CYCLICAL FEATURES OF UZAWA-LUCAS ENDOGENOUS GROWTH MODEL*

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Abstract

This paper analyzes the cyclical properties of a generalized version of Uzawa-Lucas endogenous growth model. We study the dynamic features of different cyclical components of this model characterized by a variety of decomposition methods. The decomposition methods considered can be classified in two groups. On the one hand, we consider three statistical filters: the Hodrick-Prescott filter, the Baxter-King filter and Gonzalo-Granger decomposition. On the other hand, we use four model-based decomposition methods. The latter decomposition procedures share the property that the cyclical components obtained by these methods preserve the log-linear approximation of the Euler-equation restrictions imposed by the agent's intertemporal optimization problem. The paper shows that both model dynamics and model performance substantially vary across decomposition methods. A parallel exercise is carried out with a standard real business cycle model. The results should help researchers to better understand the performance of Uzawa-Lucas model in relation to standard business cycle models under alternative definitions of the business cycle.

Key words: Endogenous growth, decomposition methods, cyclical features

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1 INTRODUCTION

Since Kydland and Prescott's (1982) influential paper, many articles in the real business cycle (RBC) literature tend to quantitatively evaluate model performance by comparing some relevant second moment statistics obtained from actual time series and synthetic data (that is, the data generated by the model). Since an aggregate time series typically involves both trend and cyclical components (usually, the seasonal components have been previously removed), RBC researchers firstly remove the trend component from the actual and synthetic data, using a decomposition method in order to carry out model evaluation. RBC researchers can be classified in two groups by the type of decomposition method they use. On the one hand, there is an important group of researchers that apply a statistical filter to extract the cyclical component from the time series data. This group consider model and filter as two basic pieces of the same measurement instrument used to analyze RBC issues (other important pieces of this instrument are the numerical algorithm used to solve the model and the selection of parameter values). On the other hand, some researchers (for instance, King, Plosser and Rebelo (1988 a,b), and King, Plosser, Stock and Watson (1991)) use model-based decompositions to study the cyclical features characterized by the model. This group of authors argues that the economic theory should specify what (it is that) constitutes a trend path, and hence a business cycle and, as a consequence, the detrended method used must be consistent with the growth theory proposed by the model studied. A model-based decomposition is therefore defined as a decomposition method having the property that, when applied to the synthetic time series obtained from a model, the cyclical components satisfy the rational expectations restrictions imposed by the agents' intertemporal optimization problem.[1,2]

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[1] Model-based decompositions are motivated by Koopmans (1947) and Singleton (1988) critique of prefiltering synthetic data when analyzing business cycles. In particular, Singleton (1988) argues that the common practice of prefiltering synthetic data will result, in general, in the violation of the rational expectations restrictions imposed by the agents' intertemporal optimization problem. This caveat might be quite disturbing, since RBC are usually viewed as the optimal responses of rational agents to stochastic shocks. In Gary Hansen's (1997, p.1005) words, "(RBC research program) *has been devoted to modeling the business cycle as an equilibrium outcome of optimizing agents responding to random changes in technology*". Of course, the reader will notice that there are also problems associated with model-based decompositions related to theoretical assumptions on the model and empirical information required to calibrate the model. Moreover, model-based decompositions are also subject to specification errors. Furthermore, certain model-based decompositions are model-dependent. Therefore, the use of these decompositions is, at least problematic, in a context where the researcher is interested in analyzing whether a particular model is better than other model in reproducing certain dynamic features observed in the data, since the different models' performance may be due to the different model-based decomposition considered in evaluating each model.

[2] Model-based decompositions are also of interest when applying GMM (Hansen (1982)) procedures to estimate aggregate models since the parameter estimates are found by minimizing a (usually quadratic) function of the orthogonality restrictions imposed by the rational expectations hypothesis. As has been emphasized by Singleton (1988), the common practice of using statistical methods to decompose time series prior to estimate leads to a violation of the moment restrictions implied by theory resulting in inconsistent estimates of the parameters.

This paper analyzes the dynamic features of an Uzawa-Lucas endogenous growth model (EG model) with human capital by using alternative decomposition methods. These dynamic features are compared with those displayed by the standard neoclassical growth model studied by King, Plosser and Rebelo (1988 a) (KPR model) under alternative decomposition methods. Uzawa-Lucas model is one of the most popular endogenous growth models. Versions of the Uzawa-Lucas framework have been adopted by many researchers aiming to explain business cycle features (Bean (1990)), long-term growth (Lucas (1988), Ladrón-de-Guevara, Ortigueira and Santos (1997)) and the money-growth relationship (Gomme (1993)). Our purpose is to provide a systematic account of the cyclical features exhibited by the EG model across several decomposition methods and examine quantitatively how model performance may vary depending on the cyclical component extracted by alternative decomposition procedures. Moreover, we want to analyze how good is the EG model in relation to the KPR model in characterizing cyclical patterns under alternative decomposition methods. As shown by Canova (1998), alternative detrending methods may provide quite different dynamic properties of macroeconomic variables, so a characterization of the different dynamic features of the EG model in relation to those exhibited by the KPR model under alternative decomposition methods will allow researchers to better understand the relative performance of these two models under alternative definitions of the business cycle.

In analyzing the EG model, we consider, on the one hand, three statistical filters: the filter suggested by Hodrick and Prescott (1980) (HP filter),[3] the Band-Pass (BP) filter proposed by Baxter and King (1995, 1999), and the multivariate decomposition suggested by Gonzalo and Granger (1995) (GG filter).[4] On the other hand, we use four model-based decomposition methods to study the EG model. Two of these model-based decompositions use a synthetic measure of human capital trend. The other two model-based decompositions consist, on the one hand, in the analysis of the great ratios and, on the other hand, in the analysis of the growth rates of non-stationary variables such as output, consumption and investment.

The analysis of the KPR model is based on three model-based decomposition methods and two statistical filters: the HP and BP filters.[5] A model-based decomposition consists in removing the linear deterministic trend postulated by the KPR model. As in the EG model, the other two model-based decompositions are based on the analyses of the great ratios and the growth rates of non-stationary variables, respectively.

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[3] See Kydland and Prescott (1990, p.8) for an exposition of HP filter properties.

[4] The GG decomposition is a reasonable filter to analyze the cyclical features of the EG model studied since the GG filter requires that the time series be difference stationary (that is, I(1) processes) and it imposes the existence of a common trend for all non-stationary variables. As shown below these two properties are consistent with the EG model: all non-stationary synthetic time series exhibit a unit root and a single common trend represented by human capital.

[5] As was pointed out above, the GG filter requires that the time series be difference stationary. Therefore, we do not use the GG filter when evaluating the KPR model since non-stationary synthetic time series obtained from this model are trend stationary.

The RBC literature implicitly assumes that the researcher does not have to worry about the specification of the trend when he is only interested in analyzing the component of the time series associated with certain business cycles frequencies. The EG model studied in this paper posits that human capital accumulation is the engine of growth and then it postulates a very different reason for non-stationarity that the standard neoclassical growth models studied in the RBC literature. The question that arises is whether including this reason is important for the features displayed by the model at alternative frequencies isolated by different decomposition methods.

The KPR model is an exogenous growth model. In the context of an exogenous growth model obtaining a model-based decomposition of observed time series data is quite simple, since it typically requires to compute the deviation from a common estimated linear trend of the log levels of all non-stationary variables of the model. However, in the context of an EG model deriving some model-based decompositions may be trickier; especially, if the engine of growth is a variable which is hard to observe, like, for example, human capital accumulation through non-market activities or learning-by-doing.

By using US post-war data, this paper shows that alternative decomposition methods lead to quite different model evaluation results either in the context of the EG model as in the KPR model. Moreover, the assessment of whether the EG model is closer to reproduce a particular cyclical feature than the KPR model many times depends on the decomposition considered. For instance, when using the BP filter one may conclude that the EG model is closer to reproduce the relative volatility between investment and output and the contemporaneus correlation between consumption and output than the KPR model. The opposite is true for the relative volatility between consumption and output, whereas the capability of the two models in reproducing the contemporaneus correlation between investment and output is similar. However, when using the HP filter the KPR model is much closer to reproduce the contemporaneus correlation between investment and output than the EG model. Other examples appears when one compares the results obtained from the great ratios and the growth rates. The analysis of the cyclical features of the consumption-output and investment-output *great ratios* and work effort shows that the EG model provides a better characterization than the KPR model in terms of the auto-correlation and cross-correlation structures of the two *great ratios* and hours observed in US data. However, the analysis of the growth rates does not show that the EG model always performs better than the KPR model. Thus, it is the case that the EG model captures remarkably well the volatility of the growth rates of output and investment whereas the KPR model is only close to capture the volatility of the output growth rate. The two models (KPR and EG model) fall short of reproducing the observed volatilities of the growth rates of consumption and work effort. Although the KPR model is closer to reproduce the volatility of consumption growth rate observed in actual data than the EG model. The opposite is true for the volatilities of the growth rates of work effort and investment.

The rest of the paper is organized as follows. Section 2 introduces an EG

model with human capital. Moreover, by using a standard calibration, the model is solved. In section 3, we propose alternative model-based decompositions which preserve the log-linear approximation of the Euler-equation restrictions imposed by the model. Section 4 presents a brief description of the KPR model. Section 5 compares model quantitative evaluation through alternative decomposition methods in order to show how the performance of the two models may change depending on the descomposition used to isolate the cyclical component. Moreover, the performance of the two models is compared under alternative decomposition methods. Section 6 extends the analysis of previous section to consider two other model-based decompositions. First, the analysis of the consumption-output and investment-output *great ratios.* Second, the analysis of the growth rates of output, consumption, investment and work effort. Finally, Section 7 concludes.

2 A GROWTH MODEL WITH HUMAN CAPITAL

We consider an EG model because in this type of models growth and cycles are determined endogenously. As pointed out by Singleton (1988), growth and cyclical components in the data may both be determined endogenously by optimal economic decision making. Therefore, the presumption underlying standard RBC models that trend components are determined by different factors from those causing business cycles may be misleading.

   In particular, this paper analyzes a stochastic discrete time version of the generalized Uzawa-Lucas framework. One of the modifications, used by King, Plosser and Rebelo (1988b), Bean (1990) and Gomme (1993), is that physical capital is included as an input of the human capital sector. The second modification is that leisure is assumed to have a positive effect in the agents' welfare. The economy is inhabited by a large number of identical households. The size of the population is assumed to be constant.

   The representative household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t h_t^\lambda), \tag{1}$$

where E_0 denotes the conditional expectation operator, $0 < \beta < 1$ is the discount factor, c_t is consumption, and $l_t h_t$ is qualified leisure, and this captures Becker(1965)'s idea that the utility of a given amount of leisure increases with the stock of human capital. In particular, we assume that the preferences of the representative household are described by the following utility function:

$$U(c_t, l_t h_t^\lambda) = \frac{[c_t^\omega (l_t h_t^\lambda)^{1-\omega}]^{1-\gamma} - 1}{1 - \gamma}, \tag{2}$$

where $0 \leq \omega \leq 1$, $\gamma > 0$ and $0 \leq \lambda \leq 1$. This type of utility function guarantees the existence of a balanced growth path for the economy, where the fraction

of time allocated to each activity remains constant and all per-capita variables grow at the same rate.[6]

There are two productive activities in this economy: the production of the final good (market sector) and the accumulation of human capital (human capital sector). At any point in time, a household has to decide what portion of its time is allocated to each of these activities, apart from the time allocated to leisure. The production function of the representative household is a neoclassical production function with constant returns to scale. Formally,

$$y_t = F^m(\phi_t k_t, n_t h_t, z_t) = A_m e^{z_t}(\phi_t k_t)^\alpha (n_t h_t)^{1-\alpha}, \tag{3}$$

where A_m is a technology parameter, ϕ_t is the fraction of physical capital stock allocated to the market sector, n_t is the fraction of time allocated to the market sector, h_t denotes the stock of human capital at the beginning of time t, α is the share of physical capital in final good production and z_t is a technology shock which follows a first-order autoregressive process $z_t = \rho z_{t-1} + \epsilon_t$, where $0 \leq \rho \leq 1$ and ϵ_t is a white noise with standard deviation σ_ϵ. The law of motion for physical capital is

$$k_{t+1} + c_t = A_m e^{z_t}(\phi_t k_t)^\alpha (n_t h_t)^{1-\alpha} + (1 - \delta_k)k_t, \tag{4}$$

where δ_k is the depreciation rate of physical capital.

The human capital sector is characterized as follows:

$$h_{t+1} = F^h[(1 - \phi_t)k_t, (1 - l_t - n_t)h_t] + (1 - \delta_h)h_t =$$

$$A_h[(1 - \phi_t)k_t]^\theta[(1 - l_t - n_t)h_t]^{1-\theta} + (1 - \delta_h)h_t, \tag{5}$$

where A_h is a technology parameter, θ is the share of physical capital stock in human capital production and δ_h is the rate of depreciation of human capital.

As is well known, the competitive equilibrium can alternatively be characterized through the first-order conditions derived from a benevolent social planner's problem in the absence of externalities and public goods. The social planner maximizes (1) subject to (4)-(5) with $k_0 > 0$ and $h_0 > 0$ given. The necessary and sufficient first-order conditions for an (interior) optimum are (for simplicity we drop all the time subscripts and a prime (') is used to denote next period values):

$$U_1 = \frac{U_2 h_t^{\lambda-1}}{F_2^m},$$

$$\frac{U_2 h_t^{\lambda-1}}{F_2^m} = \beta E_t\{\frac{U_2' h_{t+1}^{\lambda-1}}{F_2^{m'}}[F_1^{m'} + 1 - \delta_k]\},$$

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[6]See KPR (1988b, p.324) for an exposition of the restrictions one should impose on preferences in order to guarantee a constant growth rate in a steady state.

$$\frac{U_2 h_t^{\lambda-1}}{F_2^h} = \beta E_t \{ \frac{U_2' h_{t+1}^{\lambda-1}}{F_2^{h'}} [F_2^{h'}(1 - l_{t+1}) + 1 - \delta_h] + U_2' \lambda l_{t+1} h_{t+1}^{\lambda-1} \},$$

$$\frac{F_1^m}{F_2^m} = \frac{F_1^h}{F_2^h},$$

$$h_{t+1} = A_h [(1 - \phi_t) k_t]^\theta [(1 - l_t - n_t) h_t]^{1-\theta} + (1 - \delta_h) h_t,$$

$$k_{t+1} + c_t = A_m e^{z_t} (\phi_t k_t)^\alpha (n_t h_t)^{1-\alpha} + (1 - \delta_k) k_t,$$

$$\lim_{t \to \infty} E_t \beta^t U_1 k_{t+1} = 0,$$

$$\lim_{t \to \infty} E_t \beta^t U_2 \frac{h_t^{\lambda-1}}{F_2^h} h_{t+1} = 0.$$

In the steady state, the variables c_t, k_t and y_t grow at a constant rate which is equal to the rate of accumulation of human capital, and n_t, l_t and ϕ_t are constants. Therefore, the time series c_t, k_t and y_t obtained from these first-order conditions are non-stationary. In order to facilitate the use of computational techniques, it is convenient to rewrite the first-order conditions in terms of the ratios $\hat{c}_t = c_t/h_t$, $\hat{k}_t = k_t/h_t$, thus reducing the number of state variables:

$$U_1(\hat{c}_t, l_t) = \beta (\frac{h_{t+1}}{h_t})^\tau E_t \{ U_1(\hat{c}_{t+1}, l_{t+1}) [F_1^m(\phi_{t+1} \hat{k}_{t+1}, n_{t+1}) + 1 - \delta_k] \}, \qquad (6)$$

$$U_1(\hat{c}_t, l_t) = \frac{U_2(\hat{c}_t, l_t)}{F_2^m(\phi_t \hat{k}_t, n_t)}, \qquad (7)$$

$$\frac{U_2(\hat{c}_t, l_t)}{F_2^h[(1 - \phi_t)\hat{k}_t, 1 - l_t - n_t]} = \beta (\frac{h_{t+1}}{h_t})^\tau E_t \{ \frac{U_2(\hat{c}_{t+1}, l_{t+1})}{F_2^h[(1 - \phi_{t+1})\hat{k}_{t+1}, 1 - l_{t+1} - n_{t+1}]}$$

$$[F_2^{h'}(1 - l_{t+1}) + 1 - \delta_h + \lambda F_2^{h'} l_{t+1}] \}, \qquad (8)$$

$$\frac{F_1^m}{F_2^m} = \frac{F_1^h}{F_2^h}, \qquad (9)$$

$$\frac{h_{t+1}}{h_t} = A_h [(1 - \phi_t)\hat{k}_t]^\theta (1 - l_t - n_t)^{1-\theta} + 1 - \delta_h, \qquad (10)$$

$$\hat{c}_t + \hat{k}_{t+1} \frac{h_{t+1}}{h_t} = A_m e^{z_t} (\phi_t \hat{k}_t)^\alpha n_t^{1-\alpha} + (1 - \delta_k)\hat{k}_t, \qquad (11)$$

where $\tau = [\omega + \lambda(1 - \omega)](1 - \gamma) - 1$.

2.1 Uzawa-Lucas Model Calibration

Model calibration requires that values be assigned to the parameters in the model. Following Kydland and Prescott's (1982) seminal paper, we choose, on the one hand, the structural parameter values based on the existing empirical evidence obtained from micro data sets. On the other hand, we approach the steady state values of the variables by averaging the corresponding US time series. US time series are quarterly data for the period from the third quarter of 1955 to the first quarter of 1984. All series are per-capita and are described in detail in Christiano (1988).[7]

The derivation of reasonable values for the parameters describing household preferences follows standard procedures. The discount factor, β, is chosen so that the annual real interest rate is equal to 4%. The value for β is obtained from the following equation, which characterizes the steady state given the homogeneity properties of the utility function:

$$1.01 \ \beta (\frac{h_{t+1}}{h_t})^\tau = 1.$$

Mehra and Prescott (1985) establish that a reasonable value for the relative risk aversion parameter, σ, lies in the interval $[1, 2]$. We consider $\sigma = 1.3$. As shown by Barañano, Iza and Vázquez (2002), the numerical solutions obtained with $\sigma = 1.3$ are similar to those found when $\sigma = 2$ in a model which exhibits a weak propagation mechanism of the technology shocks (that is, when θ is close to zero). Since the utility function is multiplicatively separable we have that $U(c, lh^\lambda) = u(c)v(lh^\lambda)$, where $u(c)$ is homogeneous of degree $1 - \sigma$. Moreover, we follow the suggestion made by Gomme (1993) and Greenwood and Hercowitz (1991) that a reasonable value for the fraction of time allocated to the market sector is 0.24, and from this value we can derive reasonable parameter values for ω and γ using the homogeneity properties of the utility function. Finally, the choice of a parameter value for λ is not straightforward, because there is no empirical evidence. This paper considers $\lambda = 1$ (qualified leisure).[8] Looking at the market sector, the value of α is chosen so that it equals the average share of physical capital in the US GNP over the period ($\alpha = 0.36$). Since we are using quarterly data, the rate of depreciation for physical capital, δ_k, has been fixed at 0.025, which is equivalent to the 10% annual rate used by Kydland and Prescott (1982). The value for A_m is normalized to unity.

Based on first moments from the Solow residual, we follow Prescott's (1986) suggestion for ρ: $\rho = 0.95$. Moreover, the standard deviation of the innovation in the first-order autoregressive process for the technology shock, ϵ_t, is adjusted; in order that the standard deviation of per-capita US GNP be close to the standard

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[7] We restrict our sample to this period because as pointed out by McConnell and Perez-Quiros (2000) US business cycle has smoothed substantially since 1984. For instance, the standard deviation of quarterly US real GDP growth over the last seventeen years is less than half that of the rest of the post-war period. Moreover, the sample period considered allows us to compare the results found in this paper with those found in the RBC literature.

[8] As shown by Ladrón-de-Guevara, Ortigueira and Santos (1997), a value of $\lambda = 1$ guarantees the existence of a unique balanced growth path.

deviation of the synthetic output times series according to the decomposition procedures studied.

Since there is not enough empirical evidence to establish the parameter values characterizing the human capital sector, we have decided to choose parameter values in such a way that they guarantee reasonable values for steady state variables. In particular, A_h is chosen so that the growth rate of output in the steady state match the average annual growth rate of per capita US GNP, 1.4%. Moreover, we choose $\theta = 0.05$, which implies a weak internal propagation mechanism. A small value of θ is needed to mimic the cyclical features (characterized by the HP filter) displayed by standard RBC models. We focus our attention on an EG model exhibiting a weak propagation mechanism, since this is the basic assumption needed in this type of models to reproduce similar cyclical features as those implied by standard RBC models at the frequencies isolated by the HP filter. We impose this restriction in order to convince the reader that the different model evaluation results obtained in this paper using alternative decomposition methods are not driven by a model which exhibits unusual cyclical dynamics at the frequencies extracted by the HP filter. Moreover, as pointed out by Barañano, Iza and Vázquez (2002), the analysis of the EG model with a stronger propagation mechanism (induced by higher values of θ and σ) requires the use of alternative solution methods, such as parameterized expectations methods. The reason is that a log-linear solution method, as the one implemented in this paper, removes most of the nonlinearities induced by a stronger propagation mechanism. Therefore, the study of an EG model with a stronger propagation mechanism than the EG model studied in this paper would introduce an additional relevant element in the discussion (namely, the choice of the numerical solution method), but then it would be more difficult to obtain a clear cut of how sensitive are the cyclical patterns to alternative decomposition methods.[9]

2.2   Uhlig's Log-Linear Method (LLM)

In a recent paper, Uhlig (1999) proposes a simple log-linear method to solve for the dynamics of nonlinear stochastic dynamic general equilibrium models. Uhlig's method follows directly from King, Plosser and Rebelo (1988a,b) and Campbell (1994) among others. This method is used to derive the numerical solutions of the model considered in this paper.[10] Uhlig's solution method can easily be summarized in the following four steps:

Step 1: Obtain the necessary conditions that characterize the equilibrium.
Step 2: Choose the parameter values of the model and find steady state values.

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[9]Barañano, Iza and Vázquez (2002) provide an analysis of how sensitive are the cyclical patterns displayed by the EG model to some calibrated key parameters such as θ and σ.

[10]In a recent paper Barañano, Iza and Vázquez (2001) show that the second moment statistics used to evaluate model performance of the Uzawa-Lucas model that exhibits a weak internal propagation mechanism are very similar when either PEA method (den Haan and Marcet (1990)) or Uhlig's log-linear method are implemented. We carry out Uhlig's method because is much simpler than PEA method.

Step 3: Log-linearize the first-order conditions, which characterize the equilibrium of the model, in order to make all the equations approximately linear in the log-deviations from the steady state.

Step 4: Solve for the recursive equilibrium law of motion by using the method of undetermined coefficients suggested by Uhlig (1999) which is simple and of general applicability (that is, it can be implemented in models where there are more endogenous state variables than expectational equations).

Steps 1-3 involve many tedious, though simple, computations, as shown in the Appendix 1. After Step 3, it is convenient to rewrite the system of log-linearized first-order conditions in matrix form. Then, using Uhlig's notation, we have a matrix system where there is one vector of endogenous state variables x_t (size mx1), another vector containing other endogenous variables y_t (size nx1) and a third vector of exogenous stochastic variables z_t (size kx1):

$$Ax_t + Bx_{t-1} + Cy_t + Dz_t = 0, \tag{12}$$

$$E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] = 0, \tag{13}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}, \tag{14}$$

where $E_t(\epsilon_{t+1}) = 0$. It is assumed that C is of size lxn, $l \geq n$ and of rank n, l is the number of deterministic equations (i.e., the number of equations involved in (12)), F is of size $(m + n - l)$xm, and N has only stable eigenvalues. In the EG model, we have two endogenous state variables: the log-deviations from the steady-state values of $\hat{k}_t$ and $\frac{h_{t+1}}{h_t}$, respectively;[11] one exogenous variable z_t and four other (non-state) endogenous variables: the log-deviations from the steady-state values of consumption-human capital ratio, $\hat{c}_t$, the fraction of capital stock allocated to the market sector, ϕ_t, the fraction of time allocated to the market sector production, n_t, and leisure, l_t, respectively.

The log-linear solution method seeks a recursive equilibrium law of motion of the following form:

$$x_t = Px_{t-1} + Qz_t, \tag{15}$$

$$y_t = Rx_{t-1} + Sz_t, \tag{16}$$

that is, finding P, Q, R and S so that the equilibrium described by these rules is stable. In our case, $l = n$, then (see Corollary 1 of Uhlig (1999))

(i) P satisfies the following quadratic equation:

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0.$$

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[11] h_{t+1}/h_t is considered as a state variable in order to satisfy the requirement that $l \geq n$ imposed by the log-linear approximation, although it is not a state variable in the proper sense because it does not appear in the policy rules (see matrices P and R below).

Solving for P in this equation requires the use of Theorem 2 in Uhlig (1999).

(ii) R is given by

$$R = -C^{-1}(AP + B).$$

(iii) Q satisfies

$$vec(Q) = (N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A))^{-1}$$

$$vec((JC^{-1}D - L)N + KC^{-1}D - M),$$

where $vec(.)$ denotes columnwise vectorization.

(iv) S is given by

$$S = -C^{-1}(AQ + D).$$

The matrices P, R, Q and S for the model considered and the value parameters chosen are given by

$$P = \begin{pmatrix} 0.92993734 & 0 \\ 0.00592555 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0.43732421 & 0 \\ -0.21876559 & 0 \\ -0.74325643 & 0 \\ -0.11149249 & 0 \end{pmatrix},$$

$$Q' = (0.18186498, -0.01323460),$$

$$S' = (0.43001337, 0.58667992, 2.00752300, -0.05848311).$$

3 A CHARACTERIZATION OF TREND AND CYCLICAL COMPONENTS

In this subsection, we characterize the trend and cyclical components of the time series generated by the log-linear approximation of the EG model. This characterization allows us to decompose the synthetic time series in two components; trend and cycle, where the cycle component satisfies the log-linear approximation of the Euler-equation restrictions imposed by the agent's intertemporal optimization problem.

Let w_t denote a non-stationary variable of the model (for instance, output, consumption and investment).[12] We have derived above the laws of motion

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[12] From now on we shall mainly focus on these three variables as it is a common belief among RBC theorists that standard RBC models do a good job in characterizing the dynamics of these variables.

of the log-deviation from the steady-state values of output-human capital ratio, consumption-human capital ratio and investment-human capital ratio, respectively. It is therefore straightforward to obtain the log levels of output, consumption and investment since

$$log\ w_t = log\ \hat{w}_t + log\ h_t, \tag{17}$$

where $\hat{w}_t = w_t/h_t$. Notice that

$$\frac{h_{t+1}}{h_t} = \bar{HH}e^{hh_t} \approx \bar{HH}(1 + hh_t), \tag{18}$$

where $\bar{HH}$ is the steady state value of h_{t+1}/h_t and

$$hh_t = log\ \frac{h_{t+1}}{h_t} - log\ \bar{HH}.$$

From Uhlig's log-linear solution (equation (15)), we have obtained the laws of motion of the log-deviations from the steady-state values of $\hat{k}_{t+1}$ and h_{t+1}/h_t (denoted by $\hat{kk}_{t+1}$ and hh_t, respectively)

$$hh_t = p_{21}\hat{kk}_t + q_{21}z_t, \tag{19}$$

$$\hat{kk}_{t+1} = p_{11}\hat{kk}_t + q_{11}z_t, \tag{20}$$

where p_{ij} and q_{ij} denote the generic elements of matrices P and Q, respectively. As mentioned above, the log-linear solution of this model imposes that $p_{12} = p_{22} = 0$. Taking the (natural) logs in equation (18) we have that

$$log\ h_{t+1} = log\ h_t + log\ \bar{HH} + hh_t. \tag{21}$$

Using equations (19), (20) and as z_t follows a first-order autoregressive process, equation (21) can be written as

$$log\ h_{t+1} = log\ h_t + log\ \bar{HH} + p_{21}p_{11}\hat{kk}_{t-1} + (p_{21}q_{11} + q_{21}\rho)z_{t-1} + q_{21}\epsilon_t. \tag{22}$$

Equation (22) clearly shows that the synthetic time series of the log level of human capital has a unit root with drift. Therefore, from equation (17), the synthetic time series of the logs of output, consumption and investment have also a unit root. This feature of the model is consistent with the overwhelming evidence (stimulated by Nelson and Plosser (1982)) of the presence of unit roots in aggregate time series. Moreover, the synthetic time series of output, consumption and investment have a single common trend (namely, the log of human capital) and are cointegrated. Given these properties of the synthetic non-stationary variables, w_t, a model-based decomposition is obtained by eliminating the unit-root and the deterministic trend component, $log\ \bar{HH}$, from the synthetic non-stationary variables. The cyclical component of w_t obtained from

this model-based decomposition (denoted by $\tilde{w}$) is then given by the following expression:

$$log\ \tilde{w}_t = log\ w_t - log\ h_{t-1} - log\ \bar{HH}. \tag{23}$$

Notice that this model-based decomposition eliminates the unit-root and the deterministic trend component from the synthetic time series, but leaves untouched, according to equation (21), the cyclical behavior of a non-stationary time series induced by the cyclical component of human capital defined by hh_t.

Other model-based decomposition of any non-stationary variable ω_t is the one given by equation (17). One difference between these two model-based decompositions is that the cyclical component of any non-stationary time series obtained from the former decomposition (given by (23)) includes the cyclical component of human capital whereas the later decomposition (given by (17)) removes the cyclical component of human capital. Formally, $log\ \tilde{w}_t = log\ \hat{w}_t + hh_{t-1}$. Another difference is that decomposition (23) implies that the trend component is predetermined which results in that current shocks do not affect the current trend component. In spite of these qualitative differences, the two decompositions quantitatively provide, as shown below, almost identical cyclical properties.

Implementing any of these two model-based decompositions to removing the trend component from synthetic data is straightforward since a synthetic time series of h_t is obtained from the model. However, applying these model-based decompositions to actual data requires using some of additional model's structure. First, we need to obtain the time series of the technology shock z_t consistent with model's structure and the actual data set considered. Second, by using model's structure and the time series of z_t obtained in the previous step, we generate the corresponding time series for human capital that will be used in the model-based decomposition methods described above. In order to carry out the first step, we rewrite equation (20) as follows

$$\hat{kk}_t = \frac{q_{11}}{1 - p_{11}L}z_{t-1}, \tag{24}$$

where L is the lag operator. From equations (15), (16) and (24), we obtain the following expressions for consumption-human capital ratio and some other variables included in the production function (3)

$$\log \hat{c}_t = \log \overline{\hat{c}_t} + \frac{r_{11}q_{11}}{1 - p_{11}L}z_{t-1} + s_{11}z_t, \tag{25}$$

$$\log \phi_t = \log \overline{\phi} + \frac{r_{21}q_{11}}{1 - p_{11}L}z_{t-1} + s_{21}z_t, \tag{26}$$

$$\log n_t = \log \overline{n} + \frac{r_{31}q_{11}}{1 - p_{11}L}z_{t-1} + s_{31}z_t, \tag{27}$$

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$$\log \hat{k}_t = \log \overline{k} + \frac{q_{11}}{1 - p_{11}L} z_{t-1}, \tag{28}$$

where the upper bar denotes steady state values and p_{ij}, q_{ij}, r_{ij} and s_{ij} are generic elements of matrices P, Q, R and S, respectively. Note that all these matrices are known since they are defined by the parameter values of the model chosen in the calibration step.

Next we consider the following identity

$$\log\left(\frac{y_t}{c_t}\right) = \log \hat{y}_t - \log \hat{c}_t. \tag{29}$$

Taking into account the production function (3), equation (29) can be written as follows

$$\log\left(\frac{y_t}{c_t}\right) = \log[A_m e^{z_t}(\phi_t \hat{k}_t)^\alpha n_t^{1-\alpha}] - \log \hat{c}_t. \tag{30}$$

Substituting equations (25)-(28) in (30) and after some algebra we obtain that

$$(1 - p_{11}L)\log\left(\frac{y_t}{c_t}\right) = (1 - p_{11})[\log A_m + \alpha \log \overline{\phi} + \alpha \log \overline{k} + (1 - \alpha)\log \overline{n} - \log \overline{c}]$$

$$+[1 + \alpha s_{21} + (1 - \alpha)s_{31} - s_{11}](1 - p_{11}L)z_t$$

$$+q_{11}[\alpha(1 + r_{21}) + (1 - \alpha)r_{31} - r_{11}]z_{t-1}. \tag{31}$$

Taking into account the actual time series and the parameter values from the calibration step we have that in equation (31) all the variables and parameters are known except z_t and z_{t-1}. In order to obtain the time series of the technology shock z_t we use model's assumption that the technology shock follows the process $z_t = 0.95z_{t-1} + \epsilon_t$. In addition, we impose an initial condition. In particular, we assume that $\epsilon_2 = 0$, which implies that

$$z_2 = 0.95z_1. \tag{32}$$

Equation (32) and equation (31) evaluated at $t = 2$, form a pair of linear equations with two unknowns, z_1 and z_2. Solving this system for z_1 and z_2, we can then recursively obtain the time series of z_t from equation (31). Next, by using the time series of the technology shock z_t and the model we can obtain the time series for human capital to implement the model-based decompositions outlined above.

Figures 1-3 show (unfiltered) real and synthetic time series for output, investment and consumption, respectively. These figures show that the EG model captures to certain extent the timing and size of the cyclical fluctuations of

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output and investment. However, this model generates a smoother pattern for consumption than the one observed in the data.[13] Figure 4 shows the time series for human capital used to detrend both the synthetic and observed data. We observe that the cyclical component of human capital is relatively small (that is, human capital dynamics are dominated by its trend component) and this result explains why the two model-based decompositions((17) and (23)) provide, as shown below, almost identical cyclical properties.
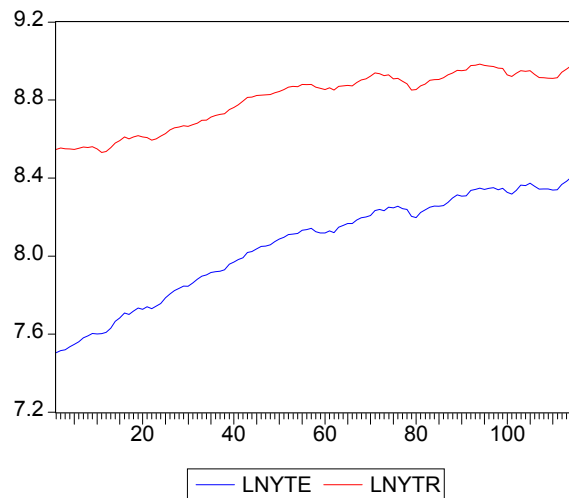
Figure 1: Log of real (LNYTR) and synthetic (LNYTE) output

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[13] The gap between observed and synthetic data in Figures 1-3 is due to the choice of initial stock values of physical and human capital used to generate synthetic data. Since this choice only affects the level of the variables and has no effect on the cyclical patterns, we maintain this gap for illustration purposes; in this way, the evolution of the two time series (actual and synthetic) can be clearly distinguished in each of the three figures.

Figure 2: Log of real (LNITR) and synthetic (LNITE) investment

Figure 3: Log of real (LNCTR) and synthetic (LNCTE) consumption
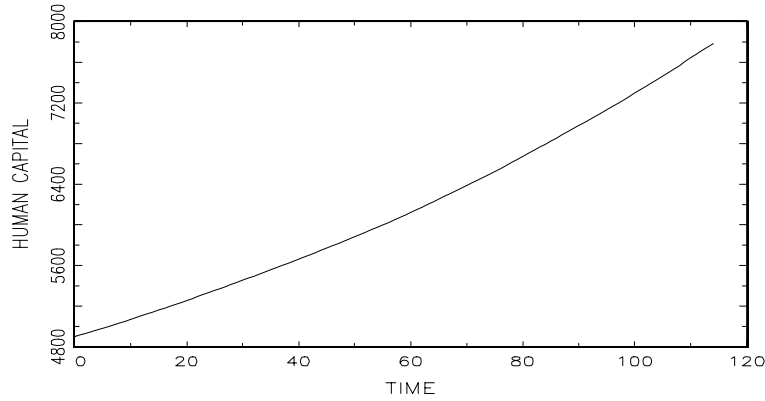
Figure 4: Synthetic Human Capital Time Series

There are other types of model-based decompositions considered in the literature. One of these (for instance, King, Plosser, Stock and Watson (1991)) consists in studying the (logs of) consumption-output and investment-output *great ratios.* As was made clear above, these ratios satisfy the log-linear approximation of the Euler-restrictions imposed by the agent's intertemporal optimization program. On the one hand, the analysis of the great ratios has the advantage that the implicit model-based decomposition is simpler to implem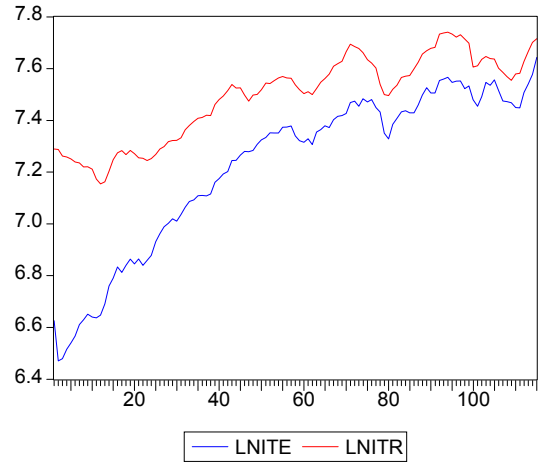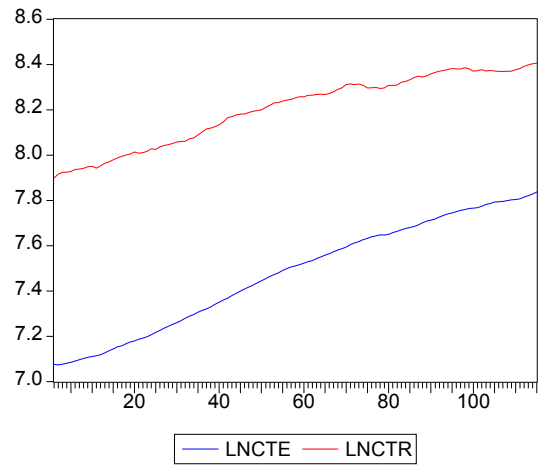ent than the model-based decompositions considered above. On the other hand, the previous model decompositions have the advantage that it allows us to analyze separately the cyclical features of output, consumption and investment. Another model based-decomposition suggested by King, Plosser and Rebelo (1988b) consists in analyzing the growth rates of output, consumption and investment. This model-based decomposition, as the one based on the analysis of the great ratios, is easy to implement and it allows to study separately the cyclical components of output, consumption and investment. As pointed out by King, Plosser and Rebelo (1988b), an inconvenient of the model-based decomposition implied by the growth rates is that mainly isolates cyclical components associated with high frequencies. An advantage of the analyses of the great ratios and the growth rates is that they can be implemented in the two growth models considered. Therefore, these two model-based decompositions can be used to compare the performance of the two models in order to reproduce the dynamic features of the great ratios and work effort and the dynamic properties of the growth rates

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of output, consumption and investment, respectively.

4   THE KING-PLOSSER-REBELO MODEL

The KPR model is a one-sector model of physical capital accumulation and labor input is a choice variable. As is well known, the competitive equilibrium of this model can be characterized by solving the following intertemporal optimization problem faced by a benevolent social planner:

$$\underset{c_t,k_t,n_t,i_t,y_t}{Max} E_t\left\{\sum_{t=0}^{\infty}\beta^t\left[\log(c_t)+A\log(l_t)\right],\right\} \tag{33}$$

subject to the following constraints

$$\begin{aligned}
&c_t+i_t=y_t,\\
&k_{t+1}=(1-\delta)k_t+i_t,\\
&y_t=Z_tk_t^\alpha(X_tn_t)^{1-\alpha},\\
&\log(\tfrac{Z_t}{\overline{Z}})=\rho\log(\tfrac{Z_{t-1}}{\overline{Z}})+\varepsilon_t,
\end{aligned} \tag{34}$$

where k_0, Z_0 and X_0 are given. The notation is standard. c_t, n_t, i_t, y_t, k_t, and Z_t denote consumption, labor input, investment, output, capital stock and technological shock at time t, respectively. β and δ are the discount factor and the depreciation rate, respectively. A, α, ρ and $\overline{Z}$ are parameters. The variable X_t is exogenously determined and it captures the growth process of the economy. It is assumed that X_t follows the process $X_t=\varphi X_{t-1}$. Therefore, the growth process is deterministic. The parameter φ determines the steady-state rate of growth.

In the steady state, the variables y_t, c_t, i_t, and k_t grow at rate φ whereas n_t and the *gross real return* of capital, denoted by R_t, are constants. In order to facilitate the use of computational techniques, it is convenient to rewrite the model in terms of stationary variables by dividing all steady-state growing variables by X_t. Then, the model can be written in terms of the variables $cx_t=c_t/X_t$, $kx_t=k_t/X_t$, $ix_t=i_t/X_t$, and $yx_t=y_t/X_t$ that are stationary. The necessary and sufficient first-order conditions for an (interior) optimum are then given by

$$Acx_tn_t=(1-\alpha)(yx_t-yx_tn_t), \tag{35}$$

$$1=\frac{\beta}{\varphi}E_t(\frac{cx_t}{cx_{t+1}}R_{t+1}), \tag{36}$$

$$R_t=\alpha\frac{yx_t}{kx_t}+(1-\delta), \tag{37}$$

$$cx_t + ix_t = yx_t, \tag{38}$$

$$yx_t = Z_t kx_t^\alpha n_t^{1-\alpha}, \tag{39}$$

$$\varphi kx_{t+1} = (1-\delta)kx_t + ix_t, \tag{40}$$

$$\lim_{t\to\infty} E_t \beta^t \frac{kx_{t+1}}{cx_t} = 0,$$

$$z_t = \rho z_{t-1} + \varepsilon_t,$$

where $z_t = \log(\frac{Z_t}{Z})$.

In order to facilitate the comparison between the two growth models, the same parameter values considered in the EG model are used in the KPR model with small differences.[14]

By using the log-linear method suggested by Uhlig (1999), the following laws of motion for the stationary variables of the model are obtained

$$
\begin{aligned}
\widetilde{kx}_{t+1} &= 0.950920\widetilde{kx}_t + 0.123720z_t, \\
\widetilde{cx}_t &= 0.551419\widetilde{kx}_t + 0.406145z_t, \\
\widetilde{yx}_t &= 0.178718\widetilde{kx}_t + 1.562405z_t, \\
\widetilde{n}_t &= -0.283253\widetilde{kx}_t + 0.878757z_t, \\
\widetilde{R}_t &= -0.028460\widetilde{kx}_t + 0.0541430z_t, \\
\widetilde{ix}_t &= -0.694592\widetilde{kx}_t + 4.271741z_t.
\end{aligned}
\tag{41}
$$

Moreover, it is assumed that

$$z_t = 0.95z_{t-1} + \varepsilon_t, \tag{42}$$

where $\varepsilon_t \sim iid(0, \sigma_\epsilon^2)$. A tilde ($\tilde{\ }$) on a variable denotes its log-deviation from the steady-state value.

Using the system of equations (41) and initial values for $\widetilde{kx}_t$ and z_t, we can obtain synthetic time series for the log-deviations from the steady-state values of the ratios $kx_t = \frac{k_t}{X_t}$, $cx_t = \frac{c_t}{X_t}$, $yx_t = \frac{y_t}{X_t}$, and $ix_t = \frac{i_t}{X_t}$, and the levels of n_t and R_t. Using the initial value X_0 and the law of motion $X_t = \varphi X_{t-1}$ the time series X_t are obtained. Next, the levels of all variables displaying a deterministic trend (y_t, c_t, i_t, and k_t) can be easily obtained using X_t. Following these procedures we have the synthetic time series of k_t, c_t, y_t and i_t in levels and the same time series filtered according to a model-based decomposition that removes the linear trend component X_t, that is, kx_t, cx_t, yx_t and ix_t.

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[14]More precisely, the parameter values chosen are $\alpha = 0.36$, $\delta = 0.025$, $\sigma = 1$, $\rho = 0.95$, $\beta = 0.99414$ and $\varphi = 1.00408067$.

The KPR model assumes a linear deterministic trend. Therefore, a model-based decomposition in this context consists in removing a linear trend from non-stationary time series (for instance, output, consumption and investment). From now on, we refer to this model-based decomposition as LT. As in the Uzawa-Lucas model, the great ratios and the growth rates of output, consumption and investment are model-based decompositions in the KPR model since these two transformations also remove the linear deterministic trend followed by X_t.

5 MODEL EVALUATION USING ALTERNATIVE DECOMPOSITION METHODS

Following the way in which the RBC literature evaluates models, we now analyze whether different cyclical features, characterized through some selected second moment statistics of the (log) levels of output, consumption and investment, change depending on the alternative decomposition methods used to obtain the cyclical components of the time series. Six decomposition procedures are considered in this section: the two model-based decompositions associated with the EG model suggested in Section 3, the model-based decomposition associated with the KPR model proposed in Section 4, the HP filter, the BP filter and the GG decomposition. As pointed above, the evaluation exercise uses quarterly US data for the period 1955:3-1984:1.

The presentation of the results has the following structure. First, we show the US cyclical features extracted from alternative decomposition methods. Second, we show the cyclical features of the KPR model and the EG model obtained from the alternative decomposition procedures. Finally, the performance of the two alternative growth models to reproduce the cyclical features observed in US data is compared across alternative decomposition methods.

5.1 US cyclical features

Table 1 shows some second moment statistics for actual US data and synthetic data (obtained from the two alternative growth models) derived from the decomposition methods considered in this paper. Before the trend component was removed from the time series and second moment statistics were calculated, time series were logged.

The top panel in Table 1 displays the cyclical properties revealed by US data. We observe that two cyclical features are qualitatively similar for all decompositions considered. First, output volatility is higher than consumption volatility and much lower than investment volatility. Second, the correlation between consumption and output is lower than the correlation between investment and output.

When comparing quantitatively the cyclical features across decomposition methods, we observe some important differences and similarities . These similarities and differences can be summarized as follows. First, the two model-based decompositions characterized by (23) (MBD$_1$) and (17) (MBD$_2$) provide two

cyclical components that share almost identical cyclical features. This result is not surprising at all since these two model-based decompositions differ only by whether or not the cyclical component of human capital, which is very small as shown above in Figure 4, is included in the cyclical component of nonstationary variables. The linear trend (LT) decomposition reveals similar cyclical features than those obtained through MBD_1 and MBD_2 except the volatilities of consumption and output. Moreover, these three model-based decompositions imply more volatile cyclical components than the other three detrending methods, where the volatility is measured by the standard deviation of the time series of output, consumption and investment, respectively.

Second, the relative volatilities of consumption and output and investment and output (measured by σ_c/σ_y and σ_i/σ_y) obtained from the HP and BP filters are quite similar. However, these statistics are different when the other decomposition methods are considered. On the one hand, the GG filter provides cyclical components characterized by a higher volatility of investment relative to output volatility and a lower volatility of consumption relative to output volatility. On the other hand, the cyclical components obtained from the two model-based decompositions associated with the EG model are characterized by a lower σ_i/σ_y and a higher σ_c/σ_y whereas the cyclical components obtained from the model-based decomposition associated with the KPR model also results in a lower σ_i/σ_y, but a similar value for σ_c/σ_y, than those obtained for the HP and BP filters.

Third, the cyclical components obtained from the HP and BP filters share similar contemporaneous correlations between consumption and output ρ_{cy} and between investment and output ρ_{iy}. In relation to the correlations found with the HP and BP filters, the cyclical components obtained from the three model-based decompositions imply a higher value for ρ_{cy} and slightly lower value for ρ_{iy}. Moreover, by using Johansen's (1988) procedure, the cointegration results obtained from actual US data show the presence of a single cointegration relationship between output, consumption and investment, and this implies the existence of two common trends.[15] This result, on the one hand, explains the perfect contemporaneous correlation between output and investment and between output and consumption when the time series are filtered using the GG filter, since this filter, by construction, imposes this cointegration restriction. On the other hand, the existence of two common trends challenges most RBC models because they typically assume the existence of a single trend. In particular, the EG model considered in this paper exhibits a single trend which is characterized by human capital accumulation.

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[15] It should be noticed that King, Plosser, Stock and Watson (1991) find, studying a similar data set, the existence of a single common trend for the same set of variables. The different results may be due to the different econometric procedures implemented to detect the cointegration relationships. We think that further research in this topic is important since the number of common trends detected in actual data may influence the evaluation results of the cyclical features of a particular model.

5.2 Cyclical features of the two growth models

The second panel in Table 1 summarizes the cyclical features exhibited by the EG model. It shows that the cyclical components extracted with the HP, BP and GG filters produce very similar volatilities for the synthetic time series of output, consumption and investment. Since the standard deviation of the technology shock process σ_ϵ is calibrated in such way that the synthetic output time series obtained from any decomposition is closer to mimicking the observed σ_y in actual US data, the standard deviations of the cyclical components of output, consumption and investment associated with the model-based decompositions are obviously higher than those associated with the HP, BP and GG filters. As for actual US data, the relative volatility between consumption and output obtained from synthetic data is higher for the cyclical components characterized by the two model-based decompositions than for the cyclical components associated with the HP, BP and GG filters. The reverse is true for the relative volatility between investment and output, but the statistics are much closer. The contemporaneous correlations between consumption and output (ρ_{cy}) and between investment and output (ρ_{iy}) change with the decomposition chosen. On the one hand, the two model-based decompositions and the BP filter provide cyclical components with similar correlation statistics and these three decompositions share the property observed in actual US data that the correlation between consumption and output is lower than the correlation between investment and output. On the other hand, the HP and GG filters result in that ρ_{cy} is higher than ρ_{iy}. Moreover, the HP filter reduces the size of the statistic ρ_{iy} in relation to the one obtained from the other decomposition methods[16] whereas the GG filter implies a larger value of ρ_{cy} than the one obtained from the other decomposition methods.

As shown by the bottom row of the second panel in Table 1, another difference arises when comparing the effects of alternative decomposition methods: σ_ϵ is different depending on the decomposition method considered. Thus, σ_ϵ has to be twice (thrice) greater when the cyclical component is characterized by the HP and BP filters (the two model-based decompositions), in order to mimicking US output volatility, than when the cyclical component is characterized by the GG filter. Computing the ratios σ_y/σ_ϵ for each alternative decomposition method provides a rough idea of how alternative decomposition methods affect the propagation mechanism of technology shocks implied by the EG model. Thus, one can easily check that the HP and BP filters induced a much weaker propagation mechanism than the other decomposition methods.

The third panel in Table 1 summarizes the cyclical features exhibited by the KPR model. As in the EG model, the standard deviation of the cyclical components of output, consumption and investment associated with the model-based decomposition LT are obviously higher than those associated with the HP

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[16] As discussed below, the fact that this model produces a low correlation between investment and output ρ_{iy} under the HP filter is quite surprising since other scenarios, such as standard RBC models filtering with the HP and the EG model considered filtering with the other decompositions, provide higher values for ρ_{iy}.

and BP filters because the standard deviation of the technology shock process σ_ϵ is calibrated in such way that the synthetic output time series obtained from any decomposition is closer to mimicking the observed σ_y in actual US data. Again, as in the EG model, the relative volatility between consumption and output is twice larger for the cyclical component associated with the model-based decomposition LT than for the cyclical components associated with the HP and BP filters. As in the EG model, the reverse is true for the relative volatility between investment and output. Contrary to the results found in the EG model, the cyclical components of the KPR model obtained from the three alternative decompositions (LT, HP and BP) share the feature observed in actual US data that the correlation between consumption and output is lower than the correlation between investment and output.

As shown by the bottom row of the third panel in Table 1 and contrary to the EG model, the parameter value of σ_ϵ barely changes with the decomposition method used.

Next, we carry out the following exercise, which is simple, in order to see whether a model-based decomposition provides a cyclical component of the synthetic time series from the EG model that is substantially different from the cyclical components obtained from alternative filters.[17] This exercise is also useful to analyze whether the EG model provides different dynamic features depending on the cyclical components obtained from alternative detrending methods. This exercise can be outlined as follows:

Step 1. We have previously solved the EG model obtaining the logarithms of the cyclical components of the following variables: output ($\tilde{y}_t$), consumption ($\tilde{c}$) and investment ($\tilde{i}_t$). These are the cyclical components characterized by model-based decomposition (23).
Step 2. The logarithms of the non-stationary synthetic time series for output (y_t), consumption (c_t) and investment (i_t) are filtered using alternative filtering techniques. As pointed out above, this paper considers two well known filters in the RBC literature, the HP and BP filters, together with the GG filter.
Step 3. We regress the logarithm of the cyclical component of output ($\tilde{y}_t$) on the logs of output time series obtained by using the three alternative filters. We do the same for the logs of $\tilde{c}_t$ and $\tilde{i}_t$.
Step 4. We repeat many times (say 500 times) Steps 1-3 for different realizations of the technology shock z_t.

Table 2 shows the average results of these regressions. On the one hand, we observe that the cyclical behavior of the output and investment associated with the model-based decomposition studied (measured by the logs of $\tilde{y}_t$ and $\tilde{i}_t$, respectively) is captured to a great extent by the GG filter. We view the R^2 coefficient obtained from any of these regressions as a rough measure of the capability of a particular filter to capture the cyclical component dynamics of the corresponding (dependent) variable characterized by the model-based

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[17]In this exercise we use the model-based decomposition characterized by (23). The results are similar when implemeting model-based decomposition (17).

decomposition considered.[18] Looking at the R^2 coefficients, we observe that 79% of the variance of output and 92% of the variance of investment are explained by the synthetic time series filtered using GG. However, when the HP and the BP filters are used the R^2 coefficients are much smaller. These results show that the explanatory power of the output and investment filtered through HP is very small compared to the explanatory power of these variables filtered using GG. The reason why the cyclical components obtained from the model-based decomposition and the GG filter are similar is that this filter requires that the time series considered be I(1) (that is, they are difference stationary) and it imposes the existence of a common trend for all non-stationary variables of the model. These two features imposed by this filter are consistent with the EG model. First, as shown above, the synthetic time series for output, consumption and investment have a unit root. Second, the EG model postulates a single source of growth represented by human capital (that is, a single common trend) for all non-stationary variables.

On the other hand, we see looking at Table 2 that the cyclical features of consumption associated with the model-based decomposition (measured by $\tilde{c}_t$) are not well captured by any of the three filters considered (the highest R^2 value is obtained for the BP filter, $R^2 = 0.35$). This result is due to the fact that the cyclical component of consumption characterized by the model-based decomposition exhibits a great deal of persistence which does not appear when the synthetic time series in levels go through the alternative filters.

5.3 A comparison of models' performance

Comparing the statistics from actual US data with the corresponding statistics from synthetic data in Table 1, we see that a model quantitative evaluation depends, to some extent, on the decomposition method considered to extract the cyclical component. Some examples are the following. First, we will say that the EG model matches the relative volatility of investment with respect to output, σ_i/σ_y, when analyzing the cyclical components characterized by the HP and BP filters. However, for the cyclical components characterized by the GG filter and the model-based decompositions associated with the EG model (MBD$_1$ and MBD$_2$), the match will not be so good.[19] Analyzing the same statistic in the KPR model, any decomposition considered (LT, HP and BP) suggests that the KPR model induces a much larger relative volatility of investment with respect to output than the one observed in US data.

Second, one may conclude that the EG model is close to match the relative volatility of consumption with respect to output, σ_c/σ_y, when analyzing the cyclical components characterized by the GG. However, this conclusion is not reached when using the other decomposition procedures since the EG model falls short of replicating the value of this statistic obtain from actual data. When

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[18] Watson (1993) also uses R^2 related measures to quantitatively evaluate model performance.

[19] Since σ_ϵ is calibrated in such way that the model is closer to mimicking the observed σ_y in actual US data, the same conclusion for σ_i can be reached.

evaluating the KPR model a conflicting result also appears: the actual value of σ_c/σ_y found in US data is larger (smaller) than the one reproduced by the KPR model using the HP and BP (LT decomposition) filters.

Third, one may conclude that the correlation between consumption and output, ρ_{cy}, is reproduced by the KPR model when the model-based decomposition LT is considered. However, by using HP and BP filters, one concludes that the KPR model induces a larger ρ_{cy} than the one observed in US data. In the EG model, on the one hand, the two model-based decompositions (MBD$_1$ and MBD$_2$) induce a smaller ρ_{cy} than the one observed in US data, whereas the opposite is true for the HP filter. On the other hand, by using the BP or GG filters one may conclude that the EG model is closer to reproduce the statistic ρ_{cy} observed in actual data than when using the other decomposition methods.

Finally, according to the two model based decompositions (MBD$_1$ and MBD$_2$), one may conclude that the correlation between investment and output, ρ_{iy}, is replicated to certain extent by the EG model. However, this conclusion is not reached using either the HP or GG filters. The KPR induces a larger ρ_{iy} than the one observed in US data independently of the decomposition considered, although the model is closer to reproduce this feature when the model-based decomposition LT is used.

In short, this section has stressed that alternative decomposition methods lead to quite different model evaluation results either in the EG model as in the KPR model. Moreover, the assessment of whether a model is closer to reproduce a particular cyclical feature than the other model in certain cases depends on the decomposition considered.[20] Thus, when using the BP filter one may conclude that the EG model is closer to reproduce the statistics σ_i/σ_y and ρ_{cy} than the KPR model. The opposite is true for σ_c/σ_y whereas for the statistic ρ_{iy} the performance of the two models is rather similar. When the HP filter is considered the comparison results between the two growth models point to the same direction as the BP filter with the exception that the KPR model is much closer to reproduce ρ_{iy} than the EG model.

6 MODEL EVALUATION BASED ON THE ANALYSIS OF THE GREAT RATIOS AND GROWTH RATES

6.1 Analysis of the great ratios

Table 3 shows some selected second moment statistics of the great ratios and the work effort for US data (upper panel) and synthetic data (bottom panel, columns 2-4 and columns 4-6 in this panel show the corresponding statistics from

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[20]Furthermore, as pointed above in the Introduction, this type of assessments make more sense when the same filter can be applied to the alternative models considered as is the case of the HP and BP filters since otherwise the different evaluation results may be due to the different decompositions used to evaluate each model.

the EG model and the KPR model, respectively). The parameter σ_ϵ has been set equal to 0.0438 (0.0058) in the EG model (KPR model) and this results in that the standard deviation of $log(i/y)$ is roughly the same for synthetic and US data. Comparing the statistics from both panels we observe that the EG model qualitatively displays the same features as those present in actual data. First, the standard deviation of $log(c/y)$ is smaller and closer to the standard deviation of the work effort, $log(n)$, than the standard deviation of $log(i/y)$. Second, $log(c/y)$ is negatively correlated with $log(i/y)$ and $log(n)$. Third, the autocorrelation functions show a slow decay. Moreover, the EG model provides some well behaved quantitative approximations of some moments observed in actual data. For instance, the autocorrelation function of $log(n)$ is quite well characterized by the model dynamics. Another example is the cross-correlation structure between $log(i/y)$ and $log(n)$. However, we also observe some important quantitative departures. First, the model shows a lower volatility for $log(c/y)$ and $log(n)$. Second, the negative contemporaneous correlation between the two ratios is almost three times larger in the EG model than the one observed in actual data. An even worse result is found for the contemporaneous correlation between $log(c/y)$ and $log(n)$. Finally, the autocorrelation functions of $log(c/y)$ and $log(i/y)$ for actual data show more persistence than those obtained from the model.

Comparing upper panel of Table 3 with the columns 4-6 in the bottom panel of Table 3, we observe that the KPR model performs much worst than the EG model when reproducing the cyclical features of the great ratios. The reason is simple. In the KPR model, the two great ratios and work effort are perfectly correlated contemporaneously and share the same autocorrelation and cross-correlation structures, since there is a stochastic singularity in the expressions that relate the two great ratios and work effort with the two state variables of the model $\widetilde{kx}_t$ and z_t. As shown by equation (41), the companion matrix associated with the bivariate system that relates the two great ratios (or alternatively, one of the great ratios and the work effort) with the two states variables is singular. This stochastic singularity implies that the two great ratios and the work effort are linear functions of the state variables where the only difference among these linear functions is a scale factor.[21] This stochastic singularity does not hold in the EG model. This less restrictive property of the EG model provides a closer match to the observed autocorrelation and cross-correlation structures of the two great ratios and work effort than those provided by the KPR model.

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[21] A similar analysis of these ratios is carried out in King, Plosser and Rebelo (1988b) in the context of an exogenous growth model in which the logarithm of total factor productivity follows a random walk. In their model the stochastic singularity occurs because the great ratios and the work effort are linear functions of only the contemporaneous deviation of the capital stock from its steady state value. The cyclical features displayed by the KPR model considered in this paper are to certain extent different from those obtained by King, Plosser and Rebelo (1988b) since certain parameter values are different. Notice that we also depart from King et al. (1988b) in that the dynamic properties of the model are analyzed by calculating the sample moments (as we have done with the actual US data) instead of the population moments as in King et al. (1988b). We proceed in this way because sample moments can substantially differ from population moments in the context of stationary, but persistent, processes.

6.2 Analysis of the growth rates

Table 4 displays some selected second moment statistics of the growth rates of output, consumption, investment and work effort. As in the analysis of the great ratios, the parameter σ_ϵ has been set equal to 0.0438 (0.0055) in the EG model (KPR model). The EG model captures remarkably well the volatility of the growth rates of output and investment whereas the KPR model is only close to capture the volatility of the output growth rate. The two models (KPR and EG models) fall short of replicating the observed volatility of the growth rates of consumption and hours. Although the KPR model is closer to reproduce the volatility of consumption growth rate observed in actual data than the EG model. The opposite is true for the volatilities of the growth rates of work effort and investment. Moreover, the two models fail in two important dimensions. First, due to the presence of a single common shock the two models imply a contemporaneous cross-correlation near unity between the growth rate of output and the growth rates of consumption, investment and work effort, respectively. Second, the two models fail to reproduce the positive serial correlation at lags one and two quarters in growth rates of output and investment present in actual data.

7   CONCLUSIONS

This paper analyzes and compares the cyclical properties of an Uzawa-Lucas endogenous growth model with those exhibited by a standard neoclassical (exogenous) growth model. We study the dynamic features of different cyclical components of these two models characterized by a variety of decomposition methods. The decomposition methods considered can be classified in two groups. On the one hand, we consider three statistical filters: the Hodrick-Prescott filter, the Baxter-King filter and the multivariate decomposition suggested by Gonzalo and Granger. On the other hand, we use a set of model-based decomposition methods. All model-based decomposition methods share the property that the cyclical components extracted by these methods preserve the log-linear approximation of the Euler-equation restrictions imposed by the agent's intertemporal optimization problem.

The quantitative evaluation of the Uzawa-Lucas model across the alternative cyclical components considered can be summarized as follows. First, the model reproduce quite well the second moments characterized by the first two model-based decompositions relatively to the other detrending methods, except the relative volatility of investment and output. Second, the model does a reasonable job in reproducing the second moments of the cyclical component characterized by the Baxter-King filter, but the relative volatility of consumption and output. The same conclusion can be reached for the Hodrick-Prescott filter except for the correlation between investment and output that it is also poorly reproduced by the model. Third, the model has some difficulties in order to mimicking the cyclical features observed in the US data through the Gonzalo-Granger filter. A

possible explanation for the poor performance of the Uzawa-Lucas model when quantitative evaluation of the cyclical component characterized by the GG filter is carried out may be due to the fact that actual US data exhibit two common trends, whereas the model only displays a single common trend (the stock of human capital). This fact shows that endogenous growth models that introduce a second source of growth can be good candidates for future research in exploring the implications of growth on cycles, and vice versa, since growth and cyclical components are likely to be linked by optimal economic decision making.

A parallel exercise is also carried out in the context of a standard neoclassical exogenous growth model. In this model, a model-based decomposition consists in removing a linear deterministic trend. The quantitative evaluation results of the exogenous growth model show that the capability of the model in characterizing certain cyclical features studied depends on the decomposition method used to isolate the cyclical component.

As pointed out above, in order to analyze whether or not the EG model is better than the KPR model in reproducing certain cyclical properties observed in actual data one should focus on decompositions that can be applied to the two models as the HP and BP filters and the model-based decompositions implied by the great ratios and the growth rates. When using the BP filter we conclude that the EG model is closer to reproduce the relative volatility between investment and output and the contemporaneous correlation between consumption and output than the KPR model. The opposite is true for the relative volatility between consumption and output whereas the performance of the two models is similar for reproducing the correlation between investment and output. When the HP filter is considered the comparison results between the two growth models point to the same direction as the BP filter with the exception that the KPR model is much closer to reproduce the correlation between investment and output than the EG model.

The analysis of the cyclical features of the consumption-output and investment-output *great ratios* and work effort shows that the generalized Uzawa-Lucas model provides a better characterization than the exogenous growth models analyzed by King, Plosser and Rebelo (1988a,b) in terms of the autocorrelation and cross-correlation structures of the two *great ratios* and hours observed in US data. Finally, the analysis of the growth rates shows that the generalized Uzawa-Lucas model provides no improvement in relation to the exogenous growth models studied by King, Plosser and Rebelo (1988a,b) in reproducing the observed autocorrelation functions of the rates of growth of output and investment. Collard (1999) has proved that a learning-by-doing endogenous growth model account much better for the dynamics of the output growth rate.

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APPENDIX 1

In this appendix we derive Uhlig's log-linear solution of the EG model.[22] This approach log-linearizes the necessary first-order conditions that characterize the equilibrium of the model. In our model we have six first-order conditions,

$$U_1(\hat{c}_t, l_t) = \frac{U_2(\hat{c}_t, l_t)}{F_2^m(\phi_t \hat{k}_t, n_t)}, \tag{A.1}$$

$$\frac{F_2^h[(1-\phi_t)\hat{k}_t, 1-l_t-n_t]}{F_2^m(\phi_t \hat{k}_t, n_t)} = \frac{F_1^h[(1-\phi_t)\hat{k}_t, 1-l_t-n_t]}{F_1^m(\phi_t \hat{k}_t, n_t)}, \tag{A.2}$$

$$\frac{h_{t+1}}{h_t} = F^h[(1-\phi_t)\hat{k}_t, 1-l_t-n_t] + 1 - \delta_h, \tag{A.3}$$

$$\frac{h_{t+1}}{h_t}\hat{k}_{t+1} = F^h(\phi_t \hat{k}_t, n_t) + (1-\delta_k)\hat{k}_t - \hat{c}_t, \tag{A.4}$$

$$U_1(\hat{c}_t, l_t) = \beta(\frac{h_{t+1}}{h_t})^{-\gamma} E_t\{U_1'(\hat{c}_t, l_t)[F_1^m(\phi_{t+1}\hat{k}_{t+1}, n_{t+1}) + 1 - \delta_k]\}, \tag{A.5}$$

$$\frac{U_2(\hat{c}_t, l_t)}{F_2^h[(1-\phi_t)\hat{k}_t, 1-l_t-n_t]} = \beta(\frac{h_{t+1}}{h_t})^{-\gamma} E_t\{\frac{U_2'(\hat{c}_t, l_t)}{F_2^h[(1-\phi_{t+1})\hat{k}_{t+1}, 1-l_{t+1}-n_{t+1}]}$$

$$[F_2^h((1-\phi_{t+1})\hat{k}_{t+1}, 1-l_{t+1}-n_{t+1}) + 1 - \delta_h]\}, \tag{A.6}$$

Substituting the functional forms chosen for the utility and production functions, we have the following expressions for the first-order conditions,

$$\hat{c}_t(1-w)n_t^\alpha = wA_m(1-\alpha)(\phi_t \hat{k}_t)^\alpha l_t Z_t, \tag{A.7}$$

$$\frac{\phi_t}{n_t} = \frac{(1-\theta)\alpha}{\theta(1-\alpha)}\frac{(1-\phi_t)}{(1-l_t-n_t)},$$

or equivalently,

[22] Similar steps have been carried out to solve the KPR model.

$$\phi_t = \frac{an_t}{1 - l_t - n_t + an_t}, \tag{A.8}$$

where $a = \frac{(1-\theta)\alpha}{\theta(1-\alpha)}$,

$$\frac{h_{t+1}}{h_t} = A_h[(1 - \phi_t)\hat{k}_t]^\theta (1 - l_t - n_t)^{1-\theta} + 1 - \delta_h, \tag{A.9}$$

$$\frac{h_{t+1}}{h_t}\hat{k}_{t+1} = A_m(\phi_t\hat{k}_t)^\alpha n_t^{1-\alpha} Z_t + (1 - \delta_k)\hat{k}_t - \hat{c}_t, \tag{A.10}$$

$$\hat{c}_t^{\omega(1-\gamma)-1} l_t^{(1-\omega)(1-\gamma)} = \beta(\frac{h_{t+1}}{h_t})^{-\gamma} E_t[\hat{c}_{t+1}^{\omega(1-\gamma)-1} l_{t+1}^{(1-\omega)(1-\gamma)}$$

$$[A_m\alpha Z_{t+1}(\phi_{t+1}\hat{k}_{t+1}^{\alpha-1})n_{t+1}^{1-\alpha} + 1 - \delta_k], \tag{A.11}$$

$$\frac{\hat{c}_t^{\omega(1-\gamma)} l_t^{(1-\omega)(1-\gamma)-1}}{[(1 - \phi_t)\hat{k}_t]^\theta (1 - l_t - n_t)^{-\theta}} = \beta(\frac{h_{t+1}}{h_t})^{-\gamma}$$

$$E_t\{\frac{\hat{c}_{t+1}^{\omega(1-\gamma)} l_{t+1}^{(1-\omega)(1-\gamma)-1}}{[(1 - \phi_{t+1})\hat{k}_{t+1}]^\theta (1 - l_{t+1} - n_{t+1})^{-\theta}}$$

$$[A_h(1 - \theta)((1 - \phi_{t+1})\hat{k}_{t+1})^\theta (1 - l_{t+1} - n_{t+1})^{-\theta} + 1 - \delta_h]\}. \tag{A.12}$$

Making the equations approximately linear in the log-deviations from the steady state as shown by Uhlig, and rearranging, we have that

$$0 = \bar{c}(1 - w)\bar{n}^\alpha(c\hat{c}_t + \alpha nn_t) - wA_m(1 - \alpha)\bar{\phi}^\alpha \bar{\hat{k}}^\alpha \bar{Z}\bar{l}[\alpha\phi\phi_t + \alpha k\hat{k}_t + z_t + ll_t], \tag{A.13}$$

where $c\hat{c}_t = log(\frac{\hat{c}_t}{\bar{c}})$, $nn_t = log(\frac{n_t}{\bar{n}})$, $\phi\phi_t = log(\frac{\phi_t}{\bar{\phi}})$, $k\hat{k}_t = log(\frac{\hat{k}_t}{\bar{\hat{k}}})$, and $ll_t = log(\frac{l_t}{\bar{l}})$;

$$0 = [a\bar{n} - (a - 1)\bar{\phi}\bar{n}]nn_t + \bar{\phi}\bar{l}ll_t - \bar{\phi}[1 - \bar{l} + (a - 1)\bar{n}]\phi\phi_t, \tag{A.14}$$

$$0 = -\bar{H}Hhh_{t+1} + A_h\left(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}}\right)^\theta\left[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}\right]^\theta(1-\bar{l}-\bar{n})\theta[\phi\phi_t + k\hat{k}_t]$$

$$-A_h\left(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}}\right)^\theta\left[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}\right]^\theta\bar{l}ll_t - A_h\left(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}}\right)^\theta\left[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}\right]^\theta[(1-\bar{l}-\bar{n})\theta+\bar{n}]nn_t, \quad \text{(A.15)}$$

$$0 = -\bar{\bar{k}}\bar{H}H[k\hat{k}_{t+1} + hh_{t+1}] + A_m\bar{Z}(\bar{\phi}\bar{\bar{k}})^\alpha\bar{n}^{1-\alpha}[\alpha\phi\phi_t + (1-\alpha)nn_t + z_t]$$

$$+[A_m\bar{Z}(\bar{\phi}\bar{\bar{k}})^\alpha\bar{n}^{1-\alpha}\alpha + (1-\delta_k)\bar{\bar{k}}]k\hat{k}_t - \bar{\bar{c}}c\hat{c}_t, \quad \text{(A.16)}$$

$$0 = E_t\{-(w(1-\gamma)-1)c\hat{c}_t - (1-w)(1-\gamma)ll_t$$

$$+\beta\bar{H}H^{-\gamma}A_m\alpha\bar{Z}(\bar{\phi}\bar{\bar{k}})^{\alpha-1}\bar{n}^{1-\alpha}[(\alpha-1)\phi\phi_{t+1} + (\alpha-1)kk_{t+1} + (1-\alpha)nn_{t+1} + z_{t+1}]$$

$$+\beta\bar{H}H^{-\gamma}(w(1-\gamma)-1)[A_m\alpha\left(\frac{\bar{\phi}\bar{\bar{k}}}{\hat{n}}\right)^{\alpha-1}\bar{Z} + (1-\delta_k)]cc\hat{}_{t+1}$$

$$+\beta\bar{H}H^{-\gamma}[(1-w)(1-\gamma)][A_m\alpha\left(\frac{\bar{\phi}\bar{\bar{k}}}{\hat{n}}\right)^{\alpha-1}\bar{Z} + (1-\delta_k)]ll_{t+1}$$

$$-\beta\bar{H}H^{-\gamma}\gamma[A_m\alpha\left(\frac{\bar{\phi}\bar{\bar{k}}}{\hat{\bar{n}}}\right)^{\alpha-1}\bar{Z} + (1-\delta_k)]hh_{t+1}\}, \quad \text{(A.17)}$$

$$0 = E_t\{-w(1-\gamma)c\hat{c}_t - ((1-w)(1-\gamma)-1)ll_t + \theta\phi\phi_t + \theta k\hat{k}_t - \theta nn_t +$$

$$\beta\bar{H}H^{-\gamma}w(1-\gamma)[A_h(1-\theta)\left(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}}\right)^\theta\left(\frac{\theta(1-\alpha)}{\alpha(1-\theta)}\right)^\theta + (1-\delta_h)]cc\hat{}_{t+1} +$$

$$\beta\bar{H}H^{-\gamma}[(1-w)(1-\gamma)-1][A_h(1-\theta)\left(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}}\right)^\theta\left(\frac{\theta(1-\alpha)}{\alpha(1-\theta)}\right)^\theta + (1-\delta_h)]ll_{t+1}$$

$$-\beta\bar{H}H^{-\gamma}\theta(1-\delta_h)[\phi\phi_{t+1} + k\hat{k}_{t+1} - nn_{t+1}]$$

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$$-\beta \bar{H}H^{-\gamma}\gamma[A_h(1-\theta)(\frac{\bar{\phi}\bar{\hat{k}}}{\bar{n}})^\theta(\frac{\theta(1-\alpha)}{\alpha(1-\theta)})^\theta + (1-\delta_h)]hh_{t+1}\}. \qquad \text{(A.18)}$$

The steady-state values are denoted with an upper bar, e.g., $\bar{H}H$ is the steady state value for $\frac{h_{t+1}}{h_t}$, i.e. the endogenous growth rate.

Using Uhlig's terminology, there is an endogenous state vector x_t, size 2x1, a list of other endogenous variables y_t, size 4x1, and a list of exogenous stochastic processes z_t, size 1x1. The equilibrium relationships between these variables can be expressed as follows

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t, \qquad \text{(A.19)}$$

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \qquad \text{(A.20)}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}, \qquad \text{(A.21)}$$

where $E_t(\epsilon_{t+1}) = 0$, $x_t' = (k\hat{k}_{t+1}, hh_{t+1})$, $y_t' = (c\hat{c}_t, \phi\phi_t, nn_t, ll_t)$, $z_t = z_t$, and matrices:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -\overline{HH} \\ -\bar{\hat{k}}\overline{HH} & -\bar{\hat{k}}\overline{HH} \end{pmatrix},$$

$$B = \begin{pmatrix} -\alpha\omega A_m(1-\alpha)(\bar{\phi}\bar{\hat{k}})^\alpha\overline{Zl} & 0 \\ 0 & 0 \\ A_h(\bar{\phi}\bar{\hat{k}})^\theta n^{-\theta}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta(1-\bar{l}-\bar{n})\theta & 0 \\ A_m\bar{Z}(\bar{\phi}\bar{\hat{k}})^\alpha\bar{n}^{1-\alpha}\alpha + (1-\delta_k)\bar{\hat{k}} & 0 \end{pmatrix}.$$

Denoting the generic element of C by C_{ij} we have that

$$C_{11} = \hat{c}(1-\omega)\bar{n}^\alpha$$

$$C_{12} = -\alpha\omega A_m(1-\alpha)(\bar{\phi}\bar{\hat{k}})^\alpha\overline{Zl},$$

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$$C_{13} = \alpha \bar{\bar{c}}(1 - \omega)\overline{n}^{\alpha}$$

$$C_{14} = -\omega A_m(1 - \alpha)(\bar{\phi}\bar{\hat{k}})^{\alpha}\overline{Z}l,$$

$$C_{21} = 0$$

$$C_{22} = -\overline{\phi}[1 - \bar{l} + (a - 1)\overline{n}],$$

$$C_{23} = a\overline{n} - \overline{\phi}(a - 1)\overline{n},$$

$$C_{24} = \overline{\phi l},$$

$$C_{31} = 0,$$

$$C_{32} = A_h(\bar{\phi}\bar{\hat{k}})^{\theta}n^{-\theta}[\frac{\theta(1 - \alpha)}{\alpha(1 - \theta)}]^{\theta}(1 - \bar{l} - \bar{n})\theta,$$

$$C_{33} = -A_h(\bar{\phi}\bar{\hat{k}})^{\theta}n^{-\theta}[\frac{\theta(1 - \alpha)}{\alpha(1 - \theta)}]^{\theta}[(1 - \bar{l} - \bar{n})\theta + \bar{n}],$$

$$C_{34} = -A_h(\bar{\phi}\bar{\hat{k}})^{\theta}n^{-\theta}\bar{l}[\frac{\theta(1 - \alpha)}{\alpha(1 - \theta)}]^{\theta},$$

$$C_{41} = -\bar{\bar{c}},$$

$$C_{42} = \alpha A_m(\bar{\phi}\bar{\hat{k}})^{\alpha}\overline{n}^{1-\alpha}\overline{Z},$$

$$C_{43} = (1 - \alpha)A_m(\bar{\phi}\bar{\hat{k}})^{\alpha}\overline{n}^{1-\alpha}\overline{Z},$$

$$C_{44} = 0,$$

$$D = \begin{pmatrix} -\omega A_m(1 - \alpha)(\bar{\phi}\bar{\hat{k}})^{\alpha}\overline{Z}l \\ 0 \\ 0 \\ A_m(\bar{\phi}\bar{\hat{k}})^{\alpha}\overline{n}^{1-\alpha}\overline{Z} \end{pmatrix},$$

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$$F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} (\alpha-1)\beta\overline{HH}^{-\gamma}(I51) & -\gamma\beta\overline{HH}^{-\gamma}(I51+I52) \\ -\theta\beta\overline{HH}^{-\gamma}(I62) & -\gamma\beta\overline{HH}^{-\gamma}(I61+I62) \end{pmatrix},$$

where

$$I_{51} = \alpha A_m (\bar{\phi}\bar{\hat{k}})^{\alpha-1}\overline{n}^{1-\alpha}\overline{Z},$$

$$I_{52} = (1-\delta_k),$$

$$I_{61} = A_h(1-\theta)(\bar{\phi}\bar{\hat{k}})^{\theta}n^{-\theta}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^{\theta},$$

$$I_{62} = (1-\delta_h),$$

$$H = \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix}.$$

Denoting the generic element of J by J_{ij} we have that

$$J_{11} = \beta\overline{HH}^{-\gamma}[\omega(1-\gamma)-1](I51+I52),$$

$$J_{12} = \beta\overline{HH}^{-\gamma}(\alpha-1)I51,$$

$$J_{13} = \beta\overline{HH}^{-\gamma}(1-\alpha)I51,$$

$$J_{14} = \beta\overline{HH}^{-\gamma}(1-\omega)(1-\gamma)(I51+I52),$$

$$J_{21} = \beta\overline{HH}^{-\gamma}\omega(1-\gamma)(I61+I62),$$

$$J_{22} = -\theta\beta\overline{HH}^{-\gamma}(I62),$$

$$J_{23} = \theta\beta\overline{HH}^{-\gamma}(I62),$$

$$J_{24} = \beta\overline{HH}^{-\gamma}[(1-\omega)(1-\gamma)-1](I61+I62),$$

$$K = \begin{pmatrix} 1-\omega(1-\gamma) & 0 & 0 & -(1-\omega)(1-\gamma) \\ -\omega(1-\gamma) & \theta & -\theta & 1-(1-\omega)(1-\gamma) \end{pmatrix}.$$

$$L = \begin{pmatrix} \beta\overline{HH}^{-\gamma}(I51) \\ 0 \end{pmatrix}.$$

$$M' = (0,0).$$

TABLE 1. Some relevant second moment statistics

US DATA	MBD$_1$	MBD$_2$	LT	HP	BP	GG
σ_y	5.56	5.56	5.12	1.98	1.82	1.74
σ_c	4.15	4.16	3.77	0.85	0.80	0.21
σ_i	7.66	7.67	7.32	4.34	4.14	5.20
σ_c/σ_y	0.75	0.75	0.43	0.43	0.44	0.12
σ_i/σ_y	1.38	1.38	1.43	2.19	2.27	2.99
ρ_{cy}	0.88	0.88	0.85	0.79	0.82	1.0
ρ_{iy}	0.91	0.91	0.90	0.93	0.94	1.0

SYNTHETIC EG model	MBD$_1$	MBD$_2$	HP	BP	GG	MBD$_3$
σ_y	5.56	5.59	1.94	1.76	1.74	1.96
σ_c	2.48	2.48	0.382	0.36	0.31	0.87
σ_i	11.59	11.59	4.45	4.02	3.99	4.01
σ_c/σ_y	0.45	0.45	0.20	0.21	0.18	0.44
σ_i/σ_y	2.09	2.07	2.29	2.28	2.30	2.05
ρ_{cy}	0.81	0.76	0.87	0.84	0.996	0.76
ρ_{iy}	0.87	0.85	0.66	0.87	0.87	0.85
σ_ϵ	0.009	0.009	0.006	0.006	0.003	0.003

SYNTHETIC KPR model	LT	HP	BP
σ_y	5.15	1.97	1.80
σ_c	3.18	0.60	0.57
σ_i	11.51	5.39	4.90
σ_c/σ_y	0.62	0.30	0.32
σ_i/σ_y	2.24	2.73	2.71
ρ_{cy}	0.84	0.89	0.88
ρ_{iy}	0.94	0.99	0.99
σ_ϵ	0.01	0.01	0.01

Notes: MBD$_1$ and MBD$_2$ denote the model-based decompositions characterized by equations (23) and (17), respectively. LT stands for the model-based decomposition associated with the KPR model, that is, the model-based decomposition obtained from removing a linear trend component from the original time series. MBD$_3$ denotes the model-based decomposition MBD$_1$ with $\sigma_\epsilon = 0.003$.

TABLE 2 Regressions results

Synthetic	Constant	GG	R^2
$\tilde{y}_t$	-0.1161	1.0181	0.7869
	(100)	(100)	
$\tilde{c}_t$	-0.6079	1.1096	0.1813
	(100)	(90)	
$\tilde{i}_t$	-1.0692	0.9918	0.9210
	(100)	(100)	

Synthetic	Constant	BP	R^2
$\tilde{y}_t$	-0.1116	1.4315	0.5068
	(100)	(100)	
$\tilde{c}_t$	-0.6048	2.3884	0.3549
	(100)	(99)	
$\tilde{i}_t$	-1.0643	1.3794	0.5543
	(100)	(100)	

Synthetic	Constant	HP	R^2
$\tilde{y}_t$	-0.1162	1.2400	0.4558
	(100)	(100)	
$\tilde{c}_t$	-0.6078	1.6864	0.1993
	(100)	(90)	
$\tilde{i}_t$	-1.0714	1.2179	0.5258
	(100)	(100)	

Notes: The numbers in parentheses represent the percentage of significance of the corresponding regression parameter for different realizations of the technology shock. A 0.05 percent significance level is considered.

TABLE 3. Sample moments for the great ratios and the work effort

Actual US data	$log(c/y)$	$log(i/y)$	$log(n)$
Standard deviation	2.743	3.517	2.379
correl. with log(c/y)	1.0	-0.232	-0.261
first-order autocorr.	0.936	0.912	0.805
second-order autocorr.	0.851	0.785	0.682
third-order autocorr.	0.760	0.630	0.611
cross-correl. with $log(n_{-12})$	0.303	-0.152	-0.061
cross-correl. with $log(n_{-8})$	0.227	-0.101	0.137
cross-correl. with $log(n_{-4})$	0.013	0.219	0.539
cross-correl. with $log(n_{-2})$	-0.143	0.477	0.682
cross-correl. with $log(n_{-1})$	-0.225	0.611	0.805
cross-correl. with log(n)	-0.261	0.673	1.0
cross-correl. with $log(n_{+1})$	-0.234	0.692	0.805
cross-correl. with $log(n_{+2})$	-0.188	0.666	0.682
cross-correl. with $log(n_{+4})$	-0.087	0.532	0.539
cross-correl. with $log(n_{+8})$	0.201	0.059	0.137
cross-correl. with $log(n_{+12})$	0.368	-0.189	-0.061

Synthetic data	EG model			KPR		
	$log(c/y)$	$log(i/y)$	$log(n)$	$log(c/y)$	$log(i/y)$	$log(n)$
Standard deviat.	1.904	3.455	1.809	1.430	3.352	1.087
correl. with log(c/y)	1.0	-0.669	-0.991	1.0	-1.0	-1.0
first-order autocorr.	0.867	0.642	0.863	0.881	0.881	0.881
second-order autocorr.	0.866	0.558	0.741	0.773	0.773	0.773
third-order autocorr.	0.718	0.474	0.630	0.674	0.674	0.674
cross-correl. with $log(n_{-12})$	-0.122	0.062	0.064	-0.130	0.130	0.130
cross-correl. with $log(n_{-8})$	-0.295	0.183	0.245	-0.316	0.316	0.316
cross-correl. with $log(n_{-4})$	-0.564	0.378	0.533	-0.586	0.586	0.586
cross-correl. with $log(n_{-2})$	-0.755	0.506	0.741	-0.773	0.773	0.773
cross-correl. with $log(n_{-1})$	-0.866	0.587	0.863	-0.881	0.881	0.881
cross-correl. with log(n)	-0.991	0.674	1.0	-1.0	1.0	1.0
cross-correl. with $log(n_{+1})$	-0.846	0.949	0.863	-0.881	0.881	0.881
cross-correl. with $log(n_{+2})$	-0.716	0.815	0.741	-0.773	0.773	0.773
cross-correl. with $log(n_{+4})$	-0.497	0.585	0.533	-0.586	0.586	0.586
cross-correl. with $log(n_{+8})$	-0.195	0.266	0.245	-0.316	0.316	0.316
cross-correl. with $log(n_{+12})$	-0.009	0.067	0.064	-0.130	0.130	0.130

TABLE 4. Sample moments for the growth rates

Actual US data	$\Delta\log(y_t)$	$\Delta\log(c_t)$	$\Delta\log(i_t)$	$\Delta\log(n_t)$
Standard deviation	1.144	0.534	2.206	1.487
correl. with $\Delta\log(y_t)$	1.0	0.517	0.792	0.353
first-order autocorr.	0.364	0.286	0.463	-0.183
second-order autocorr.	0.226	0.130	0.250	-0.139
third-order autocorr.	0.049	0.287	0.073	0.031
cross-correl. with $\Delta\log(y_{t-12})$	-0.074	-0.069	-0.113	-0.087
cross-correl. with $\Delta\log(y_{t-8})$	-0.209	-0.086	-0.174	-0.202
cross-correl. with $\Delta\log(y_{t-4})$	-0.015	0.012	-0.093	0.040
cross-correl. with $\Delta\log(y_{t-2})$	0.226	0.240	0.229	0.092
cross-correl. with $\Delta\log(y_{t-1})$	0.364	0.216	0.439	0.188
cross-correl. with $\Delta\log(y_t)$	1.0	0.517	0.792	0.353
cross-correl. with $\Delta\log(y_{t+1})$	0.364	0.405	0.471	0.232
cross-correl. with $\Delta\log(y_{t+2})$	0.226	0.190	0.219	-0.022
cross-correl. with $\Delta\log(y_{t+4})$	-0.015	0.168	0.046	-0.017
cross-correl. with $\Delta\log(y_{t+8})$	-0.209	-0.138	-0.247	-0.130
cross-correl. with $\Delta\log(y_{t+12})$	-0.074	0.033	-0.166	-0.049

Synthetic data (EG model)	$\Delta\log(y_t)$	$\Delta\log(c_t)$	$\Delta\log(i_t)$	$\Delta\log(n_t)$
Standard deviation	1.155	0.197	2.652	0.927
correl. with $\Delta\log(y_t)$	1.0	0.937	0.998	0.996
first-order autocorr.	-0.036	0.141	-0.042	-0.045
second-order autocorr.	-0.027	0.127	-0.032	-0.035
third-order autocorr.	-0.041	0.095	-0.046	-0.048
cross-correl. with $\Delta\log(y_{t-12})$	-0.018	0.015	-0.015	-0.026
cross-correl. with $\Delta\log(y_{t-8})$	-0.020	0.034	-0.032	-0.035
cross-correl. with $\Delta\log(y_{t-4})$	-0.036	0.051	-0.030	-0.058
cross-correl. with $\Delta\log(y_{t-2})$	-0.027	0.082	-0.052	-0.055
cross-correl. with $\Delta\log(y_{t-1})$	-0.036	0.085	-0.039	-0.067
cross-correl. with $\Delta\log(y_t)$	1.0	0.937	0.998	0.996
cross-correl. with $\Delta\log(y_{t+1})$	-0.036	-0.088	-0.048	-0.021
cross-correl. with $\Delta\log(y_{t+2})$	-0.027	-0.077	-0.030	-0.013
cross-correl. with $\Delta\log(y_{t+4})$	-0.036	-0.076	-0.035	-0.024
cross-correl. with $\Delta\log(y_{t+8})$	-0.020	-0.051	-0.018	-0.012
cross-correl. with $\Delta\log(y_{t+12})$	-0.018	-0.040	-0.012	-0.011

TABLE 4 (*Continued*)

Synthetic data (KPR)	$\Delta\log(y_t)$	$\Delta\log(c_t)$	$\Delta\log(i_t)$	$\Delta\log(n_t)$
Standard deviation	1.142	0.305	3.152	0.654
correl. with $\Delta\log(y_t)$	1.0	0.944	0.997	0.993
first-order autocorr.	-0.017	0.167	-0.032	-0.035
second-order autocorr.	-0.018	0.146	-0.032	-0.035
third-order autocorr.	-0.021	0.125	-0.033	-0.035
cross-correl. with $\Delta\log(y_{t-12})$	-0.012	0.028	-0.021	-0.026
cross-correl. with $\Delta\log(y_{t-8})$	-0.010	0.049	-0.024	-0.031
cross-correl. with $\Delta\log(y_{t-4})$	-0.019	0.071	-0.040	-0.051
cross-correl. with $\Delta\log(y_{t-2})$	-0.018	0.091	-0.043	-0.057
cross-correl. with $\Delta\log(y_{t-1})$	-0.017	0.103	-0.044	-0.060
cross-correl. with $\Delta\log(y_t)$	1.0	0.944	0.997	0.993
cross-correl. with $\Delta\log(y_{t+1})$	-0.017	-0.048	-0.010	-0.006
cross-correl. with $\Delta\log(y_{t+2})$	-0.018	-0.048	-0.012	-0.008
cross-correl. with $\Delta\log(y_{t+4})$	-0.019	-0.046	-0.013	-0.010
cross-correl. with $\Delta\log(y_{t+8})$	-0.010	-0.033	-0.005	-0.002
cross-correl. with $\Delta\log(y_{t+12})$	-0.012	-0.033	-0.006	-0.004