Gradu Amaierako Lana
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On the number of spatial dimensions

Egilea:
Karla Legorburu Newhouse
Zuzendaria:
José María Martín Senovilla

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Chapter 1

Introduction

Most physical theories take the number of spacetime dimensions for granted, they are constructed according to the number of dimensions perceived by us: three spatial ones, and a single time one. The aim of this study is to analyse proposals which question this assumption, either by stating that our universe could have a different number of dimensions, or by arguing that the number in fact agrees with our perception. Bearing in mind that the majority of theories involving a different dimensionality of spacetime refer to the spatial dimensions, these are the ones which will be focused on. Two radically different approaches have been chosen for this purpose. On the one hand, attempts to formulate a unified theory of physics have been considered, which often postulate additional spatial dimensions. This approach has had a significant impact on our understanding of spatial dimensions, and has led to a thorough consideration of extra compact dimensions. On the other hand, the reasoning induced by the anthropic principle will be studied, and its specific application on the number of dimensions.

Firstly, unified theories will be discussed in a chapter composed of two sections, the Kaluza-Klein theory will be explained in the former, and string theory, in the latter. The Kaluza-Klein theory postulates an extra spatial dimension, in order to construct a unified theory of electromagnetism and gravitation. As regards to string theory, the focus will be on a single calculation, due to a particularity of the theory under question: the number of spacetime dimensions is not postulated, it is a requirement of the theory.

Secondly, regarding the anthropic principle, as forementioned, the actual reasoning it entails is what the chapter will be centred on. Therefore, its historical development will be exposed, that is to say, the precedents from which it arose will be explained, and subsequently, the application of this reasoning on the number of spatial dimensions will be considered.

Chapter 2

Unified theories of physics and spatial dimensions

Our aim in this chapter is to study the changes in our understanding of the spatial dimensions (or at least, in the number of dimensions the universe is thought to have) under the influence of the development of theories seeking to give way to a unification of different branches of physics. No matter the variety of points of view on this type of theories, one cannot overlook the fact that the search for unification has very often led to valuable insight into the physical world. For instance, take the unification of electricity and magnetism achieved as a result of Maxwell and Faraday's work, and how it changed our understanding of light. Or the synthesis of the concepts of *space* and *time*, expressed in the new idea of *spacetime*, within Einstein's special relativity. Moreover, we must also refer to the further unification of those two concepts attained by means of his brilliant theory of general relativity, in which spacetime is considered to be dynamical. There are many other examples, such as the wave-particle duality in quantum physics, but what is important to us now is to study the theories which, in return for unification, involve additional spatial dimensions, that is, theories which demand a universe with more than three spatial dimensions. Therefore, the *Kaluza Klein theory* will be studied first, due to it being one of the first proposals of its kind, followed by a brief overview of the dimensionality of spacetime in *string theory*, on account of it being a contemporary proposal which may still have a broad path ahead.

2.1 Kaluza-Klein theory

Theories involving extra dimensions have been significantly developed in the last decades, becoming more and more complex, but it can be said that we may find their foundation in the Kaluza-Klein theory, named after the work of Theodor Kaluza and Oskar Klein. So as Kaluza was one of the first who made an attempt to construct such a theory, we will begin by overviewing his original paper, "On the Unification Problem in Physics" [1], which he wrote in 1919 and was published in 1921 by Albert Einstein. It is important to point out that when this paper was published, there was no knowledge whatsoever concerning the weak or strong interactions, and also that quantum mechanics had hardly been developed. Therefore, the search for unification that will be studied only concerns gravitation and electromagnetism.

Five years after Einstein published Kaluza's paper, Klein, who by that time had been able to study quantum physics, improved Kaluza's proposal by adapting it to quantum mechanics on the one hand, and by speculating about the geometry of the extra dimension on the other (Kaluza had not explicitly mentioned this aspect of the extra dimension). As to the note made about Kaluza's theory regarding the weak and strong interactions, the same holds for Klein's version: the years in which he developed his modifications were still far from the discovery of these interactions. Klein published his version of the five-dimensional theory in the articles "Quantum Theory and Five-Dimensional Relativity Theory" [2] and "The Atomicity of Electricity as a Quantum Theory Law" [3]. They were both published in 1926; the former in April, and the latter, in October.

The general procedure followed by Kaluza and Klein consists in writing the metric tensor of spacetime with an extra dimension, including an electromagnetic four-potential in it. Then, the condition known as the *cylinder condition* is applied, which will be defined later on, and the five-dimensional analogues of the following expressions are obtained: the Christoffel symbols of the first kind, which are first order derivatives of the metric; the Riemann curvature tensor, which consists of derivatives of the Christoffel symbols; the Ricci tensor, obtained by contracting the Riemann tensor; and the Ricci scalar, which is a contraction of the Ricci tensor. Kaluza and Klein show that all the resulting expressions can be rewritten in terms of the usual four-dimensional tensors, together with obtaining both the equations for electromagnetism and for gravitation; that is to say, they prove it possible to unify these two branches of physics by considering one extra spatial dimension.

In order to understand Kaluza and Klein's results, we must first revise what Maxwell's equations look like in the relativistic notation, because this is the form in which the electromagnetic equations will arise.

2.1.1 Relativistic formulation of Maxwell's equations

The source-free Maxwell equations in the Heaviside-Lorentz system of units are:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{2.1}$$

$$\nabla \cdot \vec{B} = 0. \tag{2.2}$$

The ones involving sources:

$$\nabla \cdot \vec{E} = \rho, \tag{2.3}$$

$$\nabla \times \vec{B} = \frac{1}{c}\vec{j} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \tag{2.4}$$

As the magnetic field $\vec{B}$ is divergenceless, it can be written as the curl of a vector, the well-known *vector potential* $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}. \tag{2.5}$$

Substituting this expression into (2.1),

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{2.6}$$

By setting the object between brackets equal to $-\nabla\phi$, the electric field $\vec{E}$ can be written as:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi. \tag{2.7}$$

It is worth noting that the source-free Maxwell equations are automatically satisfied with the introduction of potentials.

We construct a four-vector by combining the scalar potential ϕ with the vector potential $\vec{A}$:

$$A^{\mu} = \left(\phi, A^1, A^2, A^3\right), \tag{2.8}$$

$$A_{\mu} = \left(-\phi, A^1, A^2, A^3\right). \tag{2.9}$$

We will rewrite the equations in terms of the electromagnetic *field strength* $F_{\mu\nu}$:

$$F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \quad \text{where} \quad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}. \tag{2.10}$$

We can see that $F_{\mu\nu}$ is antisymmetric, $F_{\mu\nu} = -F_{\nu\mu}$. By computing all the components, one obtains:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2.11}$$

Considering the combination $\partial_{\lambda}F_{\mu\nu} + \partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu}$, we can see that it vanishes when $F_{\mu\nu}$ is given by (2.10) [4]. Recalling that the use of potentials to represent $\vec{E}$ and $\vec{B}$ automatically implies the source-free Maxwell equations being satisfied, it follows that they are both encoded by the following set of differential equations for the field strength:

$$\partial_{\lambda}F_{\mu\nu} + \partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} = 0. \tag{2.12}$$

A *current* four-vector must be introduced in order to describe the Maxwell equations (2.3) and (2.4),

$$j^{\mu} = \left(c\rho, j^1, j^2, j^3\right), \tag{2.13}$$

where ρ is the charge density and $\vec{j} = (j^1, j^2, j^3)$ is the current density.

The field tensor with upper indices is

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}. \tag{2.14}$$

Considering i and j as spatial indexes (they can take the values 1, 2 and 3), it can be shown that [4]:

$$F^{\mu\nu} = -F^{\nu\mu}, \tag{2.15}$$

$$F^{0i} = -F_{0i}, \quad F^{ij} = F_{ij}, \tag{2.16}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2.17}$$

(2.3) and (2.4) can now be encapsulated as

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{1}{c} j^\mu. \tag{2.18}$$

In the absence of sources, this becomes $\partial_\nu F^{\mu\nu} = 0$. In short, the equivalent of Maxwell's equations in a more compact way is given by

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{2.19a}$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{1}{c} j^\mu. \tag{2.19b}$$

2.1.2 Kaluza's theory

Kaluza's main aim is to show that the equations for both the gravitational and the electromagnetic fields could stem from a single universal tensor, by considering an extra fourth spatial dimension. He is motivated by the apparent similarity between $F_{\chi\lambda}$ and the *Christoffel symbols of the first kind* $\begin{bmatrix} i\lambda \\ \chi \end{bmatrix}$[1]:

$$\frac{1}{2} F_{\chi\lambda} = \frac{1}{2} \left(A_{\chi,\lambda} - A_{\lambda,\chi} \right), \tag{2.20}$$

$$\begin{bmatrix} i\lambda \\ \chi \end{bmatrix} = \frac{1}{2} \left(g_{i\chi,\lambda} + g_{\chi\lambda,i} - g_{i\lambda,\chi} \right). \tag{2.21}$$

He introduces the factor 1/2 in the first expression simply due to the correspondence with $\begin{bmatrix} i\lambda \\ \chi \end{bmatrix}$. In his paper, he actually uses q_χ instead of A_χ, but even so, we will still stick to A_χ. As ususal, indices after a comma represent partial derivatives with respect to one of the coordinates, for example, $A_{\chi,\lambda} = \frac{\partial A_\chi}{\partial x^\lambda}$.

In order to construct the five-dimensional theory, Kaluza sets the Christoffel symbols of the first kind just as the usual four dimensional ones, but allowing the index values to run from 0 to 4. Latin indices will run from 0 to 4, and Greek ones from 0 to 3 (the extra dimensional coordinate will be referred to as x^4)[2], which means that the Greek indices will correspond to the common spacetime dimensions, and the Latin ones, to the new five-dimensional constructions. Therefore, Kaluza writes the five-dimensional Christoffel symbols as

$$\begin{bmatrix} ik \\ l \end{bmatrix} = \frac{1}{2} \left(g_{li,k} + g_{kl,i} - g_{ik,l} \right). \tag{2.22}$$

[1]In this section, the Christoffel symbols will be referred to in accordance with the notation used by Kaluza, even though when introducing the form Γ_{ikl}, this notation and the one used today differ by a minus sign.

[2]Kaluza (and Klein) denoted the new coordinate by x^0. However, the extra dimensional component will be thought of as the fifth one, because we are currently used to x^0 denoting the time coordinate $x^0 = ct$, and also due to the fact that all subsequent reviews tend to take it to be the fifth one. As a consequence, the new coordinate will be x^4. This will also be applied in the section on Klein's work.

He denotes $\begin{bmatrix} ik \\ l \end{bmatrix}$ by $-\Gamma_{ikl}$ (as previously mentioned, Kaluza's notation differs from the one used today).

Taking into account that there has been no experimental evidence which suggests an extra dimension, Kaluza proposes the *cylinder condition*, making the derivatives of all vector and tensor quantities with respect to the new coordinate vanish (or be sufficiently small). So by applying this condition (in this case, by using $g_{ik,4} = 0$), and explicitly distinguishing the extra dimension from the ordinary ones, the new three-index quantities become

$$\Gamma_{\chi\lambda\mu} = \frac{1}{2}\left(g_{\chi\lambda,\mu} - g_{\lambda\mu,\chi} - g_{\mu\chi,\lambda}\right), \tag{2.23a}$$

$$\Gamma_{4\chi\lambda} = \frac{1}{2}\left(g_{4\chi,\lambda} - g_{4\lambda,\chi}\right), \tag{2.23b}$$

$$\Gamma_{\chi\lambda 4} = -\frac{1}{2}\left(g_{4\chi,\lambda} + g_{4\lambda,\chi}\right), \tag{2.23c}$$

$$\Gamma_{44\chi} = \frac{1}{2} g_{44,\chi}, \tag{2.23d}$$

$$\Gamma_{444} = 0. \tag{2.23e}$$

In order to keep $\Gamma_{4\chi\lambda}$ proportional to the terms of the electromagnetic field tensor, that is, to $F_{\chi\lambda}$, he sets

$$g_{4\chi} = 2\alpha A_\chi, \tag{2.24}$$

where α is just a proportionality constant. He also chooses

$$g_{44} = 2\mathfrak{g}. \tag{2.25}$$

$\mathfrak{g}$ is an additional scalar field, whose role remains undetermined for the time being[3].

Hence, the five-dimensional fundamental metric tensor is now the usual four-dimensional metric tensor framed by the electromagnetic four-potential:

$$g_{ik} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & 2\alpha A_0 \\ g_{10} & g_{11} & g_{12} & g_{13} & 2\alpha A_1 \\ g_{20} & g_{21} & g_{22} & g_{23} & 2\alpha A_2 \\ g_{30} & g_{31} & g_{32} & g_{33} & 2\alpha A_3 \\ 2\alpha A_0 & 2\alpha A_1 & 2\alpha A_2 & 2\alpha A_3 & 2\mathfrak{g} \end{pmatrix}. \tag{2.26}$$

With such a fundamental metric tensor, bearing in mind that $F_{\chi\lambda} = A_{\chi,\lambda} - A_{\lambda,\chi}$ and introducing $\Sigma_{\chi\lambda} = A_{\chi,\lambda} + A_{\lambda,\chi}$,

$$\Gamma_{4\chi\lambda} = \alpha F_{\chi\lambda}, \tag{2.27a}$$

$$\Gamma_{\chi\lambda 4} = -\alpha \Sigma_{\chi\lambda}, \tag{2.27b}$$

$$\Gamma_{44\chi} = -\Gamma_{4\chi 4} = \mathfrak{g}_{,\chi}. \tag{2.27c}$$

Now, we shall see that the following expressions arise:

$$F_{\chi\lambda,\mu} + F_{\lambda\mu,\chi} + F_{\mu\chi,\lambda} = 0 \quad \text{and} \quad \mathfrak{g}_{,\chi\lambda} = \mathfrak{g}_{,\lambda\chi}. \tag{2.28}$$

[3] In the first works based on this idea (including Klein's), $\mathfrak{g}$ is usually taken to be a constant, and normalised as $g_{44} = 1$, in spite of Kaluza's choice. Further on, the scalar field is reconsidered, and commonly denoted as ϕ.

Derivation of (2.28)

Firstly, it will be proved that $(\Gamma_{ikl}+\Gamma_{kli}+\Gamma_{lik})_{,m}=\Gamma_{mik,l}+\Gamma_{mkl,i}+\Gamma_{mli,k}$, as it is a useful expression for the derivation. [5]

$$(\Gamma_{ikl}+\Gamma_{kli}+\Gamma_{lik})_{,m}=\frac{1}{2}\left(g_{ik,l}-g_{kl,i}-g_{li,k}\right)_{,m}+\frac{1}{2}\left(g_{kl,i}-g_{li,k}-g_{ik,l}\right)_{,m}+$$
$$+\frac{1}{2}\left(g_{li,k}-g_{ik,l}-g_{kl,i}\right)_{,m}=-\frac{1}{2}\left(g_{kl,im}+g_{li,km}+g_{ik,lm}\right).$$

Assuming that the order of metric derivations is irrelevant, we will add three pairs of opposite terms:

$$(\Gamma_{ikl}+\Gamma_{kli}+\Gamma_{lik})_{,m}=\frac{1}{2}\left(-g_{kl,mi}-g_{li,mk}-g_{ik,ml}+g_{mi,kl}-g_{im,lk}-g_{km,il}+g_{mk,li}\right.$$
$$\left.-g_{lm,ki}+g_{ml,ik}\right)=\frac{1}{2}\left(g_{mi,k}-g_{ik,m}-g_{km,i}\right)_{,l}+\frac{1}{2}\left(g_{mk,l}-g_{kl,m}-g_{lm,k}\right)_{,i}+\frac{1}{2}\left(g_{ml,i}\right.$$
$$\left.-g_{li,m}-g_{im,l}\right)_{,k}=\Gamma_{mik,l}+\Gamma_{mkl,i}+\Gamma_{mli,k}.$$

Now, we shall take $m=4$ in this expression,

$$(\Gamma_{ikl}+\Gamma_{kli}+\Gamma_{lik})_{4}=\Gamma_{4ik,l}+\Gamma_{4kl,i}+\Gamma_{4li,k}.$$

Due to the cylinder condition, $(\Gamma_{ikl}+\Gamma_{kli}+\Gamma_{lik})_{4}=0$, so

$$\Gamma_{4ik,l}+\Gamma_{4kl,i}+\Gamma_{4li,k}=0,$$

which is also true if we only consider the Greek indices, hence,

$$\Gamma_{4\chi\lambda,\mu}+\Gamma_{4\lambda\mu,\chi}+\Gamma_{4\mu\chi,\lambda}=0.$$

We also know from (2.27) that $\Gamma_{4\chi\lambda}=\alpha F_{\chi\lambda}$, which means:

$$\alpha\left(F_{\chi\lambda,\mu}+F_{\lambda\mu,\chi}+F_{\mu\chi,\lambda}\right)=0.$$

$$\alpha\neq 0\rightarrow \boxed{F_{\chi\lambda,\mu}+F_{\lambda\mu,\chi}+F_{\mu\chi,\lambda}=0.}$$

If instead of taking $m=4$ we now choose $i=k=4$,

$$(\Gamma_{44l}+\Gamma_{4l4}+\Gamma_{l44})_{m}=\Gamma_{m44,l}+\Gamma_{m4l,4}+\Gamma_{ml4,4}=\Gamma_{m44,l}\rightarrow(\Gamma_{44l}+\Gamma_{4l4}+\Gamma_{l44})_{m}=\Gamma_{m44,l},$$

where the cylinder condition has been used in the second step. Further developing those four terms,

$$\Gamma_{m44}=-\frac{1}{2}g_{44,m}\overset{(2.25)}{=}-\mathfrak{g}_{,m}\quad(\text{and}\quad\Gamma_{l44}=-\mathfrak{g}_{,l}),$$
$$\Gamma_{44l}=\mathfrak{g}_{,l},$$
$$\Gamma_{4l4}=-\mathfrak{g}_{,l}.$$

Finally,

$$(\Gamma_{44l}+\Gamma_{4l4}+\Gamma_{l44})_{m}=(\mathfrak{g}_{,l}-\mathfrak{g}_{,l}-\mathfrak{g}_{,l})_{m}=-\mathfrak{g}_{,ml}\rightarrow\mathfrak{g}_{,lm}=\mathfrak{g}_{,ml}\rightarrow\boxed{\mathfrak{g}_{,\chi\lambda}=\mathfrak{g}_{,\lambda\chi}.}$$

. . . .

Therefore, it has been shown that $F_{\chi\lambda,\mu}+F_{\lambda\mu,\chi}+F_{\mu\chi,\lambda}=0$, which is precisely the compact form of two of the Maxwell equations, equations (2.1) and (2.2), as seen in the section *Relativistic Formulation of Maxwell's Equations.*

In order to continue with the calculations, Kaluza is forced to make two approximations so that the complexity is reduced: on the one hand, he only considers the case in which the fields are weak, that is to say, the case in which $g_{ik} = \eta_{ik} + h_{ik}$, where $|h| << 1$ and $\eta_{44} = 1$; and on the other hand, he also restricts himself to small velocities.[4]

Next, he calculates the components of the five-dimensional analogue of the *Ricci curvature tensor* (under the forementioned approximations), R_{ik}, which is a contraction of the (five-dimensional equivalent of the) *Riemann curvature tensor* R^i_{jkl}:

$$R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}, \tag{2.29}$$

$$R_{jk} = R^i_{jki}. \tag{2.30}$$

Kaluza then shows that the four-dimensional components of the new Ricci curvature tensor, $R_{\mu\nu}$, can lead us to the ordinary equations of gravitation, whilst $R_{0\mu}$ gives way to the remaining electromagnetic equations, and R_{00}, to an equation for the scalar field $\mathfrak{g}$ ($R_{00} = -\Box\mathfrak{g}$, where $\Box = \partial^\mu\partial_\mu$). In his book *On the unification problem in physics - the paper of Theodor Kaluza* [5], aiming for further clarification, Samir Lipovaca rewrites Kaluza's result for $R_{0\mu}$ as

$$R_{0\mu} = -\alpha F^{\mu\rho}_{,\rho}. \tag{2.31}$$

Thus, by comparing this equation to (2.18), we can see how $R_{0\mu}$ can provide the remaining Maxwell equations, i.e., (2.3) and (2.4). Kaluza also concludes that $\mathfrak{g}$ is equal to minus a gravitational potential, but it must said that further works on this issue either ignore the indicated term (that is, consider it a constant), and take it to be equal to 1, or give a completely different interpretation.

It is interesting to refer to Einstein's reaction once he had read Kaluza's paper. When he first replied to Kaluza's letter on April 21, 1919, he said:

> "The idea of achieving a unified theory by means of a five-dimensional cylinder world never dawned on me... At first glance I like your idea enormously."[a]

[a]A. Einstein in a letter to T. Kaluza as quoted in "'Subtle is the Lord...': The Science and the Life of Albert Einstein" [6], on page 330.

However, about a week later, he added [7]: "I have read through your paper and find it really interesting. Nowhere, so far, can I see an impossibility. On the other hand, I have to admit that the arguments brought forward so far do not appear convincing enough". More than two years later, on October 14, 1921:

> "I am having second thoughts about having restrained you from publishing your idea on a unification of gravitation and electricity two years ago.... If you wish, I shall present your paper to the academy after all."[a]

[a]A. Einstein in a letter to T. Kaluza as quoted in "The Hidden Dimensions of Spacetime" [7], on page 78.

[4]These approximations are not considered in Klein's work.

And we know how that ended, or else it is probable that we would not be studying this proposal in the first place! Kaluza was aware of his theory's limits considering the set of phenomena which classical physics failed to explain at the time (quantum effects), that would soon lead to the developed formulation of quantum mechanics.

"At the moment the consequence of this hypotheses is not yet examined sufficiently; other possibilities should also be searched for. After all, what threatens all the ansatz which demand universal validity is the sphinx of modern physics - quantum theory." [1]

2.1.3 Klein's improvements

As already stated, the Kaluza-Klein theory arose from Kaluza's proposal, and Oskar Klein held on to the same procedure: to consider an extra spatial dimension in order to construct a new (five-dimensional) fundamental metric tensor and thus obtain a unified theory of electromagnetism and Einstein's gravitation. Klein divided his paper "Quantum Theory and Five-Dimensional Relativity Theory" [2] into two sections: "Five-Dimensional Theory of Relativity" and "The Wave Equation of the Quantum Theory", and his paper "The Atomicity of Electricity as a Quantum Theory Law"[3] cannot be left aside, in which he refers to the geometrical aspect of the theory. The order in which his work will be addressed will therefore follow his own arrangement: firstly, we shall attempt to stress the main features developed in the two sections of his first paper on this issue, and then the geometrical side of the problem will be tackled.

Five-Dimensional Theory of Relativity

First of all, Klein slightly modifies Kaluza's theory, starting with the fundamental metric tensor.

"I begin by giving a short description of the five-dimensional relativity theory which connects closely to Kaluza's theory but differs in some points from it." [3]

In order to better visualise the new metric, a new notation will be adopted. The five-dimensional metric will be denoted by $\tilde{g}_{ik}$, and the four-dimensional one by $g_{\mu\nu}$ (Klein actually denotes the new metric by γ^{ik}).

Whilst Kaluza's five dimensional metric has the following form,

$$\tilde{g}_{ik} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & 2\alpha A_0 \\ g_{10} & g_{11} & g_{12} & g_{13} & 2\alpha A_1 \\ g_{20} & g_{21} & g_{22} & g_{23} & 2\alpha A_2 \\ g_{30} & g_{31} & g_{32} & g_{33} & 2\alpha A_3 \\ 2\alpha A_0 & 2\alpha A_1 & 2\alpha A_2 & 2\alpha A_3 & 2\mathfrak{g} \end{pmatrix}, \tag{2.32}$$

Klein revised it, suggesting what is now known as the *Kaluza Klein metric*, by also including second-order terms:

$$\tilde{g}_{ik} = \begin{pmatrix} g_{00} + \alpha\beta^2 A_0 A_0 & g_{01} + \alpha\beta^2 A_0 A_1 & g_{02} + \alpha\beta^2 A_0 A_2 & g_{03} + \alpha\beta^2 A_0 A_3 & \alpha\beta A_0 \\ g_{10} + \alpha\beta^2 A_1 A_0 & g_{11} + \alpha\beta^2 A_1 A_1 & g_{12} + \alpha\beta^2 A_1 A_2 & g_{13} + \alpha\beta^2 A_1 A_3 & \alpha\beta A_1 \\ g_{20} + \alpha\beta^2 A_2 A_0 & g_{21} + \alpha\beta^2 A_2 A_1 & g_{22} + \alpha\beta^2 A_2 A_2 & g_{23} + \alpha\beta^2 A_2 A_3 & \alpha\beta A_2 \\ g_{30} + \alpha\beta^2 A_3 A_0 & g_{31} + \alpha\beta^2 A_3 A_1 & g_{32} + \alpha\beta^2 A_3 A_2 & g_{33} + \alpha\beta^2 A_3 A_3 & \alpha\beta A_3 \\ \alpha\beta A_0 & \alpha\beta A_1 & \alpha\beta A_2 & \alpha\beta A_3 & \alpha \end{pmatrix}.$$

Or, in a more compact notation,

$$\tilde{g}_{ik} = \begin{pmatrix} g_{\mu\nu} + \alpha\beta^2 A_\mu A_\nu & \alpha\beta A_\mu \\ \alpha\beta A_\nu & \alpha \end{pmatrix}. \tag{2.33}$$

The term corresponding to $\tilde{g}_{44}$ is considered constant by Klein, he calls it α. β is also a constant, whose value is given further on. He subsequently takes α to be 1, but even so, it will maintained for the time being. The inverse metric takes the form

$$\tilde{g}^{ik} = \begin{pmatrix} g^{00} & g^{01} & g^{02} & g^{03} & -\beta A^0 \\ g^{10} & g^{11} & g^{12} & g^{13} & -\beta A^1 \\ g^{20} & g^{21} & g^{22} & g^{23} & -\beta A^2 \\ g^{30} & g^{31} & g^{32} & g^{33} & -\beta A^3 \\ -\beta A^0 & -\beta A^1 & -\beta A^2 & -\beta A^3 & \frac{1}{\alpha} + \beta^2 A_\mu A^\mu \end{pmatrix}, \tag{2.34}$$

giving rise to the correct property

$$\tilde{g}_{ik}\tilde{g}^{kl} = \delta_i^k. \tag{2.35}$$

He forms the invariant:

$$P = \tilde{g}^{ik} \left[\frac{\partial \begin{Bmatrix} i\mu \\ \mu \end{Bmatrix}}{\partial x^k} - \frac{\partial \begin{Bmatrix} ik \\ \mu \end{Bmatrix}}{\partial x^\mu} + \begin{Bmatrix} i\mu \\ \nu \end{Bmatrix} \begin{Bmatrix} k\nu \\ \mu \end{Bmatrix} - \begin{Bmatrix} ik \\ \mu \end{Bmatrix} \begin{Bmatrix} \mu\nu \\ \nu \end{Bmatrix} \right], \tag{2.36}$$

in which $\begin{Bmatrix} i\mu \\ \mu \end{Bmatrix}$ stands for the *Christoffel symbols of the second kind*,

$$\begin{Bmatrix} rs \\ i \end{Bmatrix} = \frac{1}{2}\tilde{g}^{i\mu} \left(\frac{\partial \tilde{g}_{\mu r}}{\partial x^s} + \frac{\partial \tilde{g}_{\mu s}}{\partial x^r} - \frac{\partial \tilde{g}_{rs}}{\partial x^\mu} \right). \tag{2.37}$$

Then, he considers the integral:

$$J = \int P\sqrt{-\tilde{g}}dx^0 dx^1 dx^2 dx^3 dx^4, \tag{2.38}$$

where $\tilde{g}$ is the determinant of $\tilde{g}_{ik}$. Klein naturally assumed that integral should be the five-dimensional action (P is just the five-dimensional analogue of the *scalar curvature* R). Therefore, the application of the variational principle on the action J, that is,

$$\delta J = 0, \tag{2.39}$$

leads him to the following equations:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{\alpha\beta^2}{2}T^{\mu\nu} = 0, \tag{2.40a}$$

$$\frac{\partial\sqrt{-g}F^{\nu\mu}}{\partial x^\mu} = 0. \tag{2.40b}$$

The calculations needed in order to obtain those equations are very long, they involve rewriting everything in terms of the usual four-dimensional quantities. $R^{\mu\nu}$ represent the contravariant components of the Ricci curvature tensor; R is the scalar curvature,

$R = g^{\mu\nu}R_{\mu\nu}$); $T^{\mu\nu}$ are the contravariant components of the electromagnetic *energy-momentum tensor*, in SI units, $T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\nu}F^{\alpha}_{\nu} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$; and $F^{\mu\nu}$ are the contravariant components of the electromagnetic field tensor (all of these tensors being four-dimensional). By setting

$$\frac{\alpha\beta^2}{2} = \kappa, \tag{2.41}$$

κ being the Einstein gravitational constant, the obtained equations are thus equivalent to the gravitational field equations on the one hand, and to the source-free Maxwell equations on the other[5].

The Wave Equation of the Quantum Theory

As Klein says, Kaluza connects Einstein's gravitation to the electromagnetic potentials, by considering a five dimensional spacetime. "The equations of motion of charged particles then take the form of equations of geodesic lines also in electromagnetic fields" [2]. However, Klein had been able to study Schrödinger and de Broglie's then recent work. Therefore, intending to modify Kaluza's proposal in accordance with his knowledge on the quantum theory, he continues:

> "When these are interpreted as wave equations by considering the matter as a kind of propagating wave, then one is led to a partial differential equation of second order which can be regarded as a generalization of the usual wave equation. If, now, such solutions to these equations are considered in which the fifth dimension appears with a definite period related to Planck's constant, one comes directly to the above mentioned quantum theoretical methods."

He considers the following five-dimensional differential equation as a generalisation of the wave equation:

$$\tilde{g}^{ik}\left(\frac{\partial^2\psi}{\partial x^i \partial x^k} - \begin{Bmatrix} ik \\ r \end{Bmatrix} \frac{\partial \psi}{\partial x^r}\right) = 0, \tag{2.42}$$

ψ being the five-dimensional wave function. The wave function can be written as a complex quantity for which the action S is the phase of the wave, that is,

$$\psi = Ce^{iS/\hbar}. \tag{2.43}$$

Let us consider the Lagrangian for a free particle in the five-dimensional spacetime,

$$L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k, \tag{2.44}$$

$$H = \dot{x}^i p_i - L, \quad \text{where} \quad p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{2.45}$$

In our case, the canonical momentum becomes $p_i = g_{ik}\dot{x}^k$. Due to the lack of dependence on the extra dimension, i.e., due to the cylinder condition, it follows that the extra dimension component of the momentum is constant.

[5] Just as Klein pointed out, equation (2.40b) is another way in which W. Pauli expressed (2.12) in *Theory of Relativity*, which first appeared in 1921, as can be seen on page 79 of [8].

Setting

$$p_i = \frac{\partial S}{\partial x^i}, \tag{2.46}$$

as p_4 is constant, S must satisfy $S(x_0, x_1, x_2, x_3, x_4) = p_4 x_4 + S(x_0, x_1, x_2, x_3)$. So taking (2.43) into account, the wave function may be written as

$$\psi(x_0, x_1, x_2, x_3, x_4) = e^{ip_4 x_4/\hbar} \phi(x_0, x_1, x_2, x_3). \tag{2.47}$$

Here ϕ should be regarded as the wave function in four-dimensional spacetime. Ignoring the terms proportional to $\begin{Bmatrix} ik \\ r \end{Bmatrix}$ in (2.42) [2] [9], the following equation is obtained:

$$\left[\left(\partial_\mu - \frac{i}{\hbar} e A_\mu\right)\left(\partial^\mu - \frac{i}{\hbar} e A^\mu\right) + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0, \tag{2.48}$$

which takes the well-known form of the *Klein-Gordon equation* when the vector potential is set to 0:

$$\left(\Box + m^2\right)\psi = 0, \quad \text{where} \quad \Box = \partial^\mu \partial_\mu. \tag{2.49}$$

The latter equation actually owes its name to the fact that Oskar Klein and Walter Gordon derived the equation independently in the same year. However, in spite of Klein having considered the extra dimension, Gordon did not. The equation is still used today for relativistic particles without spin.

A curled up dimension

Whilst Kaluza did not explicitly mention the geometrical aspect of the extra dimension, in 1926, Klein refers to the geometry of the extra dimension in his article “The Atomicity of Electricity as a Quantum Theory Law” [3] published in Nature. He assumes the extra dimension to be closed and periodic, and obtains the following estimated value for its length, which he denotes by l:

$$l = \frac{hc\sqrt{2\kappa}}{e} = 0.8 \times 10^{-30}\,\text{cm}. \tag{2.50}$$

> “The small value of this length together with the periodicity in the fifth dimension may perhaps be taken as a support of the theory of Kaluza in the sense that they may explain the non-appearance of the fifth dimension in ordinary experiments as the result of averaging over the fifth dimension.” [3]

Klein’s derivation of (2.50)

On the one hand, the quantization of p_4, the component of the momentum in the extra dimension, shall be studied, following from the periodicity of the extra dimension [10]. On the other hand, another expression for p_4 will be given, needed in order for the system of equations to adopt the usual form. Finally, by comparing these two expressions, the approximate value of the length corresponding to the extra dimension given by Klein will be obtained.

First of all, let us infer the implications of the extra dimension being periodic. Considering the forementioned to be closed with a period r, its coordinate may be described mathematically via the identification

$$x^4 \sim x^4 + 2\pi r. \tag{2.51}$$

On account of this, the wavefunction ψ must satisfy the boundary condition

$$\psi\left(x^4, x^\mu\right) = \psi\left(x^4 + 2\pi r, x^\mu\right). \tag{2.52}$$

As it is periodic, the following expansion for ψ can be given:

$$\psi\left(x^4, x^\mu\right) = \sum_n \psi_n\left(x^\mu\right) e^{\frac{inx^4}{r}}. \tag{2.53}$$

From quantum mechanics, the exponential term will be equal to the next expression:

$$e^{\frac{inx^4}{r}} = e^{ik_4 x^4} = e^{\frac{ip_4 x^4}{\hbar}} \rightarrow \frac{n}{r} = \frac{p_4}{\hbar}. \tag{2.54}$$

Hence, in order for the condition (2.52) to be satisfied, the momentum in the extra dimension must be quantized in this manner:

$$p_4 = n\frac{\hbar}{r}. \tag{2.55}$$

Secondly, the form that the component of the momentum in the extra dimension should take according to Klein shall be examined.

It was previously mentioned that Klein eventually takes α to be 1, and it must be noted that in this derivation (which he develops in the article [3]), he already considers it like so. The line element is $\mathrm{d}\sigma = \sqrt{\left(\mathrm{d}x^4 + \beta A_i \mathrm{d}x^i\right)^2 + g_{ik}\mathrm{d}x^i\mathrm{d}x^k}$, where the constant β is now given by $\beta = \sqrt{2\kappa}$ (the expression for the line element is obtained by choosing the positive root from $\mathrm{d}\sigma^2 = g_{ik}\mathrm{d}x^i\mathrm{d}x^k$). Taking $d\tau$ as the differential of proper time for a particle with mass m and charge q, the Lagrange function L for the geodetics representing the particle's motion:

$$L = \frac{1}{2}m\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\right)^2. \tag{2.56}$$

Momentum is defined in the ordinary way:

$$p_i = \frac{\partial L}{\partial\left(\mathrm{d}x^i/d\tau\right)}. \tag{2.57}$$

According to Klein, the system corresponding to (2.56) indeed becomes identical to the equations of motion for the particle if we set

$$p_4 = \frac{q}{\beta c}, \tag{2.58}$$

and, assuming that the charge has to be a multiple of the elementary charge, that is, assuming $q = ne$ (where n is a whole number, and e has the value of $e = 1.602 \times 10^{-19}$ C),

$$p_4 = \frac{ne}{\beta c}. \tag{2.59}$$

Finally, taking into account both (2.55) and (2.59), the size that Klein gave for the compact dimension is obtained,

$$\boxed{l = \frac{hc\sqrt{2\kappa}}{e} = 0.8 \times 10^{-30}\,\text{cm}.}$$

. . . .

Klein's suggestion concerning the geometry of the extra dimension provides a way of imagining how it could be possible for another spatial dimension to exist without us ever having noticed it. A simple analogy is often used to explain this fact: picture a cable placed far away from somebody. For such an observer, the cable will just seem a one-dimensional line. Now, imagine that an acrobat is walking along the cable (see Figure 2.1); if we asked our observer to specify the acrobat's location, they would only give us one piece of information: the distance at which we can find the acrobat, taking either the left or the right end of the cable as a reference [11]. However, we do know that the cable also has width. We shall now consider another observer, which is much closer to our setting, so nearby that this one can even identify an ant moving along the cable. The new observer will then be able not only to see how the ant can walk along the cable, but moreover, that it may also crawl around it, which means that they will perceive the cable's surface as a two-dimensional one, unlike the first observer who was placed far from it. Hence, if we now ask where the ant is, the answer we will get will include two pieces of information: not only the distance relative to one of the ends, but also the distance regarding the circular section; because the ant can move in two independent directions.[6]

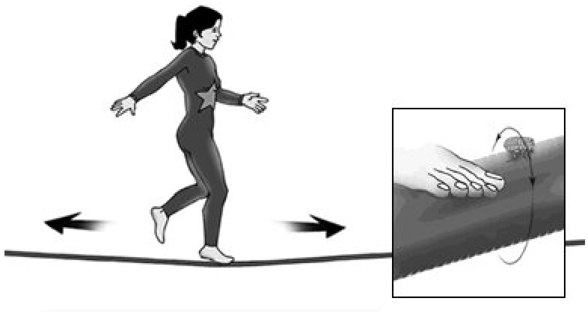


Figure 2.1: When looked at from far away, it will seem that the cable the acrobat is walking along is just a one-dimensional line, but zooming in to the insect's scale, we would be able to see how the insect can also crawl around the cable, not just along it, that is to say, the cable's second dimension would become visible to us. Taken from [12].

On that note, there is an essential difference between the two dimensions: the direction along the length of the cable is long, and easy to see. However, the direction surrounding the width of the cable is *curled up*, and the precision needed in order to spot this circular dimension is much higher. Accordingly, two types of spatial dimensions are distinguished: on the one hand, there are those which are broad and extended, whilst on

[6]We are considering that the ant is on the surface of the cable, and not anywhere inside it, otherwise we would need one more piece of information to specify its location.

the other hand, there are curled up dimensions. Within the Kaluza-Klein theory, three spatial dimensions would be extended, and the additional one would be curled up. If this extra curled up dimension were sufficiently small, we currently have no means of proving its existence to be physically impossible. Consequently, in order to picture Klein's idea of the extra dimension, we should think of all points in the three-dimensional space with an additional circular dimension which we cannot perceive. Any point in the usual space would then actually be a circle, located where the three usual coordinates lead us to.

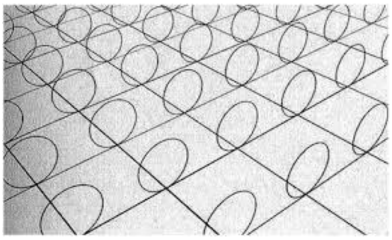


Figure 2.2: A curled up circular dimension represented on certain points of a two-dimensional plane. Taken from [11].

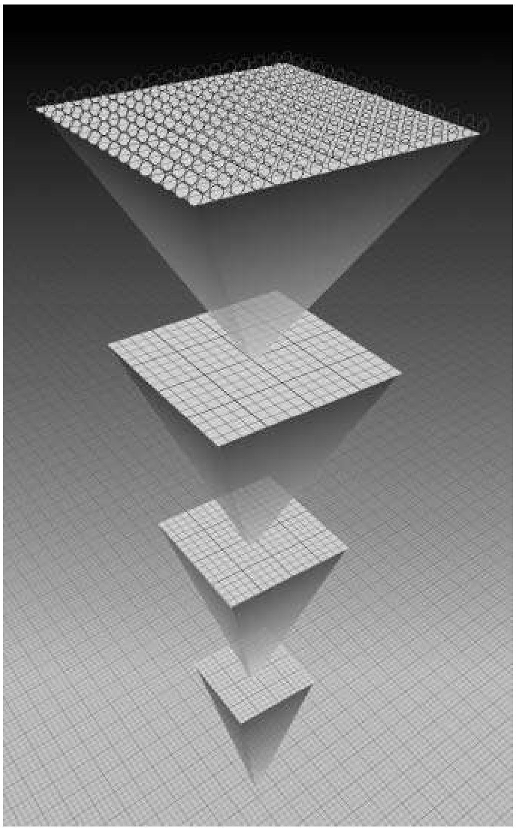


Figure 2.3: The lowest image represents the two-dimensional plane's apparent structure, and the higher the images are, they are more detailed, which means that the length scale is shorter. Taken from [11].

Obviously, it is difficult to represent the extra dimension in our three-dimensional space, that is why we limit ourselves to giving the two-dimensional example, that is, to picturing a two-dimensional plane with the extra dimension. This idea is represented in Figure 2.2: the figure should be thought of as a close-up image of the plane, in which the grid lines would be the two extended directions, and the circles drawn on the intersections of these lines should be regarded as the curled up direction. In the images there are only a few regularly spaced out representations, but they should be generalised to all points on the plane, as well as to the third extended dimension which has not been drawn; just as the cable has width in each of its lengthwise points, all spacetime points would also include this additional dimension.

Furthermore, Figure 2.3 represents the changes in the apparent geometry of the two-dimensional plane, according to the scale at which it is looked upon: at further distances, we would only be able to perceive the extended dimensions (two, in this example, but three in the Kaluza-Klein theory when speaking of spacetime), and the more we magnified the picture, the easier we would distinguish the circular dimension.

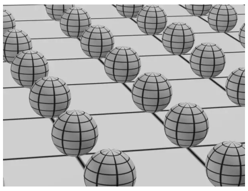
The smallest distance that has been explored with particle accelerators is roughly 10^{-16}cm [4]. As the length of the compact dimension that Klein obtained is so small that we cannot reach it experimentally, the fact that we have not detected it until today ceases to be a problem.

Despite the fact that the small size of the compact dimension could be a possible explanation for why we have not ever detected it, there are a few problems which entail a more complex theory. The first to arise was that of including matter to the theory: for example, the values obtained for the ratio between the electron's mass and its charge were far from the experimental ones [9], thereby resulting in a period in which extra-dimensional theories were mostly left aside. Nevertheless, the Kaluza-Klein theory was an important precedent of the attempts to find a unified theory of physics (which were eventually revitalised); it can be considered a fundamental start for the development of more complicated theories also involving compact extra dimensions, such as string theory.

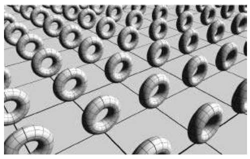
2.2 The number of spacetime dimensions in string theory

Theories speculating about extra dimensions were left aside soon after Kaluza and Klein's proposals, due to the successful development of quantum mechanics and quantum field theory. This period could be described as a time in which experimental physics was guided by theoretical advances, and vice versa; detailed properties of the atom were obtained and confirmed experimentally, leading to the formulation of the *standard model* [11]. Why would any theorist be interested in speculative theories such as extra-dimensional ones, whose validity is not even possible to test because the difference they imply would be at a length scale which is not even accessible to us? It seems natural that scientists should be interested in more reliable theories whose predictions can actually be tested experimentally.

By the early 1970s, the standard model had almost completely been crystallised theoretically, and several of its predictions would not need long to be confirmed; a great number of particle physicists believed that the remaining predictions would also inevitably be verified, leading to a complete understanding of strong, weak and electromagnetic interactions [11]. Considering the success of a quantum theory including the forementioned three forces, the task of also bringing the remaining force, gravitation, into play was resumed, and proposals such as the Kaluza-Klein theory were reconsidered. However, more than six decades had gone by since the original idea, unified theories had to face a thoroughly developed theory of quantum mechanics, together with another two forces still unknown at Kaluza's time. It seemed straightforward to at least consider more dimensions accounting for the higher amount of forces [11]. In order to understand how we could picture more than one additional dimension, once again, an illustrative example given by Brian Greene will be taken. Considering two extra dimensions and only two extended dimensions, and representing the curled up ones in regularly spaced points, Figure 2.4a and Figure 2.4b provide a way of visualising them. They are both considered to stress the fact that there are many options concerning their geometry, as long as their spatial extent is small enough so as not to have been detected yet.



(a) Two extra dimensions curled up into the surface of a sphere. Taken from [11].



(b) Two extra dimensions curled up into the surface of a torus. Taken from [11].

Figure 2.4: Different geometric configurations for two extra dimensions.

The fundamental property of string theory is that particles are replaced by elementary one-dimensional *strings*; each particle is identified with a particular vibrational mode of a string. They can be classified into *open strings*, with two endpoints, and *closed strings*, with no endpoints. On the other hand, there is an important division between *bosonic string theories* and *superstring theories*; within bosonic theories, all vibrations represent *bosons*, whilst the spectrum of superstrings also includes *fermions* [4]. Even though they are not realistic, bosonic theories will be focused on, due to their simplicity in comparison with superstrings, and only open strings will be used. The aim of this chapter is to explain that within string theory, the dimensionality of spacetime is determined by a calculation, that is, instead of postulating this number, it is a requirement of the theory. The number of dimensions obtained in each type of theory is different: $D = 26$ is obtained in bosonic theories, whilst the result using superstrings is $D = 10$, yet as already mentioned, only the first case will be developed.[7]

In an attempt to explain how these calculations arise, the procedure that appears in the textbook *A first course in string theory* [4], by Barton Zwiebach, will be followed. The set of coordinates needed to describe the strings will be introduced, followed by the equations of motion and their solution in the *light-cone gauge*, and lastly, the classical theory will be quantized. String theory is an incredibly broad field, but only the necessary

[7]Throughout this section, D will be used to refer to the number of spacetime dimensions, and d for the number of spatial ones ($D = d + 1$).

steps in order to define the so-called quantum *Lorentz generators* will be studied, which will lead to the calculation which is aimed for: the one that determines the number of spacetime dimensions (in this case, within bosonic string theories).

2.2.1 Light-cone coordinates

The *light-cone gauge* will be used, which involves using the *light-cone coordinate system* together with a set of conditions, specified later on. Therefore, the light-cone coordinate system should be defined. Two light-cone coordinates are set, x^+ and x^-, defined as independent linear combinations of the time coordinate x^0 and one spatial coordinate, usually taken to be x^1:

$$\begin{aligned} x^+ &\equiv \frac{1}{\sqrt{2}}\left(x^0 + x^1\right), \\ x^- &\equiv \frac{1}{\sqrt{2}}\left(x^0 - x^1\right). \end{aligned} \tag{2.60}$$

The remaining coordinates do not play a special role, we merely keep their usual defini-

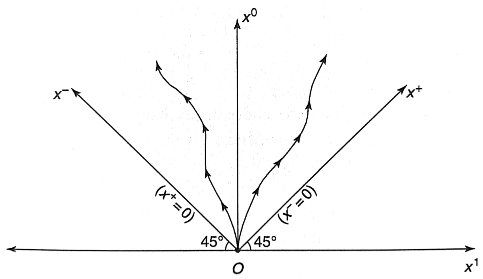


Figure 2.5: The light-cone coordinates $x^\pm$, together with x^0 and x^1.
Taken from [4].

tion. Consequently, the set of coordinates is $\left(x^+, x^-, x^2, x^3, ..., x^d\right)$, and

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{2.61}$$

Regarding the name of the two new coordinates, a brief explanation can be given. The two associated coordinate axes are actually the world-lines of beams of light emitted from the origin, along the chosen special spatial coordinate x^1. This can be seen by considering a beam of light along the positive x^1 direction; as its speed is c, $x^1 = ct$, and as usual, $x^0 = ct$, so $x^1 = x^0$, and taking into account our definitions of $x^\pm$, we have $x^- = 0$, corresponding to the x^+ axis, as can be seen in Figure 2.5. The same reasoning may be used to prove that for a beam of light moving in the negative x^1 direction, $x^+ = 0$ is obtained, corresponding to the x^- axis.

x^+ will be treated as the *light-cone time* coordinate, even though x^- could have also been chosen. The time coordinate should not be understood in the usual sense, because its fundamental property is that time goes forward with any physical motion, but considering special light rays, the light-cone time “freezes” (if x^+ is taken to be the light-cone time, it remains constant when choosing a light ray along the negative x^1 direction).

2.2.2 World-sheets

In order to study relativistic strings, just as *world-lines* are used for point particles, that is, the paths traced out by them in spacetime, the two-dimensional surface traced out by a string in spacetime is defined, known as the *world-sheet* (see Figure 2.6). As they are two-dimensional, two parameters are needed to describe world-sheets, typically denoted by τ and σ. If our usual spacetime coordinates are $x^\mu = \left(x^0, x^1, ..., x^d\right)$, the surface can be described via the mapping functions $x^\mu(\tau, \sigma)$. These functions are usually capitalised, and therefore called $X^\mu(\tau, \sigma)$, so as to avoid confusion if the (τ, σ) arguments are dropped. A point (τ, σ) in the parameter space is thus mapped to a point in spacetime, given by the coordinates $\left(X^0(\tau,\sigma), X^1(\tau,\sigma), ..., X^d(\tau,\sigma)\right)$. X^μ are called *string coordinates.*

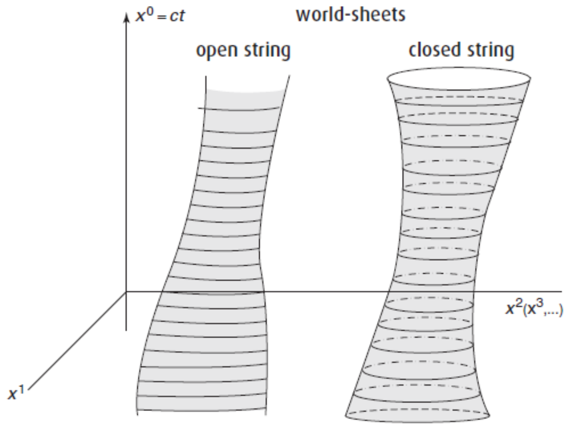


Figure 2.6: World-sheets of an open string and a closed string. Taken from [4].

A point particle's action is proportional to the proper time elapsed on its world-line, and proper length is defined by multiplying it by c. Consequently, the relativistic string action, named the *Nambu-Goto action*, is constructed proportional to the Lorentz invariant *proper area* of a world-sheet. With the notation

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma}, \tag{2.62}$$

the proper area of a world-sheet with $\sigma \in [0, \sigma_1]$ and $\tau \in [\tau_i, \tau_f]$ is taken to be [4]:

$$A = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \tag{2.63}$$

and the Nambu-Goto action is constructed by multiplying it with the appropriate constants,

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \tag{2.64}$$

where T_0 is a constant called the *string tension*. Extracting the Lagrangian density from the last equation,

$$\mathcal{L}\left(\dot{X}^\mu, X'^\mu\right) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}, \tag{2.65}$$

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \mathcal{L}\left(\dot{X}^\mu, X'^\mu\right), \tag{2.66}$$

we define

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}, \tag{2.67}$$

$$\mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}. \tag{2.68}$$

The equation of motion corresponding to both open and closed relativistic strings is obtained by varying the action,

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0. \tag{2.69}$$

For open strings one edge of the world-sheet is chosen to be the curve $\sigma = 0$ and the other one to be $\sigma = \sigma_1$, so $\sigma \in [0, \sigma_1]$. As for closed strings, an identification must be imposed in the parameter space, because the σ direction must be a circle. Consequently, the (τ, σ) space should be a cylinder. Denoting the circumference by σ_c,

$$(\tau, \sigma) \sim (\tau, \sigma + \sigma_c), \quad \sigma \in [0, \sigma_c]. \tag{2.70}$$

Choosing a σ parametrisation means constructing lines in which σ is constant, and that are perpendicular to those of constant τ. Let us show how to obtain the parametrisation of all strings, assuming that a particular one of a single string is already given. τ will be chosen as $\tau = t$, that is, $X^0(\tau, \sigma) = ct = c\tau$. Considering an open string with $t = 0$, parametrised with $\sigma \in [0, \sigma_1]$, and another one with $t = \epsilon$ (where ϵ is infinitesimal), one could picture perpendicular segments to the $t = 0$ string, that intersect the $t = \epsilon$ one, as in Figure 2.7. Any arbitrary point σ_0 on the first string is mapped to the second one with the segment going through the $t = 0$ string, and the intersection in the second string is also named σ_0. The parametrisation for the second string is obtained continuing along the first one in this manner, and the process is repeated for this last string. As a result, a set of lines is obtained which have constant σ and are perpendicular to the strings, that is, lines in which t is constant. In order to complete the parametrisation, it must still be specified for the initial string. There are many options, but this will resumed later on.

2.2.3 Conserved currents on the world-sheet

It is known that the symmetric characteristics of a dynamical system are closely linked to conserved quantities. Within our field of study, there are conserved charges that will turn out to be very useful in our brief discussion on string theory. On the one hand, a relativistic momentum p_μ will be defined, conserved for free strings, and on the other hand, the conserved charges $M_{\mu\nu}$, associated with Lorentz symmetry. With that aim, it is convenient to first revise the expressions for conserved currents and charges related to

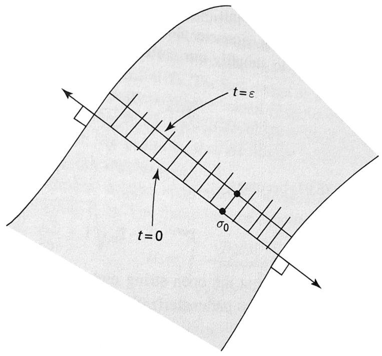


Figure 2.7: The process of constructing a σ parametrisation for all strings. Taken from [4].

the Lagrangian's symmetries.

Let us write the action as an integral of the Lagrangian density dependent on the fields $\phi^a(\xi)$ (a just labels these fields), with the set of coordinates ξ^α of the relevant world:

$$S = \int d\xi^0 d\xi^1 ... d\xi^k \mathcal{L}\left(\phi^a, \partial_\alpha \phi^a\right), \quad \partial_\alpha \phi^a = \frac{\partial \phi^a}{\partial \xi^\alpha}. \tag{2.71}$$

Consider the infinitesimal variation

$$\phi^a(\xi) \to \phi^a(\xi) + \delta\phi^a(\xi), \quad \delta\phi^a = \epsilon^i h_i^a(\phi), \tag{2.72}$$

where ϵ^i are infinitesimal constants, and i labels the parameters in the variation (it may involve several, as in spatial translations). The arguments' indices have been dropped for simplicity. If $\mathcal{L}$ is invariant under the above variation, then the currents j_i^α defined in the following way are conserved, (in the sense that $\partial_\alpha j_i^\alpha = 0$),

$$\epsilon^i j_i^\alpha \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \phi^a\right)} \delta\phi^a. \tag{2.73}$$

A conserved current implies a conserved charge; to obtain the latter, the former's zeroth component must be integrated over space:

$$Q_i = \int d\xi^1 d\xi^2 ... d\xi^k j_i^0. \tag{2.74}$$

Returning to our case, as stated earlier, the relativistic momentum p_μ will be defined, which is conserved for free strings. The action (2.65) is integrated over τ and σ, so α in (2.71) will only adopt two values,

$$S = \int d\xi^0 d\xi^1 \mathcal{L}(\partial_0 X^\mu, \partial_1 X^\mu), \quad (\xi^0, \xi^1) = (\tau, \sigma). \tag{2.75}$$

The variation $\delta X^\mu(\tau, \sigma) = \epsilon^\mu$ does not change the Lagrangian density, because $\mathcal{L}$ only depends on the derivatives of the coordinates, and their variations vanish, so retaking (2.73), α is identified as a world-sheet index, i is a spacetime index, and so is a,

$$\epsilon^\mu j_\mu^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \epsilon^\mu. \tag{2.76}$$

$$j^{\alpha}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \to (j^0_\mu, j^1_\mu) = \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \frac{\partial \mathcal{L}}{\partial X'^\mu}\right). \tag{2.77}$$

$$j^{\alpha}_{\mu} = \mathcal{P}^\alpha_\mu, \quad (j^0_\mu, j^1_\mu) = (\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu). \tag{2.78}$$

In order to get the charges, the zeroth components $\mathcal{P}^\tau_\mu$ of the currents are integrated over σ,

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu(\tau,\sigma)\, d\sigma. \tag{2.79}$$

p_μ give the spacetime momentum carried by the string, they arise from spacetime translational invariance, so $\mathcal{P}^\tau_\mu$ is the σ density of spacetime momentum carried by the string.

Secondly, the action is Lorentz invariant by construction, and the conserved charges related to this symmetry will be formulated. For this purpose, the infinitesimal form of these transformations must be used. Lorentz transformations are linear transformations, and $\eta_{\mu\nu}X^\mu X^\nu$ remains invariant under them. Infinitesimal linear transformations are written as $X^\mu \to X^\mu + \delta X^\mu$, where $\delta X^\mu = \epsilon^{\mu\nu}X_\nu$, and $\epsilon^{\mu\nu}$ is a matrix of infinitesimal constants. It can be shown that infinitesimal Lorentz transformations are characterised by an antisymmetric $\epsilon^{\mu\nu}$. Considering (2.73) to formulate the currents, we can see that $\epsilon^{\mu\nu}$ plays the role of ϵ^i:

$$\epsilon^{\mu\nu} j^\alpha_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}\delta X^\mu = \mathcal{P}^\alpha_\mu \epsilon^{\mu\nu} X_\nu. \tag{2.80}$$

Taking into account that $\epsilon^{\mu\nu}$ is antisymmetric,

$$\epsilon^{\mu\nu} j^\alpha_{\mu\nu} = \left(-\frac{1}{2}\epsilon^{\mu\nu}\right)\left(X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu\right). \tag{2.81}$$

The currents may be normalised as wished, so naming them $\mathcal{M}^\alpha_{\mu\nu}$, we define:

$$\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu. \tag{2.82}$$

Once the currents have been given, the conserved charges can be calculated. For any arbitrary curve γ on the world-sheet which connects the $\sigma = 0$ and $\sigma = \sigma_1$ boundaries, we can write

$$M_{\mu\nu} = \int_\gamma \left(M^\tau_{\mu\nu} d\sigma - M^\sigma_{\mu\nu} d\tau\right). \tag{2.83}$$

Or, using constant τ lines,

$$M_{\mu\nu} = \int \mathcal{M}^\tau_{\mu\nu}(\tau,\sigma)\, d\sigma = \int \left(X_\mu \mathcal{P}^\tau_\nu - X_\nu \mathcal{P}^\tau_\mu\right) d\sigma. \tag{2.84}$$

2.2.4 Choice of gauges

In the type of gauges that will be used, τ is set equal to a linear combination of the string coordinates (a specific choice out of these gauges defines the *light-cone gauge* that will be introduced later on). The combination of coordinates which defines these gauges can be expressed as

$$n_\mu X^\mu(\tau,\sigma) = \lambda\tau. \tag{2.85}$$

As to the σ parametrisation, one option would be setting it so that the energy density $\mathcal{P}^{\tau 0}$ is constant, that is, in order for each segment with the length σ to have the same energy. However, a more general will be considered, by demanding $n_\mu \mathcal{P}^{\tau\mu} = n \cdot \mathcal{P}^\tau$ to be constant over the strings. The first case would be equivalent to this condition with $n_\mu = (1, 0, ..., 0)$, and the new choice is valid for an arbitrary n^μ. $\sigma \in [0, \pi]$ will also be required for open strings, and $\sigma \in [0, 2\pi]$ for closed ones. It will not be proved in this work, but this can be satisfied by [4]

$$\begin{aligned} n \cdot X(\tau, \sigma) &= \beta\alpha' (n \cdot p) \tau, \\ n \cdot p &= \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau, \end{aligned} \tag{2.86}$$

where

$$\beta = \begin{cases} 2 & \text{for open strings,} \\ 1 & \text{for closed strings,} \end{cases} \tag{2.87}$$

and α' is a constant called the *slope parameter*, given by

$$\alpha' = \frac{1}{2\pi T_0 \hbar c}. \tag{2.88}$$

The choice of parametrisation will imply the following constraints on X' and $\dot{X}$ [4]:

$$\dot{X} \cdot X' = 0, \quad \text{and} \quad \dot{X}^2 + X'^2 = 0. \tag{2.89}$$

They can be put together as

$$\left(\dot{X} \pm X'\right)^2 = 0. \tag{2.90}$$

Using these constraints and the conditions (2.86), the expressions for $\mathcal{P}^{\tau\mu}$ and $\mathcal{P}^{\sigma\mu}$ given in (2.67) and (2.68) can be simplified:

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu, \tag{2.91}$$

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^\mu. \tag{2.92}$$

So the equations of motion $\partial_\tau \mathcal{P}^{\tau\mu} + \partial_\sigma \mathcal{P}^{\sigma\mu} = 0$ now become wave equations,

$$\ddot{X}^\mu - X''^\mu = 0. \tag{2.93}$$

2.2.5 Mode expansions

Regarding the boundary conditions, even though they will not be studied in detail, D-*branes* must be mentioned, which are objects on which open string endpoints must lie, characterised by their number of spatial dimensions: a Dp-brane has p spatial dimensions. It will be assumed that the endpoints satisfy free boundary conditions ($\mathcal{P}^{\sigma\mu} = 0$), which is equivalent to having a *space-filling* D-brane. The most general $X^\mu(\tau, \sigma)$ that solves the wave equation is

$$X^\mu(\tau, \sigma) = \frac{1}{2}\left(f^\mu(\tau + \sigma) + g^\mu(\tau - \sigma)\right). \tag{2.94}$$

Taking (2.92) into account, the free endpoint boundary conditions imply the Neumann boundary conditions

$$\frac{\partial X^\mu}{\partial \sigma} = 0, \quad \text{at} \quad \sigma = 0, \pi. \tag{2.95}$$

At $\sigma = 0$, the boundary condition reads

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = \frac{1}{2}\left(f'^\mu(\tau) - g'^\mu(\tau)\right) = 0. \tag{2.96}$$

Since the derivatives of f^μ and g^μ coincide, they can only differ by a constant c^μ, that is, $g^\mu = f^\mu + c^\mu$. This constant can be absorbed into the definition of f^μ, so from (2.94),

$$X^\mu(\tau, \sigma) = \frac{1}{2}\left(f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma)\right). \tag{2.97}$$

At $\sigma = \pi$,

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, \pi) = \frac{1}{2}\left(f'^\mu(\tau + \pi) - f'^\mu(\tau - \pi)\right) = 0. \tag{2.98}$$

This must hold for all τ, so f'^μ is periodic with period 2π, and the expression of the Fourier series for a periodic $f'^\mu(u)$ may be used,

$$f'^\mu(u) = f_1^\mu + \sum_{n=1}^{\infty}\left(a_n^\mu \cos nu + b_n^\mu \sin nu\right), \tag{2.99}$$

which can be integrated to get $f^\mu(u)$. Absorbing the integration constants into new coefficients,

$$f^\mu(u) = f_0^\mu + f_1^\mu u + \sum_{n=1}^{\infty}\left(A_n^\mu \cos nu + B_n^\mu \sin nu\right). \tag{2.100}$$

Substituting this back in (2.97),

$$X^\mu(\tau, \sigma) = f_0^\mu + f_1^\mu \tau + \sum_{n=1}^{\infty}\left(A_n^\mu \cos n\tau + B_n^\mu \sin n\tau\right)\cos n\sigma. \tag{2.101}$$

We now define a new set of coefficients a_n^μ,

$$\begin{aligned} A_n^\mu \cos n\tau + B_n^\mu \sin n\tau &= -\frac{i}{2}\left(\left(B_n^\mu + iA_n^\mu\right)e^{in\tau} - \left(B_n^\mu - iA_n^\mu\right)e^{-in\tau}\right) \\ &\equiv -i\frac{\sqrt{2\alpha'}}{\sqrt{n}}\left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right), \end{aligned} \tag{2.102}$$

where $*$ denotes complex conjugation. A factor was introduced to make them dimensionless. Let us develop f_1^μ.

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}f_1^\mu + \dots. \tag{2.103}$$

The dots stand for terms with $\cos n\sigma$ dependence. To find the total momentum p^μ, it must be integrated over $\sigma \in [0, \pi]$, and the terms represented by the dots do not contribute.

$$p^\mu = \int_0^\pi \mathcal{P}^{\tau\mu} d\sigma = \frac{1}{2\pi\alpha'}\pi f_1^\mu \quad \rightarrow \quad f_1^\mu = 2\alpha' p^\mu. \tag{2.104}$$

Naming $f_0^\mu = x_0^\mu$,

$$X^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'}\sum_{n=1}\left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right)\frac{\cos n\sigma}{\sqrt{n}}. \tag{2.105}$$

The terms on the right correspond to the zero mode, the momentum, and the oscillations of the string. Let us introduce the following notation,

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu, \quad \alpha_n^\mu = a_n^\mu \sqrt{n}, \quad \alpha_{-n}^\mu = a_n^{\mu *} \sqrt{n}, \quad n \geq 1. \tag{2.106}$$

There are two important notes to be made. On the one hand, $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$, and on the other, even though a_n^μ are only defined when n is a positive integer, α_n^μ are defined for all integers. We rewrite (2.105),

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau - i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu e^{in\tau} - \alpha_n^\mu e^{-in\tau} \right) \cos n\sigma. \tag{2.107}$$

This can also be written as a sum over all integers except 0:

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma, \tag{2.108}$$

and the derivatives may be calculated,

$$\dot{X}^\mu = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \cos n\sigma e^{-in\tau}, \tag{2.109}$$

$$X'^\mu = -i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \sin n\sigma e^{-in\tau}. \tag{2.110}$$

The solution of the wave equations with Neumann boundary conditions is defined once x_0^μ and α_n^μ are specified for $n \geq 0$. Having found a solution which satisfies the boundary conditions, it must still be ensured that the constraints (2.90) are satisfied. This will be achieved by using the light-cone gauge.

2.2.6 Light-cone gauge

The coordinates X^2, X^3, ..., X^d are named *transverse coordinates*, and are denoted by X^I,

$$X^I = \left(X^2, X^3, ..., X^d \right). \tag{2.111}$$

Choosing the light-cone gauge means imposing (2.86) with a choice of η^μ which implies $n \cdot X = X^+$. This is achieved by selecting

$$n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0 \right), \tag{2.112}$$

as can be seen by computing $n \cdot X$:

$$n \cdot X = \frac{X^0 + X^1}{\sqrt{2}} = X^+, \quad n \cdot p = \frac{p^0 + p^1}{\sqrt{2}} = p^+. \tag{2.113}$$

From (2.86),

$$X^+(\tau, \sigma) = \beta \alpha' p^+ \tau, \quad p^+ = \frac{2\pi}{\beta} \mathcal{P}^{\tau +}. \tag{2.114}$$

The constraints can be rewritten as

$$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'} \frac{1}{2p^+} \left(\dot{X}^I \pm X'^I \right)^2. \tag{2.115}$$

In the case of open strings, let us write the solution for the transverse coordinates X^I taking into account the general solution (2.108),

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau}\cos n\sigma. \quad (2.116)$$

As to the X^+ coordinate, the gauge condition gives

$$X^+(\tau,\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau. \quad (2.117)$$

As can be seen, the position zero mode and the oscillations of the X^+ coordinate have been set equal to zero. Being a linear combination of X^0 and X^1, X^- also satisfies the same wave equation and boundary conditions as all other coordinates. Therefore, the same expansion may be used,

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma. \quad (2.118)$$

The constraints can be used to show that the minus oscillators can be written in terms of the transverse oscillators,

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\alpha_p^I. \quad (2.119)$$

The general solution is fixed by specifying the values of p^+, x_0^-, x_0^I and all of the α_n^I. This determines $X^I(\tau,\sigma)$ and $X^+(\tau,\sigma)$. α_n^- are also fixed by (2.119), and together with x_0^-, they determine $X^-(\tau,\sigma)$. The full solution is thus constructed. The quadratic combination of oscillators that appears in (2.119) is named the *transverse Virasoro* mode $L_n^\perp$:

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp, \quad L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\alpha_p^I. \quad (2.120)$$

The particular case of $n = 0$ will be very useful later on,

$$\sqrt{2\alpha'}\alpha_0^- = \frac{1}{p^+}L_0^\perp \xrightarrow{(2.106)} 2p^+p^- = \frac{1}{\alpha'}L_0^\perp. \quad (2.121)$$

2.2.7 Relativistic quantum open string

In order to construct our quantum theory, a set of *operators* is needed, including the Hamiltonian. The procedure then involves postulating their commutation relations and checking that they generate the expected equations of motion.

With regard to the classical string, X^- has been solved in terms of X^I, only needing to specify x_0^-, and X^+ is fixed by the light-cone gauge condition. As to the momentum, the condition (2.115) can be further developed to show that $\mathcal{P}^{\tau-}$ is determined by $\mathcal{P}^{\tau I}$ and p^+. Therefore, a reasonable choice of Schrödinger operators is $X^I(\sigma)$, x_0^-, $\mathcal{P}^{\tau I}(\sigma)$, and p^+. On the other hand, regarding the commutation relations, we only expect $X^I(\sigma)$ and $\mathcal{P}^{\tau I}(\sigma)$ not to commute when they have the same σ along the string, so we set

$$\left[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma'), \quad \text{where, from (2.61),} \quad \eta^{IJ} = \delta^{IJ}, \quad (2.122)$$

$$\left[X^I(\sigma), X^J(\sigma')\right] = \left[\mathcal{P}^{\tau I}(\sigma), \mathcal{P}^{\tau J}(\sigma')\right] = 0, \quad \left[x_0^-, p^+\right] = -i. \tag{2.123}$$

The Hamiltonian should generate τ-translations. $X^+ = 2\alpha' p^+ \tau$, and p^- generates X^+-translations, therefore,

$$\frac{\partial}{\partial \tau} = \frac{\partial X^+}{\partial \tau}\frac{\partial}{\partial X^+} = 2\alpha' p^+ \frac{\partial}{\partial X^+}, \quad \text{so} \quad H = 2\alpha' p^+ p^- \xrightarrow{(2.121)} H = L_0^\perp. \tag{2.124}$$

One way of checking that our Hamiltonian generates the appropriate quantum equations of motion is to construct the Heisenberg operators, and verify that the correct time evolution is obtained for the operators. The only subtlety concerning H is that we will later subtract a constant from it.

The mode expansions studied earlier will also be used in the quantum theory, but it must be taken into account that the classical modes α_n^I now become quantum operators. It can be proved [4] that the appropriate commutation relation for them is

$$\left[\alpha_m^I, \alpha_n^J\right] = m\,\eta^{IJ}\delta_{m+n,0}. \tag{2.125}$$

By defining

$$\alpha_n^I = a_n^I\sqrt{n}, \quad \alpha_{-n}^I = a_n^{I\dagger}\sqrt{n}, \quad n \geq 1, \tag{2.126}$$

$$\left(\alpha_n^I\right)^\dagger = \alpha_{-n}^I, \quad n \in \mathbb{Z}. \tag{2.127}$$

It should be noted that α_0^I is also Hermitian because it is proportional to p^I, and the operators x_0^I and p^I are Hermitian, just as in the usual quantum mechanics. These properties mean that X^I is also Hermitian. Using (2.125),

$$\left[a_m^I, a_n^{J\dagger}\right] = \delta_{m,n}\,\eta^{IJ}, \quad \left[a_m^I, a_n^J\right] = \left[a_m^{I\dagger}, a_n^{J\dagger}\right] = 0, \tag{2.128}$$

hence, α_n^I can be thought of as *annihilation operators* and α_{-n}^I as *creation operators* of a quantum simple harmonic oscillator. As the α modes are now oscillators, instead of referring to $L_n^\perp$ as *transverse Virasoro modes*, they will be called *transverse Virasoro operators*. Taking into account that the α operators do not commute, it must checked whether the order in which they appear in the definition of $L_n^\perp$ is correct. The only case in which the commutator does not vanish is when their mode numbers add up to zero, thus we only have to check $L_0^\perp$, which is actually the Hamiltonian. Considering its action on the vacuum state, we want it to be normal-ordered, that is, the annihilation operators must appear to the right of the creation operators. From (2.120),

$$L_0^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}^I\alpha_p^I = \frac{1}{2}\alpha_0^I\alpha_0^I + \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I. \tag{2.129}$$

The first sum on the right is normal-ordered, but the last one is not, so we would like to rewrite it as

$$\begin{aligned}\frac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I = \frac{1}{2}\sum_{p=1}^{\infty}\left(\alpha_{-p}^I\alpha_p^I + \left[\alpha_p^I\,\alpha_{-p}^I\right]\right) &= \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}\sum_{p=1}^{\infty}p\,\eta^{IJ} \\ &= \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty}p.\end{aligned} \tag{2.130}$$

The last term is divergent, and it cannot simply be ignored. Because of the relation (2.121), adding a constant to $L_0^\perp$ alters the values of the masses of the string states,

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}L_0^\perp - p^Ip^I, \tag{2.131}$$

so the problematic term is dealt with in the following way: we define $L_0^\perp$ without it, and introduce an ordering constant a in (2.121),

$$L_0^\perp \equiv \frac{1}{2}\alpha_0^I\alpha_0^I + \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I = \alpha' p^Ip^I + \sum_{p=1}^{\infty} p\, a_p^{I\dagger}a_p^I, \quad 2\alpha' p^- \equiv \frac{1}{p^+}\left(L_0^\perp + a\right). \tag{2.132}$$

Instead of concluding that

$$a = \frac{1}{2}(D-2)\sum_{p=1}^{\infty} p,$$

a will be taken to be an undetermined constant, which will be fixed by the consistency of string theory.

2.2.8 Quantum Lorentz generators

The classical Lorentz generators in terms of oscillation modes can be obtained by inserting the expressions (2.91), (2.92), (2.108) and (2.109) in (2.84), and integrating,

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu\right). \tag{2.133}$$

It will be analysed whether (2.133) can be used to define the quantum Lorentz generators. When considering conserved charges in the quantum theory, they become operators that generate, via commutation, a version of the symmetry property from which they arose in the classical theory. With regard to Lorentz transformations, the generators' commutator must adopt the following form [4],

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}, \tag{2.134}$$

hence, any theory with operators $M^{\mu\nu}$ failing to satisfy (2.134) is not Lorentz invariant. For example, as to $\left[M^{-I}, M^{-J}\right]$, it must satisfy $\left[M^{-I}, M^{-J}\right] = 0$. Let us try defining M^{-I} using (2.133),

$$M^{-I} \stackrel{?}{=} x_0^- p^I - x_0^I p^- - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^-\alpha_n^I - \alpha_{-n}^I\alpha_n^-\right). \tag{2.135}$$

As x_0^I and p^- do not commute, it is not Hermitian, thus let us propose

$$M^{-I} \stackrel{?}{=} x_0^- p^I - \frac{1}{2}\left(x_0^I p^- + p^- x_0^I\right) - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^-\alpha_n^I - \alpha_{-n}^I\alpha_n^-\right). \tag{2.136}$$

The minus oscillators can be rewritten in terms of Virasoro operators by using (2.120) and (2.132),

$$\begin{aligned} M^{-I} = {} & x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^I\right) \\ & - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}\left(L_{-n}^\perp\alpha_n^I - \alpha_{-n}^I L_n^\perp\right). \end{aligned} \tag{2.137}$$

Using the Virasoro commutation relations, the result for $\left[M^{-I}, M^{-J}\right]$ given in *A First Course in String Theory* [4], and which must vanish, is

$$\begin{aligned}\left[M^{-I}, M^{-J}\right] = &-\frac{1}{\alpha' {p^+}^2}\sum_{m=1}^{\infty}\left(\alpha^I_{-m}\alpha^J_m - \alpha^J_{-m}\alpha^I_m\right) \\ &\times\left\{m\left[1-\frac{1}{24}(D-2)\right]+\frac{1}{m}\left[\frac{1}{24}(D-2)+a\right]\right\}.\end{aligned} \tag{2.138}$$

The terms $\left(\alpha^I_{-m}\alpha^J_m - \alpha^J_{-m}\alpha^I_{-m}\right)$ cannot vanish, so we must have

$$m\left[1-\frac{1}{24}(D-2)\right]+\frac{1}{m}\left[\frac{1}{24}(D-2)+a\right]=0, \quad \forall m\in\mathbb{Z}^+. \tag{2.139}$$

It is enough to consider $m=1$ and $m=2$ to see that both terms must be set to zero,

$$1-\frac{1}{24}(D-2)=0 \quad \text{and} \quad \frac{1}{24}(D-2)+a=0. \tag{2.140}$$

The first equation fixes the number of spatial dimensions,

$$\boxed{D=26,} \tag{2.141}$$

and the second one determines a,

$$a=-\frac{1}{24}(D-2)=-\frac{24}{24}=-1. \tag{2.142}$$

Only the case of open strings has been studied, yet using a similar procedure, the same value is obtained for D when considering closed strings [4]. This is actually a necessary result in order for the coexistence of open and closed strings to be possible, and taking into account that open strings can close, turning into closed strings, it is coherent to demand this condition.

There are many issues that are beyond the scope of this study, starting from the fact that even superstrings have not been studied. Could the Standard Model actually arise from them? This has not been shown in detail yet, but there are models which could give rise to it. Gauge bosons and matter particles would arise from vibrations of open strings stretching between D-branes, and the graviton, from closed strings. As to the geometric configuration, the extra dimensions are usually thought to be curled up in *Calabi-Yau manifolds*. However, compactification moduli, adjustable parameters, arise, which can be stabilized with *flux compactifications*, at the cost of forming a vast set of models with more than 10^{500} constituents, referred to as the string *landscape* [11] [13]. There may be no string model in this landscape which reproduces the Standard Model, there may be a single one that does, or there may be many models consistent with the current accuracy. [4]

It should also be mentioned that by mid 1980s, five ten-dimensional superstring theories were known, which were later found to be interrelated. The strong coupling limit of one of these theories leads to the *M-theory*, which is *eleven-dimensional*. They are all actually thought to be different limits of a single theory. [4] [11]

Chapter 3

Anthropic debate on the number of spacetime dimensions

Having studied the consequences upon our understanding of spatial dimensions (or at least, the possible number of these) caused by the attempts of unification, a radically different approach to this discussion will be considered, motivated by the *anthropic principle*. The aim is not to study specific calculations as in the previous chapter, instead, the type of *reasoning* this approach involves will be focused on, that is to say, the object of study will be the *kind of arguments* the anthropic principle provides, and how this has been applied by several authors to the dimensionality of spacetime. In very general terms, the principle fundamentally states that taking our existence into account, any physical theory must be such as for our existence to be possible, and moreover, our mere existence implies limits on the age of the universe and its fundamental constants. It arises from the questions: why does the universe have the specific characteristics needed in order for our existence to be possible? Why is it not in any other way, in which we would not be able to exist? The principle under consideration has often been classified as a tautology, or simply as a selection bias, but far from being insignificant, it may lead to very interesting debates. Proof of its contemporariness is the recently held conference "Carter Fest" [14], in which many of the talks referred to this topic.

3.1 The anthropic principle: a historical review

The anthropic principle has many precedents, which are generally reactions to a specific physical phenomenon or discovery which seems "surprising", or which seems to have a low probability. It is closely linked to believing the universe to be *fine-tuned*, although defining this property is much more complicated than it may seem at first sight. Those in favour of fine-tuning claim that if the universal constants had even a slightly different value to the experimentally observed ones, we would not be here in the first place. The problem lies in the degree of tuning; in order to affirm or reject fine-tuning, how small should the interval allowing our existence be? This interval cannot be fixed arbitrarily, the limits permitting our existence should be determined, and we should then calculate the probability of the constants adopting a value in that range, and if this is indeed a small probability, we might be able to affirm that we live in a fine-tuned universe. But in order to calculate the probability, all possible values of the constants under question should be given.

If the hypothesis of fine-tuning were to be accepted, there does not seem to be an explanation as to why the fundamental constants adopt the values they do, as it is being said that this is not likely at all. One way of interpreting this is the anthropic reasoning; we observe the values that we do because that is the only option allowing us to be observers. For example, in his book *Just Six Numbers*, Martin Rees defines six constants with finely adjusted values, including the number of spatial dimensions, for which an anthropic explanation should be given [15]. But getting to the point, let us study the forementioned precedents.

One of the first known antecedents was made by the biologist Alfred Russel Wallace, who said in 1904:

> "Such a vast and complex universe as that which we know exists around us, may have been absolutely required in order to produce a world that should be precisely adapted in every detail for the orderly development of life culminating in man." [16]

On the other hand, in the 1930s, Paul Dirac studied the coincidences between fundamental numbers of nature, inspired by Hermann Weyl [17] and Eddington's [18] work in 1919 and 1923, respectively. Considering an electron and a proton, the ratio between the electromagnetic and the gravitational force between them is a very big number, 10^{40}. The ratio between the age of the universe and the time an electron needs to complete an orbit in the fundamental state of hydrogen according to Bohr's model also turns out to be of a very similar value, 10^{39}. The age of the universe obviously increases with time, and taking into account the similarity of the two numbers, Dirac proposed an idea known as the *Dirac large number hypothesis*, according to which these coincidences would be described by a time-varying gravitational constant, inversely proportional to the age of the universe ($G \propto \frac{1}{t}$, t being the age of the universe). If not, the coincidence would only be true at a specific cosmic period. [19]

In his paper named "Dirac's Cosmology and Mach's Principle" [20], written in 1961, Robert Dicke pointed out that having included the age of the universe, the relation obtained by Dirac might just be a *selection bias*: the age of the universe is not random, because not any value would allow the existence of observers. The permitted range is related to the lifetime of *main-sequence stars*, which depends on the fundamental constants. That is, the relation between the age of the universe and the constants is no coincidence, they must be related as they are, simply on account of our existence. Dicke proceeded to give anthropic arguments to obtain upper and lower limits of the age of the universe.

As to the lower limit, minutes after the Big Bang, only hydrogen, helium and some other light elements had been produced by *primordial nucleosynthesis*. *Metallicity*[1] has had to increase sufficiently for our existence: there has had to be enough time for an early generation of stars to have formed, and for them to have turned the original hydrogen and helium into heavier elements as carbon and oxygen, to then spread them out with their death, throughout a supanova explosion and stellar winds, forming more stars and planets. Regarding the upper limit, it seems reasonable to suppose that life requires a planet orbiting around a luminous star, so we take the maximum age a star able to produce energy throughout nuclear reactions can have. The two limits obtained by Dicke are

[1] Metallicity, as used in astrophysics, refers to the level of elements that are not hydrogen and helium.

actually of the same order of magnitude, and compatible with the Dirac relation. [20]

This reasoning gave way to the talk given by Brandon Carter in the International Astronomical Union Symposium in Cracow in 1973, held for the 500th anniversary of Copernicus's birth, in which he used the term *anthropic principle* for the first time. The talk's title was "Large Number Coincidences and the Anthropic Principle in Cosmology". It was, in his words, a reaction to the "exaggerated subservience" to the Copernican principle, which states that the human being is not a privileged observer in the universe, so in the face of this he introduces his *weak anthropic principle*:

> "We must be prepared to take account of the fact that our location in the universe is necessarily privileged to the extent of being compatible with our existence as observers." [21]

It is important to note that the laws of nature and the physical constants are given beforehand within the weak anthropic principle. Carter also formulated the *strong anthropic principle*, which states that the existence of observers imposes restrictions on the fundamental constants,

> "The universe (and hence the fundamental parameters on which it depends) must be such as to admit the creation of observers within it at some stage." [21]

The difference with respect to the weak version is that it also refers to the constants (and laws) of physics, not only to our position in the universe. His talk was contemporaneous to the paper "Why Is the Universe Isotropic?" [22] by Collins and Hawking, who also raised the same issue, and referenced each other on the topic. Carter paraphrases Descartes,

> "Cogito ergo mundus talis est" [21],

translated as "I think, therefore the world is as it is". He does not refer to the cause of the universe, he does not claim that the aim of the universe is to give life to us, but that taking into account that we exist, the universe must be as it is. Regarding "cogito ergo sum" ("I think, therefore I exist"), just as according to Descartes thinking implies existence, for Carter, our existence means the universe is as we see it.

Aiming to give the strong anthropic principle an explanatory sense, he states that in order to be able to explain the conditions needed for the existence of life in a probabilistic way, an "ensemble of worlds" (universes) should be considered, in which we could find all possible initial conditions and fundamental constants. In this way, even if the conditions leading to our existence seem very carefully chosen, it should not surprise us, because all conditions are given in different universes. "It is of course philosophically possible – as a last resort, when no stronger physical argument is available – to promote a prediction based on the strong anthropic principle to the status of an explanation by thinking in terms of a 'world ensemble' "[21]. Hawking also defends this idea in *A Brief History of Time*:

> "It is a bit like the well-known horde of monkeys hammering away on typewriters – most of what they write will be garbage, but very occasionally by pure chance they will type out one of Shakespeare's sonnets." [23]

However, the anthropic principle was completely distorted with Barrow and Tipler's book *The Anthropic Cosmological Principle* [24], published in 1986. Their weak version is similar to Carter's strong one, but their strong anthropic principle states that the universe "*must* be such as to admit the creation of observers within it at some stage", which clearly has a teleological sense (in fact, Tipler was deeply religious). Together with their weak and strong anthropic variants they also introduce the *final anthropic principle*, "Intelligent information-processing must come into existence in the Universe, and, once it comes into existence, it will never die out". The order followed in the book is also very meaningful, after a historical review of more than 200 pages of teleology, we reach a chapter named "The rediscovery of the anthropic principle". When mentioning anthropic reasoning we are not referring to versions of this kind.

3.2 Dimensionality of spacetime: anthropic arguments

How does all this apply to the dimensionality of spacetime? As it shall be seen, the anthropic principle has usually been used to argue that if the number of space and time dimensions was not equal to the one we perceive, our existence would be impossible, hence, we would not ask ourselves the question "why does our universe have this amount of space and time dimensions?", which arises due to our condition as observers. Therefore, in view of that question, we would answer, "because otherwise we would not be able to raise the question in the first place". The number of spatial dimensions will be denoted by d. Consequently, anthropic arguments are used to argue that we cannot postulate a universe (with us in it) in which $d \neq 3$.

3.2.1 $d > 3$

In 1917, in an article titled "In What Way Does It Become Manifest in the Fundamental Laws of Physics that Space Has Three Dimensions?" [25], Paul Ehrenfest proved that having only one time dimension and more than three extended spatial ones, planets orbiting around the Sun would not be able to have stable orbits, which can be generalised to any star. If $d > 3$, the gravitational force between two bodies would be reduced faster than when $d = 3$. With three spatial dimensions, for example, when doubling the distance, the force is reduced by 1/4, but with four spatial dimensions it would reduce by 1/8, and with five, by 1/16.

With this reasoning, he proved that with more than three extended spatial dimensions, a slight perturbation of a circular orbit would result in the planet spiralling away from the star (or towards it, depending on the perturbation). Therefore, the orbits would not be stable. For example, if a planet orbiting a star reduced its speed slightly, instead of adopting an orbit with a smaller radius, the gravitational force would change so drastically that the planet would move spirally towards the star. The same would happen if the planet increased its speed, but it would then move away from the star. In any case, we would either freeze or burn as a result of any slight variation, hence, we would not be able to survive in such a universe. On the other hand, it would not be possible for the Sun to be in a stable state in which the pressure is in equilibrium with gravity; it would either collapse and form a black hole, or disintegrate.

The steps leading to that proof involved Ehrenfest making a generalisation of Newton's universal law of gravitation to any number N of spatial dimensions. According to Newton's gravitation, having two masses M and m at a distance r, with $\hat{r}$ being the unit vector from m to M, the force exerted on M and potential energy are given by

$$\vec{F} = \frac{-GMm}{r^2}\hat{r}, \quad F = |\vec{F}| = \frac{-GMm}{r^2}, \tag{3.1}$$

$$U_g = \frac{-GMm}{r}. \tag{3.2}$$

The generalisation given by Ehrenfest is

$$F = \frac{-GMm}{r^{d-1}}, \tag{3.3}$$

$$U_g = \frac{-GMm}{(d-2)r^{d-2}}. \tag{3.4}$$

Another author that could be mentiond is Gerald Whitrow, who published an article titled "Why Physical Space Has Three Dimensions" [26] in 1955. He begins by quoting Galileo's *Dialogue Concerning the Two Chief World Systems*, in which Salviati argues the point of view of Aristoteles on the number of spatial dimensions. In the conclusions, particularly on the last page, it can be seen that Whitrow tries to answer the question about the number of spatial dimensions with anthropic reasoning. Why are there three spatial dimensions? Because with that amount, the conditions on Earth are such that the evolution of the human being, "the formulator of the problem", as he says, has been possible.

On the other hand, it must be mentioned that the effects resulting when changing the number of spatial dimensions are not restricted to the gravitational force. For instance, when considering more than three spatial dimensions, the orbitals of the electron around the nuclei would be affected in the same way, it would be impossible for the orbitals to be stable. The electrons would escape from the atom or be attracted to it under any perturbation. As a consequence, another argument on the topic could be given by stressing that it seems impossible to think that under such conditions there could be atoms as we know them, which obviously affects our existence.

3.2.2 $d < 3$

If we considered a universe in which the number of spatial dimensions is smaller than three, any kind of gravitation would be problematic, and not only gravitation; the universe would probably be too simple as to have observers. There have been many attempts to picture what two-dimensional beings would be like if N were to be 2, for instance, there is the novel written by Edwin Abbott Abbott named *Flatland: A Romance of Many Dimensions* [27], which is in turn mentioned by Carl Sagan in Cosmos [28].

In *A brief history of time* [23], Hawking argues that two-dimensional complex observers do not seem realistic. He gives a very good visual example in Figure 3.1. Supposing there is a universe with only two spatial dimensions, he affirms that any everyday life example appears to be problematic. For example, these two-dimensional animals, living on a world whose surface has only one dimension, would have to climb one on top of another to be able to overtake each other. Even though this does not seem a crucial condition for their existence, let us consider an issue that does seem so: feeding. In the case of one of these two-dimensional animals not completely digesting anything ingested, the only way of throwing it out would be to expel it the same way out, due to the fact that if our animal had any kind of channel going through the whole of its body, it would be divided in two completely separated parts, as we can see in Figure 3.1. Following the same reasoning, it would also seem impossible to imagine how blood could circulate through our animal, because the same problem we had with the digestive tracts would hold for veins and arteries.

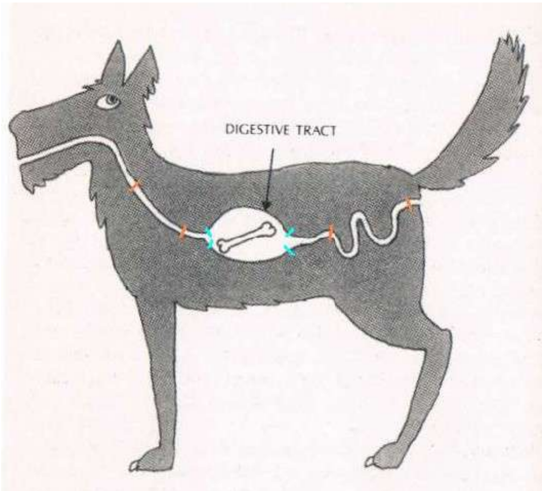


Figure 3.1: A two-dimensional dog. Taken from [23].

Chapter 4

Conclusions

Unified theories of physics have led to the understanding that it may be possible for the universe to have more spatial dimensions than we are aware of, and it has been discussed that string theory has a very interesting property, due to the number of spatial dimensions being required and not postulated, but will it ever be possible to confirm or reject these extra dimensions? Their length-scale is not fixed in string theory, and the smallest distance that has ever been explored with particle accelerators is currently about 10^{-16} cm [4]. Therefore, if there really were extra dimensions and their length were, for example, roughly equal to the Planck length (10^{-33} cm), they may never be detected directly.

However, if the extra dimensions were large, and our three-dimensional space were a D-3 brane transverse to these, they could be tested by gravitational experiments (for distances shorter than the length of the extra dimensions, the world is effectively higher-dimensional, and the force laws change). Taking into account that in most of these models electromagnetism arises from open strings whose endpoints must remain attached to the D-brane, electromagnetism itself would be bound to the brane, and not affected by the extra dimensions. Nonetheless, gravity, arising from closed strings, would be. The problem is that with it being such a weak force, it is difficult to test at small distances. Experiments up to a distance of fifty microns show that if extra dimensions exist, they must be smaller than this distance. Other possible experimental detections would be linked to *cosmic strings*, which have been exhaustively searched for, although still unsuccessfully; and supersymmetry, which would not exclusively confirm string theory, and thus, extra dimensions. [4] [11]

Lastly, it is worth mentioning that the set of possible models that may allow the existence of life within the vast string landscape has actually been referred to by many theorists, such as Leonard Susskind, as the *anthropic landscape* [13], which provides a link between the two main chapters.

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