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PH.D. THESIS

Varopoulos extensions of boundary functions in L^p and BMO in domains with Ahlfors-regular boundaries and applications

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Abstract

Varopoulos extensions of boundary functions in L^p and* BMO *in domains with Ahlfors-regular boundaries and applications

Athanasios Zacharopoulos

This thesis focuses on the construnction of Varopoulos-type extensions of L^p and BMO boundary functions in rough domains. To be more specific, let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set with s-Ahlfors regular boundary $\partial\Omega$, for some $s \in (0, n]$, such that either $s = n$ and Ω is a corkscrew domain with the pointwise John condition, or $s < n$ and $\Omega = \mathbb{R}^{n+1} \setminus E$, for some s-Ahlfors regular set $E \subset \mathbb{R}^{n+1}$. In this thesis we provide a unifying method to construct Varopoulos type extensions of L^p and BMO boundary functions. In particular, we show that a) if $f \in L^p(\partial\Omega)$, $1 < p \leq \infty$, there exists $F \in C^\infty(\Omega)$ such that the non-tangential maximal functions of F, $\operatorname{dist}(\cdot, \Omega^c)|\nabla F|$, as well as the Carleson functional of $\operatorname{dist}(\cdot, \Omega^c)^{s-n}\nabla F$ are in $L^p(\partial\Omega)$, with norms controlled by the L^p-norm of f, and $F \to f$ in some non-tangential sense $\mathcal{H}^s|_{\partial\Omega}$-almost everywhere; b) if $\bar{f} \in \mathrm{BMO}(\partial\Omega)$ there exists $\bar{F} \in C^\infty(\Omega)$ such that $\operatorname{dist}(x, \Omega^c)|\nabla \bar{F}(x)|$ is uniformly bounded in Ω and the Carleson functional of $\operatorname{dist}(x, \Omega^c)^{s-n}\nabla \bar{F}(x)$, as well the the sharp non-tangential maximal function of $\bar{F}$ are uniformly bounded on $\partial\Omega$ with norms controlled by the BMO-norm of $\bar{f}$, and $\bar{F} \to \bar{f}$ in a certain non-tangential sense $\mathcal{H}^s|_{\partial\Omega}$-almost everywhere. If, in addition, the boundary function is Lipschitz with compact support then both F and $\bar{F}$ can be constructed so that they are also Lipschitz on $\overline{\Omega}$ and converge to the boundary data continuously. The latter results hold without the additional pointwise John condition assumption. Finally, for elliptic systems of equations in divergence form with merely bounded complex-valued coefficients, we show some connections between the solvability of Poisson problems with interior data in the appropriate Carleson or tent spaces and the solvability of Dirichlet problem with L^p and BMO boundary data.

Contents

Resumen de la tesis

Esta tesis se enmarca en el área del análisis matemático. En particular, utilizando técnicas de las áreas de Análisis Armónico y Teoría de Medidas Geométricas, construimos extensiones de tipo Varopoulos de funciones de frontera L^p y BMO en “dominios salvajes”. Es decir, extensiones suaves de funciones tales que las normas L^p de su función maximal no tangencial y funcional de Carleson de sus gradientes puedan controlarse mediante la norma de los datos de frontera. Además, utilizamos estas extensiones para aplicaciones de ecuaciones en derivadas parciales y especialmente en problemas de condición de frontera.

Construir una extensión de Varopoulos de funciones $L^p/\operatorname{BMO}/\Lambda(\alpha)$ es un problema importante en Análisis Armónico con aplicaciones a Problemas de Condición de Frontera para operadores elípticos de segundo orden en forma de divergencia. En el semiespacio superior y para funciones BMO, este problema fue resuelto por Varopoulos en 1977 [Var77] y fue refinado por Garnett en 1981 [Gar81], y recientemente ha sido generalizado por Hofmann y Tapiola (2021) [HT21] a dominios “de sacacorchos” (corkscrew domains) con bordes n-UR. La versión L^p de este problema para $1 < p < \infty$ en el semiespacio superior fue demostrada por Hytönen y Rosén en 2018 [HR18].

Para ser más específico, el principal objetivo de esta tesis doctoral es construir extensiones de tipo Varopoulos de funciones definidas en el borde de un dominio Ω en $\mathbb{R}^{n+1}$ con frontera s-Ahlfors regular, donde $0 < s \leq n$. Específicamente, dada una función $f \in L^p$, para $1 < p \leq \infty$ (o $f \in \operatorname{BMO}$ o en elespacio de Campanato $\Lambda(\alpha)$ para $0 < \alpha < 1$), construimos una función suave F en Ω de manera que F converge a f casi en todas partes del borde en un cierto sentido no tangencial, mientras que la función maximal no tangencial (o la función maximal no tangencial sharp) de F y una versión modificada “ponderada” del funcional de Carleson del gradiente de F están en L^p (o acotadas uniformemente) con normas controladas por las normas L^p (o normas $\operatorname{BMO}/\Lambda(\alpha)$) de f. El segundo objetivo principal es construir extensiones de tipo Varopoulos, es decir, extensiones que satisfagan las estimaciones mencionadas anteriormente, de funciones Lipschitz con soporte compacto en el borde que sean Lipschitz en la clausura de Ω y también pertenezcan al espacio de Sobolev homogéneo (ponderado si $s < n$) $W^{1,2}(\Omega)$. Finalmente, aplicamos el segundo objetivo para obtener resultados que relacionan la solucionabilidad de Problemas de Condición de Frontera y Problemas de Poisson para sistemas elípticos en forma de divergencia con coeficientes complejos y meramente acotados. En particular, demostramos que si el problema de regularidad de Poisson o el problema de Poisson-Dirichlet con datos interiores en los espacios “correctos” de Carleson o de “tienda” (tent space) invariantes bajo escala es solucionable en Ω, entonces el problema de Dirichlet L^p/BMO también es solucionable en Ω.

La tesis comienza con la introducción donde presentamos la historia de las extensiones de Varopoulos a través de resultados conocidos y luego se procede a la presentación de los resultados obtenidos en esta tesis y las técnicas utilizadas. En el Capítulo 1 mencionamos los

preliminares y notaciones que se utilizan en el resto de la tesis. En el Capítulo 2 construimos y analizamos las propiedades de una extensión diádica suave v_f de funciones de frontera f, mientras que en el capítulo 3 construimos una descomposición en corona para funciones en L^p y BMO usando argumentos de tiempo de parada. En el capítulo 4 demostramos que la extensión diádica v_f es ε-aproximable en L^p (si $f \in L^p$) y también uniforme ε-aproximable (si $f \in$ BMO) en el dominio Ω. En el capítulo 5 construimos extensiones de tipo Varopoulos de funciones Lipschitz con soporte compacto y demostramos que nuestras extensiones también son Lipschitz en $\overline{\Omega}$. Además, en el Capítulo 6 construimos las extensiones de Varopoulos deseadas de las funciones L^p y BMO y en el capítulo 7 utilizamos las extensiones construidas en el capítulo 5 para aplicaciones a problemas de condición de frontera. Por último, al final de la tesis se ha incluido un apartado de apéndices de algunos lemas técnicos.

A continuación, analizamos cada capítulo con más detalle.

Capítulo 1: Preliminares y Notaciones

En los preliminares damos las definiciones necesarias que se utilizan en el resto de la tesis. En este capítulo también definimos algunos espacios funcionales que son cruciales para los capítulos 6 y 7. Además definimos los operadores máximos y los funcionales de Carleson y damos una breve introducción a los sistemas Elípticos y los problemas de valores en la frontera. Finalmente damos definiciones relacionadas con la geometría de los dominios que nos interesan, así como algunas construcciones técnicas como las descomposición diádica en el $\operatorname{supp} \mu$ por un s-Ahlfors medida regular μ en $\mathbb{R}^{n+1}$ y la descomposición de Whitney de los dominios $\Omega \subset \mathbb{R}^{n+1}$. En la tesis, usamos la notation $\sigma = \mathcal{H}^s|_{\partial\Omega}$.

Capítulo 2: Extensión diádica regularizada de funciones en el borde de un dominio

En este capítulo construimos la extensión diádica regularizada de la siguiente manera. A cada cubo de Whitney P le asociamos un cubo "diádico" de frontera $b(P)$ con lado de la misma longitud y de modo que $\operatorname{dist}(P, b(P)) \approx \ell(P)$. Luego definimos una versión suave de la extensión diádica de la función de borde, que denotamos como v_f, usando una partición de la unidad $\{\varphi_P\}_{P \in \mathcal{W}(\Omega)}$ subordinada a la colección de cubos de Whitney (dilatados) de Ω (siguiendo el espíritu de la extensión de Whitney) cuyos coeficientes son los promedias de f sobre $b(P)$ (ver definición 2.1).

Después, mostramos que v_f es, de hecho, una extensión de f en el sentido de que converge no tangencialmente (es decir, la convergencia tiene lugar dentro de un cono con vértice en el borde) a los datos del borde. También obtenemos algunas estimaciones locales para ∇v_f y para la función máxima no-tangencial $\mathcal{N}_\alpha(v_f)(\xi)$ (si $f \in L^p(\sigma)$).

Si f está en el espacio de Campanato $\Lambda_\beta(\partial\Omega)$ por $\beta \in [0,1)$ (ver (1.2)) obtenemos estimaciones por el functional de Carleson de ∇v_f (i.e. por $\mathcal{C}_s^{(\beta)}(\nabla v_f)(\xi)$) así como el agudo funcional maximal no-tangencial $\mathcal{N}_{\sharp,\alpha}^{\beta}(v_f)(\xi)$.

También demostramos que si la función de frontera f está en el espacio $\operatorname{Lip}_\beta(\partial\Omega)$ por $\beta \in (0,1]$, entonces $v_f \in \operatorname{Lip}_\beta(\overline{\Omega})$ con $\operatorname{Lip}_\beta(v_f) \lesssim \operatorname{Lip}_\beta(f)$. Finalmente, utilizando estos resultados demostramos el Teorema 0.4, que plantea la construcción de extensiones de función de tipo Varopoulos en el espacio de Campanato $\Lambda_\beta(\partial\Omega)$.

Capítulo 3: Una descomposición en corona para funciones en L^p o BMO

En este capítulo mediante un argumento de stopping time adecuado, construimos una descomposición de Corona en la frontera de manera que la diferencia entre el promedia f en un “top cube” y la media en el "stopping parent" no sea pequeña (en cierto sentido).

Para ser más especifico, dado cualquier cubo $R \in \mathcal{D}_\sigma$ y para fijo $\varepsilon > 0$, nosotros definimos la colección $\mathsf{Stop}(R) \subset \mathcal{D}(R)$ formado por cubos $S \in \mathcal{D}(R)$ los cuales son maximales (por lo tanto disjuntos) con respecto a la condición

$$|m_{\sigma,R}f - m_{\sigma,S}f| \geq \begin{cases} \varepsilon Mf(S) & \text{, si } f \in L^1_{\text{loc}}(\sigma) \\ \varepsilon \|f\|_{\text{BMO}(\sigma)} & \text{, si } f \in \text{BMO}(\sigma). \end{cases}$$

Primero probamos algunos lemas técnicos que se basan en resultados de Hytönen y Rosén [HR18] y luego demostramos que los cubos “superiores” satisfacen una condición de empaquetamiento de Carleson $\text{Car}(\varepsilon)$. Este es el teorema principal de este capítulo.

Capítulo 4: L^p y ε-aproximabilidad uniforme de la extensión diádica regularizada

El teorema que demostramos en este capítulo establece que para cada $\varepsilon > 0$, una versión suave del la extensión diádica υ_f de nuestra función de borde f en L^p (o BMO) se puede aproximar mediante una función w tal que la función maximal no tangencial de su diferencia en un punto del borde ξ está dominada por una constante múltiple de $\varepsilon M(f)(\xi)$, donde Mf es la función maximal de Hardy-Littlewood de f, y el funcional de Carleson modificado del gradiente de w en ξ está dominado por una constante múltiple de $M(M(M(f)))(\xi)$. Se demuestra una versión uniforme de esta aproximación si f es una función en BMO.

El esquema de prueba es el siguiente. Recuerde la construcción de la extensión diádica suave y utilizando descomposición de la corona en el borde que construimos en el capítulo 3. Definimos la función aproximada u una vez más a través de $\{\varphi_P\}_{P \in \mathcal{W}(\Omega)}$ de manera que, en pocas palabras, u es una constante (la media de f sobre el cubo superior de cada “árbol” (tree)) cuando P está asociado a un cubo diádico del borde $b(P)$ que está en el árbol pero no de manera que exista otro cubo diádico del borde Q con longitud de lado comparable que pertenezca a otro árbol y $\text{dist}(b(P), Q) \lesssim \ell(b(P))$. En el resto de los cubos, se define como la extensión diádica υ_f. Luego, mediante estimaciones sutiles que utilizan las condiciones de stopping y el teorema de embedding de Carleson discreto, podemos concluir la prueba del teorema principal de este capítulo (por el Teorema ver 4.3).

Como corolario (ver Teorema 4.4) obtenemos que si $f \in L^p(\sigma)$, $p \in (1, \infty)$, es (resp. $f \in \text{BMO}(\sigma)$), entonces υ_f es ε-aproximada en L^p (uniformemente ε-approximada).

Capítulo 5: Extensiones tipo Varopoulos de funciones de Lipschitz con soporte compacto

En este capítulo nosotros construimos extensiones de tipo Varopoulos de funciones de Lipschitz con soporte compacto en el borde que sean Lipschitz en la clausura de Ω y también pertenezcan al espacio de Sobolev homogéneo (ponderado si $s < n$) $W^{1,2}(\Omega)$.

El método que utilizamos es el siguiente. Se utiliza el primer resultado de capítulo 4 para construir la función aproximada de la extensión diádica suave de f y se define la extensión F para que sea igual a la función aproximada en todas partes, excepto en entorno del borde de “ancho” $\delta > 0$, donde se establece que sea igual a la extensión diádica suave. Dado que la extensión diádica suave de una función Lipschitz es Lipschitz en la clausura del dominio, F converge a f de manera continua. Además, las estimaciones requeridas de la función maximal no tangencial de F y del funcional de Carleson modificado del gradiente de F son consecuencias de una elección adecuada de δ. Para ser precisos, $\delta = \|f\|_{Lp}/\|f\|_{\dot{M}^{1,p}}$ para $1 < p < \infty$ o $\delta = \|f\|_{\mathrm{BMO}}/\mathrm{Lip}(f)$. Finalmente, multiplicando F con una función de cut-off relacionada con el soporte de f, se demuestra que esta nueva función sigue siendo una extensión de Varopoulos de f que es Lipschitz en la clausura de Ω y cumple las cotas de Sobolev deseadas.

En particular, probamos la extensión de Lipschitz de funciones de frontera en L^p en el Teorema 5.1. También construimos la extensión-BMO en el teorema 5.4.

Finalmente, en Teorema 5.5, modificamos la extensión construida en los teoremas anteriores para que también estén en $\dot{W}^{1,2}(\Omega)$.

Capítulo 6: Construcción de extensiones tipo Varopoulos de funciones L^p y BMO

En este capítulo, primero asumimos la condición de John puntual cuando s=n para demostrar un teorema de traza. Es decir, se mostra que existe un operador de traza sobreyectivo $\mathrm{Tr}(u)$ para todas las funciones suaves u tal que el funcional de Carleson del gradiente de u está en L^p para $1 < p \lesssim \infty$ y la función maximal no tangencial de u está en L^p, y se cumple que $\|\mathrm{Tr}(u)\|_{Lp(\sigma)} < C\|\mathcal{N}(u)\|_{L^p(\sigma)}$. Este resultado también es cierto en el caso de BMO si el funcional de Carleson del gradiente de u está uniformemente acotado y si, además, se asume que Ω cumple la condición de John local para $s = n$(que es una versión invariante bajo escala de la condición de John puntual y siempre es verdadera cuando $s < n$), el operador de traza es sobreyectivo con $\|\mathrm{Tr}(u)\|_{\mathrm{BMO}(\sigma)} < C\|\mathcal{C}(\nabla u)\|_{L^\infty(\sigma)}$. Combinando los teoremas de capítulo 5 con un argumento de iteración y el teorema de traza, las extensiones de tipo Varopoulos de funciones de borde en L^p para $1 < p \lesssim \infty$ y BMO se pueden construir. La única desventaja es que cuando $s = n$ y $f \in \mathrm{BMO}$, se debe asumir la condición de John local. Sin embargo, esto unifica el método de construcción de extensiones para funciones de borde L^p y BMO y proporciona una prueba autocontenida que aclara la verdadera naturaleza de la importante propiedad de extensión de Varopoulos y los elementos que intervienen en su prueba. Entonces podemos demostrar los Teoremas 0.5 y 0.6. Por el Teorema 0.6 es necesario el método que se utilizó anteriormente para funciones BMO cuando $s = n$ consistía en una descomposición de $f = g + b$ donde $g \in L^\infty$ y $b(\xi) = \sum a_k \mathbf{1}_{Q_k}(\xi)$, donde Q_k es una colección numerable de cubos diádicos del borde que satisfacen una condición de empaquetamiento de Carleson y $|a_k| \leq C\|f\|_{\mathrm{BMO}}$. Se puede construir una extensión B de la parte "mala" b, que es más fácil de manejar aunque aún técnica en dominios generales como los que consideramos. Luego, la dificultad está en la construcción de la extensión G de la parte "buena" g que se hacía previamente mediante métodos de EDP utilizando que las funciones armónicas acotadas son ε-aproximables en ciertos tipos de dominios (por ejemplo, con bordes UR). Uso esta descomposición de f para manejar el caso de dominios que cumplen la condición de John punto a punto pero no la condición de John local cuando $s = n$. De hecho, podemos utilizar la extensión B construida por Hofmann y Tapiola, ya que no utiliza la rectificabilidad n-uniforme del borde, sino sólo que el dominio cumple la condición de corkscrew y su borde es n-Ahlfors regular. La principal novedad aquí es que utilizo la extensión de Varopoulos de funciones L^∞ que se demostró previamente y permite de superar las restricciones geométricas provenientes de los métodos de EDP.

Capítulo 7: Aplicaciones a problemas de valores en la frontera

En este capítulo usamos los teoremas que demostramos en capítulo 5 para obtener conexiones entre los problemas Poisson y de condiciones de frontera para sistemas de ecuaciones elípticas en forma de divergencia con coeficientes de valores complejos meramente acotados. En particular, utilizamos la extensión de Varopoulos Lipschitz para mostrar que si el problema de regularidad de Poisson o el problema de Poisson-Dirichlet con datos interiores en los espacios "correctos" de Carleson o "tienda" invariantes bajo escala es resoluble en Ω, entonces el problema de Dirichlet L^p/ BMO también es resoluble en Ω. Además, demostramos desigualdades de tipo Rellich unilaterales condicionales para soluciones de ciertos problemas de condición de frontera.

Apéndice

En el apéndice pruebamos de algunos lemas técnicos que utilizamos en la tesis.

Introduction

In the present thesis we are concerned with open sets $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, which satisfy one of the following assumptions:

(a) Ω satisfies the corkscrew condition and its boundary $\partial\Omega$ is n-Ahlfors regular (see Definitions 1.1 and 1.10), or

(b) $\Omega = \mathbb{R}^{n+1} \setminus E$, for some s-Ahlfors regular set $E \subset \mathbb{R}^{n+1}$ with $s < n$.

We will call such domains AR(s) *domains* for $s \in (0, n]$. We also define $\sigma_s := \mathcal{H}^s|_{\partial\Omega}$ to be the "surface" measure of Ω, where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure.

Our first main goal is to construct, in AR(s) domains, smooth extensions $u : \Omega \to \mathbb{R}$ of boundary functions that are in $L^p(\sigma_s)$ for $p \in (1, \infty]$ (resp. in BMO(σ_s)) so that their non-tangential maximal functions defined in (1.9) (resp. sharp non-tangential maximal function defined in (1.10)) and the modified Carleson functionals (see (1.12) for the definition) of their "weighted" gradients are in $L^p(\sigma_s)$ (resp. uniformly bounded) with norms controlled by the $L^p(\sigma_s)$ (resp. BMO(σ_s)) norms of the boundary functions. The identification on the boundary is in the non-tangential convergence sense (up to a set of measure zero on the boundary). To do so, when $s = n$, we assume that Ω satisfies the *pointwise John condition* (see Definition 1.13), while no additional connectivity assumption is required for $s < n$. Let us highlight that this is the first time that such results are proved in such general geometric setting and also for $s < n$. Our second goal is to construct such extensions of Lipschitz functions with compact support on the boundary of an AR(s) domain so that they are Lipschitz on $\overline{\Omega}$ and in the weighted Sobolev space $\dot{W}^{1,2}(\Omega; \omega_s)$ as well. In fact, this is even more important due to the applications to Boundary Value Problems given in Section 7. Finally, we also prove similar extensions of boundary functions in the Campanato space $\Lambda_\beta(\partial\Omega)$ for $\beta \in (0, 1)$.

Extensions of this kind in the case $s = n$ were first constructed by Varopoulos [Var77], [Var78], in the upper half-space $\mathbb{R}^{n+1}_+$ for boundary functions in BMO, and by Hytönen and Rosén, [HR18], for boundary functions in L^p for $p \in (1, \infty)$. Hofmann and Tapiola, [HT20], showed that in corkscrew domains with uniformly n-rectifiable boundary (in the sense of David and S. Semmes [DS1], [DS2]), one can also extend BMO functions with the desired bounds. Recently, Mourgoglou and Tolsa, [MT22], constructed an *almost harmonic extension* of functions in the Hajłasz Sobolev space $\dot{M}^{1,p}(\sigma_n)$, which is the correct analogue of the L^p version of Varopoulos extension for one "smoothness level" up. To be precise, it was proved in [MT22] that the Carleson functional, defined in (1.11), of the distributional Laplacian of the almost harmonic extension is in $L^p(\sigma_n)$ and in [MPT22] that the non-tangential maximal function of its gradient is in $L^p(\sigma_n)$ with norms controlled by the $\dot{M}^{1,p}(\sigma_n)$ semi-norm of the boundary function. The almost harmonic extension and its elliptic analogue (see [MPT22]) were very important since they turned out to be the main ingredients for the solution of the L^p-Regularity problem in domains with interior big pieces of chord-arc domains ([AHMMT, Definition 2.12, p. 892]) for the Laplace operator, [MT22], and for elliptic operators satisfying the Dahlberg-Kenig-Pipher condition, [MPT22], respectively. This solved a 30 year-old question of Kenig.

To construct the extension of BMO boundary functions, Varopoulos introduced the notion of *ε-approximability* in [Var77, Var78], which was further refined by Garnett, [Gar81], who was studying the same problem, inspired by Carleson's Corona theorem and the duality between the Hardy space and BMO. The usual definition of ε-approximability for $s = n$ is the following:

We say that, for a fixed $\varepsilon > 0$, a function u is *ε-approximable* in $\Omega \subset \mathbb{R}^{n+1}$ if there exist a constant $C_\varepsilon > 0$ and a function $\varphi = \varphi^\varepsilon \in C^\infty(\Omega)$ such that $\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon$ and $\sup_{\xi \in \partial\Omega} \mathscr{C}_n(\nabla\varphi)(\xi) < C_\varepsilon$.

We will generalize the definition and say that, for a fixed $\varepsilon > 0$, a function u is *uniformly ε-approximable* in $\Omega \subset \mathbb{R}^{n+1}$ if there exist a constant $C_\varepsilon > 0$ and a function $\varphi = \varphi^\varepsilon \in C^\infty(\Omega)$ such that

$$\sup_{x \in \Omega} |u(x) - \varphi(x)| + \sup_{x \in \Omega} \delta_\Omega(x) |\nabla(u - \varphi)(x)| \lesssim \varepsilon \tag{0.1}$$

and

$$\sup_{\xi \in \partial\Omega} \mathcal{C}_{s,c}(\nabla\varphi)(\xi) \lesssim \varepsilon^{-2}, \tag{0.2}$$

where $\delta_\Omega(\cdot) = \operatorname{dist}(x\cdot, \Omega^c)$ and the implicit constants are independent of ε.

Dahlberg in [Dah80] proved that, in Lipschitz domains, every bounded harmonic function is ε-approximable, which was very useful in the solution of the Dirichlet problem with L^p boundary data for elliptic equations in [KKPT00, HKMP15]. Moreover, Hofmann, Martell, and Mayboroda, [HMM16], showed that in corkscrew domains with uniformly n-rectifiable boundary every bounded weak solution of $\operatorname{div} A\nabla u = 0$ satisfying the so-called Dahlberg-Kenig-Pipher condition is ε-approximable, while in the converse direction, Garnett, Mourgoglou and Tolsa in [GMT18] proved that if any bounded harmonic function is ε-approximable in $\Omega \in \mathrm{AR}(n)$ then $\partial\Omega$ is uniformly n-rectifiable. The latter was further generalized by the same authors along with Azzam, [AGMT], to solutions of elliptic equations with more general coefficients.

Following Hytönen and Rosén, [HR18], we generalize the definition of ε-appoximability to the case $p \in (1, \infty]$ and to domains with s-Ahlfors regular boundaries. If $\Omega \in \mathrm{AR}(s)$ then, for fixed $p \in (1, \infty]$, we say that a function u is *ε-approximable in $L^p(\sigma_s)$* if there exists a function $\varphi = \varphi^\varepsilon \in C^\infty(\Omega)$ such that

$$\|\mathcal{N}(u - \varphi)\|_{L^p(\sigma_s)} + \|\mathcal{N}(\delta_\Omega \nabla(u - \varphi))\|_{L^p(\sigma_s)} \lesssim_p \varepsilon \, \|\mathcal{N}u\|_{L^p(\sigma_s)} \tag{0.3}$$

and

$$\|\mathcal{C}_{s,c}(\nabla\varphi)\|_{L^p(\sigma_s)} \lesssim_p \varepsilon^{-2} \, \|\mathcal{N}u\|_{L^p(\sigma_s)}. \tag{0.4}$$

The concept of ε-approximability in L^p for $p \in (1, \infty)$ was introduced by Hytönen and Rosén in [HR18] who showed that the dyadic average extension operator as well as any weak solution to certain elliptic PDEs in $\mathbb{R}^{n+1}_+$ are ε-approximable in L^p for every $\varepsilon \in (0, 1)$ and $p \in (1, \infty)$. The second part of that result was extended by Hofmann and Tapiola in [HT20] to harmonic functions in $\Omega = \mathbb{R}^{n+1} \setminus E$ where $E \subset \mathbb{R}^{n+1}$ is a uniformly n-rectifiable set. The converse direction was proved by Bortz and Tapiola in [BT19].

Subsequently we formulate the original results that are proven in this thesis and we discuss the techniques used. We first prove a pointwise version of ε-approximability of a regularized version of the dyadic extension of f which we denote by υ_f (see (2.1) for the definition of υ_f).

Theorem 0.1. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$ and let $f \in L^1_{\mathrm{loc}}(\sigma_s)$. If $\varepsilon > 0$, there exist $w \in C^\infty(\Omega)$, $\alpha_0 \geq 1$, and $c_0 \in (0,1/2]$ such that for any $\alpha \geq \alpha_0$, any $c \in (0,c_0]$, and any $\xi \in \partial\Omega$,*

$$\mathcal{N}_\alpha(w-\upsilon_f)(\xi) + \mathcal{N}_\alpha(\delta_\Omega \nabla(w-\upsilon_f))(\xi) \lesssim \varepsilon\, \mathcal{M}f(\xi), \tag{0.5}$$

$$\mathcal{C}_{s,c}(\nabla w)(\xi) \lesssim \varepsilon^{-2}\left[\mathcal{M}(\widetilde{\mathcal{M}}(f))(\xi) + \mathcal{M}(\widetilde{\mathcal{M}}(\mathcal{M}f))(\xi)\right]. \tag{0.6}$$

Therefore, υ_f is ε-approximable in L^p. The implicit constants depend on n, s, and the Ahlfors regularity constants, c_0 depends on the constants of the Whitney decomposition, and α_0 depends on n and on the constants of the corkscrew condition and of the Whitney decomposition.

The following theorem shows that if $f \in \mathrm{BMO}(\sigma_s)$, then υ_f is uniformly ε-approximable.

Theorem 0.2. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$ and let $f \in \mathrm{BMO}(\sigma_s)$. If $\varepsilon > 0$ then there exist $w \in C^\infty(\Omega)$ and $c_0 \in (0,1/2]$ such that for any any $c \in (0,c_0]$, it holds that*

$$\sup_{x\in\Omega} |(w-\upsilon_f)(x)| + \sup_{x\in\Omega} \delta_\Omega(x)|\nabla(w-\upsilon_f)(x)| \lesssim \varepsilon \|f\|_{\mathrm{BMO}(\sigma_s)}, \tag{0.7}$$

$$\sup_{\xi\in\partial\Omega} \mathcal{C}_{s,c}(\nabla w)(\xi) \lesssim \varepsilon^{-2}\|f\|_{\mathrm{BMO}(\sigma_s)}. \tag{0.8}$$

The implicit constants depend on n, s, and the Ahlfors regularity constants and c_0 depends on the constants of the Whitney decomposition.

Theorems 0.1 and 0.2 are the stepping stones towards the construction of the desired extensions. Our first goal is to prove Varopoulos extensions of Lipschitz functions with compact support which are Lipschitz on $\overline{\Omega}$ and also lie in $\dot{W}^{1,2}(\Omega;\omega_s)$.

Theorem 0.3. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$. If $f \in \mathrm{Lip}_c(\partial\Omega)$ then there exist a function $F : \overline{\Omega} \to \mathbb{R}$ and $c_0 \in (0,1/2]$, such that for any $c \in (0,c_0]$, it holds that*

(i) $F \in C^\infty(\Omega) \cap \mathrm{Lip}(\overline{\Omega}) \cap \dot{W}^{1,2}(\Omega;\omega_s)$,

(ii) $\|\mathcal{N}(F)\|_{L^p(\sigma_s)} + \|\mathcal{C}_{s,c}(\nabla F)\|_{L^p(\sigma_s)} \lesssim \|f\|_{L^p(\sigma_s)}$, *for* $p \in (1,\infty]$,

(iii) $\|\mathcal{N}(\delta_\Omega \nabla F)\|_{L^p(\sigma_s)} \lesssim \|f\|_{L^p(\sigma_s)}$,

(iv) $F|_{\partial\Omega} = f$ *continuously.*

Moreover, there exist a function $\bar{F} : \overline{\Omega} \to \mathbb{R}$ and a constant $c_0 \in (0,1/2]$ such that for any $c \in (0,c_0]$ it holds that

(i) $\bar{F} \in C^\infty(\Omega) \cap \mathrm{Lip}(\overline{\Omega}) \cap \dot{W}^{1,2}(\Omega;\omega_s)$,

(ii) $\sup_{\xi\in\partial\Omega} \mathcal{N}_{\sharp,c}(\bar{F})(\xi) + \sup_{\xi\in\partial\Omega} \mathcal{C}_{s,c}(\nabla\bar{F})(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma_s)}$,

(iii) $\sup_{x\in\Omega} \delta_\Omega(x)|\nabla\bar{F}(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma_s)}$,

(iv) $\bar{F}|_{\partial\Omega} = f$ *continuously.*

The second part of the theorem above was already used without proof in connection with Boundary Value Problems (see e.g. [DaKe] and [MiTa]). To the best of our knowledge, a proof of this theorem is not available in the literature,. However, it should not be considered folklore since its proof is far from trivial (at least in our setting) and, it was neither written somewhere, nor it was known among experts.

We also prove a version of the theorem above for boundary functions in the Campanato space $\Lambda_\beta(\partial\Omega)$ for $\beta \in (0,1)$, as well as in the space $\mathrm{Lip}_\beta(\partial\Omega)$ consisting of β-Hölder continuous functions. In fact, in our setting, any function in $\Lambda_\beta(\partial\Omega)$ agrees σ_s-a.e. with a Hölder continuous function and the two semi-norms are comparable, see Remark 1.2.

Theorem 0.4. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$. If $f \in \Lambda_\beta(\partial\Omega)$ for $\beta \in (0,1)$ then there exist a function $F : \overline{\Omega} \to \mathbb{R}$ and a constant $c_0 \in (0,1/2]$, such that for any $c \in (0,c_0]$, there holds*

(i) $F \in C^\infty(\Omega)$,

(ii) $\sup_{\xi\in\partial\Omega} \mathcal{N}^{(\beta)}_{\sharp,c}(F)(\xi) + \sup_{\xi\in\partial\Omega} \mathcal{C}^{(\beta)}_{s,c}(\nabla F)(\xi) \lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}$,

(iii) $\sup_{x\in\Omega} \delta_\Omega(x)^{1-\beta}|\nabla F(x)| \lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}$,

(iv) $\text{nt-lim}_{x\to\xi} F|_{\partial\Omega}(x) = f(\xi)$ *for σ_s-a.e $\xi \in \partial\Omega$.*

Moreover, if $f \in \mathrm{Lip}_\beta(\partial\Omega)$, then $F \in \mathrm{Lip}_\beta(\overline{\Omega})$ and $F|_{\partial\Omega} = f$ continuously.

Interestingly, this is the first time that such a theorem appears in the literature although $\Lambda_\beta(\partial\Omega)$ is a natural endpoint in the interpolation scale that contains the spaces $L^p(\sigma_s)$ and $\mathrm{BMO}(\sigma_s)$. Nevertheless, its proof is surprisingly easier than the corresponding proofs for $L^p(\sigma_s)$ and $\mathrm{BMO}(\sigma_s)$ boundary functions. This is because the regularized version of the dyadic extension of the boundary data satisfies the desired properties and there is no need to use ε-approximability.

When the boundary function is discontinuous, the construction is more complicated and, for $s = n$, requires an additional mild connectivity assumption between σ_n-almost every fixed point $\xi \in \partial\Omega$ and a corkscrew point x_ξ associated to ξ by means of a "good" curve (also depending on ξ). This is necessary in order to construct a surjective trace operator given by means of (a version of) non-tangential convergence to the boundary points.

We prove the existence of an extension of a boundary function in L^p that satisfies the estimates of the one that Hytönen and Rosén built in [HR18] in $\mathbb{R}^{n+1}_+$.

Theorem 0.5. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$. If $s = n$ assume additionally that Ω satisfies the pointwise John condition. If $f \in L^p(\sigma_s)$ with $p \in (1,\infty]$, there exist $u : \Omega \to \mathbb{R}$ and $c_0 \in (0, \frac{1}{4}]$ such that, for any $c \in (0,c_0]$, it holds that*

(i) $u \in C^\infty(\Omega)$,

(ii) $\|\mathcal{N}(u)\|_{L^p(\sigma_s)} + \|\mathcal{C}_{s,c}(\nabla u)\|_{L^p(\sigma_s)} \lesssim \|f\|_{L^p(\sigma_s)}$,

(iii) $\|\mathcal{N}(\delta_\Omega \nabla u)\|_{L^p(\sigma_s)} \lesssim \|f\|_{L^p(\sigma_s)}$,

(iv) For σ_s-almost every $\xi \in \partial\Omega$,[1]

$$\fint_{B(x,\delta_\Omega(x)/2)} u(y)\,dy \to \begin{cases} f(\xi) \text{ non-tangentially}, & \text{if } s < n, \\ f(\xi) \text{ quasi-non-tangentially}, & \text{if } s = n. \end{cases}$$

For $p = \infty$, we use the sup*-norm on the right hand-side of* (ii) *and* (iii) *instead of the L^∞-norm.*

We can also prove a BMO version of the previous theorem, which is an extension that enjoys the properties of the one constructed by Varopoulos [Var77] in $\mathbb{R}^{n+1}_+$.

Theorem 0.6. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$, which, for $s = n$, assume that it additionally satisfies the pointwise John condition*[2]*. If $f \in \mathrm{BMO}(\sigma_s)$ then there exist $u : \Omega \to \mathbb{R}$ and $c_0 \in (0, \frac{1}{4}]$ such that, for any $c \leq c_0$, it holds that*

(i) $u \in C^\infty(\Omega)$,

(ii) $\sup_{\xi \in \partial\Omega} \mathcal{N}_{\sharp,c}(u)(\xi) + \sup_{\xi \in \partial\Omega} \mathcal{C}_{s,c}(\nabla u)(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma_s)}$,

(iii) $\sup_{x \in \Omega} \delta_\Omega(x)|\nabla u(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma_s)}$,

(iv) For σ_s-almost every $\xi \in \partial\Omega$,

$$\fint_{B(x,\delta_\Omega(x)/2)} u(y)\,dy \to \begin{cases} f(\xi) \text{ non-tangentially}, & \text{if } s < n, \\ f(\xi) \text{ quasi-non-tangentially}, & \text{if } s = n. \end{cases}$$

Remark 0.7. Note that the estimate (iii) of Theorem 0.6 can also be written as a non-tangential estimate. Namely, it is equivalent to the estimate

$$\sup_{\xi \in \partial\Omega} \mathcal{N}(\delta_\Omega \nabla u)(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma_s)}.$$

Remark 0.8. The proof of the existence of extensions of complex-valued boundary functions is exactly the same but for the sake of simplicity we prefer to state and prove our results for real-valued boundary functions. Moreover, if $\vec{f} : \partial\Omega \to \mathbb{C}^m$ with $\vec{f} = (f_1, \dots, f_m)$, then its extension is simply $\vec{F} = (F_1, \dots, F_m)$, where F_j is the extension of f_j for each $j \in \{1, 2, \dots, m\}$.

Let us discuss the techniques that were used to prove Theorem 0.6 by the authors in [Var77], [Var78], [Gar81], and [HT21]. For a boundary function $f \in \mathrm{BMO}(\sigma)$, the first important step was to write it as the sum of a function $g \in L^\infty(\sigma)$ and the function $b := \sum_{j\geq 1} a_j \chi_{Q_j}$, where the coefficients a_j satisfy the bound $\sup_{j\geq 1} |a_j| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$ and $\{Q_j\}_{j\geq 1}$ is a countable family of boundary cubes that satisfies a Carleson packing condition, see (3.1). Then the

[1]For the definitions of non-tangential and quasi-non-tangential convergence, see Definition 1.16.

[2]In the case that $s = n$ and Ω satisfies the pointwise John condition but not the local John condition, we assume that f is compactly supported for technical reasons.

desired extension of f is constructed as the sum of the extension G of g and the extension B of b. The extension B can be constructed by hand and although, in "rough" domains, one needs technical arguments to prove the Carleson estimate for the gradient of B and the non-tangential convergence to b (see [HT21]), this can be accomplished without resorting to deep results. On the contrary, the extension of $g \in L^\infty(\sigma)$ uses the ε-approximability of essentially bounded harmonic functions, which is an important theorem in boundary value problems for elliptic PDEs with discontinuous data. To be precise, one extends b by means of harmonic measure producing a L^∞ harmonic function, which is further approximated by a function that satisfies the desired Carleson estimate. Then the extension is constructed by an iteration method.

In light of [HMM16] and [GMT18], the notion of ε-approximability of L^∞ harmonic functions in $\mathrm{AR}(n)$ domains is equivalent to uniform n-rectifiability of $\partial\Omega$. The method described above has geometric limitations and it is natural to ask if the converse of [HT21] holds true; that is, whether the existence of a Varopoulos-type extension in $\mathrm{AR}(n)$ domains implies uniform n-rectifiability of $\partial\Omega$. Theorems 0.5, and 0.6 show that this is clearly not the case since, if E is the 4-corner Cantor set then $\Omega = \mathbb{R}^{n+1} \setminus E \in \mathrm{AR}(n)$ and it is a uniform domain (thus, it satisfies the pointwise John condition).

To tackle Theorem 0.5 and Theorem 0.6, insipred by [HR18], we use a regularized version of the standard dyadic extension v_f of a function $f \in L^1_{\mathrm{loc}}(\sigma)$ and $f \in \mathrm{BMO}(\sigma)$) respectively. We prove in Theorem 0.1 that v_f has an $L^p(\sigma)$ ε-approximator, while in Theorem 0.2 we prove that v_f has a uniform ε-approximator. This comes in contrast to the previous works in the case of $\mathrm{BMO}(\sigma)$ where one approximated a harmonic function. Our proof relies on a Corona decomposition on the boundary, see Definition 3.1, and the correct definition of the approximating function.

The scheme of the proof of Theorems 0.1 and 0.2 is the following. To each Whitney cube P we associate a boundary "dyadic" cube $b(P)$, with the same side-length so that $\mathrm{dist}(P, b(P)) \approx \ell(P)$. Then we define a regularized version of the dyadic extension of the boundary function v_f using a partition of unity $\{\varphi_P\}_{P\in\mathcal{W}(\Omega)}$ subordinated to the collection of (dilated) Whitney cubes of Ω (in the spirit of the Whitney extension) with coefficients the averages of f over $b(P)$. Subsequently we construct a Corona decomposition on the boundary, see Definition 3.1 in chapter 3, so that the difference between the average of f on a top cube and its average on its "stopping parent" is not small (in a certain sense). Define the approximating function w once again via $\{\varphi_P\}_{P\in\mathcal{W}(\Omega)}$ so that, roughly speaking, w is constant (the average of f over the top cube of each tree) when P is associated to a boundary cube $b(P)$ which is in the tree but not such that there exists another boundary dyadic cube Q with comparable side-length which belongs to another tree and $\mathrm{dist}(b(P), Q) \lesssim \ell(b(P))$. In the rest of the cubes it is defined just as the dyadic extension v_f. Then, by some subtle estimates using the stopping conditions and the discrete Carleson embedding theorem, we can conclude.

Note that even when the boundary function is in L^p we still have to overcome significant challenges due to the geometry of our domains. For instance, in $\mathbb{R}^{n+1}_+$, Hytönen and Rosén, [HR18], use the separation of variables $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ in a crucial way as they reduce their case into estimating $\mathscr{C}_n(\partial_t w)$, where $\partial_t w$ stands for the partial derivative in the transversal direction. In higher co-dimensions, even if $\Omega = \mathbb{R}^3 \setminus \mathbb{R}$, such a reduction does not seem to work. Instead, we resort to multiscale analysis to construct the approximating function. An important component is the proof of the packing condition for the top cubes, which was not shown in [HR18], however our proof relies on some of their arguments.

The proof of Theorem 0.5 (resp. 0.6) is based on a trace theorem, Theorem 0.1 (resp. Theorem 0.2), and an iteration argument. The connectivity condition (i.e., pointwise John condition) when $s = n$ is required just for the trace theorem, where we show that if $\mathcal{C}_s(\nabla u) \in L^p(\sigma_s)$ (resp. uniformly bounded) then there exists a trace operator of u on the boundary. Interestingly, to show that the trace of u is in $L^p(\sigma_s)$ we also need that the non-tangential maximal function of u is in $L^p(\sigma_s)$, while to show that it is in $\mathrm{BMO}(\sigma_n)$ we need to assume that the domain satisfies the local John condition, i.e., a stronger geometric assumption which is always satisfied when $s < n$. This is the reason why when $s = n$, $f \in \mathrm{BMO}(\sigma_n)$ and Ω has the pointwise John condition (but not the local John condition), we use the usual splitting of the $\mathrm{BMO}(\sigma_n)$ function into a $L^\infty(\sigma_n)$ function and a function of the form $\sum_{j\geq 1} a_j \chi_{Q_j}$. Then we apply Theorem 0.1 to get the extension of its L^∞ part and add it to the extension of the "bad" part as constructed in [HT21, Proposition 1.3][3]. One needs to be careful with the details of the iteration argument as well, in order to be able to define $\sum_{j\geq 0} u_j$ in a meaningful way and identify $\nabla \sum_{j\geq 0} u_j$ with $\sum_{j\geq 0} \nabla u_j$. For more details see Section 6.

To prove Theorem 0.3, for $f \in \mathrm{Lip}_c(\partial\Omega)$ first we use Theorems 0.1 and 0.2 to construct the approximating functions of the regularized dyadic extension of f. Subsequently we define the extension to be equal to the approximation everywhere, apart from a neighborhood of the boundary of "width" $\delta > 0$, where we set it to be equal to the regularized dyadic extension. Then, we choose δ to be $\|f\|_{L^p(\sigma_s)}/\|f\|_{\dot{M}^{1,p}(\sigma_s)}$ in the case of $p \in (1, \infty)$ (resp. $\|f\|_{L^\infty(\sigma_s)}/\operatorname{Lip} f$ for $p = \infty$) and $\|f\|_{\mathrm{BMO}(\sigma_s)}/\operatorname{Lip} f$ in the case of $\mathrm{BMO}(\sigma_s)$ and obtain the desired estimates. It is interesting to note that we do not construct an *a priori* extension and modify it later to obtain the Lipschitz extension; we just modify the ε-approximator of v_f. That is why we do not assume any connectivity condition as in Theorems 0.5 and 0.6. Instead, the existence of the trace is readily given by the continuity of v_f which is Lipschitz on $\overline{\Omega}$.

Finally, we use Theorem 0.3 to obtain connections between Poisson and Boundary Value Problems (see Definitions 1.6, 1.8, and 1.9) for systems of elliptic equations in divergence form with merely bounded complex-valued coefficients. In particular, we prove the following.

Theorem 0.9. *Let $\Omega \in \mathrm{AR}(n)$ and L be defined as in (1.27). If L^* is its formal adjoint then the following hold:*

1. *If (PR_p^L) is solvable in Ω for some $p > 1$ then $(\mathrm{D}_{p'}^{L^*})$ is also solvable, where $1/p + 1/p' = 1$.*
2. *If (PR_1^L) for $H = 0$ is solvable in Ω then both $(\mathrm{PD}_\infty^{L^*})$ with $H = 0$, and $(\mathrm{D}_{\mathrm{BMO}}^{L^*})$ are solvable in Ω.*
3. *If (PD_p^L) for $p \in (1, \infty)$ (resp. (PD_∞^L)) is solvable in Ω with $H = 0$, then the Dirichlet problem (D_p^L) (resp. $(\mathrm{D}_{\mathrm{BMO}}^L)$) is also solvable in Ω.*

The Poisson Dirichlet problem (PD_p^L) for $p > 1$ (resp. the Poisson regularity problem (PR_p^L)) with interior data in suitable Carleson spaces with scale-invariant estimates for the non tangential maximal function of the (resp. gradient of the) solution (see Definitions 1.8 and 1.9), was first defined and investigated in a recent work of Mourgoglou, Poggi and Tolsa [MPT22].

[3] We recall here that in [HT21], uniform n-rectifiability was only used for the extension of the L^∞ part, while the bad part can be extended in any $\mathrm{AR}(n)$ domain.

The authors were interested in the solvability of the regularity problem (see Definition 1.7) for elliptic equations satisfying the so-called Dahlberg-Kenig-Pipher condition in corkscrew domains with uniformly n-rectifiable boundaries. Due to the fact that it is not known if the layer potentials for such operators are bounded, the authors went through the solvability of the Poisson Dirichlet problem in order to circumvent the aforementioned difficulty. In particular, they show that, for such operators, $(\mathrm{PD}_{p'}^{L^*}) \Rightarrow (\mathrm{R}_p^L)$ for any $p > 1$. To do so they use the almost elliptic extension which, as mentioned before, is a one "smoothness level up" Varopoulos-type extnesion for Sobolev functions on the boundary. One of their results states the following equivalences for elliptic equations with merely bounded coefficients:

$$(\mathrm{D}_{p'}^{L^*}) \Leftrightarrow (\mathrm{PD}_{p'}^{L^*}) \Leftrightarrow (\mathrm{PR}_p^L), \qquad p \in (1, \infty).$$

The equivalence $(\mathrm{PD}_{p'}^{L^*}) \Leftrightarrow (\mathrm{PR}_p^L)$ holds for systems as well since the proof in [MPT22] does not rely on the maximum principle or the elliptic measure. Nevertheless, the rest of the results in [MPT22] exploit the connection between the weak-A_∞ condition of the elliptic measure and the solvability of $(\mathrm{D}_{p'}^{L^*})$ in a significant way and so they only hold for real equations. Inspired by the use of the almost elliptic extension in [MPT22], we utilize the Varopoulos-extension constructed in Theorem 0.3 in order to extend some of those results to the case of elliptic systems; we also obtain the endpoint results which are new even for real equations. We refer to the introduction of [MPT22] for a detailed presentation of the historically results in this area.

Related results

While writing this thesis, we were informed by Bruno Poggi and Xavier Tolsa that in collaboration with Simon Bortz and Olli Tapiola, they have independently proved in [BOPT23], a less general version of Theorem 0.6 which holds for uniform domains with n-Ahlfors regular boundaries such that there is an elliptic measure which is A_∞ with respect to surface measure. They obtain this result as a corollary of their main result studying ε-approximability of solutions to arbitrary elliptic partial differential equations. Their assumptions hold, in particular, for the complement of the 4-corner Cantor set in $\mathbb{R}^2$, thus they also show that uniform rectifiability is not a necessary condition in order to construct Varopoulos-type extensions.

Chapter 1

Preliminaries and notation

We will write $a \lesssim b$ if there is a constant $C > 0$ so that $a \leq Cb$ and $a \approx b$ if $\alpha \lesssim b$ and $b \lesssim a$. If we want to indicate the dependence of C on a certain quantity s, we write $a \lesssim_s b$. For a function space X we denote by X_c all the compactly supported functions in X.

1.1 Preliminaries

In $\mathbb{R}^{n+1}$ and for $s \in [0, n+1]$, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and assume that $\mathcal{H}^{n+1}$ is normalized so that it coincides with $\mathcal{L}^{n+1}$, the $(n+1)$-dimensional Lebesgue measure in $\mathbb{R}^{n+1}$. We also denote by $\sigma_s := \mathcal{H}^s|_{\partial\Omega}$ the surface measure of Ω. When the dimension is clear from the context we drop the dependence on s and just write σ.

Definition 1.1. If $s \in (0, n+1]$, a measure μ in $\mathbb{R}^{n+1}$ is called *s-Ahlfors regular* if there exists some constant $C_0 > 0$ such that

$$C_0^{-1}\, r^s \leq \mu(B(x,r)) \leq C_0\, r^s$$

for all $x \in \operatorname{supp}\mu$ and $0 < r < \operatorname{diam}(\operatorname{supp}\mu)$. If $E \subset \mathbb{R}^{n+1}$ is a closed set we say that E is *s-Ahlfors regular* if $\mathcal{H}^s|_E$ is s-Ahlfors regular.

1.2 Function spaces

We write $2^* = \frac{2(n+1)}{n-1}$ and $2_* = (2^*)' = \frac{2(n+1)}{n+3}$. Recall that $C_c^\infty(\Omega)$ is the space of compactly supported smooth functions in Ω. For $p \in [1, \infty)$ and a non-negative function $w \in L^1_{\text{loc}}(\Omega)$ we define the homogeneous weighted Sobolev space $\dot{W}^{1,p}(\Omega; w)$ to be the space consisting of $L^1_{\text{loc}}(\Omega)$ functions whose weak gradients exist in Ω and are in $L^p(\Omega; w)$. We also define the inhomogeneous weighted Sobolev space $W^{1,p}(\Omega; w)$ to be the space of functions in $L^p(\Omega; w)$ whose weak derivatives exist in Ω and are also in $L^p(\Omega; w)$, and $W_0^{1,p}(\Omega; w)$ to be the completion of $C_c^\infty(\Omega)$ under the norm $\|u\|_{W^{1,p}(\Omega;w)} := \|u\|_{L^p(\Omega;w)} + \|\nabla u\|_{L^p(\Omega;w)}$. Finally, we let $Y_0^{1,2}(\Omega; w)$ be the completion of $C_c^\infty(\Omega)$ under the norm $\|u\|_{Y^{1,2}(\Omega;w)} := \|u\|_{L^{2^*}(\Omega;w)} + \|\nabla u\|_{L^2(\Omega;w)}$.

Let Σ be a metric space equipped with a non-atomic doubling measure σ, which means that there is a uniform constant $C_\sigma \geq 1$ such that $\sigma(B(x, 2r)) \leq C_\sigma \sigma(B(x,r))$ for all $x \in \Sigma$ and $r > 0$. If $E \subset \Sigma$ is a Borel set such that $0 < \sigma(E) < \infty$ and $f \in L^1_{\text{loc}}(\sigma)$, we denote the average of f over E by

$$m_{\sigma,E} f := \fint_E f\, d\sigma := \frac{1}{\sigma(E)} \int_E f d\sigma. \tag{1.1}$$

If σ is the Lebesgue measure then we simply write $m_E f$.

For $\beta \in [0,1)$ we define $\Lambda_\beta(\partial\Omega)$ to be the *Campanato space* consisting of the functions $f \in L^1_{\text{loc}}(\sigma)$ satisfying

$$\|f\|_{\Lambda_\beta(\partial\Omega)} := \sup_{\substack{x \in \operatorname{supp}\sigma \\ r \in (0, 2\operatorname{diam}\partial\Omega)}} \frac{1}{r^\beta} \fint_{B(x,r)} |f(y) - m_{\sigma,B(x,r)} f| d\sigma(y) < \infty. \tag{1.2}$$

Note that $\Lambda_0(\sigma) = \text{BMO}(\sigma)$, the space of functions of *bounded mean σ-oscillation*. We also define the space of functions of *vanishing mean oscillation*[1], which we denote by $\text{VMO}(\sigma)$, to be the closure of the space of continuous functions with compact support $C_c(\Sigma)$ in the $\text{BMO}(\sigma)$ norm.

We say that α is a 2-*atom* if there exists $x \in \Sigma$ and $0 < r < \operatorname{diam}(\Sigma)$ such that

$$\operatorname{supp}\alpha \subset B(x,r), \quad \|\alpha\|_{L^2(\sigma)} \lesssim \sigma(B(x,r))^{-1/2} \quad \text{and} \quad \int \alpha\, d\sigma = 0.$$

We define the *atomic Hardy space* $H^1(\sigma)$ as follows: $f \in H^1(\sigma)$ if there exist a sequence $\lambda_j \in \mathbb{C}$ and a sequence of 2-atoms α_j such that $f = \sum_j \lambda_j \alpha_j$ in $L^1(\sigma)$; we say then that f has an atomic decomposition. $H^1(\sigma)$ is a subspace of $L^1(\sigma)$ and is a Banach space with norm

$$\|f\|_{H^1(\sigma)} := \inf \Big\{ \sum_j |\lambda_j| : \text{all atomic decompositions } f = \sum_j \lambda_j \alpha_j \Big\}.$$

By the work of Coifmann and Weiss, [CW77], we have that $(H^1(\sigma))^* = \text{BMO}(\sigma)$ and $(\text{VMO}(\sigma))^* = H^1(\sigma)$.

For $\beta \in (0,1]$ we define $\text{Lip}_\beta(\Sigma)$ to be the space of measurable functions that satisfy

$$\|f\|_{\text{Lip}_\beta(\Sigma)} := \sup_{\substack{x,y \in \Sigma \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty. \tag{1.3}$$

When $\beta = 1$ we simply write $\text{Lip}(\Sigma)$ since it is the space of Lipschitz functions. If Σ is locally compact then it holds that $\text{Lip}_c(\Sigma)$ is dense in $C_c(\Sigma)$ in the supremum norm. Therefore, it is easy to see that in that case

$$\overline{\text{Lip}_c(\Sigma)}^{\text{BMO}(\sigma)} = \text{VMO}(\sigma).$$

Remark 1.2. By a simple inspection of the proof of [MS79, Theorem 4], it is easy to see that if Σ is a metric space equipped with a measure σ which is s-Ahlfors regular and $\beta \in (0,1)$ then for every $f \in \Lambda_\beta(\Sigma)$ there exists $g \in \text{Lip}_\beta(\Sigma)$ such that $f(x) = g(x)$ for σ-a.e. $x \in \Sigma$ and $\|f\|_{\Lambda_\beta(\sigma)} \approx \|f\|_{\text{Lip}_\beta(\Sigma)}$.

Following [Ha] we will introduce the *Hajłasz's Sobolev space* on Σ. For a Borel function $f : \Sigma \to \mathbb{R}$ we say that a non-negative Borel function $g : \Sigma \to \mathbb{R}$ is a *Hajłasz upper gradient of* f if

$$|f(x) - f(y)| \leq |x-y|\,(g(x) + g(y)) \quad \text{for } \sigma\text{-a.e. } x, y \in \Sigma. \tag{1.4}$$

[1] VMO was originally introduced by Sarason in [Sar75].

We denote the collection of all the Hajłasz upper gradients of f by $D(f)$.

For $p > 0$ we denote by $\dot{M}^{1,p}(\sigma)$ the space of Borel functions f which have a Hajłasz upper gradient in $L^p(\sigma)$, and we let $M^{1,p}(\sigma)$ be the space of functions $f \in L^p(\sigma)$ which have a Hajłasz upper gradient in $L^p(\sigma)$, i.e., $M^{1,p}(\sigma) = \dot{M}^{1,p}(\sigma) \cap L^p(\sigma)$. We define the semi-norm (as it annihilates constants)

$$\|f\|_{\dot{M}^{1,p}(\sigma)} = \inf_{g \in D(f)} \|g\|_{L^p(\sigma)}. \tag{1.5}$$

If Σ is bounded then we define the norm

$$\|f\|_{M^{1,p}(\sigma)} = (\operatorname{diam} \Sigma)^{-1} \|f\|_{L^p(\sigma)} + \inf_{g \in D(f)} \|g\|_{L^p(\sigma)}, \tag{1.6}$$

while if Σ is unbounded we consider the space $M^{1,p}(\sigma) := \dot{M}^{1,p}(\sigma)/\mathbb{R}$. Observe that from the uniform convexity of $L^p(\sigma)$ for $p \in (1, \infty)$, one easily deduces that the infimum in the definition of the norms $\|\cdot\|_{\dot{M}^{1,p}(\Sigma)}$ and $\|\cdot\|_{M^{1,p}(\Sigma)}$ in (1.5) and (1.6) respectively, is attained and is unique. We denote by $\nabla_{H,p} f$ the function g which attains the infimum which we will call the *least Hajłasz upper gradient* of f.

1.3 Maximal operators and Carleson functionals

Set $\delta_\Omega(\cdot) := \operatorname{dist}(\cdot, \Omega^c)$, $B^x := B(x, \delta_\Omega(x))$, and $cB^x := B(x, c\,\delta_\Omega(x))$, for $c \in (0, \frac{1}{2}]$. For $f \in L^1_{\mathrm{loc}}(\mu)$ and $x \in \Omega$ we set

$$m_{q,c}(f)(x) := \begin{cases} m_{cB^x}(|f|^q)^{1/q} & \text{if } 1 \le q < \infty, \\ \sup\limits_{y \in cB^x} |f(y)| & \text{if } q = \infty, \end{cases}$$

and

$$m_{\sharp,c}(f)(x) := m_{\infty,c}(f - m_{cB^x} f)(x), \quad x \in \Omega \tag{1.7}$$

We define the *centered Hardy-Littlewood maximal operator* for a function $f \in L^1_{\mathrm{loc}}(\sigma)$ as

$$\mathcal{M}(f)(x) := \sup_{r>0} m_{\sigma, B(x,r)}(|f|), \quad x \in \Sigma$$

while the *non-centered Hardy-Littlewood maximal operator* is defined to be

$$\widetilde{\mathcal{M}}(f)(x) := \sup_{B \ni x} m_{\sigma,B}(|f|), \quad x \in \Sigma,$$

where the supremum is taken over all balls B containing x. The *dyadic Hardy-Littlewood maximal operator* with respect to a dyadic lattice $\mathcal{D}_\sigma$ on Σ[2] will be denoted

$$\mathcal{M}_{\mathcal{D}_\sigma} f(x) := \sup_{Q \in \mathcal{D}_\sigma, Q \ni x} m_{\sigma,Q}(|f|).$$

If the measure is clear from the context we will just write $\mathcal{M}_{\mathcal{D}} f$ in place of $\mathcal{M}_{\mathcal{D}_\sigma} f$. We also set

$$Mf(Q) := \sup_{\substack{R \in \mathcal{D}_\sigma \\ Q \subset R}} m_{\sigma,R}(|f|) \tag{1.8}$$

[2]For the construction of dyadic lattices in this setting, see e.g. [Chr90].

to be a *truncated* version of $\mathcal{M}_{\mathcal{D}} f(x)$.

From now on, we assume that $\Omega \in \mathrm{AR}(s)$ in $\mathbb{R}^{n+1}$ and $\sigma = \mathcal{H}^s|_{\partial\Omega}$.

For $\alpha > 0$ and $\xi \in \partial\Omega$ we define the *cone* with vertex ξ and aperture $\alpha > 0$ to be the set

$$\gamma_\alpha(\xi) := \{x \in \Omega : |x - \xi| < (1+\alpha)\mathrm{dist}(x, \partial\Omega)\}.$$

and for a fixed aperture $\alpha > 0$ the *non-tangential maximal operator* of a measurable function $f : \Omega \to \mathbb{R}$ by

$$\mathcal{N}_\alpha(f)(\xi) := \sup_{x \in \gamma_\alpha(\xi)} |f(x)|, \qquad \xi \in \partial\Omega. \tag{1.9}$$

By a straightforward modification of the classical proof of Feffermann and Stein [FS, Lemma 1], one can show the following.

Lemma 1.3. *For $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$ and $\beta \in [0, 1)$, there holds $\|\mathcal{N}_\alpha^{(\beta)}(f)\|_{L^p(\sigma)} \approx_{\alpha,\beta,s} \|\mathcal{N}_\beta(f)\|_{L^p(\sigma)}$ for all $\alpha, \beta > 0$ and $p \in (0, \infty)$.*

For a fixed aperture $\alpha > 0$, $\beta \in [0, 1)$, and a constant $c \in (0, \frac{1}{2}]$, we also define the *sharp non-tangential maximal opeartor* applied to a measurable function $f : \Omega \to \mathbb{R}$ by

$$\mathcal{N}_{\sharp,\alpha,c}^{(\beta)}(f)(\xi) := \sup_{x \in \gamma_\alpha(\xi)} \delta_\Omega(x)^{-\beta} m_{\sharp,c}(f)(x), \qquad \xi \in \partial\Omega. \tag{1.10}$$

Setting $\omega_s(x) := \delta_\Omega(x)^{s-n}$ for $x \in \Omega$, we define the *Carleson functional* of a function $F \in L^1_{\mathrm{loc}}\big(\Omega, \omega_s(x)\, dx\big)$ by

$$\mathscr{C}_s^{(\beta)}(F)(\xi) := \sup_{r>0} \frac{1}{r^{s+\beta}} \int_{B(\xi,r)\cap\Omega} |F(x)|\, \omega_s(x)\, dx, \qquad \xi \in \partial\Omega. \tag{1.11}$$

We define the *modified Carleson functional* of a locally bounded function F by means of

$$\mathcal{C}_{s,c}^{(\beta)}(F)(\xi) := \mathscr{C}_s^{(\beta)}\big(m_{\infty,c}(F)\big)(\xi), \qquad \xi \in \partial\Omega. \tag{1.12}$$

For $q \in [1, \infty)$, the *q-Carleson functional* of a function $F \in L^q_{\mathrm{loc}}\big(\Omega, dx\big)$ is defined to be

$$\mathcal{C}_{s,q,c}^{(\beta)}(F)(\xi) := \sup_{r>0} \frac{1}{r^{s+\beta}} \int_{B(\xi,r)\cap\Omega} m_{q,\sigma,cB^x}(|F|)\, \omega_s(x)\, dx, \qquad \xi \in \partial\Omega. \tag{1.13}$$

Lemma 1.4. *If $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$, $\beta \in [0, 1)$, $q \in [1, \infty)$, $F \in L^q_{\mathrm{loc}}\big(\Omega, dx\big)$ and $0 < c_1 < c_2 \le \frac{1}{2}$ then*

$$\mathcal{C}_{s,q,c_2}^{(\beta)}(F)(\xi) \lesssim_{c_1,c_2} \mathcal{C}_{s,q,c_1}^{(\beta)}(F)(\xi) \quad \text{for every } \xi \in \partial\Omega. \tag{1.14}$$

Proof. The case $s = n$ and $\beta = 0$ was proved in [MPT22, Lemma 2.2] while the proof in the other cases follows by a routine adaptation of the same arguments. □

If it is clear from the context and in view of Lemmas 1.3 and 1.4, we will suppress the dependence of $\mathcal{N}_{\alpha}^{(\beta)}$, $\mathcal{N}_{\sharp,\alpha,c}^{(\beta)}$, $\mathcal{C}_{s,q,c}^{(\beta)}$, and $\mathcal{C}_{s,c}^{(\beta)}$ on α and c, and write $\mathcal{N}^{(\beta)}$, $\mathcal{N}_{\sharp}^{(\beta)}$, $\mathcal{C}_{s,q}^{(\beta)}$ and $\mathcal{C}_{s}^{(\beta)}$. If $s = n$ and $\beta = 0$, we will drop the dependence on s and β as well.

For $p \in (1, \infty)$ we introduce the Banach spaces

$$\mathrm{N}^p(\Omega) := \{w : \Omega \to \mathbb{R} : w \,\text{is measurable and}\; \mathcal{N}(w) \in L^p(\sigma)\}, \tag{1.15}$$

$$\mathrm{C}^p_{s,\infty}(\Omega) := \{\vec{F} \in C(\Omega; \mathbb{R}^{n+1}) : \mathcal{C}_s(|\vec{F}|) \in L^p(\sigma)\}, \tag{1.16}$$

equipped with the norms $\|w\|_{\mathrm{N}^p(\Omega)} := \|\mathcal{N}(w)\|_{L^p(\sigma)}$, $\|\vec{F}\|_{\mathrm{C}^p_{s,\infty}(\Omega)} := \|\mathcal{C}_s(|\vec{F}|)\|_{L^p(\sigma)}$, respectively. For $p = \infty$ we define

$$\mathrm{N}^\infty(\Omega) := \{w : \Omega \to \mathbb{R} : w \,\text{is measurable and}\; \sup_{\xi\in\partial\Omega} \mathcal{N}(w)(\xi) < \infty\}, \tag{1.17}$$

$$\mathrm{C}^\infty_{s,\infty}(\Omega) := \{\vec{F} \in C(\Omega; \mathbb{R}^{n+1}) : \sup_{\xi\in\partial\Omega} \mathcal{C}_s(|\vec{F}|)(\xi) < \infty\}, \tag{1.18}$$

$$\mathrm{N}^\infty_\sharp(\Omega) := \{w : \Omega \to \mathbb{R} : w \text{ is locally bounded in } \Omega \text{ and } \sup_{\xi\in\partial\Omega} \mathcal{N}_{\sharp,c}(w)(\xi) < \infty\}, \tag{1.19}$$

and equip them, respectively, with the norms $\|w\|_{\mathrm{N}^\infty(\Omega)} := \sup_{\xi\in\partial\Omega} \mathcal{N}(w)(\xi)$, $\|\vec{F}\|_{\mathrm{C}^\infty_{s,\infty}(\Omega)} := \sup_{\xi\in\partial\Omega} \mathcal{C}_s(|\vec{F}|)(\xi)$, and with the semi-norm

$$\|w\|_{\mathrm{N}^\infty_\sharp(\Omega)} := \sup_{\xi\in\partial\Omega} \mathcal{N}_{\sharp,c}(w)(\xi) = \sup_{x\in\Omega} m_{\sharp,c}(w)(x),$$

which is a norm modulo constants. We will prove in Lemma B.1 in Appendix that the quotient space $\mathrm{N}^\infty_\sharp(\Omega)/\mathbb{R}$ is a Banach space. It is not hard to see that $\mathrm{N}^\infty(\Omega)$ and $\mathrm{C}^\infty_{s,\infty}(\Omega)$ are Banach spaces. We also define the spaces

$$\mathrm{C}^{1,p}_{s,\infty}(\Omega) := \{u \in C^1(\Omega) : \nabla u \in \mathrm{C}^p_{s,\infty}(\Omega)\}, \qquad p \in (1, \infty), \tag{1.20}$$

$$\mathrm{C}^{1,\infty}_{s,\infty}(\Omega) := \{u \in C^1(\Omega) : \nabla u \in \mathrm{C}^\infty_{s,\infty}(\Omega)\}, \tag{1.21}$$

and the semi-norms $\|u\|_{\mathrm{C}^{1,p}_{s,\infty}(\Omega)} := \|\nabla u\|_{\mathrm{C}^p_{s,\infty}(\Omega)}$ and $\|u\|_{\mathrm{C}^{1,\infty}_{s,\infty}(\Omega)} := \|\nabla u\|_{\mathrm{C}^\infty_{s,\infty}(\Omega)}$.

If $G : \Omega \to \mathbb{R}$ is a measurable function in Ω, we define the *area functional* of G, for a fixed aperture $\alpha > 0$, as

$$\mathcal{A}^{(\alpha)}G(\xi) := \int_{\gamma_\alpha(\xi)} |G(x)|\delta_\Omega(x)^{-n}\, dx, \qquad \xi \in \partial\Omega. \tag{1.22}$$

The following lemma is proved in the Appendix A.

Lemma 1.5. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$, $u \in L^1_{\mathrm{loc}}(\Omega, \omega_s)$, $p \in [1, \infty)$ and $\alpha \geq 1$. There exists $C \geq 1$ such that for any $\xi \in \partial\Omega$ and $r \in (0, 2\,\mathrm{diam}(\partial\Omega))$ there holds*

$$\|\mathcal{A}^{(\alpha)}(u\mathbf{1}_{B(\xi,r)})\|_{L^p(\sigma,B(\xi,r))} \lesssim r^\beta \|\mathscr{C}_s^{(\beta)}(u\mathbf{1}_{B(\xi,Cr)})\|_{L^p(\sigma,B(\xi,Cr))}. \tag{1.23}$$

If $\beta = 0$ we also have that

$$\|\mathcal{A}^{(\alpha)}(u)\|_{L^p(\sigma)} \lesssim \|\mathscr{C}_s(u)\|_{L^p(\sigma)}. \tag{1.24}$$

We also introduce the *modified non-tangential maximal operator* $\widetilde{\mathcal{N}}_{\alpha,c,r}$ for a given aperture $\alpha > 0$, a parameter $c \in (0, 1/2]$ and $r \geq 1$: for any $u \in L^r_{\mathrm{loc}}(\Omega)$ it is defined as

$$\widetilde{\mathcal{N}}_{\alpha,c,r} u(\xi) := \sup_{x \in \gamma_\alpha(\xi)} \left(\fint_{B(x, c\delta_\Omega(x))} |u(y)|^r \, dy \right)^{1/r}, \qquad \xi \in \partial\Omega.$$

The L^p-norms of these non-tangential maximal functions with different apertures α or averaging parameters c are comparable, see [MPT22, Lemma 2.1], and in order to simplify the notation we will just write $\widetilde{\mathcal{N}}_r = \widetilde{\mathcal{N}}_{\alpha,c,r}$ when we do not need to specify neither α nor c.

For any $q \geq 1$ and $p > 1$ we define the Banach space

$$\mathrm{C}_{s,q,p}(\Omega) := \{ H \in L^q_{\mathrm{loc}}(\Omega) : \mathcal{C}_{s,q}(H) \in L^p(\sigma) \}$$

with norm $\|H\|_{\mathrm{C}_{s,q,p}} = \|\mathcal{C}_{s.q}(H)\|_{L^p(\sigma)}$, and for $r \in [1, \infty]$, $p > 1$ we let

$$\mathrm{N}_{r,p}(\Omega) := \{ u \in L^r_{\mathrm{loc}}(\Omega) : \widetilde{\mathcal{N}}_r(u) \in L^p(\sigma) \},$$

where we identify $\widetilde{\mathcal{N}}_\infty = \mathcal{N}$ with norm $\|u\|_{\mathrm{N}_{r,p}(\Omega)} = \|\widetilde{\mathcal{N}}_r(u)\|_{L^p(\sigma)}$; By the proof of [MPT22, Proposition 2.4] it follows that if either Ω is bounded or $\partial\Omega$ is unbounded it holds $\mathrm{N}_{q,p}(\Omega) = (\mathrm{C}_{s,q',p'}(\Omega))^*$. When $s = n$ we drop the subscript s from $\mathrm{C}_{s,q,p}$.

If $\Omega \in \mathrm{AR}(s)$ we define the *tent spaces*

$$T^\infty_{s,2}(\Omega) := \{ f \in L^2_{\mathrm{loc}}(\Omega) : \mathscr{C}_s(f^2 \, \delta_\Omega^{-1}) \in L^\infty(\sigma) \} \tag{1.25}$$

and

$$T^p_2(\Omega) := \{ g \in L^2_{\mathrm{loc}}(\Omega) : \big(\mathcal{A}(g^2 \, \delta_\Omega^{-1})\big)^{1/2} \in L^p(\sigma) \}, \text{ for } p \in (0, \infty), \tag{1.26}$$

and we equip them with the respective norms

$$\|f\|_{T^\infty_{s,2}(\Omega)} = \big\| \mathscr{C}_s\big(f^2 \, \delta_\Omega^{-1}\big)^{1/2} \big\|_{L^\infty(\sigma)} \quad \text{and} \quad \|g\|_{T^\infty_2(\Omega)} = \big\| \big(\mathcal{A}(g^2 \, \delta_\Omega^{-1}\big)^{1/2} \big\|_{L^p(\sigma)}.$$

When $s = n$ we drop the subscript s from $T^\infty_{2,s}$ and just write T^∞_2.

The tent spaces were first introduced and studied in [CMS85] in the upper-half space $\mathbb{R}^{n+1}_+$ and their definition was extended to $\mathrm{AR}(n)$ domains in [MPT13]. Note that the results are stated in chord-arc domains but an easy inspection of the proofs in [MPT13] reveals that neither the Harnack chain condition nor the exterior corkscrew condition are necessary. An important result in this area is the duality between tent spaces. Namely, if $\Omega \in \mathrm{AR}(n)$ then the pairing

$$\langle f, g \rangle = \int_\Omega f(x) \, g(x) \, \frac{dx}{\delta_\Omega(x)}$$

realizes $T^\infty_2(\Omega)$ as the Banach dual of $T^1_2(\Omega)$. Moreover, for $p \in (1, \infty)$, the same pairing realizes $T^{p'}_2(\Omega)$ as the Banach dual of $T^p_2(\Omega)$, where $1/p + 1/p' = 1$. In this generality, this follows from the proof of Theorem 4.2 and Remarks 4.3 and 4.4 in [MPT13]. By an inspection of the proofs, one can easily show that if $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$, then the pairing

$$\langle f, g \rangle := \int_\Omega f(x) \, g(x) \, \frac{dx}{\delta_\Omega(x)^{n+1-s}}$$

realizes $T^\infty_{s,2}(\Omega)$ as the Banach dual of $T^1_2(\Omega)$. Analogously, for $p \in (1, \infty)$ the same pairing realizes $T^{p'}_2(\Omega)$ as the Banach dual of $T^p_2(\Omega)$, where $1/p + 1/p' = 1$.

1.4 Elliptic systems and Boundary value problems

In this section we consider domains $\Omega \in \mathrm{AR}(n)$ with $n \geq 1$. Let L be an elliptic operator acting on column vector-fields $u = (u^1, \ldots, u^m)^T$, where $u^\beta : \Omega \to \mathbb{C}$ for $\beta = 1, 2, \ldots, m$, defined as follows:

$$Lu(x) = -\sum_{i,j=1}^{n+1} \partial_i (A_{ij}(x) \partial_j u(x)) = -\sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n+1} \partial_i \big(a_{ij}^{\alpha\beta}(x)\, \partial_j u^\beta(x) \big), \tag{1.27}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq n+1$ and A_{ij} are $m \times m$ matrix-valued functions on $\mathbb{R}^{n+1}$ with entries $a_{ij}^{\alpha\beta} : \Omega \to \mathbb{C}$, $\alpha, \beta \in \{1, \ldots, m\}$ for which there exists $\lambda \in (0,1]$ such that

$$\sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n+1} |a_{ij}^{\alpha\beta}(x)|^2 \leq \lambda^{-2}, \quad \text{for a.e. } x \in \Omega \text{ and} \tag{1.28}$$

$$\mathfrak{Re} \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n+1} a_{ij}^{\alpha\beta}(x) \xi_j^\beta \overline{\xi_i^\alpha} \geq \lambda \sum_{\alpha=1}^{m} \sum_{i=1}^{n+1} |\xi_i^\alpha|^2 \quad \text{for a.e. } x \in \Omega. \tag{1.29}$$

For $m = 1$ and $a_{ij} : \Omega \to \mathbb{R}$, estimate (1.29) amounts to the standard accretivity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n+1} a_{ij}(x) \xi_i \dot{\xi}_j \quad \text{for a.e. } x \in \Omega, \;\; \xi \in \mathbb{R}^{n+1}. \tag{1.30}$$

Notice that the α-th component of the column vector Lu is given by

$$(Lu)^\alpha(x) := -\sum_{\beta=1}^{m} \sum_{i,j=1}^{n+1} \partial_i \big(a_{ij}^{\alpha\beta}(x)\, \partial_j u^\beta(x) \big). \tag{1.31}$$

We also define the adjoint operator of L by

$$L^* u(x) := -\sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n+1} \partial_i \big(\overline{a_{ji}^{\beta\alpha}}(x)\, \partial_j u^\beta(x) \big),$$

that is $L^* = -\operatorname{div} A^* \nabla$ where $A^* = (\overline{A}_{ij})^T$ or equivalently $(a_{ij}^{\alpha\beta})^* = \overline{a_{ji}^{\beta\alpha}}$.

We assume that $H : \Omega \to \mathbb{C}^m$ is given by $H = (H^1, \ldots, H^m)$ and $\boldsymbol{\Xi} : \Omega \to \mathbb{C}^{m(n+1)}$ is given by $\boldsymbol{\Xi} := (\vec{\Xi}^1, \ldots, \vec{\Xi}^m)$, where $\vec{\Xi}^\alpha : \Omega \to \mathbb{C}^{n+1}$ and $\vec{\Xi}^\alpha = (\Xi_1^\alpha, \ldots, \Xi_{n+1}^\alpha)$ for $\alpha = 1, \ldots m$. We are interested in solutions of the inhomogeneous equation $Lu = -\operatorname{div} \boldsymbol{\Xi} + H$ in Ω in the sense

$$Lu(x) = -\sum_{\alpha=1}^{m} \sum_{i=1}^{n+1} \partial_i\, \Xi_i^\alpha(x) + \sum_{\alpha=1}^{m} H^\alpha(x), \quad \text{for a.e. } x \in \Omega.$$

For $H \in L^{2_*}_{\mathrm{loc}}(\Omega; \mathbb{C}^m)$ and $\boldsymbol{\Xi} \in L^2_{\mathrm{loc}}(\Omega; \mathbb{C}^{m(n+1)})$ we say that the vector field $w \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C}^m)$ solves $Lw = H - \operatorname{div} \boldsymbol{\Xi}$ in the *weak sense*, or that w is a *weak solution* to the equation $Lw = H - \operatorname{div} \boldsymbol{\Xi}$, if for any $\Phi \in C_c^\infty(\Omega; \mathbb{C}^m)$ we have that

$$\sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n+1} \int_\Omega a_{ij}^{\alpha\beta}(x) \partial_j w^\beta\, \overline{\partial_i \Phi^\alpha} = \sum_{\alpha=1}^{m} \sum_{i=1}^{n+1} \int_\Omega \Xi_i^\alpha\, \overline{\partial_i \Phi^\alpha} + \sum_{\alpha=1}^{m} \int_\Omega H^\alpha\, \overline{\Phi^\alpha}. \tag{1.32}$$

We say that the *variational Poisson-Dirichlet problem* for L is solvable in Ω if for every $H \in L^{2_*}(\Omega;\mathbb{C}^m)$ and $\boldsymbol{\Xi} \in L^2(\Omega;\mathbb{C}^{m(n+1)})$ there exists $u \in W^{1,2}_{\text{loc}}(\Omega;\mathbb{C}^m)$ such that

$$(\mathrm{PD}^{\mathrm{L}}_{\mathrm{v}}) = \begin{cases} Lu = -\operatorname{div} A\nabla u = -\operatorname{div} \boldsymbol{\Xi} + H & \text{weakly in } \Omega, \\ u \in Y^{1,2}_0(\Omega). \end{cases} \tag{1.33}$$

By Lax-Milgram's theorem this problem is always solvable, its solution is unique and it satisfies the estimate

$$\|u\|_{Y^{1,2}_0(\Omega)} \lesssim \|H\|_{L^{2_*}(\Omega;\mathbb{C}^m)} + \|\boldsymbol{\Xi}\|_{L^2(\Omega;\mathbb{C}^{m(n+1)})}.$$

We say that the *variational Dirichlet problem* for L is solvable in Ω if for every $\varphi \in \operatorname{Lip}(\partial\Omega;\mathbb{C}^m)$ and $\Phi \in \dot W^{1,2}(\Omega;\mathbb{C}^m) \cap \operatorname{Lip}(\overline{\Omega};\mathbb{C}^m)$ satisfying $\Phi|_{\partial\Omega} = \varphi$, there exists $w \in \dot W^{1,2}(\Omega;\mathbb{C}^m)$ such that

$$(\mathrm{D}^{\mathrm{L}}_{\mathrm{v}}) = \begin{cases} Lw = -\operatorname{div} A\nabla w = 0 & \text{weakly in } \Omega, \\ w - \Phi \in Y^{1,2}_0(\Omega) & \text{on } \partial\Omega. \end{cases} \tag{1.34}$$

If u is the solution of (1.33) for $\boldsymbol{\Xi} = -A\nabla\Phi \in L^2(\Omega;\mathbb{C}^{m(n+1)}))$ and $H = 0$ then it is easy to see that $w = u + \Phi$ is the solution of (1.34).

We can consider the extended boundary $\partial\Omega_\infty := \partial\Omega \cup \{\infty\}$. Since the set of compactly supported Lipschitz functions on $\partial\Omega$ is dense in the set of compactly supported continuous functions on $\partial\Omega$, we can extend the definition of the Dirichlet problem to $C_c(\partial\Omega)$. Namely, for any $\varphi \in C_c(\partial\Omega;\mathbb{C}^m)$ and $\Phi \in \dot W^{1,2}(\Omega;\mathbb{C}^m) \cap C(\overline{\Omega};\mathbb{C}^m)$ satisfying $\Phi|_{\partial\Omega} = \varphi$, there exists $w \in \dot W^{1,2}(\Omega;\mathbb{C}^m)$ satisfying (1.34).

Definition 1.6. For $q \in (0,\infty)$ we say that the *Dirichlet problem* with L^q boundary data is solvable for L in Ω and write (D^L_q) is solvable in Ω, if there exists $C \geq 1$ so that for each $g \in \operatorname{Lip}_c(\partial\Omega)$, the solution u of (1.34) for L with boundary data g satisfies the estimate

$$\|\widetilde{\mathcal{N}}_{2^*}(u)\|_{L^q(\sigma)} \leq C\|g\|_{L^q(\sigma)}, \tag{1.35}$$

where $2^* := \frac{2(n-1)}{n+1}$. We also say that the *Dirichlet problem* with boundary data in $\mathrm{BMO}(\sigma)$ is solvable for L in Ω and write that $\mathrm{D}^L_{\mathrm{BMO}}$ is solvable in Ω, if there exists $C \geq 1$ so that for each $g \in \operatorname{Lip}_c(\partial\Omega)$, the solution u of (1.34) for L with boundary data g satisfies the estimate

$$\|\delta_\Omega \nabla u\|_{T^\infty_2(\Omega)} \leq C\|g\|_{\mathrm{BMO}(\sigma)}. \tag{1.36}$$

Definition 1.7. For $p \in (0,\infty)$ we say that the (homogeneous) *Dirichlet regularity problem* or just *regularity problem* with boundary data in $\dot M^{1,p}(\sigma)$ is solvable for L in Ω (write (R^L_p) is solvable in Ω), if there exists $C \geq 1$ so that for each $f \in \operatorname{Lip}_c(\partial\Omega)$, the solution u of (1.34) with boundary data f satisfies the estimate

$$\|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\sigma)} \leq C\|f\|_{\dot M^{1,p}(\sigma)}. \tag{1.37}$$

Following [MPT22], we introduce the Poisson-regularity problem with data in $\mathrm{C}_{q,p}(\Omega)$.

Definition 1.8. For $p \in (1,\infty)$ we say that the *Poisson-Dirichlet problem* (PD_p^L) is solvable in Ω if there exists $C>0$ so that for each $H \in L_c^\infty(\Omega;\mathbb{C}^m)$ and $\Xi \in L_c^\infty(\Omega;\mathbb{C}^{m(n+1)})$, the solution v of the problem (1.33) satisfies the estimate

$$\|\widetilde{\mathcal{N}}_{2^*}(u)\|_{L^p(\sigma)} \leq C\left(\|\mathcal{C}_{2_*}(\delta_\Omega H)\|_{L^p(\sigma)} + \|\mathcal{C}_2(\Xi)\|_{L^p(\sigma)}\right). \tag{1.38}$$

Similarly, we say that the *Poisson-Dirichlet problem* (PD_∞^L) for $H=0$ is solvable in Ω if there exists $C>0$ so that for each $\Xi \in L_c^\infty(\Omega;\mathbb{C}^{m(n+1)})$, the solution u of the problem (1.33) for $H=0$ satisfies the estimate

$$\|\delta_\Omega \nabla u\|_{T_2^\infty(\Omega)} \leq C\,\|\mathcal{C}_2(\Xi)\|_{L^\infty(\partial\Omega)}. \tag{1.39}$$

Definition 1.9. For any $p \in (1,\infty)$, we say that the *Poisson-regularity problem* (PR_p^L) is solvable in Ω if there exists $C>0$ so that for each $H \in L_c^\infty(\Omega;\mathbb{C}^m)$ and $\Xi \in L_c^\infty(\Omega;\mathbb{C}^{m(n+1)})$, the solution v of the problem (1.33) satisfies the estimate

$$\|\widetilde{\mathcal{N}}_2(\nabla v)\|_{L^p(\sigma)} \leq C\left(\|\mathcal{C}_{2_*}(H)\|_{L^p(\sigma)} + \|\mathcal{C}_2(|\Xi|/\delta_\Omega)\|_{L^p(\sigma)}\right). \tag{1.40}$$

Similarly, we say that the *Poisson-regularity problem* (PR_1^L) for $H=0$ is solvable in Ω if there exists $C>0$ so that for each $\Xi \in L_c^\infty(\Omega;\mathbb{C}^{m(n+1)})$, the solution v of the problem (1.33) for $H=0$ satisfies the estimate

$$\|\widetilde{\mathcal{N}}_2(\nabla v)\|_{L^1(\sigma)} \leq C\,\|\Xi\|_{T_2^1(\Omega)}. \tag{1.41}$$

1.5 Geometry of domains

Following Jerison and Kenig, [JK82], we introduce the corkscrew and Harnack chain conditions.

Definition 1.10. Let $c \in (0,1/2)$. We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *c-corkscrew condition* if for every ball $B(\xi,r)$ with $\xi \in \partial\Omega$ and $0<r<\operatorname{diam}(\Omega)$, there exists a point $x \in \Omega \cap B(\xi,r)$ such that $B(x,cr) \subset \Omega \cap B(\xi,r)$.

Definition 1.11. Given two points $x,x' \in \Omega$ and a pair of numbers $M,N \geq 1$, an (M,N)-*Harnack Chain connecting x to x'*, is a chain of open balls $B_1,\dots,B_N \subset \Omega$ with $x \in B_1$, $x' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ for every $k \in \{1,\cdots,N\}$ and $M^{-1}\operatorname{diam}(B_k) \leq \operatorname{dist}(B_k,\partial\Omega) \leq M\operatorname{diam}(B_k)$. We say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant M such that for any two points $x,x' \in \Omega$, there is an (M,N)-Harnack Chain connecting them, with N depending only on M and the ratio $|x-x'|/\left(\min\left(\delta_\Omega(x),\delta_\Omega(x')\right)\right)$.

It is not hard to see that if $E \subset \mathbb{R}^{n+1}$ is s-Ahlfors regular for $s \in (0,n]$ then $\mathbb{R}^{n+1}\setminus E$ satisfies the c-corkscrew condition for some $c \in (0,1/2)$ depending only on the Ahlfors regularity constants. In the case that $s<n$, the set $\mathbb{R}^{n+1}\setminus E$ satisfies the Harnack chain condition as well; see [DFM, Lemma 2.2].

Definition 1.12. Let $\lambda \in (0,1]$. A connected rectifiable curve $\gamma : [0,\ell] \to \overline{\Omega}$ connecting $\xi \in \partial\Omega$ and $x \in \Omega$, parametrized by the arc-length $s \in [0,\ell]$ and such that $\gamma(0)=\xi$ and $\gamma(\ell)=x$ is called a *λ-good curve* or a *λ-carrot path*, if $\gamma \setminus \{\xi\} \subset \Omega$ and $\delta_\Omega(\gamma(s)) > \lambda\, s$ for every $s \in (0,\ell]$.

Definition 1.13. An open set $\Omega \subset \mathbb{R}^{n+1}$ is said to satisfy the *pointwise John condition* if there exists a constant $\theta \in (0,1)$ such that for σ_n-a.e. $\xi \in \partial\Omega$, there exist $x_\xi \in \Omega$ and $r_\xi > 0$ satisfying $x_\xi \in B(\xi, 2r_\xi)$ and $\delta_\Omega(x_\xi) \geq \theta r_\xi$, and also there exists a θ-good curve $\gamma_\xi \subset \Omega \cap B(\xi, 2r_\xi)$ connecting the points ξ and x_ξ with $\ell(\gamma_\xi) \leq \theta^{-1} r_\xi$. We will write that $\xi \in \mathrm{JC}(\theta)$ if the pointwise John condition holds for the point ξ with constant $\theta \in (0,1)$.

Remark 1.14. Any domain $\Omega \in \mathrm{AR}(n)$ with n-rectifable boundary satisfies the pointwise John condition.

Following [HMT] we also introduce the notion of local John domains, which are also examples of domains satisfying the pointwise John condition.

Definition 1.15. An open set $\Omega \subset \mathbb{R}^{n+1}$ is said to satisfy the *local John condition* if there is $\theta \in (0,1)$ such that the following holds: For all $x \in \partial\Omega$ and $r \in (0, 2\,\mathrm{diam}(\Omega))$ there is $y \in B(x,r) \cap \Omega$ such that $B(y, \theta r) \subset \Omega$ with the property that for all $z \in B(x,r) \cap \partial\Omega$ one can find a rectifiable path $\gamma_z : [0,1] \to \overline{\Omega}$ with length at most $\theta^{-1}|x-y|$ such that

$$\gamma_z(0) = z, \qquad \gamma_z(1) = y, \qquad \mathrm{dist}(\gamma_z(t), \partial\Omega) \geq \theta\, |\gamma_z(t) - z| \quad \text{for all } t \in [0,1].$$

If $\Omega \in \mathrm{AR}(s)$ for $0 < s < n$ then it clearly satisfies the local John condition as it satisfies the corkscrew and the Harnack chain conditions. If $s = n$, any semi-uniform and thus any uniform domain has the local John condition.

Definition 1.16. Let Ω be a corkscrew domain, $F : \Omega \to \mathbb{R}$ and $f : \partial\Omega \to \mathbb{R}$. We say that F *converges non-tangentially* to f at $\xi \in \partial\Omega$ and write $F \to f$ n.t. at ξ, if there exists $\alpha > 0$ such that for every sequence $x_k \in \gamma_\alpha(\xi)$ for which $x_k \to \xi$ as $k \to \infty$, it holds that $F(x_k) \to f(\xi)$ as $k \to \infty$. We will also write

$$\operatorname*{nt-lim}_{x \to \xi} F(x) = f(\xi).$$

We will say that $F \to f$ *quasi-non-tangentially* at $\xi \in \partial\Omega$ and write $F \to f$ q.n.t. at ξ if there exist $r_\xi > 0$, a corkscrew point $x_\xi \in \Omega \cap B(\xi, 2r_\xi)$, and a θ-good curve $\gamma_\xi \in B(\xi, 2r_\xi)$ connecting ξ and x_ξ, such that for any $x_k \in \gamma_\xi$ converging to ξ as $k \to \infty$, it holds that $\lim_{k\to\infty} F(x_k) = f(\xi)$. We will also write

$$\operatorname*{qnt-lim}_{x \to \xi} F(x) = f(\xi).$$

1.6 Dyadic lattices

Given an s-Ahlfors-regular measure μ in $\mathbb{R}^{n+1}$ we consider the dyadic lattice of "cubes" built by David and Semmes in [DS2, Chapter 3 of Part I]. The properties satisfied by $\mathcal{D}_\mu$ are the following. Assume first, for simplicity, that $\mathrm{diam}(\mathrm{supp}\,\mu) = \infty$. Then for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_{\mu,j}$ of Borel subsets of $\mathrm{supp}\,\mu$, the dyadic cubes of the j-th generation, such that:

- (a) each $\mathcal{D}_{\mu,j}$ is a partition of $\mathrm{supp}\,\mu$, i.e. $\mathrm{supp}\,\mu = \bigcup_{Q \in \mathcal{D}_{\mu,j}} Q$ and $Q \cap Q' = \varnothing$ whenever $Q, Q' \in \mathcal{D}_{\mu,j}$ and $Q \neq Q'$;
- (b) if $Q \in \mathcal{D}_{\mu,j}$ and $Q' \in \mathcal{D}_{\mu,k}$ with $k \leq j$, then either $Q \subset Q'$ or $Q \cap Q' = \varnothing$;

(*c*) for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_{\mu,j}$, we have $2^{-j} \lesssim \operatorname{diam}(Q) \leq 2^{-j}$ and $\mu(Q) \approx 2^{-js}$;

(*d*) there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_{\mu,j}$, and $0 < \tau < 1$,

$$\begin{aligned} &\mu\big(\{x \in Q : \operatorname{dist}(x, \operatorname{supp}\mu \setminus Q) \leq \tau 2^{-j}\}\big) \\ &\qquad + \mu\big(\{x \in \operatorname{supp}\mu \setminus Q :\, \operatorname{dist}(x, Q) \leq \tau 2^{-j}\}\big) \leq C\tau^{1/C} 2^{-js}. \end{aligned} \tag{1.42}$$

This property is usually called the *small boundaries condition*. From (1.42) it follows that there is a point $x_Q \in Q$, the center of Q, such that $\operatorname{dist}(x_Q, \operatorname{supp}\mu \setminus Q) \gtrsim 2^{-j}$; see [DS2, Lemma 3.5 of Part I].

We set

$$\mathcal{D}_\mu := \bigcup_{j\in\mathbb{Z}} \mathcal{D}_{\mu,j}.$$

In case that $\operatorname{diam}(\operatorname{supp}\mu) < \infty$, the families $\mathcal{D}_{\mu,j}$ are only defined for $j \geq j_0$, with $2^{-j_0} \approx \operatorname{diam}(\operatorname{supp}\mu)$, and the same properties above hold for $\mathcal{D}_\mu := \bigcup_{j\geq j_0} \mathcal{D}_{\mu,j}$.

Given a cube $Q \in \mathcal{D}_{\mu,j}$ we say that its *side-length* is 2^{-j} and we denote it by $\ell(Q)$. Notice that $\operatorname{diam}(Q) \leq \ell(Q)$. We also denote

$$B(Q) := B(x_Q, c_1\ell(Q)), \qquad B_Q := B(x_Q, \ell(Q)), \tag{1.43}$$

where $c_1 > 0$ is some fixed constant so that $B(Q) \cap \operatorname{supp}\mu \subset Q$ for all $Q \in \mathcal{D}_\mu$. Clearly we have $Q \subset B_Q$. For $\lambda > 1$ we write

$$\lambda Q = \big\{x \in \operatorname{supp}\mu :\, \operatorname{dist}(x, Q) \leq (\lambda - 1)\,\ell(Q)\big\}.$$

The side-length of a *true cube* $P \subset \mathbb{R}^{n+1}$ is also denoted by $\ell(P)$. On the other hand, given a ball $B \subset \mathbb{R}^{n+1}$, its radius is denoted by $r(B)$. For $\lambda > 0$ the ball λB is the ball concentric to B with radius $\lambda\, r(B)$.

1.7 The Whitney decomposition

Recall that a domain is a connected open set. In this thesis, Ω will be an open set in $\mathbb{R}^{n+1}$ with $n \geq 1$. We will denote the n-Hausdorff measure on $\partial\Omega$ by σ.

We consider the following Whitney decomposition of Ω assuming $\Omega \neq \mathbb{R}^{n+1}$: we have a family $\mathcal{W}(\Omega)$ of dyadic cubes in $\mathbb{R}^{n+1}$ with disjoint interiors such that

$$\bigcup_{P\in\mathcal{W}(\Omega)} P = \Omega$$

and, moreover, there exist constants $\Lambda > 20$ and $D_0 \geq 1$ such the following conditions hold for every $P \in \mathcal{W}(\Omega)$:

(i) $10P \subset \Omega$;

(ii) $\Lambda P \cap \partial\Omega \neq \varnothing$;

(iii) there are at most D_0 cubes $P' \in \mathcal{W}(\Omega)$ such that $10P \cap 10P' \neq \varnothing$. Furthermore, for such cubes P' we have $\frac{1}{2}\ell(P') \leq \ell(P) \leq 2\ell(P')$.

From the properties (i) and (ii) it is clear that $\text{dist}(P, \partial\Omega) \approx \ell(P)$ and so there exists $\Lambda' > 20$ such that

$$\text{dist}(x, \partial\Omega) \leq \Lambda' \ell(P) \quad \text{for every } x \in P. \tag{1.44}$$

We assume that the Whitney cubes are small enough so that

$$\text{diam}(P) < \frac{1}{20} \text{dist}(P, \partial\Omega). \tag{1.45}$$

The arguments used to construct a Whitney decomposition satisfying the properties above are standard.

Suppose that $\partial\Omega$ is s-Ahlfors-regular and consider the dyadic lattice $\mathcal{D}_\sigma$ defined above. Then, for each Whitney $P \in \mathcal{W}(\Omega)$ there is some cube $Q \in \mathcal{D}_\sigma$ such that

$$\ell(Q) = \ell(P) \quad \text{and} \quad \text{dist}(P, Q) \approx \ell(Q), \tag{1.46}$$

with the implicit constant depending on the parameters of $\mathcal{D}_\sigma$ and on the Whitney decomposition. We denote such a cube by $Q = b(P)$ and we say that Q is a *boundary cube* of P. For every $P \in \mathcal{W}(\Omega)$ there is a uniformly bounded number of cubes $Q \in \mathcal{D}_\sigma$ depending on n and the s-Ahlfors regularity of $\partial\Omega$, that satisfy the properties (1.46). Conversely, given $Q \in \mathcal{D}_\sigma$, we let

$$w(Q) = \bigcup_{P \in \mathcal{W}(\Omega): Q = b(P)} P. \tag{1.47}$$

In the case of n-Ahlfors regular boundary, it is immediate to check that $w(Q)$ is made up at most of a uniformly bounded number of cubes P but it may happen that $w(Q) = \varnothing$.

In higher co-dimensions where $s < n$ it is also true that for every boundary cube $Q \in \partial\Omega$ there exists a uniformly bounded number of Whitney cubes $P \in \mathcal{W}(\Omega)$ such that $b(P) = Q$. For the proof of this fact one can see [MP, Lemma 4.16, Lemma 4.18].

We also denote the fattened Whitney region of Q by

$$\widetilde{w}(Q) = \bigcup_{P \in \mathcal{W}(\Omega): Q = b(P)} 1.1P. \tag{1.48}$$

Remark 1.17. If $x \in \bar{P} \in \mathcal{W}(\Omega)$ then there exists a constant $C_w > 1$ depending only on n, the constants of the Whitney decomposition and the s-Alhlfors regularity, so that for every $P' \in \mathcal{W}(\Omega)$ that has the property $x \in 1.1P'$, there holds

$$b(P') \subset B(x_{b(P)}, C_w \ell(P)) =: B_P.$$

Chapter 2

Regularized dyadic extension of functions on the boundary

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set and let $\mathcal{W}(\Omega)$ be the collection of Whitney cubes in which Ω is decomposed as in Subsection 1.7. Let $\{\varphi_P\}_{P\in\mathcal{W}(\Omega)}$ be a partition of unity subordinate to the open cover $\{1.1P\}_{P\in\mathcal{W}(\Omega)}$ such that

$$\varphi_P \in C^\infty(\mathbb{R}^n), \qquad |\nabla \varphi_P| \lesssim \frac{1}{\ell(P)}, \qquad \operatorname{supp}\varphi \subseteq 1.1P$$

and

$$\sum_{P\in\mathcal{W}(\Omega)} \varphi_P(x) = \mathbf{1}_\Omega(x), \qquad x \in \Omega.$$

For $f \in L^1_{\mathrm{loc}}(\sigma)$, we define the *regularized dyadic extension of* f in Ω by

$$\upsilon_f(x) := \begin{cases} \displaystyle\sum_{P\in\mathcal{W}(\Omega)} m_{\sigma,b(P)} f\, \varphi_P(x) & \text{if } x \in \Omega, \\ f(x) & \text{if } x \in \partial\Omega; \end{cases} \tag{2.1}$$

in the case that Ω is an unbounded domain with compact boundary we set $b(P) = \partial\Omega$ for every $P \in \mathcal{W}(\Omega)$ with $\ell(P) \geq \operatorname{diam}(\partial\Omega)$.

The fact that υ_f is indeed an extension of f in Ω in the non-tangential sense is proved in the following lemma.

Lemma 2.1. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$ and $f \in L^1_{\mathrm{loc}}(\sigma_s)$. There exists $\alpha > 0$ such that*

$$\operatorname*{nt-lim}_{x\to\xi} \upsilon_f(x) = f(\xi) \quad \text{for } \sigma\text{-a.e. } \xi \in \partial\Omega,$$

for some cone γ_α where $\alpha > 0$ only depends on n, the Ahlfors regularity constant and the constants of the corkscrew condition.

Proof. By [EG15, Theorem 1.33] it holds that

$$\lim_{r\to 0} m_{\sigma,B(\xi,r)}(|f - f(\xi)|) = 0, \quad \text{for } \sigma\text{-a.e. } \xi \in \partial\Omega. \tag{2.2}$$

Fix $\varepsilon > 0$ and let $\xi \in \partial\Omega$ be a point such that (2.2) holds. Let $0 < \varepsilon' < \varepsilon$ to be chosen later. Then there exists $\delta = \delta(\varepsilon', \xi) > 0$ such that $m_{\sigma,B(\xi,r)}(|f - f(\xi)|) < \varepsilon'$ for every $r < \delta$. Let now

$0 < \delta' < \delta$ be a small constant which will be chosen momentarily. For fixed $x \in \gamma_\alpha(\xi) \cap B(\xi, \delta')$, we have that

$$|\upsilon_f(x) - f(\xi)| \leq \sum_{P \in \mathcal{W}(\Omega)} |m_{\sigma, b(P)} f - f(\xi)| \, \varphi_P(x).$$

Let $P_0 \in \mathcal{W}(\Omega)$ be a fixed cube such that $x \in \bar{P}_0$. Then the only cubes $P \in \mathcal{W}(\Omega)$ that contribute to the sum are the ones for which $x \in 1.1P$ and, by Remark 1.17, $b(P) \subset B_{P_0} = B(x_{P_0}, C_w \ell(P_0))$. By the properties of the Whitney cubes there exists a constant $C_w' > C_w$ such that $B_P \subset B(\xi, C_w' \ell(P))$. Let $\delta' > 0$ be sufficiently small so that $B(\xi, C_w' \ell(P)) \subset B(\xi, \delta/2)$. We get that for any such P we have

$$|m_{\sigma, b(P)} f - f(\xi)| \lesssim m_{\sigma, B_{P_0}}(|f - f(\xi)|) \lesssim \varepsilon'$$

which, in turn, by the bounded overlap of the Whitney cubes yields that there exists a constant $C > 1$ such that

$$|\upsilon_f(x) - f(\xi)| < C \, \varepsilon'.$$

This concluses the proof of the lemma once we choose $\varepsilon = C\varepsilon'$. □

Lemma 2.2. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$. Assume that $f \in L^1_{\mathrm{loc}}(\sigma)$ and υ_f is the extension defined in (2.1). For any $P \in \mathcal{W}(\Omega)$ we have that*

$$\sup_{x \in \bar{P}} |\nabla \upsilon_f(x)| \lesssim \ell(P)^{-1} \, m_{\sigma, B_P}(|f|). \tag{2.3}$$

If additionally $f \in \Lambda_\beta(\partial\Omega)$ for $\beta \in [0, 1)$ then it holds that

$$\sup_{x \in \bar{P}} |\nabla \upsilon_f(x)| \lesssim \ell(P)^{\beta - 1} \, \|f\|_{\Lambda_\beta(\partial\Omega)}, \tag{2.4}$$

while if $f \in \dot{M}^{1,p}(\sigma)$ we get that

$$|\nabla \upsilon_f(x)| \lesssim m_{\sigma, B_P}(\nabla_{H,p} f); \tag{2.5}$$

above $\nabla_{H,p} f$ is the least Hajłasz upper gradient of f. Moreover, for any $\xi \in \partial\Omega$ we have

$$\mathcal{N}_\alpha(\upsilon_f)(\xi) \lesssim_\alpha \mathcal{M} f(\xi). \tag{2.6}$$

Proof. For fixed $x \in \bar{P} \in \mathcal{W}(\Omega)$ we have

$$|\nabla \upsilon_f(x)| \lesssim \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} m_{\sigma, b(P')}(|f|) \, \ell(P')^{-1} \lesssim_{D_0} m_{\sigma, B_P}(|f|) \, \ell(P)^{-1},$$

where we used that $\ell(P) \approx \ell(P')$ and that there are at most D_0 Whitney cubes with such property. If, in addition, $f \in \Lambda_\beta(\partial\Omega)$ then using the fact that $\nabla\big(\sum_{P \in \mathcal{W}(\Omega)} \varphi_P(x)\big) = 0$ we get

$$\begin{aligned}
|\nabla \upsilon_f(x)| &= \Big| \sum_{P' \in \mathcal{W}(\Omega)} (m_{\sigma, b(P')} f - m_{\sigma, b(P)} f) \nabla \varphi_{P'}(x) \Big| \\
&\lesssim \ell(P)^{-1} \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} |m_{\sigma, b(P')} f - m_{\sigma, b(P)} f| \\
&\lesssim \ell(P)^{-1} \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} m_{\sigma, b(P')}(|f - m_{\sigma, B_P} f|) \lesssim_{D_0} \|f\|_{\Lambda_\beta(\partial\Omega)} \ell(P)^{\beta - 1}.
\end{aligned}$$

Now, let $f \in \dot{M}^{1,p}(\sigma)$ and $x \in \bar{P}$. Then we have

$$
\begin{aligned}
|\nabla \upsilon_f(x)| &= \Big| \sum_{P' \in \mathcal{W}(\Omega)} m_{\sigma,b(P')} f \, \nabla \varphi_{P'}(x) \Big| \lesssim \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} \frac{1}{\ell(P')} |m_{\sigma,b(P')} f - m_{\sigma,b(P)} f| \\
&\lesssim \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} \frac{1}{\ell(P')} \fint_{b(P')} \fint_{b(P)} |x-y| [\nabla_{H,p} f(x) + \nabla_{H,p} f(y)] \, d\sigma(x) \, d\sigma(y) \\
&\lesssim \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} (m_{\sigma,b(P')}(\nabla_{H,p} f) + m_{\sigma,b(P)}(\nabla_{H,p} f)) \lesssim_{D_0} m_{\sigma,B_P}(\nabla_{H,p} f),
\end{aligned}
$$

where B_P was defined in Remark 1.17. Finally, fox fixed $\xi \in \partial\Omega$ and for every $x \in \gamma_\alpha(\xi)$, we have

$$
|\upsilon_f(x)| \leq \sum_{\substack{P' \in \mathcal{W}(\Omega) \\ x \in 1.1P'}} m_{\sigma,b(P)}(|f|) \, \varphi_P(x) \lesssim m_{\sigma,B_P}(|f|) \lesssim m_{\sigma,B(\xi,C'_w \ell(P))}(|f|) \leq \mathcal{M} f(\xi)
$$

for some constant $C'_w > C_w$ depending on α and the Whitney constants. This readily proves (2.6) by taking supremum over all $x \in \gamma_\alpha(\xi)$. ☐

Lemma 2.3. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$. If $f \in \Lambda_\beta(\partial\Omega)$ for $\beta \in [0,1)$, then it holds that*

$$
\sup_{\xi \in \partial\Omega} \mathcal{N}_\sharp^{(\beta)}(\upsilon_f)(\xi) \lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}. \tag{2.7}
$$

Proof. Fix $\xi \in \partial\Omega$ and take $x \in \gamma_\alpha(\xi)$ with $x \in \bar{P}_0$ for some $P_0 \in \mathcal{W}(\Omega)$. It is enough to bound the quantity

$$
\sup_{y \in B(x,c\delta_\Omega(x))} \fint_{B(x,c\delta_\Omega(x))} |\upsilon_f(y) - \upsilon_f(z)| \, dz.
$$

To this end, fix a point $z \in B(x, c\delta_\Omega(x))$ with $z \in \bar{P}_1 \in \mathcal{W}(\Omega)$ and a point $y \in B(x, c\delta_\Omega(x))$ with $y \in \bar{P}_2 \in \mathcal{W}(\Omega)$. Since $c > 0$ is small enough, the Whitney cubes P_1, P_2 and P_0 are close to each other in the sense that the intersection of a dilation of these cubes is non-empty. Then, by the properties of Whitney cubes, we get that $\ell(P_1) \approx \ell(P_2) \approx \ell(P_0) \approx \delta_\Omega(x)$. Thus, there exists a large enough constant $\Lambda_0 > 1$ such that for any $P \in \mathcal{W}(\Omega)$ with $1.1P \cap (P_1 \cup P_2) \neq \emptyset$, we have that

$$
B_{b(P)} \subset B_0 := B(x_{b(P_0)}, \Lambda_0 \ell(P_0)).
$$

It holds that

$$
\sum_{\substack{P \in \mathcal{W}(\Omega) \\ P \cap B(x,c\delta_\Omega(x)) \neq \emptyset}} (\varphi_P(z) - \varphi_P(y)) \, m_{\sigma,B_0} f = 0,
$$

and so

$$
\begin{aligned}
|\upsilon_f(y)-\upsilon_f(z)| &\leq \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(x,c\delta_\Omega(x))\neq\emptyset}} |\varphi_P(y)-\varphi_P(z)||m_{\sigma,b(P)}f-m_{\sigma,B_0}f|\\
&\lesssim \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(x,c\delta_\Omega(x))\neq\emptyset}} \|\nabla\varphi_P\|_{L^\infty}|y-z||m_{\sigma,b(P)}f-m_{\sigma,B_0}f|\\
&\lesssim \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(x,c\delta_\Omega(x))\neq\emptyset}} m_{\sigma,b(P)}(|f-m_{\sigma,B_0}f|)\lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}\ell(P_0)^\beta,
\end{aligned}
$$

since there are uniformly bounded many Whitney cubes P such that $P\cap B(x,c\delta_\Omega(x))\neq\emptyset$. This readily implies (2.7). □

Lemma 2.4. *Let $\Omega\in\mathrm{AR}(s)$ for $s\in(0,n]$. If $f\in\Lambda_\beta(\partial\Omega)$ for $\beta\in(0,1)$ then*

$$
\sup_{\xi\in\partial\Omega}\mathcal{C}_s^{(\beta)}(\nabla\upsilon_f)(\xi)\lesssim\|f\|_{\Lambda_\beta(\partial\Omega)}. \tag{2.8}
$$

Proof. By (2.4) it is easy to see that for every $\xi\in\partial\Omega$ and $r>0$, it holds that

$$
\begin{aligned}
\int_{B(\xi,r)\cap\Omega}|\nabla\upsilon_f|\,\omega_s(x)\,dx &\lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}\sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset}}\ell(P)^{\beta-1}\ell(P)^{s+1}\\
&\lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}\sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset}}\ell(b(P))^\beta\sigma(b(P))\\
&\lesssim \|f\|_{\Lambda_\beta(\partial\Omega)}\sum_{k=0}^{\infty}2^{-\beta k}r^\beta\sum_{\substack{Q\in\mathcal{D}_{k,\sigma}\\ Q\subset B(\xi,M_0r)}}\sigma(Q)\lesssim\|f\|_{\Lambda_\beta(\partial\Omega)}\,r^\beta\,r^s,
\end{aligned}
$$

which implies (2.8). □

If the boundary function is in $\mathrm{Lip}_\beta(\partial\Omega)$ for $\beta\in(0,1]$ then we can show that υ_f is $\mathrm{Lip}_\beta(\overline{\Omega})$ by arguments similar to the ones in [MT22, Lemma 4.2].

Lemma 2.5. *Let $\Omega\in\mathrm{AR}(s)$ for $s\in(0,n]$. If $f\in\mathrm{Lip}_\beta(\partial\Omega)$ for $\beta\in(0,1]$ and υ_f is the regularized dyadic extension defined in (2.1), then it holds that $\upsilon_f\in\mathrm{Lip}_\beta(\overline{\Omega})$ with $\mathrm{Lip}_\beta(\upsilon_f)\lesssim\mathrm{Lip}_\beta(f)$.*

Proof. We start by proving that $\upsilon_f\in\mathrm{Lip}_\beta(\Omega)$. Fix $x,y\in\Omega$ and let $P_1,P_2\in\mathcal{W}(\Omega)$ such that $x\in\bar{P}_1$ and $y\in\bar{P}_2$. We split into cases.

Case 1: Suppose that $2P_1\cap 2P_2\neq\emptyset$. In this case let $P_0\in\mathcal{W}(\Omega)$ be the smallest cube such that

$$
2B_{b(P)}\subset 2B_{b(P_0)},\ \text{for every } P\in\mathcal{W}(\Omega)\text{ with } 1,1P\cap(P_1\cup P_2)\neq\emptyset.
$$

By the properties of Whitney cubes then we get that $\ell(P_1) \approx \ell(P_2) \approx \ell(P_0)$. As

$$\upsilon_f(x) - \upsilon_f(y) = \sum_{P \in \mathcal{W}(\Omega)} m_{\sigma,b(P)} f \left(\varphi_P(x) - \varphi_P(y)\right)$$

and

$$\sum_{P \in \mathcal{W}(\Omega)} \left(\varphi_P(x) - \varphi_P(y)\right) m_{\sigma,b(P_0)} f = 0,$$

we can write

$$\begin{aligned} &\upsilon_f(x) - \upsilon_f(y) \\ &\quad = \sum_{P \in \mathcal{W}(\Omega)} m_{\sigma,b(P)} f \left(\varphi_P(x) - \varphi_P(y)\right) - \sum_{P \in \mathcal{W}(\Omega)} m_{\sigma,b(P_0)} f \left(\varphi_P(x) - \varphi_P(y)\right) \\ &\quad = \sum_{P \in \mathcal{W}(\Omega)} \left(\varphi_P(x) - \varphi_P(y)\right) \left(m_{\sigma,b(P)} f - m_{\sigma,b(P_0)} f\right). \end{aligned}$$

Thus we get

$$|\upsilon_f(x) - \upsilon_f(y)| \leq \sum_{P \in \mathcal{W}(\Omega)} |\varphi_P(x) - \varphi_P(y)| \left|m_{\sigma,b(P)} f - m_{\sigma,b(P_0)} f\right|. \tag{2.9}$$

Observe now that

$$|\varphi_P(x) - \varphi_P(y)| \leq |\nabla \varphi_P||x-y| \lesssim \ell(P)^{-1}|x-y|$$

while for fixed $w \in 2B_{b(P_0)}$ we can estimate

$$\begin{aligned} \left|m_{\sigma,b(P)} f - m_{\sigma,b(P_0)} f\right| &\leq m_{\sigma,b(P)}(|f(z) - f(w)|) + m_{\sigma,b(P_0)}(|f(z) - f(w)|) \\ &\lesssim \mathrm{Lip}_\beta(f)\, \ell(P_0)^\beta. \end{aligned}$$

To deal with the sum in the right hand side of the inequality (2.9), we may assume that the cubes P appearing in the sum are such that either $1.1P \cap P_1 \neq \varnothing$ or $1.1P \cap P_2 \neq \varnothing$, since otherwise the associated summand vanishes. We denote by I_0 the family of such cubes. So the cubes from I_0 are such that $B_{b(P)} \subset 2B_{b(P_0)}$ and they satisfy $\ell(P) \approx \ell(P_0)$. Combining this observation with the last two estimates and the fact that $|x-y| \lesssim \ell(P_0)$, we obtain

$$|\upsilon_f(x) - \upsilon_f(y)| \lesssim \sum_{P \in \mathcal{W}(\Omega)} \frac{1}{\ell(P)} |x-y| \mathrm{Lip}_\beta(f)\, \ell(P_0)^\beta \lesssim_{D_0} \mathrm{Lip}_\beta(f)\, |x-y|^\beta. \tag{2.10}$$

Case 2: Suppose that $2P_1 \cap 2P_2 = \emptyset$. In this case we have

$$\begin{aligned} \upsilon_f(x) - \upsilon_f(y) &= \sum_{P \in \mathcal{W}(\Omega)} m_{\sigma,b(P)} f \left(\varphi_P(x) - \varphi_P(y)\right) \\ &= \sum_{P \in \mathcal{W}(\Omega)} \varphi_P(x)(m_{\sigma,b(P)} f - m_{\sigma,b(P_1)} f) + \sum_{P \in \mathcal{W}(\Omega)} \varphi_P(y) \left(m_{\sigma,b(P_2)} f - m_{\sigma,b(P)} f\right) \\ &\qquad + \left(m_{\sigma,b(P_1)} f - m_{\sigma,b(P_2)} f\right) =: S_1 + S_2 + S_3. \end{aligned}$$

If we use that $\ell(P_i) \lesssim |x-y|$ and the fact that $|m_{\sigma,b(P)}f - m_{\sigma,b(P_i)}f| \lesssim \mathrm{Lip}_\beta(f)\,\ell(P_i)^\beta$ for $i \in \{1,2\}$, we can show that

$$|S_1| + |S_2| \lesssim \mathrm{Lip}_\beta(f)\,|x-y|^\beta.$$

It remains to bound S_3. If $w_1 \in b(P_1)$ and $w_2 \in b(P_2)$ then

$$|S_3| \leq \left|m_{\sigma,b(P_1)}f - f(w_1)\right| + |f(w_1) - f(w_2)| + \left|f(w_2) - m_{\sigma,b(P_2)}f\right|.$$

Since $f \in \mathrm{Lip}_\beta(\partial\Omega)$,

$$\begin{aligned}|f(w_1) - f(w_2)| &\leq \mathrm{Lip}_\beta(f)\,|w_1 - w_2|^\beta \leq \mathrm{Lip}_\beta(f)\left(|w_1 - x|^\beta + |x-y|^\beta + |y-w_2|^\beta\right)\\ &\lesssim \mathrm{Lip}_\beta(f)\left(\ell(P_1)^\beta + |x-y|^\beta + \ell(P_2)^\beta\right) \lesssim \mathrm{Lip}_\beta(f)\,|x-y|^\beta,\end{aligned}$$

while, for $i \in \{1,2\}$, once again using that $f \in \mathrm{Lip}_\beta(\partial\Omega)$, it is easy to see that

$$\left|m_{\sigma,b(P_i)}(f - f(w_i))\right| \leq m_{\sigma,b(P_i)}(|f - f(w_i)|) \leq \mathrm{Lip}_\beta(f)\,\ell(P_i)^\beta \lesssim \mathrm{Lip}_\beta(f)\,|x-y|^\beta.$$

This implies that $|S_3| \lesssim \mathrm{Lip}_\beta(f)|x-y|^\beta$. Combining the above estimates we get

$$|\upsilon_f(x) - \upsilon_f(y)| \leq |S_1| + |S_2| + |S_3| \lesssim \mathrm{Lip}_\beta(f)\,|x-y|^\beta,$$

in the second case as well and thus for all $x, y \in \Omega$ with $x \neq y$. This readily implies that $\upsilon_f \in \mathrm{Lip}_\beta(\Omega)$ with $\mathrm{Lip}_\beta(\upsilon_f) \leq \mathrm{Lip}_\beta(f)$.

It remains to prove that

$$|\upsilon_f(x) - \upsilon_f(y)| \lesssim \mathrm{Lip}_\beta(f)\,|x-y|^\beta \quad \text{for any} \quad x \in \partial\Omega \text{ and } y \in \Omega. \tag{2.11}$$

To this end, we fix such x and y and estimate

$$\begin{aligned}|\upsilon_f(x) - \upsilon_f(y)| = |f(x) - \upsilon_f(y)| &= \Big|f(x) - \sum_{P \in \mathcal{W}(\Omega)} m_{\sigma,b(P)}f\,\varphi_P(y)\Big|\\ &\leq \sum_{P \in \mathcal{W}(\Omega)} \varphi_P(y)|f(x) - m_{\sigma,b(P)}f| \leq \sum_{\substack{P \in \mathcal{W}(\Omega):\\ 1.1P \ni y}} \varphi_P(y)|f(x) - m_{\sigma,b(P)}f|.\end{aligned}$$

As $f \in \mathrm{Lip}_\beta(\partial\Omega)$ for every $P \in \mathcal{W}(\Omega)$ such that $y \in 1.1P$, we have that

$$|f(x) - m_{\sigma,b(P)}f| \lesssim \mathrm{Lip}_\beta(f)\,\ell(P)^\beta \approx \mathrm{Lip}_\beta(f)\,\delta_\Omega(y)^\beta \leq \mathrm{Lip}_\beta(f)\,|x-y|^\beta,$$

which implies (2.11) by the bounded overlap of the Whitney cubes that contain y, thus concluding the proof of the lemma. □

proof of Theorem 0.4. It follows by combining Lemmas 2.1, 2.2, 2.3, 2.4, and 2.5; see also Remark 1.2. □

Chapter 3

A Corona decomposition for functions in L^p or BMO

In this chapter we will assume that $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$.

We say that a family of cubes $\mathcal{F} \subset \mathcal{D}_\sigma$ satisfies a *Carleson packing condition* with constant $M > 0$, and we write $\mathcal{F} \in \mathrm{Car}(M)$, if for any $S \in \mathcal{D}_\sigma$ it holds that

$$\sum_{R \in \mathcal{F}\colon R \subset S} \sigma(R) \leq M\, \sigma(S). \tag{3.1}$$

A family $\mathcal{T} \subset \mathcal{D}_\sigma$ is a *tree* if it verifies the following properties:

1. $\mathcal{T}$ has a (unique) maximal with respect to inclusion element $Q(\mathcal{T})$ which contains all the other elements of $\mathcal{T}$ as subsets of $\mathbb{R}^{n+1}$. The cube $Q(\mathcal{T})$ is the "root" of $\mathcal{T}$ and we will call it top cube.

2. If Q, Q' belong to $\mathcal{T}$ and $Q \subset Q'$ then any cube $P \in \mathcal{D}_\sigma$ such that $Q \subset P \subset Q'$ also belongs to $\mathcal{T}$.

Definition 3.1. A *corona decomposition* of σ is a partition of $\mathcal{D}_\sigma$ into a family of *good cubes*, which we denote by $\mathcal{G}$, and a family of *bad cubes*, which we denote by $\mathcal{B}$, so that the following hold:

1. $\mathcal{D}_\sigma = \mathcal{G} \cup \mathcal{B}$;

2. there is a partition of $\mathcal{G}$ into trees, that is,

$$\mathcal{G} = \bigcup_{\mathcal{T} \subset \mathcal{G}} \mathcal{T};$$

3. the collections of the maximal cubes $Q(\mathcal{T})$ of the trees $\mathcal{T}$ satisfies (3.1) for some $M_0 > 0$;

4. the collection of cubes $\mathcal{B}$ satisfies (3.1) for some $M_1 > 0$.

We can also define a *localized Corona decomposition* in a cube $R_0 \in \mathcal{D}_\sigma$ if in the definition above we replace $\mathcal{D}_\sigma$ by $\mathcal{D}_\sigma(R_0)$.

We recall the definition of the truncated, at large scales, dyadic Hardy-Littlewood maximal function

$$Mf(Q) = \sup_{\substack{R\in\mathcal{D}_\sigma \\ Q\subset R}} m_{\sigma,R}|f|, \qquad f \in L^1_{\mathrm{loc}}(\sigma).$$

Given any $R \in \mathcal{D}_\sigma$ and a fixed $\varepsilon > 0$, we define the collection $\mathsf{Stop}(R) \subset \mathcal{D}_\sigma(R)$ consisting of cubes $S \in \mathcal{D}_\sigma(R)$ which are maximal, thus disjoint, with respect to the condition

$$|m_{\sigma,R}f - m_{\sigma,S}f| \geq \begin{cases} \varepsilon Mf(S) & \text{if } f \in L^1_{\mathrm{loc}}(\sigma) \\ \varepsilon \|f\|_{\mathrm{BMO}(\sigma)} & \text{if } f \in \mathrm{BMO}(\sigma). \end{cases} \tag{3.2}$$

We fix a cube $R_0 \in \mathcal{D}_\sigma$ and we define the family of the top cubes with respect to R_0 as follows: first we define the families $\mathsf{Top}_k(R_0)$ for $k \geq 0$ inductively. We set

$$\mathsf{Top}_0(R_0) := \{R_0\}.$$

Assuming that $\mathsf{Top}_k(R_0)$ has been defined, we set

$$\mathsf{Top}_{k+1}(R_0) = \bigcup_{R\in\mathsf{Top}_k(R_0)} \mathsf{Stop}(R),$$

and then we define

$$\mathsf{Top}(R_0) := \bigcup_{k\geq 0} \mathsf{Top}_k(R_0).$$

For $R \in \mathsf{Top}(R_0)$ we also set

$$\mathrm{Tree}(R) := \{Q \in \mathcal{D}_\sigma(R) : \nexists\, S \in \mathsf{Stop}(R) \text{ such that } Q \subset S\}.$$

Notice that

$$\mathcal{D}_\sigma(R_0) = \bigcup_{R\in\mathsf{Top}(R_0)} \mathrm{Tree}(R),$$

and this union is disjoint. This is a localized Corona decomposition in R_0 and notice that in this case we have that $\mathcal{B} = \emptyset$.

For the rest of this chapter we will devote all our efforts to proving that $\mathsf{Top}(R_0)$ satisfies a Carleson packing condition.

Proposition 3.2. *For any $R_0 \in \mathcal{D}_\sigma$, the family of cubes $\mathsf{Top}(R_0) \in \mathrm{Car}(C\varepsilon^{-2})$ for some $C > 0$ depending on the Ahlfors-regularity constants.*

To prove the proposition we first need some auxiliary lemmas.

Lemma 3.3. *Let $f \in L^1_{\mathrm{loc}}(\sigma)$ and $Q \in \mathcal{D}_\sigma$. Then it holds*

$$\frac{\sigma(Q)}{Mf(Q)^2} \leq 8 \int_Q \frac{1}{(\mathcal{M}_{\mathcal{D}_\sigma} f(x))^2}\, d\sigma(x) \tag{3.3}$$

Proof. This was proved in [HR18, Lemma 4.1] for the Lebesgue measure but the same proof works for any non-atomic Radon measure and so we skip the details. □

Recall that a nonnegative function $w \in L^1_{\text{loc}}(\sigma)$ is called a weight. For any measurable $E \subset \partial\Omega$ we let $w(E) := \int_E w \, d\sigma$. If for every cube $Q \in \mathcal{D}_\sigma$ there exists a constant c, independent of a cube Q, such that

$$m_{\sigma,Q} w \, d\sigma \leq c \operatorname*{ess\,inf}_{x \in Q} w(x),$$

we will call w an A_1 weight. For $p \in (1, \infty)$, if for every cube $Q \in \mathcal{D}_\sigma$ there exists a constant c, independent of a cube Q, such that

$$m_{\sigma,Q} w \, (m_{\sigma,Q}(w^{1-p'}))^{p-1} \leq c$$

with $1/p + 1/p' = 1$, w will be called an A_p weight. We say that $w \in A_\infty(\sigma)$ if there exists $\theta > 0$ and a positive constant $C_0 < \infty$ such that for every $Q \in \mathcal{D}_\sigma$ and every σ-measurable $E \subset Q$, it holds that

$$\frac{w(E)}{w(Q)} \leq C_0 \Big(\frac{\sigma(E)}{\sigma(Q)}\Big)^\theta.$$

Let $\mathcal{F} \subset \mathcal{D}_\sigma$ be any collection of dyadic cubes. Given any cube $Q \in \mathcal{D}_\sigma$, define its stopping parent Q^* to be the minimal $Q^* \in \mathcal{F}$ such that $Q \subsetneq Q^*$. If no such Q^* exists we set $Q^* := Q$. Define the stopped square function

$$\mathcal{S}_\mathcal{F} f(x) := \Big(\sum_{Q \in \mathcal{F}} |m_{\sigma,Q} f - m_{\sigma,Q^*} f|^2 \mathbf{1}_Q(x) \Big)^{1/2}, \qquad x \in \partial\Omega. \tag{3.4}$$

In the special case $\mathcal{F} = \mathsf{Top}(R_0)$ we will simply write $\mathcal{S}f$.

Lemma 3.4. *If $w \in A_\infty(\sigma)$ and $1 \leq p < \infty$, then*

$$\|\mathcal{S}_\mathcal{F} f\|_{L^p(\partial\Omega;w)} \lesssim \|M_\mathcal{D} f\|_{L^p(\partial\Omega;w)}$$

uniformly for any collection of dyadic cubes $\mathcal{F}$.

Proof. This was proved in [HR18, Proposition 3.2] for the Lebesgue measure but the same proof works verbatim for σ. □

We will now proceed to the proof of Proposition 3.2 for $f \in L^1_{\text{loc}}(\sigma)$ which is based on the one of [HR18, Theorem 1.2(3)].

Proof of Proposition 3.2 when $f \in L^1_{\text{loc}}(\sigma)$. We first fix $S \in \mathsf{Top}(R_0)$. As for any $R \in \mathsf{Top}(R_0)$ it holds that

$$|m_{\sigma,R} f - m_{\sigma,R^*} f| > \varepsilon M f(R),$$

we have that

$$\sum_{\substack{R \in \mathsf{Top}(R_0) \\ R \subset S}} \sigma(R) \leq \sum_{\substack{R \in \mathsf{Top}(R_0) \\ R \subset S}} \frac{|m_{\sigma,R} f - m_{\sigma,R^*} f|^2}{\varepsilon^2 M f(R)^2} \sigma(R)$$

$$\lesssim \sum_{\substack{R \in \mathsf{Top}(R_0) \\ R \subset S}} \frac{|m_{\sigma,R} f - m_{\sigma,R^*} f|^2}{\varepsilon^2} \int_S \frac{\mathbf{1}_R(x)}{\mathcal{M}_{\mathcal{D}_\sigma} f(x)^2} \, d\sigma = \int_S \frac{\mathcal{S} f(x)^2}{\varepsilon^2} \frac{d\sigma(x)}{\mathcal{M}_{\mathcal{D}_\sigma} f(x)^2},$$

where, in the second inequality, we used Lemma 3.3. We write

$$(\mathcal{M}_{\mathcal{D}_\sigma} f)^{-2} = 1 \cdot \left((\mathcal{M}_{\mathcal{D}_\sigma} f)^{2\gamma}\right)^{1-q}$$

for $\gamma \in (0,1/2)$ and $q = 1 + \frac{1}{\gamma} > 3$. Since $f \in L^1_{\text{loc}}(\sigma)$ and $2\gamma \in (0,1)$, using for example [CG, Theorem 3.4, p.158], whose proof do works for doubling Borel measures, we get that $(\mathcal{M}_{\mathcal{D}_\sigma} f)^{2\gamma} \in A_1(\sigma)$. As $1 \in A_1$ and $q > 1$ it follows from [CG, Theorem 2.16, p.407], whose proof also works for doubling Borel measures, that $1 \cdot \left((\mathcal{M}_{\mathcal{D}} f)^{2\gamma}\right)^{1-q} \in A_q(\sigma)$. Therefore, $(\mathcal{M}_{\mathcal{D}_\sigma} f)^{-2} \in A_q(\sigma) \subset A_\infty(\sigma)$. We now apply Lemma 3.4 with the collection of cubes $\widetilde{\mathcal{F}} := \{R \in \mathsf{Top}(R_0) : R \subset S\}$ to the function

$$\tilde{f}(x) := \begin{cases} f(x) - m_{\sigma,S} f & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}$$

for the weight $w := (\mathcal{M}_{\mathcal{D}_\sigma} f)^{-2}$ and $p = 2$ and obtain

$$\int |S_{\widetilde{\mathcal{F}}} \tilde{f}|^2 \, w \, d\sigma \lesssim \int |\mathcal{M}_{\mathcal{D}} \tilde{f}|^2 \, w \, d\sigma = \int_S |\mathcal{M}_{\mathcal{D}} \tilde{f}|^2 \, w \, d\sigma \lesssim \int_S |\mathcal{M}_{\mathcal{D}} f|^2 \, w \, d\sigma.$$

Thus, since $|S_{\widetilde{\mathcal{F}}} \tilde{f}(x)|^2 = |S_{\widetilde{\mathcal{F}}} f(x)|^2$ for all $x \in S$, we infer that

$$\begin{aligned} \int_S |\mathcal{S} f|^2 w \, d\sigma &\lesssim \int_S |\mathcal{S}_{\widetilde{\mathcal{F}}} f|^2 \, w \, d\sigma + \int_S |\mathcal{M}_{\mathcal{D}} f|^2 \, w \, d\sigma \leq \int |\mathcal{S}_{\widetilde{\mathcal{F}}} \tilde{f}|^2 \, w \, d\sigma + \int_S |\mathcal{M}_{\mathcal{D}} f|^2 \, w \, d\sigma \\ &\lesssim \int_S |\mathcal{M}_{\mathcal{D}} f|^2 \, w \, d\sigma = \int_S |\mathcal{M}_{\mathcal{D}} f|^2 \frac{d\sigma}{(\mathcal{M}_{\mathcal{D}} f)^2} = \sigma(S), \end{aligned}$$

proving (3.1) in the case that $S \in \mathsf{Top}(R_0)$.

If $S \in \mathcal{D}_\sigma(R_0) \setminus \mathsf{Top}(R_0)$ then we can find a maximal collection $\mathcal{F}_0$ of cubes $\widetilde{S} \in \mathsf{Top}(R_0)$ such that

$$S = \bigcup_{\widetilde{S} \in \mathcal{F}_0} \widetilde{S}.$$

Then,

$$\sum_{\substack{R \in \mathsf{Top}(R_0) \\ R \subset S}} \sigma(R) = \sum_{\widetilde{S} \in \mathcal{F}_0} \sum_{\substack{R \in \mathsf{Top}(\widetilde{S}): \\ R \subset \widetilde{S}}} \sigma(R) \lesssim \sum_{\widetilde{S} \in \mathcal{F}_0} \sigma(\widetilde{S}) = \sigma(S)$$

and the proof is complete. □

Proof of Proposition 3.2 when $f \in \mathrm{BMO}(\sigma)$. For any $R \in \mathsf{Top}(R_0)$ there holds

$$|m_{\sigma,S} f - m_{\sigma,R} f| > \varepsilon \|f\|_{\mathrm{BMO}(\sigma)}.$$

Define

$$f_R(x) := \sum_{Q \in \mathrm{Tree}(R)} \Delta_Q f(x) := \sum_{Q \in \mathrm{Tree}(R)} \sum_{Q' \in \mathrm{ch}(Q)} \left(m_{\sigma,Q'} f - m_{\sigma,Q} f\right) 1_{Q'}(x),$$

where with ch(Q) we denote the dyadic children of the cube Q. If $x \in P \in \mathsf{Stop}(R)$ we have that $f_R(x) = m_{\sigma,P} f - m_{\sigma,R} f$ and so $|f_R(x)| > \varepsilon \|f\|_{\mathrm{BMO}(\sigma)}$. This implies that

$$\begin{aligned}\varepsilon^2 \|f\|^2_{\mathrm{BMO}(\sigma)} \sum_{P \in \mathsf{Stop}(R)} \sigma(P) &\leq \sum_{P \in \mathsf{Stop}(R)} \int_P |f_R(x)|^2 \, d\sigma(x) \\ &= \int_{\bigcup_{P \in \mathsf{Stop}(R)} P} |f_R|^2 \, d\sigma \leq \int |f_R|^2 \, d\sigma.\end{aligned}$$

By the above estimate and the orthogonality of $\Delta_Q f$ we get

$$\begin{aligned}\sum_{\substack{R \in \mathsf{Top}(R_0) \\ R \subset S}} \sum_{P \in \mathsf{Stop}(R)} \sigma(P) &\leq \frac{1}{\varepsilon^2} \frac{1}{\|f\|^2_{\mathrm{BMO}(\sigma)}} \sum_{R \in \mathsf{Top}(R_0)} \|f_R\|^2_{L^2(\sigma)} \\ &= \frac{1}{\varepsilon^2} \frac{1}{\|f\|^2_{\mathrm{BMO}(\sigma)}} \sum_{R \in \mathsf{Top}(R_0)} \sum_{Q \in \mathrm{Tree}(R)} \|\Delta_Q f\|^2_{L^2(\sigma)} \\ &\leq \frac{1}{\varepsilon^2} \frac{1}{\|f\|^2_{\mathrm{BMO}(\sigma)}} \sum_{Q \in \mathcal{D}_\sigma(R_0)} \|\Delta_Q f\|^2_{L^2(\sigma)} \\ &= \frac{1}{\varepsilon^2} \frac{1}{\|f\|^2_{\mathrm{BMO}(\sigma)}} \|\mathbf{1}_{R_0}(f - m_{\sigma,R_0} f)\|^2_{L^2(\sigma)} \lesssim \varepsilon^{-2} \sigma(R_0).\end{aligned}$$

This proves (3.1) in the case that $S \in \mathsf{Top}(R_0)$. By the same argument as in the end of the proof of Proposition 3.2 when $f \in L^1_{\mathrm{loc}}(\sigma)$ we obtain (3.1) for any $S \in \mathcal{D}_\sigma(R_0)$. □

Remark 3.5. For the constructions in this chapter note that if $\operatorname{supp} \sigma$ is bounded, we can choose $R_0 = \operatorname{supp} \sigma$. In the case that $\operatorname{supp} \sigma$ is not bounded we apply a technique described in p. 38 of [DS1]: we consider a family of cubes $\{R_j\}_{j \in J} \subset \mathcal{D}_\sigma$ which are pairwise disjoint, whose union is all of $\operatorname{supp} \sigma$ and which have the property that for each k there at most C cubes from $\mathcal{D}_{\sigma,k}$ which are not contained in any cube R_j. For each R_j we construct a family $\mathsf{Top}(R_j)$ analogous to $\mathsf{Top}(R_0)$. Then we set

$$\mathsf{Top} := \bigcup_{j \in J} \mathsf{Top}(R_j)$$

and

$$\mathcal{B} := \{S \subset \mathcal{D}_\sigma : \text{there does not exist } j \in J \, \text{such that} \, S \subset R_j \in \mathsf{Top}\}.$$

One can easily check that the families Top and $\mathcal{B}$ satisfy a Carleson packing condition. See [DS1, p. 38] for the construction of the family $\{R_j\}$ and for additional details.

Chapter 4

L^p and uniform ε-approximability of the regularized dyadic extension

In this chapter we give the proof of Theorems 0.1 and 0.2;
Given $A > 1$, we say that two cubes Q_1, Q_2 are A-*close* if

$$\frac{1}{A} \operatorname{diam} Q_1 \leq \operatorname{diam} Q_2 \leq A \operatorname{diam} Q_1$$

and

$$\operatorname{dist}(Q_1, Q_2) \leq A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2).$$

The following lemma was proved in [DS2, p. 60].

Lemma 4.1. *If we have a Corona decomposition such that* $\mathsf{Top} \in \operatorname{Car}(M_0)$ *for some* $M_0 > 0$ *then the collection of cubes*

$$\begin{aligned}\mathcal{A}_0 := \big\{ Q \in \mathcal{D}_\sigma : Q \in \operatorname{Tree}(R) \text{ for some } R \in \mathsf{Top} \text{ and} \\ \exists\, Q' \in \operatorname{Tree}(R') \text{ for some } R' \in \mathsf{Top} \text{ with } R' \neq R \text{ such that } Q' \text{ is } A\text{-close to } Q \big\}\end{aligned}$$

is in $\operatorname{Car}(M_1)$ *for some* $M_1 > 0$ *depending on* M*,* A*, and the Ahlfors-regularity constants.*

Lemma 4.2 ([DS2], Lemma I.3.27, p. 59)**.** *If* $\mathcal{F} \subset \mathcal{D}_\sigma$ *is in* $\operatorname{Car}(M_1)$ *for some* $M_1 > 0$ *then the family*

$$\mathcal{F}_A := \{ Q \in \mathcal{D}_\sigma : Q \text{ is } A\text{-close to some } Q' \in \mathcal{F} \}$$

is in $\operatorname{Car}(M_2)$ *for some* $M_2 > 0$ *depending on* M_1*,* A *and the Ahlfors-regularity constants.*

Recall that $\Omega \in \operatorname{AR}(s)$ for $s \in (0, n]$ and the corona decomposition that we constructed in Chapter 3. Consider the subcollection of Whitney cubes

$$\begin{aligned}\mathcal{P}_0 := \big\{ P \in \mathcal{W}(\Omega) :\ \text{there exists } P' \in \mathcal{W}(\Omega) \text{ such that } 1.2P \cap 1.2P' \neq \emptyset \text{ and there exist} \\ R, R' \in \mathsf{Top}(R_0) \text{ with } R \neq R' \text{ such that } b(P) \in \operatorname{Tree}(R) \text{ and } b(P') \in \operatorname{Tree}(R') \big\}.\end{aligned}$$

Then, by the properties of Whitney cubes, for every $P \in \mathcal{P}_0$, the cubes $P' \in \mathcal{W}(\Omega)$ such that $b(P')$ is not in the same tree as $b(P)$ have the following properties:

- $\ell(b(P))/2 \leq \ell(b(P')) \leq 2\ell(b(P))$
- $\operatorname{dist}(b(P), b(P')) \leq C_1 \ell(b(P))$.

If for fixed $R \in \mathsf{Top}$ we define

$$\partial \text{Tree}(R) := \{Q \in \text{Tree}(R) : \text{there exists}\, P \in \mathcal{P}_0 \,\text{such that}\, b(P) = Q\}$$

then there exists $A > 1$ sufficiently large depending only on C_1 and n, such that

$$\bigcup_{R \in \mathsf{Top}} \partial \text{Tree}(R) \subset \mathcal{A}_0,$$

and, by Lemma 4.1, $\bigcup_{R \in \mathsf{Top}} \partial \text{Tree}(R) \in \text{Car}(M_1)$. If $\mathcal{F}$ is a family of "true" dyadic cubes in $\mathbb{R}^{n+1}$ we also define

$$\mathcal{N}(\mathcal{F}) := \big\{P \in \mathcal{W}(\Omega) : \text{there exists}\, P' \in \mathcal{F} \,\text{such that}\, 1.2P \cap 1.2P' \neq \emptyset\big\}.$$

Then, for $\mathcal{F} = \mathcal{P}_0$, we set

$$\partial \text{Tree}^*(R) := \big\{Q \in \mathcal{D}_\sigma : \exists\, P \in \mathcal{N}(\mathcal{P}_0) \text{ such that } Q = b(P) \in \text{Tree}(R)\big\}.$$

It is easy to see that

$$\mathcal{J} := \bigcup_{R \in \mathsf{Top}} \partial \text{Tree}^*(R) \subset \Big(\bigcup_{R \in \mathsf{Top}} \partial \text{Tree}(R)\Big)_A$$

and, by Lemma 4.2, $\mathcal{J} \in \text{Car}(M_2)$ for some $M_2 > 0$. Finally, we define

$$\mathcal{B}_0 := \{P \in \mathcal{W}(\Omega) : b(P) \in \mathcal{B}\}.$$

We are now ready to define the *approximating function* of v_f by

$$\begin{aligned} u(x) := &\sum_{S \in \mathcal{B}_0} m_{\sigma, b(S)} f \, \varphi_S(x) \\ &+ \sum_{R \in \mathsf{Top}} \Big[\sum_{\substack{P \in \mathcal{W}(\Omega) \setminus \mathcal{P}_0 \\ b(P) \in \text{Tree}(R)}} m_{\sigma, R} f \, \varphi_P(x) + \sum_{\substack{P \in \mathcal{P}_0 \\ b(P) \in \text{Tree}(R)}} m_{\sigma, b(P)} f \, \varphi_P(x) \Big], \end{aligned} \tag{4.1}$$

using the Corona decomposition constructed in Chapter 3. Note that when Ω is bounded, $\mathsf{Top} = \mathsf{Top}(\partial\Omega)$ and $\mathcal{B} = \emptyset$, while if $\partial\Omega$ is unbounded then Top and $\mathcal{B}$ are the families constructed in Remark 3.5. Finally, when Ω is an unbounded domain with compact boundary $\partial\Omega$, we modify the definition of the approximating function as follows.

$$\begin{aligned} u(x) := &\sum_{P \in \mathcal{W}(\Omega):\, \ell(P) \geq \text{diam}(\partial\Omega)} m_{\sigma, \partial\Omega} f \, \varphi_P(x) \\ &+ \sum_{R \in \mathsf{Top}} \Big[\sum_{\substack{P \in \mathcal{W}(\Omega) \setminus \mathcal{P}_0 \\ b(P) \in \text{Tree}(R)}} m_{\sigma, R} f \, \varphi_P(x) + \sum_{\substack{P \in \mathcal{P}_0 \\ b(P) \in \text{Tree}(R)}} m_{\sigma, b(P)} f \, \varphi_P(x) \Big]. \end{aligned} \tag{4.2}$$

We will now prove Theorems 0.1 and 0.2.

Theorem 4.3. *Let $f \in L^1_{\text{loc}}(\sigma)$ and $\varepsilon > 0$. There exists $\alpha_0 \geq 1$ such that for any $\alpha \geq \alpha_0$ and any $\xi \in \partial\Omega$ we have*

$$\mathcal{N}_\alpha(u - \upsilon_f)(\xi) \lesssim \varepsilon \mathcal{M} f(\xi) \tag{4.3}$$

and

$$\mathcal{C}_s(\nabla u)(\xi) \lesssim_\varepsilon \mathcal{M}(\widetilde{\mathcal{M}}(f))(\xi) + \mathcal{M}(\widetilde{\mathcal{M}}(\mathcal{M} f))(\xi). \tag{4.4}$$

Furthermore, for any $\xi \in \partial\Omega$ it holds that

$$\mathcal{N}_\alpha(\delta_\Omega \nabla u)(\xi) \lesssim \mathcal{M}(f)(\xi) + \mathcal{M}(\mathcal{M}(f))(\xi). \tag{4.5}$$

Here the ε-approximability constant c_ε is a positive and it is depending on ε and α_0 depends only on n, the Ahlfors regularity, the corkscrew, and the Whitney constants.

Let $f \in \text{BMO}(\sigma)$ and $\varepsilon > 0$. Then, for any $x \in \Omega$ it holds that

$$|u(x) - \upsilon_f(x)| \leq \varepsilon \|f\|_{\text{BMO}(\sigma)}, \tag{4.6}$$

$$\delta_\Omega(x)|\nabla u(x)| \lesssim \|f\|_{\text{BMO}(\sigma)}, \tag{4.7}$$

and for any $\xi \in \partial\Omega$

$$\mathcal{C}_s(\nabla u)(\xi) \lesssim_\varepsilon \|f\|_{\text{BMO}(\sigma)}. \tag{4.8}$$

The implicit constants depend on the dimension, the Alhfors regularity, the corkscrew condition, and the Whitney constants.

Moreover, if $f \in \text{Lip}_c(\partial\Omega)$ then $u \in \text{Lip}_{\text{loc}}(\Omega)$ and for any $x \in \Omega$ we have

$$\delta_\Omega(x)|\nabla u(x)| \lesssim \text{Lip}(f)\,\text{diam}(\text{supp}\, f). \tag{4.9}$$

Proof. We will only deal with the case that both Ω and $\partial\Omega$ are unbounded as the other cases can be treated in a similar but easier way. Note first that if we choose α_0 large enough, depending on n, the constants of the corkscrew condition and the Whitney decomposition, the cone is always non-empty and for every $Q \in \mathcal{D}_\sigma$ such that $\xi \in Q$, there exists $P \in \mathcal{W}(\Omega)$ such that $b(P) = Q$ and $P \subset \gamma_{\alpha_0}(\xi)$.

For fixed $\xi \in \partial\Omega$ we let $x \in \gamma_\alpha(\xi)$ for $a \geq \alpha_0$. There exists $P_0 \in \mathcal{W}(\Omega)$ such that $x \in \bar{P}_0$ and we either have that $P_0 \in \mathcal{B}_0$ or that there is a unique $R_0 \in \mathsf{Top}$ such that $b(P_0) \in \text{Tree}(R_0)$. If either $P_0 \in \mathcal{P}_0$ and there does not exist any $P \in \mathcal{N}(\mathcal{P}_0) \setminus \mathcal{P}_0$ such that $x \in 1.1P$, or $P_0 \in \mathcal{B}_0$, it is easy to see that $u(x) - \upsilon_f(x) = 0$. If $P_0 \in \mathcal{P}_0$ and there exists some $\widetilde{P} \in \mathcal{N}(\mathcal{P}_0) \setminus \mathcal{P}_0$ such that $x \in 1.1\widetilde{P}$, then

$$u(x) - \upsilon_f(x) = \sum_{\substack{P \in \mathcal{N}(\mathcal{P}_0) \setminus \mathcal{P}_0 \\ b(P) \in \text{Tree}(R_0)}} (m_{\sigma,R_0} f - m_{\sigma,b(P)} f)\, \varphi_P(x).$$

The same is true if $P_0 \in \mathcal{N}(\mathcal{P}_0) \setminus \mathcal{P}_0$ and there is $P \in \mathcal{P}_0$ such that $x \in 1.1P$. In any other case we have that

$$u(x) - \upsilon_f(x) = \sum_{\substack{P \in \mathcal{W}(\Omega) \\ b(P) \in \text{Tree}(R_0)}} (m_{\sigma,R_0} f - m_{\sigma,b(P)} f)\, \varphi_P(x). \tag{4.10}$$

Therefore, since $b(P) \in \text{Tree}(R_0)$, we get by (3.2) that

$$|u(x) - \upsilon_f(x)| \leq \varepsilon \sum_{\substack{P \in \mathcal{W}(\Omega) \\ b(P) \in \text{Tree}(R_0)}} Mf(b(P))\, \varphi_P(x).$$

For any $P \in \mathcal{W}(\Omega)$ such that $x \in 1.1P \cap \gamma_\alpha(\xi)$, since $|x-\xi| \approx \delta_\Omega(x) \approx \ell(P)$, it holds that $P \subset B(\xi, M\ell(P))$. The same is true for any $S \in \mathcal{D}_\sigma$ such that $b(P) \subset S$, i.e., $S \subset B(\xi, M'\ell(S))$ for a possibly larger constant $M' > 0$ depending also on the Ahlfors regularity constants. Thus

$$|u(x) - v_f(x)| \lesssim \varepsilon \sup_{S \supset b(P)} m_{\sigma, B(\xi, M'\ell(S))}(|f|) \lesssim \varepsilon \sup_{r \gtrsim \delta_\Omega(x)} m_{\sigma, B(\xi, r)}(|f|), \tag{4.11}$$

which implies (4.3) by taking supremum over all $x \in \gamma_\alpha(\xi)$. By the same arguments and the fact that $\nabla \varphi_P(x) \lesssim \ell(P)^{-1} \approx \delta_\Omega(x)^{-1}$, we conclude that

$$\nabla(u - v_f)(x) \lesssim \varepsilon \mathcal{M}(f)(\xi) \delta_\Omega(x)^{-1},$$

which implies

$$\mathcal{N}_\alpha(\delta_\Omega \nabla(u - v_f))(\xi) \lesssim \varepsilon \mathcal{M}(f)(\xi). \tag{4.12}$$

In the case that $f \in \mathrm{BMO}(\sigma)$, in view of (4.10) and the estimate $\nabla \varphi_P(x) \lesssim \ell(P)^{-1} \approx \delta_\Omega(x)^{-1}$, we have that

$$|u(x) - v_f(x)| + \delta_\Omega |\nabla(u - v_f)(x)| \lesssim \varepsilon \|f\|_{\mathrm{BMO}}. \tag{4.13}$$

We now turn our attention to the proof of (4.4) and (4.8). Let $x \in \bar{P}_0 \in \mathcal{W}(\Omega)$. Then, once again, either there exists a unique $R_0 \in \mathsf{Top}$ such that $b(P_0) \in \mathrm{Tree}(R_0)$, or there exists $B_0 \in \mathcal{B}_0$ such that $b(P_0) = B_0$. For the sake of brevity, we set

$$B_x := c\, B^x = B(x, c\, \delta_\Omega(x))$$

for a small enough constant $c > 0$ to be chosen. Fix $y \in B_x$ and if $P \in \mathcal{W}(\Omega)$ is such that $y \in 1.1P$ then $x \in 1.2P$. Indeed, by (1.44) we always have that $\mathrm{dist}(x, 1.1P) \leq |x - y| \leq c\, \delta_\Omega(x) \leq c\, \Lambda'\, \ell(P_0)$. Thus if there exists $P \in \mathcal{W}(\Omega)$ such that $y \in 1.1P$ and $x \notin 1.2P$ then it also holds that $\mathrm{dist}(x, 1.1P) \geq 0.1\ell(P)$. Now, noting that $\frac{1}{2}\ell(P_0) \leq \ell(P) \leq 2\ell(P_0)$ we get that $\frac{1}{2}\ell(P_0) \leq c\, \Lambda'\, \ell(P_0)$ and if we choose $c = \frac{1}{4\Lambda'}$ we reach a contradiction.

It is easy to see that $\nabla u(y) = 0$ if there does not exist any cube $P \in \mathcal{P}_0$ or $P \in \mathcal{B}_0$ such that $y \in 1.1P$. Using that $\sum \nabla \varphi_P(y) = 0$, we get

$$\begin{aligned} \nabla u(y) = \Big(\sum_{P \in \mathcal{B}_0} + \sum_{\substack{P \in \mathcal{N}(\mathcal{P}_0) \\ b(P) \notin \mathrm{Tree}(R_0)}} \Big) (m_{\sigma, b(P)} f - m_{\sigma, b(P_0)} f) \nabla \varphi_P(y) \\ + \sum_{\substack{P \in \mathcal{P}_0 \\ b(P) \in \mathrm{Tree}(R_0)}} (m_{\sigma, b(P)} f - m_{\sigma, b(P_0)} f) \nabla \varphi_P(y) \\ + \sum_{\substack{P \in \mathcal{N}(\mathcal{P}_0) \setminus \mathcal{P}_0 \\ b(P) \in \mathrm{Tree}(R_0)}} (m_{\sigma, R_0} f - m_{\sigma, b(P)} f) \nabla \varphi_P(y). \end{aligned} \tag{4.14}$$

Therefore, by Remark 1.17, arguing as in the proof of (4.11) and using (4.14), the fact that $\ell(P_0) \approx \ell(P)$ for any $P \in \mathcal{W}(\Omega)$ such that $1.1P \ni y$, and the Carleson packing of the cubes in $\mathcal{B}_0$, for fixed $\xi \in \partial\Omega$ and $r > 0$, we can estimate

$$\begin{aligned}
&\int_{B(\xi,r)\cap\Omega} \sup_{y\in B_x} |\nabla u(y)| \frac{dy}{\delta_\Omega(y)^{n-s}} \\
&\lesssim \Big(\sum_{P\in\mathcal{B}_0} + \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\\ b(P)\notin\mathrm{Tree}(R)}} \Big) \int_{P\cap B(\xi,r)} m_{\sigma,B(x_{b(P)},C_w\ell(P))}(|f|) \frac{dx}{\ell(P)^{n+1-s}} \\
&+ 2\varepsilon \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{P}_0\\ b(P)\in\mathrm{Tree}(R)}} \int_{P\cap B(\xi,r)} \sup_{\rho\gtrsim\ell(P)} m_{\sigma,B(x_{b(P)},\rho)}(|f|) \frac{dx}{\ell(P)^{n+1-s}} \\
&+ \varepsilon \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\setminus\mathcal{P}_0\\ b(P)\in\mathrm{Tree}(R)}} \int_{P\cap B(\xi,r)} \sup_{\rho\gtrsim\ell(P)} m_{\sigma,B(x_{b(P)},\rho)}(|f|) \frac{dx}{\ell(P)^{n+1-s}} \\
&\lesssim \Big(\sum_{\substack{P\in\mathcal{B}_0\\ P\cap B(\xi,r)\neq\emptyset}} + \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\\ b(P)\notin\mathrm{Tree}(R)\\ P\cap B(\xi,r)\neq\emptyset}} \Big) \sigma(b(P))\, m_{\sigma,B(x_{b(P)},C_w\ell(P))}(|f|) \\
&+ \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\\ b(P)\in\mathrm{Tree}(R)\\ P\cap B(\xi,r)\neq\emptyset}} \sigma(b(P)) \inf_{\zeta\in B(x_{b(P)},M\ell(P))} \mathcal{M}(f)(\zeta) \\
&\leq \sum_{\substack{Q\in\mathcal{B}\cup\mathcal{J}\\ Q\subset B(\xi,C'r)}} \sigma(Q)\, m_{\sigma,B(x_Q,C_w\ell(Q))}(|f|) + \sum_{\substack{Q\in\mathcal{J}\\ Q\subset B(\xi,C'r)}} \sigma(Q)\, m_{\sigma,B(x_Q,M\ell(Q))}(\mathcal{M}f) \\
&\lesssim \int_{B(\xi,Cr)} \sup_{Q\ni z} m_{\sigma,B(x_Q,C_w\ell(Q))}(|f|)\, d\sigma(z) + \int_{B(\xi,Cr)} \sup_{Q\ni z} m_{\sigma,B(x_Q,M\ell(Q))}(\mathcal{M}f)\, d\sigma(z) \\
&\lesssim \int_{B(\xi,Cr)} \widetilde{\mathcal{M}}(f)\, d\sigma + \int_{B(\xi,Cr)} \widetilde{\mathcal{M}}(\mathcal{M}f)\, d\sigma.
\end{aligned}$$

Above $M > 1$ is a constant possible larger than C_w and where in the antepenultimate inequality we used that if $P \cap B(\xi,r) \neq \emptyset$, then $b(P) \subset B(\xi, C'r)$ for some large constant $C' > 0$ depending on the Ahlfors-regularity and the Whitney constants, while the penultimate inequality follows from Carleson's embedding theorem, see [Tol, Theorem 5.8, p. 144], since the families $\mathcal{J} = \cup_{R\in\mathsf{Top}}\partial^*\mathrm{Tree}(R)$ and $\mathcal{B}$ are Carleson families. This concludes the proof of (4.4).

If $f \in \mathrm{BMO}(\sigma)$, using (4.14) for $\xi \in \partial\Omega$ and $r > 0$, we get

$$\begin{aligned}
\int_{B(\xi,r)\cap\Omega} \sup_{y\in B_x} |\nabla u(y)| \frac{dy}{\delta_\Omega(y)^{n-s}} &\lesssim_\varepsilon \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\\ b(P')\in\mathrm{Tree}(R)}} \int_{P\cap B(\xi,r)} \|f\|_{\mathrm{BMO}(\sigma)}\, \omega_s(y)\, dy \\
&+ \Big(\sum_{P\in\mathcal{B}_0} + \sum_{R\in\mathsf{Top}} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\\ b(P)\notin\mathrm{Tree}(R)}} \Big) \int_{P\cap B(\xi,r)} \|f\|_{\mathrm{BMO}(\sigma)} \frac{\omega_s(y)}{\ell(P)}\, dy \\
&\lesssim \|f\|_{\mathrm{BMO}(\sigma)} \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\cup\mathcal{B}_0\\ P\cap B(\xi,r)\neq\emptyset}} \sigma(b(P)) \leq \|f\|_{\mathrm{BMO}(\sigma)} \sum_{\substack{Q\in\mathcal{J}\cup\mathcal{B}\\ Q\subset B(\xi,Mr)}} \sigma(Q) \\
&\lesssim \|f\|_{\mathrm{BMO}(\sigma)}\, r^s,
\end{aligned}$$

for $M > 1$ sufficiently large depending on the Ahlfors regularity and the Whitney constants. For the last inequality we used that the families of surface cubes $\mathcal{J}$ and $\mathcal{B}$ satisfy the Carleson packing condition from Lemma 4.2. The above estimate obviously implies (4.8).

We proceed now to the proof of the estimates (4.5) and (4.7). Let $\xi \in \partial\Omega$ and $x \in \gamma_\alpha(\xi)$. There exists $P_0 \in \mathcal{W}(\Omega)$ such that $x \in \bar{P}_0$. Then using (4.14) and the bounded overlap of the Whitney cubes we get that

$$\begin{aligned} |\nabla u(x)| &\lesssim \ell(P_0)^{-1}(m_{\sigma,B_{P_0}}(|f|) + \varepsilon \sup_{\rho \gtrsim \ell(P_0)} m_{\sigma,B(x_{b(P_0)},\rho)}(|f|)) \\ &\leq \delta_\Omega(x)^{-1}\big(m_{\sigma,B(\xi,C'\ell(P_0))}(|f|) + \inf_{\zeta\in B(\xi,C'\ell(P_0))} \mathcal{M}(f)(\zeta)\big) \\ &\lesssim \delta_\Omega(x)^{-1}\big(\mathcal{M}(f)(\xi) + \mathcal{M}(\mathcal{M}(f))(\xi)\big). \end{aligned} \tag{4.15}$$

By a similar but easier argument we can show that

$$|\nabla u(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}\delta_\Omega(x)^{-1}. \tag{4.16}$$

Since $\sup_{x\in\Omega} \delta_\Omega(x)|\nabla u(x)| = \sup_{\xi\in\partial\Omega}\sup_{x\in\gamma_\alpha(\xi)} \delta_\Omega(x)|\nabla u(x)|$, it easily follows that the estimates (4.15) and (4.16) imply the estimates (4.5) and (4.7) respectively.

It remains to prove (4.9) in the case that $f \in \mathrm{Lip}_c(\partial\Omega)$. Using (4.14) and the bounded overlap of the Whitney cubes we get that for any $x \in \Omega$

$$|\nabla u(x)| \lesssim \mathrm{Lip}(f)\,\mathrm{diam}(\mathrm{supp}\, f)\ell(P)^{-1} + \varepsilon \sum_{\substack{P\in\mathcal{N}(\mathcal{P}_0)\setminus\mathcal{P}_0 \\ b(P)\in\mathrm{Tree}(R_0) \\ x\in 1.1P}} Mf(b(P))\ell(P)^{-1}. \tag{4.17}$$

Since f has compact support for $\xi_0 \notin \mathrm{supp}\, f$ and every $Q \supset b(P)$ it holds

$$m_{\sigma,Q}(|f|) = m_{\sigma,Q}(|f - f(\xi_0)|) \lesssim \mathrm{diam}(\mathrm{supp}\, f)\,\mathrm{Lip}(f).$$

Taking supremum over all cubes $Q \supset b(P)$ and using again the bounded overlaps of the Whitney cubes together with (4.17) and the fact that $\delta_\Omega(x) \approx \ell(P)$ for all $P \in \mathcal{W}(\Omega)$ such that $x \in 1.1P$, we infer that

$$|\nabla u(x)| \lesssim \mathrm{Lip}(f)\,\mathrm{diam}(\mathrm{supp}\, f)\delta_\Omega(x)^{-1}.$$

This completes the proof of (4.9) and thus that of Theorem 4.3. □

As a corollary we get that if $f \in L^p(\sigma)$, $p \in (1,\infty)$, (resp. $f \in \mathrm{BMO}(\sigma)$), then v_f is ε-approximable in L^p (uniformly ε-approximable).

Theorem 4.4. *If $f \in L^p(\sigma)$ for some $p \in (1,\infty]$ then for any $\varepsilon > 0$ there exists $u = u_\varepsilon \in C^\infty(\Omega)$, $\alpha_0 \geq 1$ and a constant $c_\varepsilon > 1$ such that for any $\alpha \geq \alpha_0$ it holds that*

$$\|\mathcal{N}_\alpha(u - v_f)\|_{L^p(\sigma)} \lesssim \varepsilon\, \|f\|_{L^p(\sigma)}, \tag{4.18}$$

$$\|\mathcal{C}_s(\nabla u)\|_{L^p(\sigma)} \lesssim \varepsilon^{-2}\, \|f\|_{L^p(\sigma)}, \tag{4.19}$$

and

$$\|\mathcal{N}_\alpha(\delta_\Omega \nabla u)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}. \tag{4.20}$$

The implicit constants depend on s, n, p, and the constants of the Ahlfors regularity, the corkscrew condition and the Whitney decomposition.

Similarly, if $f \in \mathrm{BMO}(\sigma)$ then for any $\varepsilon > 0$ there exists $u = u_\varepsilon \in C^\infty(\Omega)$, $\alpha_1 \geq 1$ and a constant $c_\varepsilon > 1$ such that, for any $\alpha \geq \alpha_1$ there holds

$$\sup_{\xi \in \partial\Omega} \mathcal{N}_\alpha(u - \upsilon_f)(\xi) \lesssim \varepsilon \|f\|_{\mathrm{BMO}(\sigma)}, \tag{4.21}$$

$$\sup_{\xi \in \partial\Omega} \mathcal{C}_s(\nabla u)(\xi) \lesssim_\varepsilon \|f\|_{\mathrm{BMO}(\sigma)}, \tag{4.22}$$

and

$$\sup_{\xi \in \partial\Omega} \mathcal{N}_\alpha(\delta_\Omega \nabla u)(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma)}. \tag{4.23}$$

The implicit constants depend on s, n and the constants of the Ahlfors regularity, the corkscrew condition and the constants of the Whitney decomposition.

Proof. The proof is an immediate consequence of (4.3), (4.4) and (4.5) of Theorem 4.3, Lemma 1.3, and the fact that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are $L^p(\sigma) \to L^p(\sigma)$-bounded for any $p \in (1, \infty)$. In the case that $f \in \mathrm{BMO}(\sigma)$ the result follows immediatelly by the estimates (4.6), (4.8) and (4.7) of Theorem 4.3. □

Remark 4.5. Note that since $L^\infty(\sigma) \subset \mathrm{BMO}(\sigma)$, the estimates (4.18), (4.19) and (4.20) for $p = \infty$ follow from (4.21), (4.22) and (4.23).

Chapter 5

Varopoulos-type extensions of compactly supported Lipschitz functions

In this chapter we provide the proof of the Theorem 0.3. We begin by constructing an extension of L^p-boundary functions in the next theorem.

Theorem 5.1. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$. If $f \in \mathrm{Lip}_c(\partial\Omega)$ then there exists a function $F : \overline{\Omega} \to \mathbb{R}$ such that for every $p \in (1, \infty]$ the following hold.*

(i) $F \in C^\infty(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$,

(ii) $F|_{\partial\Omega} = f$ *continuously.*,

(iii) $\|\mathcal{N}(F)\|_{L^p(\sigma)} + \|\mathcal{C}_s(\nabla F)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$,

(iv) $\|\mathcal{N}(\delta_\Omega \nabla F)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$.

When $p = \infty$ the norms on left hand-side of (iii) *and* (iv) *are the* sup*-norms instead of L^∞.*

Remark 5.2. The trace $F|_{\partial\Omega} = f$ continuously means that the limit of F as we approach the boundary exists and it is equal with f. This is because we construct an extension F which is Lipschitz in $\overline{\Omega}$ and equals with f on the boundary $\partial\Omega$. Note that there is no need for non-tangential limit or extra connectivity assumption for the domain in this case.

Proof. Let $\mathcal{W}(\Omega)$ be a Whitney decomposition of Ω as the one constructed in 1.7. Let $\{\varphi_P\}_{P \in \mathcal{W}(\Omega)}$ be a partition of unity of Ω so that each φ_P is supported in $1.1P$ and $\|\nabla \varphi_P\|_\infty \lesssim 1/\ell(P)$. For each $\delta \in (0, \mathrm{diam}(\Omega))$ we set

$$\mathcal{W}_\delta(\Omega) := \{P \in \mathcal{W}(\Omega) : \ell(P) \geq \delta\}$$

and

$$\varphi_\delta = \sum_{P \in \mathcal{W}_\delta(\Omega)} \varphi_P.$$

From the properties of the Whitney cubes there exists $C > 0$, depending on the parameters of the construction of the Whitney cubes, such that

$$\varphi_\delta(x) = 0 \quad \text{if } \mathrm{dist}(x, \partial\Omega) \leq \delta/C$$

and

$$\varphi_\delta(x) = 1 \quad \text{if } \operatorname{dist}(x, \partial\Omega) \geq C\delta.$$

Consequently, for a suitable constant C' depending on C, we infer that

$$\operatorname{supp}(\nabla\varphi_\delta) \subset \{x \in \Omega : \delta/C \leq \operatorname{dist}(x, \partial\Omega) \leq C\delta\} =: S_\delta \subset \bigcup_{P \in \mathcal{I}_\delta} P \tag{5.1}$$

where

$$\mathcal{I}_\delta := \{P \in \mathcal{W}(\Omega) : \frac{1}{2^{N_0+1}2^{N_1}} \leq \ell(P) \leq \frac{2^{N_1}}{2^{N_0}}\}$$

with $N_0 \in \mathbb{N}$ such that $\frac{1}{2^{N_0+1}} \leq \delta \leq \frac{1}{2^{N_0}}$ and $N_1 \in \mathbb{N}$ satisfies $2^{N_1} \leq C \leq 2^{N_1+1}$.

We define

$$F(x) := v_f(x)(1 - \varphi_\delta(x)) + u(x)\varphi_\delta(x), \tag{5.2}$$

where u is the approximation function of v_f as constructed in Theorem 4.4. It holds

$$\mathcal{C}_s(\nabla F) \leq \mathcal{C}_s(\nabla u) + \mathcal{C}_s(\nabla\varphi_\delta\,(u - v_f)) + \mathcal{C}_s(\nabla v_f(1 - \varphi_\delta)). \tag{5.3}$$

For fixed $\xi \in \partial\Omega$ and $r > 0$ we have

$$\begin{aligned}
\int_{B(\xi,r)\cap\Omega} |\nabla\varphi_\delta\,(u - v_f)| \frac{dx}{\delta_\Omega(x)^{n-s}} &\lesssim \sum_{\substack{P \in \mathcal{I}_\delta \\ P\cap B(\xi,r)\neq\emptyset}} \int_P |u - v_f| \frac{dx}{\delta_\Omega(x)^{n+1-s}} \\
&\lesssim \sum_{\substack{P \in \mathcal{I}_\delta \\ P\cap B(\xi,r)\neq\emptyset}} \ell(P)^s \inf_{\zeta\in b(P)} \mathcal{N}_\alpha(u - v_f)(\zeta) \lesssim \sum_{\substack{P \in \mathcal{I}_\delta \\ P\cap B(\xi,r)\neq\emptyset}} \int_{b(P)} \mathcal{N}(u - v_f)\, d\sigma \\
&\lesssim \sum_{k=-(N_1+1)}^{N_1} \sum_{\substack{\ell(P)=2^k/2^{N_0} \\ P\subset B(\xi,Mr)}} \int_{b(P)} \mathcal{N}_\alpha(u - v_f)\, d\sigma \lesssim_C \int_{B(\xi,Mr)} \mathcal{N}_\alpha(u - v_f)\, d\sigma,
\end{aligned}$$

for suitably chosen constants $\alpha > 1$ and $M > 1$ sufficiently large. Thus, when $p \in (1, \infty)$ we get that

$$\mathcal{C}_s(\nabla\varphi_\delta\,(u - v_f))(\xi) \lesssim \mathcal{M}(\mathcal{N}_\alpha(u - v_f))(\xi), \tag{5.4}$$

while, when $p = \infty$, we get by (4.6) that $\sup_{x\in\Omega} \sup_{y\in B_x} |u(y) - v_f(y)| \leq 2\varepsilon\|f\|_{L^\infty(\sigma)}$, which, by similar arguments as above implies the estimate

$$\begin{aligned}
\int_{B(\xi,r)\cap\Omega} |\nabla\varphi_\delta\,(u - v_f)| \frac{dx}{\delta_\Omega(x)^{n-s}} &\lesssim \|f\|_{L^\infty(\sigma)} \sum_{\substack{P \in \mathcal{I}_\delta \\ P\cap B(\xi,r)\neq\emptyset}} \ell(P)^s \\
&\lesssim \|f\|_{L^\infty(\sigma)} \sum_{k=-(N_1+1)}^{N_1} \sum_{\substack{\ell(P)=2^k/2^{N_0} \\ b(P)\subset B(\xi,Mr)}} \sigma(b(P)) \lesssim r^s \|f\|_{L^\infty(\sigma)}.
\end{aligned} \tag{5.5}$$

Thus

$$\sup_{\xi\in\partial\Omega} C_s(\nabla\varphi_\delta\,(u - v_f))(\xi) \lesssim \|f\|_{L^\infty(\sigma)}. \tag{5.6}$$

For the last term on the right hand side of (5.3) we have that $f \in \dot{M}^{1,p}(\sigma)$ whenever $f \in \mathrm{Lip}_c(\partial\Omega)$ and $p \in (1,\infty)$. So for fixed $\xi \in \partial\Omega$ and $r > 0$, if $\nabla_H f$ is the least upper gradient of f, we estimate in view of (2.5)

$$\int_{B(\xi,r)\cap\Omega} |\nabla \upsilon_f(x)||1-\varphi_\delta(x)| \frac{dx}{\delta_\Omega(x)^{n-s}} \le \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset\\ \ell(P)\lesssim\delta}} \int_P |\nabla \upsilon_f(x)| \frac{\ell(P)^s}{\ell(P)^n}\, dx$$
$$\lesssim \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset\\ \ell(P)\le C\delta}} m_{\sigma,b(P)}(\nabla_H f)\, \ell(P)^{s+1} \lesssim \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset\\ \ell(P)\le C\delta}} \ell(P)\sigma(b(P)) \inf_{\zeta\in b(P)} \mathcal{M}(\nabla_H f)(\zeta)$$
$$\le \sum_{k\ge N_0-N_1} 2^{-k} \sum_{\substack{Q\in\mathcal{D}_\sigma\\ Q\subset B(\xi,Mr)\\ \ell(Q)=2^{-k}}} \int_Q \mathcal{M}(\nabla_H f)\, d\sigma \lesssim \delta\, m_{\sigma,B(\xi,Mr)}(\mathcal{M}(\nabla_H f))\, r^s$$

which shows that

$$\mathcal{C}_s(\nabla \upsilon_f(1-\varphi_\delta))(\xi) \lesssim \delta\, \mathcal{M}(\mathcal{M}(\nabla_H f))(\xi). \tag{5.7}$$

For $p=\infty$ we use Lemma 2.5 to get

$$\mathcal{C}_s(\nabla \upsilon_f(1-\varphi_\delta))(\xi) \lesssim \mathrm{Lip}(f)\delta. \tag{5.8}$$

Indeed, for $\xi \in \partial\Omega$ and $r > 0$ we have

$$\int_{B(\xi,r)\cap\Omega} |\nabla \upsilon_f||1-\varphi_\delta| \frac{dx}{\delta_\Omega(x)^{n-s}} \lesssim \mathrm{Lip}(f) \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset\\ \ell(P)\le C\delta}} \int_P \frac{\ell(P)^s}{\ell(P)^n}\, dx$$
$$\lesssim \mathrm{Lip}(f) \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset\\ \ell(P)\le C\delta}} \ell(P)^{s+1} \le \mathrm{Lip}(f) \sum_{k\ge N_0-N_1} 2^{-k} \sum_{\substack{P\in\mathcal{W}(\Omega)\\ P\cap B(\xi,r)\neq\emptyset\\ \ell(P)=2^{-k}}} \sigma(b(P))$$

$$\lesssim \mathrm{Lip}(f) \sum_{k\ge N_0-N_1} 2^{-k} \sum_{\substack{Q\in\mathcal{D}_\sigma\\ Q\subset B(\xi,Mr)\\ \ell(Q)=2^{-k}}} \sigma(Q) \lesssim \mathrm{Lip}(f)\, \delta\, r^s,$$

for $M > 1$ a sufficiently large constant depending on the Ahlfors regularity and Whitney constants.

Combining (4.4), (5.3), (5.4), (5.6), (5.7), and (5.8) and choosing

$$\delta := \begin{cases} \|f\|_{L^\infty(\sigma)}/\,\mathrm{Lip}(f) & \text{if } p=\infty,\\ \|f\|_{L^p(\sigma)}/\|f\|_{\dot{M}^{1,p}(\sigma)} & \text{if } p\in(1,\infty), \end{cases}$$

it follows that

$$\|\mathcal{C}_s(\nabla F)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} \quad \text{for } p\in(1,\infty].$$

For the non-tangential estimate note that, since $v_f = \varphi_\delta v_f + (1-\varphi_\delta)v_f$, we can write

$$v_f - F = \varphi_\delta(v_f - u).$$

So, for every $\xi \in \partial\Omega$ we have

$$\mathcal{N}(v_f - F)(\xi) = \sup_{x\in\gamma(\xi)} |v_f(x) - F(x)| \leq \sup_{x\in\gamma(\xi)} |v_f(x) - u(x)| = \mathcal{N}(v_f - u)(\xi)$$

and by (4.18) we get

$$\|\mathcal{N}(v_f - F)\|_{L^p(\sigma)} \lesssim \varepsilon \|f\|_{L^p(\sigma)} \quad \text{for } p \in (1,\infty).$$

Using this and (2.6) we get that for $p \in (1,\infty)$ there holds

$$\|\mathcal{N}(F)\|_{L^p(\sigma)} \lesssim \|\mathcal{N}(v_f - F)\|_{L^p(\sigma)} + \|\mathcal{N}(v_f)\|_{L^p(\sigma)} \lesssim_\varepsilon \|f\|_{L^p(\sigma)}.$$

Moreover, combining the estimates (2.3), (4.3) and (4.15), the fact that $|\nabla\varphi_\delta(x)| \lesssim \delta_\Omega(x)^{-1}$, and using the L^p-boundedness of the Hardy-Littlewood maximal operator, we can easily infer that

$$\|\mathcal{N}(\delta_\Omega \nabla F)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} \quad \text{for } p \in (1,\infty).$$

The estimates for $p = \infty$ can be proved similarly and the routine details are omitted.

Note that the extension F is Lipschitz in $\overline{\Omega}$. Indeed, if $E := \operatorname{supp} f$, in light of Lemmas 2.2 and 2.5 and of Theorem 4.3, we infer that for every $x \in \Omega$ we have

$$\begin{aligned}
|\nabla v_f(x)(1-\varphi_\delta(x))| &\lesssim \operatorname{Lip}(f),\\
|\nabla u(x)\,\varphi_\delta(x)| &\lesssim \delta_\Omega(x)^{-1}\|f\|_{L^\infty(\sigma)}|\varphi_\delta(x)| \lesssim \frac{1}{\delta}\operatorname{Lip}(f)\operatorname{diam} E,\\
|(u(x) - v_f(x))\nabla\varphi_\delta(x)| &\lesssim \frac{1}{\delta}\inf_{\zeta\in B(\xi_x, 2\delta_\Omega(x))\cap\partial\Omega} \mathcal{N}(u - v_f)(\zeta) \lesssim_\varepsilon \frac{1}{\delta}\|\mathcal{M}(f)\|_{L^\infty}\\
&\lesssim \frac{\|f\|_{L^\infty(\sigma)}}{\delta} \lesssim \frac{\operatorname{Lip}(f)}{\delta}\operatorname{diam} E.
\end{aligned}$$

These estimates imply that $\|\nabla F\|_{L^\infty(\Omega)} \lesssim_{\delta,\operatorname{diam} E} \operatorname{Lip}(f)$. Moreover, since $v_f \in \operatorname{Lip}(\overline{\Omega})$, $F = v_f$ in a neighborhood of $\partial\Omega$, and $v_f|_{\partial\Omega} = f$, we deduce that $F \in \operatorname{Lip}(\overline{\Omega})$ with $F|_{\partial\Omega} = f$ and $\operatorname{Lip}(F) \lesssim_{\delta,\operatorname{diam} E} \operatorname{Lip}(f)$, concluding the proof of the theorem.

□

Remark 5.3. Note that the convergence of F to the boundary function f is inherited from the one of v_f.

We now turn our attention to the construction of an extension of BMO-boundary functions.

Theorem 5.4. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$. If $f \in \operatorname{Lip}_c(\partial\Omega)$ then there exists an extension $F : \overline{\Omega} \to \mathbb{R}$ such that*

(i) $F \in C^\infty(\Omega) \cap \operatorname{Lip}(\overline{\Omega})$,

(ii) $F|_{\partial\Omega} = f$ *continuously,*

(iii) $\sup_{\xi\in\partial\Omega} \mathcal{N}_\sharp(F)(\xi) + \sup_{\xi\in\partial\Omega} \mathcal{C}_s(\nabla F)(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$,

(iv) $\sup_{x\in\Omega} \delta_\Omega(x)|\nabla F(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$.

Proof. Let w be the approximation of v_f given by Theorem 4.4 and define

$$F(x) := v_f(x)(1-\varphi_\delta(x)) + w(x)\varphi_\delta(x). \tag{5.9}$$

Then, for any $\xi \in \partial\Omega$ there holds

$$\mathcal{C}_s(\nabla F)(\xi) \leq \mathcal{C}_s(\nabla v_f(1-\varphi_\delta))(\xi) + \mathcal{C}_s(\nabla w)(\xi) + \mathcal{C}_s((w-v_f)\nabla\varphi_\delta)(\xi). \tag{5.10}$$

For the second summand in the right hand side of (5.10) we just use (4.22), while for the first one, by (5.8), we have that

$$\mathcal{C}_s(\nabla v_f(1-\varphi_\delta))(\xi) \lesssim \delta \operatorname{Lip}(f). \tag{5.11}$$

The third summand can be bounded as in (5.5) and get

$$\sup_{\xi\in\partial\Omega} \mathcal{C}_s(\nabla\varphi_\delta(w-v_f))(\xi) \lesssim \|f\|_{\mathrm{BMO}}. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12) and choosing $\delta := \|f\|_{\mathrm{BMO}}/\operatorname{Lip}(f)$, it follows that

$$\sup_{\xi\in\partial\Omega} \mathcal{C}_s(\nabla F)(\xi) \lesssim \|f\|_{\mathrm{BMO}}.$$

For the sharp non-tangential estimate note that since $F - v_f = \varphi_\delta(w-v_f)$, using (2.7) and (4.6), we get that for every $\xi \in \partial\Omega$ it holds that

$$\mathcal{N}_\sharp(F)(\xi) \leq 2\mathcal{N}(F-v_f)(\xi) + \mathcal{N}_\sharp(v_f)(\xi) \leq 2\mathcal{N}(w-v_f)(\xi) + \mathcal{N}_\sharp(v_f)(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma)}.$$

It remains to prove that $F \in \operatorname{Lip}(\Omega)$. We first show that

$$\|\nabla F\|_{L^\infty(\sigma)} \lesssim \operatorname{Lip}(f). \tag{5.13}$$

To this end, we have by Lemma 2.5 that for every $x \in \Omega$

$$|\nabla v_f(x)(1-\varphi_\delta(x))| \lesssim \operatorname{Lip}(f).$$

By (4.6) and the fact that $\delta(x) \approx \delta$ in the support of $\nabla\varphi_\delta$, we obtain

$$|(w(x)-v_f(x))\nabla\varphi_\delta(x)| \lesssim_\varepsilon \delta(x)^{-1}\|f\|_{\mathrm{BMO}(\sigma)} \approx \delta^{-1}\|f\|_{\mathrm{BMO}(\sigma)} = \operatorname{Lip}(f).$$

On the other hand, by (4.7) it holds that

$$|\nabla w(x)\varphi_\delta(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}\, \delta_\Omega(x)^{-1}|\varphi_\delta(x)| \lesssim \delta^{-1}\|f\|_{\mathrm{BMO}(\sigma)} = \operatorname{Lip}(f),$$

which implies (5.13). By construction, F is continuous in a neighborhood of the boundary and $F|_{\partial\Omega} = f$ continuously, which implies that $F \in \operatorname{Lip}(\overline{\Omega}) \cap C^\infty(\overline{\Omega})$ with $\operatorname{Lip}(F) \lesssim \operatorname{Lip}(f)$. Moreover, combining the last two estimates above with (2.4), we get that

$$\sup_{x\in\Omega} \delta_\Omega(x)|\nabla F(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma)},$$

which concludes the proof of the theorem. □

Our last goal for this chapter is to modify that the extensions constructed in Theorems 5.1 and 5.4 so that they are also in $\dot{W}^{1,2}(\Omega;\omega_s)$. This will conclude the proof of Theorem 0.3. Recall that for $x \in \Omega$, $\omega_s(x) = \delta_\Omega(x)^{s-n}$.

Theorem 5.5. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$. If $f \in \mathrm{Lip}_c(\partial\Omega)$ then there exists an extension $F_0 \in \dot{W}^{1,2}(\Omega;\omega_s)$ (resp. $\bar{F}_0 \in \dot{W}^{1,2}(\Omega;\omega_s)$) that satisfies the conclusions (i)-(iv) of Theorem 5.1 (resp. Theorem 5.4).*

Proof. Let $f \in \mathrm{Lip}_c(\partial\Omega)$, $E := \operatorname{supp} f$, and $r_0 := \operatorname{diam} E$. Without loss of generality we may assume that $0 \in E$ and so $E \subset B(0,r_0)$. Now let $B = B(0, Mr_0)$ for some $M > 1$ large enough depending on the Whitney constants so that for every $P \in \mathcal{W}(\Omega)$ satisfying $\ell(P) \leq M^{-1} r_0$ and $1.2P \cap (2B \setminus B) \neq \emptyset$ it holds that $b(P) \cap E = \emptyset$. We denote the collection of all such Whitney cubes by $\mathscr{P}_s(E)$; s stands for small. We also denote by $\mathscr{P}_l(E)$ the collection of $P \in \mathcal{W}(\Omega)$ satisfying $\ell(P) > M^{-1} \operatorname{diam} E$ and $1.2P \cap (2B \setminus B) \neq \emptyset$; l stands for large. It is easy to see that

$$\sum_{\substack{Q \subset R:\, Q = b(P) \\ P \in \mathscr{P}_l(E)}} \sigma(Q) \lesssim r_0^s \lesssim \sigma(R). \tag{5.14}$$

Note that if $x \in (2B \setminus B) \cap \Omega$ and there exists $P \in \mathscr{P}_s(E)$ such that $x \in 1.1P$ then the extension F of Theorem 5.1 satisfies $F(x) = 0$ (resp. $\bar{F}$ of Theorem 5.4 satisfies $\bar{F}(x) = 0$). We now define the cut-off function $\psi_{r_0} \in C_c^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \psi_{r_0} \leq 1$, $\psi_{r_0} = 1$ in $\overline{B}$, $\psi_{r_0} = 0$ in $\mathbb{R}^{n+1} \setminus 2B$ and $|\nabla \psi_{r_0}| \lesssim 1/r_0$. We set

$$F_0(x) := F(x)\,\psi_{r_0}(x) \qquad \text{and} \qquad \bar{F}_0(x) := \bar{F}(x)\,\psi_{r_0}(x), \quad x \in \Omega$$

It is clear that $F_0|_{\partial\Omega} = f$ (resp. $\bar{F}_0|_{\partial\Omega} = f$) and observe that

$$\left.\begin{array}{l} \operatorname{supp}(F\nabla\psi_{r_0}) \\ \operatorname{supp}(\bar{F}\nabla\psi_{r_0}) \end{array}\right\} \subset T_{r_0} := \{x \in \Omega \cap (2B \setminus B) :\, \operatorname{dist}(x,\partial\Omega) \geq c_0 r_0\} \tag{5.15}$$

provided that$c_0 \in (0,1)$ is sufficiently small depending on M and the Whitney constants. Therefore, for any $\xi \in \partial\Omega$, if $B(\xi,r) \cap \operatorname{supp}(F\nabla\psi_{r_0}) \neq \emptyset$, we have that $r \geq c_1 \max\{r_0, \operatorname{dist}(\xi, 2B \setminus B)\}$ for some constant $c_1 \in (0,1)$ depending on c_0. Moreover,

$$|F(x)\nabla\psi_{r_0}(x)| \lesssim \delta_\Omega(x)^{-1}|F(x)| \quad \text{and} \quad |\bar{F}(x)\nabla\psi_{r_0}(x)| \lesssim \delta_\Omega(x)^{-1}|\bar{F}(x)|. \tag{5.16}$$

We will only prove the theorem for F_0 and unbounded domains Ω with unbounded boundary since for domains with compact boundary the arguments are similar.

We first prove that F_0 satisfies the conclusions of Theorem 5.1. It is easy to see that $\|\mathcal{N}(F_0)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$ since $|F_0| \leq |F|$ and the same estimate holds for F. We have that $\nabla F_0 = \nabla F\,\psi_{r_0} + F\,\nabla\psi_{r_0}$ and it is easy to see that $\|\mathcal{N}(\delta_\Omega \nabla F_0)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$ by (5.16) and the estimates in (ii) and (iii) for F in Theorem 5.1. To prove the estimate $\|\mathcal{C}_s(\nabla F_0)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$ it is enough to show that $\|\mathcal{C}_s(F\,\nabla\psi_{r_0})\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$. Thus, for any r as above, we have that

$$\int_{B(\xi,r)\cap\Omega} \sup_{y \in B_x} |F(y)\nabla\psi_{r_0}(y)|\,\omega_s(x)\,dx \lesssim r_0^{-1} \int_{B(\xi,r)\cap(2B\setminus B)\cap\Omega} \sup_{y \in B_x} |F(y)|\,\omega_s(x)\,dx.$$

By (4.1), (3.2), (5.2) and the choice of the constant M, for any $x \in B(\xi, r) \cap (2B \setminus B) \cap \Omega$ and for any $y \in B_x$, it holds that

$$\begin{aligned}|F(y)| &\leq \sum_{R \in \mathsf{Top}} \sum_{\substack{P \in \mathscr{P}_l(E) \setminus \mathcal{P}_0 \\ b(P) \in \mathrm{Tree}(R)}} |m_{\sigma,R} f - m_{\sigma,b(P)} f| \, \varphi_P(y) + \sum_{P \in \mathscr{P}_l(E)} |m_{\sigma,b(P)} f| \, \varphi_P(y) \\ &\leq \varepsilon \sum_{R \in \mathsf{Top}} \sum_{\substack{P \in \mathscr{P}_l(E) \setminus \mathcal{P}_0 \\ b(P) \in \mathrm{Tree}(R)}} M f(b(P)) \, \varphi_P(y) + \sum_{P \in \mathscr{P}_l(E)} |m_{\sigma,b(P)} f| \, \varphi_P(y),\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}r_0^{-1} \int_{B(\xi,r) \cap (2B \setminus B) \cap \Omega} \sup_{y \in B_x} |F(y)| \, \omega_s(x) \, dx &\lesssim r_0^{-1} \sum_{P \in \mathscr{P}_l(E)} \ell(P) \sigma(b(P)) M f(b(P)) \\ &\lesssim \int_{CB} \mathcal{M}(f) \, d\sigma \leq \int_{B(\xi, C'r)} \mathcal{M}(f) \, d\sigma\end{aligned}$$

for some constant $C' > 1$ depending on C and M. This readily yields that for every $\xi \in \partial\Omega$

$$\mathcal{C}_s(F \, \nabla \psi_{r_0})(\xi) \lesssim \mathcal{M}(\mathcal{M}(f))(\xi)$$

and the desired estimate follows for any $p \in (1, \infty]$.

We will show now that $F_0 \in \dot{W}^{1,2}(\Omega; \omega_s)$ since it is clear that $F_0 \in C^\infty(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$ and $F_0|_{\partial\Omega} = f$. We only show the Carleson estimate since the non-tangential function estimates are easy and we will omit their proofs. To this end, by the definition of F and the proof of its Lipschitz property, we get that

$$\begin{aligned}\int_\Omega |\nabla F_0|^2 \omega_s(x) \, dx &\leq \int_{2B \cap \Omega} |\nabla F|^2 \omega_s(x) \, dx \\ &\lesssim \left(\left(1 + \frac{r_0^2}{\delta^2} \right) \min(r_0, \delta) + \frac{r_0^3}{\delta^2} \right) r_0^s \, (\mathrm{Lip}\, f)^2.\end{aligned}$$

Moreover, using (5.15) and the fact that $\operatorname{supp} f = E$ we can show that

$$\int_\Omega |F \nabla \psi_{r_0}|^2 \omega_s(x) \, dx \lesssim r_0^{-2-n+s} \int_{T_{r_0}} |F|^2 \, dx \lesssim r_0^{s-1} \|f\|^2_{L^\infty(\partial\Omega)} \leq r_0^{s+1} \, (\mathrm{Lip}\, f)^2,$$

concluding the proof of the Theorem for L^p for $p \in (1, \infty]$.

It remains to demonstrate the theorem for $\bar{F}_0$. We will first prove the Carleson estimate. By (5.15), (4.1), (3.2), (5.9) and the choice of the constant M, for any $x \in B(\xi, r) \cap (2B \setminus B) \cap \Omega$ and every $y \in B_x$ it holds that

$$\begin{aligned}|\bar{F}(y)| &\leq \sum_{R \in \mathsf{Top}} \sum_{\substack{P \in \mathscr{P}_l(E) \setminus \mathcal{P}_0 \\ b(P) \in \mathrm{Tree}(R)}} |m_{\sigma,R} f - m_{\sigma,b(P)} f| \, \varphi_P(y) + \sum_{P \in \mathscr{P}_l(E)} |m_{\sigma,b(P)} f| \, \varphi_P(y) \\ &\leq \varepsilon \|f\|_{\mathrm{BMO}(\sigma)} + \sum_{P \in \mathscr{P}_l(E)} |m_{\sigma,b(P)} f| \, \varphi_P(y).\end{aligned}$$

It is not hard to see that for every $P \in \mathscr{P}_l(E)$ there exists $P^* \in \mathcal{W}(\Omega)$ such that $\ell(P^*) \approx \operatorname{dist}(P^*, P) \approx r_0$ and $b(P^*) \subset \partial\Omega \setminus E$, and that $m_{\sigma,b(P^*)} f = 0$. Thus, for any $x \in B(\xi, r) \cap (2B \setminus B) \cap \Omega$ we have

$$|\bar{F}(y)| \leq \varepsilon \|f\|_{\mathrm{BMO}(\sigma)} + \sum_{P \in \mathscr{P}_l(E)} |m_{\sigma,b(P)} f - m_{\sigma,b(P^*)} f| \, \varphi_P(x) \lesssim \|f\|_{\mathrm{BMO}(\sigma)},$$

which, arguing as above, implies that $\sup_{\xi \in \partial\Omega} \mathcal{C}_s(\bar{F}\, \nabla \psi_{r_0})(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$. The estimates for the non-tangential maximal functions are easy and their proofs are omitted. This finishes the proof of the theorem since the same argument as above shows that $\bar{F}_0 \in \dot{W}^{1,2}(\Omega; \omega_s)$. $\square$

Chapter 6

Construction of Varopoulos-type extensions of L^p and BMO functions

In this chapter we prove Theorems 0.5 and 0.6. Recall the definitions of the Banach spaces (1.15) and (1.16). For future reference we summarize a set of assumptions in the following hypothesis;

Hypothesis [T]

(i) There exists a bounded linear *trace* operator

$$\mathrm{Tr} : \mathrm{N}^p(\Omega) \cap \mathrm{C}^{1,p}_{s,\infty}(\Omega) \to L^p(\sigma)$$

such that $\|\mathrm{Tr}(w)\|_{L^p(\sigma)} \leq \|w\|_{\mathrm{N}^p(\Omega)}$ for every $w \in \mathrm{N}^p(\Omega) \cap \mathrm{C}^{1,p}_{s,\infty}(\Omega)$.

(ii) If υ_f the regularized dyadic extension of $f \in L^p(\sigma)$ then $\mathrm{Tr}(\upsilon_f)(\xi) = f(\xi)$ for σ-a.e. $\xi \in \partial\Omega$ and for any $w \in \mathrm{N}^p(\Omega) \cap \mathrm{C}^{p}_{s,\infty}(\Omega)$ there holds

$$\|f - \mathrm{Tr}(w)\|_{L^p(\sigma)} = \|\mathrm{Tr}(\upsilon_f - w)\|_{L^p(\sigma)} \leq \|\upsilon_f - w\|_{\mathrm{N}^p(\Omega)}. \tag{6.1}$$

The proofs of the next two lemmas are standard but for the reader's convenience we provide them in Appendix B.

Lemma 6.1. *Let $B \subset \mathbb{R}^{n+1}$ be a bounded and convex open set and let $\{f_n\}_{n\geq 1}$ be a sequence of differentiable (resp. C^1) functions in B. Let $x_0 \in B$ such that $f_n(x_0) \to f(x_0)$. If $\nabla f_n \to \vec{F}$ uniformly in B for some $\vec{F}$, then*

$$f \text{ is differentiable (resp. } C^1) \text{ at } x_0 \quad \text{and} \quad \vec{F}(x_0) = \nabla f(x_0).$$

Lemma 6.2. *If $F \in L^1_{\mathrm{loc}}(\Omega)$ then for any $x \in \Omega$ it holds that*

$$|F(x)| \lesssim \frac{1}{\delta_\Omega(x)^{1+n/p}} \|\mathcal{C}_s(F)\|_{L^p(\sigma)} \quad \text{for every } p \in (1, \infty)$$

and

$$|F(x)| \lesssim \frac{1}{\delta_\Omega(x)} \sup_{\xi \in \partial\Omega} \mathcal{C}_s(F)(\xi).$$

Theorem 6.3. *Let* $\Omega \in \mathrm{AR}(s)$ *for* $s \in (0,n]$ *satisfying* Hypothesis [T]. *If* $f \in L^p(\sigma)$ *for some* $p \in (1,\infty]$ *then there exists a function* $u : \Omega \to \mathbb{R}$ *such that*

(i) $u \in C^1(\Omega)$,

(ii) $\mathrm{Tr}(u)(\xi) = f(\xi)$ *for* σ*-a.e.* $\xi \in \partial\Omega$,

(iii) $\|\mathcal{N}(u)\|_{L^p(\sigma)} + \|\mathcal{C}_s(\nabla u)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$,

(iv) $\|\mathcal{N}(\delta_\Omega \nabla u)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$.

Proof. Fix some $p \in (1,\infty]$. To construct the desired extension of $f \in L^p(\sigma)$, we follow the inductive scheme of Hytönen and Rosén [HR18]. Fix $\varepsilon > 0$ to be chosen. By Theorem 4.4 we construct u_0, the ε-approximating function of υ_f, and by Hypothesis [T], the trace of u_0 exists and satisfies $\mathrm{Tr}(u_0) \in L^p(\sigma)$. We set

$$f_1 := f - \mathrm{Tr}(u_0) \in L^p(\sigma).$$

We then let u_1 be the ε-approximating function of υ_{f_1} and set

$$f_2 := f_1 - \mathrm{Tr}(u_1) \in L^p(\sigma).$$

Inductively, for every $k \geq 1$, we define u_k to be the ε-approximating function of υ_{f_k} and set $f_{k+1} := f_k - \mathrm{Tr}(u_k)$. Therefore, by (6.1) and (4.18) we have that

$$\|f_{k+1}\|_{L^p(\sigma)} \leq \|\upsilon_{f_k} - u_k\|_{\mathrm{N}^p(\Omega)} \leq C\,\varepsilon \|f_k\|_{L^p(\sigma)},$$

which implies that

$$\|f_{k+1}\|_{L^p(\sigma)} \leq C\,\varepsilon\,\|f_k\|_{L^p(\sigma)} \leq \cdots \leq (C\varepsilon)^{k+1}\|f\|_{L^p(\sigma)}. \tag{6.2}$$

Thus, if we choose ε so that $C\varepsilon \leq \frac{1}{2}$ and set $S_k := \sum_{j=0}^k u_j$, then for $k < m$, using (4.18), (2.6) and (6.2) we get that

$$\begin{aligned}\|S_k - S_m\|_{\mathrm{N}^p(\Omega)} &\leq \sum_{j=k+1}^{m} \big(\|\mathcal{N}(u_j - \upsilon_{f_j})\|_{L^p(\sigma)} + \|\mathcal{N}\upsilon_{f_j}\|_{L^p(\sigma)}\big) \\ &\lesssim \sum_{j=k+1}^{m} \big(\varepsilon\|f_j\|_{L^p(\sigma)} + \|f_j\|_{L^p(\sigma)}\big) \leq (1+\varepsilon)\sum_{j=k+1}^{m}(C\varepsilon)^j\|f\|_{L^p(\sigma)} \\ &\leq (2^{-k+1} - 2^{-m+1})\|f\|_{L^p(\sigma)}.\end{aligned}$$

Thus, S_k is a Cauchy sequence in $\mathrm{N}^p(\Omega)$ and since $\mathrm{N}^p(\Omega)$ is a Banach space there exists $u \in \mathrm{N}^p(\Omega)$ such that $S_k \to u$ in $\mathrm{N}^p(\Omega)$. It is easy to see that $S_k \to u$ uniformly in B_x, for any $x \in \Omega$, and so we define

$$u(x) := \sum_{k=0}^{\infty} u_k(x) \quad \text{for all } x \in \Omega. \tag{6.3}$$

Similarly, we can show that $\nabla S_k = \sum_{j=0}^k \nabla u_j$ is convergent in the Banach space $\mathrm{C}^p_{s,\infty}(\Omega)$ (resp. $\mathrm{N}^p(\Omega)$) since by (4.19) (resp. (4.20)) and (6.2) we have

$$\begin{aligned}&\|\nabla S_k - \nabla S_m\|_{\mathrm{C}^p_{s,\infty}(\Omega)} + \|\delta_\Omega \nabla S_k - \delta_\Omega \nabla S_m\|_{\mathrm{N}^p(\Omega)} \\ &\quad\leq \sum_{j=k+1}^{m} \|\mathcal{C}_s(\nabla u_j)\|_{L^p(\sigma)} + \sum_{j=k+1}^{m} \|\mathcal{N}(\delta_\Omega|\nabla u_j|)\|_{L^p(\sigma)} \leq C\,\varepsilon^{-2}\sum_{j=k+1}^{m}\|f_j\|_{L^p(\sigma)}.\end{aligned}$$

Thus, there exist $\vec{F}_1 \in \mathrm{C}^p_{s,\infty}(\Omega)$ (resp. $\vec{F}_2$ so that $\delta_\Omega \vec{F}_2 \in \mathrm{N}^p(\Omega)$) such that $\nabla S_k \to \vec{F}_1$ in $\mathrm{C}^p_{s,\infty}(\Omega)$ (resp. $\delta_\Omega \nabla S_k \to \delta_\Omega \vec{F}_2$ in $\mathrm{N}^p(\Omega)$). Hence, by Lemma 6.2 we have that for any fixed $x \in \Omega$ that

$$\sup_{y \in B_x} |\nabla S_k - F_i| \delta_\Omega(y) \to 0 \quad \text{for } i \in \{1, 2\},$$

which readily implies that $\vec{F}_1 = \vec{F}_2 =: \vec{F}$ in Ω. Thus, ∇S_k converges to $\sum_{k=0}^\infty \nabla u_k$ uniformly in B_x for every $x \in \Omega$ and by Lemma 6.1 we deduce that $u \in C^1(\Omega)$ with

$$\sum_{k=0}^\infty \nabla u_k(x) = \nabla u(x) \quad \text{for all } x \in \Omega.$$

In fact,

$$\|\mathcal{N}(u)\|_{L^p(\sigma)} + \|\mathcal{N}(\delta_\Omega \nabla u)\|_{L^p(\sigma)} + \|\mathcal{C}_s(\nabla u)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}.$$

To show that u is an extension of f note first that in light of (6.2) we have

$$0 = \lim_{k \to \infty} \|f_{k+1}\|_{L^p(\sigma)} = \lim_{k \to \infty} \|f - \operatorname{Tr}\Big(\sum_{j=0}^k u_j\Big)\|_{L^p(\sigma)}.$$

Since, by construction, $\sum_{j=0}^k u_j - u \in \mathrm{N}^p(\Omega)$, in light of Hypothesis [T] we get that

$$\|\operatorname{Tr}\Big(\sum_{j=0}^k u_j\Big) - \operatorname{Tr}(u)\|_{L^p(\sigma)} = \|\operatorname{Tr}\Big(\sum_{j=0}^k u_j - u\Big)\|_{L^p(\sigma)} \leq \|\sum_{j=0}^k u_j - u\|_{\mathrm{N}^p(\Omega)} \xrightarrow{k \to \infty} 0,$$

which entails

$$\operatorname{Tr}(u)(\xi) = f(\xi) \quad \text{for } \sigma\text{-almost every } \xi \in \partial\Omega.$$

The proof is now complete. □

We state [ST70, Theorem 2, p. 171] in the following lemma.

Lemma 6.4. *Let $E \subset \mathbb{R}^{n+1}$ be a closed set and δ_E be the distance function with respect to E. Then there exist positive constants m_1 and m_2 and a function β_E defined in E^c such that*

(i) $m_1 \delta_E(x) \leq \beta_E(x) \leq m_2 \delta_E(x)$ for every $x \in E^c$;

(ii) β_E is smooth in E^c and there exists $C_\alpha > 0$ such that

$$\Big| \frac{\partial^\alpha}{\partial x^\alpha} \beta_E(x) \Big| \leq C_\alpha \beta_E(x)^{1-|\alpha|}.$$

The constants m_1, m_2 and C_α are independent of E.

Following [HT21, Section 3], we define a kernel $\Lambda(\cdot,\cdot) : \Omega \times \Omega \to [0, \infty]$ which will be necessary in the proof of Theorem 6.6. To this end, let $\beta = \beta_\Omega$ be the function constructed in Lemma 6.4 and let $\zeta \geq 0$ be a smooth non-negative function supported on $B(0, \frac{c}{4m_2})$ and satisfying $\zeta \leq 1$ and $\int \zeta = 1$. For every $\lambda > 0$ we set

$$\zeta_\lambda(x) := \lambda^{-(n+1)} \zeta\big(x/\lambda\big)$$

and we define the mollifier

$$\Lambda(x,y) := \zeta_{\beta(x)}(x-y) = \frac{1}{\beta(x)^{n+1}}\zeta\Big(\frac{x-y}{\beta(x)}\Big).$$

Observe that, by construction, for every $x \in \Omega$ we have

$$\operatorname{supp}(\Lambda(x,\cdot)) \subset \widetilde{B}_x := B(x, c\,\delta_\Omega(x)/4) \quad \text{and} \quad \int_\Omega \Lambda(x,y)\,dy = 1. \tag{6.4}$$

Moreover, it is easy to prove that

$$\sup_{y\in B_x} \Lambda(x,y) \lesssim \delta_\Omega(x)^{-n-1} \quad \text{and} \quad \sup_{y\in B_x} |\nabla_x \Lambda(x,y)| \lesssim \delta_\Omega(x)^{-n-2}. \tag{6.5}$$

For any $F : \Omega \to \mathbb{R}$, we define the *smooth modification* of F by

$$\widetilde{F}(x) := \int_\Omega \Lambda(x,y)\,F(y)\,dy, \quad x \in \Omega. \tag{6.6}$$

The next lemma was essentially proved in [HT21, Section 3] but we include the proof for the reader's convenience. Recall the Definition 1.16 of the non-tangential and the quasi-non-tangential convergence.

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the corkscrew condition. If $F \in C^1(\Omega;\mathbb{R}^{n+1})$ and $\widetilde{F}$ is the smooth modification of F as defined in (6.6), then the following hold.*

(a) For any $x \in \Omega$,

$$|\widetilde{F}(x)| \lesssim \sup_{2B_x} |F(y)|.$$

(b) For any $x \in \Omega$,

$$|\widetilde{F}(x_1) - \widetilde{F}(x_2)| \lesssim |x_1 - x_2|\,\delta_\Omega(x)^{-1}\, m_{\sharp,c}(F)(x), \quad \text{for all } x_1, x_2 \in B_x.$$

(c) For any $x \in \Omega$,

$$m_{\sharp,c}(\widetilde{F})(x) \lesssim m_{\sharp,c}(F)(x).$$

(d) For any $\xi \in \partial\Omega$,

$$\sup_{x\in\gamma_\alpha(\xi)} \delta_\Omega(x)|\nabla \widetilde{F}(x)| \lesssim \mathscr{C}_s(\nabla \widetilde{F})(\xi).$$

(e) For any $\xi \in \partial\Omega$,

$$\mathcal{C}_{s,c}(\nabla \widetilde{F})(\xi) \lesssim \mathscr{C}_s(\nabla F)(\xi).$$

(f) If $\text{qnt-lim}_{x\to\xi} F(x) = f(\xi)$ (resp. $\text{nt-lim}_{x\to\xi} F(x) = f(\xi)$) for σ-a.e. $\xi \in \partial\Omega$, then $\text{qnt-lim}_{x\to\xi} \widetilde{F}(x) = f(\xi)$ (resp. $\text{nt-lim}_{x\to\xi} \widetilde{F}(x) = f(\xi)$) for σ-a.e. $\xi \in \partial\Omega$.

Proof. (a) follows by definition of $\widetilde{F}$. For (b), we let $x_1, x_2 \in B_x$. Then, by triangle inequality,

$$B(x_1, c\,\delta_\Omega(x_1)/4) \cup B(x_2, c\,\delta_\Omega(x_2)/4) \subset B_x. \tag{6.7}$$

Combining (6.4), (6.5), and (6.7), we get that

$$\begin{aligned}|\widetilde{F}(x_1) - \widetilde{F}(x_2)| &= \left|\int (\Lambda(x_1,y) - \Lambda(x_2,y))\,(F(y) - m_{B_x}F)\,dy\right| \\ &\leq |x_1 - x_2| \sup_{z\in B_x} |\nabla\Lambda(x,z)|\, m_{B_x}(|F - m_{B_x}F|)\,|B_x| \\ &\lesssim |x_1 - x_2|\,\delta_\Omega(x)^{-1}\, m_{\sharp,c}(F)(x),\end{aligned}$$

which proves (b) and thus (c). We turn our attention to the proof of (d) and fix $\xi \in \partial\Omega$ and $r > 0$. For any $z \in B(\xi, r) \cap \Omega$ and $x \in B_z$, using (6.4), (6.5) and the Poincaré inequality we can write

$$\begin{aligned}|\nabla\widetilde{F}(x)| = \Big|\int \nabla_x\Lambda(x,y)F(y)\,dy\Big| &= \Big|\int \nabla_x\Lambda(x,y)\big(F(y) - m_{\widetilde{B}_x}F\big)\,dy\Big| \\ \lesssim \delta_\Omega(x)^{-n-2}\int_{\widetilde{B}_x} \big|F(y) - m_{\widetilde{B}_x}F\big|\,dy &\lesssim \fint_{\widetilde{B}_x} |\nabla F|\,dy\end{aligned}$$

which immediately implies (d). To prove (e), we first define

$$\begin{aligned}A_k(\xi, r) &:= \{x \in B(\xi,r)\cap\Omega :\, 2^{-k-1}r \leq \delta_\Omega(x) < 2^{-k}r\}, \\ A_k^*(\xi, r) &:= \{x \in B(\xi,r)\cap\Omega :\, 2^{-k-2}r \leq \delta_\Omega(x) < 2^{-k+1}r\},\end{aligned}$$

and estimate

$$\int_{B(\xi,r)\cap\Omega} \sup_{y\in B_x} |\nabla\widetilde{F}(y)|\,dy \leq \sum_{k=0}^{\infty}\int_{A_k(\xi,r)} \sup_{y\in B_x} |\nabla\widetilde{F}(y)|\,dy.$$

As $\cup_{y\in B_x} B_y \subset 2B_x$, by Fubini's theorem we have

$$\int_{A_k(\xi,r)} \sup_{y\in B_x} |\nabla\widetilde{F}(y)|\,dy \lesssim \int_{A_k(\xi,r)} \fint_{2B_x} |\nabla F(y)|\,dy \lesssim \int_{A_k^*(\xi,r)} |\nabla F(y)|\,dy.$$

Summing over k and using that the sets $A_k^*(\xi, r)$ have bounded overlap we get (*e*). Finally, (f) follows from [HT21, Lemma 3.14]. □

Let us now turn our attention to the case of BMO boundary data. Recall the definition of the Banach spaces (1.17), (1.18) and (1.19). The suitable version of Hypothesis [T] in this case is given below.

Hypothesis [$\widetilde{\mathrm{T}}$]

(i) There exists a bounded linear *trace* operator

$$\mathrm{Tr} : \mathrm{C}^{1,\infty}_{s,\infty}(\Omega) \to \mathrm{BMO}(\sigma)$$

such that $\|\,\mathrm{Tr}(w)\|_{\mathrm{BMO}(\sigma)} \lesssim \|\nabla w\|_{\mathrm{C}^{\infty}_{s,\infty}(\Omega)}$.

(ii) If υ_f is the regularized dyadic extension of $f \in \mathrm{BMO}(\sigma)$ then $\mathrm{Tr}(\upsilon_f)(\xi) = f(\xi)$ for σ-a.e. $\xi \in \partial\Omega$ and for any $w \in \mathrm{C}^{\infty}_{s,\infty}(\Omega)$, it holds that

$$\|f - \mathrm{Tr}(w)\|_{\mathrm{BMO}(\sigma)} = \|\mathrm{Tr}(\upsilon_f - w)\|_{\mathrm{BMO}(\sigma)} \lesssim \sup_{x\in\Omega} |\upsilon_f(x) - w(x)|. \tag{6.8}$$

Theorem 6.6. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0,n]$ satisfying the* Hypothesis $[\widetilde{\mathrm{T}}]$. *If $f \in \mathrm{BMO}(\sigma)$ then there exist a function $u : \Omega \to \mathbb{R}$ and a constant $c_0 \in (0, \frac{1}{2}]$ such that for every $c \in (0, c_0]$ the following holds.*

(i) $u \in C^1(\Omega)$.

(ii) $\sup_{\xi\in\partial\Omega} \mathcal{N}_{\sharp,c}(u)(\xi) + \sup_{\xi\in\partial\Omega} \mathcal{C}_{s,c}(\nabla u)(\xi) \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$.

(iii) $\sup_{x\in\Omega} \delta_\Omega(x)|\nabla u(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$.

(iv) $\mathrm{Tr}(u)(\xi) = f(\xi)$ *for σ-a.e. $\xi \in \partial\Omega$.*

Proof. We will argue as in the proof of Theorem 6.3. If $f \in \mathrm{BMO}(\sigma)$ and υ_f is its regularized dyadic extension, we apply Theorem 4.4 and construct the ε-approximating function of υ_f which we denote by u_0. In light of (4.22) and Hypothesis $[\widetilde{\mathrm{T}}]$, we have that the trace $\mathrm{Tr}(u_0)$ exists and it is in $\mathrm{BMO}(\sigma)$. We set

$$f_1 := f - \mathrm{Tr}(u_0).$$

Inductively, for every $k \geq 1$, we define u_k to be the ε-approximating function of υ_{f_k} and set

$$f_{k+1} := f_k - \mathrm{Tr}(u_k).$$

Therefore, by (4.21) and (6.8) we have that

$$\|f_{k+1}\|_{\mathrm{BMO}(\sigma)} \lesssim \sup_{\xi\in\partial\Omega} \mathcal{N}(u_k - \upsilon_{f_k})(\xi) \lesssim \varepsilon \|f_k\|_{\mathrm{BMO}(\sigma)}$$

which implies that

$$\|f_{k+1}\|_{\mathrm{BMO}(\sigma)} \leq C\varepsilon \|f_k\|_{\mathrm{BMO}(\sigma)} \leq \cdots \leq (C\varepsilon)^{k+1} \|f\|_{\mathrm{BMO}(\sigma)}. \tag{6.9}$$

Assume that $C\varepsilon \leq 1/2$ and set $S_k := \sum_{j=0}^{k} u_j$ for any positive integer k. Using (4.21), (2.7) and finally (6.9), we can estimate

$$\begin{aligned}
\|S_k - S_m\|_{\mathrm{N}^\infty_\sharp(\Omega)} &\lesssim \sum_{j=k+1}^{m} \big(\sup_{\xi\in\partial\Omega} \mathcal{N}(u_j - \upsilon_{f_j})(\xi) + \sup_{\xi\in\partial\Omega} \mathcal{N}_\sharp(\upsilon_{f_j})(\xi) \big) \\
&\lesssim \sum_{j=k+1}^{m} (\varepsilon \|f_j\|_{\mathrm{BMO}(\sigma)} + \|f_j\|_{\mathrm{BMO}(\sigma)}) \lesssim \sum_{j=k+1}^{m} (C\varepsilon)^j \|f\|_{\mathrm{BMO}(\sigma)} \\
&\leq (2^{-k} - 2^{-m}) \|f\|_{\mathrm{BMO}(\sigma)}.
\end{aligned}$$

Thus, S_k is Cauchy sequence in the Banach space $\mathrm{N}^\infty_\sharp(\Omega)\setminus\mathbb{R}$ (recall definition (1.19)) and so there exists $u\in \mathrm{N}^\infty_\sharp(\Omega)$ such that $S_k\to u$ in $\mathrm{N}^\infty_\sharp(\Omega)$. In fact,

$$\widetilde{S}_k := S_k - \fint_{B_x} S_k \to u \qquad \text{uniformly in } B_x, \text{ for any } x\in\Omega.$$

By (4.22) and (4.23), for any $m>k$, we have that

$$\begin{aligned}\|\nabla S_k - \nabla S_m\|_{\mathrm{C}^\infty_{s,\infty}(\Omega)} &+ \|\delta_\Omega \nabla S_k - \delta_\Omega\nabla S_m\|_{\mathrm{N}^\infty(\Omega)}\\ &\le \sum_{j=k+1}^m \sup_{\xi\in\partial\Omega}\mathcal{C}_{s,c}(\nabla u_j)(\xi) + \sum_{j=k+1}^m \sup_{\xi\in\partial\Omega}\mathcal{N}(\delta_\Omega\nabla u_j)(\xi) \lesssim \sum_{j=k+1}^m \|f_j\|_{\mathrm{BMO}(\sigma)}\\ &\le (2^{-k}-2^{-m})\|f\|_{\mathrm{BMO}(\sigma)}.\end{aligned}$$

Thus, there exists $\vec{F}_1\in \mathrm{C}^\infty_{s,\infty}(\Omega)$ (resp. $\vec{F}_2$ so that $\delta_\Omega\vec{F}_2\in \mathrm{N}^\infty(\Omega)$) such that $\nabla S_k\to\vec{F}_1$ in $\mathrm{C}^\infty_{s,\infty}(\Omega)$ (resp. $\delta_\Omega\nabla S_k\to\delta_\Omega\vec{F}_2\in \mathrm{N}^\infty(\Omega)$). By Lemma 6.2 we have that for any fixed $x\in\Omega$,

$$\sup_{y\in B_x}|\nabla S_k - \nabla\vec{F}_i|\delta_\Omega(y)\to 0 \quad \text{for } i\in\{1,2\}$$

which implies that $\vec{F}_1=\vec{F}_2=:\vec{F}$ in Ω. Thus,

$$\nabla S_k = \nabla\widetilde{S}_k \to \vec{F} = \sum_{k=0}^\infty \nabla u_k \text{ uniformly in } B_x, \text{ for every } x\in\Omega.$$

By Lemma 6.1 we deduce that $u\in C^1(\Omega)$ and

$$\nabla u(x) = \sum_{k=0}^\infty \nabla u_k(x) \quad \text{for all } x\in\Omega.$$

Furthermore, we have that

$$\sup_{\xi\in\partial\Omega}\mathcal{C}_s(\nabla u)(\xi) \le \sum_{k=0}^\infty \sup_{\xi\in\partial\Omega}\mathcal{C}_s(\nabla u_k)(\xi) \lesssim_\varepsilon \sum_{k=0}^\infty \|f_k\|_{\mathrm{BMO}(\sigma)} \lesssim \|f\|_{\mathrm{BMO}(\sigma)}. \tag{6.10}$$

Similarly,

$$\begin{aligned}\sup_{\xi\in\partial\Omega}\mathcal{N}_\sharp(u)(\xi) &+ \sup_{\xi\in\partial\Omega}\mathcal{N}(\delta_\Omega\nabla u)(\xi)\\ &\le \sum_{k=0}^\infty\Big(\sup_{\xi\in\partial\Omega}\mathcal{N}_\sharp(u_k)(\xi) + \sup_{\xi\in\partial\Omega}\mathcal{N}(\delta_\Omega\nabla u_k)(\xi)\Big) \lesssim \|f\|_{\mathrm{BMO}(\sigma)}.\end{aligned} \tag{6.11}$$

Finally, it holds that

$$0 = \lim_{k\to\infty}\|f_{k+1}\|_{\mathrm{BMO}(\sigma)} = \|f - \sum_{j=0}^k u_j\|_{\mathrm{BMO}(\sigma)}$$

and since, by construction, $\sum_{j=0}^k u_j - u \in \mathrm{C}^{1;\infty}_{s;\infty}(\Omega)$, using the linearity of the trace and Hypothesis $[\widetilde{\mathrm{T}}]$, we have that

$$\Big\| \mathrm{Tr}(u) - \mathrm{Tr}\Big(\sum_{j=0}^k u_j\Big)\Big\|_{\mathrm{BMO}(\sigma)} \lesssim \Big\| u - \sum_{j=0}^k u_j \Big\|_{\mathrm{C}^{\infty}_{s,\infty}(\Omega)} \overset{k\to\infty}{\longrightarrow} 0.$$

This gives that u is an extension of $f \in \mathrm{BMO}(\sigma)$ with

$$\mathrm{Tr}(u)(\xi) = f(\xi) \quad \text{for } \sigma\text{-almost every } \xi \in \partial\Omega \tag{6.12}$$

and the proof is complete. □

Remark 6.7. Note that by the proof of Theorem 6.6 it is immediate that the extension u satisfies the estimate

$$\sup_{x\in\Omega} |v_f(x) - u(x)| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}. \tag{6.13}$$

Proposition 6.8. *Let $\Omega \in \mathrm{AR}(s)$ for $s \in (0, n]$. For $s = n$ assume additionally that Ω satisfies the pointwise John condition. Then, for any $p \in (1, \infty]$ there exists a bounded linear trace operator $\mathrm{Tr}_\Omega : \mathrm{N}^p(\Omega) \cap \mathrm{C}^{1,p}_{s,\infty}(\Omega) \to L^p(\sigma)$ satisfying the Hypothesis $[\mathrm{T}]$. Moreover, if Ω satisifies the local John condition for $s = n$, then there exists a bounded linear trace operator $\mathrm{Tr}_\Omega : \mathrm{C}^{1,\infty}_{s,\infty}(\Omega) \to \mathrm{BMO}(\sigma)$.*

Proof. For any $x \in \Omega$ and fixed $c \in (0, 1/2]$ we define

$$E(x) := \fint_{B(x,c\delta_\Omega(x))} u(z)\, dz. \tag{6.14}$$

Fix $\xi \in \partial\Omega$ such that $\xi \in \mathrm{JC}(\theta)$ (see Definition 1.13). Then there exist $r_\xi > 0$ and $x_\xi \in B(\xi, 2r_\xi) \cap \Omega$ such that $\delta_\Omega(x_\xi) \geq \theta r_\xi$, and also there exists a good curve (recall definition 1.12) $\gamma : [0, 1] \to \mathbb{R}$ in $B(\xi, 2r_\xi) \cap \Omega$ connecting the points ξ and x_ξ such that $|\dot\gamma(t)| = 1$ $\forall t \in [0, 1]$. For any fixed pair of points $x_1, x_2 \in \gamma$ there exist $t_1, t_2 \in [0, 1]$ such that $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$. By a change of variables and an application of the mean value theorem, we estimate

$$\begin{aligned}
|E(x_1) - E(x_2)| &= \Big| \int_{B(0,1)} \big(u(x_1 + wc\delta_\Omega(x_1)) - u(x_2 + wc\delta_\Omega(x_2))\big)\, dw \Big| \\
&= \Big| \int_{B(0,1)} \big(u(\gamma(t_1) + wc\delta_\Omega(\gamma(t_1))) - u(\gamma(t_2) + wc\delta_\Omega(\gamma(t_2)))\big)\, dw \Big| \\
&= \Big| \int_{B(0,1)} \int_{t_1}^{t_2} \nabla u(\gamma(t) + wc\delta_\Omega(\gamma(t))) \cdot \nabla\delta_\Omega(\gamma(t))\, \dot\gamma(t)\, dt\, dw \Big| \\
&\leq \int_{t_1}^{t_2} \int_{B(0,1)} \big|\nabla u(\gamma(t) + wc\delta_\Omega(\gamma(t))\big|\, dw\, dt;
\end{aligned}$$

above we used that $|\dot\gamma(t)| = 1$ and $|\nabla\delta_\Omega(\gamma(t))| \leq 1$ since the function $\mathrm{dist}(\cdot\,, \partial\Omega)$ is 1-Lipschitz. Note that for $j \in \{1, 2\}$ there exists $M_j \in \mathbb{N}$ such that $2^{-M_j} \leq t_j \leq 2^{-M_j+1}$. By the

Fundamental Theorem of Calculus, since $u \in C^1(\Omega)$ and $\gamma(t) + wc\delta_\Omega(\gamma(t))$ is $(1+c)$-Lipschitz in t for any $w \in B(0,1)$, we have that there exists $s_k \in [2^{-k}, 2^{1-k}]$ such that

$$\begin{aligned}\int_{t_1}^{t_2} \int_{B(0,1)} \big|\nabla u(\gamma(t) + wc\delta_\Omega(\gamma(t))\big| dwdt \\ &= \sum_{k=M_1}^{M_2} \int_{2^{-k}}^{2^{-k+1}} \int_{B(0,1)} \big|\nabla u(\gamma(t) + wc\delta_\Omega(\gamma(t))\big| dwdt \\ &= \sum_{k=M_1}^{M_2} 2^{-k} \int_{B(0,1)} \big|\nabla u(\gamma(s_k) + wc\delta_\Omega(\gamma(s_k))\big| dw \\ &= \sum_{k=M_1}^{M_2} 2^{-k} \int_{B(\gamma(s_k), c\delta_\Omega(\gamma(s_k)))} |\nabla u(y)| \frac{dy}{(c\delta_\Omega(\gamma(s_k)))^{n+1}} \\ &\lesssim \sum_{k=M_1}^{M_2} \int_{B(\gamma(s_k), c\delta_\Omega(\gamma(s_k)))} |\nabla u(y)| \frac{dy}{\delta_\Omega(y)^n},\end{aligned}$$

where is the last inequality we used that $\delta_\Omega(\gamma(s_k)) \approx s_k \approx 2^{-k}$ and that $\delta_\Omega(y) \lesssim \delta_\Omega(\gamma(s_k))$ for any $y \in B(\gamma(s_k), c\delta_\Omega(\gamma(s_k)))$. Therefore, there exists a cone $\gamma_\alpha(\xi)$, with apperture depending on c and θ, such that $B(\gamma(s_k), c\delta_\Omega(\gamma(s_k))) \subset \gamma_\alpha(\xi)$, and by the bounded overlap of the balls $B(\gamma(s_k), c\delta_\Omega(\gamma(s_k)))$ we infer that

$$|E(x_1) - E(x_2)| \lesssim \int_{\gamma_\alpha(\xi)\cap B(\xi, C\delta_\Omega(x_2))} |\nabla u(y)| \frac{dy}{\delta_\Omega(y)^n}.$$

By (1.24) we have that $\mathcal{A}_s^{(\alpha)}(\nabla u) \in L^p(\sigma)$ when $p \in (1,\infty)$ and $\mathcal{A}_s^{(\alpha)}(\nabla u) \in L^q_{\text{loc}}(\sigma)$ for any $q \in (1,\infty)$ when $p = \infty$. Thus, $\mathcal{A}_s^{(\alpha)}(\nabla u)(\xi) < \infty$ for σ-almost every $\xi \in \partial\Omega$ and using the fact that the above estimate holds for any pair of points $x_1, x_2 \in \gamma_\xi$, we can assume that $x_1, x_2 \in B(\xi, \varepsilon)$ for some $\varepsilon > 0$ which is small compared to r_ξ. Therefore,

$$|E(x_1) - E(x_2)| \lesssim \int_{\gamma_\alpha(\xi)\cap B(\xi, C\varepsilon)} |\nabla u(y)|\, \delta_\Omega(y)^{-n}\, dy \lesssim \mathcal{A}^{(\alpha)}(\nabla u)(\xi) < \infty.$$

By the dominated convergence theorem we get that $|E(x_1) - E(x_2)| \to 0$ as $\varepsilon \to 0$, i.e., $E(x)$ is Cauchy on γ_ξ and thus convergent. This shows that the quasi-non-tangential limit of $E(x)$ at $\xi \in \partial\Omega$ exists for σ-a.e. $\xi \in \partial\Omega$ and we can define the desired trace operator by

$$\operatorname{Tr}_\Omega(u)(\xi) := \operatorname*{qnt-lim}_{x\to\xi} E(x) \quad \text{for } \sigma\text{-a.e.}\, \xi \in \partial\Omega. \tag{6.15}$$

In the case that $s < n$, we just define $\operatorname{Tr}_\Omega(u)(\xi) = \text{nt-lim}_{x\to\xi} E(x)$ since Ω has only one connected component and any $\xi \in \partial\Omega$ can be connected to a corkscrew point by a good curve. It is clear that $\operatorname{Tr} : \mathrm{C}^{1,p}_{s,\infty}(\Omega) \to L^p(\sigma)$ is a linear operator while the fact that $\operatorname{Tr} : \mathrm{N}^p(\Omega) \cap \mathrm{C}^{1,p}_{s,\infty}(\Omega) \to L^p(\sigma)$ is bounded if $p \in (1,\infty]$ can be proved quite easily. Indeed, let $\xi \in \partial\Omega$ such that $\xi \in \mathrm{JC}(\theta)$. For fixed $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x \in B(\xi, \delta) \cap \gamma_\xi$, there holds

$$|\operatorname{Tr}_\Omega(u)(\xi)| \le |\operatorname{Tr}_\Omega(u)(\xi) - E(x)| + m_{\sigma, B(x, c\delta_\Omega(x))}(|u|) < \varepsilon + \mathcal{N}(u)(\xi).$$

Letting $\varepsilon \to 0$ we infer that $|\operatorname{Tr}_\Omega(u)(\xi)| \leq \mathcal{N}(u)(\xi)$ for σ-a.e. $\xi \in \partial\Omega$ which readily yields the validity of (i) of Hypothesis [T], while (ii) of Hypothesis [T] readily follows from Lemma 2.1.

Assume now that Ω satisfies the local John condition when $s = n$; in the case $s < n$ this is automatic. Fix $\xi \in \partial\Omega$ and $r > 0$. By the local John condition there exists a corkscrew point $x_r \in B(\xi, r)$ such that any $\zeta \in B(\xi, r) \cap \partial\Omega$ can be connected to x_r by a good curve. The existence of the trace operator follows by the same argument as above and we define it in the same way. It remains to show that $\operatorname{Tr} : \mathrm{C}^{1,\infty}_{s,\infty}(\Omega) \to \mathrm{BMO}(\sigma)$ is bounded. For $u \in \mathrm{C}^{1,\infty}_{s,\infty}(\Omega)$, if $B_r := B(x_r, c\delta_\Omega(x_r))$ is a corkscrew ball centered at x_r with radius $c\delta_\Omega(x_r) \approx r$, by the same proof as above we can show that

$$\big|\operatorname{Tr}_\Omega(u)(\zeta) - \fint_{B_r} u(y)\,dy\big| \lesssim \mathcal{A}_s(\nabla u \mathbf{1}_{B(\xi,C'r)})(\zeta), \quad \forall \zeta \in B(\xi, r) \cap \partial\Omega.$$

Thus, taking averages over the ball $B(\xi, r)$ with respect to σ and applying (1.23) in $L^1(B(\xi, C'r))$, whose proof is left to the interested reader, we conclude that

$$\fint_{B(\xi,r)} \big|\operatorname{Tr}_\Omega(u) - \fint_{B_r} u(y)\,dy\big| d\sigma \lesssim \fint_{B(\xi,C''r)} \mathcal{C}_s(\nabla u \mathbf{1}_{B(\xi,C''r)})\,d\sigma \leq \|\mathcal{C}_s(\nabla u)\|_{L^\infty(\sigma)}.$$

This readily implies that $\|\operatorname{Tr}_\Omega(u)\|_{\mathrm{BMO}(\sigma)} \lesssim \|\mathcal{C}_s(\nabla u)\|_{L^\infty(\sigma)}$ which, combined with the estimate $\|f\|_{\mathrm{BMO}(\sigma)} \leq 2\|f\|_{L^\infty(\sigma)}$, (6.13) and Hypothesis [T] (ii) (already proved above), proves Hypothesis $[\widetilde{\mathrm{T}}]$. □

Proof of Theorem 0.5. It is an immediate consequence of Theorem 6.3, Proposition 6.8 and Lemma 6.5. □

Let us now turn our attention to the proof of Theorem 0.6. When $s = n$ and Ω satisfies the pointwise John condition but not the local John condition, we will need the following generalization of Garnett's Lemma, which was proved in [HT21, Lemma 10.1].

Lemma 6.9. *Let $\Omega \in \mathrm{AR}(n)$, $Q_0 \in \mathcal{D}_\sigma$, and let $f \in \mathrm{BMO}(\sigma)$ which vanishes on $\partial\Omega \setminus Q_0$ (if it is non-empty). Then, there exists a collection of cubes $\widetilde{\mathcal{S}}(Q_0) = \{Q_j\}_j \subset \mathcal{D}(Q_0)$ and coefficients α_j such that the following hold.*

1. $\sup_j |\alpha_j| \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$.
2. $f = g + \sum_j \alpha_j \mathbf{1}_{Q_j}$, *where $g \in L^\infty(\sigma)$ with $\|g\|_{L^\infty(\sigma)} \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$.*
3. $\widetilde{\mathcal{S}}(Q_0)$ *satisfies a Carleson packing condition.*

Proof of Theorem 0.6. Recall that if $s < n$ then $\Omega \in \mathrm{AR}(s)$ is uniform and thus it satisfies the local John condition. Therefore, by Theorem 6.6, Proposition 6.8 and Lemma 6.5 we can construct the desired extension of Theorem 0.6 when either $s < n$, or $s = n$ and Ω satisfies, in addition, the local John condition. We are left with the case $s = n$ so that Ω satisfies the pointwise John condition but not the local John condition. By Lemma 6.9, if $f \in \mathrm{BMO}(\sigma)$ with compact support in $Q_0 \in \mathcal{D}_\sigma$ then there exists $g \in L^\infty(\sigma)$ and $b = \sum_j \alpha_j \mathbf{1}_{Q_j} \in \mathrm{BMO}(\sigma)$ such that $f = g + b$. We construct an extension G of g by Theorem 0.5 and so it remains to prove the existence of the extension of b. By [HT21, Proposition 1.3] there exists $B_0 : \Omega \to \mathbb{R}$ such that $\sup_{\xi\in\partial\Omega}\mathscr{C}_n(\nabla B_0)(\xi)+\sup_{x\in\Omega}\delta_\Omega(x)|\nabla B_0(x)|\lesssim\|f\|_{\mathrm{BMO}}$ and $B_0\to b$ non-tangentially for σ-a.e. $\xi\in\partial\Omega$. By Lemma 6.5, if we set $B=\widetilde{B}_0$ (as defined in (6.6)) we get the desired extension of b. The extension of f is then given by $G+B$. $\square$

Chapter 7

Applications to boundary value problems

Recall the definition of 1.27 of the elliptic operator L and that $H : \Omega \to \mathbb{C}^m$ is given by $H = (H^1, \ldots, H^m)$ and $\mathbf{\Xi} : \Omega \to \mathbb{C}^{m(n+1)}$ is given by $\mathbf{\Xi} := (\vec{\Xi}^1, \ldots, \vec{\Xi}^m)$, where $\vec{\Xi}^\alpha : \Omega \to \mathbb{C}^{n+1}$ and $\vec{\Xi}^\alpha = (\Xi_1^\alpha, \ldots, \Xi_{n+1}^\alpha)$ for $\alpha = 1, \ldots m$. We define the *variational co-normal derivative* of a solution $v \in \dot{W}^{1,2}(\Omega; \mathbb{C}^m)$ of $Lv = -\operatorname{div} \mathbf{\Xi} + H$ in Ω, and denote it by $\partial_{\nu_A} v$, to be the linear functional defined in terms of the sesquilinear form associated to L as follows:

$$\langle \partial_{\nu_A} v, \varphi \rangle := \ell_v(\varphi) := \mathbf{B}(v, \Phi) = \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n+1} \int_\Omega a_{ij}^{\alpha\beta}(x)\, \partial_j v^\beta(x) \partial_i \Phi^\alpha \, dx$$
$$- \sum_{\alpha=1}^{m} \sum_{i=1}^{n+1} \int_\Omega \Xi_i^\alpha(x) \partial_i \Phi^\alpha(x) \, dx - \sum_{\alpha=1}^{m} \int_\Omega H^\alpha(x) \Phi(x)^\alpha \, dx,$$

where $\varphi \in \operatorname{Lip}_c(\partial\Omega; \mathbb{C}^m)$ and $\Phi \in \dot{W}^{1,2}(\Omega; \mathbb{C}^m) \cap \operatorname{Lip}(\overline{\Omega}; \mathbb{C}^m)$ such that $\Phi|_{\partial\Omega} = \varphi$.

Lemma 7.1. *$\ell_v : \dot{W}^{1,2}(\Omega; \mathbb{C}^m) \cap \operatorname{Lip}(\overline{\Omega}; \mathbb{C}^m) \to \mathbb{R}$ is unambiguously defined.*

Proof. If $\Phi^1, \Phi^2 \in \dot{W}^{1,2}(\Omega; \mathbb{C}^m) \cap \operatorname{Lip}(\overline{\Omega}; \mathbb{C}^m)$ are such that $\Phi^1|_{\partial\Omega} = \Phi^2|_{\partial\Omega} = \varphi$ and $\Phi^1 \neq \Phi^2$, then $\Psi := \Phi^1 - \Phi^2 \in \dot{W}^{1,2}(\Omega : \mathbb{C}^m) \cap \operatorname{Lip}(\overline{\Omega}; \mathbb{C}^m)$ and $\Psi|_{\partial\Omega} = 0$, which implies that $\Psi \in Y_0^{1,2}(\Omega)$, see 1.2 for the definition of $Y_0^{1,2}(\Omega)$. Since $Lv = -\operatorname{div} \mathbf{\Xi} + H$, we have that $\mathbf{B}(v, \Psi) = 0$ and thus $\mathbf{B}(v, \Phi^1) = \mathbf{B}(v, \Phi^2)$. So any extension of φ belonging to $\dot{W}^{1,2}(\Omega; \mathbb{C}^m) \cap \operatorname{Lip}(\overline{\Omega}; \mathbb{C}^m)$ defines the same linear functional ℓ_v. □

From now on, we assume that $\Omega \in \mathrm{AR}(n)$, $n \geq 2$, and that either Ω is bounded or $\partial\Omega$ unbounded. This is because we will use the duality $\mathrm{N}_{q,p}(\Omega) = (\mathrm{C}_{s,q',p'}(\Omega))^*$ which is a consequence of [MPT22, Proposition 2.4].

In the sequel, we will prove for simplicity our results just for real elliptic equations (i.e., $m = 1$). Nevertheless, the proofs for $m > 1$ and complex-valued coefficients are identical, see also Remark 0.8.

7.1 Some connections between Poisson Problems and Boundary Value Problems

Proposition 7.2. *If* (PR_p^L) *is solvable in* Ω *for some* $p > 1$ *(recall definition 1.9) then its solution* u *satisfies the one-sided Rellich-type inequality*

$$\|\partial_{\nu_A} u\|_{L^p(\sigma)} \lesssim \|H\|_{\mathrm{C}_{2_*,p}(\Omega)} + \|\Xi/\delta_\Omega\|_{\mathrm{C}_{2,p}(\Omega)}. \tag{7.1}$$

Moreover, if (PR_1^L) *is solvable in* Ω *with data* $H = 0$ *and* $\Xi \in L^2_{\mathrm{loc}}(\Omega)$*, then its solution satisfies the one-sided Rellich-type inequality*

$$\|\partial_{\nu_A} u\|_{H^1(\sigma)} \lesssim \|\Xi\|_{T_2^1(\Omega)}, \tag{7.2}$$

where $H^1(\sigma)$ *is the atomic Hardy space.*

Proof. Suppose that u is the solution of (PR_p^L). Let $\varphi \in \mathrm{Lip}_c(\partial\Omega)$ and $F \in \dot{W}^{1,2}(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$ be the Varopoulos extension of the L^p boundary data φ constructed in Theorem 0.3. Then, by Lemma 7.1 we get

$$|\ell_u(\varphi)| = |\mathbf{B}(u, F)| \leq \|A\|_{L^\infty(\Omega)} \int_\Omega |\nabla u||\nabla F| + \int_\Omega |H||F| + \int_\Omega |\Xi|\,|\nabla F|.$$

By duality (see [MPT22, Proposition 2.4]), (1.41) and the properties of the extension F we infer that

$$\int_\Omega |\nabla u||\nabla F| \lesssim \|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\sigma)} \|\mathcal{C}_2(\nabla F)\|_{L^{p'}(\sigma)} \lesssim (\|H\|_{\mathrm{C}_{2_*,p}} + \|\Xi/\delta_\Omega\|_{\mathrm{C}_{2,p}(\Omega)})\|\varphi\|_{L^{p'}}.$$

By duality and using (ii) and (iii) of Theorem 0.3 we infer that

$$\begin{aligned}\int_\Omega |H||F| &+ \int_\Omega |\Xi|\,|\nabla F| \\ &\lesssim \|H\|_{\mathrm{C}_{2_*,p}} \|\mathcal{N}_{2^*}(\delta_\Omega \nabla F)\|_{L^{p'}(\sigma)} + \|\Xi/\delta_\Omega\|_{\mathrm{C}_{2,p}(\Omega)} \|\widetilde{\mathcal{N}}_2(F)\|_{L^{p'}(\sigma)} \\ &\lesssim (\|H\|_{\mathrm{C}_{2_*,p}} + \|\Xi/\delta_\Omega\|_{\mathrm{C}_{2,p}(\Omega)})\|\varphi\|_{L^{p'}(\sigma)}.\end{aligned}$$

Thus, by the above estimates, the density of $\mathrm{Lip}_c(\partial\Omega)$ in $L^p(\sigma)$ and duality we get (7.1).

For the endpoint case, let v be the solution of (PR_1^L) and let $\varphi \in \mathrm{Lip}_c(\partial\Omega)$. Arguing as above for $\widetilde{F}$ being the Varopoulos extension of BMO boundary data φ constructed in Theorem 0.3 and using Lemma 7.1 we get that

$$|\ell_v(\varphi)| \leq \|A\|_{L^\infty(\Omega)} \int_\Omega |\nabla v||\nabla \widetilde{F}| + \int_\Omega |\Xi||\nabla \widetilde{F}|.$$

By duality, (1.41) and (ii) of Theorem 0.3 for BMO we have that

$$\int_\Omega |\nabla u||\nabla \widetilde{F}| \lesssim \|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^1(\sigma)} \sup_{\xi \in \partial\Omega} \mathcal{C}(\nabla \widetilde{F})(\xi) \lesssim \|\Xi\|_{T_2^1(\Omega)} \|\varphi\|_{\mathrm{BMO}(\sigma)}.$$

For the second term, since $T_2^\infty(\Omega) = (T_2^1(\Omega))^*$ (this follows from the proof of Theorem 4.2 and Remarks 4.3 and 4.4 in [MPT13], see tent spaces in 1 for more details), it holds that

$$\int_\Omega |\Xi||\nabla \widetilde{F}| = \int_\Omega |\Xi|\,\delta_\Omega(x)|\nabla \widetilde{F}|\,\frac{dx}{\delta_\Omega(x)} \leq \|\Xi\|_{T_2^1(\Omega)} \|\delta_\Omega \nabla \widetilde{F}\|_{T_2^\infty(\Omega)}.$$

Since, by (iii) of Theorem 0.3 for BMO, we have

$$\delta_\Omega(x)|\nabla\widetilde{F}(x)|^2 \lesssim |\nabla\widetilde{F}(x)|\|\varphi\|_{\mathrm{BMO}(\sigma)},$$

it is easy to see that

$$\begin{aligned}\|\delta_\Omega\nabla\widetilde{F}\|_{T_2^\infty(\Omega)} &= \|\mathscr{C}(\delta_\Omega|\nabla\widetilde{F}|^2)\|_{L^\infty(\sigma)}^{1/2} \\ &\lesssim \|\varphi\|_{\mathrm{BMO}(\sigma)}^{1/2}\sup_{\xi\in\partial\Omega}\mathcal{C}(\nabla\widetilde{F})(\xi)^{1/2} \lesssim \|\varphi\|_{\mathrm{BMO}(\sigma)},\end{aligned} \tag{7.3}$$

where in the last inequality we used again (ii) of Theorem 0.3 for the extension $\widetilde{F}$. Thus

$$|\ell(\varphi)| \lesssim \|\mathbf{\Xi}\|_{T_2^1(\Omega)}\|\varphi\|_{\mathrm{BMO}(\sigma)}$$

which implies (7.2) since $\overline{\mathrm{Lip}_c(\partial\Omega)}^{\mathrm{VMO}(\sigma)} = \mathrm{VMO}(\sigma) = (H^1(\sigma))^*$. □

Theorem 7.3. *If* (PR_p^L) *with* $\mathbf{\Xi} = 0$ *is solvable in* Ω *for some* $p > 1$ *then* $(\mathrm{D}_{p'}^{L^*})$ *is also solvable.*

Proof. Let $f \in \mathrm{Lip}_c(\partial\Omega)$ and let u be the solution to (1.34) for L^* with data f. Using the density of $L_c^\infty(\Omega)$ in $\mathrm{C}_{2_*,p}(\Omega)$ and duality, we get that

$$\|\mathcal{N}_{2^*}(u)\|_{L^{p'}(\sigma)} \lesssim \sup_{\substack{H\in L_c^\infty(\Omega):\\ \|H\|_{\mathrm{C}_{2_*,p}(\Omega)}=1}} \Big|\int_\Omega uH\Big|.$$

Fix such an $H \in L_c^\infty(\Omega)$ and let $w \in Y_0^{1,2}(\Omega)$ be the solution to (PR_p^L) with data $\mathbf{\Xi} = 0$ and H. Then, using the fact that $L^*u = 0$ and (7.1), we estimate

$$\Big|\int_\Omega uH\Big| = \Big|-\int_\Omega \nabla u A\nabla w + \int_{\partial\Omega}\partial_{\nu_A}wf\Big| = \Big|\int_{\partial\Omega}\partial_{\nu_A}wf\Big| \lesssim \|\mathcal{C}_{2_*}(H)\|_{L^p(\sigma)}\|f\|_{L^{p'}(\sigma)}$$

which readily implies the estimate

$$\|\mathcal{N}_{2^*}(u)\|_{L^{p'}(\sigma)} \lesssim \|f\|_{L^{p'}(\sigma)},$$

thus concluding the proof of the theorem. □

Now, we turn our attention to the endpoint case of Theorem 7.3.

Theorem 7.4. *If* (PR_1^L) *with* $H = 0$ *is solvable in* Ω *then both* $(\mathrm{PD}_\infty^{L^*})$ *with* $H = 0$ *and* $(\mathrm{D}_{\mathrm{BMO}}^{L^*})$ *are solvable in* Ω.

Proof. Let v_1 be the solution of (1.33) with data $\mathbf{\Xi} \in L_c^\infty(\Omega;\mathbb{R}^{n+1})$ and $H = 0$ and let v_2 be the solution of (1.34) with data $\varphi \in \mathrm{Lip}_c(\partial\Omega)$; define $w := v_1 + v_2$. Using the tent space duality $(T_2^\infty(\Omega))^* = T_2^1(\Omega)$ along with the density of $L_c^\infty(\Omega)$ in $T_2^1(\Omega)$, we get that

$$\|\delta_\Omega\nabla w\|_{T_2^\infty(\Omega)} \approx \sup_{\substack{\mathbf{\Psi}\in L_c^\infty(\Omega):\\ \|\mathbf{\Psi}\|_{T_2^1(\Omega)}=1}} \Big|\int_\Omega \delta_\Omega(x)\nabla w\,\mathbf{\Psi}\frac{dx}{\delta_\Omega(x)}\Big| = \sup_{\substack{\mathbf{\Psi}\in L_c^\infty(\Omega):\\ \|\mathbf{\Psi}\|_{T_2^1(\Omega)}=1}} \Big|\int_\Omega \nabla w\,\mathbf{\Psi}\Big|.$$

Then, if $u \in Y_0^{1,2}(\Omega)$ is the solution of (PR_1^L) with data $\Psi \in L_c^\infty(\Omega)$ and $H = 0$, by duality and (7.2) we have

$$\begin{aligned}\Big|\int_\Omega \boldsymbol{\Psi}\,\nabla w\Big| &= \Big| -\int_\Omega A\nabla u \nabla w + \int_{\partial\Omega} \partial_{\nu_A} u f\Big| \leq \Big|\int_\Omega A^*\nabla w \nabla u\Big| + \Big|\int_{\partial\Omega} \partial_{\nu_A} u\, f\Big| \\ &= \Big|\int_\Omega \boldsymbol{\Xi}\,\nabla u\Big| + \Big|\int_{\partial\Omega} \partial_{\nu_A} u\, f\Big| \\ &\lesssim \|\mathcal{C}_2(\boldsymbol{\Xi})\|_{L^\infty(\sigma)}\|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^1(\sigma)} + \|\boldsymbol{\Psi}\|_{T_2^1(\Omega)}\|f\|_{\mathrm{BMO}(\sigma)} \\ &\lesssim \left(\|\boldsymbol{\Xi}\|_{\mathrm{C}_{2,\infty}(\Omega)} + \|f\|_{\mathrm{BMO}(\sigma)}\right)\|\boldsymbol{\Psi}\|_{T_2^1(\Omega)}\end{aligned}$$

which proves the desired estimates (1.36) and (1.39). □

Recall the definitions 1.8 and 1.6.

Theorem 7.5. *If (PD_p^L) (resp. (PD_∞^L)) is solvable in Ω with $H = 0$ for $p \in (1, \infty)$, then the Dirichlet problem (D_p^L) (resp. $(\mathrm{D}_{\mathrm{BMO}}^L)$) is also solvable in Ω.*

Proof. Let $f \in \mathrm{Lip}_c(\partial\Omega)$ and let F be the Varopoulos extensions of boundary functions $f \in L^p(\sigma)$ given by Theorem 0.3. In the construction of the solution of (1.34) with data f we can use F as the Lipschitz extension of f. If u is the aforementioned solution then $u = w + F$ where w is the solution of (1.33) with $\boldsymbol{\Xi} = -A\nabla F \in L^2(\Omega)$ and $H = 0$. Then, by (ii) of Theorem 0.3 and (1.38) we have that

$$\begin{aligned}\|\widetilde{\mathcal{N}}_{2^*}(u)\|_{L^p(\sigma)} &\leq \|\mathcal{N}(F)\|_{L^p(\sigma)} + \|\widetilde{\mathcal{N}}_{2^*}(w)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} + \|\mathcal{C}_2(\boldsymbol{\Xi})\|_{L^p(\sigma)} \\ &\lesssim \|f\|_{L^p(\sigma)} + \|A\|_{L^\infty(\Omega)}\|\mathcal{C}(\nabla F)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}.\end{aligned}$$

Thus, u is the solution of the Dirichlet problem (D_p^L). By similar arguments, using $\widetilde{F}$, the Varopoulos extension of $f \in \mathrm{BMO}(\sigma)$ given by Theorem 0.3 along with (7.3) and (1.39), we get that

$$\|\delta_\Omega \nabla u\|_{T_2^\infty(\Omega)} \leq \|\delta_\Omega \nabla F\|_{T_2^\infty(\Omega)} + \|\delta_\Omega \nabla w\|_{T_2^\infty(\Omega)} \lesssim \|f\|_{\mathrm{BMO}(\sigma)}$$

which finishes the proof of the Theorem. □

Proof of Theorem 0.9. It follows by combining Theorems 7.3, 7.4 and 7.5. □

7.2 Conditional one-sided Rellich-type inequalities

Proposition 7.6. *Suppose that (R_p^L) is solvable in Ω for some $p \geq 1$. If u is the solution of (1.34) for L^* in Ω with data $f \in \mathrm{Lip}_c(\partial\Omega)$, it holds that*

$$\|\partial_{\nu_{A^*}} u\|_{(\dot{M}^{1,p}(\sigma))^*} \lesssim \|f\|_{X_p(\sigma)} \tag{7.4}$$

where $(\dot{M}^{1,p}(\partial\Omega))^*$ stands for the Banach space dual of $\dot{M}^{1,p}(\partial\Omega)/\mathbb{R}$ and $X_p(\sigma)$ is equal to $L^{p'}(\sigma)$ if $p > 1$ and $\mathrm{BMO}(\sigma)$ if $p = 1$.

Proof. By definition we have that

$$\|\partial_{\nu_{A^*}} u\|_{(\dot{M}^{1,p}(\partial\Omega))^*} = \sup_{\substack{\varphi\in \mathrm{Lip}_c(\partial\Omega):\\ \|\varphi\|_{\dot{M}^{1,p}(\sigma)}=1}} |\langle \partial_{\nu_{A^*}} u, \varphi\rangle|.$$

Fix $\varphi \in \mathrm{Lip}_c(\partial\Omega)$ such that $\|\varphi\|_{\dot{M}^{1,p}(\sigma)} = 1$ and let w be the solution of (R_p^L) with data φ. Let also $F \in \mathrm{Lip}(\overline{\Omega})$ be the Varopoulos extension of $L^{p'}$ boundary data f as constructed in Theorem 0.3. By Lemma 7.1 we have that

$$\langle \partial_{\nu_{A^*}} u, \varphi\rangle = \int_\Omega A^* \nabla u \nabla w = \int_\Omega A\nabla w \nabla (u-F) + \int_\Omega A\nabla w\nabla F = \int_\Omega A\nabla w\nabla F$$

since $u - F \in Y_0^{1,2}(\Omega)$ and $Lw = 0$. Therefore, by duality, conclusion (ii) of Theorem 0.3 and (1.37), we infer that

$$\begin{aligned}|\langle \partial_{\nu_{A^*}} u, \varphi\rangle| = \Big|\int_\Omega A\nabla w\nabla F\Big| &\leq \|A\|_{L^\infty(\Omega)} \|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\sigma)} \|\mathcal{C}_2(\nabla F)\|_{L^{p'}} \\ &\lesssim \|\varphi\|_{\dot{M}^{1,p}(\sigma)} \|f\|_{L^{p'}(\sigma)}\end{aligned}$$

which shows (7.4) for $p > 1$. The proof in the case $p = 1$ is similar and we omit the details. □

Proposition 7.7. *Let $q \geq 1$. If u is a solution of (1.32) for $H \in L_c^\infty(\Omega)$ and $\boldsymbol{\Xi} \in L_c^\infty(\Omega;\mathbb{R}^{n+1})$ such that $\mathcal{N}_q(\nabla u) \in L^p(\sigma)$ for $p > 1$, it holds that*

$$\|\partial_{\nu_{A^*}} u\|_{L^p(\sigma)} \lesssim \|\nabla u\|_{\mathrm{N}_{q,p}(\Omega)} + \|H\|_{\mathrm{C}_{1,p}(\Omega)} + \|\boldsymbol{\Xi}/\delta_\Omega\|_{\mathrm{C}_{1,p}(\Omega)}. \tag{7.5}$$

If u is a solution of (1.32) for $H = 0$ and $\boldsymbol{\Xi} \in L_c^\infty(\Omega;\mathbb{R}^{n+1})$ such that $\mathcal{N}_q(\nabla u) \in L^1(\sigma)$, then

$$\|\partial_{\nu_{A^*}} u\|_{L^1(\sigma)} \lesssim \|\nabla u\|_{\mathrm{N}_{q,1}(\Omega)} + \|\boldsymbol{\Xi}\|_{T_2^1(\Omega)}. \tag{7.6}$$

Proof. It follows by the same arguments used in the proof of Proposition 7.2. We skip the details. □

Appendix A

Appendix to Chapter 1

A.1 Proof of Lemma 1.5

Proof. We adapt the proof of the [HR18, Proposition 2.4] and argue by duality. Indeed, let $1/p + 1/p' = 1$ and $h \in L^{p'}(\sigma)$ be a non-negative function supported in $B(\xi, r)$ and such that $\|h\|_{L^{p'}(\sigma)} = 1$. Then,

$$\begin{aligned}
\|\mathcal{A}(u\mathbf{1}_{B(\xi,r)})\|_{L^p(\sigma,B(\xi,r))} &= \int_{\partial\Omega} \Big(\int_{\gamma_\alpha(\xi)\cap B(\xi,r)} |u(y)|\delta_\Omega(y)^{-n}\, dy \Big) h(\xi)\, d\sigma(\xi) \\
&\leq \int_{B(\xi,r)} |u(y)|\delta_\Omega(y)^{s-n} \Big(\delta_\Omega(y)^{-s} \int_{B(y,\alpha\delta_\Omega(y))} h(\xi)\, d\sigma(\xi) \Big)\, dy \\
&=: \int_{B(\xi,r)} |u(y)|\delta_\Omega(y)^{s-n} H(y)\, dy \\
&= \int_0^\infty \int_{B(\xi,r)\cap\{H(y)>\lambda\}} |u(y)|\delta_\Omega(y)^{s-n}\, dy\, d\lambda,
\end{aligned}$$

where

$$H(y) := \delta_\Omega(y)^{-s} \int_{B(y,\alpha\delta_\Omega(y))} h(\xi)\, d\sigma(\xi).$$

For any $y \in \Omega \cap B(\xi, r)$ we let $\hat{y}$ be a point in $B(\xi, r) \cap \partial\Omega$ such that $|y - \hat{y}| = \delta_\Omega(y)$ and we set $B_{\hat{y}} := B(\hat{y}, (\alpha + 1)\delta(y)) \supset B(y, \alpha\delta_\Omega(y))$. Define

$$E_\lambda := \{y \in B(\xi, r) \cap \Omega : H(y) > \lambda\}$$

and note that for any $y \in E_\lambda$ it holds that $m_{\sigma,B_{\hat{y}}} h > c\lambda$ for some $c \in (0, 1)$ depending on α. If we set

$$\widehat{E}_\lambda := \{\zeta \in \partial\Omega : \zeta = \hat{y} \text{ for some } y \in E_\lambda\} \quad \text{and} \quad \mathscr{B}_\lambda = \{B_{\hat{y}} : y \in E_\lambda\},$$

then there exists a sufficiently large constant $C > 1$ such that

$$\bigcup_{\zeta\in\widehat{E}_\lambda} B_\zeta \cap \partial\Omega \subset \{\zeta \in \partial\Omega : \mathcal{M}h(\zeta) > c\lambda\} \cap B(\xi, Cr).$$

By Vitali's covering lemma there exists a subcollection $\mathscr{G}_\lambda \subset \mathscr{B}_\lambda$ of pairwise disjoint balls such that

$$\bigcup_{B' \in \mathscr{B}_\lambda} B' \subset \bigcup_{B \in \mathscr{G}_\lambda} 5B.$$

It is clear that

$$E_\lambda \subset \bigcup_{B \in \mathscr{F}_\lambda} 5B$$

and thus,

$$\begin{aligned}
\int_{E_\lambda} |u(y)| \delta_\Omega(y)^{s-n} \, dy &\leq \sum_{B \in \mathscr{F}_\lambda} \int_{5B} |u(y)| \delta_\Omega(y)^{s-n} \, dy \\
&\lesssim \sum_{B \in \mathscr{F}_\lambda} \sigma(B) \, r(B)^\beta \inf_{\zeta \in B \cap \partial\Omega} \mathscr{C}_s^{(\beta)}(u \mathbf{1}_{B(\xi, Cr)})(\zeta) \\
&\lesssim r^\beta \int_{B(\xi, Cr) \cap \{\mathcal{M}h > \lambda\}} \mathscr{C}_s^{(\beta)}(u \mathbf{1}_{B(\xi, Cr)})(\zeta) \, d\sigma(\zeta).
\end{aligned}$$

Therefore, since $\|h\|_{L^{p'}(\sigma)} = 1$, we have

$$\begin{aligned}
\|\mathcal{A}_{s,\alpha}(u \mathbf{1}_{B(\xi,r)})\|_{L^p(\sigma, B(\xi,r))} &\lesssim r^\beta \int_0^\infty \int_{B(\xi, Cr) \cap \{\mathcal{M}h > \lambda\}} \mathscr{C}_s^{(\beta)}(u \mathbf{1}_{B(\xi, Cr)})(\zeta) \, d\sigma(\zeta) \, d\lambda \\
&\lesssim r^\beta \int_{B(\xi, Cr)} \mathscr{C}_s^{(\beta)}(u \mathbf{1}_{B(\xi, Cr)})(\zeta) \, \mathcal{M}h(\zeta) \, d\sigma(\zeta) \\
&\leq r^\beta \|\mathscr{C}_s^{(\beta)}(u \mathbf{1}_{B(\xi, Cr)})\|_{L^p(\sigma, B(\xi, Cr))},
\end{aligned}$$

proving (1.23). The proof of (1.24) is similar and we omit the details. □

Appendix B

Appendix to Chapter 6

B.1 Proof of Lemma 6.1

Proof. [1]Let $x_0 \in B$ be a point such that $f_n(x_0) \to f(x_0)$ as $n \to \infty$ and write

$$\frac{f(x) - f(x_0) - \vec{F}(x_0)(x - x_0)}{|x - x_0|} = \frac{f(x) - f(x_0) - (f_n(x) - f_n(x_0))}{|x - x_0|}$$
$$+ \frac{f_n(x) - f_n(x_0) - \nabla f_n(x_0)(x - x_0)}{|x - x_0|} + \frac{(\nabla f_n(x_0) - \vec{F}(x_0))(x - x_0)}{|x - x_0|} =: I + II + III.$$

In order to control I consider the difference

$$f_m(tx + (1 - t)x_0) - f_n(tx + (1 - t)x_0), \quad t \in [0, 1],$$

for integers $m, n > 0$. By the mean value theorem there exists $t_0 \in (0, 1)$ such that

$$(f_m(x) - f_m(x_0)) - (f_n(x) - f_n(x_0)) = (\nabla f_m(z_0) - \nabla f_n(z_0))(x - x_0)$$

where $z_0 = t_0 x + (1 - t_0)x_0 \in B$. By the uniform convergence of the gradients in B we have that

$$\frac{|(f_m(x) - f_m(x_0)) - (f_n(x) - f_n(x_0))|}{|x - x_0|} = \frac{|(\nabla f_m(z_0) - \nabla f_n(z_0))(x - x_0)|}{|x - x_0|} \leq 2\varepsilon.$$

Since B is bounded and the sequence $\{f_n\}$ converges at x_0, there exists an integer $n_1 = n_1(\varepsilon, x_0) > 0$ such that for every $n \geq n_1$ the above inequality implies that

$$|f_m(x) - f_n(x)| \leq |f_m(x_0) - f_n(x_0)| + |x - x_0| 2\varepsilon \leq M\varepsilon,$$

for $M > 1$ a sufficiently large constant. So, $\{f_n\}$ is a uniformly Cauchy sequence and thus it converges uniformly to function f. Letting $m \to \infty$, we get that for every $n \geq n_1$ it holds

$$|I| = \frac{|f(x) - f(x_0) - (f_n(x) - f_n(x_0))|}{|x - x_0|} \leq 2\varepsilon.$$

Using the fact that f is differentiable in x_0 we get that there exists an integer $n_2 = n_2(\varepsilon, x_0) > 0$ and $\delta = \delta(\varepsilon) > 0$ such that $|II| \leq 2\varepsilon$ for all $x \in B$ with $|x - x_0| \leq \delta$ and all $n > n_2$. Moreover,

[1]This proof was given by Professor Giovanni Leoni at math.stackexchange/gradient convergence.

the sequence $\{\nabla f_n\}_{n\geq 1}$ converges to $\vec{F}$ uniformly in B. Thus, there exists an integer $n_3 > 0$ such that, for all $n \geq n_3$, we have

$$|III| \leq |\nabla f_n(x_0) - \vec{F}(x_0)| \leq \varepsilon.$$

Finally, setting $n_0 := \max\{n_1, n_2, n_3\}$ and using the above estimates we get that for every $x \in B$ with $|x - x_0| \leq \delta$ and $n \geq n_0$, it holds

$$\Big|\frac{f(x) - f(x_0) - \vec{F}(x_0)(x - x_0)}{|x - x_0|}\Big| \leq 5\varepsilon,$$

which implies that f if differentiable in x_0, and thus in B, with $\nabla f(x_0) = \vec{F}(x_0)$. Since this holds for all $x_0 \in B$ and f_n is differentiable in B, it is easy to see that f is differentiable in B. □

B.2 Proof of Lemma 6.2

Proof. Fix $x \in \Omega$ and note that if $c' = \frac{c}{c+1}$ then for any $z \in B(x, c'\delta_\Omega(x))$ we have that

$$|z - x| \le c' \operatorname{dist}(x, \partial\Omega) \le c' \operatorname{dist}(z, \partial\Omega) + c' \, |z - x|,$$

which implies that $|z - x| \le c\, \delta_\Omega(z)$, i.e., $x \in B_z$. If $\xi_x \in \partial\Omega$ is the point such that $\delta_\Omega(x) = |x - \xi_x|$ then it is clear that $B_x \subset B(\xi, 3\delta_\Omega(x)) \cap \Omega$ for every $\xi \in B(\xi_x, \delta_\Omega(x))$, and so

$$\begin{aligned}
\|\mathcal{C}_s(F)\|_{L^p(\sigma)} &= \Big(\int_{\partial\Omega} \Big[\sup_{r>0} \frac{1}{r^n} \int_{B(\xi,r)\cap\Omega} \sup_{y \in B_z} |F(y)| \, dz \Big]^p \, d\sigma(\xi) \Big)^{1/p} \\
&\gtrsim \Big(\int_{B(\xi_x, \delta_\Omega(x))\cap\partial\Omega} \Big[\frac{1}{\delta_\Omega(x)^n} \int_{B(\xi, 3\delta_\Omega(x))} \sup_{y \in B_z} |F(y)| \, dz \Big]^p \, d\sigma(\xi) \Big)^{1/p} \\
&\ge \Big(\int_{B(\xi_x, \delta_\Omega(x))\cap\partial\Omega} \Big[\frac{1}{\delta_\Omega(x)^n} \int_{B(x, c'\delta_\Omega(x))} \sup_{y \in B_z} |F(y)| \, dz \Big]^p \, d\sigma(\xi) \Big)^{1/p} \\
&\gtrsim \delta_\Omega(x)^{1+\frac{n}{p}} |F(x)|.
\end{aligned}$$

Note that for $p = \infty$ the same argument implies that

$$\sup_{\xi \in \partial\Omega} \mathcal{C}_s(F)(\xi) \gtrsim \delta_\Omega(x) |F(x)|.$$

□

Lemma B.1. *The quotient space $\mathrm{N}^\infty_\sharp(\Omega)/\mathbb{R}$ is a Banach space.*

Proof. Note that the space $\mathrm{N}^\infty_\sharp(\Omega)/\mathbb{R}$ can be written as the direct product $\bigotimes_{x\in\Omega} \Lambda_x/\mathbb{R}$ equipped with the sup norm, where

$$\Lambda_x := \big\{ u : \Omega \to \mathbb{R} : u \in L^1_{\text{loc}}(\Omega) \text{ and } \sup_{y \in B_x} \big| u(y) - \fint_{B_x} u \big| < \infty \big\},$$

is equipped with the semi-norm $\|u\|_{\Lambda_x} := \sup_{y\in B_x} |u(y) - \fint_{B_x} u|$ which is a norm modulo constants. By the theory of Banach spaces it is enough to prove that the space $\Lambda_x/\mathbb{R}$ is Banach for any fixed $x \in \Omega$[2]. To this end, let $x \in \Omega$, $\varepsilon > 0$, and take $\{u_n\}_{n\in\mathbb{N}}$ to be a Cauchy sequence in $\Lambda_x/\mathbb{R}$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n, m \ge n_0$ it holds that $\|u_n - u_m\|_{\Lambda_x} < \varepsilon$. So, for any $y \in B_x$ we have

$$\big| u_n(y) - u_m(y) - \fint_{B_x} (u_n(z) - u_m(z)) \, dz \big| < \varepsilon,$$

which means that the sequence $\{u_n - \fint_{B_x} u_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy in B_x and so it converges to some u locally uniformly in B_x. Thus there exists a positive integer $m_0 = m_0(\varepsilon, x)$ such that for any $m > m_0$ we have

$$\sup_{B_x} |u_m(y) - \fint_{B_x} u_m - u(y)| < \varepsilon/2.$$

[2]We would like to thank Professor Pandelis Dodos for providing us with a reference about this fact. See math.stackexchange/infinite direct sum.

Thus, for any $n > \max\{n_0, m_0\}$ we have

$$\begin{aligned}
&\sup_{B_x} |u_n(y) - u(y) - \fint_{B_x} (u_n - u)| = \\
&\sup_{B_x} \Big|u_n(y) - \fint_{B_x} u_n - u(y)\Big| + \Big|\fint_{B_x} (u(z) + u_m(z) - u_m(z))\, dz\Big| \\
&\leq \sup_{B_x} |u_n(y) - \fint_{B_x} u_n - u(y)| + \fint_{B_x} |u_m(z) - \fint_{B_x} u_m - u(z)|\, dz < \varepsilon.
\end{aligned}$$

We conclude that $\|u_n - u\|_{\Lambda_x} < \varepsilon$ which means that $u_n \to u$ in $\Lambda_x/\mathbb{R}$ with respect to the norm $\|\cdot\|_{\Lambda_x}$ and so $\Lambda_x/\mathbb{R}$ is a Banach space for every $x \in \Omega$. As a result, the space $\mathrm{N}^\infty_\sharp(\Omega)/\mathbb{R}$ is Banach. $\square$

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