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Anatomy of the Attraction Basins: Breaking with the Intuition

Leticia Hernando[1], Alexander Mendiburu[2], and Jose A. Lozano[1,3]

[1]Intelligent Systems Group, Department of Computer Science and Artificial Intelligence, University of the Basque Country UPV/EHU, 20018 San Sebastián, Spain
[2]Intelligent Systems Group, Department of Computer Architecture and Technology, University of the Basque Country UPV/EHU, 20018 San Sebastián, Spain
[3]Basque Center for Applied Mathematics (BCAM), 48009 Bilbao, Spain

Abstract

Solving combinatorial optimization problems efficiently requires the development of algorithms that consider the specific properties of the problems. In this sense, local search algorithms are designed over a neighborhood structure that partially accounts for these properties. Considering a neighborhood, the space is usually interpreted as a natural landscape, with valleys and mountains. Under this perception, it is commonly believed that, if maximizing, the solutions located in the slopes of the same mountain belong to the same attraction basin, with the peaks of the mountains being the local optima. Unfortunately, this is a widespread erroneous visualization of a combinatorial landscape. Thus, our aim is to clarify this aspect, providing a detailed analysis of, first, the existence of plateaus where the local optima are involved, and second, the properties that define the topology of the attraction basins, picturing a reliable visualization of the landscapes. Some of the features explored in this paper have never been examined before. Hence, new findings about the structure of the attraction basins are shown. The study is focused on instances of permutation-based combinatorial optimization problems considering the 2-exchange and the insert neighborhoods. As a consequence of this work, we break away from the extended belief about the anatomy of attraction basins.

Keywords

Permutation-based combinatorial optimization problems, local optima, attraction basins, local search, landscape visualization

1 Introduction

Local search algorithms have been one of the most developed metaheuristics used to solve combinatorial optimization problems (COPs). These algorithms are defined to work under a neighborhood structure built, on most occasions, by an operator. Therefore, their behavior will be conditioned by the properties that this neighborhood imposes on the search space, namely the landscape properties. Thus, many authors have already noticed the importance of studying these landscape features in order to predict, in advance, the performance of local search algorithms, or to use them in the development of new proposals (Albrecht et al., 2008, 2010; Alyahya and Rowe, 2014; Caruana and Mullin, 1999; Chicano et al., 2012; Fonlupt et al., 1999; Hernando et al., 2011; Mattfeld and Bierwirth, 1999; Merz, 2004; Merz and Freisleben, 2001; Ochoa et al., 2012, 2014; Prügel-Bennett and Tayarani-Najaran, 2012; Reeves, 1999; Reeves and Aupetit-Bélaidouni, 2004; Tayarani-Najaran and Prügel-Bennett, 2014, 2015a,b; Tomassini et al., 2008; Verel et al., 2011b; Watson, 2010; Moser et al., 2016). Some of these works assume that the topology defined by the neighborhoods in the combinatorial spaces is analogous to that found in the continuous domain. In the continuous space, there always exists a ball of radius $r > 0$ and centered at each of the local optima which is included in its attraction basin. This intuition has been transferred to the combinatorial space. Therefore, it is commonly believed that local search algorithms draw paths in a mountainous landscape, and, depending on the neighborhood, we could find a different number of mountains with different heights and sizes (Hernando et al., 2013b, 2016b; Mattfeld and Bierwirth, 1999; Tayarani-Najaran and Prügel-Bennett, 2014).

Although this thought is widespread across the literature and it has usually been accepted by the research community, other authors have started to notice a contradiction with experimental results. For example, in Tomassini et al. (2008), a representation of the landscape as local optima networks was proposed, that is, a graph where the nodes were the local optima and the edges accounted for the probabilities of connecting the different attraction basins. They showed that the number of edges was extremely large, that is, each of the attraction basins connects with almost all of the remaining attraction basins (Daolio et al., 2010, 2014). This result led them to think of *a different landscape picture than the smooth standard representation of 2D landscapes, where the basins of attraction are visualized as real mountains*. In fact, the first study that warned about this visualization of the combinatorial landscape was published in 1999 (Fonlupt et al., 1999). In that work, the authors already stated that *basins of attraction seem to be highly intertwined, giving a canyon-like structure to the landscape, rather than a crater-like structure*. Nevertheless, this finding was ignored by those works referring to it (Stützle, 2006; Preux and Talbi, 1999; Angel and Zissimopoulos, 2002; Bouziri et al., 2009). For instance, seven years after that publication, the author in Stützle (2006) contradicted it: *Intuitively, the search landscape can be imagined as a (multi-dimensional) mountainous region with hills, craters, and valleys*. The problem is that even in more recent works, these kinds of declarations are still found, as in Tayarani-Najaran and Prügel-Bennett (2015b) which, for minimization problems, says: *each local optimum has a bowl shape basin of attraction*.

As can be observed, different intuitions of the combinatorial landscapes coexist in the literature. The reason for this is the difficulty in understanding the topology of the combinatorial spaces. In all these papers, the authors try to provide an interpretation about the combinatorial landscape, giving insights according to the observed results collected from landscape features analyses, but without a solid study of the topology of the space under a specific neighborhood. Precisely, this is the objective of this paper: to examine topological features of the attraction basins of the local optima, in order to provide a reliable comprehension. Each local optimum has an attraction basin associated; however, it is already known that many different neighboring local optima with the same fitness could appear in a given instance. In this case, the local optima belong to a plateau and we could consider that they share the attraction basin. Thus, the study of the plateaus containing local optima is the starting point for the analysis of the topology of the attraction basins (Hoos and Stützle, 2004; Watson, 2010). In the literature, the graphs composed by neighboring solutions with the same fitness value, have been sometimes known as neutral networks; and when these solutions are local optima, they are called the local optimum neutral networks. This concept of neutral network emerged in the literature less than 10 years ago, and it has been analyzed for some problems: the NK landscapes (Verel et al., 2011b), the permutation flowshop problem (Marmion et al., 2011b; Daolio et al., 2014), or the traveling salesman problem (Ochoa and Veerapen, 2017). Unfortunately, the plateaus containing local optima are not detected by the common local search algorithms, even though their behavior is highly conditioned by the presence of these plateaus, as they are trapped inside them. So, in the last decade, some authors identified this situation, and modified common local search algorithms, such as the hill-climbing algorithm, in order to escape from the plateaus (Sutton et al., 2010; Marmion et al., 2011a; Tayarani-Najaran and Prügel-Bennett, 2014, 2015b,a; Prügel-Bennett and Tayarani-Najaran, 2012; Humeau et al., 2013).

The final purpose of this paper is, by studying topological features of the attraction basins, to shed light on the different misunderstandings and contradictions that have arisen in the last two decades in our research community. We focus on COPs based on permutations, that is, those where the search space is the set of permutations of size n. Particularly, we work with the permutation flowshop scheduling problem (PFSP), the linear ordering problem (LOP) and the quadratic assignment problem (QAP). First, we examine the locally optimal solutions returned by a classical hill climbing algorithm with the aim of providing knowledge related to the plateaus. Secondly, we study those features of the attraction basins that we consider essential in order to understand their topology: the roundness of the attraction basins, the centrality of the local optima, and the interior and frontier of the attraction basins. Although the frontier of the attraction basins has already been explored in previous works (Fonlupt et al., 1999; Tomassini et al., 2008; Verel et al., 2008; Ochoa et al., 2014), in this paper we take a step forward, studying the evolution of the neighboring solutions in the same attraction basin with the increase of the distance to the local optimum. To the best of our knowledge, this property, together with the roundness of the attraction basins and the centrality of the local optima, has never been analyzed before. We include a deep discussion about the obtained results, emphasizing the importance of the work presented in this paper by giving clues of how to use these discoveries in, for example, the design of efficient local search algorithms, and pointing out the potential uses these new findings could have in the combinatorial optimization field. We also present two examples for the graphical representation of the attraction basins.

The rest of the paper is organized as follows. The concepts of local optimum, plateau and attraction basin are formally introduced in Section 2. The topological features considered in the analysis are explained in Section 3. In Section 4, we carry out the experiments in order to, first, classify the solutions returned by the hill-climbing algorithm in terms of plateaus; and secondly, examine the properties of the attraction basins necessary to understand their topology. Some examples of the visualization of the anatomy of the attraction basins are shown in Section 5. Finally, in Section 6, the conclusions and future work are presented.

2 Definitions

2.1 Neighborhood

A combinatorial optimization problem (COP) consists of finding the optimal solutions of (from now on, maximizing) a function

$$\begin{array}{cccc} f: & \Omega & \longrightarrow & \mathbb{R} \\ & \sigma & \longmapsto & f(\sigma) \end{array}$$

where the search space, Ω, is a finite or countable infinite set. Specifically, we work with instances of permutation-based COPs. So, from now on, Ω is the set of permutations of size n, thus, the search space is of size $n!$.

We focus on local search algorithms, which are one of the most developed metaheuristics used to solve COPs. These algorithms are defined to work under a neighborhood structure. A **neighborhood** $\mathcal{N}$ in a search space Ω is a mapping that assigns, to each solution $\sigma \in \Omega$, a non-empty set of neighboring solutions $\mathcal{N}(\sigma)$:

$$\begin{array}{rccc} \mathcal{N}: & \Omega & \longrightarrow & P(\Omega)\backslash\emptyset \\ & \sigma & \longmapsto & \mathcal{N}(\sigma) \end{array}$$

where $P(\Omega)$ is the powerset of Ω.

Adding the concept of neighborhood to the instance of a COP, we define a **landscape** as the triple $(f, \Omega, \mathcal{N})$, where f is the objective function to optimize (Reidys and Stadler, 2002). Two examples of the most commonly used neighborhoods in the space of permutations are given by the 2-exchange and the insert operators (Taillard, 1990; Angel and Zissimopoulos, 2002; Schiavinotto and Stützle, 2005; Chicano et al., 2012). Given a permutation $\sigma = (\sigma(1) \cdots \sigma(n))$, one swap movement between the i-th and j-th items, $S_i^j(\sigma)$, is defined as follows:

$$\begin{array}{rcl} S_i^j(\sigma) & = & S_i^j\left(\sigma(1)\cdots\sigma(i-1)\sigma(i)\sigma(i+1)\cdots\sigma(j-1)\sigma(j)\sigma(j+1)\cdots\sigma(n)\right) = \\ & & (\sigma(1)\cdots\sigma(i-1)\sigma(j)\sigma(i+1)\cdots\sigma(j-1)\sigma(i)\sigma(j+1)\cdots\sigma(n)) \end{array}$$

Thus, the **2-exchange neighborhood**, $\mathcal{N}_S$, considers that two solutions are neighbors if one is generated by swapping two elements of the other:

$$\mathcal{N}_S(\sigma) = \bigcup_{i=1}^{n-1} \bigcup_{j=i+1}^{n} S_i^j(\sigma)$$

Given a permutation $\sigma = (\sigma(1) \cdots \sigma(n))$, the insertion of the i-th item into the j-th position, for $i > j$, is defined as follows:

$$\begin{array}{rcl} I_{i>j}^j(\sigma) & = & I_{i>j}^j\left(\sigma(1)\cdots\sigma(j-1)\sigma(j)\sigma(j+1)\cdots\sigma(i-1)\sigma(i)\sigma(i+1)\cdots\sigma(n)\right) = \\ & & (\sigma(1)\cdots\sigma(j-1)\sigma(i)\sigma(j)\sigma(j+1)\cdots\sigma(i-1)\sigma(i+1)\cdots\sigma(n)) \end{array}$$

while for $i < j$:

$$\begin{array}{rcl} I_{i<j}^j(\sigma) & = & I_{i<j}^j\left(\sigma(1)\cdots\sigma(i-1)\sigma(i)\sigma(i+1)\cdots\sigma(j-1)\sigma(j)\sigma(j+1)\cdots\sigma(n)\right) = \\ & & (\sigma(1)\cdots\sigma(i-1)\sigma(i+1)\cdots\sigma(j-1)\sigma(j)\sigma(i)\sigma(j+1)\cdots\sigma(n)) \end{array}$$

Thus, in the **insert neighborhood**, $\mathcal{N}_I$, two solutions are neighbors if one is the result of moving an element of the other to a different position:

$$\mathcal{N}_I(\sigma) = \left\{ \bigcup_{i=2}^{n} \bigcup_{j<i} I_{i>j}^j(\sigma) \right\} \cup \left\{ \bigcup_{i=1}^{n-2} \bigcup_{j>i+1} I_{i<j}^j(\sigma) \right\}$$

These two neighborhoods fulfill the symmetric property, that is,

$$\forall \sigma, \sigma' \in \Omega\,, \quad \sigma' \in \mathcal{N}(\sigma) \Longleftrightarrow \sigma \in \mathcal{N}(\sigma').$$

2.2 Local optima and plateaus

Assuming a maximization problem, a solution $\sigma^* \in \Omega$ is a **strict local optimum** under a neighborhood $\mathcal{N}$ if

$$f(\sigma^*) > f(\sigma),\ \forall \sigma \in \mathcal{N}(\sigma^*).$$

Allowing for the equality in the above equation, we have the definition of relaxed local optimum. In the literature, this is commonly known as local optimum. That is, a solution $\sigma^* \in \Omega$ is a **relaxed local optimum**, or simply, **local optimum** if

$$f(\sigma^*) \geq f(\sigma),\ \forall \sigma \in \mathcal{N}(\sigma^*).$$

We define a **connecting path** $\mathcal{L}(\sigma_1, \sigma_r)$ between a solution $\sigma_1 \in \Omega$ and another solution $\sigma_r \in \Omega$ as a sequence $\mathcal{L}(\sigma_1, \sigma_r) = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ where

$$\sigma_2 \in \mathcal{N}(\sigma_1), \sigma_3 \in \mathcal{N}(\sigma_2), \ldots, \sigma_r \in \mathcal{N}(\sigma_{r-1}).$$

A **plateau** is a set of solutions $\mathcal{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ that fulfills (Hoos and Stützle, 2004; Watson, 2010; Sutton et al., 2010):

(i) $\forall \sigma_i, \sigma_j \in \mathcal{P}$, $f(\sigma_i) = f(\sigma_j)$.

(ii) $\forall \sigma_i, \sigma_j \in \mathcal{P}$, $\exists \mathcal{L}(\sigma_i, \sigma_j) \subseteq \mathcal{P}$.

(iii) $\mathcal{P}$ is maximal, i.e., $\nexists \mathcal{Q}$ complying (i) and (ii) such that $\mathcal{P} \subset \mathcal{Q}$.

A plateau $\mathcal{P}$ is a **local optimal plateau** or **closed plateau** (Hoos and Stützle, 2004; Watson, 2010; Sutton et al., 2010) if all the solutions in $\mathcal{P}$ are local optima. We denote by $\mathcal{P}^*$ the local optimal plateaus. Thus, being $\mathcal{P}^*$ a local optimal plateau:

$$\bigcup_{\substack{\sigma^* \in \mathcal{P}^* \\ \sigma^* is\ loc\ opt.}} \{\sigma^*\} = \mathcal{P}^*,$$

The plateaus $\mathcal{P}$ that contain local optima as well as solutions that are not local optima are usually known in the literature as **benches** (Watson, 2010) or as **open plateaus** (Hoos and Stützle, 2004; Sutton et al., 2010). Therefore, if $\mathcal{P}$ is an open plateau:

$$\bigcup_{\substack{\sigma^* \in \mathcal{P} \\ \sigma^* is\ loc\ opt.}} \{\sigma^*\} \subset \mathcal{P}.$$

Given a plateau P we will denote by $O^*_{\mathcal{P}}$ the set composed by all the local optima found in the plateau:

$$O^*_{\mathcal{P}} = \bigcup_{\substack{\sigma^* \in \mathcal{P} \\ \sigma^* is\ loc\ opt.}} \{\sigma^*\}$$

Finally, there are also kinds of plateaus that do not contain any local optimum, called **unoptimal plateaus**.

In Figure 1 we represent four different structures found in a figurative combinatorial space where the neighboring solutions are joined by an edge. These examples show a strict local optimum 1(a), a local optimal plateau 1(b), an open plateau 1(c) and an unoptimal plateau 1(d). The strict and relaxed local optima are in red. So, notice that all the solutions that form the local optimal plateau (1(b)) are relaxed local optima.

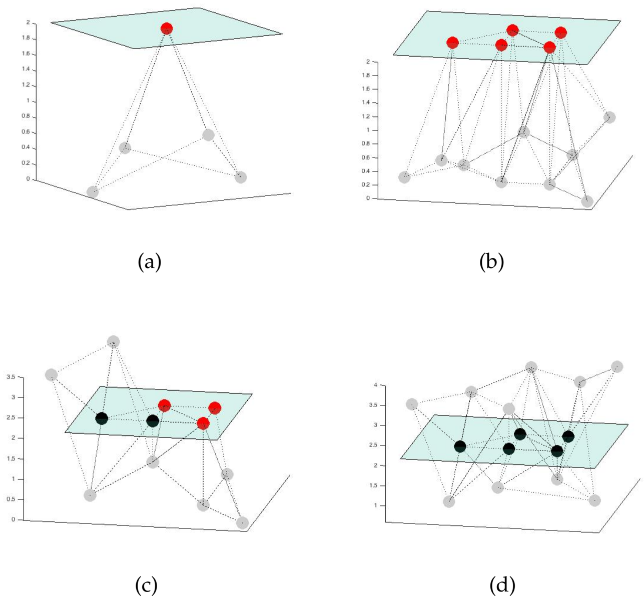


Figure 1: Representation of a strict local optimum (a), a local optimal plateau (b), an open plateau (c), and an unoptimal plateau (d) in a figurative combinatorial landscape when considering a maximization problem. The nodes represent the solutions of the search space and they are connected according to a neighborhood. The height indicates the objective function values. The solutions in red are the local optima, the non-optimal solutions that belong to a plateau are in black, while the rest of the solutions are in gray.

However, in an open plateau (1(c)), we do find solutions that are not local optima (in black), and also relaxed local optima. In both cases, 1(b) and 1(c), the red points form the set $O^*_{\mathcal{P}}$. In the unoptimal plateaus 1(d), no local optima are found.

It is very important to note that all local optimum σ^* will necessarily belong to one, and only one, of the three following cases:

1. Be a strict local optimum.
2. Belong to a local optimal plateau.
3. Belong to an open plateau.

2.3 Attraction basin of a local optimum

The attraction basin of a local optimum σ^*, $\mathcal{B}(\sigma^*)$, is composed of all the solutions which lead to the local optimum σ^*, after applying a greedy local search algorithm to them. We denote by $\mathcal{H}$ the operator that associates, to each solution σ, the local optimum obtained after applying the algorithm. Different definitions could be given for the attraction basin depending on the nature of the operator $\mathcal{H}$ (see for example Verel et al. (2011a); Watson (2010) for stochastic operators). We work with a deterministic $\mathcal{H}$; so, the **attraction basin of a local optimum**, $\mathcal{B}(\sigma^*)$, is the set that can be defined in the following way:

$$\mathcal{B}(\sigma^*) = \{\sigma \in \Omega \mid \mathcal{H}(\sigma) = \sigma^*\}\,.$$

It is also important to define the **attraction basin of a plateau**. We define this set as the union of the attraction basins of all the local optima that belong to such a plateau

(Klemm et al., 2014):

$$\mathcal{B}(\mathcal{P}) = \bigcup_{\sigma^* \in O^*_{\mathcal{P}}} \mathcal{B}(\sigma^*).$$

Of course, we can not refer to an attraction basin of an unoptimal plateau, as it does not contain any local optimum.

This definition of attraction basin of a plateau is consistent with the definitions found in the literature (Verel et al., 2011b; Daolio et al., 2014). In these works, the plateaus formed by local optima were defined as local optima neutral networks (LONN). The attraction basin of a LONN was defined as the set of all the solutions of the search space that belong to the attraction basin of any of the local optima that compose the LONN.

When two neighboring solutions belong to different attraction basins, it is said that the attraction basins are neighbors, that is, we define $\mathcal{B}(\sigma_1^*)$ and $\mathcal{B}(\sigma_2^*)$ as **neighboring attraction basins** if

$$\exists \sigma \in \mathcal{B}(\sigma_1^*) \text{ and } \exists \sigma' \in \mathcal{B}(\sigma_2^*) \text{ such that } \sigma' \in \mathcal{N}(\sigma).$$

In the same way, the attraction basins of two different plateaus, $\mathcal{P}_1$ and $\mathcal{P}_2$, are neighbors if

$$\exists \sigma \in \mathcal{B}(\mathcal{P}_1) \text{ and } \exists \sigma' \in \mathcal{B}(\mathcal{P}_2) \text{ such that } \sigma' \in \mathcal{N}(\sigma).$$

3 Topology of the attraction basins

The attraction basin of a local optimum depends on the algorithm used. Furthermore, when using a deterministic algorithm, an important property is derived from the concept of attraction basins of the local optima: they define a partition of Ω.

The distance d defined by both the swap and the insert neighborhoods is a metric, that is, $\forall \sigma_1, \sigma_2, \sigma_3 \in \Omega$:

(i) $d(\sigma_1, \sigma_2) = 0 \iff \sigma_1 = \sigma_2$

(ii) $d(\sigma_1, \sigma_2) = d(\sigma_2, \sigma_1)$

(iii) $d(\sigma_1, \sigma_3) \leq d(\sigma_1, \sigma_2) + d(\sigma_2, \sigma_3)$

Now, we can study topological features of the attraction basins. Particularly, we focus on three properties: the roundness of the attraction basins, the centrality of the local optima within the attraction basins, and the interior and frontier of the attraction basins.

3.1 Roundness of the attraction basins

We define the roundness of the attraction basins using the concept of closed ball. A **closed ball** $B_r(\pi)$ of radius $r > 0$ and centered at a permutation $\pi \in \Omega$ is:

$$B_r(\pi) = \{\sigma \in \Omega | d(\sigma, \pi) \leq r\}.$$

Considering the closed ball $B_{r^*}(\sigma^*)$ centered at the local optimum σ^* and with a radius $r^* = \max\limits_{\sigma_i \in \mathcal{B}(\sigma^*)} d(\sigma_i, \sigma^*)$, we say that an attraction basin $\mathcal{B}(\sigma^*)$ is round if

$$\mathcal{B}(\sigma^*) = B_{r^*}(\sigma^*).$$

Likewise, we define a closed ball $B_r(\mathcal{P})$ of radius $r > 0$ and centered at a plateau $\mathcal{P}$ as:

$$B_r(\mathcal{P}) = \{\sigma \in \Omega | d_{min}(\sigma, \mathcal{P}) \leq r\},$$

where $d_{min}(\sigma, \mathcal{P}) = \min_{\sigma^* \in O^*_{\mathcal{P}}} d(\sigma, \sigma^*)$.

In this case, we consider that the attraction basin of a plateau $\mathcal{B}(\mathcal{P})$ is round if

$$\mathcal{B}(\mathcal{P}) = B_{r^*}(\mathcal{P}),$$

where $r^* = \max_{\sigma_i \in \mathcal{B}(\mathcal{P})} d_{min}(\sigma_i, \mathcal{P})$.

3.2 Centrality of the local optima

Let us suppose σ^* to be a local optimum and $\sigma \in \mathcal{B}(\sigma^*)$, and let us consider the measure $\mathcal{D}_{\sigma^*}(\sigma)$ as the sum of the distances between σ and the rest of the permutations in $\mathcal{B}(\sigma^*)$:

$$\mathcal{D}_{\sigma^*}(\sigma) = \sum_{\sigma_i \in \mathcal{B}(\sigma^*)} d(\sigma_i, \sigma).$$

If

$$\sigma^* = \arg\min_{\sigma_i \in \mathcal{B}(\sigma^*)} \mathcal{D}_{\sigma^*}(\sigma_i)$$

then we say that the local optimum is centered within the attraction basin.

In the case of having a set of local optima belonging to a plateau $\mathcal{P}$, we calculate the measure $\mathcal{D}$ as follows.

If $\sigma \in \mathcal{B}(\mathcal{P})$ and $\sigma \notin O^*_{\mathcal{P}}$

$$\mathcal{D}_{\mathcal{P}}(\sigma) = \min_{\sigma_i \in O^*_{\mathcal{P}}} d(\sigma_i, \sigma) + \sum_{\substack{\sigma_i \in \mathcal{B}(\mathcal{P}) \\ \sigma_i \notin O^*_{\mathcal{P}}}} d(\sigma_i, \sigma),$$

and

$$\mathcal{D}_{\mathcal{P}}(O^*_{\mathcal{P}}) = \sum_{\substack{\sigma_i \in \mathcal{B}(\mathcal{P}) \\ \sigma_i \notin O^*_{\mathcal{P}}}} \min_{\sigma_j \in O^*_{\mathcal{P}}} d(\sigma_j, \sigma_i).$$

As in the previous case, we say that the plateau is centered within the attraction basin if

$$\forall \sigma \in \mathcal{B}(\mathcal{P}), \sigma \notin O^*_{\mathcal{P}}, \qquad \mathcal{D}_{\mathcal{P}}(O^*_{\mathcal{P}}) < \mathcal{D}_{\mathcal{P}}(\sigma).$$

3.3 Interior and frontier of the attraction basins

We differentiate the interior and the frontier of an attraction basin set. The **interior of the attraction basin of a local optimum** σ^*, $\mathcal{I}(\mathcal{B}(\sigma^*))$, is the subset composed of all the solutions in $\mathcal{B}(\sigma^*)$ which also have all their neighbors in $\mathcal{B}(\sigma^*)$:

$$\mathcal{I}(\mathcal{B}(\sigma^*)) = \{\sigma \mid \sigma \in \mathcal{B}(\sigma^*) \wedge \mathcal{N}(\sigma) \subset \mathcal{B}(\sigma^*)\}.$$

So, the **frontier of the attraction basin of a local optimum** σ^*, $\mathcal{F}(\mathcal{B}(\sigma^*))$, is composed of all the solutions in the attraction basin $\mathcal{B}(\sigma^*)$ that are not located in its interior, $\mathcal{F}(\mathcal{B}(\sigma^*)) = \mathcal{B}(\sigma^*) \setminus \mathcal{I}(\mathcal{B}(\sigma^*))$:

$$\mathcal{F}(\mathcal{B}(\sigma^*)) = \{\sigma \mid \sigma \in \mathcal{B}(\sigma^*) \wedge \mathcal{N}(\sigma) \not\subset \mathcal{B}(\sigma^*)\}.$$

Similarly, we define the **interior of the attraction basin of a plateau** $\mathcal{P}$,

$$\mathcal{I}(\mathcal{B}(\mathcal{P})) = \{\sigma \mid \sigma \in \mathcal{B}(\mathcal{P}) \ \wedge \ \mathcal{N}(\sigma) \subset \mathcal{B}(\mathcal{P})\},$$

and the **frontier of the attraction basin of a plateau** $\mathcal{P}$,

$$\mathcal{F}(\mathcal{B}(\mathcal{P})) = \{\sigma \mid \sigma \in \mathcal{B}(\mathcal{P}) \ \wedge \ \mathcal{N}(\sigma) \not\subset \mathcal{B}(\mathcal{P})\}.$$

4 Experiments

The topological features of the attraction basins described in Section 3 are analyzed in order to connect the observed results with a comprehension of the structure of the attraction basins. This will help us to contradict the commonly found extrapolation from the continuous domain to the combinatorial space, which is responsible for a visualization of the landscape full of mountains and valleys: the assumption that a ball of radius $r > 0$ and centered at the local optima is included in their attraction basins. For this purpose, we focus on permutation-based problems, and divide our analysis in two parts: the study of the type of solution returned by a classical deterministic hill-climbing algorithm in terms of plateaus, and the exploration of the topology of the attraction basins regarding the roundness, the centrality of the local optima and the interior and frontier of the attraction basins. First, the experimental design is detailed, and then the results are shown, followed by a thorough discussion relating them to the topology defined.

4.1 Experimental design

We work with instances of the permutation flowshop scheduling problem (PFSP), the linear ordering problem (LOP), and the quadratic assignment problem (QAP), which are known to be NP-hard problems (Garey et al., 1976; Mishra and Sikdar, 2004; Sahni and Gonzalez, 1976). The flowshop scheduling problem can be stated as follows: there are b jobs to be scheduled in c machines. A job consists of c operations and the j-th operation of each job must be processed on machine j for a specific processing time without interruption. We consider that the jobs are processed in the same order on different machines. Generally, the objective of the PFSP is to find the order in which the jobs have to be scheduled on the machines, minimizing the total flow time.

In the LOP, given a matrix $B = [b_{ij}]_{n \times n}$ of numerical entries, we have to find a simultaneous permutation of the rows and columns of B, such that the sum of the entries above the main diagonal is maximized (or equivalently, the sum of the entries below the main diagonal is minimized).

The QAP consists of allocating a set of facilities to a set of locations, with a cost function associated to the distance and the flow between the facilities. The objective is to assign each facility to a location such that the total cost is minimized. Specifically, we are given two $n \times n$ input matrices with real values $\mathbf{H} = [h_{ij}]$ and $\mathbf{D} = [d_{kl}]$, where h_{ij} is the flow between facility i and facility j, and d_{kl} is the distance between location k and location l. As PFSP and QAP are minimization problems, they have been transformed into maximization problems by simply reversing the sign of the cost function.

The solutions of these three problems are coded as permutations of size n, so the search space is of size $n!$. We use a deterministic best-improvement local search (see Algorithm 1) to solve the instances. It is important to notice that the neighbors are evaluated in a specific order, so that, in the case of two neighbors having the same function value, the algorithm will always choose the first encountered. Specifically, for the 2-exchange, the neighbors are explored swapping the items i and j in ascending order: the i-th item increases from 1 to $n-1$ and, for each value of i, the j-th item goes from $i+1$ to n. For the insert, the neighborhood is evaluated taking the i-th item and inserting it in the j-th position, also in order, with i increasing from 1 to n, and for each value of i, the j-th item also increases from 1 to n, always considering $j \neq i$.

Algorithm 1 Deterministic Best-Improvement Local Search Algorithm

```
Choose an initial solution σ ∈ Ω
repeat
  σ* = σ
  for each σ_k ∈ N(σ*) do
    if f(σ_k) > f(σ) then
      σ = σ_k
    end if
  end for
until σ = σ*
```

The instances used in the experiments have been taken from three well-known benchmarks. The PFSP instances were obtained from the Taillard's benchmark (Taillard, 1993), the LOP instances from the xLOLIB benchmark (Schiavinotto and Stützle, 2005) and the QAP instances from the QAPLIB (Burkard et al., 1997). First, we choose 8 instances for each problem (numbered from 1 to 8) for which the number of local optima and their attraction basins are exhaustively computed according to the 2-exchange ($\mathcal{N}_S$) and the insert ($\mathcal{N}_I$) neighborhoods. That is, Algorithm 1 is applied to each solution of the search space, for both neighborhoods separately. As this implies a high computational cost, the original instances were reduced in order to work with permutations of size $n = 10$, so that the experimentation is computationally affordable: $|\Omega| = 10! \approx 3.63 \cdot 10^6$. In the case of the PFSP, we reduce the instances considering 10 jobs and 5 machines. Secondly, in order to check if those properties observed in the small instances are also shared by higher dimensions, we work with 4 original instances of each problem of size $n = 50$ (50 jobs and 10 machines in the case of the PFSP instances). For these instances, Algorithm 1 is applied to a sample of initial solutions of size $M = 500$. Thus, a sample of local optima is obtained. In each run of the algorithm, a sample of solutions belonging to the attraction basin of each local optimum is also encountered: the initial solution plus all the solutions found in the path to the local optimum. We analyze the different features of the attraction basins of this sample of local optima. Of course, we do not aim to guarantee that all the different landscapes have the same properties as those observed in this paper.

In summary, we use $8+4$ instances for each problem, taking into account the 2-exchange and the insert neighborhoods, that is, a total of 72 different landscapes are analyzed. The specific instances used in the experimentation are available in the website[1].

4.2 Local optima and local optimal plateaus

Algorithm 1 returns a local optimum $\sigma^* \in \Omega$ such that, by definition, $f(\sigma^*) \geq f(\sigma), \forall \sigma \in \mathcal{N}(\sigma^*)$. As previously mentioned, this local optimal solution σ^*, will necessarily match one, and only one, of the three following options:

(i) Be a strict local optimum.

(ii) Belong to a local optimal plateau.

[1] http://www.sc.ehu.es/ccwbayes/members/leticia/AnatomyOfAB/Instances.html

(iii) Belong to an open plateau.

Papers about neutral networks can be found for NK landscapes (Verel et al., 2011b) and PFSP (Marmion et al., 2011b; Daolio et al., 2014). To the best of our knowledge, there are no analyses about the neutrality carried out for the LOP and the QAP. In the case of PFSP, while Marmion et al. (2011b); Daolio et al. (2014) used makespan as the objective function, our approach focuses on the total flow time. In fact, in the literature, the only work analyzing the neutrality for the total flow time is Hernando et al. (2017). Moreover, in this paper, we explicitly present the number and sizes of the plateaus, distinguishing between the local optimal and the open plateaus, which has not been reflected in any previous work. Our aim, in this section, is to explore those plateaus formed by local optima, as a first step in our topological analysis of the basins of attraction.

We report in Table 1, for the different instances of the problems and the different neighborhoods, in the columns labeled **LO**, the number of different solutions output by Algorithm 1. In the case of the small instances ($n = 10$), this is the total number of local optima, while for the larger instances ($n = 50$), it reflects the number of different solutions obtained after applying Algorithm 1 to the sample of 500 random initial solutions. Then, we show in the **strict LO**, **LO plateau** and **open plateau** columns, (i) the number of strict local optima, (ii) the number of local optimal plateaus and (iii) the number of open plateaus, respectively. Notice that the unoptimal plateaus are not included in this analysis because no local optimum belongs to them. These plateaus of the smaller, as well as the larger, instances have been calculated starting with each of the observed local optima and exploring recursively the neighboring solutions until no more solutions with the same fitness are found. Notice that the sum of the **strict LO**, **LO plateau** and **open plateau** columns is not necessarily the same as the number of **LO** (more than one **LO** can belong to the same plateau). In fact, we add, in parenthesis, the average and the standard deviation of the number of solutions forming the plateaus. Remember that the local optimal plateaus are composed only of local optima, while the open plateaus include other non-optimal solutions. We emphasize in bold, for the smaller instances, whether the global optimum is a strict local optimum or whether it belongs to a local optimal plateau. Obviously, the global optima can never belong to an open plateau. However, in the case of multiple global optima, some of them could be strict local optima while others could form one or more plateaus, meaning that the global optima of the instance belong to two different classes of local optima: (i) and (ii).

Regarding the results, very different values are obtained for instances of the same problem. In general, the presence of plateaus is remarkable. The average size of all the plateaus found in the PFSP instances, under both neighborhoods, is around 2 (with a low standard deviation). In the smaller LOP instances, for the 2-exchange neighborhood, the number of plateaus is higher than for the insert neighborhood. For the larger LOP instances, the number of plateaus is high and their size is considerably large. Most of the QAP smaller instances contain a large number of plateaus under both neighborhoods, but, in general, for the larger instances, except for the *Wil50* instance, the presence of plateaus is small. Finally, for all the PFSP instances the global optima are strict local optima. However, in LOP and QAP, strict global optima as well as global optimal plateaus are found, according to the instance.

In general, the presence of local optima belonging to plateaus is a tangible aspect of permutation-based combinatorial problems, or, at least, one should not work with these problems assuming that they do not exist. One of the main conclusions derived from this study is that, usually, Algorithm 1 gets trapped inside the plateaus. Although we find instances with plateaus composed by just two solutions, Algorithm 1 is not designed to detect and escape from them. Of course, we can not overlook the fact that some authors have already studied the number and extension of the plateaus. In Marmion et al. (2011b), the authors concluded that plateaus with local optima are numerous and large for the PFSP. However, the objective function to minimize considered in that work was the makespan, instead of the total flow time, as in this paper. According to a recent work (Hernando et al., 2017) there is a lower number of local optima sharing the fitness value when minimizing the total flow time than when minimizing the makespan. Indeed, Table 1 shows that, for the PFSP instances, the presence of local optima belonging to plateaus is not so high.

Once the plateaus are explored, we take into account that those local optima belonging to a plateau share the same attraction basin. An attraction basin, thus, will be the attraction basin of a strict local optima or that of a plateau (optimal or open).

Table 1: For each instance of the PFSP, the LOP and the QAP considering the 2-exchange and the insert neighborhoods, the different number of solutions in which the Algorithm 1 stops (**LO**) is reported. From these solutions, the number of strict local optima (**strict LO**), the number of local optimal plateaus (**LO plateau**) and open plateaus (**open plateau**) are detailed, indicating in parenthesis the average / standard deviation of the number of solutions forming the plateaus. For the smaller instances, the type of solutions where the global optima are found is emphasized in **bold**.

		2-EXCHANGE NEIGHBORHOOD				INSERT NEIGHBORHOOD			
		LO	**strict LO**	**LO plateau**	**open plateau**	**LO**	**strict LO**	**LO plateau**	**open plateau**
PFSP	Inst 1	11	**11**	0 (0.00/0.00)	0 (0.00/0.00)	3	**2**	0 (0.00/0.00)	1 (2.00/0.00)
	Inst 2	24	**22**	0 (0.00/0.00)	2 (2.00/0.00)	7	**7**	0 (0.00/0.00)	0 (0.00/0.00)
	Inst 3	58	**36**	9 (2.00/0.00)	4 (2.25/0.50)	8	**6**	1 (2.00/0.00)	0 (0.00/0.00)
	Inst 4	83	**67**	2 (2.00/0.00)	12 (2.08/0.29)	8	**6**	0 (0.00/0.00)	2 (2.00/0.00)
	Inst 5	117	**91**	6 (2.00/0.00)	14 (2.29/0.82)	16	**9**	3 (2.00/0.00)	1 (2.00/0.00)
	Inst 6	158	**145**	1 (2.00/0.00)	11 (2.00/0.00)	31	**27**	1 (2.00/0.00)	2 (2.00/0.00)
	Inst 7	225	**194**	4 (2.00/0.00)	21 (2.71/1.19)	43	**39**	1 (2.00/0.00)	2 (3.50/2.12)
	Inst 8	295	**274**	1 (2.00/0.00)	19 (2.16/0.50)	24	**23**	0 (0.00/0.00)	1 (2.00/0.00)
	tai50_10.00	500	454	27 (2.00/0.00)	19 (2.16/0.50)	500	462	23 (2.13/0.46)	15 (2.40/0.74)
	tai50_10.01	500	443	34 (2.03/0.17)	23 (2.26/0.75)	500	457	22 (2.00/0.00)	21 (2.33/0.97)
	tai50_10.02	500	462	17 (2.00/0.00)	21 (2.00/0.00)	500	458	22 (2.09/0.43)	20 (2.50/1.39)
	tai50_10.03	500	442	30 (2.23/0.63)	28 (2.39/0.79)	500	457	22 (2.10/0.31)	21 (2.20/0.52)
LOP	Inst 1	13	**13**	0 (0.00/0.00)	0 (0.00/0.00)	1	**1**	0 (0.00/0.00)	0 (0.00/0.00)
	Inst 2	24	**20**	0 (0.00/0.00)	4 (2.00/0.00)	1	**1**	0 (0.00/0.00)	0 (0.00/0.00)
	Inst 3	112	**1**	**4** **(2.25/0.50)**	49 (10.61/9.13)	3	0	**1** **(3.00/0.00)**	0 (0.00/0.00)
	Inst 4	129	5	**8** **(2.75/1.04)**	44 (7.98/7.84)	2	0	**1** **(2.00/0.00)**	0 (0.00/0.00)
	Inst 5	171	22	**28** **(2.89/1.23)**	31 (4.77/2.54)	9	0	**3** **(3.00/1.00)**	0 (0.00/0.00)
	Inst 6	226	**9**	9 (3.11/1.05)	79 (6.99/4.90)	4	**1**	0 (0.00/0.00)	1 (4.00/0.00)
	Inst 7	735	0	**2** **(15.00/0.00)**	34 (76.79/97.36)	15	0	**1** **(15.00/0.00)**	0 (0.00/0.00)
	Inst 8	8652	0	**1** **(735.00/0.00)**	22 (13805.41/10621.15)	2399	0	**1** **(735.00/0.00)**	4 (2996.00/2103.94)
	N-be75eec	500	0	137 (13579.04/89.64)	363 (88620.04/102.77)	500	0	97 (8522.36/598.33)	403 (14522.36/645.00)
	N-be75np	500	0	201 (22540.79/249.82)	299 (19979.36/425.13)	467	0	445 (692.18/296.55)	22 (864.91/426.84)
	N-be75oi	500	0	185 (45978.01/304.33)	315 (50003.21/297.99)	500	0	105 (34450.12/654.37)	395 (81657.98/675.54)
	N-be75tot	500	0	175 (94561.57/245.74)	325 (65265.79/351.20)	400	0	287 (312.44/103.83)	113 (559.93/142.42)
QAP	Inst 1	19	**19**	0 (0.00/0.00)	0 (0.00/0.00)	2713	**2713**	0 (0.00/0.00)	0 (0.00/0.00)
	Inst 2	108	94	**5** **(2.00/0.00)**	4 (2.00/0.00)	4464	4374	**32** **(2.00/0.00)**	26 (2.00/0.00)
	Inst 3	165	**68**	10 (2.30/0.67)	63 (7.54/11.09)	2975	**1725**	99 (2.14/0.43)	972 (4242.04/11133.48)
	Inst 4	474	**474**	0 (0.00/0.00)	0 (0.00/0.00)	16433	**16412**	6 (2.00/0.00)	9 (2.00/0.00)
	Inst 5	476	**288**	0 (0.00/0.00)	152 (6.95/9.24)	5352	**3388**	234 (2.12/0.48)	1334 (184.51/1715.27)
	Inst 6	598	**296**	25 (2.20/0.58)	223 (3.95/3.25)	15394	**10690**	863 (2.09/0.37)	2666 (208.01/2138.66)
	Inst 7	752	0	**34** **(7.76/5.30)**	70 (2916.57/4455.86)	9112	**1560**	**1858** **(2.50/1.10)**	1742 (6149.37/14113.83)
	Inst 8	1840	0	**26** **(12.92/10.99)**	192 (22591.33/36992.05)	12360	0	**1060** **(2.74/1.46)**	3580 (90831.07/88548.56)
	Lipa50b	474	471	1 (2.00/0.00)	2 (2.00/0.00)	500	499	0 (0.00/0.00)	1 (2.00/0.00)
	Tai50a	500	499	0 (0.00/0.00)	1 (2.00/0.00)	500	498	0 (0.00/0.00)	2 (2.00/0.00)
	Tai50b	500	500	0 (0.00/0.00)	0 (0.00/0.00)	500	500	0 (0.00/0.00)	0 (0.00/0.00)
	Wil50	500	164	92 (2.51/1.08)	244 (5.65/8.43)	500	264	120 (2.47/1.04)	116 (3.99/4.20)

4.3 Topology of the attraction basins

From our point of view, the structure of the attraction basins can be characterized by these principal aspects: roundness of the attraction basins, centrality of the local optima, and interior and frontier of the attraction basins. We examine these topological features of the attraction basins of the strict local optima, the open plateaus and the local optimal plateaus.

4.3.1 Roundness of the attraction basins

As was explained in Section 3.1, an attraction basin is considered to be round if all the solutions at distance 1, 2, ... until a certain distance r from the local optimum or the plateau are within the attraction basin. We record, for the smaller instances, for each local optimum, the proportion of solutions belonging to its attraction basin that are at different distances from it. Just as a reference, in Table 2, we show the total number of solutions at different distances from any solution in the space of permutations of size 10. Notice that these values differ from the number of solutions at different distances when referring to a plateau. The distance between one plateau and one solution is the minimal distance between this solution and all the solutions in the plateau. For example, if a plateau is formed by two solutions, the number of permutations at distance one is the sum of the number of the neighbors of both solutions, eliminating repetitions and both solutions themselves. Furthermore, in this case, the maximum reachable distance would be 8. For the larger instances, we take those local optima obtained in the sample of size 500, and, as it is impossible to check the solutions at all the possible distances from them, we focus on those at distance 1. So, we analyze if the solutions at distance 1 from each of the local optima (or distance 1 from the plateaus) belong to its same attraction basin. We also record, among the solutions found in the sample for each local optimum, the maximum distance at which a solution is within the attraction basin.

In Figure 2, the results are plotted distinguishing between problems and neighborhoods. In the X axis of the figures the distance to the local optimum is indicated and the Y axis shows the percentage of solutions that belong to their attraction basins. For the smaller instances (bars in gray), the average value obtained in each landscape and the maximum and minimum percentages found are represented. Those instances of the LOP that present just one local optimum under the insert neighborhood have been removed from Figure 2. The average percentage of solutions in the attraction basin decreases with the distance to the local optimum. The maximum and the minimum percentage found for a local optimum also decreases with the distance to the local optimum for the three problems considering the 2-exchange neighborhood. This also happens for the QAP and the insert neighborhood. However, for the PFSP and LOP under the insert neighborhood, we find that the maximum percentage encountered increases when reaching the longest distances. The reason for this phenomenon is that, for the insert neighborhood, the number of possible solutions decreases quickly at long distances and, for example, for those strict local optima, there is just 1 possible solution at distance 9 (see Table 2). Therefore, if we find a strict local optimum whose solution at the maximum distance belongs to its attraction basin, we will find that 100% of the solutions (just one) at this distance belongs to it, as observed for the PFSP.

Regarding the smaller instances, the average percentage of solutions at distance 1 from the local optima is lower than 100. This means that, on average, we can find neighboring solutions of the local optima that belong to different attraction basins. Obviously, in all the problems and neighborhoods we find at least one local optimum with all its neighbors belonging to its own attraction basin: the global optima or the global

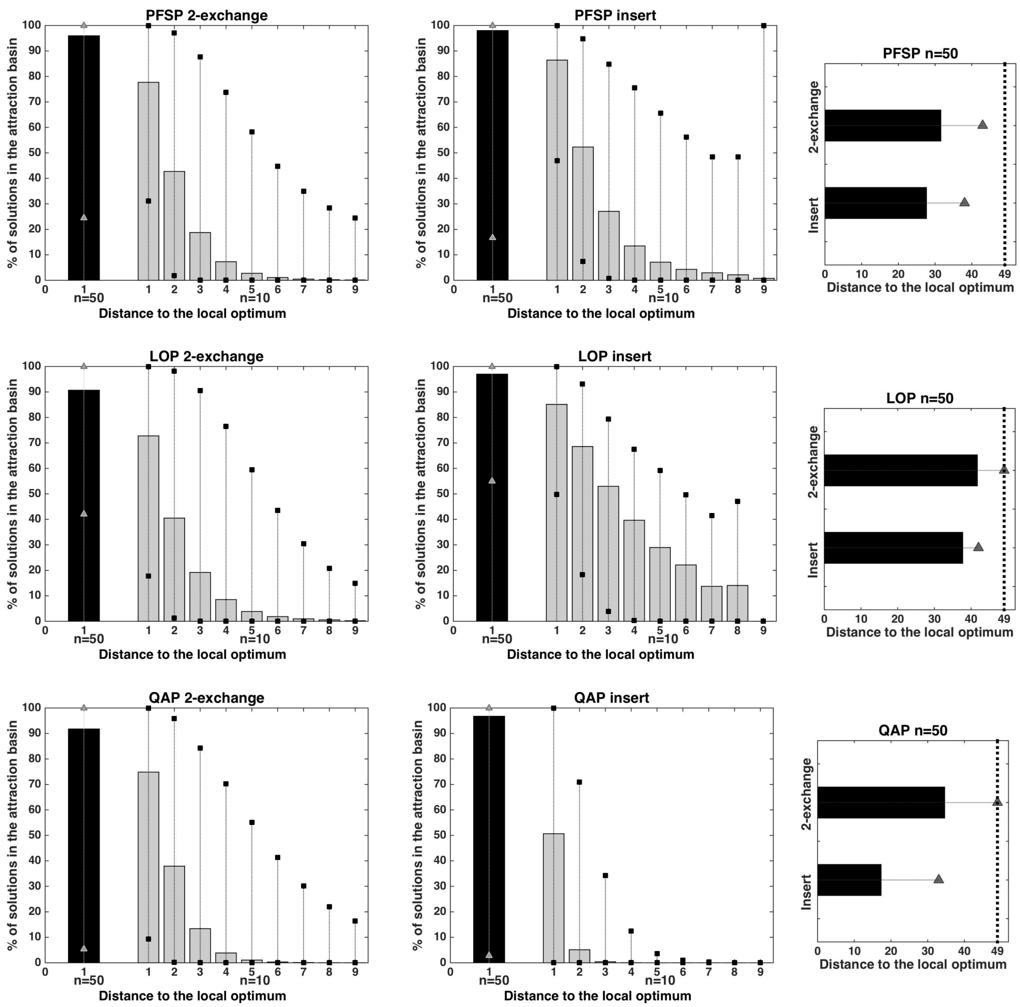


Figure 2: Average, maximum and minimum percentage of solutions that are at different distances from the local optima and that belong to its attraction basin, for the instances of the (from top to bottom) PFSP, LOP, and QAP, considering the 2-exchange and the insert neighborhoods. For the instances with $n = 10$ (bars in gray), the results at all the possible distances are shown, while for the instances with $n = 50$ (bars in black), just those at distance 1, which is the only computationally affordable in a reasonable time. The graphs on the right represent the average (and maximum) of the maximal distances found in the sample obtained for the attraction basins of the instances with $n = 50$.

Table 2: Number of permutations of size 10 at the different distances from a given solution according to the type of neighborhood.

	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8	d=9
2-exchange	45	870	9450	63273	269325	723680	1172700	1026576	362880
Insert	81	2521	38281	296326	1100902	1604098	569794	16795	1

optimal plateaus. The results obtained for the larger instances (bars in black) confirm this result. Taking into account the local optima found in the sample, on average, there is a probability higher than 0 of finding a neighboring solution of the local optima belonging to a different attraction basin. In all the scenarios, we find local optima whose attraction basins contain solutions far from them. In the case of the smaller instances, solutions at distance 8 or 9 to the local optima are found in the attraction basins. For the larger instances, in Figure 2 we add a graph showing the average of the maximum distances from the local optima at which a solution inside the attraction basin has been found. The maximum distance encountered in all the attraction basins is also included (the gray triangles). The dashed line indicates the maximum possible distance (49). According to these plots, solutions far from the local optima are also inside their attraction basins. Notice that the maximum distances presented here are lower bounds of the real ones, this is because we are exploring just a sample of the solutions of the attraction basins. However, in this sample for the LOP and QAP using the 2-exchange neighborhood, attraction basins with solutions at the maximum possible distance from the local optima have already been found.

The percentage of the solutions in an attraction basin is related with the escape rate analyzed in a number of works, such as, Merz (2004); Daolio et al. (2014). The escape rate is a measure that gives the probability of reaching a different attraction basin. According to all those works, the escape rate increases with the distance to the local optima, that is precisely what can be observed in Figure 2: the decrease of the percentage of the solutions in the attraction basins means the growth of the percentage of the solutions in different attraction basins. Although examining the roundness of the attraction basins and delving into the escape rate produce closely related information, we consider that our perspective is more suitable for giving explicit knowledge about the topology of the attraction basins and helps in the visualization of the structure of these sets.

In general, Figure 2 reveals that, on average, the local optima are located in the frontier of the attraction basins, as they have a number of neighboring solutions belonging to a different attraction basin. However, on average, we also find solutions at the longest distances from them that do belong to their attraction basins. This structure clearly differs from the concept of roundness.

4.3.2 Centrality of the local optima

We aim to study the position of the local optima within the attraction basins. For this purpose, we focus on the centrality of the local optima inside the attraction basins.

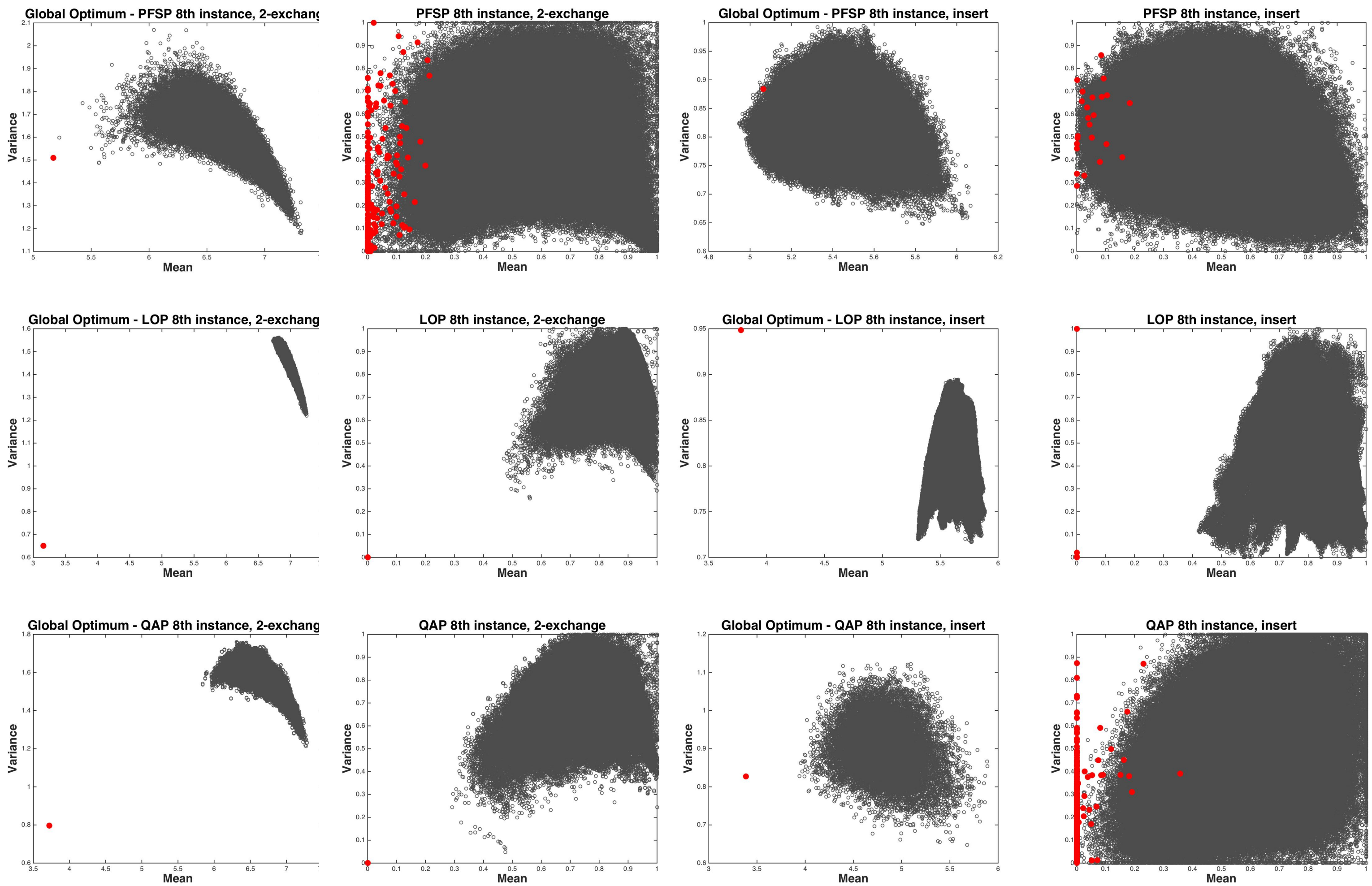


Figure 3: For each solution of the attraction basin, the average and variance of the distances to the rest of the solutions in the attraction basin are shown for the $8th$ instance of the PFSP, the $8th$ instance of the LOP and the $8th$ instance of the QAP (from top to bottom, respectively), using the 2-exchange and insert neighborhoods (from left to right, respectively). The results corresponding to the local optima are in red, while those of the rest of solutions are in gray.

As seen in Section 3.2, the local optima (or the plateaus) are considered to be centered if they minimize the average distance to the rest of the solutions in the attraction basin. In order to examine this point, we calculate, for each solution in each attraction basin, the previously defined measure $\mathcal{D}$. As computing this experiment for the larger instances is unaffordable, it has been done just for those smaller instances. For this experiment, we show, as representative cases, one instance of each problem (8th instance of PFSP, 8th instance of LOP and 8th instance of QAP) and we report, in Figure 3, the results obtained for the attraction basins of the global optima as well as the results obtained for all the attraction basins of all the local optima of each of the selected instances. The figures with the distances of the remaining instances are available at the website[2]. In the X axis of Figure 3 the average distance, that is, $\frac{\mathcal{D}_{\sigma^*}}{|\mathcal{B}(\sigma^*)|}$ or $\frac{\mathcal{D}_{\mathcal{P}}}{|\mathcal{B}(\mathcal{P})|}$, is indicated, while its variance is represented in the Y axis. So, each point in gray shows the average/variance of the distances for those solutions that are not local optima, and the points in red represent the local optima. The three figures in the first and third columns display the distances for the solutions of the attraction basins of the global optima under the 2-exchange and the insert neighborhoods, respectively. In the second and fourth columns, the distances of the different solutions of each attraction basin have been scaled to the interval [0,1] in order to compare different results belonging to different attraction basins. Each of these plots shows the scaled average and variance of the distance from each solution of the search space to the remaining solutions in its same attraction basin. So, a total of $10!$ points are plotted, highlighting in red those referring to the local optima.

All the plots are very similar for all the problem instances. The global optima have a really small average distance to the rest of the permutations in their attraction basins. From those graphs that represent all the solutions of the search space, we can deduce that not only the global optima, but also all of the local optima of the instance have a lower average distance than the rest of the solutions of the attraction basins. In fact, a large proportion of the local optima have their average scaled values equal to zero. We conclude that the local optima are located close to the barycenter of the attraction basins, as they have the minimal (or almost the minimal) average distances to the rest of the solutions: in general, they are almost centered within the attraction basins. The large variance observed in some of the local optima (particularly, for the insert neighborhood), corresponds with the behavior observed in Figure 2: not all the solutions close to the local optima are in their attraction basins, and, at the longest distances, some solutions are found which belong to them.

4.3.3 Interior and frontier of the attraction basins

According to the study about the roundness of the attraction basins (Section 4.3.1), in most cases, the local optima are located in the frontier of the attraction basins. We develop two different analyses for a deep examination regarding the interior and frontier of the attraction basins. First, we check whether each solution in the attraction basins is located in the interior, and for each attraction basin we record the number of neighboring attraction basins. Next, we explore the evolution of the number of neighboring attraction basins and the evolution of the neighboring solutions that share the same attraction basin, as the distance to the local optimum increases.

I. The interior and the neighboring relations of the attraction basins

In Table 3 we report, for each instance, distinguishing between the 2-exchange and

[2] http://www.sc.ehu.es/ccwbayes/members/leticia/AnatomyOfAB/centralityLO.html

Table 3: Percentage of the number of solutions that fall in the interior of the attraction basins and, for the smaller instances, the last column shows the average number of neighboring attraction basins.

			strict LO + LO plateau + plateau	% in the interior of the attraction basins Mean (Variance)	Maximum	Minimum	Average number of neighboring att. bas.
2-EXCHANGE	PFSP	Inst 1	11	0.0392 (0.0037)	0.1531	0.0000	10.00
		Inst 2	24	0.0169 (0.0021)	0.2026	0.0000	23.00
		Inst 3	49	0.0020 (0.0000)	0.0387	0.0000	47.76
		Inst 4	81	0.0016 (0.0000)	0.0420	0.0000	75.73
		Inst 5	111	0.0010 (0.0000)	0.0214	0.0000	106.41
		Inst 6	157	0.0007 (0.0000)	0.0231	0.0000	146.82
		Inst 7	219	0.0012 (0.0000)	0.0221	0.0000	200.34
		Inst 8	294	0.0007 (0.0000)	0.0288	0.0000	260.86
		tai50_10.00	500	0.0159 (0.0422)	2.7027	0.0000	
		tai50_10.01	500	0.0000 (0.0000)	0.0000	0.0000	
		tai50_10.02	500	0.0071 (0.0255)	3.5714	0.0000	
		tai50_10.03	500	0.0205 (0.0775)	5.0000	0.0000	
	LOP	Inst 1	13	0.0075 (0.0003)	0.0458	0.0000	12.00
		Inst 2	24	0.0320 (0.0241)	0.7606	0.0000	23.00
		Inst 3	54	0.0002 (0.0000)	0.0026	0.0000	51.41
		Inst 4	57	0.0003 (0.0000)	0.0110	0.0000	54.04
		Inst 5	81	0.0008 (0.0000)	0.0087	0.0000	78.91
		Inst 6	97	0.0002 (0.0000)	0.0117	0.0000	89.67
		Inst 7	36	0.0013 (0.0000)	0.0166	0.0000	33.83
		Inst 8	23	0.0180 (0.0052)	0.3420	0.0000	20.26
		N-be75eec	500	0.0000 (0.0000)	0.0000	0.0000	
		N-be75np	500	0.0085 (0.0195)	2.7397	0.0000	
		N-be75oi	500	0.0000 (0.0000)	0.0000	0.0000	
		N-be75tot	500	0.0000 (0.0000)	0.0000	0.0000	
	QAP	Inst 1	19	0.0095 (0.0010)	0.1311	0.0000	18.00
		Inst 2	103	0.0036 (0.0001)	0.0588	0.0000	97.07
		Inst 3	141	0.0008 (0.0000)	0.0288	0.0000	130.68
		Inst 4	474	0.0011 (0.0000)	0.0304	0.0000	398.33
		Inst 5	440	0.0005 (0.0000)	0.0190	0.0000	377.24
		Inst 6	544	0.0003 (0.0000)	0.0180	0.0000	423.57
		Inst 7	104	0.0015 (0.0000)	0.0441	0.0000	91.77
		Inst 8	218	0.0001 (0.0000)	0.0059	0.0000	152.19
		Lipa50b	474	0.3846 (18.2420)	67.4419	0.0000	
		Tai50a	500	0.0887 (0.4069)	7.4074	0.0000	
		Tai50b	500	0.0714 (0.4059)	8.4746	0.0000	
		Wil50	500	0.1747 (0.8887)	11.8644	0.0000	
INSERT	PFSP	Inst 1	3	0.0628 (0.0037)	0.1212	0.0000	2.00
		Inst 2	7	0.0166 (0.0008)	0.0616	0.0000	6.00
		Inst 3	7	0.0167 (0.0020)	0.1171	0.0000	6.00
		Inst 4	8	0.0064 (0.0002)	0.0447	0.0000	7.00
		Inst 5	13	0.0284 (0.0105)	0.3691	0.0000	12.00
		Inst 6	30	0.0002 (0.0000)	0.0024	0.0000	29.00
		Inst 7	42	0.0002 (0.0000)	0.0068	0.0000	40.81
		Inst 8	24	0.0010 (0.0000)	0.0094	0.0000	23.00
		tai50_10.00	500	0.0000 (0.0000)	0.0000	0.0000	
		tai50_10.01	500	0.0000 (0.0000)	0.0000	0.0000	
		tai50_10.02	500	0.0045 (0.0103)	2.2727	0.0000	
		tai50_10.03	500	0.0096 (0.0232)	2.5641	0.0000	
	LOP	Inst 1	1	100.0000 (0.0000)	100.0000	100.0000	—
		Inst 2	1	100.0000 (0.0000)	100.0000	100.0000	—
		Inst 3	1	100.0000 (0.0000)	100.0000	100.0000	—
		Inst 4	1	100.0000 (0.0000)	100.0000	100.0000	—
		Inst 5	3	0.1612 (0.0052)	0.2216	0.0818	2.00
		Inst 6	2	0.4072 (0.1448)	0.6763	0.1381	1.00
		Inst 7	1	100.0000 (0.0000)	100.0000	100.0000	—
		Inst 8	5	0.7944 (3.1556)	3.9722	0.0000	3.60
		N-be75eec	500	0.0524 (0.3991)	10.0000	0.0000	
		N-be75np	432	2.7516 (37.4291)	29.4118	0.0000	
		N-be75oi	500	0.0305 (0.1204)	4.6512	0.0000	
		N-be75tot	400	0.1546 (1.1971)	16.9811	0.0000	
	QAP	Inst 1	2713	0.0000 (0.0000)	0.0199	0.0000	558.08
		Inst 2	4432	0.0002 (0.0000)	0.0876	0.0000	1167.52
		Inst 3	2796	0.0001 (0.0000)	0.0606	0.0000	771.59
		Inst 4	16427	0.0009 (0.0001)	0.4348	0.0000	1453.54
		Inst 5	4956	0.0001 (0.0000)	0.0358	0.0000	1214.20
		Inst 6	14219	0.0005 (0.0001)	0.2283	0.0000	1356.76
		Inst 7	5160	0.0000 (0.0000)	0.0000	0.0000	1180.41
		Inst 8	4640	0.0000 (0.0000)	0.0000	0.0000	909.66
		Lipa50b	500	0.0000 (0.0000)	0.0000	0.0000	
		Tai50a	500	0.0000 (0.0000)	0.0000	0.0000	
		Tai50b	500	0.0000 (0.0000)	0.0000	0.0000	
		Wil50	500	0.0000 (0.0000)	0.0000	0.0000	

the insert neighborhoods, the percentage of the solutions found in the attraction basins that have all their neighboring solutions in their same attraction basin (interior solutions). For the smaller instances, all the solutions of the search space are checked, while, for the larger instances, the percentages refer to the solutions found in the sample (not only the 500 initial random solutions, but also those solutions that the algorithm finds in the paths until it reaches the local optima). The first column in Table 3 shows the sum of the number of **strict LO**, **LO plateau** and **plateau** seen in Table 1. In the second column of Table 3, the average percentage of the interior solutions with its variance is reported, and the third and fourth columns provide the maximum and the minimum percentages found for each of the instances.

In general, the percentages are really small, with the exception of those instances that have just one local optimum, for which, obviously, 100% of the solutions are located in the interior of the only attraction basin (all of the solutions of the search space are in the same attraction basin). We also find some high maximum percentages for the QAP larger instances under the 2-exchange neighborhood and for the LOP larger instances under the insert neighborhood. However, the average percentages obtained for these instances are very low. This result matches with some previous works where it was found that, for different optimization problems, most of the solutions were located in the frontier of the attraction basins (Fonlupt et al., 1999; Tomassini et al., 2008; Verel et al., 2008; Ochoa et al., 2014).

The fact that there is a really low number of solutions in the interior of the attraction basins leads us to report in the last column, for the smaller instances, the average number of neighboring attraction basins. Surprisingly, for most of the instances of all problems, and considering both neighborhoods, we find that this average value is really high: close to the total number of local optima. Basically, we could conclude that, for each pair of attraction basins, we can find two neighboring solutions belonging to each of them: almost all of the attraction basins are neighboring attraction basins. This result reveals that all the attraction basins are intertwined in the search space.

II. Evolution of the neighboring attraction basins with the distance to the local optima

We still do not know if each of these solutions in the frontier touches just one or more attraction basins. Furthermore, we do not know if the distance from the frontier solution to the local optimum affects this number of neighboring attraction basins. So, in order to take a step forward in the analysis about the frontier of the attraction basins, we delve into the neighboring relations of the attraction basins as the distance to the local optima increases.

In this sense, for each solution, we study (i) the number of its neighbors that belong to its same attraction basin, and (ii) for all those neighbors that do not belong to that attraction basin, we record the number of different attraction basins observed. Notice that, when working with $n = 10$, each solution can have a maximum of 45 and 81 different neighboring attraction basins, for the 2-exchange and the insert neighborhoods, respectively (the size of the neighborhood in each case). In the case of $n = 50$, this maximum number is 1225 and 2401, respectively. In Figure 4 the results are distinguished between problems and neighborhoods. The results are also separated for the larger and the smaller instances, indicating $n = 10$ and $n = 50$ in each case. In all the graphs, the X axis indicates the distance of the

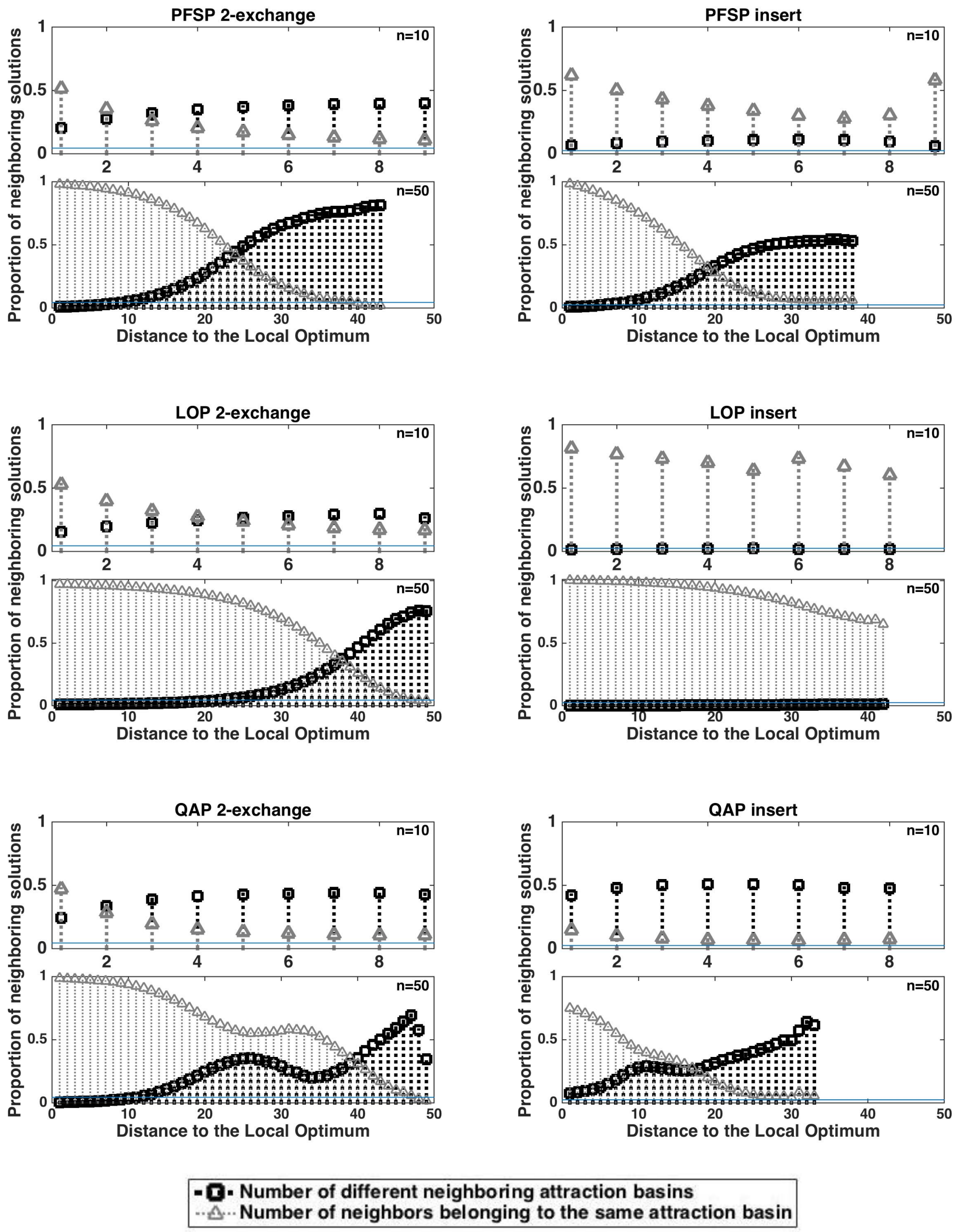


Figure 4: Average of the proportion of the number of different neighboring attraction basins and of the proportion of the number of neighbors belonging to the same attraction basin, according to the distance of the solutions to the local optimum. The different graphs show the results for the instances of the PFSP, LOP and QAP (from top to bottom), when considering the 2-exchange (left) and the insert (right) neighborhoods. Inside each of these graphs, the smaller ($n = 10$) and the larger (n=50) instances are distinguished.

solutions in the attraction basins to the local optima. For $n = 10$, this distance goes from 1 to 9 (the maximum possible distance for both neighborhoods), and for $n = 50$ it goes from 1 to 49. In each graph, the Y axis shows the following:

(i) in gray, the average of the proportion of neighboring solutions that fall in the same attraction basin.

(ii) in black, the average of the proportion of the number of different neighboring attraction basins per number of neighboring solutions.

For the three problems a general behavior is observed. The average proportion of the number of neighbors belonging to the same attraction basin decreases with the distance to the local optima, while the average proportion of the number of different neighboring attraction basins increases. That is, those solutions that are close to the local optimum have a large proportion of their neighbors inside the same attraction basin. The solutions that are far from the local optimum have a small number of neighbors in the same attraction basin. At the same time, the number of different neighboring attraction basins is large. It seems that if we take all the solutions of an attraction basin, we will find that the connectivities with other different attraction basins are higher for those solutions at long distances from the local optima. The behavior for the LOP instances with the insert neighborhood is different than in the rest of the landscapes. We could say that in this scenario, the attraction basins are less interconnected with each other than in the others.

We should take into account that the attraction basins are built by the local search algorithm. It draws paths that start at a certain point and continue from neighboring to neighboring solutions, ending at the local optimum. Therefore, the starting point has at least one neighbor in the same attraction basin, and the rest of the points visited across the path have at least two neighbors in the same attraction basin. In the case of a solution having three or more neighbors belonging to the attraction basin, it means that either the paths are not completely disconnected and they share a number of solutions, or they are so close to each other that the solutions in one path are neighbors of those of the other paths. In order to examine this point, we add a blue line in all the graphs of Figure 4. This blue line indicates the proportion of 2 divided by the size of the neighborhood; that is, $2/45$ and $2/81$ in the case of $n = 10$ for the 2-exchange and the insert neighborhoods, respectively, and $2/1225$ and $2/2401$ for $n = 50$ for the 2-exchange and the insert neighborhoods, respectively. So, if the gray triangle is above the blue line, it means that, on average, the solutions have three or more neighbors in the same attraction basin. For the instances of size $n = 10$, the blue line is always below the average proportion of the number of neighbors in the same attraction basin, even for those solutions at the longest distances. For the instances of size $n = 50$, for the insert neighborhood, this average proportion is also higher than the blue line for all the distances. Considering the 2-exchange neighborhood, just for those solutions at the longest distances, the average of the number of neighbors in the same attraction basin is equal to or lower than 2. Under this result, we can assert that the paths drawn by the local search algorithm are interconnected with each other or, at least, they are close to each other.

4.4 Discussion

The attraction basins can be understood as long intertwined rivers, that flow into the different local optima, instead of being mountains in a landscape. In fact, each attraction basin is composed of several of those rivers ending at the same local optimum, while at the same time, each of them could have different tributaries. Moreover, the end of those rivers can be made up of more than one local optimum that have the same objective function value, forming a plateau. The local optima or the plateaus composed by local optima are centered in the attraction basin. Nevertheless, we should be cautious with this perception, because by understanding the combinatorial optimization landscapes as if we were in a 3D natural landscape, we could be misunderstanding the real anatomy.

Two main conclusions regarding the performance of the algorithms based on local search could be derived from this understanding of the combinatorial landscapes. First, we have seen that the local search is sometimes trapped inside a plateau which has a better neighboring solution, giving the possibility of reaching a better local optimum. Therefore, the number and extension of the plateaus containing local optima should be taken into account when referring to the difficulty of the algorithm for a specific instance. Secondly, until now, it was believed that the larger an attraction basin, the further we should go from the local optimum to escape from it and find a different one. However, as can be observed in this study, on average, just one movement in the neighborhood is enough to find a new attraction basin, because, usually, the local optima are located in the frontier of the attraction basins. Obviously, the probability of finding different attraction basins increases with the distance to the local optimum.

All the information gathered in this paper could be utilized for the design of new algorithms. Particularly, iterated local search (ILS) algorithms, which are built to escape from the local optima (Lourenço et al., 2003, 2010). Most of these implementations use a specific operator in the local search and apply a different operator to escape from the local optima. Some other authors design ILS algorithms, which escape from the local optima by applying a large number of movements of the same operator used in the local search. Nevertheless, with this new knowledge about the attraction basins, we could find other more simple and effective ways of escaping from the attraction basins. It just depends on the aim of the researcher: exploitation versus exploration. On the one hand, we know that, on average, once a local optimum is found, just by checking all the solutions at distance one, we will find at least one solution belonging to the attraction basin of a different local optimum. Furthermore, this new local optimum has a better objective function value than the previous one because that solution has a neighbor with better fitness than the local optimum, and, therefore, the new local optimum will also have better fitness than the previous one. In this case, we do not need to check any acceptance criterion for the new local optimum, as we are sure that this new local optimum improves the value of the previous one. However, as previously mentioned, the probability of finding new attraction basins at distance one from the local optima is lower than at higher distances. That is why, on the other hand, when we are interested in finding a high number of different attraction basins (exploration), the results of this paper recommend looking among those solutions in the attraction basins that are far from the local optima. Inside the set of their neighboring solutions, there is a high probability of finding a large number of different attraction basins. Nevertheless, in this case, the acceptance criterion should be evaluated, as we do not have information about the goodness of those new local optima.

The applicability of our results is not limited to the design of algorithms that solve instances of COPs; indeed, these results are also useful for improving the development of the study of other features of the landscape. Firstly, having information about the structure of the attraction basins helps in the estimation of their size, as was seen in

Hernando et al. (2016a). Secondly, this anatomy of the attraction basins can be a useful tool to estimate the number of local optima of the instance, as it is already known that some of the existing methods to estimate the number of local optima are based on the distribution of the sizes of the attraction basins (Hernando et al., 2013a). Furthermore, we can discover from this study those regions of the search space that, with high probability, contain different local optima or contain solutions belonging to different attraction basins. This information could somehow be used to estimate the number of non-observed local optima in the sample.

5 Visualizing the attraction basins

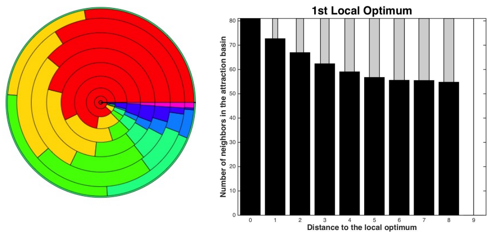
We show two examples of the visualization of the attraction basins. First, we choose a specific instance, and we show, in a more visual way however, all the information analyzed in the paper, in order to provide the reader with an intuition about the connections between the different attraction basins of one instance. Secondly, we choose a specific attraction basin of an instance, and give a representation by means of a network showing all the paths encountered until the local optimum is reached. We consider that these two examples help to understand the conclusions collected from the results obtained in the previous sections. The specific examples presented here have been chosen as representative cases. These are small but large enough to visualize the studied features. More examples of similar representations for other different instances and attraction basins are available at the website[3].

5.1 Example 1: Interrelation of the attraction basins of one instance.

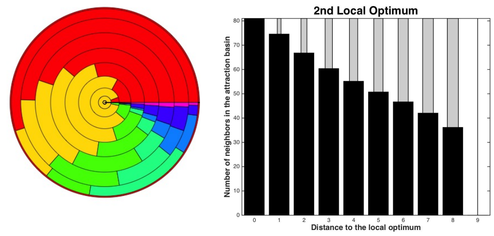
In order to provide an example of how all the attraction basins found in an instance are interrelated, for each of the 7 local optima of the $2nd$ PFSP instance under the insert neighborhood, we plot the proportion of the number of solutions at the different distances from each of the local optima that belong to each of the different attraction basins. That is, plots (a)-(g) in Figure 5 show the results obtained for the 1^{st} to the 7^{th} local optima, respectively. Notice that, in this instance, all the local optima are strict local optima. The local optima are sorted according to their objective function value (the 1^{st} local optimum being the global optimum) and each color refers to the attraction basin of each of the local optima, maintaining the relation between colors and local optima in the 7 plots. Specifically, the proportion in red, yellow, green, aquamarine, blue, indigo, and purple will refer to the attraction basins from the 1^{st} to the 7^{th} local optima, respectively, in all the 7 plots. Without wanting to lose perspective and being aware of the non-round shape of the attraction basins, we represent them as bullseyes. The corresponding local optimum is centered in the bullseye and 9 concentric rings represent the solutions at the 9 possible distances. So, at each of these distances, the proportion of solutions belonging to the distinct attraction basins are plotted with different colors.

By showing these bullseyes, we provide an intuition on how solutions belonging to different attraction basins appear while going further from the local optima. However, all the solutions inside a specific color are not necessarily all together as in the rings of the bullseyes. That is, as previously mentioned, the attraction basins are interrelated with many connectivities appearing among them. So, a figure in which all the colors mixed with each other would be more realistic. However, we found this representation a bit awkward and, therefore, in order to complete this information, we decided to include a graph accompanying each bullseye. We gather the solutions of

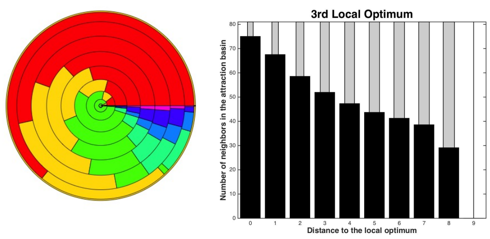
[3]http://www.sc.ehu.es/ccwbayes/members/leticia/AnatomyOfAB/visualization.html



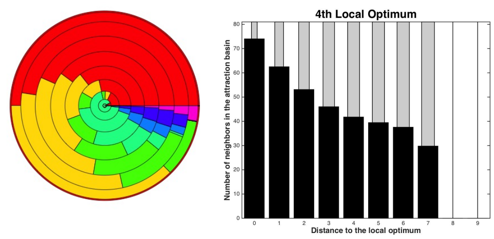
(a) Global optimum in red



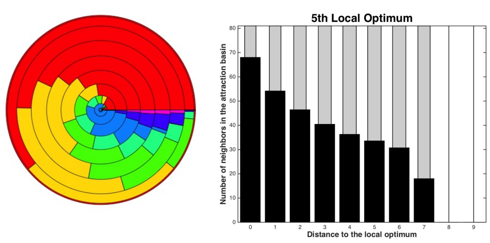
(b) Local optimum in yellow



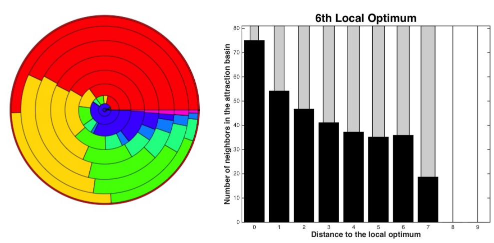
(c) Local optimum in green



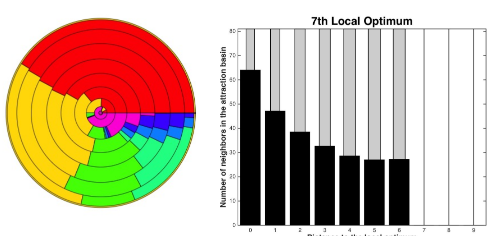
(d) Local optimum in aquamarine



(e) Local optimum in blue



(f) Local optimum in indigo



(g) Local optimum in purple

Figure 5: The bullseyes represent, for the 1st to the 7th local optimum (for (a)-(g), respectively) the proportion of the number of solutions that belong to the different attraction basins (plotted in different colors) according to the distance from the local optimum (located in the center of the bullseye). For the solutions of each of the attraction basins, the bars in black show the average proportion of neighbors belonging to the same attraction basin according to the distance to the local optimum. The width of the bars in gray is proportional to the number of the different neighboring attraction basins.

each attraction basin according to their distance to the local optimum. Then, for each of these solutions, we record the number of its neighbors that belong to the same attraction basin. The black bars represent the average number of the neighboring solutions in the same attraction basin according to the distance from the local optimum (given by the X axis). Notice that, the maximum value of the bars is 81, because under the insert neighborhood, this is the number of neighboring solutions. The width of the gray bars is proportional to the average number of different attraction basins found at each distance. According to these figures, all the local optima of this instance, except the two local optima with highest fitness, always have neighboring permutations in a different attraction basin. Moreover, for all the local optima (including the global optimum) we find solutions at distance one that have neighboring solutions belonging to different attraction basins. This means that paths belonging to other different attraction basins are mixed with them from a distance of just 1.

5.2 Example 2: The attraction basin of one local optimum.

The visualization of an attraction basin is not an easy task. The large number of permutations in the search space and the considerably large number of neighbors for each solution, together with the non-ordered permutation space, means that any plot in 2D or 3D should be interpreted with caution. Here, we propose a way of visualizing the attraction basins by means of directed graphs. Each node of the graph represents one solution belonging to the attraction basin. Edges between nodes indicate that the node at the end of the edge is the best neighbor of the node at the start of the edge. The color of the nodes changes with the distance to the local optimum. Particularly, red, yellow, green, light blue, dark blue and purple are used to represent the solutions at distances 0 (the local optima), 1, 2, 3, 4 and 5, respectively. The size of the nodes and the width of the edges also decrease as the distance to the local optimum increases.

Figure 6 presents two different graphs illustrating the same attraction basin of a local optimal plateau of the $7th$ instance of the PFSP when using the 2-exchange neighborhood. Both graphs have been created using the *igraph* package in the R programming language (Csárdi and Nepusz, 2006). Figure 6 (left) represents this attraction basin considering the steps that the algorithm takes until it reaches the local optimal plateau (and not the distances between the solutions). For this purpose, we add the *layout_with_kk* option when plotting the graph in R. In this case, there are two local optima (the nodes in red) forming the local optimal plateau. This visualization could lead us to think that the structure defined by an attraction basin is perfectly ordered in the search space and close to the convexity, roundness and symmetry. However, as previously mentioned, in this plot, the distances between each pair of solutions in the attraction basin have not been considered. Moreover, we should point out that the number of edges that each of the nodes of the graph, and especially the local optima, have is really small compared with the maximum possible number (45 neighbors). This means that solutions belonging to different attraction basins are mixed, as can be seen in Example 1.

In an attempt to visualize this attraction basin in a more realistic way, we force the graph to take into account the distances between all the solutions that belong to the attraction basin (Figure 6 (right)). We use the *sammon* function also found in R (Venables and Ripley, 2002), which maps a high-dimensional space to a 2D space, trying to preserve those distances in the high-dimensional space (Sammon, 1969). Of course, the resulting graph does not fulfill exactly all the distances between each pair of nodes, because visualizing the permutation space in 2D is impossible. However, it allows us to observe that the real structure of the attraction basins is more complex than one could

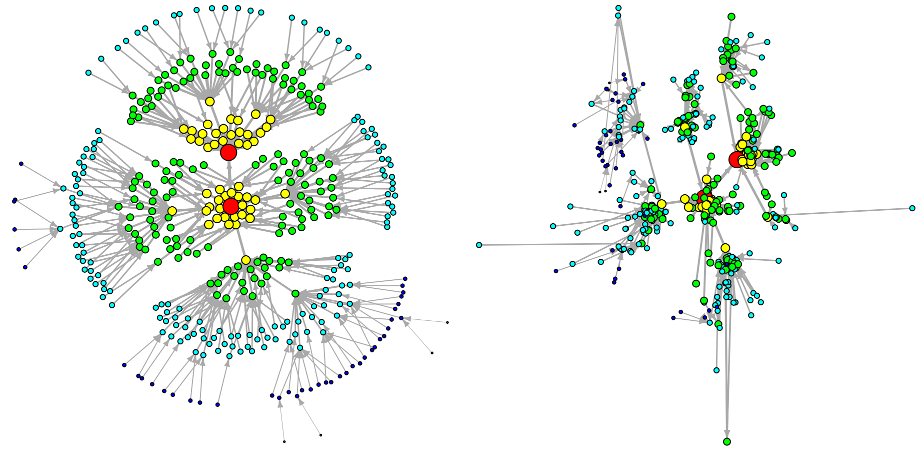


Figure 6: Visualization of an attraction basin of a local optimal plateau found in the $7th$ instance of the PFSP with the 2-exchange neighborhood, considering only the steps of the algorithm (left) and taking into account, as far as possible, the distances between all the solutions (right).

try to imagine, and, of course, much more complex than that shown in Figure 6 (left). The visualization in the 3D space can be found at the website[4].

6 Conclusions and Future Work

The landscape of combinatorial problems has been described by many authors as a set of mountains and valleys, resembling the continuous domain. This widespread idea has been the basis for the design of algorithms. However, as a few authors already noticed, some results found for the features of the attraction basins contradict this. Unfortunately, these works have been ignored in practice. Therefore, our aim in this paper has been to analyze a number of topological properties of the attraction basins in order to provide knowledge about their structure and to break with the extended and mistaken intuition about the anatomy of the attraction basins.

A deep analysis of the features of the attraction basins has been developed. The fact of having local optima forming plateaus is a remarkable aspect of the instances of permutation-based COPs. Thus, we usually find that the algorithm gets trapped inside the plateaus. For this reason, they could be seen as a complexity measure for the instances, or, at least, the number and extension of plateaus could be a supplement to the number of local optima when dealing with the difficulty of the instances. It is fundamental to take the plateaus into account when analyzing the landscape features or when proposing a new algorithm. We used this information about plateaus to gather all the local optima that belong to a plateau, so as to consider that they share the same attraction basin. It was found that, on average, there are neighboring solutions of the local optima or of the plateaus that do not belong to their attraction basins. However, at the same time, there are usually some solutions far from the local optima which still belong to its attraction basin. This information totally differs from the concept of roundness. However, the local optima are located almost centered within the attraction

[4]http://www.sc.ehu.es/ccwbayes/members/leticia/AnatomyOfAB/visualization/visualizeOneAB.html

basins. Moreover, not only the local optima but also an incredibly high percentage of the solutions of the attraction basins fall into the frontier. So, we could say that the interior of the attraction basins is almost empty. The attraction basins are neighbors of a large number of the remaining attraction basins. In this sense, the solutions far from the local optima and inside their attraction basins present a larger number of neighboring attraction basins than those solutions nearer to the local optima. In fact, each of the attraction basin can be seen as a number of long rivers, with tributaries, that flow into the local optimum (or local optimal plateau). The different rivers are interconnected with each other or, at least, very close to each other.

The understanding of the landscapes in combinatorial optimization has been one of the main challenges when developing and improving algorithms. This work not only breaks with an erroneous extended belief about the attraction basin shapes, but also provides valuable information for the design of new algorithms based on local search. For example, this work clearly helps in the development of Iterated Local Search algorithms when looking for an effective way to escape from the local optima. In fact, we plan to add information about the objective function value of the local optima to our analysis of the neighboring relations between the attraction basins. In this sense, we hope that we will find regularities of the distances to the local optima where they have more connectivities with other better local optima. Moreover, this paper also provides useful knowledge to improve those methods that estimate the number of local optima or the sizes of the attraction basins.

7 Acknowledgements

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References

Albrecht, A., Lane, P., and Steinhofel, K. (2008). Combinatorial landscape analysis for k-SAT instances. In *Evolutionary Computation, 2008. CEC 2008. (IEEE World Congress on Computational Intelligence). IEEE Congress on*, pages 2498 –2504.

Albrecht, A., Lane, P., and Steinhofel, K. (2010). Analysis of Local Search Landscapes for k-SAT Instances. *Mathematics in Computer Science*, 3(4):465–488.

Alyahya, K. and Rowe, J. E. (2014). *Evolutionary Computation in Combinatorial Optimisation: 14th European Conference, EvoCOP 2014, Granada, Spain, April 23-25, 2014, Revised Selected Papers*, chapter Phase Transition and Landscape Properties of the Number Partitioning Problem, pages 206–217. Springer Berlin Heidelberg, Berlin, Heidelberg.

Angel, E. and Zissimopoulos, V. (2002). On the hardness of the quadratic assignment problem with metaheuristics. *Journal of Heuristics*, 8(4):399–414.

Bouziri, H., Mellouli, K., and Talbi, E.-G. (2009). The k-coloring fitness landscape. *Journal of Combinatorial Optimization*, 21(3):306–329.

Burkard, R. E., Karisch, S. E., and Rendl, F. (1997). Qaplib – a quadratic assignment problem library. *Journal of Global Optimization*, 10(4):391–403.

Caruana, R. and Mullin, M. (1999). Estimating the Number of Local Minima in Big, Nasty Search Spaces. In *In Proceedings of IJCAI-99 Workshop on Statistical Machine Learning for Large-Scale Optimization*.

Chicano, F., Daolio, F., Ochoa, G., Verel, S., Tomassini, M., and Alba, E. (2012). Local optima networks, landscape autocorrelation and heuristic search performance. In *Proceedings of Parallel Problem Solving from Nature, PPSN XII*, volume 7492 of *Lecture Notes in Computer Science*, pages 337–347. Springer.

Csárdi, G. and Nepusz, T. (2006). The igraph software package for complex network research. *InterJournal*, Complex Systems(1695):1–9.

Daolio, F., Verel, S., Ochoa, G., and Tomassini, M. (2010). Local optima networks of the quadratic assignment problem. In *IEEE Congress on Evolutionary Computation (CEC 2010)*, pages 2145–3152.

Daolio, F., Verel, S., Ochoa, G., and Tomassini, M. (2014). Local optima networks of the permutation flow-shop problem. In *Artificial Evolution, EA 2013, Lecture Notes in Computer Science 8752*, pages 41–52.

Fonlupt, C., Robilliard, D., Preux, P., and Talbi, E.-G. (1999). Fitness Landscapes And Performance of Meta-Heuristics. In *Meta-Heuristics: Advances and Trends in Local Search Paradigms for Optimization*, pages 257–268. Kluwer Academic Publishers.

Garey, M. R., Johnson, D. S., and Sethi, R. (1976). The complexity of flowshop and jobshop scheduling. *Mathematics of Operations research*, 1(2):117–129.

Hernando, L., Daolio, F., Veerapen, N., and Ochoa, G. (2017). Local optima networks of the permutation flowshop scheduling problem: Makespan vs. total flow time. In *IEEE Congress on Evolutionary Computation (CEC 2017)*, San Sebastian, Spain.

Hernando, L., Mendiburu, A., and Lozano, J. A. (2013a). An evaluation of methods for estimating the number of local optima in combinatorial optimization problems. *Evolutionary Computation*, 21(4):625–658.

Hernando, L., Mendiburu, A., and Lozano, J. A. (2013b). Generating customized landscapes in permutation-based combinatorial optimization problems. In Springer, Berlin, H., editor, *International Conference on Learning and Intelligent Optimization*, pages 299–303.

Hernando, L., Mendiburu, A., and Lozano, J. A. (2016a). *Estimating Attraction Basin Sizes*, pages 458–467. Springer International Publishing, Cham.

Hernando, L., Mendiburu, A., and Lozano, J. A. (2016b). A tunable generator of instances of permutation-based combinatorial optimization problems. *IEEE Transactions on Evolutionary Computation*, 20(2):165–179.

Hernando, L., Pascual, J. A., Mendiburu, A., and Lozano, J. A. (2011). A study on the complexity of TSP instances under the 2-exchange neighbor system. In *Foundations of Computational Intelligence (FOCI)*, pages 15–21.

Hoos, H. H. and Stützle, T. (2004). *Stochastic Local Search: Foundations and Applications*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA.

Humeau, J., Liefooghe, A., Talbi, E.-G., and Verel, S. (2013). ParadisEO-MO: From Fitness Landscape Analysis to Efficient Local Search Algorithms. *Journal of Heuristics*, 19(6):881–915.

Klemm, K., Qin, J., and Stadler, P. F. (2014). *Recent Advances in the Theory and Application of Fitness Landscapes*, chapter Geometry and Coarse-Grained Representations of Landscapes. Springer Berlin Heidelber.

Lourenço, H. R., Martin, O. C., and Stützle, T. (2003). *Handbook of Metaheuristics*, chapter Iterated Local Search, pages 320–353. Springer US, Boston, MA.

Lourenço, H. R., Martin, O. C., and Stützle, T. (2010). *Handbook of Metaheuristics*, chapter Iterated Local Search: Framework and Applications, pages 363–397. Springer US, Boston, MA.

Marmion, M.-E., Dhaenens, C., Jourdan, L., Liefooghe, A., and Verel, S. (2011a). Nils: A neutrality-based iterated local search and its application to flowshop scheduling. In Merz, P. and Hao, J.-K., editors, *Evolutionary Computation in Combinatorial Optimization: 11th European Conference, EvoCOP 2011*, pages 191–202, Torino, Italy. Springer Berlin Heidelberg.

Marmion, M.-E., Dhaenens, C., Jourdan, L., Liefooghe, A., and Verel, S. (2011b). On the neutrality of flowshop scheduling fitness landscapes. In *Learning and Intelligent OptimizatioN Conference (LION 5)*, pages 238–252, Rome, Italy.

Mattfeld, D. C. and Bierwirth, C. (1999). A Search Space Analysis of the Job Shop Scheduling Problem. *Annals of Operations Research*, 86:441–453.

Merz, P. (2004). Advanced fitness landscape analysis and the performance of memetic algorithms. *Evolutionary Computation*, 12(3):303–325.

Merz, P. and Freisleben, B. (2001). Memetic algorithms for the travelling salesman problem. *Complex Systems*, 13(4):297–345.

Mishra, S. and Sikdar, K. (2004). On approximability of linear ordering and related NP-optimization problems on graphs. *Discrete Applied Mathematics*, 136(2-3):249–269.

Moser, I., Gheorghita, M., and Aleti, A. (2016). Identifying features of fitness landscapes and relating them to problem difficulty. *Evolutionary computation*.

Ochoa, G. and Veerapen, N. (2017). Mapping the global structure of TSP fitness landscapes. *Journal of Heuristics*.

Ochoa, G., Verel, S., Daolio, F., and Tomassini, M. (2012). Clustering of local optima in combinatorial fitness landscapes. In *Learning and Intelligent Optimization*, volume 6683, pages 454–457. Springer Berlin Heidelberg.

Ochoa, G., Verel, S., Daolio, F., and Tomassini, M. (2014). *Recent Advances in the Theory and Application of Fitness Landscapes*, volume 6 of *Emergence, Complexity and Computation*, chapter Local optima networks: a new model of combinatorial fitness landscapes, pages 233–262. Springer Berlin Heidelber.

Preux, P. and Talbi, E. G. (1999). Towards hybrid evolutionary algorithms. *International Transactions in Operational Research*, 6(6):557–570.

Prügel-Bennett, A. and Tayarani-Najaran, M. H. (2012). Maximum satisfiability: Anatomy of the fitness landscape for a hard combinatorial optimization problem. *IEEE Transactions on Evolutionary Computation*, 16(3):319–338.

Reeves, C. and Aupetit-Bélaidouni, M. (2004). Estimating the Number of Solutions for SAT problems. In Yao, X., Burke, E., Lozano, J., Smith, J., Merelo-Guervós, J., Bullinaria, J., Rowe, J., Tino, P., Kabán, A., and Schwefel, H.-P., editors, *Parallel Problem Solving from Nature - PPSN VIII*, volume 3242 of *Lecture Notes in Computer Science*, pages 101–110. Springer Berlin / Heidelberg.

Reeves, C. R. (1999). Landscapes, operators and heuristic search. *Annals of Operations Research*, 86:473–490.

Reidys, C. M. and Stadler, P. F. (2002). Combinatorial landscapes. *SIAM Review*, 44(1):3–54.

Sahni, S. and Gonzalez, T. (1976). P-complete approximation problems. *Journal of the ACM (JACM)*, 23(3):555–565.

Sammon, J. W. (1969). A nonlinear mapping for data structure analysis. *IEEE Transactions on Computers*, C-18(5):401–409.

Schiavinotto, T. and Stützle, T. (2005). The Linear Ordering Problem: Instances, Search space analysis and algorithms. *Journal of Mathematical Modelling and Algorithms*, 3(4):367–402.

Stützle, T. (2006). Iterated local search for the quadratic assignment problem. *European Journal of Operational Research*, 174(3):1519 – 1539.

Sutton, A. M., Howe, A. E., and Whitley, L. D. (2010). Directed plateau search for max-k-sat. In *Proceedings of SoCS*, pages 90–97, Atlanta, GA, USA.

Taillard, E. (1990). Some efficient heuristic methods for the flow shop sequencing problem. *European Journal of Operational Research*, 47(1):65 – 74.

Taillard, E. (1993). Benchmarks for basic scheduling problems. *European Journal of Operational Research*, 64(2):278 – 285.

Tayarani-Najaran, M. H. and Prügel-Bennett, A. (2014). On the landscape of combinatorial optimization problems. *IEEE Transactions on Evolutionary Computation*, 18(3):420–434.

Tayarani-Najaran, M. H. and Prügel-Bennett, A. (2015a). Anatomy of the fitness landscape for dense graph-colouring problem. *Swarm and Evolutionary Computation*, 22:47–65.

Tayarani-Najaran, M. H. and Prügel-Bennett, A. (2015b). Quadratic assignment problem: a landscape analysis. *Evolutionary Intelligence*, 8(4):165–184.

Tomassini, M., Vérel, S., and Ochoa, G. (2008). Complex-network analysis of combinatorial spaces: The NK landscape case. *Physical Review E*, 78(6):66–114.

Venables, W. N. and Ripley, B. D. (2002). *Modern Applied Statistics with S*. Springer, New York, fourth edition.

Verel, S., Daolio, F., Ochoa, G., and Tomassini, M. (2011a). Local optima networks with escape edges. In *International Conference on Artificial Evolution (EA-2011)*, pages 10–23, Angers, France.

Verel, S., Ochoa, G., and Tomassini, M. (2008). The connectivity of nk landscapes' basins: A network analysis. In *Artificial Life XI*.

Verel, S., Ochoa, G., and Tomassini, M. (2011b). Local optima networks of NK landscapes with neutrality. *IEEE Transactions on Evolutionary Computation*, 15(6):783–797.

Watson, J.-P. (2010). *Handbook of Metaheuristics*, volume 146, chapter An introduction to Fitness Landscape Analysis and Cost Models for Local Search, pages 599–623. Springer.