SoftwareX 19 (2022) 101111

Contents lists available at ScienceDirect

SoftwareX

journal homepage: www.elsevier.com/locate/softx

Original software publication

ADITU: A mesh-free formulation for the solution of Helmholtz equation in bounded and unbounded domains

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A R T I C L E I N F O

Article history:
Received 19 January 2022
Received in revised form 4 May 2022
Accepted 19 May 2022

Keywords:
Helmholtz equation
Acoustic radiation
Acoustic scattering
Unbounded domains
Fluid–fluid coupling

A B S T R A C T

This work describes a mesh-free collocation formulation for the acoustic fluid–fluid coupling among non-homogeneous bounded and unbounded fluid domains. The proposed formulations use series of functions that are generated using the analytical solutions of Helmholtz equation in spherical coordinates, and coefficients of the series are calculated from applying the boundary conditions. The technique is able to calculate the acoustic pressure in any bounded or unbounded non-spherical domain. The accuracy, efficiency and robustness of the model is tested by using as benchmark classical analytic solutions of radiation and scattering from spheres.

Code metadata

Current code version	v1.0
Permanent link to code/repository used for this code version	https://github.com/ElsevierSoftwareX/SOFTX-D-22-00022
Code versioning system used	git
Legal Code License	GPLv2
Software code languages, tools, and services used	OOP in C++
Computing platforms/Operating Systems	Linux, Microsoft Windows
Compilation requirements, operating environments & dependencies	None
Support email for questions	gorka.garate@ehu.eus

1. Introduction

Traditionally, the method for solving scattering problems have been FEM (Finite Element Method) [1–3] and BEM (Boundary Element Method) [4–6]. The great advantage of BEM over FEM is that, instead of the whole domain 3D mesh, only a 2D mesh at the boundary is needed, substantially reducing the size of the problem. Its main shortcomings are that they are computationally costly, that they present non-uniqueness of the solution at certain characteristic frequencies and, what is more important for the problems here posed, that they cannot handle systems with objects of different mass properties.

Another strategy is the use of IEM (Infinite Element Method) [7–9] or hybrid FEM/IEM, dividing the whole domain into two: a bounded one, meshed using classical 3D FEM, and the outer domain, meshed using specific 3D meshes (Mapped IEM [10,11], Wave Envelope IEM [12,13], etc.). The advantage is that the techniques of FEM can be applied (variational formulation, basis functions, numerical integration, etc.). Recently, Isogeometric Analysis has been coupled to IEM [14], and IEM still remains a field of research. IEM methods do have the capacity of dealing with systems with objects of different mass properties; but, whenever a new object is added, it must be included using a 3D finite mesh, increasing significantly the number of degrees of freedom of the global system to be solved. Another increase in the size of the system is due to the inclusion of additional degrees of freedom in the radial direction at the infinite elements.

Due to the imperative of reducing the global size of the system, several methods have followed the strategy of truncating the domain and including artificial Absorbing Boundary Conditions (ABC) [15–19], Non-Reflecting Boundary Conditions (NRBC) [20] or Perfectly Matched Layers (PML) [21,22]. PMLs are typically

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https://doi.org/10.1016/j.softx.2022.101111

formulated in Cartesian, cylindrical, spherical and recently ellipsoidal geometries and coupled to FE [23,24]. Although all three methods do reduce the computational cost and storage requirements, all of them lack inherently the capacity of calculating pressure in the domain outside the artificial boundary or layer.

More recently, Scaled Boundary Finite Element Method (SBFEM) has been applied to 3D unbounded vibro-acoustic domains [25–27]. To deal with unbounded domains, the whole domain is usually split into a bounded and a unbounded domain, but both domains are coupling the domains via the nodal flux on a outside boundary that is necessarily spherical. The method [28] is able to deal with complex geometries, but still, even if for the outer unbounded domain 2D elements can used, the inner bounded domain must be meshed using 3D elements.

The work here presented is the sequel of a previous work [29]. A collocation method using function series is presented. The main advantages of the method are:

- Only 2D meshes at the boundaries of the objects are needed, and it uses fewer degrees of freedom that other methods.
- It is able to solve Helmholtz equation in bounded and unbounded domains containing objects not only with different properties, but with different formulations (series expansions) as well.

The method is based on classical theoretical acoustics [30,31]. In sound radiation, when theoretical physics analyzes the case of radiation from spheres, it studies sound waves caused by the vibration pattern of a sphere. The radiated pressure is expressed as a series of Hankel and Legendre functions; the coefficients of the series are determined using the known radiation pattern on the sphere. Some researchers have already been developing models to obtain the exact solutions of the scattering by elastic spheres [32].

ADITU implements a numerical method for the calculation of sound pressure levels in bounded and unbounded fluid domains. It is an Object-Oriented C++ software that stores objects as ASCII text files. Each `ObjektuPuntu` class object is generated from an individual file with a list of points and properties generated by the pre/postprocessor GiD.[1] The software is split into two sequential applications:

- The first one (`aurre.cpp`) takes as input several files of extension `pun` generated by preprocessor GiD corresponding the coupled fluid objects of the model, and it creates and stores an `ObjektuPuntu` class object for each file. After the calculations its output consists of an `ObjektuAdi` class object for each of the fluid objects. These `ObjektuAdi` class objects contain the result for each object as a series expansion of the solution.
- The second one (`ondoren.cpp`) takes as input the `ObjektuAdi` class objects generated and generates the solution based on series expansions. Its output consists of a text file of extension `flavia.res` that can be directly postprocessed by GiD.

All headers with the class definitions as well as examples are provided within the code at directory [33].

The main contribution of the method presented here is that it splits the entire domain into subdomains (called objects) and proposes a different type of series expansion for the acoustic pressure in each of the objects. The method then imposes the fulfillment of the Boundary Conditions (BC) at the boundary of each of the objects and enforces the Coupling Conditions (CC) of the acoustic pressure at the interfaces between objects. The resulting code presented in this work can be found at this web repository[2]:

1 https://www.gidhome.com.

2 https://github.com/GorkaGarate01/Helmholtz-Series-Formulation.

The paper is structured as follows. Section 2 shows the proposed solution of Helmholtz equation for bounded and unbounded domains. The method takes as input two types of data:

- A set of n_{obj} objects, which are domains in space, bounded (finite) or unbounded (infinite), together with the specific physical properties of the object. In the case here presented of homogeneous, isotropic fluids the specific physical properties are wave speed c and density ρ.
- the explicit BCs at the boundaries of the objects and the CCs among the set of objects of the model.

Section 3 contrasts the numerical results obtained by the model with two benchmark analytical results: the radiation of a pulsating sphere and the scattering of a plane wave that strikes a rigid sphere.

Finally, Section 4 enumerates the main conclusions and the future lines of work of the authors.

2. Analytical solution in fluid domains

When acoustic waves propagate in a homogeneous, isotropic fluid, the classical general analytical solution of the homogeneous Helmholtz equation in spherical coordinates [31] is based on a separation of variables r, θ and φ. The general solution is

$$\begin{aligned} p(r,\theta,\varphi) = & \sum_{m=0}^{\infty} H_m^1(k,r) \sum_{n=0}^{m} P_m^n(\cos\theta) \\ & \times \left(a_{mn}^1 \cos(n\varphi) + b_{mn}^1 \sin(n\varphi)\right) \\ & + \sum_{m=0}^{\infty} H_m^2(k,r) \sum_{n=0}^{m} P_m^n(\cos\theta) \\ & \times \left(a_{mn}^2 \cos(n\varphi) + b_{mn}^2 \sin(n\varphi)\right), \end{aligned} \tag{1}$$

where H_m^1 stands for the spherical Hankel function of the first kind and H_m^2 stands for the spherical Hankel function of the second kind (their arguments, (k, r), have been dropped for the sake of brevity), and P_m^n stands for the nth derivative of the Legendre polynomial of degree m (its argument, $\cos\theta$, has been dropped for the sake of brevity). a_{mn}^1, a_{mn}^2, b_{mn}^1 and b_{mn}^2 are coefficients to be determined.

2.1. Formulation for bounded fluid objects

The Boundary-Value problem that must be solved for the bounded fluid object i is

$$\begin{cases} \nabla^2 p^i + k^2 p^i = 0 & \text{(a)} \\ p^i = g \text{ at } \Gamma_g & \text{(b)} \\ p^i_{,j} n_j = h \text{ at } \Gamma_h & \text{(c)} \\ p^i = p^l \text{ at } \Gamma_{il} & \text{(d)} \\ p^i_{,j} = p^l_{,j} \text{ at } \Gamma_{il}, & \text{(e)} \end{cases} \tag{2}$$

where (2a) is the governing Helmholtz equation inside the domain Ω, (2b) stands for the enforcement of the essential BCs applied on the part of the boundary Γ_g, (2c) stands for the enforcement of the natural BCs applied on the part of the boundary Γ_h and (2d) and (2e) stand for the enforcement of the CC conditions in the part of the boundary Γ_{kl} common to objects i and l (C^1 continuity). Index l is to be extended to d objects coupled to object i. That is, $\Gamma = \Gamma_g \cup \Gamma_h \cup \sum_{l=1}^{r} \Gamma_{il}$. The numerical solution proposed for the acoustic pressure in the bounded fluid object i uses pattern (1) to build for each object a series of functions such as

$$\begin{aligned} p^i = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \{ & \left[H_m^1 P_m^n \cos n\varphi\right] a_{mn}^1 + \left[H_m^2 P_m^n \cos n\varphi\right] a_{mn}^2 + \\ & \left[H_m^1 P_m^n \sin n\varphi\right] b_{mn}^1 + \left[H_m^2 P_m^n \sin n\varphi\right] b_{mn}^2 \}, \end{aligned} \tag{3}$$

which fulfills automatically Helmholtz Eq. (2a). Partial derivatives of the acoustic pressure can be calculated differentiating series (3):

Series (3) and their partial derivatives are used to enforce the specific BCs (2b) and (2c) and the CCs (2d) and (2e) on the boundary of each bounded fluid object.

2.2. Formulation for unbounded fluid objects

Similarly as in bounded fluid objects, the Boundary Differential problem that must be solved for the unbounded fluid object i is

$$\begin{cases} \nabla^2 p + k^2 p = 0 & \text{(a)} \\ \left| \dfrac{\partial p}{\partial \vec{n}} - \mathbf{i}kp \right| = o\left(\dfrac{1}{r^2}\right) & \text{(b)} \\ p^i = g \text{ at } \Gamma_g & \text{(c)} \\ p^i_{,j} n_j = h \text{ at } \Gamma_h & \text{(d)} \\ p^i = p^l \text{ at } \Gamma_{il} & \text{(e)} \\ p^i_{,j} = p^l_{,j} \text{ at } \Gamma_{il}. & \text{(f)} \end{cases} \tag{4}$$

In the case of unbounded objects, the only difference with bounded objects is the Sommerfeld radiation condition (4b), which removes the Hankel functions of the second kind. Therefore, the solution proposed for the acoustic pressure in the unbounded object i is similar to (3), using in the function series only the Hankel functions of the first kind.

$$p^i = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \left[H^1_m P^n_m \cos n\varphi \right] a_{mn} + \left[H^1_m P^n_m \sin n\varphi \right] b_{mn}, \tag{5}$$

which fulfills automatically Helmholtz Eq. (4a) and Sommerfeld radiation condition (4b). The partial derivatives of the acoustic pressure can be calculated differentiating series (5):

Series (5) and their derivatives are used to enforce the specific BCs (4c) and (4d) and the CCs (4e) and (4f) on the boundary of each unbounded fluid object. Post-processing of acoustic pressure (i.e. intensity, directivity, etc.) at any point inside the corresponding object can be done calculating series (3), (5) and their partial derivatives using the calculated coefficients a^1_{mn}, a^2_{mn}, b^1_{mn} and b^2_{mn} of the bounded objects and a_{mn} and b_{mn} of the unbounded objects.

2.3. System of equations

The particularity of the method lies on being partly analytic. The analytic part lies in the fact that any combination of the series (3) or (5) fulfills automatically Helmholtz Eq. (2a) or (4a). Therefore, the only system of equations that must be solved is the one generated by the enforcement of the BCs and the CCs for all objects of the model. This enforcement leads eventually to a complex, non-symmetric and linear global system of equations $A\mathbf{x} = \mathbf{c}$ of size s, where s is the total number of collocation points at which BCs and CCs are applied. The number of collocation points is considerably greater than the number of coefficients ($s \gg nc$). The vector of nc unknowns, $\mathbf{x}$, contains the coefficients for the whole set of objects of the model, a^1_{mn}, a^2_{mn}, b^1_{mn} and b^2_{mn} for the bounded finite objects and a_{mn} and b_{mn} for the unbounded fluid objects.

As each column of the matrix of the system of equations contains the values of a specific function assigned to a specific coefficient, the iterative technique of resolution takes the following steps:

Previously, a minimum acceptable residual error

$$\mathbf{norm}\left[\mathbf{c} - A\mathbf{x}\right] / \mathbf{norm}\left[\mathbf{c}\right] \tag{6}$$

is preset, and the iteration variable h is set to 0. h stands for the harmonic at which all series will be truncated (m in Eqs. (3) and (5)).

- At each iteration h an $s \times nq$ rectangular submatrix A_{nq} is calculated. s stands for the total number of collocation points for all the BCs and CCs for all objects of the whole model. Each row represents either the enforcement of one of the BCs at the collocation point on the boundary of one of the objects, either the enforcement of one of the CCs between two of the objects of the system at the collocation point on the boundary common to both. nq stands for the total number of terms (coefficients) of series (3) and (5) up to spherical harmonic of truncation h, corresponding to all the objects of the model.
- The overdetermined rectangular complex non-symmetric system $A_{nq}\mathbf{x} \approx \mathbf{c}$ is solved by the (discrete) least squares method, using an direct LU decomposition with partial pivoting in the square complex symmetric system $\left(A^T_{nq} A_{nq}\right) \mathbf{x}_{nq} = A^T_{nq}\mathbf{c}$.
- The residual (6) for solution $\mathbf{x}_{nq}$ is calculated.
- The harmonic of truncation h is increased by one and the iteration continues until the residual error (6) is smaller than the acceptable minimum preset, until the velocity of convergence is smaller than a given minimum or until nq reaches size s of the whole system, whichever happens first.

After having solved the global system for the coefficients a^1_{mn}, a^2_{mn}, b^1_{mn}, b^2_{mn} (for the bounded fluid objects) and a_{mn} and b_{mn} (for the unbounded fluid objects), each set of coefficients is stored separately for each object of the model for further postprocessing.

3. Numerical experiments

Two theoretical benchmark models have been used to test the accuracy of the method: one of radiation, the simple pulsating sphere, and another of scattering, the scattering of a plane wave that encounters on its way a rigid sphere.

3.1. Simple pulsating sphere

In the case of a simple pulsating sphere, the particle velocity in a fluid of density ρ at the surface of the source is equal to the velocity of the sphere at its boundary. Taking a sphere of radius r_0 and a boundary velocity $u_0 \exp \mathbf{i}\omega t$ the boundary condition is

$$\frac{\partial p}{\partial r} = \rho u_0 \exp\left(\mathbf{i}\omega t\right). \tag{7}$$

3.1.1. Analytical solution

Establishing the time dependence of the boundary velocity of the sphere as $\exp \mathbf{i}\omega t$ and using the general solution for an outgoing wave, the pattern of the acoustic pressure in the fluid is

$$p = -\frac{\rho \mathbf{i}\omega A}{r} \exp\left(\mathbf{i}\left(\omega t + kr\right)\right), \tag{8}$$

where $k = \omega/c$ is the wave number in the fluid. Applying boundary condition (7) yields

$$A = \frac{u_0 r_0^2 \exp\left(-\mathbf{i}kr_0\right)}{1 - \mathbf{i}kr_0}. \tag{9}$$

For the results of the present analytical model we will set $\mathbf{i}\rho\omega u_0 = 1$. This yields the following pressure in the radial direction for $r \geq r_0$:

$$p\left(r, t\right) = -\frac{r_0^2 \exp\left(-\mathbf{i}kr_0\right)}{r\left(1 - \mathbf{i}kr_0\right)} \exp\left(\mathbf{i}\left(\omega t + kr\right)\right). \tag{10}$$

3.1.2. Numerical models used

To solve the simple pulsating sphere problem two different experimental models have been used. The first one contains one object of air of infinite fluid type (in Section 2.2) whose domain is the outside of a sphere of radius 1 m. To apply Neumann BC (7) at the sphere $r = 1$ m 4050 points on the sphere have been chosen. Translating the center of the spherical coordinate system to point (0.3,0,0), we can test the ability of the method to solve problems posed on domains *without* spherical symmetry comparing the numerical results with the analytical results of Section 3.1.

The second experimental model consists in a coupled model with two objects of air of different types: one of finite fluid type (in Section 2.1), whose domain is the space between an inner sphere of radius 1 m and an outer sphere of radius 1.1 m, and a second object of infinite fluid type (in Section 2.2) whose domain is the outside of a sphere of radius 1.1 m. A total of 8100 points have been chosen to apply the BCs, 4050 points at the inner surface and other 4050 points at the outer surface. Besides Neumann BC (7) at the sphere $r = 1$ m, at the coupling sphere $r = 1.1$ m equality of acoustic pressures and gradients of the acoustic pressure in the two objects has been enforced. Again, in order to check the ability of the method to solve problems posed on domains *without* spherical symmetry, the center of the spherical coordinate system is translated to the point (0.3,0,0) for the first object and to point (0,0.3,0). Thus, we can test comparing the numerical results with the analytical results of Section 3.1.

3.1.3. Results obtained

For the first model with a single object, frequencies ranging from 16 Hz ($k = 0.29$) to 2 kHz ($k = 36.62$) have been tested. Compared to the analytical solution, the model has attained an maximum relative error of 0.04% at frequency 1 kHz. Fig. 1 displays the relative errors obtained in percentage for the SPL (Sound Pressure Level) in decibel scale (dB) at frequencies 1 kHz and 2 kHz, respectively.

For the second model with two objects coupled to each other, the range of frequencies from 16 Hz ($k = 0.29$) to 1 kHz ($k = 18.31$) has been tested. Compared to the analytical solution, the model has attained an maximum relative error of 1.87% at frequency 1 kHz. Fig. 2 displays the relative errors obtained in percentage for the SPL in dB at frequencies 500 Hz and 1 kHz, respectively.

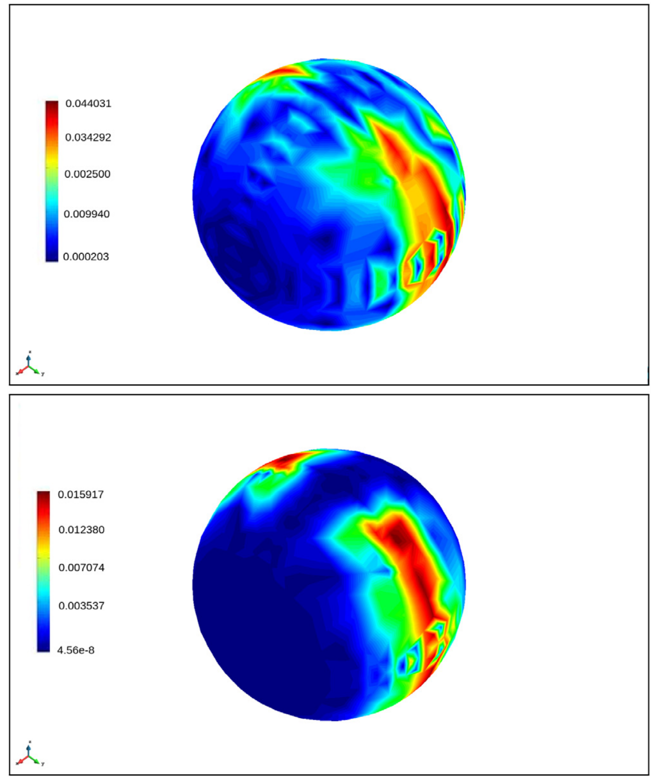

Fig. 1. Graphical representation of the relative errors in percentage obtained for the SPL (Sound Pressure Level) in decibel scale (dB) at frequencies 1 kHz and 2 kHz, respectively.

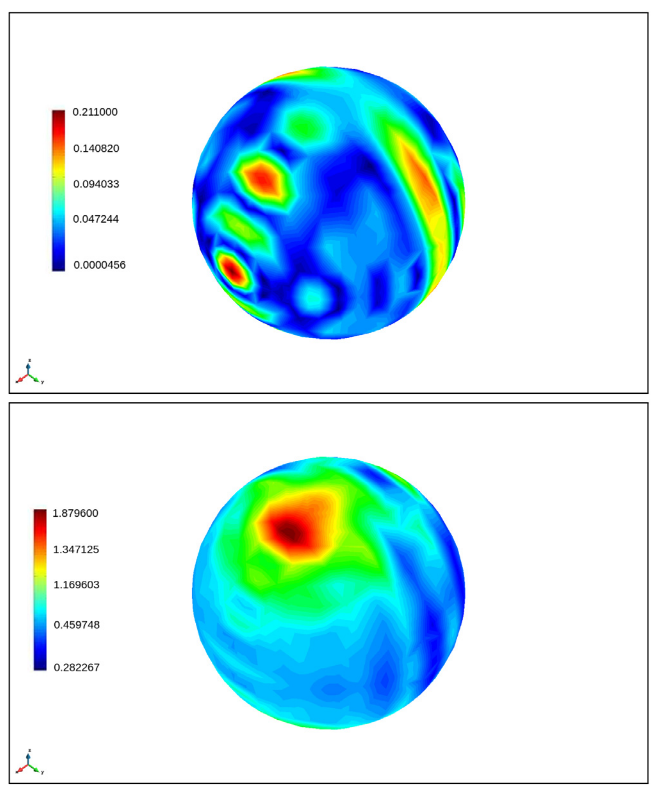

Fig. 2. Relative error (%) in pressure values for $f = 500$ Hz and $f = 1$ kHz.

3.2. Scattering from a rigid sphere

3.2.1. Analytical solution

In this section, the analytical model used to verify the results given bay the method is the scattering of an incident plane wave that encounters in its way a rigid sphere of radius 1. Supposing the plane wave to travel in the z-direction, the goal is to find the acoustic pressure of the scattered wave p^s.

For a plane wave of amplitude 1, the acoustic pressure is [31]

$$p^{inc}(r,\theta) = \sum_{m=0}^{\infty} (2m+1)\,\mathbf{i}^m P_m(\cos\theta)\, j_m(k,r), \tag{11}$$

where $P_m(\cos\theta)$ is the Legendre polynomial of degree m and $j_m(k,r)$ is the mth spherical Bessel function of the first kind.

The relation between the incident and the scattered wave is

$$-\frac{\partial p^s}{\partial r} = \frac{\partial p^{inc}}{\partial r} \text{ at } r = 1, \tag{12}$$

where

$$\begin{aligned} \left.\frac{\partial p^{inc}}{\partial r}\right|_{r=1} &= \sum_{m=0}^{\infty} (2m+1)\,\mathbf{i}^m P_m(\cos\theta) \left.\frac{\partial j_m(k,r)}{\partial r}\right|_{r=1} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} P_m^n(\cos\theta)\left(\widetilde{a}_{mn}\cos(n\varphi) + \widetilde{b}_{mn}\sin(n\varphi)\right). \end{aligned} \tag{13}$$

Taking into account the Neumann boundary condition and the orthogonality properties that Legendre functions $P_m^n(\cos\theta)$, cos

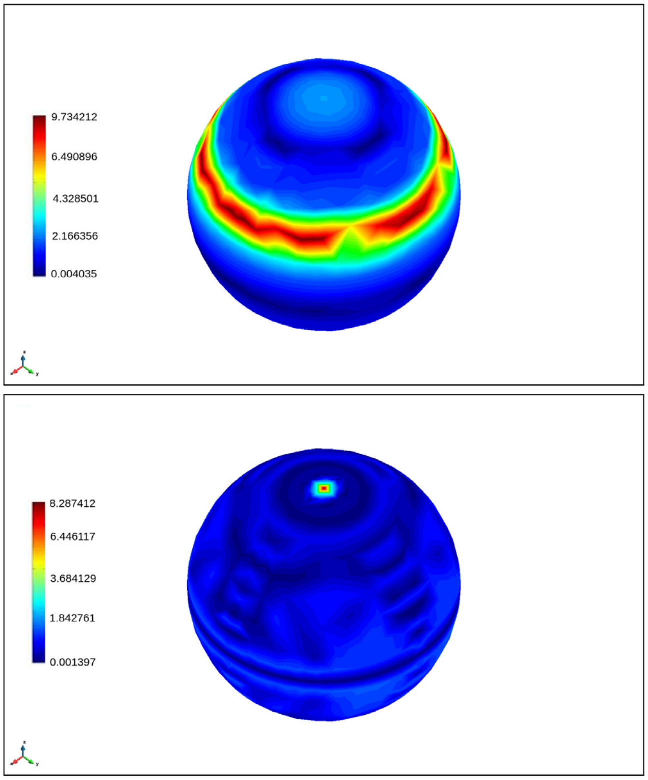

Fig. 3. Relative error (%) in pressure values for $f = 250$ Hz and $f = 1$ kHz.

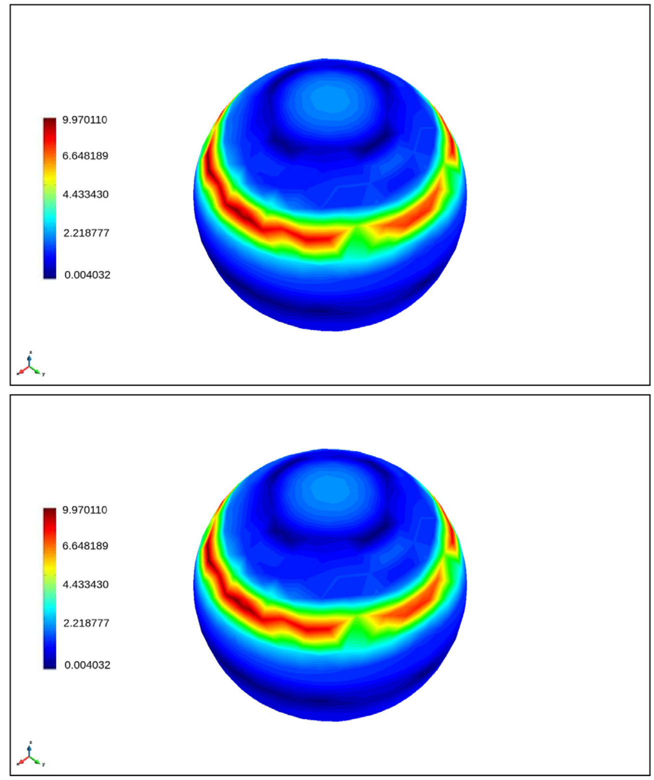

Fig. 4. Relative error (%) in pressure values for $f = 250$ Hz and $f = 1$ kHz.

and sin functions fulfill, the following values of the coefficients are obtained:

$$\widetilde{b}_{mn} = 0 \ \forall m, n$$
$$\widetilde{a}_{mn} = 0 \ \forall n \geq 1$$
$$\widetilde{a}_{m0} = \widetilde{a}_m = (2m+1)\,\mathbf{i}^m \left.\frac{\partial j_m(k,r)}{\partial r}\right|_{r=1} \quad \forall m \geq 1$$

Finally, recalling Eq. (1), we can write down the expression of acoustic pressure the scattered wave as

$$p^s(r,\theta) = \sum_{m=0}^{\infty} H_m^1(k,r)\, P_m(\cos\theta)\, a_m, \tag{14}$$

where

$$a_m = \frac{-(2m+1)\,\mathbf{i}^m \left.\frac{\partial j_m(k,r)}{\partial r}\right|_{r=1}}{\left.\frac{\partial H_m^1(k,r)}{\partial r}\right|_{r=1}} \ \forall m \geq 1.$$

3.2.2. Models used

The second check problem of scattering of an incident plane wave uses the same two models used in Section 3.1.2. The difference is that the origins of the spherical coordinate systems used are (0.2,0,0) for the model with an only object and (0.2,0,0) and (0,0.2,0), respectively, for the second model with two objects. In both cases the sums of Eqs. (11) and (14) have been truncated at $m = 15$.

3.2.3. Results obtained

For the first model with a single object, frequencies ranging from 16 Hz ($k = 0.29$) to 1 kHz ($k = 18.31$) have been tested. Fig. 3 displays the errors obtained in percentage for the SPL in dB at frequencies 250 Hz and 1 kHz. Compared to the analytical solution, the model has attained an maximum relative error of 9.73% at frequency 250 Hz.

For the second model with two objects coupled to each other, frequencies ranging from 16 Hz ($k = 0.29$) to 1 kHz ($k = 18.31$) have been tested. Fig. 4 displays the relative errors obtained in percentage for the SPL in dB at frequencies 250 Hz and 1 kHz. Compared to the analytical solution, the model has attained an maximum error of 9.55% at frequency 250 Hz.

4. Conclusions

This paper discusses a meshless global collocation formulation for the acoustic fluid–fluid coupling intended to solve problems of radiation and scattering in bounded and unbounded domains.

The main outcome of the study here presented is that the patterns of solutions proposed fulfill automatically Helmholtz equation and all BC for bounded and unbounded domains, including Sommerfeld BC. This means that all is reduced to the task of solving a linear system of equations.

Furthermore, the following observations are made:

- The only pre-processing needed is a classified list of points lying on the surfaces on the boundaries of the objects of the model; therefore, no 3D meshing is needed, and the number of degrees of freedom is substantially reduced, which results in a lower computational cost than that of element-based methods. The presented two formulations work well on complex acoustic scattering models at high frequencies using few degrees of freedom.
- In the examples presented, for the highest frequency analyzed, $k = 18.31$, corresponding to a wavelength of 0.343 m, the method has been able to obtain good results using a minimum of 4 degrees of freedom per wavelength.

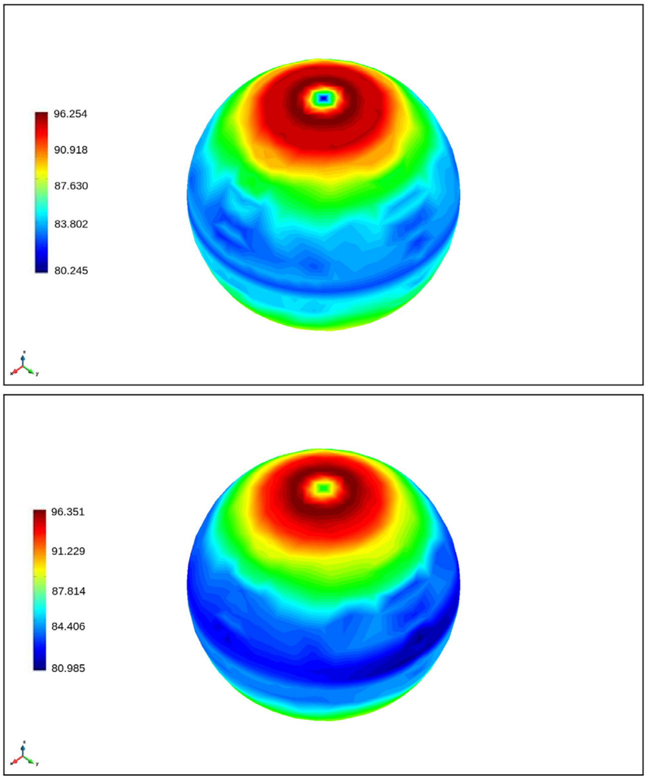

Fig. 5. Exact and calculated modulus for $f = 1$ kHz.

- The maximum relative errors take place at points where the value of acoustic pressure is small, and the patterns of modulus and phase are very similar, even for the example of scattering from a rigid sphere (see Fig. 5).

The main open issues of the method are, first, an analysis of the error propagation in the outer domain for complex patterns of acoustic pressure; second, a faster resolution of the rectangular complex, dense and non-symmetrical system of equations. The authors are already working on the former, and the latter should be explored in future research.

Another line of work of the authors is to extend the method here proposed to propagation of elastic waves in other media, such as structures.

5. Impact overview

- The method is an alternative to Finite Element Method, with the advantage that it uses fewer degrees of freedom and therefore is less hardware-demanding.
- The premises of the work are, first, to minimize the size of the system using the fewer degrees of freedom as possible; second, to be able to solve Helmholtz equation in bounded and unbounded domains containing objects with different mass properties (different fluids such as air, water, etc.).
- The software is a further development of the previous work published [29].

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Gorka Garate reports article publishing charges was provided by Ministry of Science and Innovation project PID2020-116346GB-I00.

Acknowledgment

Project partially funded by grant KK-2021/00070 of the Elkartek program of the Basque Government, Spain.

References

[1] Gómez-Revuelto I, García-Castillo LE, Salazar-Palma M, Sarkar Tapan K. Fully coupled hybrid method FEM/High-Frequency technique for the analysis of radiation and scattering problems. Microw Opt Technol Lett 2005;47:104–7. http://dx.doi.org/10.1002/mop.21094.

[2] Wu Haijun. Pre-asymptotic error analysis of CIP-fem and FEM for the Helmholtz equation with high wave number. Part I: Linear version. IMA J Numer Anal 2014;34:1266–88. http://dx.doi.org/10.1093/imanum/drt033.

[3] Zhang LP, Li ZC, Huang HT, Wei Y. The modified method of fundamental solutions for exterior problems of the Helmholtz equation; spurious eigenvalues and their removals. Appl Numer Math 2019;145:236–60.

[4] Grigoriev MM, Dargush GF. A fast multi-level boundary element method for the Helmholtz equation. Comput Methods Appl Mech Engrg 2004;193:165–203. http://dx.doi.org/10.1016/j.cma.2003.09.004.

[5] Blyth MG, Pozridikis C. A comparative study of the boundary and finite element methods for the Helmholtz equation in two dimensions. Eng Anal Bound Elem 2007;31:35–49. http://dx.doi.org/10.1016/j.enganabound.2006.07.005.

[6] Ghangale Dhananjay, Colaço Aires, Costa Pedro Alves, Arco Robert. A methodology based on structural finite element method-boundary element method and acoustic boundary element method models in 2.5D for the prediction of reradiated noise in railway-induced ground-borne vibration problems. J Vib Acoust 2019;141(3). http://dx.doi.org/10.1115/1.4042518.

[7] Bettess P. Infinite elements. Sunderland, U.K.: Penshaw Press; 1992.

[8] Gerdes K, Demkowicz L. Solution of 3D-laplace and Helmholtz equations in exterior domains using hp-infinite elements. Comput Methods Appl Mech Engrg 1996;137(1):239–73. http://dx.doi.org/10.1016/0045-7825(95)00987-6.

[9] Gerdes K. A summary of infinite element formulations for exterior Helmholtz problems. Comput Methods Appl Mech Engrg 1998;164:95–105. http://dx.doi.org/10.1016/S0045-7825(98)00048-6.

[10] Cipolla J. Acoustic infinite elements with improved robustness. In: Proceedings of ISMA 2002: international conference on noise and vibration engineering, vol. 1–5. 2002.

[11] Li LX, Sun JS, Sakamoto H. A generalized infinite element for acoustic radiation. J Vib Acoust-Trans ASME 2005;127. http://dx.doi.org/10.1115/1.1855927.

[12] Cremers L, Fyfe KR. On the use of variable order infinite wave envelope elements for acoustic radiation and scattering. J Acoust Soc Am 1995;97(4). http://dx.doi.org/10.1121/1.411994.

[13] Autrique JC, Magoules F. Analysis of a conjugated infinite element method for acoustic scattering. Comput Struct 2007;85(9). http://dx.doi.org/10.1016/j.compstruc.2006.08.038.

[14] Venås Jon Vegard, Kvamsdal Trond, Jenserud Trond. Isogeometric analysis of acoustic scattering using infinite elements. Comput Methods Appl Mech Engrg 2018;335. http://dx.doi.org/10.1016/j.jsv.2017.08.006.

[15] Antoine X, Darbas M, Lu Y. An improved surface radiation condition for high-frequency acoustic scattering problems. Comput Methods Appl Mech Engrg 2006;Volume 195:4060–74. http://dx.doi.org/10.1016/j.cma.2005.07.010.

[16] Hagstrom T, Assaf MO, Givoli D. High-order local absorbing conditions for the wave equation: Extensions and improvements. J Comput Phys 2008;227:3322–57. http://dx.doi.org/10.1016/j.jcp.2007.11.040.

[17] Thirunavukkarasu S, Murthy N. Absorbing boundary conditions for time harmonic wave propagation in discretized domains. Comput Methods Appl Mech Engrg 2011;200:2483–97. http://dx.doi.org/10.1016/j.cma.2011.04.021.

[18] Takekawa Junichi, Mikada Hitoshi. An absorbing boundary condition for acoustic-wave propagation using a mesh-free method. Geophysics 2016;81. http://dx.doi.org/10.1190/geo2015-0315.1.

[19] Moreira Roger M, Cetale Santos Marco A, et al. Frequency-domain acoustic-wave modeling with hybrid absorbing boundary conditions. Geophysics 2016;79. http://dx.doi.org/10.1190/geo2014-0085.1.

[20] Falletta S, Monegato G, Scuderi L. On the discretization and application of two space-time boundary integral equations for 3D wave propagation problems in unbounded domains. Appl Numer Math 2018;124. http://dx.doi.org/10.1016/j.apnum.2017.10.001.

[21] Shirron JJ, Giddings TE. A finite element model for acoustic scattering from objects near a fluid-fluid interface. Comput Methods Appl Mech Engrg 2006;196:279–88. http://dx.doi.org/10.1016/j.cma.2006.07.009.

[22] Jiang X, Weiying Z. Adaptive perfectly matched layer method for multiple scattering problems. Comput Methods Appl Mech Engrg 2012;201–204:42–52. http://dx.doi.org/10.1016/j.cma.2011.09.013.

[23] Harari H, Albocher U. Studies of FE/PML for exterior problems of time-harmonic elastic waves. Comput Methods Appl Mech Engrg 2006;195:3854–79. http://dx.doi.org/10.1016/j.cma.2005.01.024.

[24] Bunting Gregory, Prakash Arun, Walsh Timothy, Dohrmann Clark. Parallel ellipsoidal perfectly matched layers for acoustic Helmholtz problems on exterior domains. J Theor Comput Acoust 2018;26(2). http://dx.doi.org/10.1142/S2591728518500159.

[25] Chongmin S, Wolf JP. The scaled boundary finite element method: Analytical solution in frequency domain. Comput Methods Appl Mech Engrg 1998;164:249–64.

[26] Lehmann L, Langer S, Clasen D. Scaled boundary finite element method for acoustics. J Comput Acoust 2006;14. http://dx.doi.org/10.1142/S0218396X06003141.

[27] Li Boning, Cheng Liang, Deeks Andrew J, Zhao Ming. A semi-analytical solution method for two-dimensional Helmholtz equation. Appl Ocean Res 2006;28. http://dx.doi.org/10.1016/j.apor.2006.06.003.

[28] Liu Lei, Zhang Junqi, Song Chongmin, Birk Carolin, Gao Wei. An automatic approach for the acoustic analysis of three-dimensional bounded and unbounded domains by scaled boundary finite element method. Int J Mech Sci 2019;151. http://dx.doi.org/10.1016/j.ijmecsci.2018.12.018.

[29] Garate G, Vadillo EG, Santamaria J, Pardo D. Solution of 3D-Helmholtz equation in exterior domains using spherical harmonic decomposition. Comput Math Appl 2012;64:2520–43.

[30] Morse PM, Feshbach H. Methods of theoretical physics. McGraw-Hill Book Company; 1953.

[31] Morse PM, Ingard KU. Theoretical acoustics, vols. 1 and 2. McGraw-Hill Book Company; 1968.

[32] Venås Jon Vegard, Jenserud Trond. Exact 3D scattering solutions for spherical symmetric scatterers. J Sound Vib 2019;440. http://dx.doi.org/10.1016/j.jsv.2017.08.006.

[33] https://github.com/GorkaGarate01/Helmholtz-Series-Formulation.