Games and Economic Behavior 136 (2022) 524–541

Contents lists available at ScienceDirect

Games and Economic Behavior

journal homepage: www.elsevier.com/locate/geb

Probabilistic Owen-Shapley spatial power indices

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ARTICLE INFO

Article history:
Received 21 April 2021
Available online 26 October 2022

JEL classification:
C71

Keywords:
Power index
The spatial Owen-Shapley index
Probabilistic spatial power indices

ABSTRACT

In this paper we study probabilistic Owen-Shapley spatial power indices, which are generalizations of the Owen-Shapley spatial power index (1977). We provide an explicit formula for calculating these spatial indices for unanimity games and give an axiomatic characterization of the family of probabilistic Owen-Shapley spatial power indices. We employ an equal power change property, a spatial dummy property, anonymity, a positional invariance property, and a positional continuity property. Some examples are also given.

1. Introduction

Cooperative game theory has been applied successfully to measure the power of agents in voting situations, which are represented by simple games. A winning coalition is assigned a worth of one, and a losing one a worth of zero. The Shapley-Shubik index (1954) and the Banzhaf index (1965) can be seen as the best known indices for measuring power. They both take into account whether the presence of an agent changes a losing coalition into a winning one, i.e. whether an agent is pivotal. In defining the Shapley-Shubik index, all possible orderings of agents are considered. Each ordering has a pivotal agent associated with it: the one whose addition to the coalition formed by the previous agents changes that coalition from losing to winning. When all orderings are equally probable, the probability of an agent being pivotal is by definition his or her Shapley-Shubik index. Orderings are not considered in defining the Banzhaf index. The index of an agent is the number of coalitions in which he or she is pivotal.

A paper by Owen (1971) inspired Shapley (1977) to propose a spatial power index: the Owen-Shapley power index (see also Owen and Shapley, 1989). In this new model, ideological differences between the agents can be taken into account. It is formalized by means of a spatial game, which is a simple game together with a constellation of agent profiles, i.e., a set of vectors in the Euclidean space $\mathbb{R}^m$ that represents the ideological locations of voters. The different dimensions can be seen as ideological considerations or criteria, so each position represents the "ideal point" (of highest preference) in the space. Shapley (1977) writes that the use of the Euclidean space $\mathbb{R}^m$ "seems to leave us ample scope for capturing many kinds of political and ideological parameters without an excess of abstraction and generality".

An issue is formalized by Shapley (1977) through a vector $r \in \mathbb{R}^m$. A player in position x is more in favor of r than a player in position y if the scalar product $r \cdot x$ is less than or equal to $r \cdot y$. Therefore, players can be ordered from the most to the least enthusiastic with respect to an issue, which implies that one of them is pivotal in that ordering. When all issues are equally likely, the probability of a player being pivotal is his or her Owen-Shapley spatial power index.

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https://doi.org/10.1016/j.geb.2022.10.004
0899-8256/

Peters and Zarzuelo (2017) study the Owen-Shapley spatial power index when there are two dimensions. They provide a formula for calculating the index for unanimity games and give an axiomatic characterization by means of a transfer axiom, a dummy axiom, anonymity and two invariant positional axioms (reflection invariance and positional invariance).

A natural variation of the Owen-Shapley spatial power index is to consider that not all issues are equally probable. For example, think about a regional parliament of a country, in which local issues are more relevant than the state ones (see Section 6). Barr and Passarelli (2009) consider that there is a probability distribution over issues, defined by a continuous density function. They analyze the distribution of power in the Council of the EU with two dimensions. The stances toward the EU on international issues and domestic issues are the two dimensions that are taken into account.

In this paper we also consider the variation of the Owen-Shapley spatial power index with two dimensions when there is a(ny) probability distribution over issues. We call these spatial power indices *probabilistic Owen-Shapley spatial power indices*. We give a formula for calculating the indices for unanimity games. Therefore, the index of any spatial game can be easily calculated by means of linear combinations of indices of unanimity games. We conduct an axiomatic study and prove that the family of probabilistic Owen-Shapley spatial power indices can be obtained by means of the axioms employed by Peters and Zarzuelo (2017), dropping reflection invariance and adding continuity. Reflection invariance requires the index not to change when the constellation of agents is shifted or rotated. However, as mentioned above, in a regional parliament, where local issues are more relevant than the state ones, all issues are not regarded equally likely. Consider also the possible presence of the agenda setter effect, which influences the importance or likelihood of the different issues at stake. Within the EU, the Commission would play the role of agenda setter in the Council (see Passarelli and Barr, 2007). Thus, reflection invariance should not be imposed. On the contrary, positional invariance changes relative positions of agents and is satisfied by probabilistic Owen-Shapley spatial power indices. Therefore, a transfer axiom, a dummy axiom, positional invariance and continuity generate the family of probabilistic Owen-Shapley spatial power indices. We also give some illustrative examples, including an application to the Basque Parliament.

Other spatial power indices have also been introduced over the years. For example Shenoy (1982) extends the Banzhaf index to the spatial setting when voters are represented by points in $\mathbb{R}^m$. Passarelli and Barr (2007) employ the multilinear extension approach for cooperative games (Owen, 1972) to define a spatial power index when issues belong to $\mathbb{R}^m$. Alonso-Meijide et al. (2011) define a spatial power index taking into account lengths of paths connecting players' positions. Benati and Marzetti (2013) obtain a family that includes both the Shapley-Shubik index and the Owen-Shapley spatial power index modeling voters' propensity to support an issue through a random utility function. Multinomial values are introduced by Albina-Puente and Carreras (2015) to model different tendencies of agents. Blockmans and Guerry (2015) study the impact of issue saliences and distance selection on the family introduced by Benati and Marzetti (2013). Martin et al. (2017) propose a generalization of the Owen spatial power index (1971).

There are also other studies. Álvarez-Mozos et al. (2017) address the problem of extending the Shapley-Shubik index to the class of simple games with externalities. Karos and Peters (2018) develop a class of power indices for effectivity functions. An issue based power index is also introduced by Kong and Peters (2021) by means of orderings of issues.

The structure of the paper is as follows. Section 2 gives notation and preliminaries. Section 3 presents probabilistic Owen-Shapley spatial power indices. Section 4 calculates these spatial power indices for unanimity games. Section 5 presents the axiomatic characterization of the family of probabilistic Owen-Shapley spatial power indices. We also show the independence of the axioms employed. Some examples can be found in Section 6. Finally, Section 7 gives some concluding remarks and pointers for future work.

2. Preliminaries

2.1. Notation

Given $x, y \in \mathbb{R}^2$ such that $x \neq y$, the half-line which starts at x and passes through y is denoted by $[x, y, \rightarrow)$, the segment with endpoints x and y by $[x, y]$, and we write $(x, y) = [x, y] \setminus \{x, y\}$. The perpendicular bisector line of $[x, y]$ is the line perpendicular to the line through x and y that passes through $\frac{1}{2}x + \frac{1}{2}y$. Given $x \in \mathbb{R}^2$ and a line ℓ in $\mathbb{R}^2$ such that $x \notin \ell$, x^ℓ denotes the reflection of x with respect to ℓ. Notice that ℓ is the perpendicular bisector line of $[x, x^\ell]$. If $x \in \ell$, then $x^\ell = x$.

The projection of $x \in \mathbb{R}^2$ on a line ℓ in $\mathbb{R}^2$, i.e. $\frac{1}{2}x + \frac{1}{2}x^\ell$, is denoted by $\bar{x}^\ell$. Given $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, $x \cdot y = x_1 y_1 + x_2 y_2$. Given $X \subseteq \mathbb{R}^2$, $co(X)$ refers to the convex hull of X.

2.2. Spatial games and spatial power indices

Let U be a set, the universe of players. A *coalition* is a finite nonempty subset of U. A *transferable utility (TU) game* is a pair (N, v) such that N is a coalition and $v : 2^N \rightarrow \mathbb{R}^N$, $v(\emptyset) = 0$. A *simple game* is a TU game (N, v) such that $v(S) \in \{0, 1\}$ for all $S \in 2^N$, $v(N) = 1$ and $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$ (that is, v is monotonic). Let (N, v) be a simple game. A coalition S is *winning* in (N, v) if $v(S) = 1$; otherwise S is *losing*. A *minimal winning* coalition is a winning coalition with no proper winning subcoalition. If $v(S) = 1$ and $v(S \setminus \{i\}) = 0$, player i is said to be *pivotal in* S.

Denote by G^N the set of all TU games with set of players N and by $\mathcal{S}^N$ the subset of all simple games.

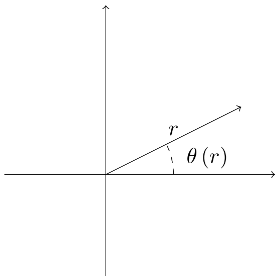

Fig. 1. Polar angle $\theta(r)$.

A *constellation* for a set of players N is a vector $p = (p_i)_{i \in N} \in (\mathbb{R}^2)^N$ such that $p_i \neq p_j$ for all $i, j \in N$ with $i \neq j$. The set of all constellations for N is denoted by C^N. Given $p \in C^N$ and $S \subseteq N$, define $p_S \in (\mathbb{R}^2)^S$ by $(p_S)_i = p_i$ for all $i \in S$. With a slight abuse of notation, $co(p_S)$ denotes the convex hull of the set $\{p_i\}_{i \in S}$.

Given a line ℓ in $\mathbb{R}^2$ and $p \in C^N$, denote by p^ℓ the constellation which is the reflection of p with respect to ℓ, i.e. $p^\ell = (p_i^\ell)_{i \in N}$.

A *spatial game* with a set of players N is a triple (N, v, p) such that $(N, v) \in \mathcal{S}^N$ and $p \in C^N$.

A *spatial power index* on $A \subseteq \mathcal{S}^N$ is a function $\varphi : A \to \mathbb{R}^N$ such that for all $(N, v, p) \in A$ it holds that $\varphi_i(N, v, p) \geq 0$ for all $i \in N$ and $\sum_{i \in N} \varphi_i(N, v, p) = 1$.

The Owen-Shapley spatial power index is defined as follows.

Let (N, v, p) be a spatial game and consider $r \in \mathbb{R}^2$ such that $||r|| = 1$, where $||r||$ denotes the Euclidean length of r. Each r represents a possible issue to be treated by the agents in N. At each r the agent $i \in N$ who is pivotal in $\{j \in N \mid r \cdot p_j \leq r \cdot p_i\}$ is calculated and i is said to be pivotal at r. Notice that i is unique except for a finite number of issues r. We assume that all issues are equally likely. The Owen-Shapley spatial power index of i, denoted by $\Phi_i(N, v, p)$, is the probability of i being pivotal at an issue r.

We take into account the following geometrical consideration. Given an issue r, let ℓ be the line with direction vector r, and for each $j \in N$ the projection $\bar{p}_j^\ell$ on ℓ. We say that $j \in N$ precedes $k \in N$ on ℓ if $\bar{p}_j^\ell$ precedes $\bar{p}_k^\ell$ in the direction of r. Thus, $i \in N$ is pivotal at r if and only if i is pivotal in the set of agents who precede him or her (including himself or herself) on ℓ.

2.3. Dummy player in spatial games

Player $i \in N$ is a *dummy* (Peters and Zarzuelo, 2017) in a spatial game (N, v, p) if $p_i \in co\left(p_{S \setminus \{i\}}\right)$ for every coalition S in which i is pivotal. Observe that if a player is not pivotal in any coalition, then he or she is a dummy. And if a dummy player is pivotal in a coalition S, then he or she is surrounded, according to p, by players in S, and it can therefore be assumed that the dummy player can not take advantage of his/her pivotal power in S.

Observe also that if i is a dummy, then i is never pivotal at any issue r, and therefore, $\Phi_i(N, v, p) = 0$. There can be two situations. If i is not pivotal in any coalition, i clearly cannot be pivotal at any issue r. If i is pivotal in a coalition S, given that i is a dummy, $p_i \in co\left(p_{S \setminus \{i\}}\right)$. As a consequence, given any issue r and a line ℓ with direction vector r, there is always a player j in S such that $\bar{p}_i^\ell$ precedes $\bar{p}_j^\ell$. And hence, i cannot be pivotal in the set of the agents who precede him or her (including i). Therefore, nor is i pivotal at r.

3. Probabilistic Owen-Shapley spatial power indices

Let $\mathcal{B}$ be the σ-field generated by subintervals in $\Omega = (0, 2\pi]$. The Owen-Shapley spatial power index arises when the Lebesgue probability measure λ is considered on $\mathcal{B}$. Indeed, there is a bijection from the set of issues, $\left\{r \in \mathbb{R}^2 : ||r|| = 1\right\}$, into $(0, 2\pi]$ that associates each issue r with its polar angle $\theta(r) \in (0, 2\pi]$, as depicted in Fig. 1. In defining the Owen-Shapley spatial power index, if $\alpha, \beta \in (0, 2\pi]$, $\alpha < \beta$, the probability of the issues r satisfying $\alpha < \theta(r) \leq \beta$ is given by $\lambda(\alpha, \beta] = (\beta - \alpha)/2\pi$.

But not all the areas for the issues might be equally probable when their (Lebesgue) measure is the same. All possible situations are represented by probability measures P on $\mathcal{B}$. Moreover, the probability of single issues r is required to be zero, i.e. P must be non-atomic, as happens for the Lebesgue probabilistic measure.

A spatial power index φ is said to be *associated with a probability* P on $\mathcal{B}$ if for each spatial game (N, v, p) and $i \in N$, $\varphi_i(N, v, p)$ is the probability of i being pivotal at an issue r, when the probability for the issues r, which are represented by $\theta(r) \in (0, 2\pi]$, is given by P. We write $\varphi = \Phi^P$ and we say that φ is a *probabilistic Owen-Shapley spatial power index*.

As pointed out above, this is restricted to non-atomic probability measures.

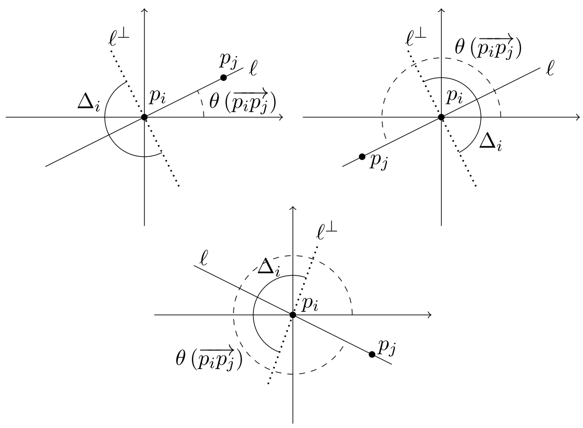

Fig. 2. Different positions of $\overrightarrow{p_ip_j}$.

4. Probabilistic Owen-Shapley spatial power indices for unanimity games

First note that if Δ_i denotes the union of the angle or angles formed by issues r at which i is pivotal, then

$$\Phi_i^P(N, v, p) = P(\Delta_i).$$

Let $v = u_S$, $\emptyset \neq S \subseteq N$, be the unanimity game on S, i.e.,

$$u_S(T) = \begin{cases} 1 \text{ if } T \supseteq S, \\ 0 \text{ otherwise.} \end{cases}$$

Observe that if $i \notin S$, $\Phi_i^P(N, v, p) = 0$, since i is not pivotal at any issue r.

If $|S| = 1$, it is clear that $\Phi_i^P(N, v, p) = 1$ when $i \in S$.

If $|S| > 1$, let

$$S(p) = \left\{i \in S : p_i \notin co\left(p_{S\setminus\{i\}}\right)\right\}, \tag{1}$$

that is, the set of non dummy players in (N, u_S, p). If $i \in S \setminus S(p)$, then i cannot be pivotal at any issue r, and therefore $\Phi_i^P(N, v, p) = 0$. For $i \in S(p)$ take into account the following. Let $i, j \in S$ and an issue r, which can be assumed to start from p_i (we can consider r as a free vector). Let ℓ be the line through p_i and p_j, $\ell^{\perp}$ be the perpendicular line to ℓ through p_i and ℓ' be a line in the direction of r (Fig. 2). Notice that $\bar{p}_j^{\ell'}$ precedes $\bar{p}_i^{\ell'}$ in the direction of r if and only if r is pointing into the halfplane outside p_j with contour $\ell^{\perp}$.

Two cases are distinguished. If $|S(p)| = 2$ and $i \in S(p)$, Δ_i has three expressions according to different positions of $\overrightarrow{p_ip_j}$, as depicted in Fig. 2, and accordingly,

$$\Phi_i^P(N, v, p) = \begin{cases} P\left(\theta\left(\overrightarrow{p_ip_j}\right) + \frac{\pi}{2}, \theta\left(\overrightarrow{p_ip_j}\right) + \frac{3\pi}{2}\right] & \text{if } \theta\left(\overrightarrow{p_ip_j}\right) \leq \frac{\pi}{2}, \\ P\left(0, \theta\left(\overrightarrow{p_ip_j}\right) - \frac{\pi}{2}\right] + P\left(\theta\left(\overrightarrow{p_ip_j}\right) + \frac{\pi}{2}, 2\pi\right] & \text{if } \frac{\pi}{2} < \theta\left(\overrightarrow{p_ip_j}\right) \leq \frac{3\pi}{2}, \\ P\left(\theta\left(\overrightarrow{p_ip_j}\right) - \frac{3\pi}{2}, \theta\left(\overrightarrow{p_ip_j}\right) - \frac{\pi}{2}\right] & \text{if } \frac{3\pi}{2} < \theta\left(\overrightarrow{p_ip_j}\right). \end{cases}$$

If $|S(p)| > 2$, consider $co\left(p_{S(p)}\right)$, which is, by definition, the smallest convex set that contains all points p_i such that $i \in S(p)$. By definition of $S(p)$, the boundary of $co\left(p_{S(p)}\right)$ is the polygon whose vertices are all points p_i such that $i \in S(p)$. Thus, for $i \in S(p)$, there are two players $j, k \in S(p) \setminus \{i\}$ such that p_j and p_k are adjacent vertices to p_i in $co\left(p_{S(p)}\right)$ (Fig. 3). Consider ℓ (resp. $\widetilde{\ell}$) to be the line through p_i and p_j (resp. p_k); $\ell^{\perp}$ (resp. $\widetilde{\ell}^{\perp}$) the line perpendicular to ℓ (resp. $\widetilde{\ell}$) through p_i, and ℓ' to be a line in a direction of an issue r. It turns out that $\bar{p}_{k'}^{\ell'}$ precedes $\bar{p}_i^{\ell'}$ in the direction of r for all $k' \in S$ if and only if r is pointing outwards from $co\left(p_{S(p)}\right)$ between $\ell^{\perp}$ and $\widetilde{\ell}^{\perp}$. These issues form an arc Δ_i that has several expressions depending on the positions of vectors $\overrightarrow{p_ip_j}$ and $\overrightarrow{p_ip_k}$. For the sake of simplicity we write $\theta = \theta\left(\overrightarrow{p_ip_j}\right)$ and $\theta' = \theta\left(\overrightarrow{p_ip_k}\right)$. Without loss of generality we assume that $\theta < \theta'$. Firstly, note that $\theta' - \theta \neq \pi$, since, as written above, p_i, p_j and p_k are vertices of a polygon. Two main cases are distinguished.

1) $\theta' - \theta > \pi$. Thus,

$$\Phi_i^P(N, v, p) = P\left(\theta + \frac{\pi}{2}, \theta' - \frac{\pi}{2}\right].$$

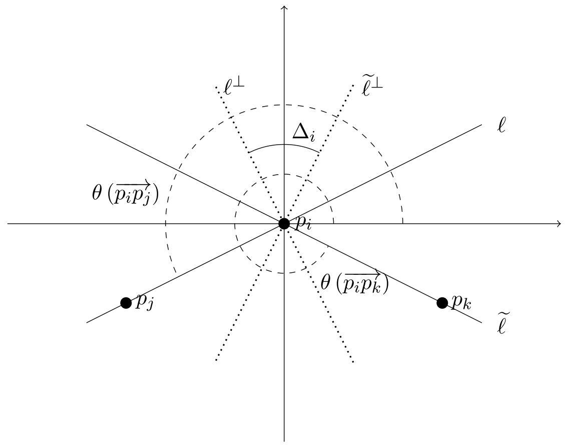

Fig. 3. Δ_i when $|S(p)| > 2$.

2) $\theta' - \theta < \pi$. Thus,

$$\Phi_i^P(N, v, p) = \begin{cases} P\left(\theta' + \frac{\pi}{2}, \theta + \frac{3\pi}{2}\right] & \text{if } \theta \leq \pi/2, \\ P\left(0, \theta - \frac{\pi}{2}\right] + P\left(\theta' + \frac{\pi}{2}, 2\pi\right] & \text{if } \theta > \pi/2 \text{ and } \theta' < 3\pi/2, \\ P\left(\theta' - \frac{3\pi}{2}, \theta - \frac{\pi}{2}\right] & \text{if } \theta > \pi/2 \text{ and } \theta' \geq 3\pi/2. \end{cases}$$

Note that the picture in Fig. 3 is case 2) when $\theta > \pi/2$ and $\theta' \geq 3\pi/2$.

5. Axiomatic characterization of the family of probabilistic Owen-Shapley spatial power indices

Peters and Zarzuelo (2017) characterize the Owen-Shapley spatial power index by means of five axioms: Equal Power Change, Dummy Property, Anonymity, Positional Invariance and Reflection Invariance. Probabilistic Owen-Shapley spatial power indices satisfy all but the last of them, which requires symmetry for the issues. The whole family is characterized by adding one axiom: a continuity axiom.

The first is equivalent to the transfer axiom of Dubey (1975), as remarked in Dubey et al. (2005).

Equal Power Change (EPC) For all set of players N, all $p \in C^N$, and all $v, v', w, w' \in \mathcal{S}^N$, if $v - v' = w - w' \geq 0$, then

$$\varphi(N, v, p) - \varphi(N, v', p) = \varphi(N, w, p) - \varphi(N, w', p).$$

According to this axiom, if the same winning coalitions are added when going from v to v' as when going from w to w', then the change in power for the players when going from v to v' is also the same as when going from w to w'.

In the second axiom, if $(N, v) \in \mathcal{S}^N$ is such that $|N| \geq 2$ and $i \in N$, then $(N \setminus \{i\}, v_{-i}) \in \mathcal{S}^{N\setminus\{i\}}$ is defined by

$$v_{-i}(S) = \begin{cases} v(S \cup \{i\}) & \text{if } \emptyset \neq S \subseteq N \setminus \{i\}, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Game $(N \setminus \{i\}, v_{-i})$ can be seen as the resulting game when player i leaves and gives consent to others, since any winning coalition in (N, v) containing i continues being winning in $(N \setminus \{i\}, v_{-i})$ when player i is no longer present.

Dummy Property (DP) For every spatial game (N, v, p) and every dummy i in (N, v, p),

$$\varphi_j(N, v, p) = \varphi_j(N \setminus \{i\}, v_{-i}, p_{N\setminus\{i\}})$$

for all $j \in N \setminus \{i\}$.

Therefore, the presence of a dummy player does not affect the power of the other players; and the power of a dummy player is zero.

Anonymity is required, i.e. power is independent of the names of the players. Given a spatial game (N, v, p) and an injective function $\sigma : N \to U$, define the spatial game $(\sigma(N), \sigma v, \sigma p)$ by $\sigma v(\sigma(S)) = v(S)$ for all $S \subseteq N$ and $(\sigma p)_{\sigma(i)} = p_i$ for all $i \in N$.

Anonymity (AN) For every spatial game (N, v, p) and every injective function $\sigma : N \to U$,

$$\varphi_{\sigma(i)}(\sigma(N), \sigma v, \sigma p) = \varphi_i(N, v, p),$$

for all $i \in N$.

The following axiom, which is also satisfied by the Owen-Shapley spatial power index, is a spatial invariant axiom. If the relative positions of the agents do not change with respect to a given agent, the index of that agent does not change.

Positional Invariance (PI) For all player sets N and $i \in N$, if $p, p' \in C^N$ satisfy $p_i = p'_i$ and $p'_j \in [p_i, p_j, \rightarrow)$ for all $j \in N \setminus \{i\}$, then

$$\varphi_i(N, v, p) = \varphi_i(N, v, p').$$

The last axiom is a continuity axiom with respect to constellations, which is introduced by Peters and Zarzuelo (2017).

Positional Continuity (PC) Let (N, v, p) be a spatial game and $\{p^m\}$ be a sequence of constellations $p^m \in C^N$ such that $p^m \rightarrow p$. Then,

$$\lim_{m\rightarrow\infty} \varphi(N, v, p^m) = \varphi(N, v, p).$$

Theorem 1. *A spatial power index φ satisfies EPC, DP, AN, PI and PC if and only if there exists a non-atomic probability measure P on $\mathcal{B}$ such that $\varphi = \Phi^P$.*

It is worth noting that Peters and Zarzuelo (2017) gave a second characterization adding PC to the five aforementioned axioms, weakening DP. The weak DP axiom requires dummy players to have zero power. In the case of the probabilistic Owen-Shapley spatial power indices, DP can not be weakened, as shown by this counterexample. Let $\widetilde{P}$ be a probability measure on $\mathcal{B}$ such that $\widetilde{P}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right] = 1$. Consider the spatial power index that satisfies EPC and coincides with Φ for unanimity games u_S such that $|S| = 2$ and with $\Phi^{\widetilde{P}}$ when $|S| > 2$. This spatial power index satisfies EPC, AN, PI, PC and the weak DP, and it is not a probabilistic Owen-Shapley spatial power index.

We prove Theorem 1 in three steps (Propositions 1, 2 and 3). In the first step we employ this lemma, which is used by Peters and Zarzuelo (2017). It follows from Lemma 2.3 in Einy (1987), see also Einy and Haimanko (2011).

Lemma 1. *Let φ be a spatial power index that satisfies EPC and (N, v, p) be a spatial game such that $S_1, ..., S_k$ are the minimal winning coalitions of (N, v). Then,*

$$\varphi(N, v, p) = \sum_{\emptyset \neq I \subseteq \{1,...,k\}} (-1)^{|I|+1} \varphi(N, u_{\bigcup_{k'\in I} S_{k'}}, p).$$

Proposition 1. *If a spatial power index φ satisfies EPC, DP, AN, PI and PC, then, for every $x \in \mathbb{R}^2$ there exists a non-atomic probability measure P_x on $\mathcal{B}$ such that*

$$\varphi_i(N, v, p) = \Phi_i^{P_x}(N, v, p)$$

if $p_i = x$.

Proof. Let φ be a spatial power index that satisfies EPC, DP, AN, PI and PC, and $x \in \mathbb{R}^2$. A probability measure P_x^φ on $\mathcal{B}$ must be found such that

$$\varphi_i(N, v, p) = \Phi_i^{P_x}(N, v, p)$$

if $p_i = x$. Since P_x^φ is a probability measure, $P_x^\varphi(\emptyset) = 0$. Consider now subintervals in $(0, 2\pi]$. If $\alpha, \beta \in (0, 2\pi]$ satisfy $\alpha < \beta$ and $\beta - \alpha < \pi$, let $i, j, k \in U$ and $p \in C^{\{i,j,k\}}$ such that $p_i = x$, the polar angle $\theta\left(\overrightarrow{p_i p_j}\right)$ is smaller than $\theta\left(\overrightarrow{p_i p_k}\right)$,

$$\theta\left(\overrightarrow{p_i p_j}\right) = \begin{cases} \beta + \frac{\pi}{2} & \text{if } \beta + \frac{\pi}{2} \leq 2\pi, \\ \beta - \frac{3\pi}{2} & \text{if } \beta + \frac{\pi}{2} > 2\pi, \end{cases}$$

and

$$\theta\left(\overrightarrow{p_i p_k}\right) = \begin{cases} \alpha + \frac{3\pi}{2} & \text{if } \alpha + \frac{3\pi}{2} \leq 2\pi, \\ \alpha - \frac{\pi}{2} & \text{if } \alpha + \frac{3\pi}{2} > 2\pi. \end{cases}$$

Fig. 4 shows the case in which $\beta + \frac{\pi}{2} \leq 2\pi$ and $\alpha + \frac{3\pi}{2} \leq 2\pi$. Define

$$P_x^\varphi(\alpha, \beta] = \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p).$$

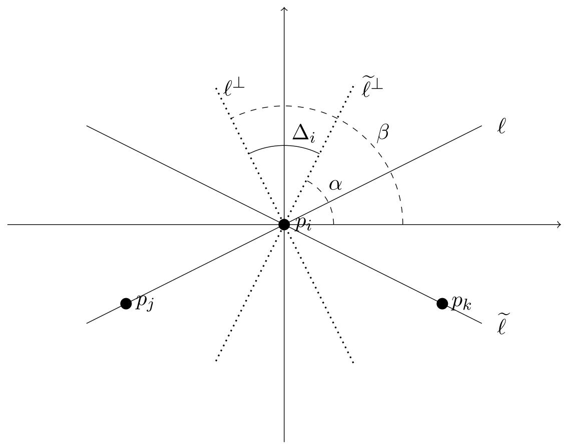

Fig. 4. The case in which $\beta + \frac{\pi}{2} \leq 2\pi$ and $\alpha + \frac{3\pi}{2} \leq 2\pi$.

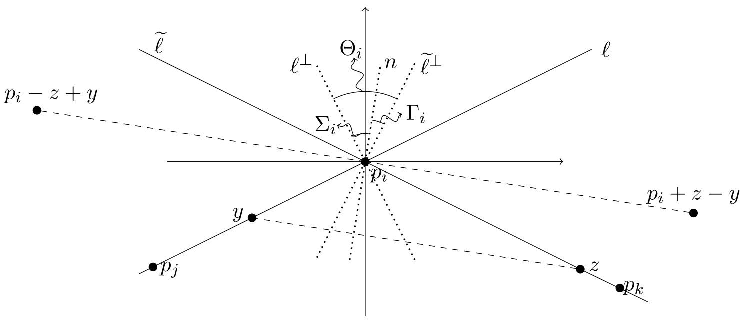

Fig. 5. Graphic for Θ_i, Σ_i and Γ_i.

This is well defined. Indeed, by PI, j and k can be located nearer to or further from i in a straight line. And by AN, this index does not depend on the names of the players.

If $\alpha, \beta \in (0, 2\pi]$ satisfy $\alpha < \beta$ and $\beta - \alpha \geq \pi$, then there exists $\gamma \in (0, 2\pi]$ such that $\alpha < \gamma < \beta$, $\gamma - \alpha < \pi$ and $\beta - \gamma < \pi$, and define

$$P_x^{\varphi}(\alpha, \beta] = P_x^{\varphi}(\alpha, \gamma] + P_x^{\varphi}(\gamma, \beta].$$

To prove that it is also well defined take $\gamma' \in (0, 2\pi]$ such that $\alpha < \gamma' < \beta$, $\gamma' - \alpha < \pi$ and $\beta - \gamma' < \pi$, and prove that

$$P_x^{\varphi}\left(\alpha, \gamma'\right] + P_x^{\varphi}\left(\gamma', \beta\right] = P_x^{\varphi}(\alpha, \gamma] + P_x^{\varphi}(\gamma, \beta]. \tag{2}$$

Assume, without loss of generality, that $\gamma < \gamma'$; then the above equality becomes

$$P_x^{\varphi}\left(\alpha, \gamma'\right] = P_x^{\varphi}(\alpha, \gamma] + P_x^{\varphi}\left(\gamma, \gamma'\right]. \tag{3}$$

By definition of P_x^{φ}, there exist $i, j, k \in U$, and $p \in C^{\{i,j,k\}}$ such that $p_i = x$ and

$$P_x^{\varphi}\left(\alpha, \gamma'\right] = \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p).$$

Let n be the line that passes through p_i and has a direction vector with polar angle γ (Fig. 5). There exist $y \in \left(p_i, p_j\right)$ and $z \in (p_i, p_k)$ such that the line that passes through y and z is perpendicular to n. Applying ii) of Lemma 2 in the Appendix,

$$\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p) = \varphi_{i_0}(\{i_0, i_1, j\}, u_{\{i_0,i_1,j\}}, \widetilde{q}) + \varphi_{i_1}(\{i_0, i_1, k\}, u_{\{i_0,i_1,k\}}, \widehat{q}),$$

where $\widetilde{q} \in C^{\{i_0,i_1,j\}}$ and $\widehat{q} \in C^{\{i_0,i_1,k\}}$ satisfy

$$\widetilde{q}_{i_0} = p_i,\ \widetilde{q}_j = p_j,\ \widetilde{q}_{i_1} = p_i + z - y,$$

and

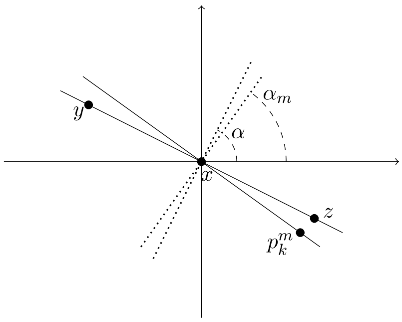

Fig. 6. Graphic for $x, y, z \in \mathbb{R}^2$, the polar angle α and the sequences α_m and p^m.

$$\widehat{q}_{i_1} = p_i,\ \widehat{q}_k = p_k,\ \widehat{q}_{i_0} = p_i + y - z.$$

And taking into account the definition of P_x^φ, equality (3) is proved. Substituting in (2) gives

$$P_x^\varphi\left(\gamma, \gamma'\right] + P_x^\varphi\left(\gamma', \beta\right] = P_x^\varphi\left(\gamma, \beta\right],$$

which is true because $\beta - \gamma < \pi$ and can be proved with the same reasoning as above.

It can be proved similarly that if $\alpha, \beta \in (0, 2\pi]$ satisfy $\alpha < \beta$, then

$$P_x^\varphi\left(\alpha, \beta\right] = \sum_{k=1}^{m} P_x^\varphi\left(\alpha_k, \beta_k\right], \tag{4}$$

when

$$(\alpha, \beta] = \bigcup_{k=1}^{m} (\alpha_k, \beta_k]$$

and the intervals $(\alpha_k, \beta_k]$ are pairwise disjoint.

Lemma 4 in the Appendix proves that P_x^φ has a unique extension (written also P_x^φ) on $\mathcal{B}$ that is a probability measure.

We now prove that P_x^φ is non-atomic, that is, $P_x^\varphi(\alpha) = 0$ for all $\alpha \in (0, 2\pi]$.

Take $x, y, z \in \mathbb{R}^2$ such that $x \in (y, z)$ and let α be the polar angle of a direction vector of the line that passes through x and is perpendicular to the line that passes through y and z (as depicted in Fig. 6). Let α_m be a sequence in $(0, 2\pi]$ such that $\{\alpha_m\} \to \alpha$, $i, j, k \in U$ and $p^m \in C^{\{i,j,k\}}$ be a sequence such that

$$p_i^m = x,\ p_j^m = y,\ \left\{p_k^m\right\} \to z$$

and

$$P_x^\varphi\left(\alpha_m, \alpha\right] = \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p^m), \tag{5}$$

where α_m is the polar angle of a direction vector of the line that passes through x and is perpendicular to the line that passes through x and p_k^m. Thus,

$$\lim_{m\to\infty} \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p^m) = \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, (x, y, z)) = 0,$$

where the first equality holds by PC and the second by DP.

Therefore, by (5),

$$\lim_{m\to\infty} P_x^\varphi\left(\alpha_m, \alpha\right] = 0.$$

Since P_x^φ is a probability measure,

$$\lim_{m\to\infty} P_x^\varphi\left(\alpha_m, \alpha\right] = P_x^\varphi(\alpha),$$

and hence, $P_x^\varphi(\alpha) = 0$.

Finally, we prove that

$$\varphi_i(N, v, p) = \Phi_i^{P_x}(N, v, p)$$

if $p_i = x$. Since φ satisfies EPC, by Lemma 1 it is sufficient to consider the equality for unanimity games. By construction, the equality holds for (N, u_S, p) if $|S| > 2$ and there are at least three non-dummy players in S. It obviously holds when $|S| = 1$. Therefore, by DP, it is sufficient to consider (S, u_S, p) where $|S| = 2$, i.e., $S = \{i, j\} \subseteq N$.

It must now be proved that $\varphi_i(\{i, j\}, u_{\{i,j\}}, p)$ coincides with $P^{\varphi}_x(\Delta_i)$ (see Fig. 2). Let $y \in (p_i, p_j)$ and $\{y_m\} \subseteq \mathbb{R}^2$ be a sequence such that $\{y_m\} \to y$. Let $k \in N \setminus \{i, j\}$ and $p^m \in C^{\{i,j,k\}}$ be a sequence such that $p^m_{\{i,j\}} = p_{\{i,j\}}$ and $p^m_k = y_m$. Thus, PC and DP give

$$\varphi_i(\{i, j\}, u_{\{i,j\}}, p) = \lim_{m\to\infty} \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p^m). \tag{6}$$

By definition of P^{φ}_x, it follows that $\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p^m) = P^{\varphi}_x(\Delta^m_i)$, where Δ^m_i is a sequence of intervals or unions of two disjoint intervals in $(0, 2\pi]$ whose limit is Δ_i. Since P^{φ}_x is a probability measure,

$$\lim_{m\to\infty} P^{\varphi}_x\left(\Delta^m_i\right) = P^{\varphi}_x(\Delta_i),$$

which, together with (6), implies the required result. □

Proposition 2. *If a spatial power index φ satisfies DP, AN, PI and PC, then, for every $x, y \in \mathbb{R}^2$,*

$$P^{\varphi}_x = P^{\varphi}_y.$$

Proof. i) Let $\alpha, \beta \in (0, 2\pi]$. We prove that $P^{\varphi}_x(\Delta) = P^{\varphi}_y(\Delta)$ when $\Delta = (\alpha, \beta] \subseteq (0, 2\pi]$, where $\beta - \alpha = \pi$, or $\Delta = (\alpha, 2\pi] \cup (0, \beta] \subseteq (0, 2\pi]$, where $2\pi - \alpha + \beta = \pi$.

Let $i, j \in U$ and $p \in C^{\{i,j\}}$ such that $p_i = x$, $p_j = y$ and $\Delta = \Delta_i$, where Δ_i is the set formed by the polar angles of the vectors that start at p_i and point to the half-plane with contour $\ell^{\perp}$ that does not contain p_j (see Fig. 2). Since φ is a spatial power index,

$$\varphi_i(\{i, j\}, u_{\{i,j\}}, p) + \varphi_j(\{i, j\}, u_{\{i,j\}}, p) = 1.$$

By Proposition 1,

$$\varphi_i(\{i, j\}, u_{\{i,j\}}, p) = P^{\varphi}_x(\Delta_i)$$

and

$$\varphi_j(\{i, j\}, u_{\{i,j\}}, p) = P^{\varphi}_y\left(\Delta_j\right),$$

where $\Delta_j = (0, 2\pi] \setminus \Delta_i$. Substituting the two equalities in the first one and taking into account that P^{φ}_y is a probability measure on $\mathcal{B}$, $P^{\varphi}_x(\Delta_i) = P^{\varphi}_y(\Delta_i)$ is proved.

ii) Let $\alpha, \beta \in (0, 2\pi] \subseteq (0, 2\pi]$. Let $\Theta = (\alpha, \beta] \subseteq (0, 2\pi]$, where $\beta - \alpha < \pi$, or $\Theta = (\alpha, 2\pi] \cup (0, \beta] \subseteq (0, 2\pi]$, where $2\pi - \alpha + \beta < \pi$. Let (see Fig. 7) $i, j, k, i_0, i_1 \in U$, $p \in C^{\{i,j,k\}}$ such that $p_i = x$, $y \in (p_i, p_j)$, $z \in (p_i, p_k)$ and Θ is the set of polar angles of the vectors that start at p_i and point to the intersection of the half-plane with contour $\ell^{\perp}$ that does not contain y and the half-plane with contour $\widetilde{\ell}^{\perp}$ that does not contain z. By i) of Lemma 2,

$$\begin{aligned}&\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p)\\ &= \varphi_{i_0}(\{i_0, i_1, j, k\}, u_{\{i_0,i_1,j,k\}}, q) + \varphi_{i_1}(\{i_0, i_1, j, k\}, u_{\{i_0,i_1,j,k\}}, q),\end{aligned} \tag{7}$$

where $q \in C^{\{i_0,i_1,j,k\}}$ satisfies $q_{\{j,k\}} = p_{\{j,k\}}$, $q_{i_0} = y$ and $q_{i_1} = z$. Moving p_k closer to y in a straight line, by PI, the first index on the right-hand side of (7) does not change. Moreover, when $p_k \in co\left(\{y, p_j, z\}\right)$, player k becomes a dummy player. Thus, DP implies

$$\varphi_{i_0}(\{i_0, i_1, j, k\}, u_{\{i_0,i_1,j,k\}}, q) = \varphi_{i_0}(\{i_0, i_1, j\}, u_{\{i_0,i_1,j\}}, q_{\{i_0,i_1,j\}}).$$

Similarly, approaching p_j closer to z in a straight line, by PI, the second index on the right-hand side of (7) does not change. And player j becomes a dummy player when $p_j \in co(\{y, p_k, z\})$. Again by DP,

$$\varphi_{i_1}(\{i_0, i_1, j, k\}, u_{\{i_0,i_1,j,k\}}, q) = \varphi_{i_1}(\{i_0, i_1, k\}, u_{\{i_0,i_1,k\}}, q_{\{i_0,i_1,k\}}).$$

Then, (7) turns into

$$\begin{aligned}&\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p)\\ &= \varphi_{i_0}(\{i_0, i_1, j\}, u_{\{i_0,i_1,j\}}, q_{\{i_0,i_1,j\}}) + \varphi_{i_1}(\{i_0, i_1, k\}, u_{\{i_0,i_1,k\}}, q_{\{i_0,i_1,k\}}).\end{aligned}$$

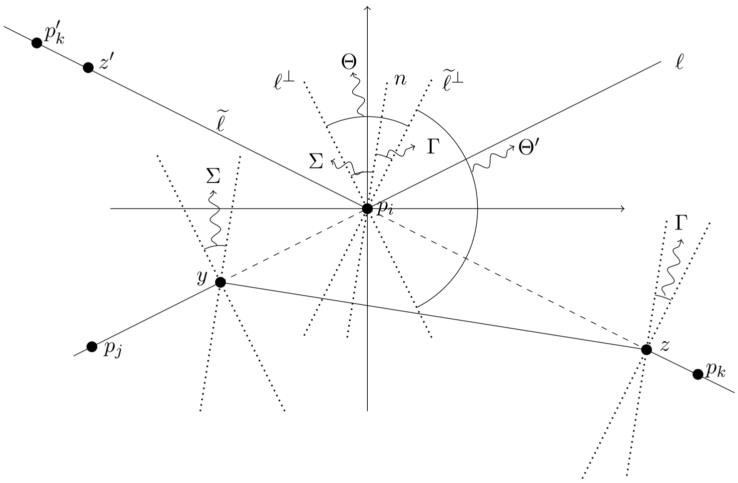

Fig. 7. Graphic aid for Proposition 2, ii).

Let n be the line that passes through p_i and is perpendicular to the line that passes through y and z. By Proposition 1, the above equality turns into

$$P_x^\varphi(\Theta) = P_y^\varphi(\Sigma) + P_z^\varphi(\Gamma),$$

where Σ and Γ are as follows. On the one hand, Σ is the set of polar angles of the vectors that start at y and point to the translation to y of the intersection of the half-plane with contour $\ell^\perp$ that does not contain y and the half-plane with contour n that does not contain z. On the other hand, Γ is the set of polar angles of the vectors that start at z and point to the translation to z of the intersection of the half-plane with contour $\widetilde{\ell}^\perp$ that does not contain z and the half-plane with contour n that does not contain y.

Now, instead of z and p_k, take z_m and $\widetilde{z}_m$. Consider the sequence $z_m = z + m(p_k - z)$ and $\widetilde{z}_m = p_k + m(p_k - z)$ and the associated sequence of constellations $p^m \in C^{\{i,j,k\}}$ and $q^m \in C^{\{i_0,i_1,j,k\}}$ such that $p^m_{\{i,j\}} = p_{\{i,j\}}$, $p^m_k = \widetilde{z}_m$, $q_{\{j,k\}} = p_{\{j,k\}}$, $q_{i_0} = y$ and $q_{i_1} = z_m$. For every m, we have the sets Θ, Σ^m and Γ^m defined similarly as above, and then

$$P_x^\varphi(\Theta) = P_y^\varphi\left(\Sigma^m\right) + P_{z_m}^\varphi\left(\Gamma^m\right). \tag{8}$$

Note that the associated n approaches $\widetilde{\ell}^\perp$ when m goes to infinity, and therefore, Σ^m goes to Θ and Γ^m goes to the empty set.

If instead of z and p_k, we take $z' = p_i - (z - p_i)$ and $p'_k = p_i - (p_k - p_i)$, likewise,

$$P_x^\varphi\left(\Theta'\right) = P_y^\varphi\left(\Sigma'\right) + P_{z'}^\varphi\left(\Gamma'\right),$$

where Θ' is the set of polar angles of the vectors that start at p_i and point to the intersection of the half-plane with contour $\ell^\perp$ that does not contain y and the half-plane with contour $\widetilde{\ell}^\perp$ that does not contain z'. If n' is the line that passes through p_i and is perpendicular to the line that passes through y and z', Σ' is the set of polar angles of the vectors that start at y and point to the translation to y of the intersection of the half-plane with contour $\ell^\perp$ that does not contain y and the half-plane with contour n' that does not contain z'. Similarly, Γ' is the set of polar angles of the vectors that start at z' and point to the translation to z' of the intersection of the half-plane with contour $\widetilde{\ell}^\perp$ that does not contain z' and the half-plane with contour n' that does not contain y.

And if we take the sequence $z'_m = z' + m\left(p'_k - z'\right)$ and $\widetilde{z}'_m = p'_k + m\left(p'_k - z'\right)$, likewise, we have

$$P_x^\varphi\left(\Theta'\right) = P_y^\varphi\left(\left(\Sigma'\right)^m\right) + P_{z'_m}^\varphi\left(\left(\Gamma'\right)^m\right). \tag{9}$$

When m goes to infinity, then $\left(\Sigma'\right)^m$ goes to Θ' and $\left(\Gamma'\right)^m$ goes to a singleton set.

If $\Delta = \Theta \cup \Theta'$, adding equalities (8) and (9) we get

$$P_x^\varphi(\Delta) = P_y^\varphi\left(\Sigma^m\right) + P_y^\varphi\left(\left(\Sigma'\right)^m\right) + P_{z_m}^\varphi\left(\Gamma^m\right) + P_{z'_m}^\varphi\left(\left(\Gamma'\right)^m\right). \tag{10}$$

Since P_y^φ is a probability measure and Σ^m is a sequence of intervals or unions of two disjoint intervals in $(0, 2\pi]$ such that $\lim_{m\to\infty} \Sigma^m = \Theta$, then

$$\lim_{m\to\infty} P_y^\varphi\left(\Sigma^m\right) = P_y^\varphi(\Theta). \tag{11}$$

In the same way, since $\lim_{m\to\infty}\left(\Sigma'\right)^m=\Theta'$, then $\lim_{m\to\infty}P_y^{\varphi}\left(\Sigma'\right)^m=P_y^{\varphi}\left(\Theta'\right)$. Therefore,

$$\lim_{m\to\infty}\left(P_y^{\varphi}\left(\Sigma^m\right)+P_y^{\varphi}\left(\left(\Sigma'\right)^m\right)\right)=P_y^{\varphi}\left(\Theta\right)+P_y^{\varphi}\left(\Theta'\right)=P_y^{\varphi}\left(\Delta\right),$$

which, jointly with equality (10), implies

$$\lim_{m\to\infty}\left(P_{z_m}^{\varphi}\left(\Gamma^m\right)+P_{z'_m}^{\varphi}\left(\left(\Gamma'\right)^m\right)\right)=P_x^{\varphi}\left(\Delta\right)-P_y^{\varphi}\left(\Delta\right).$$

Since Δ belongs to case i) w.r.t. x and y, the right-hand side is equal to zero, and consequently,

$$\lim_{m\to\infty}\left(P_{z_m}^{\varphi}\left(\Gamma^m\right)+P_{z'_m}^{\varphi}\left(\left(\Gamma'\right)^m\right)\right)=0.$$

And taking into account that both $P_{z_m}^{\varphi}\left(\Gamma^m\right)$ and $P_{z'_m}^{\varphi}\left(\left(\Gamma'\right)^m\right)$ are non-negative, because $P_{z_m}^{\varphi}$ and $P_{z'_m}^{\varphi}$ are probability measures, then

$$\lim_{m\to\infty}P_{z_m}^{\varphi}\left(\Gamma^m\right)=0.$$

Therefore, taking limits in both sides of (8) and taking (11) also into account,

$$P_x^{\varphi}\left(\Theta\right)=\lim_{m\to\infty}P_y^{\varphi}\left(\Sigma^m\right)+\lim_{m\to\infty}P_{z_m}^{\varphi}\left(\Gamma^m\right)=P_y^{\varphi}\left(\Theta\right),$$

and the required result is obtained. □

Proposition 3. *If P is a non-atomic probability measure P on $\mathcal{B}$, then Φ^P is a spatial power index that satisfies EPC, DP, AN, PI and PC.*

Proof. Axioms EPC, DP, AN and PI are satisfied as in Peters and Zarzuelo (2017). As for PC, it is immediate since, in the definition of Φ^P, the probability for the issues r is given by the non-atomic probability measure P. □

The following counterexamples show that the axioms in Theorem 1 are independent. Counterexamples 1 and 2 are used by Peters and Zarzuelo (2017).

(1) Let

$$\xi^1(N,v,p)=\frac{1}{2^k-1}\sum_{\emptyset\neq I\subseteq\{1,\dots,k\}}\Phi(N,u_{\bigcup_{k'\in I}S_{k'}},p),$$

where $S_1,\dots,S_k$ are the minimal winning coalitions of (N,v). This spatial power index satisfies all the axioms in Theorem 1 except EPC.

(2) Let $\xi^2(N,v,p)=Sh(v)$, where Sh denotes the Shapley-Shubik index. Then, ξ^2 satisfies all the axioms except DP.

(3) For every $i\in U$, take $a_i>0$, not all equal, and define

$$\xi_i^3(N,v,p)=\frac{a_i\Phi_i(N,v,p)}{\sum_{j\in N}a_j\Phi_j(N,v,p)}.$$

This spatial power index satisfies all the axioms but AN.

(4) Let $\mathcal{R}$ be the set formed by all the rectangles R with sides parallels to the axes of coordinates in $\mathbb{R}^2$. Every rectangle $R\in\mathcal{R}$ that contains $\{(0,0),(0,1),(1,1),(1,0)\}$ is assigned a probability measure p_R in a continuous way w.r.t. those rectangles R, not all those probabilities being equal. For every spatial game (N,u_S,p), let $R_{S(p)}$ be the smallest $R\in\mathcal{R}$ that contains $S(p)\cup\{(0,0),(0,1),(1,1),(1,0)\}$, where $S(p)$ is defined in formula (1). Let ξ^4 be a spatial power index that satisfies EPC and such that $\xi^4(N,u_S,p)=\Phi^{P_{R_{S(p)}}}(N,u_S,p)$. This index satisfies all the axioms except PI.

(5) Let P be a probability measure on $\mathcal{B}$ other than the Lebesgue measure and let ξ^5 be a spatial power index that satisfies EPC and

$$\xi^5(N,u_S,p)=\begin{cases}\Phi(N,u_S,p) & \text{if } |S(p)|\le 2,\\ \Phi^P(N,u_S,p) & \text{if } |S(p)|>2.\end{cases}$$

Thus, ξ^5 satisfies all the axioms except PC.

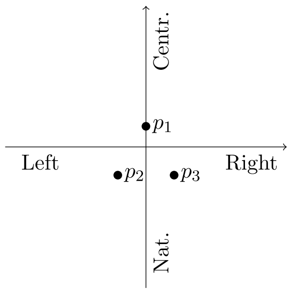

Fig. 8. The constellation in Example 1.

6. Examples

Example 1. Consider a regional parliament with three parties: 1, 2 and 3. There are 9 seats, thus, 5 are needed for a majority. Assume that the parties are represented on the following axes: left-right and nationalist-centralist. The seats and the positions of the three parties are the following ones:

$*$	1	2	3
Seats	2	4	3
Left-right	0	-1	1
Nat-Centralist	$\sqrt{3}-1$	-1	-1

Therefore, they form an equilateral triangle in $\mathbb{R}^2$, as depicted in Fig. 8. Parties 2 and 3 are nationalist, while party 1 is centralist. Parties 1, 2 and 3 are the centrist, left and right parties, respectively.

The Owen-Shapley spatial power index of all parties in the majority spatial game is the same, that is, each index is equal to 1/3. This is due to the symmetry of parties in the majority spatial game and because all issues are regarded equally likely.

However, assume that the proposals or bills that are decided in the parliament have a stronger nationalist component, which are represented by issues r such that $\theta(r) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right]$. Furthermore, assume that these proposals are of equal importance. Hence, consider the uniform probability measure $\widetilde{P}$ on $\mathcal{B}$ such that $\widetilde{P}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right] = 1$. In the majority spatial game, Party 2 is pivotal at issues r such that $\theta(r) \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$, while Party 3 is pivotal at issues r such that $\theta(r) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Since

$$\widetilde{P}\left(\frac{\pi}{2}, \frac{3\pi}{4}\right] = \frac{1}{2} = \widetilde{P}\left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

then $\Phi_2^{\widetilde{P}}(N, v, p) = 0.5 = \Phi_3^{\widetilde{P}}(N, v, p)$ and $\Phi_1^{\widetilde{P}}(N, v, p) = 0$, where (N, v, p) denotes the majority spatial game. Therefore, when all issues are not regarded equally likely and non-nationalist issues are negligible, Party 2 has less power than the others. In fact, Party 2 is powerless, as it is not pivotal at any nationalist issue. Note also that Party 2 is not a dummy player in (N, v, p) since Party 2 is pivotal, for example, in $S = \{1, 2\}$ and $p_2 \notin co(p_1)$. The spatial indices in this example can also be calculated using Lemma 1, as in the following one.

Example 2. Consider the Basque Parliament elected in 2020. There are 6 parties: Eusko Alderdi Jeltzailea/Partido Nacionalista Vasco (EAJ/PNV), Euskal Herria Bildu (EHB), Partido Socialista de Euskadi (PSE), Elkarrekin Podemos (EP), Partido Popular (PP) and Vox. There are 75 seats, therefore, 38 seats are needed for a majority. From the Euskobarometer (2019), a political research magazine of the University of the Basque Country, we can represent these political parties on the following axes: left-right and nationalist-centralist (see Fig. 9). These are the seats and positions.

$*$	EAJ/PNV (1)	EHB (2)	PSE (3)	EP (4)	PP (5)	Vox (6)
Seats	31	21	10	6	6	1
Left-right	5.5	2.1	4.1	2.8	7.8	9.3
Nat-Centralist	3.3	1.9	6.3	5.4	8.5	9.5

Let (N, v, p), where $N = \{1, 2, 3, 4, 5, 6\}$, v is the majority game and p is the constellation of parties. The minimal winning coalitions of the majority game are: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4, 5\}$, $\{1, 4, 6\}$, $\{1, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$ and $\{2, 3, 5, 6\}$. Thus, if P is a probability measure on $\mathcal{B}$, Lemma 1 implies

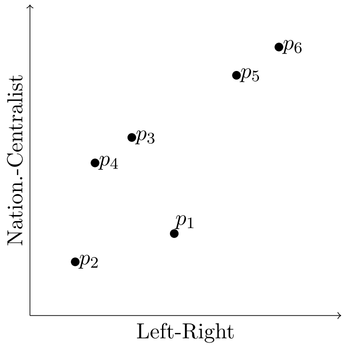

Fig. 9. The constellation in Example 2.

$$\Phi^P(N,v,p) = \Phi^P(N,u_{\{1,2\}},p) + \Phi^P(N,u_{\{1,3\}},p) - \Phi^P(N,u_{\{1,2,3\}},p) + \Phi^P(N,u_{\{1,4,5\}},p)$$
$$-\Phi^P(N,u_{\{1,2,4,5\}},p) - \Phi^P(N,u_{\{1,3,4,5\}},p) + \Phi^P(N,u_{\{2,3,4,5\}},p) + \Phi^P(N,u_{\{1,4,6\}},p)$$
$$-\Phi^P(N,u_{\{1,2,4,6\}},p) - \Phi^P(N,u_{\{1,3,4,6\}},p) + \Phi^P(N,u_{\{2,3,4,6\}},p) + \Phi^P(N,u_{\{1,5,6\}},p)$$
$$-\Phi^P(N,u_{\{1,2,5,6\}},p) - \Phi^P(N,u_{\{1,3,5,6\}},p) + \Phi^P(N,u_{\{2,3,5,6\}},p) - 2\Phi^P(N,u_{\{1,4,5,6\}},p)$$
$$+2\Phi^P(N,u_{\{1,2,4,5,6\}},p) + 2\Phi^P(N,u_{\{1,3,4,5,6\}},p) - 2\Phi^P(N,u_{\{2,3,4,5,6\}},p).$$

As in Example 1, consider the Lebesgue probability measure and $P = \widetilde{P}$. The above equality gives the following indices:

$*$	EAJ/PNV (1)	EHB (2)	PSE (3)	EP (4)	PP (5)	Vox (6)
Φ	0.76	0.15	0	0	0.09	0
$\Phi^{\widetilde{P}}$	0.75	0.25	0	0	0	0

For example, 5 is a dummy player in (N, u_S, p) when $5 \notin S$ and when

$$S \in \{1356,\ 2356,\ 1456,\ 12456,\ 13456,\ 23456\}.$$

Therefore, the above expression reduces to

$$\Phi_5^P(N,v,p) = \Phi_5^P(N,u_{\{1,4,5\}},p) - \Phi_5^P(N,u_{\{1,2,4,5\}},p)$$
$$-\Phi_5^P(N,u_{\{1,3,4,5\}},p) + \Phi_5^P(N,u_{\{2,3,4,5\}},p) + \Phi_5^P(N,u_{\{1,5,6\}},p) - \Phi_5^P(N,u_{\{1,2,5,6\}},p).$$

And by PI and DP,

$$\Phi_5^P(N,v,p)$$
$$= -\Phi_5^P(N,u_{\{1,3,5\}},p) + \Phi_5^P(N,u_{\{2,3,5\}},p) + \Phi_5^P(N,u_{\{1,5,6\}},p) - \Phi_5^P(N,u_{\{2,5,6\}},p).$$

If $\Phi^P = \Phi$, this expression is equal to 0.09. If $\Phi^P = \Phi^{\widetilde{P}}$, then

$$\Phi_5^P(N,u_{\{1,3,5\}},p) = \Phi_5^P(N,u_{\{2,3,5\}},p) \text{ and } \Phi_5^P(N,u_{\{1,5,6\}},p) = \Phi_5^P(N,u_{\{2,5,6\}},p),$$

and hence $\Phi_5^{\widetilde{P}}(N,v,p) = 0$.

The governing party (EAJ/PNV) is by far the party with the largest power for the two indices, 0.76 for Φ and 0.75 for $\Phi^{\widetilde{P}}$. The main opposition party (EHB) has some power in both cases. However, the Owen-Shapley spatial power index of PP, which is clearly a centralist party, is not 0. Meanwhile, the other parties are powerless. If we assume that the issues of the Basque Parliament are nationalist issues, then the index of PP decreases to 0. Moreover, the power of EHB, which is the most nationalist one, has increased.

Neither PSE, EP, PP nor Vox are dummy players in (N, v, p). Indeed, PSE, EP and PP are pivotal, for example, in $S = \{2,3,4,5\}$ and $p_i \notin co\left(p_{S\setminus\{i\}}\right)$ if $i \in \{3,4,5\}$. Moreover, Vox is pivotal in $S = \{2,3,5,6\}$ and $p_6 \notin co\left(p_{\{2,3,5\}}\right)$. Nevertheless, PSE, EP and Vox are powerless for the two spatial indices, since they are not pivotal at any issue. Vox, with only 1 seat, is not pivotal at any issue because it is far from the other parties. PSE and EP are not pivotal at nationalist or centralist issues (represented by r such that $\theta(r) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right]$ or $\theta(r) \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right]$) because EAJ/PNV or EHB are the pivotal parties. Note that PSE, EP, PP and Vox add up to 23 seats, which are not sufficient for them to be pivotal at these issues. As for left or right issues (represented by r such that $\theta(r) \in \left(\frac{7\pi}{4}, 2\pi\right] \cup \left(0, \frac{\pi}{4}\right]$ or $\theta(r) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right]$), EHB, PSE and EP add up to 37 seats, and therefore, neither PSE nor EP is pivotal at these issues. Although PP has some power with the Owen-Shapley spatial power index, it is powerless with $\Phi^{\widetilde{P}}$, since, as written above, PP is not pivotal at any nationalist issue.

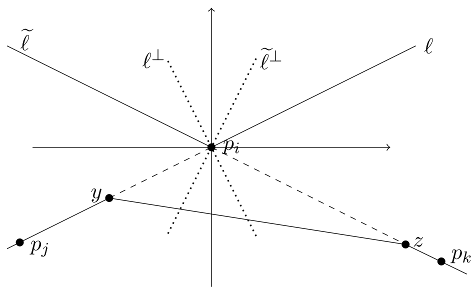

Fig. 10. Graphic aid for Lemma 2, i).

7. Concluding remarks

In this paper, we provide an axiomatic characterization of a probabilistic family of spatial power indices when there are two dimensions. To that end, we use variations of some well known axioms in cooperative game theory, as well as specific spatial axioms leading to the family of spatial power indices under study.

As future work, the case with more than two dimensions can be studied. The axioms of our characterization are satisfied if more than two dimensions are considered, but are they enough to characterize the family? We do not know the answer.

We can also look at the case with a finite number of issues and agents that form a constellation in $\mathbb{R}^2$. The probabilistic Owen-Shapley spatial power indices can also be defined if the preferences of agents are different with respect to the issues. The probability measure is finite and all our axioms except continuity are satisfied. However, they do not seem enough to obtain the family of the probabilistic Owen-Shapley spatial power indices. What else is needed? We propose a weaker definition of dummy player. If R denotes the finite set of issues, a player i is R-dummy in (N, v, p) if i is not pivotal at any $r \in R$. Our guess is that EPC, DP for R-dummy players and AN characterize the family of probabilistic Owen-Shapley spatial power indices associated with R. We will address this issue in future work.

Finally, we believe that this work might be useful for the study of other spatial power indices.

Declaration of competing interest

There is no conflict of interest.

Acknowledgments

This work has been partially supported by the Ministry of Science and Innovation (PID2019-105291GB-I00), and by UPV/EHU (GIU20/019).

Appendix A

Lemma 2. *Let $i, j, k, i_0, i_1 \in U$, $p \in C^{\{i,j,k\}}$ such that p_j, p_i and p_k do not belong to the same line, $y \in \left(p_i, p_j\right)$ and $z \in (p_i, p_k)$.*

i) If φ is a spatial power index that satisfies DP, AN and PI, then (see Fig. 10)

$$\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p) = \varphi_{i_0}(\{i_0, i_1, j, k\}, u_{\{i_0,i_1,j,k\}}, q) + \varphi_{i_1}(\{i_0, i_1, j, k\}, u_{\{i_0,i_1,j,k\}}, q),$$

where $q \in C^{\{i_0,i_1,j,k\}}$ satisfies $q_{\{j,k\}} = p_{\{j,k\}}$, $q_{i_0} = y$ and $q_{i_1} = z$.

ii) If φ is a spatial power index that satisfies DP, AN, PI and PC, then (see Fig. 5)

$$\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p) = \varphi_{i_0}(\{i_0, i_1, j\}, u_{\{i_0,i_1,j\}}, \widetilde{q}) + \varphi_{i_1}(\{i_0, i_1, k\}, u_{\{i_0,i_1,k\}}, \widehat{q}),$$

where $\widetilde{q} \in C^{\{i_0,i_1,j\}}$ and $\widehat{q} \in C^{\{i_0,i_1,k\}}$ satisfy

$$\widetilde{q}_{i_0} = p_i,\ \widetilde{q}_j = p_j,\ \widetilde{q}_{i_1} = p_i + z - y,$$

and

$$\widehat{q}_{i_1} = p_i,\ \widehat{q}_k = p_k,\ \widehat{q}_{i_0} = p_i + y - z.$$

Proof. i)

Let $\widetilde{p} \in C^{\{i,j,k,i_1\}}$ such that $\widetilde{p}_{\{i,j,k\}} = p$ and $\widetilde{p}_{i_1} = z$. Since i_1 is a dummy player in $(\{i, j, k, i_1\}, u_{\{i,j,k,i_1\}}, \widetilde{p})$, by DP,

$$\varphi_j(\{i, j, k\}, u_{\{i,j,k\}}, p) = \varphi_j(\{i, j, k, i_1\}, u_{\{i,j,k,i_1\}}, \widetilde{p}),$$

and moving p_i to y, PI implies

$$\varphi_j(\{i,j,k,i_1\},u_{\{i,j,k,i_1\}},\widetilde{p})=\varphi_j(\{i,j,k,i_1\},u_{\{i,j,k,i_1\}},q'),$$

where $q'_{\{j,k,i_1\}}=q$ and $q'_i=y$. And applying AN,

$$\varphi_j(\{i,j,k,i_1\},u_{\{i,j,k,i_1\}},q')=\varphi_j(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q).$$

Therefore, the above three equalities imply

$$\varphi_j(\{i,j,k\},u_{\{i,j,k\}},p)=\varphi_j(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q). \tag{12}$$

Similarly, we get

$$\varphi_k(\{i,j,k\},u_{\{i,j,k\}},p)=\varphi_k(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q). \tag{13}$$

Equalities (12) and (13) imply

$$\begin{aligned}&\varphi_i(\{i,j,k\},u_{\{i,j,k\}},p)\\&=1-\sum_{l\in\{j,k\}}\varphi_l(\{i,j,k\},u_{\{i,j,k\}},p)=1-\sum_{l\in\{j,k\}}\varphi_l(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q)\\&=\varphi_{i_0}(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q)+\varphi_{i_1}(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q),\end{aligned}$$

where the first and third equality hold because φ is a spatial power index, and the required result is obtained.

ii) For every $m\in\mathbb{N}$, let $y_m=y+\left(1-\frac{1}{m}\right)(p_i-y)=\left(1-\frac{1}{m}\right)p_i+\frac{1}{m}y$, and $z_m\in\mathbb{R}^2$ be the intersection of the line that passes through p_i and z and the line with direction vector $z-y$ that passes through y_m. By i),

$$\begin{aligned}&\varphi_i(\{i,j,k\},u_{\{i,j,k\}},p)\\&=\varphi_{i_0}(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q^m)+\varphi_{i_1}(\{i_0,i_1,j,k\},u_{\{i_0,i_1,j,k\}},q^m),\end{aligned}$$

where $q^m\in C^{\{i_0,i_1,j,k\}}$ satisfies $q^m_{\{j,k\}}=p_{\{j,k\}}$, $q^m_{i_0}=y_m$ and $q^m_{i_1}=z_m$. For the first index on the right-hand side of the above equality, apply PI (moving p_k closer to y_m in a straight line) and DP (k becomes a dummy player). And again PI (approaching p_j closer to z_m in a straight line) and DP (j becomes a dummy player) for the second index. Then

$$\begin{aligned}&\varphi_i(\{i,j,k\},u_{\{i,j,k\}},p)\\&=\varphi_{i_0}(\{i_0,i_1,j\},u_{\{i_0,i_1,j\}},q^m_{\{i_0,i_1,j\}})+\varphi_{i_1}(\{i_0,i_1,k\},u_{\{i_0,i_1,k\}},q^m_{\{i_0,i_1,k\}}).\end{aligned}$$

And again applying PI (moving y_m to y_m+z-y for the first index on the right-hand side, and z_m to z_m+y-z for the second index),

$$\begin{aligned}&\varphi_i(\{i,j,k\},u_{\{i,j,k\}},p)\\&=\varphi_{i_0}(\{i_0,i_1,j\},u_{\{i_0,i_1,j\}},\widetilde{q}^m)+\varphi_{i_1}(\{i_0,i_1,k\},u_{\{i_0,i_1,k\}},\widehat{q}^m),\end{aligned} \tag{14}$$

where $\widetilde{q}^m_{\{i_0,j\}}=q^m_{\{i_0,j\}}$, $\widetilde{q}^m_{i_1}=y_m+z-y$, $\widehat{q}^m_{\{i_1,k\}}=q^m_{\{i_1,k\}}$ and $\widehat{q}^m_{i_0}=z_m+y-z$.

Since $\{y_m\}\longrightarrow p_i$, it follows that $\left\{\widetilde{q}^m\right\}\longrightarrow\widetilde{q}$, and therefore PC implies

$$\lim_{m\to\infty}\varphi_{i_0}(\{i_0,i_1,j\},u_{\{i_0,i_1,j\}},\widetilde{q}^m)=\varphi_{i_0}(\{i_0,i_1,j\},u_{\{i_0,i_1,j\}},\widetilde{q}).$$

Similarly, since $\{z_m\}\longrightarrow p_i$, it follows that $\left\{\widehat{q}^m\right\}\longrightarrow\widehat{q}$, and hence PC implies

$$\lim_{m\to\infty}\varphi_{i_1}(\{i_0,i_1,k\},u_{\{i_0,i_1,k\}},\widehat{q}^m)=\varphi_{i_1}(\{i_0,i_1,k\},u_{\{i_0,i_1,k\}},\widehat{q}).$$

Taking limits on both sides of (14) the required result is obtained. □

Lemma 3. *If φ is a spatial power index that satisfies DP, AN, PI and PC, then, for every $x\in\mathbb{R}^2$, $P^{\varphi}_x(0,2\pi]=1$.*

Proof. For every $m\in\mathbb{N}$, let $x_m=x+\frac{1}{m}(1,0)$, $y_m=x+\frac{1}{m}(1,1)$ and $z_m=x+\frac{1}{m}(0,1)$. Let $N=\{i_0,i_1,i_2,i_3\}\subseteq U$ and $p^m\in C^N$ be such that $p^m_{i_0}=x$, $p^m_{i_1}=x_m$, $p^m_{i_2}=y_m$ and $p^m_{i_3}=z_m$. Since φ is a spatial power index,

$$\sum_{j=0}^{3}\varphi_{i_j}\left(N,u_N,p^m\right)=1. \tag{15}$$

By definition of P_x^{φ}, and applying PI (approaching y_m closer to x in a straight line) and DP (i_2 becomes a dummy player),

$$\varphi_{i_0}\left(N, u_N, p^m\right) = P_x^{\varphi}\left(\pi, \frac{3\pi}{2}\right].$$

Applying PI (moving z_m closer to x_m in a straight line) and DP (i_3 becomes a dummy player),

$$\varphi_{i_1}\left(N, u_N, p^m\right) = \varphi_{i_1}\left(\{i_0, i_1, i_2\}, u_{\{i_0,i_1,i_2\}}, p^m_{\{i_0,i_1,i_2\}}\right). \tag{16}$$

And by PI (moving $p^m_{i_0}$ and $p^m_{i_2}$ further from $p^m_{i_1}$ in straight lines),

$$\varphi_{i_1}\left(\{i_0, i_1, i_2\}, u_{\{i_0,i_1,i_2\}}, p^m_{\{i_0,i_1,i_2\}}\right) = \varphi_{i_1}\left(\{i_0, i_1, i_2\}, u_{\{i_0,i_1,i_2\}}, \left(p^{i_1}\right)^m\right), \tag{17}$$

where $\left(p^{i_1}\right)^m_{i_1} = p^m_{i_1}$, $\left(p^{i_1}\right)^m_{i_0} = p^m_{i_1} - (1,0)$ and $\left(p^{i_1}\right)^m_{i_2} = p^m_{i_1} + (0,1)$.

Since $\left\{p^m_{i_1}\right\} \to x$, it follows that $\left(p^{i_1}\right)^m \to p^{i_1}$, where $p^{i_1} \in C^{\{i_0,i_1,i_2\}}$ satisfies $\left(p^{i_1}\right)_{i_1} = x$, $\left(p^{i_1}\right)_{i_0} = x - (1,0)$ and $\left(p^{i_1}\right)_{i_2} = x + (0,1)$. And therefore, PC implies

$$\lim_{m\to\infty} \varphi_{i_1}\left(\{i_0, i_1, i_2\}, u_{\{i_0,i_1,i_2\}}, \left(p^{i_1}\right)^m\right) = \varphi_{i_1}\left(\{i_0, i_1, i_2\}, u_{\{i_0,i_1,i_2\}}, p^{i_1}\right),$$

which, together with (16) and (17), implies

$$\lim_{m\to\infty} \varphi_{i_1}\left(N, u_N, p^m\right) = \varphi_{i_1}\left(\{i_0, i_1, i_2\}, u_{\{i_0,i_1,i_2\}}, p^{i_1}\right).$$

By definition of P_x^{φ}, the right-hand side of this equality equals to $P_x^{\varphi}\left(\frac{3\pi}{2}, 2\pi\right]$, and hence,

$$\lim_{m\to\infty} \varphi_{i_1}\left(N, u_N, p^m\right) = P_x^{\varphi}\left(\frac{3\pi}{2}, 2\pi\right].$$

Similarly, we have

$$\lim_{m\to\infty} \varphi_{i_2}\left(N, u_N, p^m\right) = P_x^{\varphi}\left(0, \frac{\pi}{2}\right]$$

and

$$\lim_{m\to\infty} \varphi_{i_3}\left(N, u_N, p^m\right) = P_x^{\varphi}\left(\frac{\pi}{2}, \pi\right].$$

Consequently, taking limits on both sides of (15),

$$P_x^{\varphi}\left(\pi, \frac{3\pi}{2}\right] + P_x^{\varphi}\left(\frac{3\pi}{2}, 2\pi\right] + P_x^{\varphi}\left(0, \frac{\pi}{2}\right] + P_x^{\varphi}\left(\frac{\pi}{2}, \pi\right] = 1.$$

That is, by (4), $P_x^{\varphi}(0, 2\pi] = 1$. □

Lemma 4. *If φ is a spatial power index that satisfies DP, AN, PI and PC, then, for every $x \in \mathbb{R}^2$, P_x^{φ} is a probability measure on $\mathcal{B}$.*

Proof. We prove that P_x^{φ} is a probability measure on the field of finite unions of intervals, denoted by $\mathcal{B}_0$. The result follows since any probability measure on a field has a unique extension that is a probability measure on the associated σ-field.

P_x^{φ} has to satisfy several properties in order to be a probability measure on $\mathcal{B}_0$.

i) We prove in the previous lemma that $P_x^{\varphi}(0, 2\pi] = 1$.

ii) P_x^{φ} is countably additive on $\mathcal{B}_0$. We prove this in two steps.

• First, on the class of intervals. Let $(\alpha_m, \beta_m]$ be a finite or infinite sequence of pairwise disjoint intervals. It needs to be proved that if

$$(\alpha, \beta] = \bigcup_{m=1}^{\infty} (\alpha_m, \beta_m],$$

then

$$P_x^{\varphi}(\alpha, \beta] = \sum_{m=1}^{\infty} P_x^{\varphi}(\alpha_m, \beta_m].$$

The finite case is already proven (expression (4)), so now the infinite case is considered. Since $P_x^{\varphi}(\alpha_m, \beta_m] \geq 0$ for all $m \in \mathbb{N}$, the terms in this series can be rearranged so that $\beta_{m+1} < \beta_m$ for all $m \in \mathbb{N}$ and $\beta_1 = \beta$ (note also that $\{\alpha_m\} \to \alpha$). Hence, taking into account (4),

$$\sum_{m=1}^{M} P_x^{\varphi}(\alpha_m, \beta_m] = P_x^{\varphi}(\alpha_M, \beta]$$

holds for all $M \in \mathbb{N}$, so the equality to be proved is

$$P_x^{\varphi}(\alpha, \beta] = \lim_{M\to\infty} P_x^{\varphi}(\alpha_M, \beta].$$

We distinguish two cases.

If $\beta - \alpha < \pi$, let $i, j, k \in N$ and $p \in C^{\{i,j,k\}}$ such that $p_i = x$ and

$$P_x^{\varphi}(\alpha, \beta] = \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p), \tag{18}$$

where α (resp. β) is the polar angle of a direction vector of the line which is perpendicular to the line that passes through p_i and p_k (resp. p_j) (Fig. 4). Let $\{y_m\} \subseteq \mathbb{R}^2$ be a sequence such that $\{y_m\} \to p_k$ and α_m (for big enough $m \in \mathbb{N}$) is the above polar angle associated with p_i and y_m. Let $p^m \in C^{\{i,j,k\}}$ such that $p^m_{\{i,j\}} = p_{\{i,j\}}$ (observe that $p^m_i = x$) and $p^m_k = y_m$. Thus, PC implies

$$\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p) = \lim_{m\to\infty} \varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p^m). \tag{19}$$

By the definition of P_x^{φ},

$$\varphi_i(\{i, j, k\}, u_{\{i,j,k\}}, p^m) = P_x^{\varphi}(\alpha_m, \beta],$$

so this equality, together with (18) and (19), implies the required result.

If $\beta - \alpha \geq \pi$, there exists γ such that $\alpha < \gamma < \beta$, $\gamma - \alpha < \pi$, (and by (4))

$$P_x^{\varphi}(\alpha, \beta] = P_x^{\varphi}(\alpha, \gamma] + P_x^{\varphi}(\gamma, \beta],$$

and

$$P_x^{\varphi}(\alpha_m, \beta] = P_x^{\varphi}(\alpha_m, \gamma] + P_x^{\varphi}(\gamma, \beta].$$

Hence, the equality to be proved reduces to

$$P_x^{\varphi}(\alpha, \gamma] = \lim_{M\to\infty} P_x^{\varphi}(\alpha_M, \gamma],$$

which is true because this is the first case again.

• Now we prove that P_x^{φ} is countably additive on $\mathcal{B}_0$.

First, if

$$A = \bigcup_{k=1}^{m} I_k \in \mathcal{B}_0,$$

where I_k are pairwise disjoint intervals, define

$$P_x^{\varphi}(A) = \sum_{k=1}^{m} P_x^{\varphi}(I_k).$$

This is well defined because if

$$A = \bigcup_{l=1}^{m'} I'_l,$$

then

$$A = \bigcup_{k=1}^{m} \bigcup_{l=1}^{m'} \left(I_k \cap I'_l\right),$$

and therefore,

$$P_x^{\varphi}(A) = \sum_{k=1}^{m} \sum_{l=1}^{m'} P_x^{\varphi}\left(I_k \cap I'_l\right),$$

and by (4), this double addition coincides with both

$$\sum_{k=1}^{m} P_x^{\varphi}(I_k) \text{ and } \sum_{l=1}^{m'} P_x^{\varphi}\left(I'_l\right).$$

To prove that P_x^{φ} is countably additive on $\mathcal{B}_0$, let A_k be a sequence of pairwise disjoint elements in $\mathcal{B}_0$ such that

$$A = \bigcup_{k=1}^{\infty} A_k \in \mathcal{B}_0.$$

Since $A \in \mathcal{B}_0$ and $A_k \in \mathcal{B}_0$, it follows that

$$A = \bigcup_{l=1}^{m} I_l \text{ and } A_k = \bigcup_{l'=1}^{m_k} I_{l'}^k,$$

where I_l and $I_{l'}^k$ are pairwise disjoint intervals. Thus, since P_x^{φ} is countably additive on the class of intervals,

$$P_x^{\varphi}(A) = \sum_{l=1}^{m} P_x^{\varphi}(I_l) = \sum_{l=1}^{m} \sum_{k=1}^{\infty} \sum_{l'=1}^{m_k} P_x^{\varphi}\left(I_l \cap I_{l'}^k\right) = \sum_{k=1}^{\infty} \sum_{l'=1}^{m_k} P_x^{\varphi}\left(I_{l'}^k\right) = \sum_{k=1}^{\infty} P_x^{\varphi}(A_k).$$

iii) Taking into account that φ is non-negative, $P_x^{\varphi}(A) \geq 0$ for all $A \in \mathcal{B}_0$ and $P_x^{\varphi}(A) \leq 1$ for all $A \in \mathcal{B}_0$ (since $P_x^{\varphi}(0, 2\pi] = 1$). $\square$

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