\tau) &:= \big\{\varphi \in H^1(\Omega)\otimes\mathbb{C}^4 : \ \varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi \ \text{ on } \partial\Omega\big\}, \\ \mathcal{H}_\tau\varphi &:= (-i\alpha\cdot\nabla + m\beta)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_\tau), \end{aligned}$$

where $-i\alpha\cdot\nabla + m\beta =: \mathcal{H}$ denotes the differential expression which gives the action of the free Dirac operator on $\mathbb{R}^3$. More precisely, $m \geq 0$ denotes the mass of the particle, $\alpha := (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3, \quad \text{and} \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

are the $\mathbb{C}^{4\times4}$-valued Dirac matrices, I_d denotes the identity matrix in $\mathbb{C}^{d\times d}$ (it will also be denoted by 1 when no confusion arises), and

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. As customary, we use the notation $\alpha\cdot X := \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3$ for $X = (X_1, X_2, X_3)$, and analogously for $\sigma\cdot X$ with $\sigma := (\sigma_1, \sigma_2, \sigma_3)$.

The family $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ naturally arises in the context of confining δ-shell interactions. In the last decade, Dirac operators coupled with δ-shell potentials have been investigated from a mathematical perspective: their self-adjointness and spectral properties [9, 10, 16, 17, 19, 20, 22, 39, 57, 68], the case of rough domains [27, 26], and their approximations and other asymptotic regimes [33, 50, 59, 58, 61]; we refer to the survey [63] for further details on the state of the art of shell interactions for Dirac operators. Several of these works addressed singular perturbations of the form

$$-i\alpha\cdot\nabla + m\beta + (\lambda_e I_4 + \lambda_s\beta)\delta_{\partial\Omega}, \tag{1.1}$$

which correspond to the free Dirac operator on $\mathbb{R}^3$ coupled with electrostatic and Lorentz scalar δ-shell potentials with strengths λ_e and λ_s, respectively. Here, the δ-shell distribution acts as $\delta_{\partial\Omega}(\varphi) = \frac{1}{2}(\varphi_+ + \varphi_-)$, where $\varphi_\pm$ denotes the boundary values of $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ when one

approaches $\partial\Omega$ from inside/outside Ω. It is well known that the operator associated to (1.1) decouples as the orthogonal sum of two operators, one acting in $L^2(\Omega)\otimes\mathbb{C}^4$ and the other in $L^2(\mathbb{R}^3\setminus\overline{\Omega})\otimes\mathbb{C}^4$, if and only if $\lambda_e^2-\lambda_s^2=-4$. This has important consequences in the time-dependent scenario: the Hamiltonian (1.1) generates confinement if and only if $\lambda_e^2-\lambda_s^2=-4$, meaning that a particle which is initially located inside/outside Ω will remain inside/outside Ω for all time. Under the confining relation $\lambda_e^2-\lambda_s^2=-4$, the boundary condition for the operator acting in $L^2(\Omega)\otimes\mathbb{C}^4$ is

$$\varphi=\frac{i}{2}(\lambda_e-\lambda_s\beta)(\alpha\cdot\nu)\varphi\quad\text{on }\partial\Omega; \tag{1.2}$$

recall that the MIT bag boundary condition corresponds to (1.2) with $\lambda_e=0$ and $\lambda_s=2$. Hence, if we set

$$\tau\mapsto\lambda_e(\tau):=2\sinh\tau\quad\text{and}\quad\tau\mapsto\lambda_s(\tau):=2\cosh\tau\quad\text{for }\tau\in\mathbb{R}, \tag{1.3}$$

we obtain a parametrization of the whole branch of the hyperbola $\lambda_e^2-\lambda_s^2=-4$ that contains the MIT bag boundary condition, which is attained through the parametrization at $\tau=0$. Observe that the boundary condition used in the definition of $\mathrm{Dom}(\mathcal{H}_\tau)$ is simply the combination of (1.2) and (1.3). That is, from the singular perturbations point of view, $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ is the restriction to Ω of the branch of confining electrostatic plus Lorentz scalar δ-shell interactions (of constant strength in $\partial\Omega$) that contains the MIT bag model.

In two dimensions, a parametrization similar to $\tau\mapsto\mathcal{H}_\tau$ was used in [23] to describe graphene quantum dots. In there, the boundary conditions are given by δ-shell potentials of Lorentz scalar plus magnetic type. In our three-dimensional framework, the situation analogous to [23] would be to couple the free Dirac operator with potentials of the form $(\lambda_s\beta+\lambda_a i(\alpha\cdot\nu)\beta)\delta_{\partial\Omega}$ with $\lambda_s^2+\lambda_a^2=4$, instead of the ones given by (1.1). Of course, both shell interactions agree for $\lambda_s=2$ (hence $\lambda_e=\lambda_a=0$), and they lead to the MIT bag operator. We refer to [33, Sections 2.3 and 9] for more details.

The operator $\mathcal{H}_\tau$ is self-adjoint. Its spectrum $\sigma(\mathcal{H}_\tau)$ is purely discrete and is contained in $\mathbb{R}\setminus[-m,m]$. That is, for every $\tau\in\mathbb{R}$, $\sigma(\mathcal{H}_\tau)$ is a sequence $\{\lambda_k^\pm(\tau)\}_{k\geq1}\subset\mathbb{R}$ such that

$$\ldots\leq\lambda_2^-(\tau)\leq\lambda_1^-(\tau)<-m\leq m<\lambda_1^+(\tau)\leq\lambda_2^+(\tau)\leq\ldots$$

In addition, it holds that λ is an eigenvalue of $\mathcal{H}_\tau$ if and only if $-\lambda$ is an eigenvalue of $\mathcal{H}_{-\tau}$ —in particular $\sigma(\mathcal{H}_0)$ is symmetric. All these properties are gathered in Lemma 1.2 below. Our main goal in this work is to describe the *eigenvalue curves* of the family $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$, that is, the mappings $\tau\mapsto\lambda_k^\pm(\tau)$, $k\geq1$, as τ ranges over all $\mathbb{R}$. We pay special attention to the study of the *first eigenvalues*, and by this we mean the ones whose absolute value is closest to m, i.e., $\lambda_1^-(\tau)$ and $\lambda_1^+(\tau)$. By the odd symmetry with respect to the parameter τ mentioned above, it is enough to study $\lambda_1^+(\tau)$, which will be called in the sequel the *first positive eigenvalue* of $\mathcal{H}_\tau$.

In Section 1.1 we investigate how the eigenvalue curves look like when Ω is a ball, a case where explicit formulas involving Bessel functions are available. This is our starting point for the spectral analysis in the general case: the evidences observed on the ball provide us with clues of what could be expected to hold on any domain. Our main results for general domains are described in Section 1.2.

Before getting into more details, a few words on our motivation for the results presented in this work are in order. From a general perspective, there is a large body of literature on the spectral analysis of differential operators on domains with parameter-dependent boundary conditions. The Robin Laplacian is a very remarkable example; the interested reader may look

at [31] and the references therein. However, as far as we know, the type of perturbative analysis carried out in the present work has not been considered so far in the context of shell interactions for Dirac operators, except for [11]. In there, the monotonicity of the eigenvalues with respect to the parameter that defines the electrostatic δ-shell interaction is used to optimize the threshold of admissible strengths that yield nontrivial point spectrum, and to characterize the optimal domains. Roughly speaking, a property of the parameter-dependent family of operators (the monotonicity) is successfully used to solve a shape optimization problem for a certain quantity of spectral nature (the threshold of strengths).

One may consider the Dirac operators on domains as the relativistic counterpart of Laplacians with boundary conditions, as for example of Robin type. In this way, the study carried out in the present work has its own interest from the point of view of perturbation theory. However, the main motivation that originated this article was to address the shape optimization problem for the spectral gap of the MIT bag operator, which consists of minimizing the first squared eigenvalue of $\mathcal{H}_0$ among all domains with prescribed volume. The analogous question in the two-dimensional framework —the optimization of the spectral gap for Dirac operators with infinite mass boundary conditions— is considered a hot open problem in spectral geometry [54]. More generally, the quest of geometrical upper and lower bounds for the spectral gap is a trending topic of research [5, 24, 30, 56]. This quest is also addressed in the differential geometry literature for Dirac operators on spin manifolds, where sharp inequalities for spectral gaps in terms of geometric quantities are shown [2, 4, 12, 13, 35, 45, 46, 47, 53]. Despite the amount of works available on this topic, for the case of bounded domains in euclidean spaces the problem of minimizing the spectral gap under a volume constraint (and with no further restrictions on the geometry of the boundary) remains open.

Following the line of our comments about [11], in order to address the shape optimization problem for the MIT bag operator it could be useful to take benefit from the inclusion of $\mathcal{H}_0$ in the family $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$, and to exploit the connection of $\mathcal{H}_\tau$ with the Dirichlet Laplacian as $\tau \to \pm\infty$; see Theorem 1.4 below. In this regard, Corollary 1.6 shows the optimality of the ball in the asymptotic regime $\tau \uparrow +\infty$. In addition, Theorem 1.7 draws a path to address the optimality of the ball in the asymptotic regime $\tau \downarrow -\infty$. As we said, we expect that this information for $|\tau|$ large enough will be useful to deal with the optimality of the ball in the general case of $\tau \in \mathbb{R}$ and, in particular, for $\tau = 0$.

We close this introductory section clarifying some conventions that we are going to use in the sequel. Besides an amount of standard notation, as well as further shorthand that will be introduced in due time, we use the following notation throughout the paper:

E^d	tensor product $E \otimes \mathbb{C}^d$ of a vector space E with $\mathbb{C}^d$
$A^\intercal$	transpose of a matrix A
$\varphi = (u, v)^\intercal$	decomposition of $\varphi \in \mathbb{C}^4$ in upper and lower components, that is, $u = (\varphi_1, \varphi_2)^\intercal$ and $v = (\varphi_3, \varphi_4)^\intercal$ for $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^\intercal$
$\langle \cdot, \cdot \rangle_{L^2(\Omega)^d}$	scalar product in $L^2(\Omega)^d$ given by $\langle u, v \rangle_{L^2(\Omega)^d} := \int_\Omega u \cdot \overline{v}\, dx$
$\langle \cdot, \cdot \rangle_{L^2(\partial\Omega)^d}$	scalar product in $L^2(\partial\Omega)^d$ given by $\langle u, v \rangle_{L^2(\partial\Omega)^d} := \int_{\partial\Omega} u \cdot \overline{v}\, d\sigma$
$H^1(\Omega)$	Sobolev space of functions in $L^2(\Omega)$ with first weak partial derivatives in $L^2(\Omega)$
σ	3-tuple $(\sigma_1, \sigma_2, \sigma_3)$ of Pauli matrices
$\sigma(T)$	spectrum of the operator T
$\{T_1, T_2\}$	anticommutator $T_1T_2 + T_2T_1$ of operators T_1 and T_2

1.1. **Eigenvalue curves for a ball.** We present here a brief summary of the spectral study of $\mathcal{H}_\tau$ in the case $\Omega = B_R \subset \mathbb{R}^3$, the ball of radius $R > 0$ centered at the origin. In this radially symmetric domain we can use separation of variables ($r \in [0, R)$ and $\theta \in \mathbb{S}^2$) and, thanks to the spherical harmonic spinors, obtain explicit equations for the eigenvalues and eigenfunctions of $\mathcal{H}_\tau$. The analysis done for the ball provides some intuition on which kind of situations one can expect when studying the operator in a general domain $\Omega \subset \mathbb{R}^3$, for which no explicit formulas are available. A more detailed analysis including the proofs of the facts stated in this section can be found in the Appendix B.

To deal with the problem in the ball, it is suitable to use the decomposition

$$L^2(\mathbb{R}^3)^4 = \bigoplus_{j=1/2}^{+\infty} \bigoplus_{\mu_j=-j}^{j} L^+_{j,\mu_j} \oplus L^-_{j,\mu_j},$$

where $L^\pm_{j,\mu_j}$ for $j = 1/2,\, 3/2,\, \ldots$, and $\mu_j = -j,\, -j+1,\, \ldots,\, j-1,\, j$, are invariant spaces under the action of $\mathcal{H}$, and are defined in terms of the spherical harmonic spinors; see Appendix B for the explicit expressions. Thanks to the above decomposition, the eigenvalue problem for $\mathcal{H}_\tau$ can be reduced to a system of two Bessel-type ODE. After imposing the boundary conditions, for each invariant space $L^\pm_{j,\mu_j}$ we obtain the eigenvalue equation

$$0 = e^\tau J_{\ell+1/2}(\sqrt{\lambda^2 - m^2}R) \pm \frac{\sqrt{\lambda^2 - m^2}}{\lambda + m} J_{\ell'+1/2}(\sqrt{\lambda^2 - m^2}R), \tag{1.4}$$

where J_k is the k-th Bessel function of the first kind, $\ell = j \pm 1/2$, and $\ell' = j \mp 1/2$. This equation already appears in [39, formula (6.3)]. Note that the index μ_j does not appear in (1.4), meaning that for a given half-integer j there are $2j + 1$ linearly independent eigenfunctions associated to the same eigenvalue. Therefore, all the eigenvalues have even multiplicity.

If we solve numerically (1.4) for some choice of indexes, we obtain the plot shown in Figure 1.

As we see, the eigenvalues can be parametrized in terms of τ, obtaining a family of increasing functions. To show rigorously that this parametrization can be done, it suffices to write the eigenvalue equation (1.4) as

$$e^\tau = \mp \frac{\sqrt{\lambda^2 - m^2}}{\lambda + m} \frac{J_{\ell'+1/2}(\sqrt{\lambda^2 - m^2}R)}{J_{\ell+1/2}(\sqrt{\lambda^2 - m^2}R)} =: h(\lambda) \tag{1.5}$$

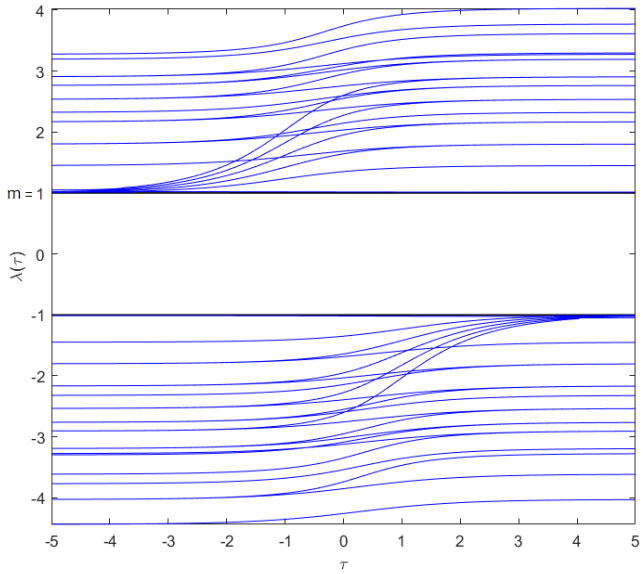

FIGURE 1. Some eigenvalue curves $\tau \mapsto \lambda(\tau)$ of $\mathcal{H}_\tau$ on B_R for $R = 3$ and $m = 1$.

and invert the function h in suitable intervals I. This provides, for each of these intervals, a parametrization of an eigenvalue given by $\tau \mapsto \lambda(\tau) = h^{-1}(e^\tau) \in I$. By the distribution of the zeroes and singularities of h, it can be seen that h is only invertible in intervals I in which the above quotient of Bessel functions does not change sign. These maximal intervals have the form $(-z_{\ell'}, -z_\ell)$, $(-z_{\ell'}, -m)$, (m, z_ℓ), or $(z_{\ell'}, z_\ell)$, where z_k denotes a positive zero of the function $J_{k+1/2}(\sqrt{(\cdot)^2 - m^2}R)$. In each of these intervals, $h : I \to (0, +\infty)$ is strictly increasing and surjective, and thus it can be inverted. From this it follows that $\lambda(\tau)$ is also strictly increasing. Proposition B.2 gathers all these considerations.

In addition, the limit of $|\lambda(\tau)|$ as $\tau \to \pm\infty$ must be at the boundary of the interval I, and thus be either m or a positive zero of $J_{k+1/2}(\sqrt{(\cdot)^2 - m^2}R)$ for some $k = 0, 1, 2, \ldots$; note that each of these zeroes corresponds to the square root of a Dirichlet eigenvalue of $-\Delta + m^2$ in B_R. Since the same Bessel function may appear in the denominator of the expression of h in (1.5) for different choices of the index j , more than one eigenvalue curve may converge to the same value as $\tau \uparrow +\infty$, and analogously as $\tau \downarrow -\infty$. This is illustrated in Figure 1 and, more explicitly, in Figures 4 and 5.

Remark 1.1. Note that (1.5) can be rewritten as

$$e^\tau = \mp \operatorname{sign}(\lambda + m)\sqrt{\frac{\lambda - m}{\lambda + m}}\frac{J_{\ell'+1/2}(\sqrt{\lambda^2 - m^2}R)}{J_{\ell+1/2}(\sqrt{\lambda^2 - m^2}R)} =: h(\lambda).$$

From this, it follows that $\lambda(\tau)$ is an eigenvalue curve corresponding to the subspace $L^{\pm}_{j,\mu_j}$ if and only if $-\lambda(-\tau)$ is an eigenvalue curve associated to $L^{\mp}_{j,\mu_j}$. This fact is illustrated in Figures 2 and 3. As we will see, the odd symmetry of the eigenvalue curves with respect to τ holds as well for a general domain Ω; see Lemma 1.2 (*iii*). Therefore, it will be enough to study the positive spectrum of $\mathcal{H}_\tau$.

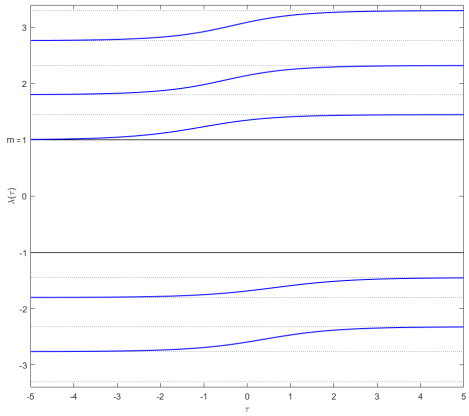

FIGURE 2. Eigenvalue curves for $L^-_{1/2,\mu_{1/2}}$.

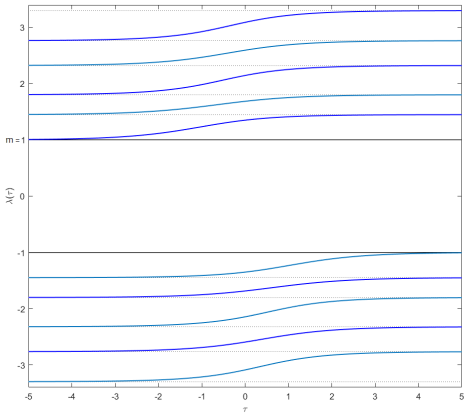

FIGURE 3. Eigenvalue curves for $L^-_{1/2,\mu_{1/2}}$ and $L^+_{1/2,\mu_{1/2}}$.

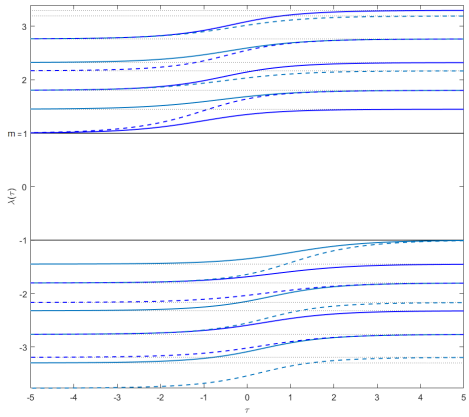

FIGURE 4. Eigenvalue curves for $j = 1/2, 3/2$.

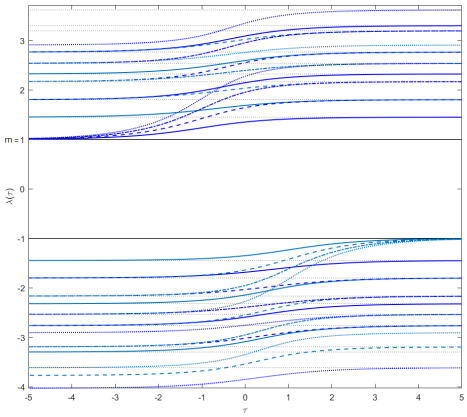

FIGURE 5. Eigenvalue curves for $j = 1/2, 3/2, 5/2, 7/2$.

To summarize, all the eigenvalues of $\mathcal{H}_\tau$ in B_R can be represented as a set of monotone increasing curves parametrized by $\tau \in \mathbb{R}$, which may cross among them; see Figure 5. For a given curve $\tau \mapsto \lambda(\tau) \in \sigma(\mathcal{H}_\tau) \cap (m, +\infty)$, we know which are the possible limits of $\lambda(\tau)$ as $\tau \to \pm\infty$: with the only exception of some eigenvalues that, as $\tau \downarrow -\infty$, converge to m, the limiting values of $\lambda(\tau)$ are of the form $\sqrt{\Lambda + m^2}$, where Λ is an eigenvalue of the Dirichlet Laplacian in B_R. In the next sections we will show that all these properties hold for every bounded domain $\Omega \subset \mathbb{R}^3$ with C^2 boundary; see Theorems 1.3 and 1.4. In addition, we will use the explicit information available for the ball to study a shape optimization problem for the first positive eigenvalue of $\mathcal{H}_\tau$ when $|\tau|$ is large enough.

1.2. **Main results.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. In this section we state our main results on the spectral study of the operator $\mathcal{H}_\tau$ in terms of the parameter $\tau \in \mathbb{R}$. The study is carried out taking into account the different phenomena observed in Figure 1, where the case of a ball is illustrated.

Recall that

$$\mathrm{Dom}(\mathcal{H}_\tau) := \big\{\varphi \in H^1(\Omega)^4 : \varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi \text{ on } \partial\Omega\big\},$$
$$\mathcal{H}_\tau\varphi := \mathcal{H}\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_\tau),$$

and $\mathcal{H} := -i\alpha\cdot\nabla + m\beta$. The fact that $\mathcal{H}_\tau$ is self-adjoint follows by [21, Proposition 5.15]. Let us begin by recalling several known properties of the spectrum of $\mathcal{H}_\tau$. We gather them in the following lemma, whose proof is given in Appendix A for the sake of completeness.

Lemma 1.2. *For every $\tau \in \mathbb{R}$, the following holds:*

(i) The spectrum $\sigma(\mathcal{H}_\tau)$ is a sequence of real eigenvalues that only accumulate at $\pm\infty$.
(ii) $\sigma(\mathcal{H}_\tau) \subset (-\infty, -m) \cup (m, +\infty)$.
(iii) $\lambda \in \sigma(\mathcal{H}_\tau)$ if and only if $-\lambda \in \sigma(\mathcal{H}_{-\tau})$. In particular, $\sigma(\mathcal{H}_0)$ is symmetric.
(iv) Every $\lambda \in \sigma(\mathcal{H}_\tau)$ has finite and even multiplicity.

As we see from Lemma 1.2 (i), for every $\tau \in \mathbb{R}$ the spectrum $\sigma(\mathcal{H}_\tau)$ only contains real eigenvalues with no accumulation points. It is then natural to investigate how the eigenvalues depend on the parameter τ. Our first result in this direction is that they can be parametrized by means of strictly increasing real analytic functions of τ whose graphs may cross at locally finitely many exceptional values of τ. Note that these crossings indeed occur in the case of the ball, as shown in Figures 1 and 5. The following proposition gives the precise statement about the existence of the mentioned parametrization; see Section 3 for a proof.

Theorem 1.3. *Given $\tau_0 \in \mathbb{R}$, let $\{\lambda_k(\tau_0)\}_{k\in\mathbb{Z}\setminus\{0\}}$ be an enumeration of the eigenvalues of $\mathcal{H}_{\tau_0}$ (each repeated according to its finite algebraic multiplicity). Let $\varphi_k(\tau_0)$ be an eigenfunction associated to $\lambda_k(\tau_0)$, that is, $\varphi_k(\tau_0) \in \mathrm{Dom}(\mathcal{H}_{\tau_0}) \setminus \{0\}$ and $\mathcal{H}_{\tau_0}\varphi_k(\tau_0) = \lambda_k(\tau_0)\varphi_k(\tau_0)$.*

Then, each $\lambda_k(\tau_0)$ can be extended to a real analytic function $\tau \mapsto \lambda_k(\tau)$ in such a way that

$$\sigma(\mathcal{H}_\tau) = \textstyle\bigcup_{k\in\mathbb{Z}\setminus\{0\}}\{\lambda_k(\tau)\} \quad \text{for all } \tau \in \mathbb{R}.$$

Similarly, each eigenfunction $\varphi_k(\tau_0)$ can be extended to a real analytic function $\tau \mapsto \varphi_k(\tau)$ such that, for every $\tau \in \mathbb{R}$, $\varphi_k(\tau)$ is an eigenfunction associated to the eigenvalue $\lambda_k(\tau)$.

All the functions $\lambda_k(\tau)$ are strictly increasing in $\mathbb{R}$ and, up to repetition (due to the even multiplicity), for each eigenvalue curve $\lambda_k(\tau)$ there are only finitely many eigenvalue curves meeting at locally finitely many crossing points with $\lambda_k(\tau)$.

In the last statement of this theorem we mean that, modulo eigenvalue curves which are exactly the same for all $\tau \in \mathbb{R}$, in every compact set each eigenvalue curve can cross other curves only a finite number of times, and in each crossing it will coincide only with a finite number of eigenvalue curves. It is also worth pointing out that since the eigenvalue curves are continuous, $\sigma(\mathcal{H}_0)$ is symmetric, and $\sigma(\mathcal{H}_\tau) \subset (-\infty, -m) \cup (m, +\infty)$ for all τ, then it holds that $\sigma(\mathcal{H}_\tau) \cap (m, +\infty) \neq \emptyset$ and $\sigma(\mathcal{H}_\tau) \cap (-\infty, -m) \neq \emptyset$ for all $\tau \in \mathbb{R}$.

The next step in our analysis is to address the asymptotic behavior of the eigenvalue curves as $\tau \to \pm\infty$. Thanks to the odd symmetry of the eigenvalues with respect to the parameter τ shown in Lemma 1.2 (iii), it is enough to consider only positive eigenvalues, i.e., the elements of $\sigma(\mathcal{H}_\tau) \cap (m, +\infty)$. In the following result we describe their asymptotic behavior in terms of $\sigma(-\Delta_D)$, the spectrum of the Dirichlet Laplacian $-\Delta_D$. The proof is given in Section 4.

Theorem 1.4. *Let $\tau \mapsto \lambda(\tau) \in \sigma(\mathcal{H}_\tau) \cap (m, +\infty)$ be a continuous function defined on an interval $I \subset \mathbb{R}$. The following holds:*

(*i*) *If* $I = (-\infty, \tau_0)$ *for some* $\tau_0 \in \mathbb{R}$, *then* $\lambda(-\infty) := \lim_{\tau\downarrow-\infty} \lambda(\tau)$ *exists and belongs to* $[m, +\infty)$. *In addition,* $\lambda(-\infty) = m$ *if* $\lambda(\tau)^2 - m^2 \leq \min \sigma(-\Delta_D)$ *for some* $\tau \in I$, *and* $\lambda(-\infty)^2 - m^2 \in \sigma(-\Delta_D)$ *otherwise.*

(*ii*) *If* $I = (\tau_0, +\infty)$ *for some* $\tau_0 \in \mathbb{R}$, *then* $\lambda(+\infty) := \lim_{\tau\uparrow+\infty} \lambda(\tau)$ *exists as an element of the set* $(m, +\infty]$. *In addition, if* $\lambda(+\infty) < +\infty$ *then* $\lambda(+\infty)^2 - m^2 \in \sigma(-\Delta_D)$.

The function $\lambda_1^+ : \mathbb{R} \to (m, +\infty)$ defined by

$$\tau \mapsto \lambda_1^+(\tau) := \min(\sigma(\mathcal{H}_\tau) \cap (m, +\infty)) \tag{1.6}$$

assigns to every $\tau \in \mathbb{R}$ the first (smallest) positive eigenvalue of $\mathcal{H}_\tau$, as Lemma 1.2 (*ii*) shows. Combining Theorems 1.3 and 1.4, we deduce several properties of the function λ_1^+ which are gathered in the following result; see Section 4.1 for a proof. Analogous conclusions hold for the mapping $\tau \mapsto \lambda_1^-(\tau) := \max(\sigma(\mathcal{H}_\tau) \cap (-\infty, -m))$, which assigns to every $\tau \in \mathbb{R}$ the first (largest) negative eigenvalue of $\mathcal{H}_\tau$. This is because $\lambda_1^-(\tau) = -\lambda_1^+(-\tau)$ by Lemma 1.2 (*iii*).

Theorem 1.5. *The function* λ_1^+ *defined by* (1.6) *is continuous and strictly increasing in* $\mathbb{R}$, *and satisfies*

$$\lim_{\tau\downarrow-\infty} \lambda_1^+(\tau) = m \quad \text{and} \quad \lim_{\tau\uparrow+\infty} \lambda_1^+(\tau)^2 - m^2 \in \sigma(-\Delta_D) \cup \{+\infty\}. \tag{1.7}$$

In addition, λ_1^+ *is real analytic in* $\mathbb{R} \setminus E$, *where* $E \subset \mathbb{R}$ *is some set such that* $E \cap [-R, R]$ *is finite for all* $R > 0$.

In Remark 4.4 we make a comment on the possibility of having $\lim_{\tau\uparrow+\infty} \lambda_1^+(\tau) = +\infty$. See also Remark 4.5 for a comment on the set E.

Theorem 1.5 has some consequences on a shape optimization problem for the first positive eigenvalue of $\mathcal{H}_\tau$ when τ is large enough. In order to highlight the dependence of $\mathcal{H}_\tau$ on the domain $\Omega \subset \mathbb{R}^3$ —which we assume through the paper that has a C^2 boundary—, let us now denote the first positive eigenvalue of $\mathcal{H}_\tau$ by $\lambda_\Omega(\tau)$ (that is, we set $\lambda_\Omega := \lambda_1^+$). Then, using (1.7) and Faber-Krahn inequality, we get the following result, whose proof is given in Section 4.1.

Corollary 1.6. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* C^2 *boundary, and let* $B \subset \mathbb{R}^3$ *be a ball such that* $|\Omega| = |B|$. *If* Ω *is not a ball, then there exists* $\tau_0 \in \mathbb{R}$ *depending on* Ω *such that* $\lambda_B(\tau) < \lambda_\Omega(\tau)$ *for all* $\tau \geq \tau_0$.

This means that, if Ω is not a ball, for large values of τ the first positive eigenvalue of $\mathcal{H}_\tau$ on Ω is strictly larger than the one on a ball with the same volume as Ω. Then, a natural question is whether the analogous result for τ tending to $-\infty$ holds or not. Recall that the first positive eigenvalue of $\mathcal{H}_\tau$ tends to m as $\tau \downarrow -\infty$ independently of the shape of Ω, by (1.7). Hence, in order to find the optimal shape of Ω for τ negative and far from the origin, one is forced to study the asymptotic expansion of $\lambda_\Omega(\tau) - m$ as $\tau \downarrow -\infty$. Our next result goes in this direction, but requires some preliminaries to state it properly.

Let $L_\Omega : \mathbb{R} \to (0, +\infty)$ be defined by

$$\tau \mapsto L_\Omega(\tau) := (\lambda_\Omega(\tau) - m)e^{-\tau}, \tag{1.8}$$

thus $\lambda_\Omega(\tau) = m + e^\tau L_\Omega(\tau)$. In Lemma 4.9 we will prove that L_Ω is strictly decreasing in $\mathbb{R}$. Therefore, the limit

$$L_\Omega^\star := \lim_{\tau\downarrow-\infty} L_\Omega(\tau)$$

exists as an element of the set $(0, +\infty]$. Moreover, in Proposition B.3 we will also show that $L_B^\star < +\infty$ for every ball $B \subset \mathbb{R}^3$. This in particular suggests that $L_\Omega^\star$ is the natural quantity to look at when addressing the asymptotic expansion of $\lambda_\Omega(\tau) - m$ as $\tau \downarrow -\infty$.

Assume now for a while that $\Omega \subset \mathbb{R}^3$ is a domain such that $L_\Omega^\star = +\infty$, and let $B \subset \mathbb{R}^3$ be a ball, not necessarily with the same volume as Ω. Since $L_B^\star < +\infty$, there exists $\tau_0 \in \mathbb{R}$ such that $L_\Omega(\tau) > L_B^\star \geq L_B(\tau)$ for all $\tau \leq \tau_0$. Therefore,

$$\lambda_\Omega(\tau) = m + e^\tau L_\Omega(\tau) > m + e^\tau L_B(\tau) = \lambda_B(\tau) \tag{1.9}$$

for all $\tau \leq \tau_0$. That is, if $L_\Omega^\star = +\infty$, then the first positive eigenvalue of $\mathcal{H}_\tau$ on Ω is strictly greater than the one on any ball B whenever $-\tau$ is large enough depending on Ω and B. As a consequence, in order to look for the optimal shape of Ω (under a volume constraint) for τ negative and far from the origin, we can assume without loss of generality that $L_\Omega^\star < +\infty$.

Our last result in this work provides a lower bound for $L_\Omega^\star$ in terms of the optimization, among functions in a boundary Hardy space, of a Rayleigh quotient which involves the single layer potential for the Laplacian. We expect that this lower bound, which is attained if Ω is a ball, will be useful to solve the above-mentioned optimization problem for τ negative and far from the origin. On the one hand, the single layer potential appears in trace form as the operator $K : L^2(\partial\Omega)^2 \to L^2(\partial\Omega)^2$ defined by

$$Ku(x) := \frac{1}{4\pi} \int_{\partial\Omega} \frac{u(y)}{|x-y|}\, d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega.$$

It is well known that K is a bounded, self-adjoint, compact, positive, and injective operator in $L^2(\partial\Omega)^2$. In particular, K diagonalizes in an $L^2(\partial\Omega)^2$-orthonormal basis of eigenfunctions and all its eigenvalues are strictly positive real numbers. On the other hand, the boundary Hardy space referred above is the subspace $P(L^2(\partial\Omega)^2)$ of $L^2(\partial\Omega)^2$, where $P : L^2(\partial\Omega)^2 \to L^2(\partial\Omega)^2$ is defined by $P := \frac{1}{2} + iW(\sigma \cdot \nu)$, and

$$Wu(x) := \lim_{\epsilon \downarrow 0} \frac{i}{4\pi} \int_{\{y \in \partial\Omega :\, |x-y| > \epsilon\}} \Big(\sigma \cdot \frac{x-y}{|x-y|^3}\Big) u(y)\, d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega.$$

It is well known that W is a bounded operator in $L^2(\partial\Omega)^2$, and that P is a projection. The subspace $P(L^2(\partial\Omega)^2)$ arises as the trace space on $\partial\Omega$ of null-solutions of $\sigma\cdot\nabla$ in Ω; see Section 4.2 for more details (by a matter of notation, in Sections 4.2 and 4.3 the operators K and W are denoted by K_m and W_m, respectively, and P by P_+).

Before stating our result regarding the lower bound for $L_\Omega^\star$, let us briefly explain the heuristics behind it. Recall that $L_\Omega^\star$ arises when considering the behavior of $\lambda_\Omega(\tau)$ as $\tau \downarrow -\infty$. If one looks at the associated eigenfunction for $\lambda_\Omega(\tau)$, in the limit $\tau \downarrow -\infty$ one is led to an eigenvalue-type problem of the form

$$P^* u = L_\Omega^\star (\sigma \cdot \nu) K (\sigma \cdot \nu) u \quad \text{for some } u \in P(L^2(\partial\Omega)^2) \setminus \{0\}, \tag{1.10}$$

where P^* denotes the adjoint of P. Hence, in order to estimate $L_\Omega^\star$ it is natural to investigate the set

$$\mathcal{L}_\Omega := \Big\{ L \in \mathbb{C} : \text{ there exists } u \in P(L^2(\partial\Omega)^2) \setminus \{0\} \text{ with } P^* u = L(\sigma \cdot \nu) K (\sigma \cdot \nu) u \Big\}. \tag{1.11}$$

By definition, we have $L_\Omega^\star \in \mathcal{L}_\Omega$. Despite that we initially think of $\mathcal{L}_\Omega$ as a subset of $\mathbb{C}$, it turns out that $\mathcal{L}_\Omega$ only contains positive real numbers. Actually, given $L \in \mathcal{L}_\Omega$ and an associated function u as in (1.11), it is not hard to show (see the proof of Lemma 4.11) that

$$\|u\|^2_{L^2(\partial\Omega)^2} = L \langle (\sigma \cdot \nu) K (\sigma \cdot \nu) u, u \rangle_{L^2(\partial\Omega)^2}. \tag{1.12}$$

This suggests considering the functional

$$\mathscr{R}(u) := \frac{\langle (\sigma\cdot\nu)K(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2}}{\|u\|^2_{L^2(\partial\Omega)^2}} \quad \text{for } u \in L^2(\partial\Omega)^2 \setminus \{0\}, \tag{1.13}$$

so that for L and u as in (1.12) we get $L = 1/\mathscr{R}(u)$. Since K is bounded in $L^2(\partial\Omega)^2$ and is strictly positive, we have $0 < \mathscr{R}(u) \leq \|K\|_{L^2(\partial\Omega)^2\to L^2(\partial\Omega)^2}$ for all $u \in L^2(\partial\Omega)^2 \setminus \{0\}$, and this yields

$$\mathcal{L}_\Omega \subset \big[1/\|K\|_{L^2(\partial\Omega)^2\to L^2(\partial\Omega)^2}, +\infty\big) \subset \mathbb{R}$$

by (1.12). Therefore, it is reasonable to use $\inf\mathcal{L}_\Omega$ to bound $L^\star_\Omega$ from below. Finally, recalling the relation $L = 1/\mathscr{R}(u)$ given by (1.12), and defining

$$\mathscr{R}_\Omega := \sup_{u\in P(L^2(\partial\Omega)^2)\setminus\{0\}} \mathscr{R}(u), \tag{1.14}$$

we end up with

$$L^\star_\Omega \geq \inf\mathcal{L}_\Omega \geq 1/\mathscr{R}_\Omega.$$

This is the lower bound for $L^\star_\Omega$ in terms of the optimization of a Rayleigh quotient that we were referring to as our last main result in this work. At this point the reader might think that substantial information may have been lost during all these steps, and that in the end we have $L^\star_\Omega > 1/\mathscr{R}_\Omega$ for all Ω. However, we will show that the equality holds if Ω is a ball, hence the lower bound that we found is sharp. Furthermore, we will prove that $1/\mathscr{R}_\Omega = \min\mathcal{L}_\Omega$ for every Ω, which means that $1/\mathscr{R}_\Omega$ is actually the smallest admissible value in the eigenvalue-type problem (1.10). It would be very interesting to know whether the equality $1/\mathscr{R}_\Omega = L^\star_\Omega$ holds in general or not.

We gather all these considerations in the following theorem, which is our last main result of this work. It will be proven in Section 4.3.

Theorem 1.7. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with C^2 boundary. The following holds:*

- *(i) If $L^\star_\Omega < +\infty$ then $L^\star_\Omega \in \mathcal{L}_\Omega$.*
- *(ii) $\mathcal{L}_\Omega \subset \mathbb{R}$, and $1/\mathscr{R}_\Omega \leq L$ for all $L \in \mathcal{L}_\Omega$.*
- *(iii) The supremum in (1.14) is attained. Moreover, if $u \in P(L^2(\partial\Omega)^2)\setminus\{0\}$ is such that $\mathscr{R}(u) = \mathscr{R}_\Omega$, then $P^+u = \frac{1}{\mathscr{R}_\Omega}(\sigma\cdot\nu)K(\sigma\cdot\nu)u$. In particular, $1/\mathscr{R}_\Omega \in \mathcal{L}_\Omega$.*

As a consequence of (i), (ii), and (iii), it holds that $1/\mathscr{R}_\Omega = \min\mathcal{L}_\Omega \leq L^\star_\Omega$ for every bounded domain $\Omega\subset\mathbb{R}^3$ with C^2 boundary. Furthermore, $1/\mathscr{R}_B = L^\star_B$ for every ball $B\subset\mathbb{R}^3$.

In view of Theorem 1.7, the question that we would like to answer at this point is whether the ball is the unique maximizer of $\mathscr{R}_\Omega$ among all bounded and smooth domains Ω of the same volume. The fact that Hardy spaces come into play in (1.14) makes this question difficult and challenging, and it requires further study to be answered. By the same argument as in (1.9), but using also Theorem 1.7, an affirmative answer would yield that if Ω is not a ball then for $-\tau$ large enough the first positive eigenvalue of $\mathcal{H}_\tau$ on Ω is strictly greater than the one on a ball with the same volume as Ω. That is, we would get the analogue of Corollary 1.6 in the regime $\tau\downarrow-\infty$.

Having in mind the results presented above (and with a bit of positive thinking), we would like to finish this introduction by posing the following

Conjecture 1.8. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with C^2 boundary, and let $B\subset\mathbb{R}^3$ be a ball such that $|\Omega| = |B|$. If Ω is not a ball, then $\lambda_B(\tau) < \lambda_\Omega(\tau)$ for all $\tau\in\mathbb{R}$.*

A positive answer to this conjecture for $\tau = 0$ would give the solution to the shape optimization problem for the spectral gap of the MIT bag operator on smooth domains.

Remark 1.9. Conjecture 1.8 also makes sense in the two-dimensional framework, that is, for a bounded domain $\Omega \subset \mathbb{R}^2$. The corresponding Dirac operator would be $-i(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3$ acting on functions $\varphi \in H^1(\Omega)^2$ such that

$$\varphi = i(\sinh\tau - \cosh\tau\,\sigma_3)(\sigma_1\nu_1 + \sigma_2\nu_2)\varphi \quad \text{on } \partial\Omega.$$

When $\tau = 0$, this is the so-called *Dirac operator with infinite mass boundary conditions*.

The techniques used in this work are not specific of the three-dimensional framework. They mainly depend on the algebraic structure of the operator and its boundary conditions. Therefore, they would also apply in the two-dimensional framework, taking into account the obvious modifications which arise when replacing α by (σ_1, σ_2) and β by σ_3. In this regard, we expect that the main results presented in Section 1.2 also hold true for the two-dimensional Dirac operator.

Organization of the paper. In Section 2 we present some preliminary results and introduce the boundary integral operators associated to $\mathcal{H}$, proving some regularity estimates. With these tools at hand, in Section 3 we establish Theorem 1.3, proving that the eigenvalues of $\mathcal{H}_\tau$ can be parametrized as increasing functions of $\tau \in \mathbb{R}$. Then, in Section 4 we study the asymptotic behavior of the eigenvalue curves. First, in Section 4.1 we prove Theorem 1.4 regarding the limits of the eigenvalue curves, Theorem 1.5, on the first positive eigenvalue, and Corollary 1.6, the shape optimization result. Next, in Section 4.2 we introduce skew projections onto Hardy spaces, which are used in Section 4.3 to establish Theorem 1.7.

This article contains two appendixes. In Appendix A we establish the main properties of the spectrum of $\mathcal{H}_\tau$, gathered in Lemma 1.2. In there the reader can also find a formula relating $\mathcal{H}_\tau$ with the mean curvature of $\partial\Omega$. Finally, Appendix B is devoted to the spectral analysis when the underlying domain is a ball, and we provide an extensive and explicit description of the eigenvalue curves.

2. Regularity estimates

In this section, after presenting some preliminary results, we will introduce boundary integral operators associated to $\mathcal{H}$. We will use them to rewrite the eigenvalue equation for $\mathcal{H}_\tau$ as an integral equation on $\partial\Omega$; see Proposition 2.9 below. Then, we will establish regularity estimates for solutions to this boundary eigenvalue equation (stated in Lemma 2.10) which will be crucial in the following sections when using compactness arguments.

2.1. **Preliminaries.** The contents of this section are known results from the literature and simple computations that will be used in the sequel. Let us begin by recalling a result from [64] which explores the relation between a function in Ω an its trace on $\partial\Omega$. Given $u \in L^2(\Omega)^2$ such that $\sigma \cdot \nabla u \in L^2(\Omega)^2$, in general one cannot assure that $u \in H^1(\Omega)^2$. Nevertheless, as the following lemma shows, if the trace of u belongs to $H^{1/2}(\partial\Omega)^2$, then $u \in H^1(\Omega)^2$. Here, $H^s(\partial\Omega)^d$ with $s \in (0,1)$ and $d > 0$ integer denotes the fractional Sobolev space of functions $f \in L^2(\partial\Omega)^d$ such that

$$\|f\|_{H^s(\partial\Omega)^d} := \left(\int_{\partial\Omega} |f|^2 \, d\sigma + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x) - f(y)|^2}{|x-y|^{2+2s}} \, d\sigma(y)\, d\sigma(x) \right)^{1/2} < +\infty.$$

We will also use the symbol $H^{-s}(\partial\Omega)^d$ to denote the continuous dual of $H^s(\partial\Omega)^d$. Recall that

$$\|\cdot\|_{H^{-s}(\partial\Omega)^d} \leq \|\cdot\|_{L^2(\partial\Omega)^d} \leq \|\cdot\|_{H^s(\partial\Omega)^d} \quad \text{for all } s \in (0,1).$$

Lemma 2.1. *Let $u \in L^2(\Omega)^2$ be such that $\sigma \cdot \nabla u \in L^2(\Omega)^2$. Then,*

(i) the trace of u belongs to $H^{-1/2}(\partial\Omega)^2$, and satisfies

$$\|u\|_{H^{-1/2}(\partial\Omega)^2} \leq C\left(\|u\|_{L^2(\Omega)^2} + \|\sigma\cdot\nabla u\|_{L^2(\Omega)^2}\right)$$

for some $C > 0$ depending only on Ω.

(ii) Assume in adittion that $u \in H^{1/2}(\partial\Omega)^2$. Then, $u \in H^1(\Omega)^2$ and

$$\|u\|_{H^1(\Omega)^2} \leq C\left(\|u\|_{L^2(\Omega)^2} + \|\sigma\cdot\nabla u\|_{L^2(\Omega)^2} + \|u\|_{H^{1/2}(\partial\Omega)^2}\right)$$

for some $C > 0$ depending only on Ω.

Proof. The proof of these results follows similar arguments as the ones in [64, Section 2], and therefore some details may be omitted. Throughout the proof, we shall denote

$$H(\sigma,\Omega) := \left\{u \in L^2(\Omega)^2 : \sigma\cdot\nabla u \in L^2(\Omega)^2\right\},$$

which is a Hilbert space equipped with the scalar product

$$\langle u, v\rangle_{H(\sigma,\Omega)} := \langle u, v\rangle_{L^2(\Omega)^2} + \langle \sigma\cdot\nabla u, \sigma\cdot\nabla v\rangle_{L^2(\Omega)^2}$$

and the associated norm $\|u\|_{H(\sigma,\Omega)} := \left(\|u\|^2_{L^2(\Omega)^2} + \|\sigma\cdot\nabla u\|^2_{L^2(\Omega)^2}\right)^{1/2}$.

Statement (i) simply means that the trace operator is bounded from $H(\sigma,\Omega)$ into $H^{-1/2}(\partial\Omega)^2$, and it is proved exactly as [64, Proposition 2.1]. Hence, we will only address the proof of (ii). Given $u \in H(\sigma,\Omega)$ such that $u \in H^{1/2}(\partial\Omega)^2$, set

$$\tilde{u} := u - E(u),$$

where $E : H^{1/2}(\partial\Omega)^2 \to H^1(\Omega)^2$ is a bounded extension operator, i.e.,

$$\|E(u)\|_{H^1(\Omega)^2} \leq C\,\|u\|_{H^{1/2}(\partial\Omega)^2} \tag{2.1}$$

for some $C > 0$ depending only on Ω. Note that the new function $\tilde{u}$ has zero trace on $\partial\Omega$ and, therefore, denoting by u_0 its extension by 0 in $\mathbb{R}^3 \setminus \Omega$, we have that

$$\sigma\cdot\nabla u_0 = (\sigma\cdot\nabla\tilde{u})\chi_\Omega \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where χ_Ω denotes the characteristic function of the set Ω; for a rigorous proof of (2.2) see for instance the proof of [64, Proposition 2.16].

We claim that

$$\|\tilde{u}\|_{H^1(\Omega)^2} = \|\tilde{u}\|_{H(\sigma,\Omega)}\,. \tag{2.3}$$

Note first that $\|\tilde{u}\|_{H^1(\Omega)^2} = \|u_0\|_{H^1(\mathbb{R}^3)^2}$. Using the fact that $H^1(\mathbb{R}^3)^2 = H(\sigma,\mathbb{R}^3)$, which can be shown using integration by parts, we obtain $\|\tilde{u}\|_{H^1(\Omega)^2} = \|u_0\|_{H(\sigma,\mathbb{R}^3)}$. Thus, the claim follows from (2.2).

Finally, noting that

$$\|\tilde{u}\|_{H(\sigma,\Omega)} \leq \|u\|_{H(\sigma,\Omega)} + \|E(u)\|_{H(\sigma,\Omega)} \leq \|u\|_{H(\sigma,\Omega)} + \|E(u)\|_{H^1(\Omega)^2}$$

and that

$$\|\tilde{u}\|_{H^1(\Omega)^2} \geq \|u\|_{H^1(\Omega)^2} - \|E(u)\|_{H^1(\Omega)^2}\,,$$

from (2.3) we get

$$\|u\|_{H^1(\Omega)^2} \leq \|u\|_{H(\sigma,\Omega)} + 2\,\|E(u)\|_{H^1(\Omega)^2}\,,$$

and using (2.1) we conclude the proof of (ii). □

Using Lemma 2.1 we can easily get estimates for the trace of the eigenfunctions of $\mathcal{H}_\tau$, once they are written component-wise. Given $\tau, \lambda \in \mathbb{R}$, by definition we have that $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ satisfies $\mathcal{H}_\tau \varphi = \lambda \varphi$ if and only if $\varphi \in H^1(\Omega)^4$ and

$$\begin{cases} \mathcal{H}\varphi &= \lambda\varphi & \text{in } \Omega, \\ \varphi &= i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

If we write $\varphi = (u, v)^\intercal$ with $u, v \in H^1(\Omega)^2$, and we recall that $\mathcal{H} = -i\alpha\cdot\nabla + m\beta$, then (2.4) is equivalent to

$$\begin{cases} -i\sigma\cdot\nabla v &= (\lambda - m)u & \text{in } \Omega, \\ -i\sigma\cdot\nabla u &= (\lambda + m)v & \text{in } \Omega, \\ v &= ie^\tau(\sigma\cdot\nu)u & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

In particular, since $(-i\sigma\cdot\nabla)^2 = -\Delta$, from the first two equations in (2.5) we deduce that

$$-\Delta u = (\lambda^2 - m^2)u \quad \text{and} \quad -\Delta v = (\lambda^2 - m^2)v \quad \text{in } \Omega. \tag{2.6}$$

In addition, combining (2.5) with Lemma 2.1 (i) we obtain the following result.

Lemma 2.2. *Let $u, v \in H^1(\Omega)^2$ be such that $\varphi = (u, v)^\intercal \in H^1(\Omega)^4$ solves (2.4). Then,*

$$\|u\|_{H^{-1/2}(\partial\Omega)^2} \le C\left(\|u\|_{L^2(\Omega)^2} + |\lambda + m|\|v\|_{L^2(\Omega)^2}\right),$$
$$\|v\|_{H^{-1/2}(\partial\Omega)^2} \le C\left(|\lambda - m|\|u\|_{L^2(\Omega)^2} + \|v\|_{L^2(\Omega)^2}\right)$$

for some $C > 0$ depending only on Ω.

Proof. Using Lemma 2.1 (i) we get $\|u\|_{H^{-1/2}(\partial\Omega)^2} \le C(\|u\|_{L^2(\Omega)^2} + \|\sigma\cdot\nabla u\|_{L^2(\Omega)^2})$. The equality $-i\sigma\cdot\nabla u = (\lambda + m)v$ from (2.5) yields the first estimate. The second one is analogous. □

We finish this section by giving the relation between the L^2-norm of u on $\partial\Omega$ and the L^2-norms of u and v on Ω. An analogous formula for the L^2-norm of v on $\partial\Omega$ can be obtained using the last equality in (2.5).

Lemma 2.3. *Let $u, v \in H^1(\Omega)^2$ such that $\varphi = (u, v)^\intercal \in H^1(\Omega)^4$ solves (2.4). Then,*

$$e^\tau\|u\|^2_{L^2(\partial\Omega)^2} = e^{-\tau}\|v\|^2_{L^2(\partial\Omega)^2} = (\lambda - m)\|u\|^2_{L^2(\Omega)^2} - (\lambda + m)\|v\|^2_{L^2(\Omega)^2}.$$

Proof. Note that $e^\tau\|u\|^2_{L^2(\partial\Omega)^2} = e^{-\tau}\|v\|^2_{L^2(\partial\Omega)^2}$ because $v = ie^\tau(\sigma\cdot\nu)u$ by (2.5). Now, if we multiply the first equation in (2.5) by $\overline{u}$, we integrate by parts in Ω, and we use the other two equations in (2.5), we get

$$\begin{aligned}(\lambda - m)\int_\Omega |u|^2 &= \int_\Omega (-i\sigma\cdot\nabla v)\cdot\overline{u} = \int_\Omega v\cdot\overline{(-i\sigma\cdot\nabla u)} - i\int_{\partial\Omega}(\sigma\cdot\nu)v\cdot\overline{u}\,d\sigma \\ &= (\lambda + m)\int_\Omega |v|^2 + e^\tau\int_{\partial\Omega}|u|^2\,d\sigma.\end{aligned}$$

□

2.2. **Boundary integral operators.** In this section we introduce boundary integral operators associated to $\mathcal{H}$. We will use them to rewrite the eigenvalue problem as an integral equation on $\partial\Omega$ (see Proposition 2.9 below), and to prove regularity estimates for the eigenfunctions of $\mathcal{H}_\tau$ that will be used in the following sections. The spectral analysis using boundary integral operators is an approach already used in [9, 10, 11]. It represents the Birman-Schwinger principle on our setting.

A fundamental solution of $\mathcal{H}-\lambda$ for $\lambda\in\mathbb{R}$ is given by

$$\phi_\lambda(x):=\frac{e^{-i\sqrt{\lambda^2-m^2}|x|}}{4\pi|x|}\left[\lambda+m\beta+\left(1+i\sqrt{\lambda^2-m^2}|x|\right)i\alpha\cdot\frac{x}{|x|^2}\right]\quad\text{for all } x\in\mathbb{R}^3\setminus\{0\}$$

if $\lambda\in(-\infty,-m)\cup(m,+\infty)$, and by

$$\phi_\lambda(x):=\frac{e^{-\sqrt{m^2-\lambda^2}|x|}}{4\pi|x|}\left[\lambda+m\beta+\left(1+\sqrt{m^2-\lambda^2}|x|\right)i\alpha\cdot\frac{x}{|x|^2}\right]\quad\text{for all } x\in\mathbb{R}^3\setminus\{0\}\qquad(2.7)$$

if $\lambda\in[-m,m]$. This means that $(\mathcal{H}-\lambda)\phi_\lambda=\delta_0 I_4$ in the sense of distributions, where δ_0 denotes the Dirac delta measure centered at the origin of coordinates. Therefore, given $g\in L^2(\partial\Omega)^4$, if we set

$$\Phi_\lambda g(x):=\int_{\partial\Omega}\phi_\lambda(x-y)g(y)\,d\sigma(y)\quad\text{for all } x\in\Omega,\qquad(2.8)$$

then $(\mathcal{H}-\lambda)\Phi_\lambda g=0$ in Ω. Reciprocally, null solutions of $\mathcal{H}-\lambda$ in Ω can be expressed by means of ϕ_λ, as the following result shows.

Lemma 2.4 (Reproducing formula)**.** *Let $\lambda\in\mathbb{R}$ and $\varphi\in H^1(\Omega)^4$ be such that $(\mathcal{H}-\lambda)\varphi=0$ in Ω. Then, $\varphi=\Phi_\lambda(i(\alpha\cdot\nu)\varphi)$ in Ω. That is,*

$$\varphi(x)=\int_{\partial\Omega}\phi_\lambda(x-y)i(\alpha\cdot\nu(y))\,\varphi(y)\,d\sigma(y)\quad\textit{for all } x\in\Omega.\qquad(2.9)$$

Proof. Note that if $\mathcal{H}\varphi=\lambda\varphi$ in Ω then $-\Delta\varphi=(\lambda^2-m^2)\varphi$ in Ω. By standard regularity theory, φ is infinitely differentiable in Ω. In particular, it makes sense to write "for all $x\in\Omega$" in (2.9).

The proof of the lemma follows the same lines as in [9, Lemma 3.3]: given $x\in\Omega$, one combines that ϕ_λ is a fundamental solution of $\mathcal{H}-\lambda$ with an integration by parts in $\Omega\setminus\{y\in\mathbb{R}^3:\,|x-y|<\epsilon\}$, and then one lets $\epsilon\downarrow 0$ to obtain (2.9). We omit the details. □

In the following lemma, the boundary trace of the function $\Phi_\lambda g$ given in (2.8) is described in terms of a singular integral operator, and a trace identity for null solutions of $\mathcal{H}-\lambda$ is deduced thanks to Lemma 2.4. This operator is defined, for every $g\in L^2(\partial\Omega)^4$, by

$$\mathscr{C}_\lambda g(x):=\lim_{\epsilon\downarrow 0}\int_{\{y\in\partial\Omega:\,|x-y|>\epsilon\}}\phi_\lambda(x-y)g(y)\,d\sigma(y)\quad\text{for } \sigma\text{-a.e. } x\in\partial\Omega.$$

The same arguments as in [9, Lemma 3.3] show that $\mathscr{C}_\lambda$ is bounded in $L^2(\partial\Omega)^4$. In addition, since $\mathscr{C}_\lambda$ is defined by means of the kernel (2.7) for $\lambda\in[-m,m]$, $\mathscr{C}_\lambda$ is self-adjoint in $L^2(\partial\Omega)^4$ for $\lambda\in[-m,m]$.

Lemma 2.5 (Plemelj-Sokhotski jump formula)**.** *Let $\lambda\in\mathbb{R}$. Then, for every $g\in L^2(\partial\Omega)^4$,*

$$\lim_{z\in\Omega,\,z\to x}\Phi_\lambda g(z)=-\frac{i}{2}(\alpha\cdot\nu(x))g(x)+\mathscr{C}_\lambda g(x)\quad\textit{for } \sigma\textit{-a.e. } x\in\partial\Omega,\qquad(2.10)$$

where the limit on the left-hand side of (2.10) *is taken nontangentially. As a consequence, if $\varphi\in H^1(\Omega)^4$ is such that $\mathcal{H}\varphi=\lambda\varphi$ in Ω, then $\varphi=2i\mathscr{C}_\lambda(\alpha\cdot\nu)\varphi$ σ-almost everywhere on $\partial\Omega$.*

Proof. The identity (2.10) is proved exactly as in [9, Lemma 3.3]. Now, let $\varphi\in H^1(\Omega)^4$ be such that $\mathcal{H}\varphi=\lambda\varphi$ in Ω. From Lemma 2.4 it follows that $\varphi=\Phi_\lambda(i(\alpha\cdot\nu)\varphi)$ in Ω. Taking the nontangential limit towards $\partial\Omega$, and using (2.10) and that $(\alpha\cdot\nu)^2=1$, we have

$$\varphi=-\frac{i}{2}(\alpha\cdot\nu)(i(\alpha\cdot\nu)\varphi)+i\mathscr{C}_\lambda(\alpha\cdot\nu)\varphi=\frac{1}{2}\varphi+i\mathscr{C}_\lambda(\alpha\cdot\nu)\varphi\quad\sigma\text{-a.e. on }\partial\Omega.$$

□

Remark 2.6. Note that the left-hand side of (2.10) concerns a pointwise nontangential limit. If $\Phi_\lambda g \in H^1(\Omega)$, this limit coincides σ-a.e. on $\partial\Omega$ with the standard trace of functions in Sobolev spaces defined through a trace operator; see [18, Lemma 3.1] for example.

In Proposition 2.9 below we will use Lemma 2.5 to rewrite the eigenvalue problem $\mathcal{H}_\tau\varphi = \lambda\varphi$ as an integral equation for φ on $\partial\Omega$. To do so, it is convenient to express the operator $\mathscr{C}_\lambda$ in terms of its action on the components of $\varphi \in L^2(\partial\Omega)^4$, namely,

$$\mathscr{C}_\lambda\varphi = \begin{pmatrix} (\lambda+m)K_\lambda & W_\lambda \\ W_\lambda & (\lambda-m)K_\lambda \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{where } \varphi = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } u, v \in L^2(\partial\Omega)^2. \tag{2.11}$$

The operators K_λ and W_λ are defined, for every $u \in L^2(\partial\Omega)^2$ and σ-a.e. $x \in \partial\Omega$, by

$$\begin{aligned} K_\lambda u(x) &:= \frac{1}{4\pi}\int_{\partial\Omega} k_\lambda(x-y)u(y)\,d\sigma(y), \\ W_\lambda u(x) &:= \lim_{\epsilon\downarrow 0}\frac{1}{4\pi}\int_{\{y\in\partial\Omega:\,|x-y|>\epsilon\}} w_\lambda(x-y)u(y)\,d\sigma(y), \end{aligned} \tag{2.12}$$

where the kernels k_λ and w_λ are given, for $x \in \mathbb{R}^3 \setminus \{0\}$, by

$$k_\lambda(x) := \frac{e^{-i\sqrt{\lambda^2-m^2}|x|}}{|x|}, \quad w_\lambda(x) := e^{-i\sqrt{\lambda^2-m^2}|x|}\left(1 + i\sqrt{\lambda^2-m^2}|x|\right) i\sigma\cdot\frac{x}{|x|^3} \tag{2.13}$$

if $\lambda \in (-\infty,-m)\cup(m,+\infty)$, and by (2.13) replacing $i\sqrt{\lambda^2-m^2}$ by $\sqrt{m^2-\lambda^2}$ if $\lambda \in [-m,m]$. The fact that $\mathscr{C}_\lambda$ is bounded in $L^2(\partial\Omega)^4$ yields the boundedness of K_λ and W_λ in $L^2(\partial\Omega)^2$. In addition, since $\mathscr{C}_\lambda$ is self-adjoint in $L^2(\partial\Omega)^4$ for $\lambda \in [-m,m]$, we deduce that K_λ and W_λ are self-adjoint in $L^2(\partial\Omega)^2$ for $\lambda \in [-m,m]$.

The combination of Lemmas 2.4 and 2.5 leads to the important identity $(\mathscr{C}_\lambda(\alpha\cdot\nu))^2 = -1/4$ as operators in $L^2(\partial\Omega)^4$. This, apart from showing that $(\mathscr{C}_\lambda)^{-1} = -4(\alpha\cdot\nu)\mathscr{C}_\lambda(\alpha\cdot\nu)$, has some consequences on K_λ and W_λ that will be used in the proof of Lemma 4.13. We gather them in the following result.

Lemma 2.7. *For every $\lambda \in \mathbb{R}$, the following holds:*

(i) $(\mathscr{C}_\lambda(\alpha\cdot\nu))^2 = -1/4$ *as operators in* $L^2(\partial\Omega)^4$.
(ii) $(W_\lambda(\sigma\cdot\nu))^2 + (\lambda^2-m^2)(K_\lambda(\sigma\cdot\nu))^2 = -1/4$ *as operators in* $L^2(\partial\Omega)^2$.
(iii) $\{K_\lambda(\sigma\cdot\nu), W_\lambda(\sigma\cdot\nu)\} = 0$ *as operators in* $L^2(\partial\Omega)^2$.

Proof. Let us show *(i)*. Given $g \in C^\infty(\partial\Omega)^4$, set $\varphi = \Phi_\lambda g$. Then, $\varphi \in H^1(\Omega)$ and $(H-\lambda)\varphi = 0$ in Ω. Thus, by Lemma 2.4 and (2.10) we get that

$$\Phi_\lambda g = \varphi = \Phi_\lambda(i(\alpha\cdot\nu)\varphi) = \Phi_\lambda\Big(i(\alpha\cdot\nu)\big(-\tfrac{i}{2}(\alpha\cdot\nu)g + \mathscr{C}_\lambda g\big)\Big) = \Phi_\lambda\Big(\big(\tfrac{1}{2} + i(\alpha\cdot\nu)\mathscr{C}_\lambda\big)g\Big)$$

in Ω. Taking traces on $\partial\Omega$ and using once again (2.10), we deduce that

$$\begin{aligned} \big(-\tfrac{i}{2}(\alpha\cdot\nu) + \mathscr{C}_\lambda\big)g &= \big(-\tfrac{i}{2}(\alpha\cdot\nu) + \mathscr{C}_\lambda\big)\big(\tfrac{1}{2} + i(\alpha\cdot\nu)\mathscr{C}_\lambda\big)g \\ &= \big(-\tfrac{i}{4}(\alpha\cdot\nu) + \mathscr{C}_\lambda + i\mathscr{C}_\lambda(\alpha\cdot\nu)\mathscr{C}_\lambda\big)g. \end{aligned}$$

From here, and by the density of $C^\infty(\partial\Omega)^4$ in $L^2(\partial\Omega)^4$, we get $-\frac{1}{4}(\alpha\cdot\nu) = \mathscr{C}_\lambda(\alpha\cdot\nu)\mathscr{C}_\lambda$, which proves *(i)*. Then, *(ii)* and *(iii)* follow by *(i)* writing $\mathscr{C}_\lambda$ as in (2.11), in terms of K_λ and W_λ. □

Let us now establish a regularity result that will be a key tool in the compactness arguments carried out in Section 4.1. It concerns regularity estimates for the operators K_λ and $\{W_\lambda, \sigma\cdot\nu\}$.

Lemma 2.8. *For every $\lambda \in \mathbb{R}$ there exists $C_\lambda > 0$ depending only on Ω and $\lambda^2 - m^2$ such that*

$$\|K_\lambda u\|_{H^{1/2}(\partial\Omega)^2} \leq C_\lambda \|u\|_{H^{-1/2}(\partial\Omega)^2}$$

and

$$\|\{W_\lambda, \sigma\cdot\nu\}u\|_{H^{1/2}(\partial\Omega)^2} \leq C_\lambda \|u\|_{H^{-1/2}(\partial\Omega)^2}$$

for all $u \in H^{-1/2}(\partial\Omega)^2$. The constant C_λ can be chosen in such a way that for every $C_\circ > 0$ there exists $C' > 0$ depending only on Ω and $C_\circ$ such that $C_\lambda < C'$ for all $\lambda \in \mathbb{R}$ with $|\lambda^2 - m^2| < C_\circ$.

Proof. The result essentially follows from the fact that both operators K_λ and $\{W_\lambda, \sigma\cdot\nu\}$ are integral operators given by kernels whose singularity is comparable to $1/|x|$, and which are regular enough far from the singularity. The proof is similar to that of [64, Proposition 2.8], and thus we will omit some details. In particular, we will only address the proof of the proposition for $|\lambda| \geq m$; the proof for $|\lambda| < m$ follows by similar arguments. Through the proof we will make use of the Sobolev space $H^1(\partial\Omega)^2$ as well as its (continuous) dual $H^{-1}(\partial\Omega)^2$, which are defined in the standard way thanks to the fact that Ω has a C^2 boundary.

Given $b \in \mathbb{R}$, consider the bounded operators in $L^2(\partial\Omega)^2$ defined, for $u \in L^2(\partial\Omega)^2$, by

$$\mathcal{T}_b u(x) := \int_{\partial\Omega} T_b(x-y)u(y)\,d\sigma(y) \quad \text{with} \quad T_b(x) := \frac{e^{-ib|x|}}{|x|},$$

and

$$\mathcal{S}_b u(x) := \lim_{\epsilon\downarrow 0} \int_{\{y\in\partial\Omega:\,|x-y|>\epsilon\}} S_b(x-y)u(y)\,d\sigma(y) \quad \text{with} \quad S_b(x) := e^{-ib|x|}(1+ib|x|)i\sigma\cdot\frac{x}{|x|^3}.$$

It is easy to check that $\overline{T_b(-x)} = (T_{-b}(x))^\intercal$ and $\overline{S_b(-x)} = (S_{-b}(x))^\intercal$. Thus,

$$(\mathcal{T}_b)^* = \mathcal{T}_{-b} \quad \text{and} \quad (\mathcal{S}_b)^* = \mathcal{S}_{-b}. \tag{2.14}$$

Let us also define $\mathcal{A}_b := \{\mathcal{S}_b, \sigma\cdot\nu\}$. By (2.14), we have

$$(\mathcal{A}_b)^* = \mathcal{A}_{-b}. \tag{2.15}$$

In addition, $\mathcal{A}_b$ can be written as the integral operator

$$\mathcal{A}_b u(x) = \int_{\partial\Omega} A_b(x,y)u(y)\,d\sigma(y)$$

with kernel

$$A_b(x,y) := e^{-ib|x-y|}(1+ib|x-y|)i\Big[(\sigma\cdot\nu(x))\Big(\sigma\cdot\frac{x-y}{|x-y|^3}\Big) + \Big(\sigma\cdot\frac{x-y}{|x-y|^3}\Big)(\sigma\cdot\nu(y))\Big].$$

As we will see from (2.16) below, the singularity of the kernel A_b does not require to write the integral operator $\mathcal{A}_b$ as an integral in the principal value sense.

Note that the operators $\mathcal{T}_b$ and $\mathcal{A}_b$ coincide, respectively, with K_λ and $\{W_\lambda, \sigma\cdot\nu\}$ (up to a multiplicative constant) for $b = \sqrt{\lambda^2 - m^2}$. We will establish the regularity estimates for $\mathcal{T}_b$ and $\mathcal{A}_b$, since the notation is slightly more convenient thanks to (2.14) and (2.15).

First, let us prove that $\mathcal{T}_b$ and $\mathcal{A}_b$ are bounded operators from $L^2(\partial\Omega)^2$ into $H^1(\partial\Omega)^2$ for every $b \in \mathbb{R}$, with a norm uniformly controlled for all b such that $|b|^2 < C_\circ$. To prove it for $\mathcal{T}_b$, note that its kernel T_b can be written as

$$T_b(x-y) = \frac{1}{|x-y|} + h_b(x-y)$$

for some smooth function h_b. This entails that the kernel T_b is pseudo-homogeneous of class -1 in the sense of [62, §4.3.3], and thus [62, Theorem 4.3.2] yields that $\mathcal{T}_b$ is bounded from $L^2(\partial\Omega)^2$ into $H^1(\partial\Omega)^2$. Note that the dependence on b in the norm of the operator comes from the kernel h_b, which has been obtained after considering the Taylor expansion of $e^{ib|x-y|}$. Therefore, it follows readily that it can be uniformly controlled for all b such that $|b|^2 < C_\circ$. The result for $\mathcal{A}_b$ is obtained using the same argument, provided that we show that

$$\widetilde{A}_b(x,y) := (\sigma\cdot\nu(x))\Big(\sigma\cdot\frac{x-y}{|x-y|^3}\Big) + \Big(\sigma\cdot\frac{x-y}{|x-y|^3}\Big)(\sigma\cdot\nu(y)) = O(|x-y|^{-1}) \quad \text{as } |x-y|\to 0. \tag{2.16}$$

To see this, note first that, for every $p, q \in \mathbb{R}^3$,

$$(\sigma\cdot p)(\sigma\cdot q) = p\cdot q + i\sigma\cdot(p\times q) = -q\cdot p - i\sigma\cdot(q\times p) + 2p\cdot q = -(\sigma\cdot q)(\sigma\cdot p) + 2p\cdot q.$$

Thus

$$(\sigma\cdot\nu(x))(\sigma\cdot(x-y)) = -(\sigma\cdot(x-y))(\sigma\cdot\nu(x)) + 2\nu(x)\cdot(x-y),$$

which yields

$$\widetilde{A}_b(x,y) = \Big(\sigma\cdot\frac{x-y}{|x-y|^3}\Big)\big(\sigma\cdot(\nu(y)-\nu(x))\big) + 2\nu(x)\cdot\frac{x-y}{|x-y|^3}.$$

Then, (2.16) follows by using that $|\nu(x)-\nu(y)| = O(|x-y|)$ and $|\nu(x)\cdot(x-y)| = O(|x-y|^2)$ because $\partial\Omega$ is of class C^2; see [62, Example 4.5] or [41, Lemma 3.15].

Finally, since $\mathcal{T}_b$ and $\mathcal{A}_b$ are linear bounded operators from $L^2(\partial\Omega)^2$ into $H^1(\partial\Omega)^2$ for every $b\in\mathbb{R}$, by (2.14), (2.15), and duality it follows that $\mathcal{T}_b$ and $\mathcal{A}_b$ are also bounded from $H^{-1}(\partial\Omega)^2$ into $L^2(\partial\Omega)^2$, with the same norm. From this, we obtain the desired estimates by using classical interpolation results; see for instance [69, Propositions 2.1.62 and 2.4.3]. □

We finish this section by showing a reformulation of the eigenvalue equation

$$\mathcal{H}_\tau\varphi = \lambda\varphi, \quad \varphi\in\mathrm{Dom}(\mathcal{H}_\tau)$$

in terms of boundary integral equations which involve the components of the trace of φ on $\partial\Omega$ and the operators K_λ and W_λ introduced in (2.12). Despite that φ belongs to $H^1(\Omega)^4$ by assumption, and thus the trace of its components belong to $H^{1/2}(\partial\Omega)^2$, we will see that it is equivalent to pose the corresponding boundary integral equations in the larger space $L^2(\partial\Omega)^2$. This is because they indeed force the solution to belong to $H^{1/2}(\partial\Omega)^2$ thanks to Lemma 2.8, as we will see.

Proposition 2.9. *Let $\tau,\lambda\in\mathbb{R}$, let $\varphi:\Omega\to\mathbb{C}^4$, and let Φ_λ be as in* (2.8)*. Then, the following are equivalent:*

(i) $\varphi\in\mathrm{Dom}(\mathcal{H}_\tau)$ *and* $\mathcal{H}_\tau\varphi=\lambda\varphi$.
(ii) $\varphi = \Phi_\lambda(i(\alpha\cdot\nu)g)$ *in* Ω, *where*

$$g = \begin{pmatrix} u \\ ie^\tau(\sigma\cdot\nu)u\end{pmatrix} \tag{2.17}$$

for some $u\in L^2(\partial\Omega)^2$ *such that*

$$\begin{aligned} u &= \big(2iW_\lambda(\sigma\cdot\nu) - 2(\lambda+m)e^\tau K_\lambda\big)u \quad \text{and}\\ u &= \big(2i(\sigma\cdot\nu)W_\lambda + 2(\lambda-m)e^{-\tau}(\sigma\cdot\nu)K_\lambda(\sigma\cdot\nu)\big)u \end{aligned} \tag{2.18}$$

in $L^2(\partial\Omega)^2$.

In addition, if φ is as in (ii), then $\varphi = g$ in $L^2(\partial\Omega)^4$.

To prove Proposition 2.9 we will use the following regularity result, which will be also used in the compactness arguments carried out in Sections 3 and 4.1. It will be specially useful when studying the asymptotic behavior of the eigenvalue curves as $\tau \to \pm\infty$, since a precise control over the dependence on τ in the constants involved in the estimates is needed.

Lemma 2.10. *Let $u \in L^2(\partial\Omega)^2$ be a solution to (2.18), and set $v = ie^{\tau}(\sigma\cdot\nu)u$. Then,*

$$u = \big(i\{W_\lambda, \sigma\cdot\nu\} - (\lambda+m)e^{\tau}K_\lambda + (\lambda-m)e^{-\tau}(\sigma\cdot\nu)K_\lambda(\sigma\cdot\nu)\big)u \quad \text{and} \tag{2.19}$$

$$v = \big(i\{W_\lambda, \sigma\cdot\nu\} - (\lambda+m)e^{\tau}(\sigma\cdot\nu)K_\lambda(\sigma\cdot\nu) + (\lambda-m)e^{-\tau}K_\lambda\big)v \tag{2.20}$$

σ-a.e. on $\partial\Omega$. As a consequence, $u, v \in H^{1/2}(\partial\Omega)^2$, and

$$\begin{aligned} \|u\|_{H^{1/2}(\partial\Omega)^2} &\leq C(1+|\lambda+m|e^{\tau}+|\lambda-m|e^{-\tau})C_\lambda \|u\|_{H^{-1/2}(\partial\Omega)^2}, \\ \|v\|_{H^{1/2}(\partial\Omega)^2} &\leq C(1+|\lambda+m|e^{\tau}+|\lambda-m|e^{-\tau})C_\lambda \|v\|_{H^{-1/2}(\partial\Omega)^2}, \end{aligned} \tag{2.21}$$

where $C>0$ only depends on Ω, and C_λ is as in Lemma 2.8.

Proof. By adding the two equations from (2.18) we directly get (2.19) in the $L^2(\partial\Omega)^2$ sense, and therefore also σ-a.e. on $\partial\Omega$. Then, plugging $v = ie^{\tau}(\sigma\cdot\nu)u$ in (2.19) we obtain (2.20).

We now prove that (2.19) yields $u \in H^{1/2}(\partial\Omega)^2$ and the estimate for $\|u\|_{H^{1/2}(\partial\Omega)^2}$; the proof for v is analogous. Using that ν is of class C^1 on $\partial\Omega$ (which gives that $\sigma\cdot\nu$ is a bounded operator in $H^{\pm1/2}(\partial\Omega)^2$; see [21, Lemma A.2] for the case $H^{1/2}(\partial\Omega)^2$) and Lemma 2.8, we have

$$\begin{aligned} \|u\|_{H^{1/2}(\partial\Omega)^2} &\leq \|\{W_\lambda,\sigma\cdot\nu\}u\|_{H^{1/2}(\partial\Omega)^2} + |\lambda+m|e^{\tau}\|K_\lambda u\|_{H^{1/2}(\partial\Omega)^2} \\ &\quad + C|\lambda-m|e^{-\tau}\|K_\lambda(\sigma\cdot\nu)u\|_{H^{1/2}(\partial\Omega)^2} \\ &\leq C_\lambda\|u\|_{H^{-1/2}(\partial\Omega)^2} + |\lambda+m|e^{\tau}C_\lambda\|u\|_{H^{-1/2}(\partial\Omega)^2} \\ &\quad + C|\lambda-m|e^{-\tau}C_\lambda\|(\sigma\cdot\nu)u\|_{H^{-1/2}(\partial\Omega)^2}. \end{aligned}$$

□

Using the previous result we can now establish Proposition 2.9.

Proof of Proposition 2.9. We first show that (i) implies (ii). Let $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ such that $\mathcal{H}_\tau\varphi = \lambda\varphi$. Then $\varphi \in H^1(\Omega)^4 \subset H^{1/2}(\partial\Omega)^4 \subset L^2(\partial\Omega)^4$, $\varphi = \Phi_\lambda(i(\alpha\cdot\nu)\varphi)$ in Ω by Lemma 2.4, and $\varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi$ on $\partial\Omega$. Writing this boundary equation in terms of the components of φ, we get

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} := \varphi &= i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi \\ &= i\begin{pmatrix} -e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix}\begin{pmatrix} 0 & \sigma\cdot\nu \\ \sigma\cdot\nu & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = i\begin{pmatrix} -e^{-\tau}(\sigma\cdot\nu)v \\ e^{\tau}(\sigma\cdot\nu)u \end{pmatrix}. \end{aligned} \tag{2.22}$$

This shows (2.17). Hence, it only remains to prove that u satisfies (2.18). Thanks to Lemma 2.5, we have $\varphi = 2i\mathscr{C}_\lambda(\alpha\cdot\nu)\varphi$ on $\partial\Omega$. Writing this equation in terms of u and v using (2.11) we get

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= 2i\begin{pmatrix} (\lambda+m)K_\lambda & W_\lambda \\ W_\lambda & (\lambda-m)K_\lambda \end{pmatrix}\begin{pmatrix} 0 & \sigma\cdot\nu \\ \sigma\cdot\nu & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= 2i\begin{pmatrix} (\lambda+m)K_\lambda(\sigma\cdot\nu)v + W_\lambda(\sigma\cdot\nu)u \\ W_\lambda(\sigma\cdot\nu)v + (\lambda-m)K_\lambda(\sigma\cdot\nu)u \end{pmatrix} = \begin{pmatrix} -2(\lambda+m)e^{\tau}K_\lambda u + 2i\,W_\lambda(\sigma\cdot\nu)u \\ -2e^{\tau}W_\lambda u + 2i(\lambda-m)K_\lambda(\sigma\cdot\nu)u \end{pmatrix}, \end{aligned} \tag{2.23}$$

where in this last equality we used the identity $v = ie^{\tau}(\sigma\cdot\nu)u$ proved in (2.22). Therefore, the first equation in (2.18) is established. To prove the second one we simply have to multiply the equation

$$ie^{\tau}(\sigma\cdot\nu)\,u = v = -2e^{\tau}W_\lambda u + 2i(\lambda - m)K_\lambda(\sigma\cdot\nu)u$$

by $-ie^{-\tau}(\sigma\cdot\nu)$ from the left.

We now show that (ii) implies (i). Let $u \in L^2(\partial\Omega)^2$ be a solution to (2.18), set $v = ie^{\tau}(\sigma\cdot\nu)u$, g as in (2.17), and define $\varphi = \Phi_\lambda(i(\alpha\cdot\nu)g)$ in Ω. We want to prove that $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ and that $\mathcal{H}_\tau\varphi = \lambda\varphi$. Note that the eigenvalue equation $\mathcal{H}_\tau\varphi = \lambda\varphi$ is automatically satisfied once we know that $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$, since we have set $\varphi = \Phi_\lambda(i(\alpha\cdot\nu)g)$ and ϕ_λ is a fundamental solution of $\mathcal{H} - \lambda$. Therefore, it is enough to prove that $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$. For this purpose, let us first show that $\varphi = g$ on $\partial\Omega$, which is the last statement of the proposition. Since u solves (2.18) and we have set $v = ie^{\tau}(\sigma\cdot\nu)u$, (2.23) yields $g = 2i\mathscr{C}_\lambda(\alpha\cdot\nu)g$ in $L^2(\partial\Omega)^4$. Then, if we take traces on both sides of the identity $\varphi = \Phi_\lambda(i(\alpha\cdot\nu)g)$, Lemma 2.5 gives

$$\varphi = -\frac{i}{2}(\alpha\cdot\nu)i(\alpha\cdot\nu)g + \mathscr{C}_\lambda i(\alpha\cdot\nu)g = \frac{1}{2}g + \mathscr{C}_\lambda i(\alpha\cdot\nu)g = \frac{1}{2}g + \frac{1}{2}g = g \quad \text{on } \partial\Omega,$$

as desired. At this point, that φ solves the boundary equation $\varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi$ on $\partial\Omega$ is straightforward arguing as in (2.22).

It only remains to prove that $\varphi \in H^1(\Omega)^4$. We already know that $\varphi = g = (u, v)^{\intercal}$ on $\partial\Omega$. Abusing notation, let us also denote by u and v the components of φ as functions defined on Ω. It is known that Φ_λ is a bounded operator from $L^2(\partial\Omega)^4$ into $L^2(\Omega)^4$; one can argue as in the proof of [9, Lemma 2.1] but using that Ω is bounded instead of the exponential decay assumed there. Hence, $\varphi = \Phi_\lambda(i(\alpha\cdot\nu)g) \in L^2(\Omega)^4$, which means that $u, v \in L^2(\Omega)^2$. Then, using the equation $\mathcal{H}\varphi = \lambda\varphi$ once written component-wise as in (2.5), we deduce that $\sigma\cdot\nabla u, \sigma\cdot\nabla v \in L^2(\Omega)^2$. Note also that $u, v \in H^{1/2}(\partial\Omega)^2$ by Lemma 2.10. Therefore, thanks to Lemma 2.1 (ii), we conclude that $u, v \in H^1(\Omega)^2$, that is, $\varphi \in H^1(\Omega)^4$. □

3. Parametrization and monotonicity

In this section we study the dependence of the eigenvalues of $\mathcal{H}_\tau$, and the associated eigenfunctions, on the parameter τ, establishing Theorem 1.3. Note that even if the dependence on τ in $\mathcal{H}_\tau$ is expressed through the smooth functions $\tau \mapsto \sinh\tau$ and $\tau \mapsto \cosh\tau$, the proof of Theorem 1.3 is not straightforward and requires some ingredients. First, we will prove that the eigenvalues can be parametrized locally; see Lemma 3.1 below. Once this is done, we will show that the local parametrization can be extended to a global one. To accomplish this, we obtain an explicit formula for the derivative of the eigenvalues with respect to τ (see Lemma 3.3). It will provide a growth estimate for the eigenvalue curves which will be crucial to eventually establish Theorem 1.3.

Let us first prove that the eigenvalues of the operator $\mathcal{H}_\tau$ can be parametrized locally. The dependence of $\mathcal{H}_\tau$ on τ appears in the boundary conditions, and hence, in the domain of definition of the operator. Therefore, we cannot apply directly the very extensive theory of [52] devoted to the so-called holomorphic families of operators of type (A) —in which the domain of the operators is independent of the parameter. The usual strategy in these cases consists of passing to the weak formulation of the problem, constructing the associated bilinear form in which one of its terms account for the boundary condition —and thus the domain of definition may be independent of the parameter. However, this method does not work in our context: the

theory developed in [52] is devoted to sectorial operators and this is not our case (the spectrum of $\mathcal{H}_\tau$ cannot be included into a sector of the complex plane, since it accumulates at $\pm\infty$).

Despite these difficulties, we can still use some of the theory developed in [52], since it can be shown that the resolvent of $\mathcal{H}_\tau$ is analytic in $\tau \in \mathbb{R}$ (using crucially a Krein-type resolvent formula from [21]). Once this is proved, it readily follows that the eigenvalues can be locally parametrized, as stated in the following result.

Lemma 3.1. *Let $\tau_0 \in \mathbb{R}$ and let $\lambda_\star$ be an eigenvalue of $\mathcal{H}_{\tau_0}$ with multiplicity $\mu_\star$. Then, $\lambda_\star$ splits into one or several eigenvalue curves $\lambda_k(\tau)$ (with $k = 1, \dots, k_\star$ for some $k_\star \leq \mu^\star$) which are real analytic close to $\tau = \tau_0$, and such that $\lambda_k(\tau) \in \sigma(\mathcal{H}_\tau)$. Each of these eigenvalues has constant multiplicity μ_k and the sum of all the multiplicities is equal to $\mu_\star$. Moreover, for each k there exist μ_k linearly independent eigenfunctions $\varphi_k^j(\tau)$ associated to $\lambda_k(\tau)$ which depend analytically on τ and such that $\{\varphi_k^j(\tau_0) : k = 1, \dots, k_\star,\ j = 1, \dots, \mu_k\}$ forms an orthogonal basis of the eigenspace associated to $\lambda_\star$.*

Proof. Although in the rest of the article we will consider $\tau \in \mathbb{R}$, in this proof we may assume that $\tau \in \mathbb{C}$, since the results from perturbation theory that we will use are more naturally presented in this setting. Then, the result will follow restricting τ to $\mathbb{R}$.

We claim that for $\zeta \in \mathbb{C}\setminus\mathbb{R}$, the resolvent operator $(\mathcal{H}_\tau - \zeta)^{-1}$ is bounded and holomorphic in τ in a neighborhood of τ_0. To prove this claim, we will use a resolvent formula from [21] which we recall now. For $\zeta \in \mathbb{C} \setminus \mathbb{R}$, consider the γ-field $\gamma(\zeta)$ and the Weyl function $M(\zeta)$ associated to the quasi boundary triple introduced in [21] for $\mathcal{H}$ acting on $H^1(\Omega)^4$. These operators are defined in terms of Φ_λ and $\mathscr{C}_\lambda$ from Section 2.2 taking $\lambda = \zeta \in \mathbb{C} \setminus \mathbb{R}$. Their explicit expression can be found in [21, Proposition 4.2], but for our purposes it suffices to use that $\gamma(\zeta) : \mathcal{G}_\Omega^{1/2} \to H^1(\Omega)^4$, $M(\zeta) : \mathcal{G}_\Omega^{1/2} \to \mathcal{G}_\Omega^{1/2}$, and $\gamma(\overline{\zeta})^* : L^2(\Omega)^4 \to \mathcal{G}_\Omega^{1/2}$ are bounded operators, where $\mathcal{G}_\Omega^{1/2} := \big(1 + i\beta(\alpha\cdot\nu)\big)\big(H^{1/2}(\partial\Omega)^4\big)$. From [21, Theorem 5.9 and Proposition 5.15], for every $\zeta \in \rho(\mathcal{H}_\tau) \cap \rho(\mathcal{H}_0)$ the following resolvent formula holds:

$$(\mathcal{H}_\tau - \zeta)^{-1} = \big(\mathcal{H}_0 - \zeta\big)^{-1} + \gamma(\zeta)\big(I_4 - \omega(\tau)M(\zeta)\big)^{-1}\omega(\tau)\gamma(\overline{\zeta})^*, \quad \text{where } \ \omega(\tau) := -\frac{\sinh\tau}{1+\cosh\tau}.$$

Note that ω is holomorphic in τ wherever the denominator does not vanish. From this and using that $\big(\mathcal{H}_0 - \zeta\big)^{-1}$, $\gamma(\zeta)$, and $\gamma(\overline{\zeta})^*$ are bounded, the holomorphy of $(\mathcal{H}_\tau - \zeta)^{-1}$ will follow if we prove that $(I_4 - \omega(\tau)M(\zeta))^{-1}$ is a bounded operator holomorphic in τ. From [21, Lemma 4.4] it follows that, for every $\omega_0 \in \mathbb{R}$ such that $|\omega_0| \neq 1$ and for every $\zeta \in \mathbb{C} \setminus \mathbb{R}$, the operator $I_4 - \omega_0 M(\zeta)$ has a bounded and everywhere defined inverse in $\mathcal{G}_\Omega^{1/2}$. Therefore, using this last assertion with $\omega_0 := \omega(\tau_0) \in \mathbb{R}$ (note that $|\omega(\tau_0)| < 1$ for every $\tau_0 \in \mathbb{R}$) together with the identity

$$I_4 - \omega(\tau)M(\zeta) = \Big(I_4 - \big(\omega(\tau) - \omega(\tau_0)\big)M(\zeta)\big(I_4 - \omega(\tau_0)M(\zeta)\big)^{-1}\Big)\big(I_4 - \omega(\tau_0)M(\zeta)\big),$$

and a Neumann series argument, it follows that $(I_4 - \omega(\tau)M(\zeta))^{-1}$ is a bounded operator holomorphic in τ for $\tau \in \mathbb{C}$ close enough to $\tau_0 \in \mathbb{R}$; see the comments in [52, VII-§1.1] for more details. Hence, the claim is proved.

Once we know that the resolvent of $\mathcal{H}_\tau$ is bounded and holomorphic[1] in τ, by classical results in perturbation theory [52, Theorems VII.1.3 and VII.1.8] it follows that every isolated eigenvalue $\lambda_\star$ of $\mathcal{H}_{\tau_0}$ splits into one or several eigenvalues $\lambda_k(\tau)$ of $\mathcal{H}_\tau$ which depend holomorphically

[1]From the proof of [52, Theorem VII.1.3], it is enough to have $(\mathcal{H}_\tau - \zeta)^{-1}$ holomorphic in τ for a single $\zeta \in \rho(\mathcal{H}_0) \cap \rho(\mathcal{H}_\tau)$, thus we can take any $\zeta \in \mathbb{C} \setminus \mathbb{R}$.

on τ (for τ close to τ_0), and the same is true for the corresponding eigenprojections. As a consequence, if μ_k is the multiplicity of each eigenvalue $\lambda_k(\tau)$, there exist μ_k linearly independent associated eigenfunctions $\varphi_k^j(\tau)$, with $j = 1, \dots, \mu_k$, which depend holomorphically on τ; see [52, II-§4] for more details on this. Note that μ_k is constant for τ close to τ_0, and the sum of all these multiplicities is equal to the multiplicity of $\lambda_\star$. Moreover, all the functions $\varphi_k^j(\tau_0)$ form an orthogonal basis of the eigenspace associated to $\lambda_\star$. □

Remark 3.2. If $\lambda_\star$ is an eigenvalue of $\mathcal{H}_{\tau_0}$ with multiplicity $\mu_\star$, then using Lemma 3.1 we can define $\mu_\star$ eigenvalue curves (for τ close to τ_0), each one associated to a single eigenfunction. Some of these eigenvalue curves may be equal, accounting for the multiplicity of the corresponding eigenvalue of $\mathcal{H}_\tau$. This setting will be the appropriate one to establish Theorem 1.3 below.

With a local parametrization of the eigenvalue curves at hand, our next result shows their monotonicity with respect to the parameter τ. In addition, we also provide an explicit expression for the derivative of the eigenvalues with respect to τ. Note that the monotone behavior was expected if one looks at the curves plotted in Figure 1 for the case of a ball.

Lemma 3.3. *Let $\tau \mapsto \lambda(\tau) \in \sigma(\mathcal{H}_\tau)$ and $\tau \mapsto \varphi_\tau \in \mathrm{Dom}(\mathcal{H}_\tau) \setminus \{0\}$ be differentiable functions on an interval $I \subset \mathbb{R}$, with φ_τ such that $\mathcal{H}_\tau \varphi_\tau = \lambda(\tau)\varphi_\tau$ for all $\tau \in I$. Then*

$$\lambda'(\tau) = e^\tau \frac{\|u_\tau\|^2_{L^2(\partial\Omega)^2}}{\|\varphi_\tau\|^2_{L^2(\Omega)^4}} > 0 \quad \text{for all } \tau \in I, \tag{3.1}$$

where $\varphi_\tau = (u_\tau, v_\tau)^\intercal$ with $u_\tau, v_\tau \in H^1(\Omega)^2$. In particular, λ is strictly increasing on I. Moreover,

$$\lambda'(\tau) = (\lambda(\tau) - m)\frac{\|u_\tau\|^2_{L^2(\Omega)^2}}{\|\varphi_\tau\|^2_{L^2(\Omega)^4}} - (\lambda(\tau) + m)\frac{\|v_\tau\|^2_{L^2(\Omega)^2}}{\|\varphi_\tau\|^2_{L^2(\Omega)^4}}. \tag{3.2}$$

Proof. For simplicity of notation, in the following we will write φ and φ' instead of φ_τ and $\partial_\tau \varphi_\tau$, respectively, and analogously for u_τ and v_τ. Differentiating the equation $\mathcal{H}\varphi = \lambda\varphi$ with respect to τ, we get $\mathcal{H}\varphi' = \lambda'\varphi + \lambda\varphi'$. Multiplying this equation by $\overline{\varphi}$ and integrating by parts in Ω we obtain

$$\begin{aligned} \lambda' \int_\Omega |\varphi|^2 + \lambda \int_\Omega \varphi' \cdot \overline{\varphi} = \int_\Omega \mathcal{H}\varphi' \cdot \overline{\varphi} &= \int_\Omega \varphi' \cdot \overline{\mathcal{H}\varphi} - i \int_{\partial\Omega} (\alpha \cdot \nu)\varphi' \cdot \overline{\varphi}\, d\sigma \\ &= \lambda \int_\Omega \varphi' \cdot \overline{\varphi} + \int_{\partial\Omega} \varphi' \cdot \overline{i(\alpha \cdot \nu)\varphi}\, d\sigma. \end{aligned} \tag{3.3}$$

Here we used that λ is real-valued by Lemma 1.2, and that $(\alpha \cdot \nu)$ is hermitian. Thus, from (3.3) we deduce that

$$\lambda' = \|\varphi\|^{-2}_{L^2(\Omega)^4} \int_{\partial\Omega} \varphi' \cdot \overline{i(\alpha \cdot \nu)\varphi}\, d\sigma. \tag{3.4}$$

Now, using the boundary condition $(\alpha \cdot \nu)\varphi = i(\sinh\tau + \cosh\tau\, \beta)\varphi$ given by the fact that $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ (recall that $(\alpha \cdot \nu)^2 = 1$ and that $\alpha_j \beta = -\beta\alpha_j$ for $j = 1, 2, 3$), we have

$$\begin{aligned} \int_{\partial\Omega} \varphi' \cdot \overline{i(\alpha \cdot \nu)\varphi}\, d\sigma &= -\int_{\partial\Omega} \varphi' \cdot (\sinh\tau + \cosh\tau\, \beta) \cdot \overline{\varphi}\, d\sigma \\ &= -e^\tau \int_{\partial\Omega} u' \cdot \overline{u}\, d\sigma + e^{-\tau} \int_{\partial\Omega} v' \cdot \overline{v}\, d\sigma. \end{aligned} \tag{3.5}$$

Let us rewrite the last term on the right-hand side in terms of u. Recall that the boundary condition defining $\mathrm{Dom}(\mathcal{H}_\tau)$ is expressed component-wise as $v = ie^\tau(\sigma\cdot\nu)u$. If we differentiate this identity with respect to τ, we get

$$v' = ie^\tau(\sigma\cdot\nu)(u+u'),$$

and thus

$$e^{-\tau}v'\cdot\overline{v} = e^\tau(\sigma\cdot\nu)(u+u')\cdot\overline{(\sigma\cdot\nu)u} = e^\tau(|u|^2 + u'\cdot\overline{u}).$$

Therefore,

$$e^{-\tau}\int_{\partial\Omega} v'\cdot\overline{v}\,d\sigma = e^\tau\|u\|^2_{L^2(\partial\Omega)^2} + e^\tau\int_{\partial\Omega} u'\cdot\overline{u}\,d\sigma,$$

and inserting this into (3.5) we obtain

$$\int_{\partial\Omega}\varphi'\cdot\overline{i(\alpha\cdot\nu)\varphi}\,d\sigma = e^\tau\|u\|^2_{L^2(\partial\Omega)^2}.$$

Plugging this last expression into (3.4) we obtain (3.1). Finally, (3.2) is obtained combining (3.1) with Lemma 2.3. □

Once we have established the monotonicity of the eigenvalues, we can finally prove Theorem 1.3. To do it, it will be crucial to use the explicit formula (3.2), which provides a growth estimate for the eigenvalue curves $\tau\mapsto\lambda(\tau)$, preventing them to escape to infinity at finite values of the parameter τ.

Proof of Theorem 1.3. Let $\tau_0\in\mathbb{R}$ and let $\lambda_\star\in\sigma(\mathcal{H}_{\tau_0})$ be an eigenvalue with multiplicity $\mu_\star$. Note that thanks to Lemma 1.2 (*iii*) we can assume without loss of generality that $\lambda_\star > m$. Then, by Lemma 3.1 and taking into account Remark 3.2, there exist $\mu_\star$ real analytic functions $\lambda_1(\tau),\dots,\lambda_{\mu_\star}(\tau)$ defined in a neighborhood of τ_0, such that $\lambda_j(\tau)\in\sigma(\mathcal{H}_\tau)$ and $\lambda_j(\tau_0)=\lambda_\star$ for $j=1,\dots,\mu_\star$. Each of these functions $\lambda_j(\tau)$, some of them possibly equal in a neighborhood of τ_0, is associated to a different eigenfunction which also depends analytically on τ.

Let $\lambda(\tau)$ be one of these analytic eigenvalue curves and let φ_τ be its associated eigenfunction, which can be taken to have norm equal to 1 in $L^2(\Omega)^4$. The curve $\lambda(\tau)$ can be continued analytically in a maximal interval I in which $\lambda(\tau)$ represents an eigenvalue of $\mathcal{H}_\tau$ (this is true even when the graph of $\lambda(\tau)$ crosses the graph of another such eigenvalue curve, see the comments in [52, VII-§3.2]). Our goal is to prove that $I=\mathbb{R}$.

By contradiction, let us assume that $I\neq\mathbb{R}$, and let $\tau^\star$ be a finite end of the maximal interval I —that is, $\tau^\star\in\partial I$ and $|\tau^\star|<+\infty$. Then, we set $\lambda^\star := \lim_{\tau\to\tau^\star}\lambda(\tau)$, which exists by the monotonicity of $\lambda(\tau)$ shown in Lemma 3.3. We claim that $\lambda^\star$ is finite. Indeed, by the formula for $\lambda'(\tau)$ given by (3.2) and using that $\lambda(\tau)>m$, we get $(\lambda(\tau)-m)'\le\lambda(\tau)-m$, and therefore $m<\lambda(\tau)\le C(1+e^\tau)$ for $\tau\in I$, with some constant C depending on m and τ_0. As a consequence, since $\tau^\star<+\infty$, $\lambda(\tau)$ cannot tend to $+\infty$ as $\tau\to\tau^\star$.

We shall prove now that $\lambda^\star$ is an eigenvalue of $\mathcal{H}_{\tau^\star}$. To do it, recall that the eigenfunctions $\varphi_\tau = (u_\tau, v_\tau)^\intercal$ are normalized in $L^2(\Omega)^4$. Our goal is to bound them uniformly in $H^1(\Omega)^4$. To accomplish this, note first that by Lemma 2.1 we have

$$\|u_\tau\|_{H^1(\Omega)^2}\le C\left(\|u_\tau\|_{L^2(\Omega)^2} + \|\sigma\cdot\nabla u_\tau\|_{L^2(\Omega)^2} + \|u_\tau\|_{H^{1/2}(\partial\Omega)^2}\right).$$

Since $\|\varphi_\tau\|^2_{L^2(\Omega)^4}=1$, using the equation $-\sigma\cdot\nabla u_\tau = (\lambda(\tau)+m)v_\tau$ and the bounds $m<\lambda(\tau)\le\lambda^\star+1$ for τ close enough to $\tau^\star$, it follows that

$$\|u_\tau\|_{H^1(\Omega)^2}\le C\left(1+\|u_\tau\|_{H^{1/2}(\partial\Omega)^2}\right)$$

for some constant C depending only on Ω, m, and $\lambda^\star$. It remains to estimate $\|u_\tau\|_{H^{1/2}(\partial\Omega)^2}$. To do it, we use (2.21) to obtain

$$\|u_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C\|u_\tau\|_{H^{-1/2}(\partial\Omega)^2},$$

for some constant C depending only on Ω, m, and $\lambda^\star$, provided that $\tau \in I$ is close to $\tau^\star$. Finally, note that, under the normalization hypothesis, by Lemma 2.2 we have $\|u_\tau\|_{H^{-1/2}(\partial\Omega)^2} \leq C$ for some constant depending only on Ω, m, and $\lambda^\star$. Therefore, for some other constant C depending only on the same quantities, we get

$$\|u_\tau\|_{H^1(\Omega)^2} \leq C,$$

provided that $\tau \in I$ is close to $\tau^\star$. Using a similar argument for v_τ, we obtain the uniform bound $\|\varphi_\tau\|_{H^1(\Omega)^4} \leq C$ for $\tau \in I$ close to $\tau^\star$. Therefore, by the compact embedding of $H^1(\Omega)^4$ into $L^2(\Omega)^4$ and of $H^{1/2}(\partial\Omega)^4$ into $L^2(\partial\Omega)^4$, and by weak* compactness on $H^1(\Omega)^4$, we can find a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ with $\lim_{k\uparrow+\infty}\tau_k = \tau^\star$ such that φ_{τ_k} converges in $L^2(\Omega)^4$ and in $L^2(\partial\Omega)^4$ to some function $\varphi^\star \in H^1(\Omega)^4$ as $k \uparrow +\infty$. Since φ_{τ_k} are normalized in $L^2(\Omega)^4$, we have $\varphi^\star \not\equiv 0$. Moreover, by the convergence in $L^2(\partial\Omega)^4$, it follows readily that $\varphi^\star \in \mathrm{Dom}(\mathcal{H}_{\tau^\star})$. Writing the eigenvalue equation for φ_{τ_k} in weak form and taking the limit $k \uparrow +\infty$, we obtain that $\varphi^\star$ solves weakly an eigenvalue equation with associated eigenvalue $\lambda^\star$. Since $\varphi^\star \in H^1(\Omega)^4$ (by the weak* compactness), a standard density argument shows that $\mathcal{H}\varphi^\star = \lambda^\star\varphi^\star$ in Ω.

Finally, once we have proven that $\lambda^\star \in \sigma(\mathcal{H}_{\tau^\star})$, we can apply again Lemma 3.1 to this eigenvalue. By doing this, we extend analytically the curve $\lambda(\tau)$ to a bigger interval, contradicting the maximality of the interval I. As a consequence, we have shown that any eigenvalue curve $\lambda(\tau)$ can be defined for all $\tau \in \mathbb{R}$ as a real analytic function.

To conclude the proof, we should show that, for the given $\tau_0 \in \mathbb{R}$, the extensions of all the eigenvalues $\{\lambda_k(\tau_0)\}_{k\in\mathbb{Z}\setminus\{0\}}$ exhaust the spectrum of $\mathcal{H}_\tau$ for every $\tau \in \mathbb{R}$. Indeed, if there were some $\tau_\star$ and an eigenvalue $\lambda_\star \in \sigma(\mathcal{H}_{\tau_\star})$ which did not lie on any of the eigenvalue curves found before, then $\lambda_\star$ could itself be extended to an analytic eigenvalue curve on $\mathbb{R}$ by the previous arguments, and in particular we would have an eigenvalue $\lambda_\star(\tau_0)$ not included in $\{\lambda_k(\tau_0)\}_{k\in\mathbb{Z}\setminus\{0\}}$, contradicting the assumption that $\{\lambda_k(\tau_0)\}_{k\in\mathbb{Z}\setminus\{0\}}$ (counting multiplicities) is the totality of the spectrum of $\mathcal{H}_{\tau_0}$. □

4. Asymptotic behavior

In this section we study the asymptotic behavior of the eigenvalue curves as $\tau \to \pm\infty$. First, we will establish Theorem 1.4, regarding the possible limits of the eigenvalue curves, and we will provide a finer description of the curve corresponding to the first positive eigenvalue of $\mathcal{H}_\tau$, as stated in Theorem 1.5. Furthermore, we will prove the shape optimization result for large values of τ stated in Corollary 1.6. After that, we will focus on the first order asymptotics as $\tau \downarrow -\infty$. To do so, we will first need to introduce skew projections onto Hardy spaces, and then we will address the proof of Theorem 1.7.

4.1. Limits of eigenvalue curves as $\tau \to \pm\infty$.

In this section we prove Theorems 1.4 and 1.5 and Corollary 1.6. We begin by showing two compactness results that will be used in the proof of Theorem 1.4, and also Theorem 1.7. Roughly speaking, Theorem 1.4 is based on the following argument: every $\varphi_\tau = (u_\tau, v_\tau)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau)$ satisfies the boundary condition $v_\tau = ie^\tau(\sigma\cdot\nu)u_\tau$ on $\partial\Omega$. If φ_τ is an eigenfunction, as $\tau \downarrow -\infty$ the boundary condition forces the trace of v_τ on $\partial\Omega$ to vanish, and as $\tau \uparrow +\infty$ the boundary condition forces the trace of u_τ to vanish. Combining this with uniform estimates on the $H^1(\Omega)^2$-norm of the components of φ_τ and a compactness

argument, we end up with an eigenfunction of the Dirichlet Laplacian. From here we can then find a set of candidates to be the limit of the eigenvalue associated to φ_τ as $\tau \to \pm\infty$. This, in particular, will lead to the proof of Theorem 1.4.

Let us first address the compactness result for v_τ as $\tau \downarrow -\infty$. The crucial point is to establish a uniform bound for the $H^1(\Omega)^2$-norm of v_τ, (4.1) below. The proof of this estimate will be a slight modification of some ideas that were used to prove Theorem 1.3, but since now we do not have control on $e^{-\tau}$ as $\tau \downarrow -\infty$, we also need to use some estimates for u_τ and the boundary condition.

Proposition 4.1. *Let $\mathcal{T} = \{\tau_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\lim_{k\uparrow+\infty} \tau_k = -\infty$. For every $\tau \in \mathcal{T}$, let $\lambda(\tau) \in \sigma(\mathcal{H}_\tau) \cap (m, +\infty)$ and $\varphi_\tau = (u_\tau, v_\tau)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau)$ such that $\mathcal{H}_\tau\varphi_\tau = \lambda(\tau)\varphi_\tau$ and $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$. Assume that $\lambda(\tau) \leq C_\circ$ for some $C_\circ > 0$ and all $\tau \in \mathcal{T}$. Then,*

$$\|v_\tau\|_{H^1(\Omega)^2} \leq C \quad \text{for all } \tau \in \mathcal{T}, \tag{4.1}$$

where $C > 0$ depends only on m, $C_\circ$, $\max \mathcal{T}$, and Ω.

As a consequence, there exists a subsequence $\{\tau_{k_j}\}_{j\in\mathbb{N}} \subset \mathcal{T}$ for which the limit

$$\lambda_\star := \lim_{j\uparrow+\infty} \lambda(\tau_{k_j}) \in [m, C_\circ]$$

exists and such that $v_{\tau_{k_j}}$ converges in $L^2(\Omega)^2$ to a function $v_\star \in H^2(\Omega)^2$ satisfying

$$\begin{cases} -\Delta v_\star &= (\lambda_\star^2 - m^2)v_\star & \text{in } \Omega, \\ v_\star &= 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Throughout the proof we will use the letter C to denote different constants depending only on m, $C_\circ$, $\max \mathcal{T}$, and Ω.

First, note that by Lemma 2.1 (ii) we have

$$\|v_\tau\|_{H^1(\Omega)^2} \leq C \left(\|v_\tau\|_{L^2(\Omega)^2} + \|\sigma \cdot \nabla v_\tau\|_{L^2(\Omega)^2} + \|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \right).$$

Since $\|u_\tau\|^2_{L^2(\Omega)^2} + \|v_\tau\|^2_{L^2(\Omega)^2} = \|\varphi_\tau\|^2_{L^2(\Omega)^4} = 1$, using the equation $-i\sigma \cdot \nabla v_\tau = (\lambda(\tau) - m)u_\tau$ from (2.5) and the upper bound $\lambda(\tau) \leq C_\circ$, it follows that

$$\|v_\tau\|_{H^1(\Omega)^2} \leq C \left(1 + \|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \right) \quad \text{for all } \tau \in \mathcal{T}.$$

Therefore, to prove (4.1) we only need to estimate $\|v_\tau\|_{H^{1/2}(\partial\Omega)^2}$ uniformly in $\tau \in \mathcal{T}$.

From the boundary condition $v_\tau = ie^\tau(\sigma \cdot \nu)u_\tau$, and since ν is of class C^1 on $\partial\Omega$ (which yields that $\sigma \cdot \nu$ is a bounded operator in $H^{1/2}(\partial\Omega)^2$), we see that

$$\|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq Ce^\tau \|u_\tau\|_{H^{1/2}(\partial\Omega)^2}. \tag{4.2}$$

Looking at u_τ, if we combine Proposition 2.9 with (2.21) we get

$$\|u_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C \left(1 + |\lambda(\tau) + m|e^\tau + |\lambda(\tau) - m|e^{-\tau} \right) C_{\lambda(\tau)} \|u_\tau\|_{H^{-1/2}(\partial\Omega)^2}.$$

Now, using that $\lambda(\tau) \leq C_\circ$, it follows that

$$\|u_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C(1 + e^{-\tau})\|u_\tau\|_{H^{-1/2}(\partial\Omega)^2} \quad \text{for all } \tau \in \mathcal{T}.$$

Here we have used once again that ν is of class C^1 on $\partial\Omega$. Applying Lemma 2.2 and using that $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$, we obtain $\|u_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C(1 + e^{-\tau})$ for all $\tau \in \mathcal{T}$. Combining this with (4.2) we deduce that $\|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C$ for all $\tau \in \mathcal{T}$. This concludes the proof of (4.1).

Let us now address the proof of the statement regarding the function $v_\star$. Firstly, since $\lambda(\tau) \in (m, C_\circ]$ for all $\tau \in \mathcal{T}$, there exists a subsequence $\mathcal{T}' = \{\tau_{k_j}\}_{j\in\mathbb{N}} \subset \mathcal{T}$ such that the limit $\lambda_\star := \lim_{j\uparrow+\infty} \lambda(\tau_{k_j})$ exists and satisfies $m \leq \lambda_\star \leq C_\circ$. Secondly, the normalization $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$ together with Lemma 2.3 yield $e^{-\tau}\|v_\tau\|^2_{L^2(\partial\Omega)^2} \leq C$ for all $\tau \in \mathcal{T}$, which leads to

$$\lim_{j\uparrow+\infty} \|v_{\tau_{k_j}}\|_{L^2(\partial\Omega)^2} = 0. \tag{4.3}$$

Now, combining these two ingredients with the uniform estimate (4.1), we can show the existence of $v_\star$. More precisely, by the compact embedding of $H^1(\Omega)^2$ into $L^2(\Omega)^2$ and of $H^{1/2}(\partial\Omega)^2$ into $L^2(\partial\Omega)^2$, and by weak* compactness on $H^1(\Omega)^2$, we can find a subsequence of $\mathcal{T}'$, which we denote again by $\{\tau_{k_j}\}_{j\in\mathbb{N}}$, for which $v_{\tau_{k_j}}$ converges in $L^2(\Omega)^2$ to a function $v_\star \in H^1_0(\Omega)^2$ ($v_\star$ has zero trace thanks to (4.3)) satisfying

$$-\Delta v_\star = (\lambda_\star^2 - m^2)v_\star$$

in the weak sense in Ω —recall that $-\Delta v_\tau = (\lambda(\tau)^2 - m^2)v_\tau$ in Ω by (2.6). Finally, standard elliptic estimates show that $v_\star \in H^2(\Omega)^2$; see [40, Theorem 4 in §6.3.2] for example. □

We now address the compactness result related to the upper component u_τ of the eigenfunction φ_τ as $\tau \uparrow +\infty$. That is, the analogue of Proposition 4.1 for $\tau \uparrow +\infty$.

Proposition 4.2. *Let $\mathcal{T} = \{\tau_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\lim_{k\uparrow+\infty} \tau_k = +\infty$. For every $\tau \in \mathcal{T}$, let $\lambda(\tau) \in \sigma(\mathcal{H}_\tau) \cap (m, +\infty)$ and $\varphi_\tau = (u_\tau, v_\tau)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau)$ such that $\mathcal{H}_\tau\varphi_\tau = \lambda(\tau)\varphi_\tau$ and $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$. Assume that $\lambda(\tau) \leq C_\circ$ for some $C_\circ > 0$ and all $\tau \in \mathcal{T}$. Then,*

$$\|u_\tau\|_{H^1(\Omega)^2} \leq C \quad \text{for all } \tau \in \mathcal{T}, \tag{4.4}$$

where $C > 0$ depends only on m, $C_\circ$, $\min \mathcal{T}$, and Ω.

As a consequence, there exists a subsequence $\{\tau_{k_j}\}_{j\in\mathbb{N}} \subset \mathcal{T}$ for which the limit

$$\lambda^\star := \lim_{j\uparrow+\infty} \lambda(\tau_{k_j}) \in [m, C_\circ]$$

exists and such that $u_{\tau_{k_j}}$ converges in $L^2(\Omega)^2$ to a function $u^\star \in H^2(\Omega)^2$ satisfying

$$\begin{cases} -\Delta u^\star &= ((\lambda^\star)^2 - m^2)u^\star & \text{in } \Omega, \\ u^\star &= 0 & \text{on } \partial\Omega, \end{cases}$$

Proof. The proof follows the same lines as the one of Proposition 4.1. By Lemma 2.1 (*ii*),

$$\|u_\tau\|_{H^1(\Omega)^2} \leq C\left(\|u_\tau\|_{L^2(\Omega)^2} + \|\sigma\cdot\nabla u_\tau\|_{L^2(\Omega)^2} + \|u_\tau\|_{H^{1/2}(\partial\Omega)^2}\right).$$

Since $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$, using the equation $-i\sigma\cdot\nabla u_\tau = (\lambda(\tau) + m)v_\tau$ from (2.5) and the upper bound $\lambda(\tau) \leq C_\circ$, we have $\|u_\tau\|_{H^1(\Omega)^2} \leq C(1 + \|u_\tau\|_{H^{1/2}(\partial\Omega)^2})$. Now, $v_\tau = ie^\tau(\sigma\cdot\nu)u_\tau$ yields

$$\|u_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq Ce^{-\tau}\|v_\tau\|_{H^{1/2}(\partial\Omega)^2}. \tag{4.5}$$

Looking at v_τ, if we combine Proposition 2.9 with (2.21) we get

$$\|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C\left(1 + |\lambda(\tau) + m|e^\tau + |\lambda(\tau) - m|e^{-\tau}\right)C_{\lambda(\tau)}\|v_\tau\|_{H^{-1/2}(\partial\Omega)^2}.$$

Using that $\lambda(\tau) \leq C_\circ$, it follows that $\|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C(1 + e^\tau)\|v_\tau\|_{H^{-1/2}(\partial\Omega)^2}$ for all $\tau \in \mathcal{T}$. Applying Lemma 2.2 and using that $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$, we obtain $\|v_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C(1 + e^\tau)$ for all $\tau \in \mathcal{T}$. Combining this with (4.5), we get $\|u_\tau\|_{H^{1/2}(\partial\Omega)^2} \leq C$ for all $\tau \in \mathcal{T}$, which proves (4.4). Once we have this uniform bound we proceed as in the proof of Proposition 4.1, using now that $e^\tau\|u_\tau\|^2_{L^2(\partial\Omega)^2} \leq C$ by Lemma 2.3 to get that $u^\star = 0$ on $\partial\Omega$. □

The following lemma will be used in the proof of Theorem 1.4. It assures that $\|u_\tau\|_{L^2(\Omega)^2}$ does not tend to zero as $\tau \uparrow +\infty$. This fact will be used to show that the limit function $u^\star$ found in Proposition 4.2 is not identically zero and, therefore, that $(\lambda^\star)^2 - m^2$ is an eigenvalue of the Dirichlet Laplacian on Ω. As we will see in the proof of Theorem 1.4, to show an analogous nondegeneracy for $v^\star$ we will need a different argument based on formula (3.2).

Lemma 4.3. *Let $\tau \in \mathbb{R}$, $\lambda > 0$, and $\varphi = (u, v)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau) \setminus \{0\}$ be such that $\mathcal{H}_\tau \varphi = \lambda \varphi$. Then,*

$$\frac{\|u\|_{L^2(\Omega)^2}}{\|\varphi\|_{L^2(\Omega)^4}} > \frac{1}{\sqrt{2}} > \frac{\|v\|_{L^2(\Omega)^2}}{\|\varphi\|_{L^2(\Omega)^4}}.$$

Proof. Thanks to Theorem 1.3, we can take a smooth parametrization of the eigenvalue $\lambda = \lambda(\tau)$ and the associated eigenfunction $\varphi = \varphi_\tau = (u_\tau, v_\tau)^\intercal$ in a neighborhood of τ (indeed in the whole real line). Set

$$\gamma(\tau) := \frac{\|u_\tau\|^2_{L^2(\Omega)^2}}{\|\varphi_\tau\|^2_{L^2(\Omega)^4}}.$$

Then $1 - \gamma(\tau) = \|v_\tau\|^2_{L^2(\Omega)^2} / \|\varphi_\tau\|^2_{L^2(\Omega)^4}$. Hence, it suffices to prove that $\gamma(\tau) > 1/2$. From (3.2) and the fact that $\lambda'(\tau) > 0$ it follows that $(\lambda(\tau) - m)\gamma(\tau) > (\lambda(\tau) + m)(1 - \gamma(\tau))$, which entails $\lambda(\tau)(2\gamma(\tau) - 1) > m \geq 0$. Since $\lambda(\tau) > 0$, we conclude that $\gamma(\tau) > 1/2$. □

With the previous results at hand, we can now establish Theorem 1.4.

Proof of Theorem 1.4. In the following, for each $\tau \in I$ let $\varphi_\tau = (u_\tau, v_\tau)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau)$ be such that $\mathcal{H}_\tau \varphi_\tau = \lambda(\tau)\varphi_\tau$ and $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$.

Let us first prove (ii), which is shorter than proving (i). Since $\tau \mapsto \lambda(\tau) \in \sigma(\mathcal{H}_\tau) \cap (m, +\infty)$ is assumed to be continuous, the monotonicity of the eigenvalue curves proved in Theorem 1.3 assures that λ is strictly increasing on I, hence the limit $\lambda(+\infty)$ exists and satisfies $m < \lambda(+\infty) \leq +\infty$.

Assume that $\lambda(+\infty) < +\infty$. Then $\lambda(\tau) \leq \lambda(+\infty) < +\infty$ for all $\tau \in (\tau_0, +\infty)$. By Proposition 4.2, there exists a sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset (\tau_0, +\infty)$ with $\lim_{k\uparrow+\infty} \tau_k = +\infty$ for which u_{τ_k} converges in $L^2(\Omega)^2$ to a function $u^\star \in H^2(\Omega)^2$ satisfying

$$\begin{cases} -\Delta u^\star &= \big(\lambda(+\infty)^2 - m^2\big) u^\star & \text{in } \Omega, \\ u^\star &= 0 & \text{on } \partial\Omega. \end{cases}$$

Now, since $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$ for all τ, Lemma 4.3 gives that $\|u_{\tau_k}\|_{L^2(\Omega)^2} > 1/2$ for all k, thus $\|u^\star\|_{L^2(\Omega)^2} > 1/2$ by the convergence in $L^2(\Omega)^2$. Therefore, $u^\star$ is an eigenfunction of the Dirichlet Laplacian on Ω, which yields $\lambda(+\infty)^2 - m^2 \in \sigma(-\Delta_D)$, as desired.

Let us now address the proof of (i). We are assuming that $\tau \mapsto \lambda(\tau)$ is a continuous function on $(-\infty, \tau_0)$. Then, thanks to Theorem 1.3, λ is indeed real analytic everywhere on $(-\infty, \tau_0)$ except possibly on countable many exceptional points where the graph of λ may change from one real analytic eigenvalue curve to another one through a crossing point. Theorem 1.3 actually shows that on every compact set of $(-\infty, \tau_0)$ there are only a finite number of these exceptional points. Moreover, by the monotonicity shown in Theorem 1.3, $\lambda' > 0$ wherever λ is differentiable (see Lemma 3.3), and thus the limit $\lambda(-\infty)$ exists and satisfies $m \leq \lambda(-\infty) < +\infty$. All these considerations justify the identities

$$\int_{-\infty}^{\tau_0 - 1} |\lambda'| = \int_{-\infty}^{\tau_0 - 1} \lambda' = \lambda(\tau_0 - 1) - \lambda_\star.$$

Since $\lambda(\tau_0 - 1) - \lambda_\star < +\infty$, we deduce that λ' is absolutely integrable in $(-\infty, \tau_0 - 1)$. In particular, there exists a sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset (-\infty, \tau_0 - 1)$ such that $\lim_{k\uparrow+\infty} \tau_k = -\infty$ and

$$\lim_{k\uparrow+\infty} \lambda'(\tau_k) = 0. \tag{4.6}$$

Set $\gamma(\tau) := \|u_\tau\|^2_{L^2(\Omega)^2}$. Then, $1 - \gamma(\tau) = \|v_\tau\|^2_{L^2(\Omega)^2}$ by the fact that $\|\varphi_\tau\|_{L^2(\Omega)^4} = 1$, and

$$\gamma(\tau) > \frac{1}{2} \tag{4.7}$$

for all $\tau \in I$ by Lemma 4.3. Combining (4.6) with (3.2) we deduce that

$$\lim_{k\uparrow+\infty} \Big(\big(\lambda(\tau_k) - m\big)\gamma(\tau_k) - \big(\lambda(\tau_k) + m\big)\big(1 - \gamma(\tau_k)\big)\Big) = \lim_{k\uparrow+\infty} \lambda'(\tau_k) = 0. \tag{4.8}$$

This limit will be the key point in the proof of (ii).

Recall that $m \leq \lambda(-\infty) < +\infty$. The next step is to show that if $m < \lambda(-\infty)$ then $\lambda(-\infty)^2 - m^2 \in \sigma(-\Delta_D)$. Note that if $m < \lambda(-\infty)$ then the fact that λ is increasing and (4.7) yield that $(\lambda(\tau_k) - m)\gamma(\tau_k) \geq (\lambda(-\infty) - m)/2 > 0$ for all $k \in \mathbb{N}$. Then, using (4.8) we get that $(\lambda(\tau_k) + m)(1 - \gamma(\tau_k)) > C$ for some $C > 0$ and all k big enough. In particular, since $\lambda(-\infty) < +\infty$, we deduce that

$$\|v_{\tau_k}\|^2_{L^2(\Omega)^2} = (1 - \gamma(\tau_k)) > C \tag{4.9}$$

for some $C > 0$ and all k big enough. At this point, we simply have to use Proposition 4.1 on the sequence $\{\tau_k\}_{k\geq k_0}$ for some k_0 big enough to find a function $v_\star \in H^2(\Omega)^2 \setminus \{0\}$ satisfying

$$\begin{cases} -\Delta v_\star &= \big(\lambda(-\infty)^2 - m^2\big) v_\star & \text{in } \Omega, \\ v_\star &= 0 & \text{on } \partial\Omega. \end{cases}$$

Note that $v_\star$ is not identically zero thanks to (4.9) and the convergence in $L^2(\Omega)^2$ of the subsequence of $\{v_{\tau_k}\}_{k\geq k_0}$ given by Proposition 4.1. From here, we conclude that

$$\text{if } \lambda(-\infty) > m \text{ then } \lambda(-\infty)^2 - m^2 \in \sigma(-\Delta_D). \tag{4.10}$$

Finally, assume that $\lambda(\tau)^2 - m^2 \leq \min \sigma(-\Delta_D)$ for some $\tau \in I$. Since λ is strictly increasing on I, we deduce that $\lambda(-\infty)^2 - m^2 < \min \sigma(-\Delta_D)$, which leads to $\lambda(-\infty)^2 - m^2 \notin \sigma(-\Delta_D)$. This, together with (4.10) and the fact that $m \leq \lambda(-\infty) < +\infty$, entails $\lambda(-\infty) = m$. On the contrary, if $\lambda(\tau)^2 - m^2 > \min \sigma(-\Delta_D)$ for all $\tau \in I$ then $\lambda(-\infty)^2 - m^2 \geq \min \sigma(-\Delta_D) > 0$, and thus $\lambda(-\infty)^2 - m^2 \in \sigma(-\Delta_D)$ by (4.10). □

We now establish Theorem 1.5, concerning the first positive eigenvalue.

Proof of Theorem 1.5. Let us first show that $\tau \mapsto \lambda_1^+(\tau) := \min(\sigma(\mathcal{H}_\tau) \cap (m, +\infty))$ is continuous and strictly increasing on $\mathbb{R}$. To do so, we will show that λ_1^+ is continuous and strictly increasing on $[0, +\infty)$, the proof for $(-\infty, 0]$ is analogous.

Let $\{\tau \mapsto \lambda_k(\tau)\}_{k\in\mathbb{Z}\setminus\{0\}}$ be the family of eigenvalues curves associated to the mapping $\tau \mapsto \mathcal{H}_\tau$ given by Theorem 1.3. Note that this family contains pairs of curves whose graphs coincide. This is due to the fact that in the statement of Theorem 1.3 the eigenvalues were repeated according to their algebraic multiplicity. In order to avoid this repetition, let us remove from the family $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ any curve λ_k whose graph coincides with the graph of λ_j for some $j < k$. In this way we get a new family of curves, still denoted by $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, such that $\sigma(\mathcal{H}_\tau) = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} \lambda_k(\tau)$ for all $\tau \in \mathbb{R}$, and such that the graphs of λ_j and λ_k differ whenever $j \neq k$. Moreover, thanks to Theorem 1.3 the following holds: if $P \subset \mathbb{R}^2$ denotes the union of all the intersection points among the graphs of the curves in $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, then $P \cap K$ is finite

for every compact set $K \subset \mathbb{R}^2$, and for each $p \in P$ there are only a finite number of curves whose graph intersects p.

From the previous considerations, and recalling that $\lambda_1^+(0) = \min(\sigma(\mathcal{H}_0) \cap (m, +\infty))$, we see that there exist only finitely many curves $\lambda_{k_1}, \dots, \lambda_{k_J} \in \{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ whose graphs intersect the point $p_0 = (0, \lambda_1^+(0)) \in \mathbb{R}^2$. Furthermore, in a neighborhood of p_0 these curves only intersect at p_0. Hence, there exists $k' \in \{k_1, \dots, k_J\}$ and $\epsilon > 0$ such that

$$\lambda_{k'}(\tau) < \lambda_{k_j}(\tau) \quad \text{for all } j = 1, \dots, J \text{ with } k_j \neq k', \text{ and all } \tau \in (0, \epsilon). \tag{4.11}$$

Moreover, the curves $\lambda_{k_j}(\tau)$ do not intersect any other eigenvalue curve for $\tau \in [0, \epsilon)$ —since in that interval all the other eigenvalue curves lie either below $-m$ or above the second positive eigenvalue of $\mathcal{H}_0$ (by monotonicity). This shows that $\lambda_1^+(\tau) = \lambda_{k'}(\tau)$ for all $\tau \in [0, \epsilon)$. Then, since $\lambda_{k'}$ is real analytic and strictly increasing by Theorem 1.3, the same holds for λ_1^+ on $(0, \epsilon)$. Now, since $\tau \mapsto \lambda_{k'}(\tau)$ is defined for all $\tau \in \mathbb{R}$, we can increase τ starting from $\tau = 0$ in order to move us to the right along the graph of $\lambda_{k'}$. Regarding the family of curves $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, only two situations can happen. Either

- (i) there exists $\tau_1 > 0$ such that the graph of $\lambda_{k'}$ does not intersect any other graph for any $\tau \in (0, \tau_1)$, but it intersects the graphs of (at most) finitely many curves at the point $p_1 = (\tau_1, \lambda_{k'}(\tau_1)) \in \mathbb{R}^2$, or
- (ii) the graph of $\lambda_{k'}$ does not intersect the graph of any other curve for any $\tau \in (0, +\infty)$.

We claim that if (ii) holds then $\lambda_{k'} = \lambda_1^+$ on $[0, +\infty)$. Clearly, $\lambda_{k'} \geq \lambda_1^+$ on $[0, +\infty)$ by the definition of λ_1^+, the fact that $\lambda_{k'}$ is continuous, and that $\lambda_{k'}(0) = \lambda_1^+(0)$. To prove the claim, assume by contradiction that $\lambda_{k'}(\tau_0) > \lambda_1^+(\tau_0)$ for some $\tau_0 > 0$. We know that there exists $\lambda \in \{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ such that $\lambda(\tau_0) = \lambda_1^+(\tau_0)$. Since $\lambda(0) \geq \lambda_1^+(0) = \lambda_{k'}(0)$, by (ii) and continuity we deduce that $\lambda(0) = \lambda_{k'}(0)$ and that $\lambda(\tau) < \lambda_{k'}(\tau)$ for all $\tau \in [0, \tau_0]$, but this contradicts (4.11). Therefore, $\lambda_{k'} = \lambda_1^+$ on $[0, +\infty)$ if (ii) holds.

Assume now that (i) holds. Arguing as in (ii) we see that $\lambda_{k'} = \lambda_1^+$ on $[0, \tau_1]$. Then, we can proceed exactly as we did for the point $p_0 = (0, \lambda_1^+(0)) \in \mathbb{R}^2$ but now for the point $p_1 = (\tau_1, \lambda_1^+(\tau_1)) \in \mathbb{R}^2$. We would see that either λ_1^+ coincides with some curve in $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ on $[\tau_1, +\infty)$ or there exists, as in (i), a new intersection point $p_2 = (\tau_2, \lambda_1^+(\tau_2)) \in \mathbb{R}^2$ with $\tau_2 > \tau_1$ associated to the curve in $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ that coincides with λ_1^+ on $[\tau_1, \tau_2]$. Iterating this argument, in the worst case we would get an infinite sequence of intersection points $p_j = (\tau_j, \lambda_1^+(\tau_j)) \in \mathbb{R}^2$ for $j = 1, 2, \dots$ such that $\tau_j < \tau_{j+1}$ for all j. However, recall from the beginning of the proof that, if $P \subset \mathbb{R}^2$ denotes the union of all the intersection points among the graphs of the curves in $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, then $P \cap K$ is finite for every compact set $K \subset \mathbb{R}^2$. This yields that the set $\{\tau_j\}_j \cap [0, R]$ is finite for all $R > 0$, which in particular means that $\lim_{j\uparrow+\infty} \tau_j = +\infty$. Therefore, in this worst case we still get a covering of the graph of λ_1^+ by the graphs of the curves in $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ in a locally finite way.

In conclusion, from how we described λ_1^+ in terms of the eigenvalue curves $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, and since these curves are real analytic and strictly increasing on $\mathbb{R}$, we deduce that λ_1^+ is continuous and strictly increasing on $\mathbb{R}$, and real analytic on $\mathbb{R} \setminus E$, where $E \subset \mathbb{R}$ is some set such that $E \cap [-R, R]$ is finite for all $R > 0$.

It only remains to prove (1.7). That $\lim_{\tau\uparrow+\infty} \lambda_1^+(\tau)^2 - m^2 \in \sigma(-\Delta_D) \cup \{+\infty\}$ follows directly from Theorem 1.4 (ii), hence we only need to prove that $\lim_{\tau\downarrow-\infty} \lambda_1^+(\tau) = m$. To show this, by Theorem 1.4 (i) it is enough to check that $\lambda_1^+(\tau)^2 - m^2 \leq \min \sigma(-\Delta_D)$ for some $\tau \in \mathbb{R}$. The idea will be to bound from above the first eigenvalue of the positive operator $\mathcal{H}_0^2$ by the one of $-\Delta_D$, and then to use that $\mathcal{H}_0$ diagonalizes in a basis of eigenvectors to bring this estimate to

$\lambda_1^+(0)$. The operator $\mathcal{H}_0^2$ is defined by

$$\begin{aligned}\mathrm{Dom}(\mathcal{H}_0^2) &:= \{\varphi \in \mathrm{Dom}(\mathcal{H}_0) : \mathcal{H}_0\varphi \in \mathrm{Dom}(\mathcal{H}_0)\},\\ \mathcal{H}_0^2\varphi &:= \mathcal{H}^2\varphi = (-\Delta + m^2)\varphi \quad \text{for all } \varphi \in \mathrm{Dom}(\mathcal{H}_0^2).\end{aligned}$$

From (the proof of) Lemma 1.2, we know that $\mathcal{H}_0$ diagonalizes in an $L^2(\Omega)^4$-orthonormal basis of eigenfunctions, and that $\sigma(\mathcal{H}_0)$ is symmetric. This yields that

$$\langle \mathcal{H}_0^2\varphi, \varphi\rangle_{L^2(\Omega)^4} \geq \lambda_1^+(0)^2 \|\varphi\|^2_{L^2(\Omega)^4}$$

for all $\varphi \in \mathrm{Dom}(\mathcal{H}_0^2)$, and the equality holds if $\mathcal{H}_0\varphi = \pm\lambda_1^+(0)\varphi$. Therefore,

$$\begin{aligned}\lambda_1^+(0)^2 &= \min_{\varphi\in\mathrm{Dom}(\mathcal{H}_0^2)\setminus\{0\}} \frac{\langle \mathcal{H}_0^2\varphi, \varphi\rangle_{L^2(\Omega)^4}}{\|\varphi\|^2_{L^2(\Omega)^4}} = \min_{\varphi\in\mathrm{Dom}(\mathcal{H}_0^2)\setminus\{0\}} \frac{\langle (-\Delta + m^2)\varphi, \varphi\rangle_{L^2(\Omega)^4}}{\|\varphi\|^2_{L^2(\Omega)^4}}\\ &\leq \inf_{\varphi\in C_c^\infty(\Omega)^4\setminus\{0\}} \frac{\langle (-\Delta + m^2)\varphi, \varphi\rangle_{L^2(\Omega)^4}}{\|\varphi\|^2_{L^2(\Omega)^4}} \leq m^2 + \inf_{\phi\in C_c^\infty(\Omega)\setminus\{0\}} \frac{\|\nabla\phi\|^2_{L^2(\Omega)}}{\|\phi\|^2_{L^2(\Omega)}}\\ &= m^2 + \min\sigma(-\Delta_D).\end{aligned}$$

where we used that $C_c^\infty(\Omega)^4 \subset \mathrm{Dom}(\mathcal{H}_0^2)$, and the Rayleigh-Ritz principle in the last equality above. With this estimate at hand, Theorem 1.4 (i) shows that $\lim_{\tau\downarrow-\infty} \lambda_1^+(\tau) = m$. □

Remark 4.4. Despite that from our arguments we cannot assure that $\lim_{\tau\uparrow+\infty} \lambda_1^+(\tau) < +\infty$ in (1.7), we believe that indeed

$$\lim_{\tau\uparrow+\infty} \lambda_1^+(\tau) = \sqrt{\min\sigma(-\Delta_D) + m^2} \tag{4.12}$$

for every bounded domain $\Omega \subset \mathbb{R}^3$, as Proposition B.3 shows in the case of a ball. The reason for this belief is the following one: in view of the boundary equation $v = ie^\tau(\sigma\cdot\nu)u$ in (2.5), we expect that, as $\tau \downarrow -\infty$, $\mathcal{H}_\tau$ tends to the Dirac operator A_m with zigzag type boundary conditions studied in [49], whose negative eigenvalue with the smallest modulus is $-\sqrt{\min\sigma(-\Delta_D) + m^2}$. If, for example, the convergence of $\mathcal{H}_\tau$ to A_m as $\tau \downarrow -\infty$ is in the strong resolvent sense, an application of Lemma 1.2 (iii) would yield that $\sqrt{\min\sigma(-\Delta_D) + m^2} = \lim_{\tau\uparrow+\infty} \lambda(\tau)$ for some $\lambda(\tau) \in \sigma(\mathcal{H}_\tau)\cap(m, +\infty)$. Then, the fact that we must have $\lambda(\tau) = \lambda_1^+(\tau)$ for all τ big enough, which would lead to (4.12), should follow from the monotonicity of the eigenvalue curves and the fact that $-\sqrt{\min\sigma(-\Delta_D) + m^2}$ is the negative eigenvalue of A_m with the smallest modulus.

In order to use this argument to get (4.12), the convergence of $\mathcal{H}_\tau$ in a resolvent sense as $\tau \to \pm\infty$ must be studied. This question requires further work, and it will not be addressed in this article, since Theorem 1.5 suffices to establish the shape optimization result stated in Corollary 1.6.

Remark 4.5. As seen from the proof of Theorem 1.5, the existence of the set E in which λ_1^+ is not analytic depends on the possible crossing which an eigenvalue curve $\tau \mapsto \lambda(\tau)$, locally representing λ_1^+, may have with other eigenvalue curves. This, in turn, depends on the fact of λ_1^+ having constant multiplicity for all $\tau \in \mathbb{R}$. Although in the case of Ω being a ball we know from Proposition B.3 that λ_1^+ has always multiplicity 2 (and thus $E = \emptyset$ in this case), these questions remain open for a general domain Ω.

We conclude the section by proving our shape optimization result for large values of the parameter τ.

Proof of Corollary 1.6. Denote by $\lambda_\Omega^D := \min\sigma(-\Delta_D)$ the first eigenvalue of the Dirichlet Laplacian in Ω. From the Faber-Krahn inequality we know that if Ω is not a ball then

$$\lambda_\Omega^D > \lambda_B^D, \tag{4.13}$$

since $\partial\Omega$ is regular enough; see [44, Remark 3.2.2]. Now, on the one hand, Theorem 1.5 yields

$$\lim_{\tau\uparrow+\infty} \lambda_\Omega(\tau) \geq \sqrt{\lambda_\Omega^D + m^2} \tag{4.14}$$

and, on the other hand, Proposition B.3 shows that

$$\lim_{\tau\uparrow+\infty} \lambda_B(\tau) = \sqrt{\lambda_B^D + m^2}. \tag{4.15}$$

Combining (4.13), (4.14), and (4.15) we deduce that $\lim_{\tau\uparrow+\infty}\lambda_\Omega(\tau) > \lim_{\tau\uparrow+\infty}\lambda_B(\tau)$, from which the corollary follows. □

4.2. **Skew projections onto Hardy spaces.** In this section we introduce skew projections of $L^2(\partial\Omega)^2$ onto Hardy spaces. They are used in Section 4.3 to prove Theorem 1.7, a result which addresses the asymptotic expansion of $\lambda_\Omega(\tau) - m$ as $\tau \downarrow -\infty$. For a more general perspective on this topic from the point of view of Clifford algebras and the Cauchy-Clifford operator, the reader may look at [48].

Let $P_\pm : L^2(\partial\Omega)^2 \to L^2(\partial\Omega)^2$ be defined by

$$P_\pm := \frac{1}{2} \pm iW_m(\sigma\cdot\nu),$$

where W_m is defined in (2.12), and let $(P_\pm)^* : L^2(\partial\Omega)^2 \to L^2(\partial\Omega)^2$ be the adjoint operators with respect to $\langle\cdot,\cdot\rangle_{L^2(\partial\Omega)^2}$, namely,

$$(P_\pm)^* := \frac{1}{2} \mp i(\sigma\cdot\nu)W_m$$

(recall that both W_m and $\sigma\cdot\nu$ are bounded self-adjoint operators in $L^2(\partial\Omega)^2$).

Lemma 4.6. *P_+ and P_- are complementary projections of $L^2(\partial\Omega)^2$. More precisely,*

(i) $P_+ + P_- = 1$,
(ii) $P_\pm P_\mp = 0$,
(iii) $P_\pm P_\pm = P_\pm$,

as bounded operators in $L^2(\partial\Omega)^2$. The same holds replacing $P_\pm$ by $(P_\pm)^*$.

Proof. Statement *(i)* is obvious. To show *(ii)*, recall that $(W_m(\sigma\cdot\nu))^2 = -1/4$ by Lemma 2.7 *(ii)*, and thus

$$P_\pm P_\mp = (\tfrac{1}{2} \pm iW_m(\sigma\cdot\nu))(\tfrac{1}{2} \mp iW_m(\sigma\cdot\nu)) = \tfrac{1}{4} + (W_m(\sigma\cdot\nu))^2 = 0.$$

Then, combining *(i)* and *(ii)* we see that $P_\pm P_\pm = P_\pm(P_\pm + P_\mp) = P_\pm$, which proves *(iii)*. Finally, taking adjoints in these identities, we get the same conclusions for $(P_\pm)^*$. □

The previous lemma shows that $P_\pm$ are skew projections of $L^2(\partial\Omega)^2$ parallel to (with kernel) $P_\mp(L^2(\partial\Omega)^2)$, and analogously for their adjoints. The subspaces $P_\pm(L^2(\partial\Omega)^2)$ of $L^2(\partial\Omega)^2$ are the so-called boundary Hardy (or Smirnov) spaces obtained by taking traces on $\partial\Omega$ of inner/outer null-solutions of $\sigma\cdot\nabla$: setting $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^3\setminus\overline{\Omega}$, from the reproducing formula for $\mathbb{C}^2$-valued functions analogous to (2.9) one sees that if $u_\pm \in H^1(\Omega_\pm)^2$ are such that $(\sigma\cdot\nabla)u_\pm = 0$ in $\Omega_\pm$, then their traces on $\partial\Omega$ satisfy $P_\pm u_\pm = u_\pm$ as functions in $L^2(\partial\Omega)^2$, which is equivalent to say that $u_\pm \in P_\pm(L^2(\partial\Omega)^2) = \mathrm{Ker}(P_\mp)$.

Despite being projections, P_+ and P_- are not orthogonal projections in general. Indeed, the fact of being orthogonal characterizes the shape of Ω, as the following result shows. The reader should also look at [48] for a deeper treatment of the interplay between the geometry of Hardy spaces and the geometry of the underlying domain Ω. In [48] the authors consider the general framework of domains with locally finite perimeter. For the sake of simplicity, in the following lemma we only focus on bounded regular domains.

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary. The following are equivalent:*

(i) P_+ and P_- are self-adjoint operators in $L^2(\partial\Omega)^2$.
(ii) P_+ and P_- are orthogonal projections of $L^2(\partial\Omega)^2$.
(iii) $\{W_m, \sigma\cdot\nu\} = 0$ as operators in $L^2(\partial\Omega)^2$.
(iv) Ω is a ball.

Proof. It is obvious that (i) is equivalent to (iii). We first prove that (ii) is equivalent to (iii). Then we prove that, if Ω is bounded, (iii) is equivalent to (iv).

On the one hand, if P_+ and P_- are orthogonal projections of $L^2(\partial\Omega)^2$ then

$$0 = \langle P_\pm u, P_\mp v\rangle_{L^2(\partial\Omega)^2} = \langle (P_\mp)^* P_\pm u, v\rangle_{L^2(\partial\Omega)^2}$$

for all $u, v \in L^2(\partial\Omega)^2$. This is equivalent to say that

$$\begin{aligned} 0 = (P_\mp)^* P_\pm &= (\tfrac{1}{2} \pm i(\sigma\cdot\nu)W_m)(\tfrac{1}{2} \pm iW_m(\sigma\cdot\nu)) \\ &= \tfrac{1}{4} - (\sigma\cdot\nu)W_m W_m(\sigma\cdot\nu) \pm \tfrac{i}{2}\{W_m, \sigma\cdot\nu\}. \end{aligned}$$

Subtracting both expressions, we conclude that $\{W_m, \sigma\cdot\nu\} = 0$. On the other hand, if (iii) holds then $(P_\pm)^* = P_\pm$ by (i), thus $(P_\mp)^* P_\pm = P_\mp P_\pm = 0$ by Lemma 4.6. This shows that (ii) and (iii) are equivalent.

We now prove that (iii) and (iv) are equivalent statements. Recall that

$$W_m u(x) = \lim_{\epsilon\downarrow 0} \frac{i}{4\pi} \int_{\partial\Omega\cap\{|x-y|>\epsilon\}} |x-y|^{-3}\big(\sigma\cdot(x-y)\big)u(y)\, d\sigma(y);$$

see (2.12). Therefore, $\{W_m, \sigma\cdot\nu\} = 0$ if and only if

$$\big(\sigma\cdot(x-y)\big)\big(\sigma\cdot\nu(y)\big) = -\big(\sigma\cdot\nu(x)\big)\big(\sigma\cdot(x-y)\big) \quad \text{for } \sigma\text{-a.e. } x, y \in \partial\Omega. \tag{4.16}$$

Since N is continuous because $\partial\Omega$ is of class C^1, we can replace "for σ-a.e. $x, y \in \partial\Omega$" by "for all $x, y \in \partial\Omega$" in (4.16). Recall now that $(\sigma\cdot a)(\sigma\cdot b) = a\cdot b + i\sigma\cdot(a\times b)$ for all $a, b \in \mathbb{R}^3$. Therefore, multiplying by $\sigma\cdot(x-y)$ from the left both hand sides of (4.16), we get that $\{W_m, \sigma\cdot\nu\} = 0$ if and only if

$$|x-y|^2\big(\sigma\cdot\nu(y)\big) = -\big(\sigma\cdot(x-y)\big)\big(\sigma\cdot\nu(x)\big)\big(\sigma\cdot(x-y)\big) \quad \text{for all } x, y \in \partial\Omega. \tag{4.17}$$

Observe also that, for every $a, b \in \mathbb{R}^3$,

$$(\sigma\cdot a)(\sigma\cdot b) = a\cdot b + i\sigma\cdot(a\times b) = -b\cdot a - i\sigma\cdot(b\times a) + 2a\cdot b = -(\sigma\cdot b)(\sigma\cdot a) + 2a\cdot b,$$

which yields

$$\begin{aligned} (\sigma\cdot a)(\sigma\cdot b)(\sigma\cdot a) &= -(\sigma\cdot b)(\sigma\cdot a)(\sigma\cdot a) + 2(a\cdot b)(\sigma\cdot a) \\ &= -|a|^2(\sigma\cdot b) + 2(a\cdot b)(\sigma\cdot a) = \sigma\cdot\big(-|a|^2 b + 2(a\cdot b)a\big). \end{aligned}$$

Using this formula on the right-hand side of (4.17) taking $a = x-y$ and $b = N(x)$ we get

$$\sigma\cdot\big(|x-y|^2 N(y)\big) = \sigma\cdot\big(|x-y|^2 N(x) - 2((x-y)\cdot\nu(x))(x-y)\big). \tag{4.18}$$

Now, since the Pauli matrices together with the identity matrix form a basis for the real vector space of 2×2 Hermitian matrices, from (4.17) and (4.18), we see that $\{W_m, \sigma \cdot \nu\} = 0$ if and only if

$$N(y) = N(x) - \frac{2(x-y)\cdot \nu(x)}{|x-y|^2}(x-y) \quad \text{for all } x, y \in \partial\Omega \text{ with } x \neq y. \tag{4.19}$$

Finally, since Ω is bounded and with C^1 boundary, the Reflection Lemma (see, for example, [36, Lemma 5.3 on page 45]) shows that (4.19) holds if and only if Ω is a ball. □

Remark 4.8. From the previous proof it follows that the equivalence between (*iii*) and (*iv*) in Lemma 4.7 holds not only for W_m, but for W_λ with $\lambda \in \mathbb{R}$ —recall that W_λ is defined in (2.12).

4.3. **First order asymptotics as $\tau \downarrow -\infty$.** Through this section, in order to highlight the dependence of $\mathcal{H}_\tau$ on the domain $\Omega \subset \mathbb{R}^3$, we will denote by $\lambda_\Omega(\tau)$ the first positive eigenvalue of $\mathcal{H}_\tau$ (that is, we set $\lambda_\Omega := \lambda_1^+$). Recall that $\lim_{\tau\downarrow-\infty} \lambda_\Omega(\tau) = m$ by Theorem 1.5. The purpose of this section is to address the asymptotic expansion of $\lambda_\Omega(\tau) - m$ as $\tau \downarrow -\infty$. To do it, in (1.8) we introduced the function $L_\Omega : \mathbb{R} \to (0, +\infty)$ defined by

$$\tau \mapsto L_\Omega(\tau) := (\lambda_\Omega(\tau) - m)e^{-\tau}.$$

With this notation, $\lambda_\Omega(\tau) = m + e^\tau L_\Omega(\tau)$ for all $\tau \in \mathbb{R}$.

Lemma 4.9. *The function L_Ω is strictly decreasing on $\mathbb{R}$.*

Proof. Since λ_Ω is differentiable everywhere except possibly at countable many points by Theorem 1.5, L_Ω too. Therefore, the lemma follows if we show that $L'_\Omega < 0$ at the points of differentiability.

For every $\tau \in \mathbb{R}$, let $\varphi_\tau = (u_\tau, v_\tau)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau) \setminus \{0\}$ such that $\mathcal{H}_\tau \varphi_\tau = \lambda_\Omega(\tau)\varphi_\tau$. We claim that $\|v_\tau\|_{L^2(\Omega)^2} > 0$ for all $\tau \in \mathbb{R}$. To see this, assume that $\|v_\tau\|_{L^2(\Omega)^2} = 0$. Then, (2.5) shows that $-\Delta u_\tau = (-i\sigma\cdot\nabla)^2 u_\tau = (-i\sigma\cdot\nabla)(\lambda_\Omega(\tau)+m)v_\tau = 0$ in Ω, and $u_\tau = -ie^{-\tau}(\sigma\cdot\nu)v_\tau = 0$ on $\partial\Omega$. Hence, $\|u_\tau\|_{L^2(\Omega)^2} = \|v_\tau\|_{L^2(\Omega)^2} = 0$. This implies that $\|\varphi_\tau\|_{L^2(\Omega)^4} = 0$, which contradicts the fact that $\varphi_\tau \in \mathrm{Dom}(\mathcal{H}_\tau) \setminus \{0\}$.

Now, set $\gamma(\tau) := \|u_\tau\|^2_{L^2(\Omega)^2}/\|\varphi_\tau\|^2_{L^2(\Omega)^4} = 1 - \|v_\tau\|^2_{L^2(\Omega)^2}/\|\varphi_\tau\|^2_{L^2(\Omega)^4}$. We have shown that $\gamma(\tau) < 1$ for all $\tau \in \mathbb{R}$. Then, using (3.2) we have

$$\begin{aligned} L'_\Omega(\tau) &= \lambda'_\Omega(\tau)e^{-\tau} - L_\Omega(\tau) \\ &= (\lambda_\Omega(\tau) - m)\gamma(\tau)e^{-\tau} - (\lambda_\Omega(\tau)+m)(1-\gamma(\tau))e^{-\tau} - L_\Omega(\tau) \\ &= L_\Omega(\tau)\gamma(\tau) - (\lambda_\Omega(\tau)+m)(1-\gamma(\tau))e^{-\tau} - L_\Omega(\tau) \\ &= (\gamma(\tau)-1)\big(L_\Omega(\tau) + (\lambda_\Omega(\tau)+m)e^{-\tau}\big) < 0, \end{aligned}$$

establishing the result. □

Thanks to the monotonicity of L_Ω proved in Lemma 4.9, we get that the limit

$$L^\star_\Omega := \lim_{\tau\downarrow-\infty} L_\Omega(\tau)$$

exists as an element of $(0, +\infty]$. In particular, in the case that $L^\star_\Omega < +\infty$, we deduce that $\lambda_\Omega(\tau)$ behaves like $m + L^\star_\Omega e^\tau$ as $\tau \downarrow -\infty$. That is, $L^\star_\Omega$ quantifies the speed of convergence of $\lambda_\Omega(\tau)$ towards m as $\tau \downarrow -\infty$. Our purpose now is to prove Theorem 1.7, where we give a lower bound for $L^\star_\Omega$ (which is sharp if Ω is a ball) in terms of an optimization problem posed on the boundary Hardy space $P_+(L^2(\partial\Omega)^2)$. For the convenience of the reader, we first recall

the definitions of $\mathcal{L}_\Omega$, $\mathscr{R}_\Omega$, and $\mathscr{R}$ given in (1.11), (1.14), and (1.13), respectively. Under the notation used in Section 4.2, the set $\mathcal{L}_\Omega$ is defined by

$$\mathcal{L}_\Omega := \Big\{ L \in \mathbb{C} : \exists\, u \in L^2(\partial\Omega)^2 \setminus \{0\} \text{ with } P_- u = 0,\ (P_+)^* u = L(\sigma\cdot\nu) K_m (\sigma\cdot\nu) u \Big\}, \quad (4.20)$$

the functional $\mathscr{R}$ is given by

$$\mathscr{R}(u) := \frac{\langle (\sigma\cdot\nu) K_m (\sigma\cdot\nu) u, u\rangle_{L^2(\partial\Omega)^2}}{\|u\|^2_{L^2(\partial\Omega)^2}} \quad \text{for } u \in L^2(\partial\Omega)^2 \setminus \{0\},$$

and

$$\mathscr{R}_\Omega := \sup_{u\in L^2(\partial\Omega)^2\setminus\{0\},\, P_- u = 0} \mathscr{R}(u).$$

Below (1.13) we mentioned that $0 < \mathscr{R}(u) < \|K_m\|_{L^2(\partial\Omega)^2\to L^2(\partial\Omega)^2}$ for all $u \in L^2(\partial\Omega)^2 \setminus \{0\}$. This combined with the fact that $P_+(L^2(\partial\Omega)^2) = \mathrm{Ker}(P_-) \neq \{0\}$ —for instance, the constants belong to $P_+(L^2(\partial\Omega)^2)$— yields

$$0 < \mathscr{R}_\Omega \leq \|K_m\|_{L^2(\partial\Omega)^2\to L^2(\partial\Omega)^2}. \quad (4.21)$$

The proof of Theorem 1.7 will be divided into several steps. First, in Lemma 4.10 we will show that $L^\star_\Omega \in \mathcal{L}_\Omega$ whenever $L^\star_\Omega < +\infty$. Then, in Lemma 4.11 we will prove that $\mathcal{L}_\Omega \subset \mathbb{R}$ and that $1/\mathscr{R}_\Omega \leq L$ for all $L \in \mathcal{L}_\Omega$. The proof of the fact that $\mathscr{R}_\Omega$ is attained is given in Lemma 4.12, and in Lemma 4.13 we will see that the maximizers for $\mathscr{R}_\Omega$ are in the kernel of $(P_+)^* - \frac{1}{\mathscr{R}_\Omega}(\sigma\cdot\nu)K_m(\sigma\cdot\nu)$. Finally, the proof of the fact that the equality in $1/\mathscr{R}_\Omega \leq L^\star_\Omega$ is attained if Ω is a ball is given in Proposition B.3.

Lemma 4.10. *If $L^\star_\Omega < +\infty$ then $L^\star_\Omega \in \mathcal{L}_\Omega$.*

Proof. From Proposition 2.9 we see that the eigenvalue equation $\mathcal{H}_\tau \varphi_\tau = \lambda_\Omega(\tau)\varphi_\tau$ is equivalent the to system of equations

$$\begin{aligned} u_\tau &= \big(2iW_{\lambda_\Omega(\tau)}(\sigma\cdot\nu) - 2(\lambda_\Omega(\tau)+m)e^\tau K_{\lambda_\Omega(\tau)}\big)u_\tau, \\ u_\tau &= \big(2i(\sigma\cdot\nu)W_{\lambda_\Omega(\tau)} + 2(\lambda_\Omega(\tau)-m)e^{-\tau}(\sigma\cdot\nu)K_{\lambda_\Omega(\tau)}(\sigma\cdot\nu)\big)u_\tau, \end{aligned} \quad (4.22)$$

where $\varphi_\tau = (u_\tau, ie^\tau(\sigma\cdot\nu)u_\tau)^\intercal$ on $\partial\Omega$. Proposition 2.9 also shows that φ_τ vanishes identically on Ω if and only if u_τ vanishes identically on $\partial\Omega$. Hence, by homogeneity we can assume that $\|u_\tau\|_{L^2(\partial\Omega)^2} = 1$ for all $\tau \in \mathbb{R}$. Now, using that $\lim_{\tau\downarrow-\infty}\lambda_\Omega(\tau) = m$, that $L^\star_\Omega < +\infty$, and (2.21), we see that there exist $\tau_0 \in \mathbb{R}$ and $C_\circ > 0$ such that

$$\begin{aligned} \|u_\tau\|_{H^{1/2}(\partial\Omega)^2} &\leq C(1 + |\lambda_\Omega(\tau)+m|e^\tau + |\lambda_\Omega(\tau)-m|e^{-\tau})C_{\lambda_\Omega(\tau)}\,\|u_\tau\|_{H^{-1/2}(\partial\Omega)^2} \\ &\leq C\,\|u_\tau\|_{H^{-1/2}(\partial\Omega)^2} \leq C_\circ\,\|u_\tau\|_{L^2(\partial\Omega)^2} = C_\circ \end{aligned}$$

for all $\tau < \tau_0$. From this uniform estimate and the compact embedding of $H^{1/2}(\partial\Omega)^2$ into $L^2(\partial\Omega)^2$, we get the existence of a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ with $\lim_{k\uparrow+\infty}\tau_k = -\infty$ for which u_{τ_k} converges in $L^2(\partial\Omega)^2$ as $k\uparrow+\infty$ to some $u_\star \in L^2(\partial\Omega)^2$ with $\|u_\star\|_{L^2(\partial\Omega)^2} = 1$. With this limit function at hand, we now consider (4.22) for $\tau = \tau_k$. It is an exercise to show that the operators W_λ and K_λ, as bounded operators in $L^2(\partial\Omega)^2$, depend continuously on the parameter $\lambda \in \mathbb{R}$; recall (2.12) and (2.13). Therefore, taking the limit $k\uparrow+\infty$ in (4.22) with $\tau = \tau_k$, we get that

$$u_\star = 2iW_m(\sigma\cdot\nu)u_\star, \quad (4.23)$$

$$u_\star = 2i(\sigma\cdot\nu)W_m u_\star + 2L^\star_\Omega(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_\star \quad (4.24)$$

in $L^2(\partial\Omega)^2$. Since (4.23) is equivalent to $P_- u_\star = 0$, and (4.24) can be rewritten as $(P_+)^* u_\star = L^\star_\Omega(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_\star$, we conclude that $L^\star_\Omega \in \mathcal{L}_\Omega$. □

Lemma 4.11. *$\mathcal{L}_\Omega \subset \mathbb{R}$, and $1/\mathscr{R}_\Omega \leq L$ for all $L \in \mathcal{L}_\Omega$.*

Proof. Let $L \in \mathcal{L}_\Omega$. By definition of $\mathcal{L}_\Omega$, there exists $u \in L^2(\partial\Omega)^2 \setminus \{0\}$ such that $P_- u = 0$ and $(P_+)^* u = L(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u$, that is,

$$u = 2iW_m(\sigma\cdot\nu)u, \tag{4.25}$$
$$u = 2i(\sigma\cdot\nu)W_m u + 2L(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u. \tag{4.26}$$

Multiplying (4.26) by $\overline{u}$, integrating on $\partial\Omega$, using that $(\sigma\cdot\nu)$ and W_m are self-adjoint operators on $L^2(\partial\Omega)^2$, and using (4.25) we obtain

$$\begin{aligned}\|u\|^2_{L^2(\partial\Omega)^2} &= \langle 2i(\sigma\cdot\nu)W_m u, u\rangle_{L^2(\partial\Omega)^2} + 2L\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2}\\ &= -\langle u, 2iW_m(\sigma\cdot\nu)u\rangle_{L^2(\partial\Omega)^2} + 2L\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2}\\ &= -\|u\|^2_{L^2(\partial\Omega)^2} + 2L\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2}.\end{aligned}$$

From this we deduce that

$$L = \frac{\|u\|^2_{L^2(\partial\Omega)^2}}{\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2}} = \frac{1}{\mathscr{R}(u)}. \tag{4.27}$$

On the one hand, using that $(\sigma\cdot\nu)K_m(\sigma\cdot\nu)$ is self-adjoint we get that $L \in \mathbb{R}$, which proves that $\mathcal{L}_\Omega \subset \mathbb{R}$. On the other hand, using that $P_- u = 0$, (4.27) also yields $L \geq 1/\mathscr{R}_\Omega$. □

Lemma 4.12. *There exists $u \in L^2(\partial\Omega)^2 \setminus \{0\}$ such that $P_- u = 0$ and $\mathscr{R}(u) = \mathscr{R}_\Omega$.*

Proof. We take $u_j \in L^2(\partial\Omega)^2 \setminus \{0\}$ such that $P_- u_j = 0$ for all $j \in \mathbb{N}$ and $\lim_{j\uparrow+\infty}\mathscr{R}(u_j) = \mathscr{R}_\Omega$. From the fact that $\mathscr{R}$ is homogeneous, we can assume that $\|u_j\|_{L^2(\partial\Omega)^2} = 1$ for all j. Then, since K_m is a compact operator in $L^2(\partial\Omega)^2$, up to a subsequence, there exists

$$g := \lim_{j\uparrow+\infty}(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_j \quad \text{in } L^2(\partial\Omega)^2. \tag{4.28}$$

Also, since $\|u_j\|_{L^2(\partial\Omega)^2} = 1$ for all j, by Banach-Alaoglu theorem there exists

$$u := \lim_{j\uparrow+\infty} u_j \quad \text{in the weak}^* \text{ topology of } (L^2(\partial\Omega)^2)^* \cong L^2(\partial\Omega)^2, \tag{4.29}$$

up to a subsequence. This means that $u \in L^2(\partial\Omega)^2$ and $\lim_{j\uparrow+\infty}\langle u_j, w\rangle_{L^2(\partial\Omega)^2} = \langle u, w\rangle_{L^2(\partial\Omega)^2}$ for all $w \in L^2(\partial\Omega)^2$. In particular, for every $w \in L^2(\partial\Omega)^2$ with $\|w\|_{L^2(\partial\Omega)^2} = 1$ we have

$$1 = \lim_{j\uparrow+\infty}\|u_j\|_{L^2(\partial\Omega)^2} = \lim_{j\uparrow+\infty}\sup_{\|v\|_{L^2(\partial\Omega)^2}=1}|\langle u_j, v\rangle_{L^2(\partial\Omega)^2}| \geq \lim_{j\uparrow+\infty}|\langle u_j, w\rangle_{L^2(\partial\Omega)^2}| = |\langle u, w\rangle_{L^2(\partial\Omega)^2}|.$$

Taking the supremum of $|\langle u, w\rangle_{L^2(\partial\Omega)^2}|$ among all $w \in L^2(\partial\Omega)^2$ with $\|w\|_{L^2(\partial\Omega)^2} = 1$, we get

$$\|u\|_{L^2(\partial\Omega)^2} \leq 1. \tag{4.30}$$

The next step is to relate u and g. Since K_m and $\sigma\cdot\nu$ are self-adjoint in $L^2(\partial\Omega)^2$, for every $w \in L^2(\partial\Omega)^2$ we have

$$\begin{aligned}\langle g, w\rangle_{L^2(\partial\Omega)^2} &= \lim_{j\uparrow+\infty}\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_j, w\rangle_{L^2(\partial\Omega)^2} = \lim_{j\uparrow+\infty}\langle u_j, (\sigma\cdot\nu)K_m(\sigma\cdot\nu)w\rangle_{L^2(\partial\Omega)^2}\\ &= \langle u, (\sigma\cdot\nu)K_m(\sigma\cdot\nu)w\rangle_{L^2(\partial\Omega)^2} = \langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, w\rangle_{L^2(\partial\Omega)^2},\end{aligned}$$

which yields

$$g = (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u \quad \text{in } L^2(\partial\Omega)^2. \tag{4.31}$$

Let us now prove that $P_-u = 0$. Using that $P_-u_j = 0$ for all j and (4.29) we see that, for every $w \in L^2(\partial\Omega)^2$,

$$0 = \lim_{j\uparrow+\infty} \langle P_-u_j, w\rangle_{L^2(\partial\Omega)^2} = \lim_{j\uparrow+\infty} \langle u_j, (P_-)^*w\rangle_{L^2(\partial\Omega)^2} = \langle u, (P_-)^*w\rangle_{L^2(\partial\Omega)^2} = \langle P_-u, w\rangle_{L^2(\partial\Omega)^2},$$

which leads to

$$P_-u = 0 \quad \text{in } L^2(\partial\Omega)^2. \tag{4.32}$$

We also claim that u is not identically zero. If it was, we would get $g = 0$ by (4.31) and, thus, $\mathscr{R}_\Omega = \lim_{j\uparrow+\infty} \mathscr{R}(u_j)$ would also be zero by (4.28). However, this would contradict (4.21).

Finally, since $\|u_j\|_{L^2(\partial\Omega)^2} = 1$ for all j, we have

$$|\langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_j - g, u_j\rangle_{L^2(\partial\Omega)^2}| \leq \|(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_j - g\|_{L^2(\partial\Omega)^2},$$

which yields $\lim_{j\uparrow+\infty}\langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_j - g, u_j\rangle_{L^2(\partial\Omega)^2} = 0$ by (4.28). Combining this with (4.31), (4.29), (4.30), and (4.32), we conclude that

$$\begin{aligned} 0 < \mathscr{R}_\Omega &= \lim_{j\uparrow+\infty} \mathscr{R}(u_j) = \lim_{j\uparrow+\infty} \langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_j, u_j\rangle_{L^2(\partial\Omega)^2} = \lim_{j\uparrow+\infty} \langle g, u_j\rangle_{L^2(\partial\Omega)^2} \\ &= \lim_{j\uparrow+\infty} \langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u_j\rangle_{L^2(\partial\Omega)^2} = \langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2} \\ &\leq \frac{\langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2}}{\|u\|^2_{L^2(\partial\Omega)^2}} = \mathscr{R}(u) \leq \mathscr{R}_\Omega, \end{aligned}$$

which leads to $\mathscr{R}(u) = \mathscr{R}_\Omega$. As a byproduct, we also deduce that $\|u\|_{L^2(\partial\Omega)^2} = 1$. □

Lemma 4.13. *If $u \in L^2(\partial\Omega)^2 \setminus \{0\}$ is such that $P_-u = 0$ and $\mathscr{R}(u) = \mathscr{R}_\Omega$, then*

$$(P_+)^*u = \frac{1}{\mathscr{R}_\Omega}(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u. \tag{4.33}$$

Proof. Given $\epsilon \in \mathbb{R}$ and $v \in L^2(\partial\Omega)^2$, set $u_\epsilon = u + \epsilon P_+v$. Then, $P_-u_\epsilon = P_-u + \epsilon P_-P_+v = 0$ by Lemma 4.6 (*ii*). In addition, $\|u_\epsilon\|_{L^2(\partial\Omega)^2} > 0$ if $|\epsilon|$ is small enough because $u \in L^2(\partial\Omega)^2 \setminus \{0\}$ by assumption. Therefore, $\mathscr{R}(u) = \mathscr{R}_\Omega \geq \mathscr{R}(u_\epsilon)$ for all $|\epsilon|$ small enough, which entails

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathscr{R}(u_\epsilon) = 0. \tag{4.34}$$

On the one hand,

$$\begin{aligned} \|u_\epsilon\|^2_{L^2(\partial\Omega)^2} &= \langle u + \epsilon P_+v, u + \epsilon P_+v\rangle_{L^2(\partial\Omega)^2} \\ &= \|u\|^2_{L^2(\partial\Omega)^2} + 2\epsilon\Re\langle u, P_+v\rangle_{L^2(\partial\Omega)^2} + \epsilon^2\|P_+v\|^2_{L^2(\partial\Omega)^2}, \end{aligned}$$

which yields $\frac{d}{d\epsilon}\big|_{\epsilon=0}(\|u_\epsilon\|^2_{L^2(\partial\Omega)^2}) = 2\Re\langle u, P_+v\rangle_{L^2(\partial\Omega)^2}$. On the other hand, since K_m and $\sigma\cdot\nu$ are self-adjoint,

$$\begin{aligned} \langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_\epsilon, u_\epsilon\rangle_{L^2(\partial\Omega)^2} &= \langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)(u + \epsilon P_+v), u + \epsilon P_+v\rangle_{L^2(\partial\Omega)^2} \\ &= \langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, u\rangle_{L^2(\partial\Omega)^2} \\ &\quad + 2\epsilon\Re\langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, P_+v\rangle_{L^2(\partial\Omega)^2} \\ &\quad + \epsilon^2\langle (\sigma\cdot\nu)K_m(\sigma\cdot\nu)P_+v, P_+v\rangle_{L^2(\partial\Omega)^2}, \end{aligned}$$

which yields $\frac{d}{d\epsilon}\big|_{\epsilon=0}\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u_\epsilon, u_\epsilon\rangle_{L^2(\partial\Omega)^2} = 2\Re\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, P_+v\rangle_{L^2(\partial\Omega)^2}$. Therefore, plugging this in (4.34) and using that $\mathscr{R}(u) = \mathscr{R}_\Omega$, we deduce that

$$\begin{aligned} 0 &= \frac{1}{2}\|u\|^2_{L^2(\partial\Omega)^2}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathscr{R}(u_\epsilon) \\ &= \Re\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u, P_+v\rangle_{L^2(\partial\Omega)^2} - \mathscr{R}(u)\Re\langle u, P_+v\rangle_{L^2(\partial\Omega)^2} \\ &= \Re\langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u - \mathscr{R}_\Omega u, P_+v\rangle_{L^2(\partial\Omega)^2} \end{aligned} \tag{4.35}$$

for all $v \in L^2(\partial\Omega)^2$. Replacing v by iv in (4.35), we conclude that

$$0 = \langle(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u - \mathscr{R}_\Omega u, P_+v\rangle_{L^2(\partial\Omega)^2}$$

for all $v \in L^2(\partial\Omega)^2$, which leads to

$$(P_+)^*\big((\sigma\cdot\nu)K_m(\sigma\cdot\nu) - \mathscr{R}_\Omega\big)u = 0 \quad \text{in } L^2(\partial\Omega)^2. \tag{4.36}$$

Since $(P_+)^*(P_+)^* = (P_+)^*$ by Lemma 4.6 (iii), (4.36) can be rewritten as

$$(P_+)^*\big((\sigma\cdot\nu)K_m(\sigma\cdot\nu) - \mathscr{R}_\Omega(P_+)^*\big)u = 0 \quad \text{in } L^2(\partial\Omega)^2. \tag{4.37}$$

Let us now show that we also have $(P_-)^*\big((\sigma\cdot\nu)K_m(\sigma\cdot\nu) - \mathscr{R}_\Omega(P_+)^*\big)u = 0$. Using that $W_m(\sigma\cdot\nu)K_m = -K_m(\sigma\cdot\nu)W_m$ by Lemma 2.7 (iii), we have

$$\begin{aligned} (P_-)^*(\sigma\cdot\nu)K_m(\sigma\cdot\nu) &= (\tfrac{1}{2} + i(\sigma\cdot\nu)W_m)(\sigma\cdot\nu)K_m(\sigma\cdot\nu) \\ &= (\sigma\cdot\nu)K_m(\sigma\cdot\nu)(\tfrac{1}{2} - iW_m(\sigma\cdot\nu)) \\ &= (\sigma\cdot\nu)K_m(\sigma\cdot\nu)P_-. \end{aligned} \tag{4.38}$$

Combining (4.38) with the fact that $(P_-)^*(P_+)^* = 0$ by Lemma 4.6 (ii), and that $P_-u = 0$, we deduce that

$$\begin{aligned} (P_-)^*\big((\sigma\cdot\nu)K_m(\sigma\cdot\nu) - \mathscr{R}_\Omega(P_+)^*\big)u &= (P_-)^*(\sigma\cdot\nu)K_m(\sigma\cdot\nu)u \\ &= (\sigma\cdot\nu)K_m(\sigma\cdot\nu)P_-u = 0 \quad \text{in } L^2(\partial\Omega)^2. \end{aligned} \tag{4.39}$$

Finally, since $(P_+)^* + (P_-)^* = 1$ by Lemma 4.6 (i), summing (4.37) and (4.39) we arrive to

$$\big((\sigma\cdot\nu)K_m(\sigma\cdot\nu) - \mathscr{R}_\Omega(P_+)^*\big)u = 0 \quad \text{in } L^2(\partial\Omega)^2,$$

from which (4.33) follows. □

Combining the previous lemmas we establish Theorem 1.7.

Proof of Theorem 1.7. Lemma 4.10 proves (i), and Lemma 4.11 yields (ii). The proof of (iii) follows from Lemmas 4.12 and 4.13. Proposition B.3 shows the last statement in the theorem regarding the ball, using Lemmas 4.10 and 4.11. □

Appendix A. Properties of the spectrum

Here we prove the properties of the spectrum of $\mathcal{H}_\tau$ collected in Lemma 1.2. For a shorter notation, we will use

$$\mathcal{B} := -i\beta(\alpha\cdot\nu),$$

which defines a self-adjoint operator in $L^2(\partial\Omega)^4$ such that $\mathcal{B}^2 = I_4$. For every $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ it holds $\varphi = \sinh\tau\mathcal{B}\beta\varphi + \cosh\tau\mathcal{B}\varphi$ on $\partial\Omega$.

Proof of Lemma 1.2. From [21, Proposition 5.15] we know that $\mathcal{H}_\tau$ is self-adjoint in $L^2(\Omega)^4$. The proof of (i) follows from this and the compact embedding of $H^1(\Omega)^4$ into $L^2(\Omega)^4$.

Let us now show (ii). We first claim that for every $\varphi = (u, v)^\intercal \in \mathrm{Dom}(\mathcal{H}_\tau)$ it holds

$$\|\mathcal{H}\varphi\|^2_{L^2(\Omega)^4} = \|\alpha\cdot\nabla\varphi\|^2_{L^2(\Omega)^4} + m^2\|\varphi\|^2_{L^2(\Omega)^4} + me^{\tau}\|u\|^2_{L^2(\partial\Omega)^2} + me^{-\tau}\|v\|^2_{L^2(\partial\Omega)^2}. \tag{A.1}$$

To prove this formula, note first that expanding the square we have

$$\begin{aligned}\|\mathcal{H}\varphi\|^2_{L^2(\Omega)^4} &= \langle -i\alpha\cdot\nabla\varphi, -i\alpha\cdot\nabla\varphi\rangle_{L^2(\Omega)^4} + m^2\langle\beta\varphi,\beta\varphi\rangle_{L^2(\Omega)^4} + 2m\Re\langle\beta\varphi, -i\alpha\cdot\nabla\varphi\rangle_{L^2(\Omega)^4}\\ &= \|\alpha\cdot\nabla\varphi\|^2_{L^2(\Omega)^4} + m^2\|\varphi\|^2_{L^2(\Omega)^4} + 2m\Re\langle\beta\varphi, -i\alpha\cdot\nabla\varphi\rangle_{L^2(\Omega)^4}.\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}\langle\beta\varphi, -i\alpha\cdot\nabla\varphi\rangle_{L^2(\Omega)^4} &= \langle -i\alpha\cdot\nabla(\beta\varphi),\varphi\rangle_{L^2(\Omega)^4} - \langle -i(\alpha\cdot\nu)\beta\varphi,\varphi\rangle_{L^2(\partial\Omega)^4}\\ &= -\langle\beta(-i\alpha\cdot\nabla)\varphi,\varphi\rangle_{L^2(\Omega)^4} + \langle\mathcal{B}\varphi,\varphi\rangle_{L^2(\partial\Omega)^4},\end{aligned}$$

and using that β is hermitian, we obtain $2\Re\langle\beta\varphi, -i\alpha\cdot\nabla\varphi\rangle_{L^2(\Omega)^4} = \langle\mathcal{B}\varphi,\varphi\rangle_{L^2(\partial\Omega)^4}$. Thus,

$$\|\mathcal{H}\varphi\|^2_{L^2(\Omega)^4} = \|\alpha\cdot\nabla\varphi\|^2_{L^2(\Omega)^4} + m^2\|\varphi\|^2_{L^2(\Omega)^4} + m\langle\mathcal{B}\varphi,\varphi\rangle_{L^2(\partial\Omega)^4}. \tag{A.2}$$

Now, using that $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$, we see that

$$\begin{aligned}\langle\mathcal{B}\varphi,\varphi\rangle_{L^2(\partial\Omega)^4} &= \langle\mathcal{B}\varphi, \sinh\tau\mathcal{B}\beta\varphi + \cosh\tau\mathcal{B}\varphi\rangle_{L^2(\partial\Omega)^4} = \langle\varphi, \sinh\tau\mathcal{B}^2\beta\varphi + \cosh\tau\mathcal{B}^2\varphi\rangle_{L^2(\partial\Omega)^4}\\ &= \langle\varphi, \sinh\tau\beta\varphi + \cosh\tau\varphi\rangle_{L^2(\partial\Omega)^4},\end{aligned}$$

where we have used that $\mathcal{B}$ is self-adjoint and that $\mathcal{B}^2 = I_4$. From here, and writing $\varphi = (u, v)^\intercal$, we easily see that

$$\langle\mathcal{B}\varphi,\varphi\rangle_{L^2(\partial\Omega)^4} = e^{\tau}\|u\|^2_{L^2(\partial\Omega)^2} + e^{-\tau}\|v\|^2_{L^2(\partial\Omega)^2}.$$

Plugging this into (A.2) we obtain (A.1), proving the claim. Now, let $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)\setminus\{0\}$ be such that $\mathcal{H}\varphi = \lambda\varphi$ in Ω. Note that, by Lemma 2.4, φ cannot vanish identically on $\partial\Omega$. Therefore, using (A.1) we obtain

$$\lambda^2\|\varphi\|^2_{L^2(\Omega)^4} = \|\mathcal{H}\varphi\|^2_{L^2(\Omega)^4} > m^2\|\varphi\|^2_{L^2(\Omega)^4},$$

which yields $|\lambda| > m$.

We finally prove (iii) and (iv). First, by the compact embedding of $H^1(\Omega)^4$ into $L^2(\Omega)^4$ we have that the resolvent of $\mathcal{H}_\tau$ is a compact operator, which yields that every eigenvalue has finite multiplicity. Now, given $\psi \in \mathbb{C}^4$, consider the charge conjugation operator

$$\mathcal{C}\psi := i\beta\alpha_2\overline{\psi}$$

and the time reversal-symmetry operator

$$T\psi := -i\gamma_5\alpha_2\overline{\psi}, \quad \text{where} \quad \gamma_5 := \begin{pmatrix} 0 & I_2\\ I_2 & 0\end{pmatrix}. \tag{A.3}$$

Then, simple computations show that $\mathcal{H}T = T\mathcal{H}$, $\mathcal{H}\mathcal{C} = -\mathcal{C}\mathcal{H}$, and $T\mathcal{C} = \mathcal{C}T$. In addition, setting

$$\mathcal{B}_\tau := \sinh\tau\mathcal{B}\beta + \cosh\tau\mathcal{B} = i(\sinh\tau - \cosh\tau\beta)(\alpha\cdot\nu),$$

it is also easy to check that $\mathcal{B}_\tau T = T\mathcal{B}_\tau$ and $\mathcal{B}_\tau\mathcal{C} = \mathcal{C}\mathcal{B}_{-\tau}$. Note that for every function $\varphi \in \mathrm{Dom}(\mathcal{H}_\tau)$ it holds $\varphi = \mathcal{B}_\tau\varphi$ on $\partial\Omega$. As a consequence, given an eigenfunction φ of $\mathcal{H}_\tau$ with eigenvalue λ, $T\varphi$ is also an eigenfunction of $\mathcal{H}_\tau$ with eigenvalue λ. Furthermore, $\mathcal{C}\varphi$ and $T\mathcal{C}\varphi$ are eigenfunctions of $\mathcal{H}_{-\tau}$ with eigenvalue $-\lambda$. □

To conclude this section, we establish a formula which relates the $L^2(\Omega)^4$-norms of $\nabla\varphi$ and $\alpha\cdot\nabla\varphi$ for functions $\varphi\in\mathrm{Dom}(\mathcal{H}_\tau)$. Although we do not use this formula in this article, we think that it has its own interest, and it may be useful to present it here for future reference. The formula is a generalization of [7, formula (1.3)], in which the case $\tau=0$ is considered, and the proof follows the same lines.

Lemma A.1. *Let $\tau\in\mathbb{R}$ and $\varphi\in\mathrm{Dom}(\mathcal{H}_\tau)\cap H^1(\partial\Omega)^4$. Then,*

$$\|\alpha\cdot\nabla\varphi\|^2_{L^2(\Omega)^4}=\|\nabla\varphi\|^2_{L^2(\Omega)^4}+\frac{1}{2}\int_{\partial\Omega}\kappa|\varphi|^2d\sigma+\sinh\tau\langle\gamma_5(\alpha\cdot\nu)\varphi,\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4},$$

where κ denotes the mean curvature of $\partial\Omega$.

Proof. First, for every $\varphi\in H^2(\Omega)^4$, it holds

$$\|\alpha\cdot\nabla\varphi\|^2_{L^2(\Omega)^4}=\|\nabla\varphi\|^2_{L^2(\Omega)^4}-\langle\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4},\tag{A.4}$$

where γ_5 is defined in (A.3). This is proved in [7, Appendix A.2]. By density, it also holds for all $\varphi\in H^1(\Omega)^4\cap H^1(\partial\Omega)^4$.

Let us now investigate the boundary term in the above expression. The crucial point is to use that the mean curvature of $\partial\Omega$ arises in our context through the formula

$$[-i\alpha\cdot(\nu\times\nabla),\mathcal{B}]=-\kappa\gamma_5\mathcal{B},$$

where $[\cdot,\cdot]$ denotes the commutator of two operators, i.e., $[S,T]:=ST-TS$; see [7, Lemma A.3]. Using this and the boundary condition for $\varphi\in\mathrm{Dom}(\mathcal{H}_\tau)\cap H^1(\partial\Omega)^4$ we get

$$\begin{aligned}
&\langle\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\quad=\sinh\tau\langle\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\mathcal{B}\beta\varphi\rangle_{L^2(\partial\Omega)^4}+\cosh\tau\langle\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\mathcal{B}\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\quad=\sinh\tau\langle\gamma_5\varphi,[-i\alpha\cdot(\nu\times\nabla),\mathcal{B}]\beta\varphi\rangle_{L^2(\partial\Omega)^4}+\sinh\tau\langle\gamma_5\varphi,\mathcal{B}(-i\alpha\cdot(\nu\times\nabla))\beta\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\qquad+\cosh\tau\langle\gamma_5\varphi,[-i\alpha\cdot(\nu\times\nabla),\mathcal{B}]\varphi\rangle_{L^2(\partial\Omega)^4}+\cosh\tau\langle\gamma_5\varphi,\mathcal{B}(-i\alpha\cdot(\nu\times\nabla))\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\quad=-\sinh\tau\langle\gamma_5\varphi,\kappa\gamma_5\mathcal{B}\beta\varphi\rangle_{L^2(\partial\Omega)^4}+\sinh\tau\langle\mathcal{B}\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\beta\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\qquad-\cosh\tau\langle\gamma_5\varphi,\kappa\gamma_5\mathcal{B}\varphi\rangle_{L^2(\partial\Omega)^4}+\cosh\tau\langle\mathcal{B}\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\quad=-\langle\gamma_5\varphi,\kappa\gamma_5(\sinh\tau\mathcal{B}\beta+\cosh\tau\mathcal{B})\varphi\rangle_{L^2(\partial\Omega)^4}+\sinh\tau\langle\mathcal{B}\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\beta\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\qquad+\cosh\tau\langle\mathcal{B}\gamma_5\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\quad=-\langle\gamma_5\varphi,\kappa\gamma_5\varphi\rangle_{L^2(\partial\Omega)^4}-\sinh\tau\langle\gamma_5\mathcal{B}\varphi,-i\alpha\cdot(\nu\times\nabla)\beta\varphi\rangle_{L^2(\partial\Omega)^4}\\
&\qquad-\cosh\tau\langle\gamma_5\mathcal{B}\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}.
\end{aligned}$$

Here we have used that $(\alpha\cdot x)\gamma_5=\gamma_5(\alpha\cdot x)$ for all $x\in\mathbb{R}^3$ (see [7, Lemma A.1]) and that β anticommutes with γ_5, thus $\mathcal{B}\gamma_5=-\gamma_5\mathcal{B}$. Now, using that $(\alpha\cdot x)\beta=-\beta(\alpha\cdot x)$ for every $x\in\mathbb{R}^3$, and that β anticommutes with $\mathcal{B}$ and γ_5, we have

$$\begin{aligned}
\langle\gamma_5\mathcal{B}\varphi,-i\alpha\cdot(\nu\times\nabla)\beta\varphi\rangle_{L^2(\partial\Omega)^4}&=-\langle\gamma_5\mathcal{B}\varphi,\beta(-i\alpha\cdot(\nu\times\nabla))\varphi\rangle_{L^2(\partial\Omega)^4}\\
&=-\langle\beta\gamma_5\mathcal{B}\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}=-\langle\gamma_5\mathcal{B}\beta\varphi,-i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}.
\end{aligned}$$

Hence,

$$\begin{aligned}\langle\gamma_5\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4} &= -\langle\gamma_5\varphi, \kappa\gamma_5\varphi\rangle_{L^2(\partial\Omega)^4} + \sinh\tau\langle\gamma_5\mathcal{B}\beta\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\\ &\quad - \cosh\tau\langle\gamma_5\mathcal{B}\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\\ &= -\langle\varphi,\kappa\varphi\rangle_{L^2(\partial\Omega)^4} + 2\sinh\tau\langle\gamma_5\mathcal{B}\beta\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\\ &\quad - \langle\gamma_5(\sinh\tau\mathcal{B}\beta+\cosh\tau\mathcal{B})\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}\end{aligned}$$

and thus, using the boundary condition, we get

$$\langle\gamma_5\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4} = -\frac{1}{2}\int_{\partial\Omega}\kappa|\varphi|^2 d\sigma + \sinh\tau\langle\gamma_5\mathcal{B}\beta\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4},$$

which combined with (A.4) gives

$$\|\alpha\cdot\nabla\varphi\|^2_{L^2(\Omega)^4} = \|\nabla\varphi\|^2_{L^2(\Omega)^4} + \frac{1}{2}\int_{\partial\Omega}\kappa|\varphi|^2 d\sigma - \sinh\tau\langle\gamma_5\mathcal{B}\beta\varphi, -i\alpha\cdot(\nu\times\nabla)\varphi\rangle_{L^2(\partial\Omega)^4}.$$

Finally, using again that $(\alpha\cdot x)\beta = -\beta(\alpha\cdot x)$ for every $x\in\mathbb{R}^3$, we have $\mathcal{B}\beta = i\alpha\cdot\nu$, and inserting this into the above identity we conclude the proof. □

Appendix B. The ball

In this appendix we present a more explicit spectral analysis in the case that $\Omega\subset\mathbb{R}^3$ is a ball of radius $R>0$ centered at the origin, which will be denoted by B_R. To study this radially symmetric case we introduce spherical coordinates: if $x\in\mathbb{R}^3$ we write $x = r\theta$ with $r=|x|\in[0,+\infty)$ and $\theta = x/|x|\in\mathbb{S}^2$. Using separation of variables and the spherical harmonic spinors, we give the explicit equations for the eigenvalues and eigenfunctions of $\mathcal{H}_\tau$.

B.1. **Decomposition using spherical harmonic spinors.** Let Y_n^ℓ be the usual spherical harmonics on $\mathbb{S}^2$; here $n=0,1,2,\dots$ and $\ell=-n,-n+1,\dots,n+1,n$. They satisfy $\Delta_{\mathbb{S}^2}Y_n^\ell = -n(n+1)Y_n^\ell$, where $\Delta_{\mathbb{S}^2}$ denotes the usual spherical Laplacian. Moreover, Y_n^ℓ form a complete orthonormal set in $L^2(\mathbb{S}^2)$.

Following [71, Section 4.6.4], the spherical harmonic spinors are defined as follows: for $j = 1/2, 3/2,\dots$ and $\mu_j = -j, -j+1,\dots,j-1,j$, set

$$\psi^{\mu_j}_{j-1/2} = \frac{1}{\sqrt{2j}}\begin{pmatrix}\sqrt{j+\mu_j}\,Y^{\mu_j-1/2}_{j-1/2}\\ \sqrt{j-\mu_j}\,Y^{\mu_j+1/2}_{j-1/2}\end{pmatrix}\quad\text{and}\quad \psi^{\mu_j}_{j+1/2} = \frac{1}{\sqrt{2j+2}}\begin{pmatrix}\sqrt{j+1-\mu_j}\,Y^{\mu_j-1/2}_{j+1/2}\\ -\sqrt{j+1+\mu_j}\,Y^{\mu_j+1/2}_{j+1/2}\end{pmatrix}.$$

As shown in [71, Section 4.6.5], one can decompose the space $L^2(\mathbb{R}^3)^4$ —and analogously $L^2(B_R)^4$— as

$$L^2(\mathbb{R}^3)^4 = \bigoplus_{j=1/2}^{+\infty}\bigoplus_{\mu_j=-j}^{j} L^+_{j,\mu_j}\oplus L^-_{j,\mu_j},$$

where

$$L^\pm_{j,\mu_j} := \left\{\varphi\in L^2(\mathbb{R}^3)^4 :\ \varphi(r\theta) = \begin{pmatrix} i\tilde f(r)\psi^{\mu_j}_{j\pm1/2}(\theta)\\ \tilde g(r)\psi^{\mu_j}_{j\mp1/2}(\theta)\end{pmatrix}\ \text{with}\ \tilde f,\tilde g\in L^2(\mathbb{R}_+,r^2dr)\right\}.$$

In each subspace define the mapping $U^\pm_{j,\mu_j}: L^\pm_{j,\mu_j}\to L^2(\mathbb{R}_+)^2$ by

$$(U^\pm_{j,\mu_j}\varphi)(r) = \begin{pmatrix} r\tilde f(r)\\ r\tilde g(r)\end{pmatrix} =: \begin{pmatrix} f(r)\\ g(r)\end{pmatrix},$$

and also define the differential operator (see [71, equation (4.129)])

$$\widehat{H}_{j,\pm} := \begin{pmatrix} m & -\partial_r + \kappa_{j,\pm}/r \\ \partial_r + \kappa_{j,\pm}/r & -m \end{pmatrix}, \quad \text{where } \kappa_{j,\pm} := \pm(j+1/2).$$

Then, the differential operator $\mathcal{H} = -i\alpha\cdot\nabla + m\beta$ decomposes into the orthogonal sum of the operators $(U^{\pm}_{j,\mu_j})^{-1}\widehat{H}_{j,\pm}U^{\pm}_{j,\mu_j}$. In particular, if $\phi = (f,g)^\intercal$ satisfies $\widehat{H}_{j,\pm}\phi = \lambda\phi$ in $(0,R)$, then

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{if(r)}{r}\psi^{\mu_j}_{j\pm 1/2} \\ \dfrac{g(r)}{r}\psi^{\mu_j}_{j\mp 1/2} \end{pmatrix} \tag{B.1}$$

satisfies $\mathcal{H}\varphi = \lambda\varphi$ in $B_R\setminus\{0\}$. As we will see, by further imposing that $f(0)$ is finite, we can guarantee that $\mathcal{H}\varphi = \lambda\varphi$ holds across the origin.

B.2. **Eigenvalue equations.** Our first goal is to find solutions to $\widehat{H}_{j,\pm}(f,g)^\intercal = \lambda(f,g)^\intercal$. This equation rewrites as the system of ODE

$$\begin{cases} -g' + \frac{\kappa}{r}g &= (\lambda - m)f, \\ f' + \frac{\kappa}{r}f &= (\lambda + m)g, \end{cases}$$

where $\kappa := \kappa_{j,\pm} := \pm(j+1/2)$. For simplicity, let us assume first that $\kappa = j+1/2$. To solve the system, note that from the second ODE we get

$$g = \frac{1}{\lambda+m}\Big(f' + \frac{\kappa}{r}f\Big) \tag{B.2}$$

and, thus, inserting this into the first one we get the Bessel-type ODE

$$f'' + \Big(\lambda^2 - m^2 - \frac{\kappa^2+\kappa}{r^2}\Big)f = 0.$$

Therefore, f is of the form

$$f(r) = b_1\sqrt{r}J_{\kappa+1/2}(\sqrt{\lambda^2-m^2}r) + b_2\sqrt{r}Y_{\kappa+1/2}(\sqrt{\lambda^2-m^2}r),$$

where $b_1, b_2 \in \mathbb{C}$, and $J_{\kappa+1/2}$ and $Y_{\kappa+1/2}$ denote the Bessel functions of the first and second kind of order $\kappa+1/2$; see [1, Chapters 9 and 10]. Since the eigenfunctions are not allowed to be singular at $r=0$ (as the corresponding φ given by (B.1) must solve an elliptic equation across the origin), we deduce that $b_2 = 0$, and thus f is of the form

$$f(r) = b_1\sqrt{r}J_{\kappa+1/2}(\sqrt{\lambda^2-m^2}r).$$

Now, note that for every real index p, one has the relation

$$\partial_r[J_p(\sqrt{\lambda^2-m^2}r)] = \sqrt{\lambda^2-m^2}J_{p-1}(\sqrt{\lambda^2-m^2}r) - \frac{p}{r}J_p(\sqrt{\lambda^2-m^2}r);$$

see [1, formula (9.1.27)]. Using this and (B.2), we see that

$$g(r) = b_1\frac{\sqrt{\lambda^2-m^2}}{\lambda+m}\sqrt{r}J_{\kappa-1/2}(\sqrt{\lambda^2-m^2}r).$$

The case $\kappa = -j-1/2$ follows by similar arguments. One isolates f instead of g and uses that, for a positive integer p, $J_{-(p+1/2)} = (-1)^{p+1}Y_{p+1/2}$ and $Y_{-(p+1/2)} = (-1)^p J_{p+1/2}$.

As a conclusion, we obtain that every eigenfunction of $\widehat{H}_{j,\pm}$ with eigenvalue λ is, up to a multiplicative constant, of the form

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \sqrt{r} \begin{pmatrix} J_{\ell+1/2}(\sqrt{\lambda^2-m^2}r) \\ \pm \dfrac{\sqrt{\lambda^2-m^2}}{\lambda+m} J_{\ell'+1/2}(\sqrt{\lambda^2-m^2}r) \end{pmatrix},$$

where $\ell = j \pm 1/2$ and $\ell' = j \mp 1/2$.

To obtain the equation (1.4) that relates λ and τ by means of Bessel functions, it only remains to impose the boundary condition $v = ie^{\tau}(\sigma \cdot \nu)u$ on ∂B_R for $\varphi = (u,v)^{\intercal}$ as in (B.1) and satisfying $\mathcal{H}\varphi = \lambda\varphi$ in B_R. Since

$$(\sigma \cdot \nu)\psi^{\mu_j}_{j\pm 1/2} = \psi^{\mu_j}_{j\mp 1/2},$$

by [71, equation (4.121)], it follows from (B.1) that the boundary condition relating f and g is

$$g(R) = -e^{\tau} f(R).$$

Therefore, for each $j = 1/2, 3/2, \ldots$, each $\mu_j = -j, -j+1, \ldots, j$, and each subspace $L^{\pm}_{j,\mu_j}$, we obtain the eigenvalue equation

$$e^{\tau} J_{\ell+1/2}(\sqrt{\lambda^2-m^2}R) \pm \frac{\sqrt{\lambda^2-m^2}}{\lambda+m} J_{\ell'+1/2}(\sqrt{\lambda^2-m^2}R) = 0, \tag{B.3}$$

where $\ell = j \pm 1/2$ and $\ell' = j \mp 1/2$. This corresponds to (1.4). Note that the equation is independent of the indexes μ_j, accounting for the multiplicity of the eigenvalues.

B.3. **Parametrization of the eigenvalues.** Our goal now is to exploit the eigenvalue equations given by (B.3) to prove that the eigenvalues of $\mathcal{H}_{\tau}$ can be parametrized in terms of $\tau \in \mathbb{R}$, obtaining a family of increasing curves whose limits as $\tau \to \pm\infty$ are related with the zeroes of the Bessel functions (and thus with the eigenvalues of the Dirichlet Laplacian). In the following lemma we collect the results on the Bessel functions that we will use.

Lemma B.1. *Let J_p be the Bessel function of the first kind of order $p > 0$, and denote the k-th positive zero of this function by $z_{p,k}$.*

Then,

(*i*) *the positive zeroes of J_p are simple and form an infinite increasing sequence,*

(*ii*) *the zeroes of two consecutive Bessel functions are interlaced, meaning that*

$$0 < z_{p,1} < z_{p+1,1} < z_{p,2} < z_{p+1,2} < \ldots,$$

(*iii*) *the quotient of two consecutive Bessel functions can be expressed as*

$$\frac{J_{p+1}(x)}{J_p(x)} = \sum_{k\geq 1} \frac{2x}{z_{p,k}^2 - x^2} \qquad \text{for } x \in \mathbb{R} \setminus \{z_{p,k}\}_{k\in\mathbb{N}}. \tag{B.4}$$

As a consequence, J_{p+1}/J_p is odd, strictly increasing in each interval contained in $\mathbb{R} \setminus \{z_{p,k}\}_{k\in\mathbb{N}}$, it is positive in the intervals $(0, z_{p,1})$ and $(z_{p+1,k}, z_{p,k+1})$ for $k \geq 1$, and negative in the intervals $(z_{p,k}, z_{p+1,k})$ for $k \geq 1$.

Proof. The first two statements are shown in [72, Chapter XV]. Note that for $p \geq -1$ the zeroes of J_p are real, and thus we can order them; see also [1, p. 372]. Last, (B.4) follows from formula (1) in [72, p. 498]. Note that this yields that J_{p+1}/J_p is an infinite sum of functions with singularities at $\pm z_{p,k}$ which have strictly positive derivative in their domain of definition. As a consequence, in each interval of the form $(z_{p,k}, z_{p,k+1})$ for $k \in \mathbb{N}$, the function J_{p+1}/J_p is

well defined, smooth, and strictly increasing, and therefore has a unique zero which necessarily is $z_{p+1,k}$. □

With the help of the previous lemma we can now establish the following result on the parametrization of the eigenvalues.

Proposition B.2. *For each index $j = 1/2, 3/2, \dots$ there exists an infinite number of smooth and strictly increasing functions $\{\tau \mapsto \boldsymbol{\lambda}^{\pm}_{j,k}(\tau)\}_{k\in\mathbb{Z}}$ such that, for each $\tau \in \mathbb{R}$, the real number $\boldsymbol{\lambda}^{\pm}_{j,k}(\tau)$ is an eigenvalue of $\mathcal{H}_\tau$. The functions $\boldsymbol{\lambda}^{\pm}_{j,k}$ are surjectively defined by*

$$\begin{array}{rccc} \boldsymbol{\lambda}^{\pm}_{j,k}: & \mathbb{R} & \to & I^{\pm}_{j,k} \\ & \tau & \mapsto & \boldsymbol{\lambda}^{\pm}_{j,k}(\tau), \end{array}$$

with

- $I^{-}_{j,0} = \big(m, \sqrt{(z_{j,1}/R)^2 + m^2}\,\big)$,
- $I^{-}_{j,k} = \big(\sqrt{(z_{j+1,k}/R)^2 + m^2}, \sqrt{(z_{j,k+1}/R)^2 + m^2}\,\big)$ *for* $k = 1, 2, \dots$,
- $I^{-}_{j,k} = \big(-\sqrt{(z_{j+1,|k|}/R)^2 + m^2}, -\sqrt{(z_{j,|k|}/R)^2 + m^2}\,\big)$ *for* $k = -1, -2, \dots$,

and

- $I^{+}_{j,0} = \big(-\sqrt{(z_{j,1}/R)^2 + m^2}, -m\big)$,
- $I^{+}_{j,k} = \big(-\sqrt{(z_{j,k+1}/R)^2 + m^2}, -\sqrt{(z_{j+1,k}/R)^2 + m^2}\,\big)$ *for* $k = 1, 2, \dots$,
- $I^{+}_{j,k} = \big(\sqrt{(z_{j,|k|}/R)^2 + m^2}, \sqrt{(z_{j+1,|k|}/R)^2 + m^2}\,\big)$ *for* $k = -1, -2, \dots$,

where $z_{p,k}$ denotes the k-th positive zero of J_p, the Bessel function of order p. As a consequence, for every $\tau \in \mathbb{R}$, the function

$$\varphi_\tau := \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \frac{1}{\sqrt{r}} \begin{pmatrix} iJ_{\ell+1/2}\left(\sqrt{\boldsymbol{\lambda}^{\pm}_{j,k}(\tau)^2 - m^2}\, r\right) \psi^{\mu_j}_{j\pm 1/2}(\theta) \\ \pm \dfrac{\sqrt{\boldsymbol{\lambda}^{\pm}_{j,k}(\tau)^2 - m^2}}{\boldsymbol{\lambda}^{\pm}_{j,k}(\tau) + m} J_{\ell'+1/2}\left(\sqrt{\boldsymbol{\lambda}^{\pm}_{j,k}(\tau)^2 - m^2}\, r\right) \psi^{\mu_j}_{j\mp 1/2}(\theta) \end{pmatrix}$$

with $j = 1/2, 3/2, \dots$, $\ell = j \pm 1/2$, $\ell' = j \mp 1/2$, $\mu_j = -j, -j+1, \dots, j$, and $k \in \mathbb{Z}$, belongs to $L^{\pm}_{j,\mu_j}$ and is an eigenfunction of $\mathcal{H}_\tau$ with eigenvalue $\boldsymbol{\lambda}^{\pm}_{j,k}(\tau)$.

Note that the superindex in $\boldsymbol{\lambda}^{\pm}_{j,k}$ indicates to which invariant subspace belongs the associated eigenfunction. It should not be confused with the sign of the eigenvalue (as the superindex in $\lambda^{\pm}_1$ denotes, hence the different typography $\boldsymbol{\lambda}$ vs. λ).

Proof of Proposition B.2. From the arguments already presented in Appendix B.1 it only remains to show that from (B.3) we can obtain the aforementioned infinite number of parametrizations of λ in terms of $\tau \in \mathbb{R}$. To do it, the idea is to rewrite the eigenvalue equation (B.3) as

$$e^\tau = \mp \frac{\sqrt{\lambda^2 - m^2}}{\lambda + m} \frac{J_{\ell'+1/2}(\sqrt{\lambda^2 - m^2}R)}{J_{\ell+1/2}(\sqrt{\lambda^2 - m^2}R)} =: h(\lambda).$$

Then, our goal will be to invert h in suitable intervals to get $\lambda = \lambda(\tau) := h^{-1}(e^\tau)$.

First, note that we can restrict ourselves to the subspaces L^{-}_{j,μ_j}, thanks to the odd symmetry mentioned in Remark 1.1. In this case the eigenvalue equation is written as

$$e^\tau = \operatorname{sign}(\lambda + m) \sqrt{\frac{\lambda - m}{\lambda + m}} \frac{J_{j+1}(\sqrt{\lambda^2 - m^2}R)}{J_j(\sqrt{\lambda^2 - m^2}R)} = h(\lambda).$$

Due to the fact that $e^\tau > 0$ for all $\tau \in \mathbb{R}$, we are forced to work with h only on intervals $I \subset \mathbb{R}$ such that $h(I) \subset (0, +\infty)$ and where h is invertible. Since $\frac{\lambda-m}{\lambda+m}$ is positive and strictly increasing for $\lambda \in (m, +\infty)$ and for $\lambda \in (-\infty, -m)$, I must be such that

$$\operatorname{sign}(\lambda+m)\frac{J_{j+1}(\sqrt{\lambda^2-m^2}R)}{J_j(\sqrt{\lambda^2-m^2}R)} \text{ is positive and strictly increasing for } \lambda \in I.$$

Then, Lemma B.1 yields that all the possible intervals I are:

- $I = (m, \sqrt{(z_{j,1}/R)^2 + m^2})$,
- $I = (\sqrt{(z_{j+1,k}/R)^2 + m^2}, \sqrt{(z_{j,k+1}/R)^2 + m^2})$ for $k \geq 1$,
- $I = (-\sqrt{(z_{j+1,k}/R)^2 + m^2}, -\sqrt{(z_{j,k}/R)^2 + m^2})$ for $k \geq 1$.

In each of these intervals I the function $h : I \to (0, +\infty)$ is of class C^∞, strictly increasing, and surjective. Therefore, it can be inverted, obtaining a C^∞ function $\tau \mapsto \lambda(\tau) := h^{-1}(e^\tau)$ which maps $\mathbb{R}$ into I surjectively and corresponds, for each $\tau \in \mathbb{R}$, to an eigenvalue of $\mathcal{H}_\tau$. In addition, the monotonicity of $\lambda \mapsto \tau = \tau(\lambda) := \log(h(\lambda))$ yields that $\tau \mapsto \lambda(\tau)$ is also strictly increasing. □

The previous result yields that, for any given eigenvalue curve $\tau \mapsto \lambda(\tau)$, $\lim_{\tau\to\pm\infty} |\lambda(\tau)|$ is either m or a positive zero of the function $J_{k+1/2}(\sqrt{(\cdot)^2 - m^2}R)$ for some $k = 0, 1, 2, \ldots$; note that each of these zeroes corresponds to the square root of a Dirichlet eigenvalue of $-\Delta + m^2$ in B_R. The monotonicity and limiting values of $\tau \mapsto \lambda(\tau)$ was observed in the curves plotted in Figure 1. Furthermore, the alternation (with respect to the zeroes of the Bessel function) between positive and negative eigenvalue curves, which is given by the intervals $I^\pm_{j,k}$ defined in Proposition B.2, was already shown numerically in Figures 2 and 3.

B.4. **The first positive eigenvalue.** In this section we focus on the first positive eigenvalue of $\mathcal{H}_\tau$ when $\Omega = B_R$. We provide a fine description of the associated eigenvalue curve, whose main properties are summarized in the following proposition. The reader may compare it with Theorem 1.5, which is the analogous result for general domains; see also Theorem 1.7 regarding (B.5).

Proposition B.3. *The function $\tau \mapsto \lambda_1^+(\tau) = \min(\sigma(\mathcal{H}_\tau) \cap (m, +\infty))$ is of class C^∞ in $\mathbb{R}$, and satisfies*

$$\lim_{\tau\downarrow-\infty} \lambda_1^+(\tau) = m \quad \text{and} \quad \lim_{\tau\uparrow+\infty} \lambda_1^+(\tau) = \sqrt{\pi^2/R^2 + m^2} = \sqrt{\min\sigma(-\Delta_D) + m^2},$$

where $\min\sigma(-\Delta_D)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in B_R. In addition, the corresponding eigenspace associated to $\lambda_1^+(\tau)$ has always dimension 2.

Furthermore,

$$L^\star_{B_R} := \lim_{\tau\downarrow-\infty} (\lambda_1^+(\tau) - m)e^{-\tau} = \frac{3}{R} = \frac{1}{\mathscr{R}_{B_R}}, \tag{B.5}$$

where $\mathscr{R}_{B_R}$ is defined in (1.14).

Proof. Let $\lambda : \mathbb{R} \to \left(m, \sqrt{\pi^2/R^2 + m^2}\right)$ be the eigenvalue curve corresponding to $\boldsymbol{\lambda}^-_{1/2,0}$ in the notation of Proposition B.2. This eigenfunction is associated to the two subspaces $L^-_{1/2,\mu_{1/2}}$, with either $\mu_{1/2} = 1/2$ or $\mu_{1/2} = -1/2$, and solves the implicit equation

$$e^\tau = \frac{\sqrt{\lambda^2-m^2}}{\lambda+m}\frac{J_{3/2}(\sqrt{\lambda^2-m^2}R)}{J_{1/2}(\sqrt{\lambda^2-m^2}R)}. \tag{B.6}$$

We will show that $\lambda = \lambda_1^+$. The upper bound for $\lambda(\tau)$ (and limit as $\tau \to +\infty$) is given by the fact that $z_{1/2,1} = \pi$, which follows from the explicit expression of the Bessel functions involved in the above equation:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \quad \text{and} \quad J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin(x)}{x} - \cos(x)\right).$$

As a matter of fact, these expressions can be used to show —after a tedious computation and using that $x > \sin(x)$ for all $x \in (0,\pi)$— that the right-hand side of (B.6) is a strictly increasing function of λ for $\lambda \in \left(m, \sqrt{\pi^2/R^2 + m^2}\right)$ without making use of Lemma B.1 (as done in the proof of Proposition B.2).

Since $\pi = z_{1/2,1}$ is the smallest positive zero among all the positive zeroes of the Bessel functions of half-integer index —as shown by Lemma B.1 (ii)—, it follows that $\lambda(\tau)$ coincides with $\lambda_1^+(\tau)$ at least for big enough values of τ. To show that indeed $\lambda(\tau)$ is the first positive eigenvalue $\lambda_1^+(\tau)$ for all $\tau \in \mathbb{R}$, it suffices to show that λ cannot cross any other eigenvalue curve. On the one hand, taking into account the possible intervals $I_{j,k}^+$ given in Proposition B.2, it follows that any positive eigenvalue curve associated to the spaces L_{j,μ_j}^+ must lie above $\sqrt{\pi^2/R^2 + m^2}$, and thus it cannot cross $\lambda(\tau)$. On the other hand, if there was a crossing between $\lambda(\tau)$ and another positive eigenvalue curve associated to $L_{j_\circ,\mu_{j_\circ}}^-$ for some half-integer $j_\circ > 1/2$, then by (B.6) there would exist a point $x_\circ \in (0, z_{1/2,1})$ such that

$$\frac{J_{3/2}(x_\circ)}{J_{1/2}(x_\circ)} = \frac{J_{j_\circ+1}(x_\circ)}{J_{j_\circ}(x_\circ)}. \tag{B.7}$$

However, since the zeroes of the Bessel functions are ordered (see Lemma B.1), for every half-integer $j \geq 1/2$ it follows that $z_{j,k} < z_{j+1,k}$ for all $k \geq 1$, and therefore

$$\frac{2x}{z_{j,k}^2 - x^2} > \frac{2x}{z_{j+1,k}^2 - x^2} \quad \text{for every } x \in (0, z_{1/2,1}).$$

Hence, by (B.4) it follows that

$$\frac{J_{j+1}(x)}{J_j(x)} > \frac{J_{j+2}(x)}{J_{j+1}(x)} \quad \text{for every } x \in (0, z_{1/2,1}) \text{ and for every half-integer } j \geq 1/2,$$

thus there cannot exist such a $x_\circ \in (0, z_{1/2,1})$ satisfying (B.7). In conclusion, λ does not cross any other eigenvalue curve. Therefore, $\lambda(\tau)$ is the first positive eigenvalue of $\mathcal{H}_\tau$ for all $\tau \in \mathbb{R}$. As a byproduct, since $\lambda(\tau)$ is associated to $L_{1/2,\mu_{1/2}}^-$ with either $\mu_{1/2} = 1/2$ or $\mu_{1/2} = -1/2$, it follows that the first positive eigenvalue of $\mathcal{H}_\tau$ has multiplicity 2 for all $\tau \in \mathbb{R}$.

To conclude the proof, we are only left to show that $L_{B_R}^* = 3/R = 1/\mathscr{R}_{B_R}$. First, note that from the rescaling properties of the operators K_m and W_m defined in (2.12), it follows readily that $\mathscr{R}_{B_R} = R\mathscr{R}_{B_1}$. Moreover, $L_{B_R}^\star = L_{B_1}^\star/R$. To show this second equality, one notices that if u_1 and u_R denote the boundary values of the upper component of the first eigenfunction in B_1 and B_R respectively, both associated to the same subspace $L_{1/2,\mu_{1/2}}^-$, then after a normalization one can choose them in such a way that $u_R(\cdot) = u_1(\cdot/R)$. Hence, from (2.19) and taking the limit $\tau \downarrow -\infty$, using again the scaling of K_m, and that $\{W_m, \sigma\cdot\nu\} = 0$ by Lemma 4.7, it follows that $L_{B_R}^\star = L_{B_1}^\star/R$; see the argument to get to (B.9) below for more details. As a consequence of all this, it is enough to prove the result for $R = 1$.

Recall that $\lambda(\tau)$ is associated, for all $\tau \in \mathbb{R}$, to the two subspaces $L_{1/2,\mu_{1/2}}^-$ with either $\mu_{1/2} = 1/2$ or $\mu_{1/2} = -1/2$. We will work in one of these two subspaces (the precise choice will

be completely irrelevant for the rest of the argument), and therefore, after a normalization, we can take an eigenfunction $\varphi_\tau = (u_\tau, v_\tau)^\intercal$ associated to $\lambda(\tau)$ such that u_τ at the boundary of B_1 is given by $\psi_0^{\mu_{1/2}}$ for all $\tau \in \mathbb{R}$. By Proposition 2.9, it holds

$$\begin{aligned} \Big(\frac{1}{2} - iW_\lambda(\sigma\cdot\nu)\Big)\psi_0^{\mu_{1/2}} &= -(\lambda+m)e^{\tau} K_\lambda \psi_0^{\mu_{1/2}} \quad \text{and} \\ \Big(\frac{1}{2} - i(\sigma\cdot\nu)W_\lambda\Big)\psi_0^{\mu_{1/2}} &= (\lambda-m)e^{-\tau}(\sigma\cdot\nu)\, K_\lambda(\sigma\cdot\nu)\psi_0^{\mu_{1/2}} \end{aligned} \tag{B.8}$$

in $L^2(\mathbb{S}^2)^2$. We will take the limit $\tau \downarrow -\infty$ in (B.8), taking into account that $\lambda \downarrow m$ as $\tau \downarrow -\infty$. Letting $\tau \downarrow -\infty$ in (B.8), and using that then $K_\lambda \to K_m$ and $W_\lambda \to W_m$ as bounded operators in $L^2(\mathbb{S}^2)^2$ (which follows from the explicit expressions of the operators), we obtain

$$P_-\,\psi_0^{\mu_{1/2}} = 0 \quad \text{and} \quad (P_+)^*\,\psi_0^{\mu_{1/2}} = L^\star_{B_1}(\sigma\cdot\nu)\, K_m(\sigma\cdot\nu)\psi_0^{\mu_{1/2}} \quad \text{in } L^2(\mathbb{S}^2)^2, \tag{B.9}$$

where $L^\star_{B_1} = \lim_{\tau\downarrow-\infty}(\lambda(\tau)-m)e^{-\tau}$, and $P_- = \frac{1}{2} - iW_m(\sigma\cdot\nu)$ and $(P_+)^* = \frac{1}{2} - i(\sigma\cdot\nu)W_m$, as defined in Section 4.2. In particular, (B.9) shows that $L^\star_{B_1}$ is finite, since all the involved operators are bounded and K_m is injective. It is worth pointing out that this last argument leading to $L^\star_{B_1} < +\infty$ works thanks to the fact that, on $\partial\Omega$, the eigenfunction $u_\tau = \psi_0^{\mu_{1/2}}$ is indeed independent of τ, something that may not be guaranteed on a general domain Ω.

We will use (B.9) to establish (B.5) for $R = 1$. First, let $L \in \mathcal{L}_{B_1}$ —recall that $\mathcal{L}_{B_1}$ is defined in (1.11), see also (4.20). We claim that $1/L$ is an eigenvalue of K_m. This claim can be proven by adding the two equations from which $\mathcal{L}_{B_1}$ is defined and using that $\{W_m, \sigma\cdot\nu\} = 0$ as operators in $L^2(\mathbb{S}^2)^2$; see Lemma 4.7. As a consequence, and since $1/\mathscr{R}_\Omega = \min \mathcal{L}_\Omega$ for every $\Omega \subset \mathbb{R}^3$ by Theorem 1.7, $\mathscr{R}_{B_1}$ is the maximum of the eigenvalues of K_m among eigenfunctions of the form $(\sigma\cdot\nu)u$ with $u \in P_+(L^2(\mathbb{S}^2)^2)$, where $P_+ = \frac{1}{2} + iW_m(\sigma\cdot\nu)$.

Let us now compute explicitly the eigenvalues of K_m as an operator in $L^2(\mathbb{S}^2)^2$. Using [10, Lemma 4.3] and [58, Theorem 3.6] on K_a for $a > 0$ and letting[2] $a \uparrow m$, it follows that the spectrum of K_m is given by a sequence $\{d_{j\pm1/2}\}_{j=1/2,\ 3/2,\ldots}$ whose associated eigenfunctions are $\psi^{\mu_j}_{j\pm1/2}$, i.e.,

$$K_m\psi^{\mu_j}_{j\pm1/2} = d_{j\pm1/2}\,\psi^{\mu_j}_{j\pm1/2}, \tag{B.10}$$

and the eigenvalues $d_{j\pm1/2}$ are given by the expression

$$d_{j\pm1/2} = \lim_{t\downarrow0} \mathcal{I}_{j\pm1/2+1/2}(t)\mathcal{K}_{j\pm1/2+1/2}(t).$$

Here $\mathcal{I}_\kappa$ and $\mathcal{K}_\kappa$ are the modified Bessel functions of the first and second kind of order κ, respectively. By [1, formulas 9.6.7 and 9.6.9] we have

$$\lim_{t\downarrow0} \mathcal{I}_\kappa(t)\mathcal{K}_\kappa(t) = \frac{\Gamma(\kappa)}{2\Gamma(\kappa+1)} = \frac{1}{2\kappa} \quad \text{for } \kappa > 0.$$

Hence, for $k = 0, 1, \ldots$, we have

$$d_k = \frac{1}{2k+1}. \tag{B.11}$$

We claim that $(\sigma\cdot\nu)\psi_0^{\mu_j} = \psi_1^{\mu_j}$ is orthogonal to $P_+(L^2(\mathbb{S}^2)^2)$. Once this is proved, it follows that $1/d_0 \notin \mathcal{L}_{B_1}$ and, as a consequence, $\mathscr{R}_{B_1} \le d_1 = 1/3$. To prove the claim, notice that for all $\tau \in \mathbb{R}$, after a normalization, $\psi_1^{\mu_j}$ is the upper component of the eigenfunction of $\mathcal{H}_\tau$ associated to $-\lambda(-\tau)$, i.e., the first (larger) negative eigenvalue of $\mathcal{H}_\tau$. Indeed, this can be shown with

[2]Note that the kernel k_m of K_m does not depend on m. Hence, we can assume here that $m > 0$ to cover as well the case $m = 0$.

exactly the same arguments as we did at the beginning of the proof of the proposition for the first positive eigenvalue, considering in this case the subspace $L^+_{1/2,\mu_{1/2}}$; see also Remark 1.1. Therefore, since such eigenvalue converges to $-m$ as $\tau \uparrow +\infty$, using the second equation in (2.18) with $u = \psi_1^{\mu_j}$ and taking the limit $\tau \uparrow +\infty$ —analogously as how we proceeded before to show (B.9)—, it follows that $(P_+)^*\psi_1^{\mu_j} = 0$, establishing our claim and, as a byproduct, the inequality $\mathscr{R}_{B_1} \leq d_1 = 1/3$.

We shall now prove that $L^\star_{B_1} = 1/d_1 = 3$, which will finish the proof of (B.5) since $1/\mathscr{R}_{B_1} \leq L^*_{B_1}$ by Lemmas 4.10 and 4.11. By adding the two equations in (B.9) and using again that $\{W_m, \sigma\cdot\nu\} = 0$ (or equivalently, that $(P_+)^* = P_+$ by Lemma 4.7 since the underlying domain is a ball), we obtain

$$\psi_0^{\mu_{1/2}} = L^\star_{B_1}(\sigma\cdot\nu)\, K_m(\sigma\cdot\nu)\psi_0^{\mu_{1/2}} \quad \text{in } L^2(\mathbb{S}^2)^2.$$

Finally, using that $(\sigma\cdot\nu)\psi^{\mu_{1/2}}_{j\pm1/2} = \psi^{\mu_{1/2}}_{j\mp1/2}$ for all half-integers j (see [71, equation (4.121)]), and that $\psi_1^{\mu_{1/2}}$ is an eigenfunction of K_m with eigenvalue d_1 by (B.10), we get

$$\psi_0^{\mu_{1/2}} = L^\star_{B_1} d_1 \psi_0^{\mu_{1/2}} \quad \text{in } L^2(\mathbb{S}^2)^2.$$

Therefore, we have $L^\star_{B_1} = 1/d_1 = 3$ by (B.11). This concludes the proof. □

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