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PhD Thesis

Cohomological uniqueness of finite groups of prime power order

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"Why, anybody can have a brain. That's a very mediocre commodity. Every pusillanimous creature that crawls on the earth – or slinks through slimy seas has a brain! Back where I come from we have universities, seats of great learning – where men go to become great thinkers. And when they come out, they think deep thoughts – and with no more brains than you have... But! They have one thing you haven't got! A diploma!"

The Wizard to the Scarecrow (*The Wizard of Oz* 1939 film)

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Contents

Introduction

Group cohomology originates, along with homological algebra, from algebraic topology and the study of cohomology groups of certain topological spaces. Given a finite group G, we can associate to it a classifying space BG, which satisfies that its first homotopy group is isomorphic to G, and its higher order homotopy groups are trivial. We can then define the cohomology groups $\mathrm{H}^n(G, V)$ of G with coefficients on an RG-module V, where R is a commutative ring, to be the cohomology groups of its classifying space BG, see [Hat02]. It is also possible to give a purely algebraic definition of the cohomology groups of G in terms of derived functors, and in fact this is the approach that we will follow throughout this thesis.

Subsequently, the study of group cohomology has developed into a deep and vibrant area of research in its own right. If we take the direct sum $\mathrm{H}^\bullet(G, R)$ of all cohomology groups of the finite group G with coefficients on the trivial module R equipped with the so-called cup product, we obtain a finitely generated graded-commutative ring [Eve91, Chapter 3]. This shows that group cohomology possesses a rich algebraic structure that may be exploited to glean a substantial amount of information, such as the minimal number of generators and relations in a presentation of the group. It also has countless applications outside group theory, in areas such as number theory and algebraic geometry, see [Gui18] and [Sil13].

An important feature of group cohomology is the fact that it admits multiple characterizations, all of them contributing to giving us a fuller picture of the subject. Indeed, cohomology can be described in terms of extensions of both modules and groups. On the one hand, the characterization of the Ext functor in terms of extensions of modules, originally due to Yoneda [Yon92], allows us to describe the elements of $\mathrm{H}^n(G, V)$ as equivalence classes of extensions of RG-modules of the form

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow R \longrightarrow 0.$$

On the other hand, it is well known that the second cohomology group $\mathrm{H}^2(G, V)$ classifies, up to equivalence, the group extensions of the form

$$0 \longrightarrow V \longrightarrow E \longrightarrow G \longrightarrow 1.$$

This can be generalized as in [Hol79] to higher degree cohomology groups, so that the elements of $\mathrm{H}^n(G, V)$ can be seen as equivalence classes of crossed extensions of the form

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

Both of these characterizations of group cohomology share many of their fundamental features. When studied side by side, they provide a much more natural, rather than computational, description of not only the cohomology groups themselves, but also their functorial properties, connecting homomorphisms and the cup product. Furthermore, these characterizations also help us construct explicit cohomology classes with specific properties that would otherwise be very difficult to find.

We will be most interested in mod-p cohomology rings of finite p-groups, as the computation of cohomology rings of finite groups can actually be reduced to that setting. Indeed, cohomology is in general easier to compute when working with coefficients over a field K due to results such as the Künneth Formula and the Universal Coefficient Theorem, see [Eve91, Section 2.5]. Moreover, we can use Maschke's Theorem [Ben91, Corollary 3.6.12] to show that the cohomology of G is trivial unless the characteristic of K divides the order of G. It is then possible to assume that $K = \mathbb{F}_p$, with p a prime factor of the order of G, using the Universal Coefficient Theorem. Finally, it can be shown that the mod-p cohomology of G embeds into the mod-p cohomology of any of its Sylow p-subgroups [Bro82, Section III.10].

In general, it is not possible to determine if two given finite groups are isomorphic just by looking at their cohomology rings, i.e. the family of finite groups does not possess the property of cohomological uniqueness. Indeed, if p is an odd prime and G is a finite abelian p-group, the isomorphism type of the mod-p cohomology ring of G only depends on the minimal number of generators of G, see [Eve91, Section 3.5]. In particular, any two non-isomorphic cyclic p-groups will have isomorphic mod-p cohomology rings.

We may nevertheless restrict our attention to specific families of finite p-groups, and study whether the groups in these families can be distinguished by their mod-p cohomologies. We then ask the following questions: Is a given family of groups cohomologically unique? And conversely, which families of groups cannot be distinguished cohomologically, because their cohomology rings are only finitely many up to isomorphism? As we have already explained, the answer to the first question is positive for the family of finite elementary abelian p-groups. On the other hand, if we take all finite abelian p-groups of a fixed rank, then all the groups in this family have the same mod-p cohomology ring.

In our quest to determine the cohomological uniqueness of certain families of groups, we encounter one major challenge. In general, it is extremely difficult to compute the cohomology ring of a given group, let alone a family of groups. In fact, there are very few examples, outside of the ones already mentioned, of explicit computations of cohomology rings in the literature. It is for this reason that we may choose, on a related note, to focus our attention on computing certain invariants of the cohomology, rather than trying to determine the full structure of the ring.

It is then interesting to study the algebraic invariants of the cohomology ring of G, and how these relate to the group theoretic structure of G. For example, the Krull dimension of the mod-p cohomology of G can be easily computed, thanks to a result by Quillen [CTVZ03, Corollary 8.4.7], as the rank of a maximal elementary abelian p-subgroup of G. This remarkable result completely determines the Krull dimension of the cohomology in terms of the subgroup structure of the group.

Attempts to do the same for another algebraic invariant, the depth, have so far proven much less successful. Although closely related to the Krull dimension, the depth is considerably harder to compute. So far, only upper [CTVZ03, Proposition 12.2.5] and lower bounds [Duf81] have been found. Nevertheless, Carlson stated in [Car95] a conjecture characterizing the depth of the mod-p cohomology of G by looking at how well said cohomology can be detected by restricting to the cohomologies of certain subgroups of G.

The main obstacle in the study of this conjecture is that, typically, we would need to first compute the cohomology ring and then use computational methods to determine the depth. The lack of examples in the literature, nonetheless, makes this approach futile for the study of the depth in infinite families of p-groups. For p odd, we consider the pro-p group of maximal nilpotency class G, which has a unique finite quotient G_r of order p^{r+1} for each integer $r \geq 2$. Using the aforementioned bounds on the depth of $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$, we are able to determine that its value is either 1 or 2, for every $r \geq 2$. It is now that we may employ the characterization of group cohomology in terms of extensions. In order to compute the depth when $r \leq p-2$, we construct a non-trivial cohomology class in $\mathrm{H}^3(G_r, \mathbb{F}_p)$ as a product of a Yoneda extension in $\mathrm{H}^1(G_r, \mathbb{F}_p)$ and a crossed extension in $\mathrm{H}^2(G_r, \mathbb{F}_p)$, and prove that it restricts trivially to every subgroup in a particular family of subgroups of G_r. This allows us to use Carlson's results from

[Car95] to show that the depth of $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ is 1 for $2 \leq r \leq p-2$, meaning that these groups satisfy Carlson's depth conjecture. Crucially, we are able to compute the value of the depth without first computing the cohomology rings themselves. These results published in [GGG22], along with others by Garaialde Ocaña [Gar18], suggest that the cohomology rings of the finite quotients of G are either identical or extremely similar.

We now turn our attention to the matter of computing the cohomology rings themselves. Doing so for infinite families of groups is only possible through a detailed study of the specific groups under consideration. Spectral sequences have proven to be an extraordinarily powerful tool in the computation of cohomology of finite groups, and have become one of the main techniques employed with this goal. They appear mainly in the shape of the Lyndon-Hochschild-Serre (LHS) spectral sequence [Eve91, Section 7.2], which allows us to somehow approximate the cohomology of a group G that can be obtained as a group extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

starting with the cohomologies of the quotient Q and the normal subgroup N, and repeatedly computing cohomology groups. The main drawback of this approach is that it relies on the computation of certain differentials, of which we usually possess little to no information. This problem can be solved under certain circumstances, when it is possible to find explicit formulas that will aid us in computing said differentials. Such is the case when G is a split extension of Q by N, as shown by Charlap and Vasquez [CV69] and later adapted by Siegel in [Sie96] to the specific case when Q is a cyclic p-group of order p. In [GG23], we generalize this result of Siegel for when Q is a cyclic p-group of any order.

Also in [Sie96], Siegel uses his results about differentials in the computation of the LHS spectral sequence of the Heisenberg group $\mathrm{Heis}(p)$ modulo p for $p \geq 3$. We are able to follow the same argument as Siegel in order to compute the LHS spectral sequences of the Heisenberg groups $\mathrm{Heis}(p^n)$ modulo p^n for $p \geq 5$ and $n \geq 2$, and show that they are all isomorphic starting with the second page. This implies that, in this infinite family of groups, only a finite number of isomorphism classes of cohomology rings appear. The previous results have been published in [GG23].

The structure of the thesis is as follows:

In Chapter 1, we introduce the main concepts and results from homological algebra that we will be using throughout this thesis. We begin by introducing chain and cochain complexes of modules, as well as projective modules and projective resolutions, which we use to define the Tor and Ext functors. Afterwards, we define the cohomology groups of finite groups and review some of its basic properties. We construct the bar resolution of a finite group and compute the lower degree cohomology groups with it. Then, we recall the structure of cohomology as a graded-commutative ring given by the cup product. Furthermore, we give a detailed description of the Bockstein homomorphisms, which we later use to classify the central extensions of elementary abelian p-groups of rank two with cyclic kernel of order p. We conclude the chapter with an introduction to spectral sequences, focusing in particular on the Lyndon-Hochschild-Serre spectral sequence and its main properties.

In Chapter 2, we describe the cohomology of a finite group in terms of extensions. First, we give the classical description of Ext using Yoneda extensions. Afterwards, we introduce crossed extensions in order to describe cohomology groups. We then define a product of Yoneda extensions with crossed extensions that coincides with the usual cup product in cohomology.

In Chapter 3, we introduce the concept of depth for the mod-p cohomology ring of a finite group and state Carlson's depth conjecture. Afterwards, we compute the depth of the mod-p cohomology rings of certain quotients of the pro-p group of maximal class that, moreover, satisfy Carlson's depth conjecture.

In Chapter 4, we state a theorem by Charlap and Vasquez regarding the computation of the second differential of the Lyndon-Hochschild-Serre spectral sequence associated to a split extension of finite groups. Afterwards, we introduce a generalization of a result by Siegel that can be used to compute the differentials appearing in the spectral sequence associated to a split extension of finite groups with cyclic quotient of prime power order.

In Chapter 5, we compute the Lyndon-Hochschild-Serre spectral sequence of a family of finite Heisenberg groups of prime power order, up to the infinity page. We begin by computing the second page of the spectral sequence and its structure as an algebra, before putting the results from the last chapter to use in the computation of the second differential. Afterwards, we determine the third page and show that it is at this point that the spectral sequence collapses. In so doing, we provide one of the first infinite families of groups of prime power order whose associated LHS spectral sequences collapse in the same page and are isomorphic. Finally, we compute the Poincaré series of the cohomology rings.

1 Review of group cohomology

In this chapter, we will introduce the main concepts and results from homological algebra that we will be using throughout this thesis. These notions are standard, and the chapter is mainly intended as a brief introduction in order to fix notation. We begin by introducing chain and cochain complexes of modules, as well as projective modules and projective resolutions, which we use to define the Tor and Ext functors. Afterwards, we define the cohomology groups of finite groups and review some of its basic properties. We construct the bar resolution of a finite group and compute the lower degree cohomology groups with it. Then, we recall the structure of cohomology as a graded-commutative ring given by the cup product. Furthermore, we give a detailed description of the Bockstein homomorphisms, which we later use to classify the central extensions of elementary abelian p-groups of rank two with cyclic kernel of order p. We conclude the chapter with an introduction to spectral sequences, focusing in particular on the Lyndon-Hochschild-Serre spectral sequence and its main properties. For a detailed overview of the concepts exposed in this chapter, see [Bro82], [Eve91] and [Wei94].

1.1 Basic homological algebra

Throughout this chapter, R will denote a commutative ring with unity. Let G be a finite group. We consider the group algebra RG as the free R-module with basis

G with the product induced by the product in G. The group algebra RG is an augmented R-algebra, i.e. there is an R-algebra homomorphism $\varepsilon\colon RG \longrightarrow R$, called the *augmentation*, which is defined by $\varepsilon(g) = 1$ for $g \in G$.

All the modules we consider will be left modules unless otherwise stated. In general, we will not need to make the distinction because every right RG-module V can be turned into a left RG-module by setting $gv = vg^{-1}$ for $g \in G$ and $v \in V$.

1.1.1 Chain and cochain complexes

Definition 1.1. A *chain complex* of RG-modules is a sequence of RG-modules of the form

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

with RG-module homomorphisms $\partial_n\colon C_n \longrightarrow C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. We denote this chain complex by $C_\bullet$. Analogously, a *cochain complex* of RG-modules is a sequence of RG-modules of the form

$$\cdots \longleftarrow C^{n+1} \xleftarrow{\partial^n} C^n \xleftarrow{\partial^{n-1}} C^{n-1} \longleftarrow \cdots$$

with RG-module homomorphisms $\partial^n\colon C^n \longrightarrow C^{n+1}$ such that $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{Z}$. We denote this cochain complex by $C^\bullet$.

Definition 1.2. Given a chain complex of RG-modules $C_\bullet$, we define for each $n \in \mathbb{Z}$ the n-th *homology group* of $C_\bullet$ as

$$\mathrm{H}_n(C_\bullet) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}}.$$

Analogously, given a cochain complex of RG-modules $C^\bullet$, we define the n-th *cohomology group* of $C^\bullet$ as

$$\mathrm{H}^n(C^\bullet) = \frac{\operatorname{Ker}\partial^n}{\operatorname{Im}\partial^{n-1}}.$$

Definition 1.3. Let $A_\bullet$ and $B_\bullet$ be chain complexes of RG-modules. A *chain morphism* $f\colon A_\bullet \longrightarrow B_\bullet$ is a collection of RG-module homomorphisms $f_n\colon A_n \longrightarrow B_n$ such that

$$\partial^B_{n+1} \circ f_{n+1} = f_n \circ \partial^A_{n+1}$$

for all $n \in \mathbb{Z}$.

Whenever we have a chain morphism $f\colon A_\bullet \longrightarrow B_\bullet$, it is easy to see that it induces an RG-module homomorphism $f^n_*\colon \mathrm{H}_n(A_\bullet) \longrightarrow \mathrm{H}_n(B_\bullet)$.

Definition 1.4. Given chain morphisms $f, g\colon A_\bullet \longrightarrow B_\bullet$, a *chain homotopy* from f to g is a collection of RG-module homomorphisms $h\colon A_n \longrightarrow B_{n+1}$ such that

$$f_n - g_n = h_{n-1} \circ \partial_n^A + \partial_{n+1}^B \circ h_n$$

for every $n \in \mathbb{Z}$. If there is a chain homotopy from f to g, we say that f and g are *chain homotopic*.

The importance of chain homotopies lies in the fact that homotopic chain maps induce the same morphisms in homology, see [Wei94, Lemma 1.4.5].

It is possible to construct new chain complexes from old chain complexes by taking tensor products and Hom functors. Given RG-modules V and W, the tensor product $V \otimes_R W$ is also an RG-module, with the action of G given by

$$g(v \otimes w) = gv \otimes gw$$

for $g \in G$, $v \in V$ and $w \in W$. We can also define an RG-module structure on $\operatorname{Hom}_R(V, W)$ by setting

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for $g \in G$, $v \in V$ and $\varphi \in \operatorname{Hom}_R(V, W)$. Now, if $A_\bullet$ and $B_\bullet$ are chain complexes of RG-modules with differentials ∂^A and ∂^B, respectively, we can define the chain complex of RG-modules $A_\bullet \otimes_R B_\bullet$ with

$$(A_\bullet \otimes_R B_\bullet)_n = \bigoplus_{r+s=n} A_r \otimes_R B_s$$

and differential $\partial_n\colon (A_\bullet \otimes_R B_\bullet)_n \longrightarrow (A_\bullet \otimes_R B_\bullet)_{n-1}$ given by

$$\partial_n(a \otimes b) = \partial_r^A(a) \otimes b + (-1)^r a \otimes \partial_s^B(b) \tag{1.1}$$

for $a \in A_r$ and $b \in B_s$ with $r + s = n$. We can also define the chain complex of RG-modules $\operatorname{Hom}_R(A_\bullet, B_\bullet)$ with

$$\operatorname{Hom}_R(A_\bullet, B_\bullet)_n = \prod_{s-r=n} \operatorname{Hom}_R(A_r, B_s)$$

and differential $\partial_n\colon \operatorname{Hom}_R(A_\bullet, B_\bullet)_n \longrightarrow \operatorname{Hom}_R(A_\bullet, B_\bullet)_{n-1}$ given by

$$(\partial_n f)(a) = \partial_s^B\big(f(a)\big) - (-1)^n f\big(\partial_{r+1}^A(a)\big)$$

for $f \in \operatorname{Hom}_R(A_r, B_s)$ with $s - r = n$ and $a \in A_{r+1}$.

1.1.2 Projective modules and resolutions

In this section, we will recall the concepts of projective and injective modules, after which we introduce projective resolutions and some of their properties.

Definition 1.5. Let P be an RG-module. We say that P is *projective* if, given a surjective RG-module homomorphism $g\colon V \longrightarrow W$, any RG-module homomorphism $f\colon P \longrightarrow W$ can be lifted to an RG-module homomorphism $h\colon P \longrightarrow V$ making the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\ V \arrow[r, "g"] & W \arrow[r] & 0 \end{tikzcd}$$

It is easily checked that free modules are projective, although the converse need not be true in general. Dual to the concept of projective module, we have that of injective module.

Definition 1.6. Let I be an RG-module. We say that I is *injective* if, given an injective RG-module homomorphism $g\colon V \longrightarrow W$, any RG-module homomorphism $f\colon V \longrightarrow W$ can be extended to an RG-module homomorphism $h\colon W \longrightarrow I$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "g"] \arrow[d, "f"] & W \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd}$$

We can now define resolutions of modules and their main properties.

Definition 1.7. Given an RG-module V, we refer to an exact sequence of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

as a *resolution* of V, and we denote this resolution by $P_\bullet \longrightarrow V$. We say that $P_\bullet \longrightarrow V$ is a *projective resolution* if P_n is a projective RG-module for every $n \geq 0$.

It is well known that every RG-module V admits a projective resolution (see [Wei94, Lemma 2.2.5]). The following result, which can be found in [Wei94, Theorem 2.2.6], shows that projective resolutions are unique up to homotopy equivalence.

Theorem 1.8 (Comparison Theorem)**.** *Let V and W be RG-modules, $P_\bullet \longrightarrow V$ be a chain complex of projective RG-modules and $Q_\bullet \longrightarrow W$ be a resolution of W. Given an RG-module homomorphism $\alpha\colon V \longrightarrow W$, there is a chain morphism $f\colon P_\bullet \longrightarrow Q_\bullet$ such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & V \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

Furthermore, f is unique up to chain homotopy.

Definition 1.9. A projective resolution $X_\bullet \longrightarrow V$ is said to be *minimal* if we have that $\operatorname{Ker} \partial_n \subseteq JX_n$ for all $n \geq 0$. Here, J denotes the Jacobson radical of RG, i.e. the intersection of all maximal left ideals of RG.

Given a projective resolution $P_\bullet \longrightarrow V$ and a minimal resolution $X_\bullet \longrightarrow V$, we have that $P_\bullet = X_\bullet \oplus Q_\bullet$, where $Q_\bullet \longrightarrow 0$ is a projective resolution of the zero module. Furthermore, if V is a simple RG-module, then both $X_\bullet \otimes_{RG} V$ and $\operatorname{Hom}_{RG}(X_\bullet, V)$ have zero differential, see [Eve91, Section 2.4].

1.1.3 Ext and Tor functors

We will now define the Tor and Ext functors.

Definition 1.10. Let V and W be RG-modules and $P_\bullet \longrightarrow V$ be a projective resolution. For each $n \geq 0$, we can define the n-th Tor *group*

$$\operatorname{Tor}_n^{RG}(V, W) = \operatorname{H}_n(P_\bullet \otimes_{RG} W).$$

Similarly, we define the n-th Ext *group*

$$\operatorname{Ext}_R^n(V, W) = \operatorname{H}^n\left(\operatorname{Hom}_{RG}(P_\bullet, W)\right).$$

We have that $\operatorname{Tor}_n^{RG}(-,-)$ is a bifunctor covariant in both arguments, whereas $\operatorname{Ext}_{RG}^n(-,-)$ is a bifunctor contravariant in the first argument and covariant in the second.

We will focus on studying the Ext functor. One of the main properties of Ext is that it can be applied to short exact sequences of modules in order to obtain long exact sequences of Ext groups. In order to state this fact more precisely, we will introduce a new concept that will be useful in the next chapter when studying different characterizations of Ext.

Definition 1.11. Let $\mathrm{E}^\bullet$ be a family of covariant functors $\mathrm{E}^n\colon \mathbf{Mod}_{RG} \longrightarrow \mathbf{Ab}$ for $n \geq 0$ such that, given a short exact sequence of RG-modules

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

we obtain a long exact sequence

$$\begin{tikzcd} \mathrm{E}^0(V_1) \arrow[r] & \mathrm{E}^0(V_2) \arrow[r] & \mathrm{E}^0(V_3) \arrow[dll, dashed] \\ \mathrm{E}^n(V_1) \arrow[r] & \mathrm{E}^n(V_2) \arrow[r] & \mathrm{E}^n(V_3) \arrow[dll, "\delta"] \\ \mathrm{E}^{n+1}(V_1) \arrow[r] & \mathrm{E}^{n+1}(V_2) \arrow[r] & \mathrm{E}^{n+1}(V_3) \arrow[r] & \cdots \end{tikzcd}$$

such that given a morphism of short exact sequences of RG-modules of the form

$$\begin{tikzcd} 0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & V_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V_1' \arrow[r] & V_2' \arrow[r] & V_3' \arrow[r] & 0 \end{tikzcd}$$

we have a commutative diagram as follows:

$$\begin{tikzcd} \mathrm{E}^n(V_3) \arrow[r] \arrow[d] & \mathrm{E}^{n+1}(V_1) \arrow[d] \\ \mathrm{E}^n(V_3') \arrow[r] & \mathrm{E}^{n+1}(V_1') \end{tikzcd}$$

We say that $(\mathrm{E}^\bullet, \delta)$ is a *cohomological δ-functor*, and the homomorphism δ is the *connecting morphism*. Furthermore, if for every injective RG-module I and integer $n > 0$ we have that $\mathrm{E}^n(I) = 0$, then $(\mathrm{E}^\bullet, \delta)$ is a *universal δ-functor*.

If $(\mathrm{E}^\bullet, \delta)$ is a universal δ-functor, given any other universal δ-functor $(\mathrm{E}_0^\bullet, \delta_0)$ with a natural isomorphism of functors $\mathrm{E}_0^0 \cong \mathrm{E}^0$, we have natural isomorphisms $\mathrm{E}_0^n \cong \mathrm{E}^n$ of functors that commute with the connecting morphisms for all $n \geq 0$, see [Bro82, Theorem III.7.5].

Theorem 1.12 ([Mac63, Theorem 10.2])**.** *For a fixed RG-module B, the functor $\mathrm{Ext}^\bullet_{RG}(B, -)$ is a universal δ-functor.*

1.2 Group cohomology

In this section, we will introduce group cohomology and some of its main properties.

Definition 1.13. Let G be a finite group and V be an RG-module. For every integer $n \geq 0$ we define the n-th *homology group* of G with coefficients in V by

$$\mathrm{H}_n(G,V) = \mathrm{Tor}_n^{RG}(R,V),$$

and the n-th *cohomology group* of G with coefficients in V by

$$\mathrm{H}^n(G,V) = \mathrm{Ext}^n_{RG}(R,V).$$

Definition 1.14. Let R be a commutative ring. We say that an R-module V is *graded* if there are R-submodules $V^n \leq V$ for all integers $n \geq 0$ such that $V = \bigoplus_{n=0}^{\infty} V^n$. The elements of V^n for some $n \geq 0$ are called *homogeneous* and, given $v \in V^n \setminus \{0\}$, we write $\deg v = n$ and say that v has *degree* n.

An R-algebra A is *graded* if it is graded as an R-module and $A^nA^m \subseteq A^{n+m}$ for all $n, m \geq 0$.

Every cohomology group $\mathrm{H}^n(G,V)$ is actually an R-module. We can thus consider the direct sum of cohomology groups

$$\mathrm{H}^\bullet(G,V) = \bigoplus_{n=0}^{\infty} \mathrm{H}^n(G,V),$$

which is a graded R-module. We will be particularly interested in the case when G is a finite p-group and $R = \mathbb{F}_p$, and we usually refer to $\mathrm{H}^\bullet(G,\mathbb{F}_p)$ as the *mod-p cohomology* of G.

Remark 1.15. In principle, the ring R on which we take the coefficients matters, but as can be seen in [Eve91, Section 1], if V is an RG-module we have isomorphisms

$$\mathrm{H}^n(G,V) = \mathrm{Ext}^n_{RG}(R,V) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},V).$$

Thus, there is no ambiguity when we write $\mathrm{H}^n(G,V)$ without specifying the base ring.

When the ring of coefficients is a field, homology is the dual of cohomology (see [Eve91, Section 2.5] for more details).

Theorem 1.16 (Universal Coefficient Theorems)**.** *Let G be a finite group, K be a field and V be a trivial KG-module. Then, there are natural isomorphisms of graded K-vector spaces*

$$\mathrm{H}^\bullet(G,V) \cong \mathrm{H}^\bullet(G,K) \otimes_K V \cong \mathrm{Hom}_K\big(\mathrm{H}_\bullet(G,K), V\big).$$

Cohomology is functorial on the group. Indeed, given a group homomorphism $\alpha\colon G_1 \longrightarrow G_2$, for any RG_2-module V, we can turn V into an RG_1-module with action given by $gv = \alpha(g)v$ for $g \in G_1$ and $v \in V$. Thus, there is an induced graded R-module homomorphism $\alpha^\bullet\colon \mathrm{H}^\bullet(G_2, V) \longrightarrow \mathrm{H}^\bullet(G_1, V)$. We will be particularly interested in the morphisms induced by subgroup inclusions and quotient projections. If $H \leq G$, the homomorphism $\iota^\bullet\colon \mathrm{H}^\bullet(G, V) \longrightarrow \mathrm{H}^\bullet(H, V)$ induced by the inclusion $\iota\colon H \longrightarrow G$ is called the *restriction* from G to H, and is denoted by $\mathrm{res}_{G\to H}$. If $N \trianglelefteq G$ and $Q = G/N$, the homomorphism $\pi^\bullet\colon \mathrm{H}^\bullet(Q, V) \longrightarrow \mathrm{H}^\bullet(G, V)$ induced by the projection $\pi\colon G \longrightarrow Q$ is called the *inflation* from Q to G, and is denoted by $\mathrm{inf}_{Q\to G}$.

Examples 1.17. (i) For finite cyclic groups, there is a particularly simple free resolution of the trivial module. Let $C_n = \langle\sigma\rangle$ denote the finite cyclic group of order $n \geq 1$. Consider $\mathcal{S}_k C_n = RC_n e_k$ for any $k \geq 0$, where e_k is the basis element of $\mathcal{S}_k C_n$ as a free RC_n-module, and $\partial_k\colon \mathcal{S}_k C_n \longrightarrow \mathcal{S}_{k-1} C_n$ given by

$$\partial_k(e_k) = \begin{cases} (\sigma - 1)e_{k-1}, & \text{if } k \text{ is odd,} \\ T(\sigma)e_{k-1}, & \text{if } k \text{ is even,} \end{cases}$$

where $T(\sigma) = 1 + \sigma + \cdots + \sigma^{n-1} \in RC_n$. It is easy to see that $\mathcal{S}_\bullet C_n \longrightarrow R$ is an RC_n-projective resolution [Eve91, Section 2.1]. We refer to $\mathcal{S}_\bullet C_n$ as the *special resolution* of C_n. Using this resolution, we can easily compute the homology and cohomology groups of C_n with coefficients in any module. For any RC_n-module V and any element $\alpha \in RC_n$, consider the α-torsion submodule

$$V[\alpha] = \{v \in V \mid \alpha v = 0\}.$$

Then, the cohomology groups of C_n with coefficients on V are

$$\mathrm{H}^r(C_n, V) = \begin{cases} V[\sigma - 1], & \text{if } r = 0, \\ \dfrac{V\big[T(\sigma)\big]}{(\sigma - 1)V}, & \text{if } r > 0 \text{ is odd,} \\ \dfrac{V[\sigma - 1]}{T(\sigma)V}, & \text{if } r > 0 \text{ is even.} \end{cases} \tag{1.2}$$

It is of particular interest to us the case when we have a cyclic group of prime power order C_{p^n} with $n \geq 1$, and $R = \mathbb{F}_p$. From (1.2), we can compute the mod-p cohomology groups

$$\mathrm{H}^r(C_{p^n}, \mathbb{F}_p) = \mathbb{F}_p$$

for all $r \geq 0$. Furthermore, in this case the special resolution $\mathcal{S}_\bullet C_{p^n} \longrightarrow \mathbb{F}_p$ is minimal and satisfies the properties described at the end of Section 1.1.2, a fact that will become useful in Section 4.

(ii) Let G be a finite group and K be a field of characteristic p, such that p does not divide $|G|$. Then, the group algebra KG is semisimple by Maschke's Theorem (see [Ben91, Corollary 3.6.12]), and so every KG-module is projective. In particular, the trivial KG-module K has a trivial projective resolution. Therefore, for any KG-module V, we have that $\mathrm{H}^n(G, V) = 0$ for all $n \geq 1$.

1.2.1 Bar resolution and low dimensional cohomology groups

The problem of computing cohomology of groups is reduced to finding projective resolutions of modules, which is not a trivial matter in general. Nevertheless, there is a certain resolution that we can always construct, namely the bar resolution that we will now describe.

Let $\mathcal{B}_n G$ be the free RG-module with basis

$$G^{(n)} = \{[g_1|\cdots|g_n] \mid g_i \in G\}$$

and let $\partial_n \colon \mathcal{B}_n G \longrightarrow \mathcal{B}_{n-1} G$ be the RG-module homomorphism defined by

$$\begin{aligned}\partial_n[g_1|\cdots|g_n] =& g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] \\ &+ (-1)^n[g_1|\cdots|g_{n-1}].\end{aligned}$$

It is possible to check that $\mathcal{B}_\bullet G \longrightarrow R$ is a free resolution of the trivial RG-module [Eve91, Section 2.3], called the *bar resolution* of G. When applying the Hom functor with an RG-module V, we obtain the cochain complex

$$C^n(G, V) = \mathrm{Hom}_{RG}(\mathcal{B}_n G, V) = \{\text{maps } f \colon G^{(n)} \longrightarrow V\}$$

with differential $\partial^n \colon C^n(G, V) \longrightarrow C^{n+1}(G, V)$ given by

$$\begin{aligned}\partial^n f[g_1|\cdots|g_{n+1}] =& g_1 f[g_2|\cdots|g_{n+1}] + \sum_{i=1}^{n}(-1)^i f[g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_{n+1}] \\ &+ (-1)^{n+1}[g_1|\cdots|g_n].\end{aligned}$$

The elements of $C^n(G, V)$ are called *n-cochains* of G with coefficients in V. Also denote by $Z^n(G, V) = \mathrm{Ker}\, \mathrm{d}^n$ and $B^n(G, V) = \mathrm{Im}\, \mathrm{d}^{n-1}$ for $n \geq 1$ and $B^0(G, M) = 0$.

The elements of $Z^n(G,V)$ and $B^n(G,V)$ are called *n-cocycles* and *n-coboundaries* of G with coefficients in V, respectively. Using the bar resolution to compute the cohomology of G with coefficients on V, we obtain that

$$\mathrm{H}^n(G,V) = \frac{Z^n(G,V)}{B^n(G,V)}$$

for all $n \geq 0$.

Sometimes, it is also useful to consider the *normalized bar resolution* of G, the chain complex denoted by $\bar{\mathcal{B}}_\bullet G$ with $\bar{\mathcal{B}}_n G$ the free RG-module with basis

$$\{[g_1|\cdots|g_n] \mid g_i \neq 1 \text{ for all } 1 \leq i \leq n\}$$

and the same differential as before. We have that $\bar{\mathcal{B}}_\bullet G \longrightarrow R$ is also an RG-free resolution. Also denote by $\bar{C}^n(G,V)$, $\bar{Z}^n(G,V)$ and $\bar{B}^n(G,V)$ the normalized n-cochains, n-cocycles and n-coboundaries, respectively, defined analogously to the non-normalized case.

Using the bar resolution, we can easily identify the low degree cohomology groups. First, we have that

$$\mathrm{H}^0(G,V) = V^G = \{v \in V \mid gv = v \text{ for all } g \in G\},$$

the submodule of invariant elements. If the module of coefficients V is trivial, we have that

$$\mathrm{H}^1(G,V) = \mathrm{Hom}(G,V).$$

When G is a p-group and $V = \mathbb{F}_p$, a minimal generating set $\{g_1,\ldots,g_k\} \subseteq G$ projects to an $\mathbb{F}_p$-basis of $G/G^p[G,G]$, and we can write

$$\mathrm{H}^1(G,\mathbb{F}_p) = \mathrm{Hom}\left(\frac{G}{G^p[G,G]},\mathbb{F}_p\right) = \langle g_1^*,\ldots,g_k^*\rangle, \tag{1.3}$$

where $g_i^*\colon G \longrightarrow \mathbb{F}_p$ is defined by $g_i^*(g_i) = 1$ and $g_i^*(g_j) = 0$ for $j \neq i$.

The description of the second cohomology group is more involved. We will consider group extensions of the form

$$0 \longrightarrow V \longrightarrow E \longrightarrow G \longrightarrow 1$$

with conjugation of V inside E induced by the action of G. We will say that two extensions E_1 and E_2 are *equivalent*, and write $E_1 \equiv E_2$, if there exists a group homomorphism $f\colon E_1 \longrightarrow E_2$ making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "f"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & V \arrow[r] & E_2 \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Note that, in this case, the homomorphism f must be an isomorphism. Nevertheless, it is possible to have isomorphic extensions that are not equivalent. Define the set

$$\mathrm{XExt}^1_R(G,V) = \{0 \longrightarrow V \longrightarrow E \longrightarrow G \longrightarrow 1\}/\equiv$$

of equivalence classes of extensions of G by V. Then, it can be shown that there is a one-to-one correspondence

$$\mathrm{H}^2(G,M) \cong \mathrm{XExt}^1_R(G,V).$$

The correspondence between $\mathrm{H}^2(G,V)$ and $\mathrm{XExt}^1_R(G,V)$ can be described using the normalized bar resolution. We will explain how to obtain an extension from a 2-cocycle, for a description of the inverse process see [Bro82, Section IV.3]. Given a normalized 2-cocycle $\varphi \in \bar{C}^2(G,V)$, we can construct a group E_φ with underlying set $G \times V$ and multiplication given by

$$(g,v)(h,w) = \big(gh, h^{-1}v + w + \varphi(g,h)\big)$$

for $g,h \in G$ and $v,w \in M$. The group E_φ fits into the extension

$$0 \longrightarrow V \longrightarrow E_\varphi \longrightarrow G \longrightarrow 1,$$

and cocycles in the same cohomology class give rise to isomorphic extensions. Note that in his book, Brown uses conjugation on the left, whereas we write conjugation on the right, explaining why we write the factors of $G \times V$ in that order, and why we need to take the inverse when defining the product in E_φ. It is possible to describe all higher degree cohomology groups in terms of group extensions, as we will do in Chapter 2.

1.2.2 Cup product

The key feature of group cohomology distinguishing it from homology is the fact that we can define a product of cohomology classes. Equipped with this product, cohomology acquires the structure of a ring.

Assume that A is an RG-module that is also an R-algebra, with the product in A compatible with the action of G. Then, we have a product

$$\mathrm{H}^n(G,A) \otimes_R \mathrm{H}^m(G,A) \longrightarrow \mathrm{H}^{n+m}(G,A)$$

called *cup product*, which for cohomology classes $\varphi \in \mathrm{H}^n(G,A)$ and $\psi \in \mathrm{H}^m(G,A)$ is denoted by $\varphi \smile \psi \in \mathrm{H}^{n+m}(G,A)$, see [Eve91, Chapter 3] for the details. The cup product turns $\mathrm{H}^\bullet(G,A)$ into a graded-commutative R-algebra, i.e. given homogeneous elements $\varphi, \psi \in \mathrm{H}^\bullet(G,A)$, we have that $\varphi \smile \psi = (-1)^{\deg\varphi \deg\psi}\psi \smile \varphi$. Furthermore, induced morphisms in cohomology are compatible with the cup product. When R is a noetherian ring, $\mathrm{H}^\bullet(G,R)$ is a finitely generated R-algebra (see [Eve91, Corollary 7.4.6]).

Remark 1.18. Graded-commutative rings of characteristic 2 are commutative in the classical sense. Otherwise, when the characteristic is not 2, every homogeneous element of odd degree in a graded-commutative ring is nilpotent.

Examples 1.19. (i) When using the bar resolution, the cup product can be easily computed. Let $\varphi \in Z^r(G,R)$ and $\psi \in Z^s(G,R)$ represent two cohomology classes in $\mathrm{H}^\bullet(G,R)$. Their cup product is given by

$$(\varphi \smile \psi)[g_1|\cdots|g_{r+s}] = \varphi[g_1|\cdots|g_r]\psi[g_{r+1}|\cdots|g_{r+s}].$$

(ii) Let $C_n = \langle\sigma\rangle$ be the finite cyclic group of order $n \geq 1$ and A be an RC_n-module with a compatible product over R. After having computed the cohomology groups of C_n in Example 1.17, following [Eve91, Section 3.2] we can compute its cohomology ring. The cup product of cohomology classes $\bar\varphi \in \mathrm{H}^r(G,A)$ and $\bar\psi \in \mathrm{H}^s(G,A)$, represented by elements $\varphi, \psi \in A$, is given by

$$\bar\varphi \smile \bar\psi = \begin{cases} \overline{\varphi\psi}, & \text{if } r \text{ or } s \text{ is even,} \\ \displaystyle\sum_{0\leq i\leq j<n} \overline{(\sigma^i\varphi)(\sigma^j\psi)}, & \text{if } r, s \text{ are odd.} \end{cases}$$

Assume that we have a cyclic group of prime power order C_{p^n} with $n \geq 1$, and take $R = \mathbb{F}_p$. From the previous discussion, we can compute the mod-p cohomology algebra

$$\mathrm{H}^\bullet(C_{p^n}, \mathbb{F}_p) = \begin{cases} \mathbb{F}_2[x], & \text{if } p = 2 \text{ and } n = 1, \\ \Lambda(x) \otimes_{\mathbb{F}_p} \mathbb{F}_p[y], & \text{if } p > 2 \text{ or } n > 1, \end{cases}$$

where $\deg x = 1$ and $\deg y = 2$. Here, $\Lambda(x)$ denotes the exterior $\mathbb{F}_p$-algebra generated by x.

In the case when the base ring is a field, it is possible to compute the cohomology of direct products in terms of the cohomologies of the factors. See [Eve91, Section 2.5] for a more general version of the following result.

Theorem 1.20 (Künneth Formula)**.** *Let G_1 and G_2 be finite groups and K be a field. There is a natural isomorphism of graded K-algebras*

$$\mathrm{H}^\bullet(G_1 \times G_2, K) \cong \mathrm{H}^\bullet(G_1, K) \otimes_K \mathrm{H}^\bullet(G_2, K).$$

We will now give some examples of mod-p cohomology rings of finite p-groups.

Examples 1.21. (i) Let $G = C_{p^{n_1}} \times \cdots \times C_{p^{n_r}}$ be an abelian p-group of rank r, and let s be the rank of G^2. Using the description of the mod-p cohomology ring of cyclic p-groups in Example 1.19(ii) and Theorem 1.20, we can see that the mod-p cohomology ring of G is given by

$$\mathrm{H}^\bullet(G, \mathbb{F}_p) = \begin{cases} \Lambda(x_1, \ldots, x_s) \otimes \mathbb{F}_2[y_1, \ldots, y_s, x_{s+1}, \ldots, x_r], & \text{if } p = 2, \\ \Lambda(x_1, \ldots, x_r) \otimes \mathbb{F}_p[y_1, \ldots, y_r], & \text{if } p > 2, \end{cases}$$

with $\deg x_i = 1$ and $\deg y_i = 2$ for $i = 1, \ldots, r$.

(ii) In [Wei00], Weigel gave a characterization, for p odd, of the finite p-groups that have the same mod-p cohomology algebra as a finite abelian p-group. Those are precisely the powerful p-central p-groups with the Ω-extension property.

(iii) Let D_{2^n} be the dihedral group of order 2^n with $n \geq 3$. Then, its mod-2 cohomology ring is given by

$$\mathrm{H}^\bullet(D_{2^n}, \mathbb{F}_2) = \frac{\mathbb{F}_2[x, y, z]}{(xy)}$$

with $\deg x = \deg y = 1$ and $\deg z = 2$, see [AM03, Theorem IV.2.7].

(iv) Let Q_{2^n} be the generalized quaternion group of order 2^n with $n \geq 4$. Then, its mod-2 cohomology ring is given by

$$\mathrm{H}^\bullet(Q_{2^n}, \mathbb{F}_2) = \frac{\mathbb{F}_2[x, y, z]}{(xy, x^3 + y^3)}$$

with $\deg x = \deg y = 1$ and $\deg z = 4$, see [AM03, Theorem IV.2.11].

(v) Let $G = C_3 \ltimes (C_3 \times C_3)$ be the extraspecial 3-group of order 27 and exponent 3. The explicit mod-3 cohomology ring of G was computed by Leary in [Lea92, Theorem 7], where he gave a presentation of $\mathrm{H}^\bullet(G, \mathbb{F}_3)$ as a graded-commutative $\mathbb{F}_3$-algebra consisting of 9 generators subject to 23 relations.

1.2.3 Bockstein homomorphisms

Group cohomology is defined as an Ext functor and, as such, it possesses the ability to turn short exact sequences of modules into long exact sequences of cohomology groups. As a consequence of Theorem 1.12, given a finite group G and a short exact sequence of RG-modules

$$0 \longrightarrow V_1 \xrightarrow{\iota} V_2 \xrightarrow{\pi} V_3 \longrightarrow 0,$$

we obtain a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(G,V_1) \arrow[r, "\iota_*"] & \mathrm{H}^0(G,V_2) \arrow[r, "\pi_*"] & \mathrm{H}^0(G,V_3) \arrow[dll, dashed] \\ & \mathrm{H}^n(G,V_1) \arrow[r, "\iota_*"] & \mathrm{H}^n(G,V_2) \arrow[r, "\pi_*"] & \mathrm{H}^n(G,V_3) \arrow[dll, "\delta"] \\ & \mathrm{H}^{n+1}(G,V_1) \arrow[r, "\iota_*"] & \mathrm{H}^{n+1}(G,V_2) \arrow[r, "\pi_*"] & \mathrm{H}^{n+1}(G,V_3) \arrow[r] & \cdots \end{tikzcd}$$

where for each $n \geq 0$, the connecting homomorphism $\delta\colon \mathrm{H}^n(G,V_3) \longrightarrow \mathrm{H}^n(G,V_1)$ is obtained as follows (see [Wei94, Section 1.3]). Let $P_\bullet \longrightarrow R$ be a projective resolution of the trivial RG-module. Given a cohomology class in $\varphi \in \mathrm{H}^n(G,V_3)$ represented by a homomorphism $\varphi\colon P_n \longrightarrow V_3$, we can lift it to a homomorphism $\hat{\varphi}\colon P_n \longrightarrow V_2$ such that $\pi \circ \hat{\varphi} = \varphi$, although $\hat{\varphi}$ need not represent a cohomology class in $\mathrm{H}^n(G,V_2)$. Now, we can define

$$\delta(\varphi) = \iota^{-1} \circ \mathrm{d}(\hat{\varphi}). \tag{1.4}$$

A family of important examples of connecting homomorphisms are the Bockstein homomorphisms. Given $n \geq 1$, consider the short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\cdot p^n} \mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0.$$

This induces a long exact sequence in cohomology with connecting homomorphism $\beta_n\colon \mathrm{H}^r(G,\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathrm{H}^{r+1}(G,\mathbb{F}_p)$ called the n-th order *Bockstein homomorphism*. If $n=1$, the first Bockstein is simply refered to as the *Bockstein homomorphism* and denoted by β. Observe that $\beta\colon \mathrm{H}^r(G,\mathbb{F}_p) \longrightarrow \mathrm{H}^{r+1}(G,\mathbb{F}_p)$ is defined on the elements of $\mathrm{H}^\bullet(G,\mathbb{F}_p)$, and it is one of the Steenrod operations (see [CTVZ03, Chapter 7]). If $n \geq 2$, from the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{F}_p \arrow[r, "\cdot p^{n-1}"] \arrow[d, equal] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d, "\cdot p"] & \mathbb{Z}/p^{n-1}\mathbb{Z} \arrow[r] \arrow[d, "\cdot p"] & 0 \\ 0 \arrow[r] & \mathbb{F}_p \arrow[r, "\cdot p^n"] & \mathbb{Z}/p^{n+1}\mathbb{Z} \arrow[r] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

we obtain that

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^r(G,\mathbb{Z}/p^n\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^r(G,\mathbb{Z}/p^{n-1}\mathbb{Z}) \arrow[r, "\beta_{n-1}"] \arrow[d] & \mathrm{H}^{r+1}(G,\mathbb{F}_p) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^r(G,\mathbb{Z}/p^{n+1}\mathbb{Z}) \arrow[r] & \mathrm{H}^r(G,\mathbb{Z}/p^n\mathbb{Z}) \arrow[r, "\beta_n"] & \mathrm{H}^{r+1}(G,\mathbb{F}_p) \arrow[r] & \cdots
\end{tikzcd}$$

and an element $\varphi \in \operatorname{Ker}\beta_{n-1}$ can be lifted to an element $\tilde{\varphi} \in \mathrm{H}^\bullet(G,\mathbb{Z}/p^n\mathbb{Z})$ such that $\pi_*(\hat{\varphi}) = \varphi$. In this case, we can define $\beta_n(\varphi) = \beta_n(\hat{\varphi})$. Therefore, given an element $\varphi \in \mathrm{H}^r(G,\mathbb{F}_p)$, we can define inductively $\beta_n(\varphi) \in \mathrm{H}^{r+1}(G,\mathbb{F}_p)$ whenever $\beta_i(\varphi)$ is well defined for all $i = 1,\ldots,n-1$ and $\varphi \in \operatorname{Ker}\beta_{n-1}$, in which case we can lift φ to some element $\tilde{\varphi} \in \mathrm{H}^r(G,\mathbb{Z}/p^n\mathbb{Z})$ such that $\pi_*(\tilde{\varphi}) = \varphi$, and write $\beta_r(\varphi) = \beta_r(\tilde{\varphi})$.

The Bockstein homomorphisms allow us to give a more explicit description of the mod-p cohomology algebra of a cyclic p-group. Indeed, write $C_{p^n} = \langle \sigma \rangle$ for $n \geq 1$. We have that $\mathrm{H}^1(C_{p^n},\mathbb{F}_p) = \langle x \rangle$, where $x = \sigma^*$. It is possible to lift x to some element $\tilde{x} \in \mathrm{H}^1(C_{p^n},\mathbb{Z}/p^i\mathbb{Z})$ such that $\pi \circ \tilde{x} = x$ if and only if $i \leq n$. Therefore, $\beta_i(x) = 0$ for all $i = 1,\ldots,n-1$ and $\beta_n(x) \neq 0$. Using the bar resolution, $\beta_n(x)$ can be represented explicitly by the 2-cocycle given by

$$\beta_n(x)(\sigma^i,\sigma^j) = p^{-n}\big(i + j - (i + j \mod p^n)\big).$$

Here, the exponents $i,j \in \{0,\ldots,p^n-1\}$ should be considered integers when doing the computations. Thus, we have that $\mathrm{H}^2(C_{p^n},\mathbb{F}_p) = \langle \beta_n(x) \rangle$ and, for $p \geq 3$ and $n \geq 1$, or $p = 2$ and $n \geq 2$, we can write

$$\mathrm{H}^\bullet(C_{p^n},\mathbb{F}_p) = \Lambda(x) \otimes_{\mathbb{F}_p} \mathbb{F}_p\big[\beta_n(x)\big].$$

For $p = 2$ and $n = 1$, the first Bockstein is given by

$$\beta_1(x) = x^2.$$

1.2.4 Extensions of elementary abelian p-groups

We will now classify the extensions of an elementary abelian p-group with kernel $\mathbb{F}_p$ using the knowledge we have obtained from its mod-p cohomology algebra. For simplicity, we will restrict our attention to elementary abelian p-groups of rank two.

Let p be an odd prime and

$$G = C_p \times C_p = \langle a, b \mid a^p = b^p = [a,b] = 1 \rangle$$

be an elementary abelian p-group of rank two. Then, if we take $a^*, b^* \in \mathrm{Hom}(G, \mathbb{F}_p)$ as in (1.3), we have that

$$\mathrm{H}^2(G, \mathbb{F}_p) = \langle \beta(a^*), \beta(b^*), a^* \smile b^* \rangle.$$

We can classify the different types of extensions in $\mathrm{XExt}^1_{\mathbb{F}_p}(G, \mathbb{F}_p)$ up to isomorphism. We will use multiplicative notation for the trivial $\mathbb{F}_pG$-module $V = C_p = \langle c \rangle$. Consider a cohomology class

$$\varphi = \lambda\beta(a^*) + \mu\beta(b^*) + \nu a^* \smile b^* \in \mathrm{H}^2(G, \mathbb{F}_p)$$

with $\lambda, \mu, \nu \in \mathbb{F}_p$. As we discussed in Section 1.2.1, the extension E_φ has underlying set $G \times V$ and, using multiplicative notation, the group operation is given by

$$(a^{i_1}b^{j_1}, c^{k_1})(a^{i_2}b^{j_2}, c^{k_2}) = (a^{i_1+i_2}b^{j_1+j_2}, c^{k_1+k_2+\varphi(a^{i_1}b^{j_1}, a^{i_2}b^{j_2})})$$

for $0 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq p-1$. Consider the elements

$$\tilde{a} = (a, 1), \quad \tilde{b} = (b, 1), \quad \tilde{c} = (1, c).$$

Firstly, it is clear that

$$\tilde{c}^p = [\tilde{a}, \tilde{c}] = [\tilde{b}, \tilde{c}] = 1.$$

Furthermore, for $0 \leq i \leq p-1$ we can compute

$$\varphi(a^i, a) = \lambda\beta(a^*)(a^i, a) = \begin{cases} 0, & \text{if } 0 \leq i \leq p-2, \\ \lambda, & \text{if } i = p-1, \end{cases}$$

which allows us to show inductively that

$$\tilde{a}^i = (a^i, 1), \quad \tilde{a}^p = (1, c^\lambda) = \tilde{c}^\lambda.$$

We can analogously show that $\tilde{b}^p = \tilde{c}^\mu$. Finally, we have that

$$\tilde{a}\tilde{b} = (a, 1)(b, 1) = (ab, c^\nu) = (b, 1)(a, 1)(1, c) = \tilde{b}\tilde{a}\tilde{c}^\nu,$$

and hence $[\tilde{a}, \tilde{b}] = \tilde{c}^\nu$. Consequently, we can write

$$E_\varphi = \langle \tilde{a}, \tilde{b}, \tilde{c} \mid \tilde{c}^p = [\tilde{a}, \tilde{c}] = [\tilde{b}, \tilde{c}] = 1,\ \tilde{a}^p = \tilde{c}^\lambda,\ \tilde{b}^p = \tilde{c}^\mu,\ [\tilde{a}, \tilde{b}] = \tilde{c}^\nu \rangle. \tag{1.5}$$

There are four possible isomorphism types for the group E_φ depending on the values of the parameters $\lambda, \mu, \nu \in \mathbb{F}_p$. This can be seen by comparing the presentation (1.5) to the four different isomorphism types of p-groups of order p^3 and exponent at most p^2, see [Bur11, Section VIII.112].

(i) If $\lambda = \mu = \nu = 0$, then

$$E_\varphi \cong C_p \times C_p \times C_p = \langle x, y, z \mid x^p = y^p = z^p = [x, y] = [x, z] = [y, z] = 1\rangle.$$

(ii) If $\lambda \neq 0$ or $\mu \neq 0$, and $\nu = 0$, then

$$E_\varphi \cong C_{p^2} \times C_p = \langle x, y \mid x^{p^2} = y^2 = [x, y] = 1\rangle.$$

(iii) If $\lambda = \mu = 0$ and $\nu \neq 0$, then

$$E_\varphi \cong C_p \ltimes (C_p \times C_p) = \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = 1,\ [x, y] = z\rangle.$$

(iv) If $\lambda \neq 0$ or $\mu \neq 0$, and $\nu \neq 0$, then

$$E_\varphi \cong C_p \ltimes C_{p^2} = \langle x, y \mid x^p = y^{p^2} = 1,\ [x, y] = y^p\rangle.$$

1.3 Introduction to spectral sequences

We will end the chapter by briefly introducing spectral sequences, which constitute extremely powerful computational tools from homological algebra with applications in a multitude of areas throughout mathematics. As such, their definition and initial properties can be rather generic and abstract. For this reason, after introducing the most important notation and terminology, we will quickly move on to discussing the Lyndon-Hochschild-Serre spectral sequence. This specific spectral sequence is one of the main tools for computing group cohomology, and will take center stage during Chapters 4 and 5. For a more detailed discussion of spectral sequences, see [McC01] and [Eve91, Chapter 7].

Definition 1.22. We say that an R-module V is *bigraded* if there are R-submodules $V^{r,s} \leq V$ for all integers $r, s \geq 0$ such that $V = \bigoplus_{r,s\geq 0} V^{r,s}$. The elements of $V^{r,s}$ for some $r, s \geq 0$ are called *homogeneous* and, given $v \in V^{r,s} \setminus \{0\}$, we write $\deg v = r + s$ and say that v has *(total) degree* $r + s$.

An R-algebra A is *bigraded* if it is bigraded as an R-module and we have that $A^{r,s}A^{r',s'} \subseteq A^{r+r',s+s'}$ for all $r, r', s, s' \geq 0$.

Definition 1.23. A *(first quadrant) (cohomological) spectral sequence* is a family of bigraded R-modules $\{(E_k, \mathrm{d}_k) \mid k \geq 0\}$ where $\mathrm{d}_k \colon E_k^{r,s} \longrightarrow E_k^{r+k,s-k+1}$ satisfies that $\mathrm{d}_k \circ \mathrm{d}_k = 0$, and such that $E_{k+1} = \mathrm{H}^\bullet(E_k, \mathrm{d}_k)$ for all $k \geq 0$. For each $k \geq 0$, we say that E_k is the k-th *page* of the spectral sequence and d_k is the k-th *differential*.

We will usually omit the differentials, and simply write E instead of $\{(E_k, \mathrm{d}_k)\}$ for the spectral sequence. Our spectral sequence E is concentrated in the first quadrant, i.e. $E_k^{r,s} \neq 0$ only if $r, s \geq 0$. Thus, for each $r, s \geq 0$ there is some $k_{r,s} \geq 0$ such that $E_k^{r,s} = E_{k_{r,s}}^{r,s}$ for all $k \geq k_{r,s}$. We will write $E_\infty^{r,s} = E_{k_{r,s}}^{r,s}$ and say that E_∞ is the *infinity page* of E. Furthermore, we will say that E *collapses* if there is some $k_0 \geq 0$ such that $\mathrm{d}_k = 0$ for all $k \geq 0$, in which case $E_{k_0} = E_\infty$.

Definition 1.24. Let $H^\bullet$ be a graded R-module. We say that the spectral sequence E *converges* to $H^\bullet$ if there is a filtration of graded R-submodules $FH^\bullet$ of $H^\bullet$ with

$$0 = F^{n+1}H^n \subseteq F^n H^n \subseteq \cdots \subseteq F^0 H^n = H^n$$

for all $n \geq 0$, such that

$$E_\infty^{r,s} = F^r H^{r+s} / F^{r+1} H^{r+s} \tag{1.6}$$

for all $r, s \geq 0$. We write

$$E_2^{r,s} \Longrightarrow H^{r+s}.$$

In general terms, the fact that the spectral sequence E converges to $H^\bullet$ means that the infinity page E_∞ is the graded object associated to some filtration of $H^\bullet$. The philosophy of working with spectral sequences is that we want to compute a certain object which is hard to compute, and we find a spectral sequence whose second page is easy to compute, and that converges to the object in which we are interested. This is nonetheless notoriously difficult, for two main reasons. Firstly, we usually have no information on the differentials, nor how to compute them, making it a considerable challenge to obtain the infinity page. Even after we have solved this problem, there is still a lifting problem (going from the graded object to the filtered one) stemming from (1.6). In principle, we might have multiple non-isomorphic filtered modules whose associated graded object is the same. Nonetheless, if we are working over a field K, this problem disappears, and we can completely recover the graded K-vector space $H^\bullet$ from the infinity page with

$$H^n \cong \bigoplus_{r=0}^{r} E_\infty^{r,n-r}$$

for every $n \geq 0$.

1.3.1 Lyndon-Hochschild-Serre spectral sequence

We are interested in computing group cohomology and, therefore, we will focus on the Lyndon-Hochschild-Serre spectral sequence, which helps us compute the cohomology of a group that can be obtained as an extension. See [Eve91, Section 7.2] for a detailed exposition. Let

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1 \tag{1.7}$$

be an extension of finite groups.

Theorem 1.25. *There is a spectral sequence associated to the extension* (1.7) *of the form*

$$E_2^{r,s} = E_2^{r,s}(G) = \mathrm{H}^r\left(Q, \mathrm{H}^s(N,V)\right) \Longrightarrow \mathrm{H}^{r+s}(G,V).$$

The spectral sequence in Theorem 1.25 is called the *Lyndon-Hochschild-Serre spectral sequence* (LHS spectral sequence for short) of the extension (1.8). Even though we tend to start at the second page, the zeroth and first pages of E can be identified as follows. Let $Y_\bullet \longrightarrow R$ be a projective RQ-resolution and $P_\bullet \longrightarrow R$ be a projective RG-resolution. Then, we have that

$$\begin{aligned} E_0^{r,s} &= \mathrm{Hom}_{RQ}\big(Y_r, \mathrm{Hom}_{RN}(P_s,V)\big), \\ E_1^{r,s} &= \mathrm{Hom}_{RQ}\big(Y_r, \mathrm{H}^s(N,V)\big). \end{aligned} \tag{1.8}$$

Assume that V is a noetherian R-module. Then, E_k is a noetherian R-module for all $k \geq 2$ [Eve91, Lemma 7.4.3]. Furthermore, the spectral sequence E always collapses [Eve91, Lemma 7.4.4].

When taking coefficients on the trivial module, the LHS spectral sequence not only preserves the structure as an R-module of $\mathrm{H}^\bullet(G,R)$, but also the cup product. Indeed, for every $k \geq 2$, the page E_k is a bigraded R-algebra, the differential d_k satisfies that

$$d_k(xy) = \mathrm{d}_k(x)y + (-1)^{\deg x} x\, \mathrm{d}_k(y)$$

for all homogeneous $x, y \in E_k$, and the product on E_{k+1} is induced by the one on E_k. This in turn induces a product on E_∞ that coincides with the one coming from the filtration on the ring $\mathrm{H}^\bullet(G,R)$, which also preserves the product. Moreover, the product $E_2^{r,s} \otimes E_2^{r',s'} \longrightarrow E_2^{r+r',s+s'}$ on the second page is given by $\varphi\varphi' = (-1)^{r's}\varphi \smile \varphi'$, where $\smile$ denotes the ordinary cup product in cohomology.

As mentioned earlier, if K is a field, we can recover the graded K-vector space $\mathrm{H}^\bullet(G,K)$ from just E_∞. It is in general not possible to compute the cup product on $\mathrm{H}^\bullet(G,K)$ just from the spectral sequence, without additional information. Nevertheless if K is a finite field, then the ring structure of $\mathrm{H}^\bullet(G,K)$ is determined by that of E_∞ within a finite number of possibilities [Car05, Theorem 2.1].

We will focus on the problem of computing the differentials in the LHS spectral sequence in Chapters 4 and 5, where we will see some results that give us explicit ways to compute the second differential of some specific types of group extensions. Nonetheless, we will finish by mentioning some general properties of the LHS spectral sequence that can be exploited to compute the differentials of some specific elements. The differential $\mathrm{d}_2^{0,1}\colon E_2^{0,1} \longrightarrow E_2^{2,0}$ can be computed explicitly from the class of the extension (1.7), see [Eve91, Lemma 7.3.1] for details. In particular, if the extension is split then $\mathrm{d}_2^{0,1} = 0$. Another property that will be useful when computing differentials is the fact that the LHS spectral sequence is functorial, in the following sense: given a morphism of group extensions of the form

$$\begin{tikzcd}
1 \arrow[r] & N_1 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N_2 \arrow[r] & G_2 \arrow[r] & Q_2 \arrow[r] & 1
\end{tikzcd}$$

and an RG_2-module V, for every $k \geq 2$ there are induced homomorphisms of bigraded R-algebras $E_k(G_2) \longrightarrow E_k(G_1)$ that commute with the differentials. These homomorphisms coincide with the corresponding induced morphisms on cohomology.

2

Cohomology via extensions

In this chapter, we will describe the cohomology of a finite group in terms of extensions. First, we will give the classical description of Ext using Yoneda extensions. Afterwards, we will introduce crossed extensions in order to describe cohomology groups. We will then combine both concepts by defining a product of Yoneda extensions with crossed extensions that coincides with the usual cup product in cohomology. For a more detailed account of these topics, we refer to [Mac63, Chapter III] and [Niw92] for information about Yoneda extensions; and [Hol79], [Hue80] and [Niw92] for a discussion of crossed extensions.

2.1 Yoneda extensions

Let G be a finite group and R be a commutative ring with unity. In this section, we will describe the classical characterization of the Ext functor using extensions of modules, originaly due to Yoneda [Yon92].

Definition 2.1. Let V and W be RG-modules. For every integer $n \geq 1$, a *Yoneda n-fold extension φ of W by V* is an exact sequence of RG-modules of the form

$$\varphi\colon\ 0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow W \longrightarrow 0.$$

We can define an equivalence relation between Yoneda extensions as follows. Let V and W be RG-modules and $n \geq 1$ be an integer. Given two n-fold Yoneda extensions

φ and φ' of W by V, we write $\varphi \implies \varphi'$ if there is morphism of Yoneda n-fold extensions $\varphi \longrightarrow \varphi'$ of the following form:

$$\begin{tikzcd}
\varphi\colon\ 0 \arrow[r] & V \arrow[r] \arrow[d, equal] & M_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
\varphi'\colon\ 0 \arrow[r] & V \arrow[r] & M'_n \arrow[r] & \cdots \arrow[r] & M'_1 \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

More generally, we say that φ is *equivalent* to φ', and write $\varphi \equiv \varphi'$, if there is a chain of Yoneda n-fold extensions of W by V and morphisms of the form

$$\varphi \implies \varphi_1 \impliedby \varphi_2 \implies \cdots \impliedby \varphi_{r-1} \implies \varphi_r \impliedby \varphi'.$$

It is not difficult to see that this does indeed define an equivalence relation on the set of Yoneda n-fold extensions of W by V, and the set of all Yoneda n-fold extensions of W by V up to equivalence will be denoted by $\mathrm{YExt}^n_{RG}(W, V)$.

In fact, the equivalence between two extensions can be described in a simpler way.

Proposition 2.2. *Let V and W be RG-modules and $n \geq 1$ be an integer. Given two Yoneda n-fold extensions φ and φ' of V by W, we have that $\varphi \equiv \varphi'$ if and only if there is a Yoneda n-fold extension φ_1 of V by W such that $\varphi \impliedby \varphi_1 \implies \varphi'$.*

Proof. We only need to show that given Yoneda n-fold extensions $\varphi, \varphi', \varphi'', \varphi_1, \varphi_2$ of W by V such that $\varphi \impliedby \varphi_1 \implies \varphi'$ and $\varphi' \impliedby \varphi_2 \implies \varphi''$, there is another Yoneda n-fold extension φ_3 of W by V such that $\varphi \impliedby \varphi_3 \implies \varphi''$. Suppose then that we have $\varphi \impliedby \varphi_1 \implies \varphi'$ and $\varphi' \impliedby \varphi_2 \implies \varphi''$, which is summarised in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\rho_n"] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r, "\rho_0"] & W \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, "\chi_n"] \arrow[u, equal] \arrow[d, equal] & X_n \arrow[r] \arrow[u, "\lambda_n"'] \arrow[d, "\mu_n"] & \cdots \arrow[r] & X_1 \arrow[r, "\chi_0"] \arrow[u, "\lambda_1"'] \arrow[d, "\mu_1"] & W \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r, "\tau_n"] & N_n \arrow[r] & \cdots \arrow[r] & N_1 \arrow[r, "\tau_0"] & W \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, "\sigma_n"] \arrow[u, equal] \arrow[d, equal] & Y_n \arrow[r] \arrow[u, "\eta_n"'] \arrow[d, "\nu_n"] & \cdots \arrow[r] & Y_1 \arrow[r, "\sigma_0"] \arrow[u, "\eta_1"'] \arrow[d, "\nu_1"] & W \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r, "\kappa_n"] & L_n \arrow[r] & \cdots \arrow[r] & L_1 \arrow[r, "\kappa_0"] & W \arrow[r] & 0
\end{tikzcd}$$

Consider, for each $i = 1, \dots, n$, the RG-module

$$Z_i = \big\{(x_i, y_i) \in X_i \oplus Y_i \mid \mu_i(x_i) = \eta_i(y_i)\big\}.$$

Then, can write the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\rho_n"] \arrow[d, equal] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r, "\rho_0"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r, "\chi_n \times \sigma_n"] \arrow[d, equal] & Z_n \arrow[r] \arrow[u, "\lambda_n"'] \arrow[d, "\nu_n"] & \cdots \arrow[r] & Z_1 \arrow[r, "\chi_0"] \arrow[u, "\lambda_1"'] \arrow[d, "\nu_1"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r, "\kappa_n"] & L_n \arrow[r] & \cdots \arrow[r] & L_1 \arrow[r, "\kappa_0"] & W \arrow[r] & 0
\end{tikzcd}$$

We have thus constructed a Yoneda n-fold extension φ_3 of W by V with morphisms $\varphi \Longleftarrow \varphi_3 \Longrightarrow \varphi''$. $\square$

Remark 2.3. It is well known that $\mathrm{YExt}^n_{RG}(W, V) \cong \mathrm{Ext}^n_{RG}(W, V)$, but we will use the notation YExt to emphasize that we are considering the description of Ext in terms of Yoneda extensions.

We can see that $\mathrm{YExt}_{RG}(-,-)$ is functorial on both components. Let V, V', W and W' be RG-modules and $n \geq 1$ be an integer. Consider the n-fold Yoneda extension $\varphi \in \mathrm{YExt}^n_{RG}(W, V)$ represented by

$$0 \longrightarrow V \xrightarrow{\rho_n} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} W \longrightarrow 0.$$

On the one hand, given an RG-module homomorphism $\alpha \colon V \longrightarrow V'$, we say that $\varphi' \in \mathrm{YExt}^n_{RG}(W, V')$ is a *pushout* of φ via α if there is a morphism $\varphi \longrightarrow \varphi'$ of the following form:

$$\begin{tikzcd}
\varphi\colon & 0 \arrow[r] & V \arrow[r] \arrow[d, "\alpha"] & M_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
\varphi'\colon & 0 \arrow[r] & V' \arrow[r] & M'_n \arrow[r] & \cdots \arrow[r] & M'_1 \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

We can always construct the pushout $\alpha_*\varphi \in \mathrm{YExt}^n_{RG}(W, V')$ represented by the extension

$$0 \longrightarrow V' \longrightarrow N \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow W \longrightarrow 0,$$

where

$$N = \frac{M_n \times V'}{\big\{\big(\rho_n(v), -\alpha(v)\big) \mid v \in V\big\}}.$$

On the other hand, given an RG-module homomorphism $\zeta\colon W' \longrightarrow W$, we say that $\varphi' \in \mathrm{YExt}^n_{RG}(W', V)$ is a *pullback* of φ via ζ if there is a morphism $\varphi \longrightarrow \varphi'$ of the following form:

$$\begin{tikzcd}
\varphi\colon\ 0 \arrow[r] & V \arrow[r] \arrow[d, equal] & M_n' \arrow[r] \arrow[d] & \cdots \arrow[r] & M_1' \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d, "\zeta"] & 0 \\
\varphi'\colon\ 0 \arrow[r] & V \arrow[r] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

We can always take the pullback $\zeta^*\varphi \in \mathrm{YExt}^n_{RG}(W', V)$ represented by the extension

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow N \longrightarrow W' \longrightarrow 0,$$

where

$$N = \big\{(x, w') \in M_1 \times W' \mid \rho_0(x) = \zeta(w')\big\}.$$

Pushouts and pullbacks always exist as we have already mentioned, and are unique up to equivalence of extensions, see [Mac63, Lemmas 1.2 and 1.4, and Section III.5]. Therefore, given $\alpha\colon V \longrightarrow V'$, we can define a morphism $\alpha_*\colon \mathrm{YExt}^\bullet_{RG}(W, V) \longrightarrow \mathrm{YExt}^\bullet_{RG}(W, V')$ via the pushout construction. Analogously, given $\zeta\colon W' \longrightarrow W$, there is a morphism $\zeta^*\colon \mathrm{YExt}^\bullet_{RG}(W, V) \longrightarrow \mathrm{YExt}^\bullet_{RG}(W', V)$ defined via the pullback construction.

Additionally, given RG-modules V and W, the set $\mathrm{YExt}^n_{RG}(W, V)$ can be given the structure of an abelian group for every $n \geq 1$. Suppose that we have two n-fold Yoneda extensions $\varphi, \varphi' \in \mathrm{YExt}^n_{RG}(W, V)$. Then, we can construct its direct product $\varphi \times \varphi' \in \mathrm{YExt}^n_{RG}(W \times W, V \times V)$ in the obvious way. Now, consider the diagonal homomorphism $\Delta_W\colon W \longrightarrow W \times W$ defined by

$$\Delta_W(w) = (w, w)$$

for $b \in W$, and the codiagonal homomorphism $\nabla_V\colon V \times V \longrightarrow V$ defined by

$$\nabla_V(v_1, v_2) = v_1 + v_2$$

for $v_1, v_2 \in V$. We define the *Baer sum* of φ and φ' to be the Yoneda extension

$$\varphi + \varphi' = (\nabla_V)_*(\Delta_W)^*(\varphi \times \varphi') \in \mathrm{YExt}^n_{RG}(W, V).$$

Equipped with the Baer sum, the set $\mathrm{YExt}^n_{RG}(W,V)$ acquires the structure of an abelian group for every $n \geq 1$ [Mac63, Theorem 5.3]. Furthermore, the zero element $0 \in \mathrm{YExt}^n_{RG}(W,V)$ is the Yoneda extension

$$0 \longrightarrow V \longrightarrow V \times W \longrightarrow W \longrightarrow 0$$

for $n = 1$, and

$$0 \longrightarrow V \longrightarrow V \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow W \longrightarrow W \longrightarrow 0$$

for $n > 1$. If $\varphi \in \mathrm{YExt}^n_{RG}(W,V)$ is the Yoneda extension

$$0 \longrightarrow V \xrightarrow{\rho_n} M_n \xrightarrow{\rho_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\rho_0} W \longrightarrow 0,$$

then its opposite $-\varphi \in \mathrm{YExt}^n_{RG}(W,V)$ is the extension

$$0 \longrightarrow V \xrightarrow{-\rho_n} M_n \xrightarrow{\rho_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\rho_0} W \longrightarrow 0.$$

Theorem 2.4 ([Mac63, Theorem 6.4])**.** *Let V and W be RG-modules. Then, for every $n \geq 1$ there is a group isomorphism $\mathrm{Ext}^n_{RG}(W,V) \cong \mathrm{YExt}^n_{RG}(W,V)$ that is natural in both V and W.*

Remark 2.5. We can describe the isomorphism $\Psi\colon \mathrm{Ext}^n_{RG}(W,V) \longrightarrow \mathrm{YExt}^n_{RG}(W,V)$ explicitly. Let $P_\bullet \longrightarrow W$ be a projective resolution of W, and consider the RG-module

$$\Omega_n = \frac{P_n}{\operatorname{Ker} \partial_n} \cong \operatorname{Im} \partial_n. \tag{2.1}$$

An element $\xi \in \mathrm{Ext}^n_{RG}(W,V)$ can thus be represented by an RG-module homomorphism $\xi\colon \Omega_n \longrightarrow V$. Consider the n-fold Yoneda extension $\psi \in \mathrm{YExt}^n_{RG}(W,\Omega_n)$ represented by

$$0 \longrightarrow \Omega_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow W \longrightarrow 0. \tag{2.2}$$

Then, we define $\Psi(\xi) = \xi_*(\psi)$.

We can use Theorem 2.4 to give an alternative characterization of the cohomology of a finite group.

Corollary 2.6. *Let G be a finite group and V be an RG-module. Then, for every $n \geq 1$ there is a group isomorphism $\mathrm{H}^n(G,V) \cong \mathrm{YExt}^n_{RG}(R,V)$ that is natural in V.*

As an example, we can describe explicitly how to construct the 1-fold Yoneda extension in $\mathrm{YExt}^1_{RG}(R,R)$ associated to a cohomology class in $\mathrm{H}^1(G,R)$.

Example 2.7. Let G be a finite group, and take $\varphi \in \mathrm{H}^1(G,R) = \mathrm{Hom}(G,R)$. Let $\Psi\colon \mathrm{Ext}^1_{RG}(R,R) \longrightarrow \mathrm{YExt}^1_{RG}(R,R)$ be the isomorphism in Remark 2.5. The 1-fold Yoneda extension $\Psi(\xi)$ is of the form

$$0 \longrightarrow yR \longrightarrow M_1 \longrightarrow xR \longrightarrow 0, \tag{2.3}$$

where x and y are the R-basis elements in the corresponding modules. As an R-module, M_1 is free of rank two, and we can write $M_1 = xR \oplus_R yR$. In order to determine the structure of M_1 as an RG-module, the only thing left to do is to compute the action of G on $x, y \in M_1$. Because G acts trivial on both $xR = M_1/yR$ and $yR \leq M_1$, the action of $g \in G$ on M_1 is given by

$$g \cdot x = x + \lambda y, \quad g \cdot y = y$$

with $\lambda \in R$. Now, consider the bar resolution $\mathcal{B}_\bullet G \longrightarrow R$ as defined in Section 1.2.1. Then, $P_0 = RG$ and the RG-module Ω_1 defined as in (2.1) is nothing more than the augmentation ideal $(G-1)RG \leq RG$. Thus, the pushout of the extension (2.2) by φ is given by the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_1 \arrow[r] \arrow[d, "\varphi"] & RG \arrow[r, "\varepsilon"] \arrow[d, "\varphi \oplus \varepsilon"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & yR \arrow[r] & M_1 \arrow[r] & xR \arrow[r] & 0
\end{tikzcd}$$

From this, we can easily deduce that $\lambda = \varphi(g)$, and so the action of $g \in G$ on M_1 is given by

$$g \cdot x = x + \varphi(g)y, \quad g \cdot y = y.$$

We will now see how to define a product of Yoneda extensions. Let V, W and U be RG-modules and $n, m \geq 1$. Take $\varphi \in \mathrm{YExt}^n_{RG}(W,V)$ and $\varphi' \in \mathrm{YExt}^m_{RG}(U,W)$. Then, if $\varphi \in \mathrm{YExt}^n_{RG}(W,V)$ is of the form

$$0 \longrightarrow V \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow W \longrightarrow 0$$

and $\varphi' \in \mathrm{YExt}^m_{RG}(U,W)$ is of the form

$$0 \longrightarrow W \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow U \longrightarrow 0,$$

we define their *Yoneda product* $\varphi \smile \varphi' \in \mathrm{YExt}^{n+m}_{RG}(U,V)$ as the extension

$$0 \longrightarrow V \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow U \longrightarrow 0.$$

Proposition 2.8. *Let V, W and U be RG-modules and $n, m \geq 1$. Then, the Yoneda product gives a bilinear pairing*

$$\mathrm{YExt}^n_{RG}(W, V) \otimes \mathrm{YExt}^m_{RG}(U, W) \longrightarrow \mathrm{YExt}^{n+m}_{RG}(U, V).$$

In particular, given an RG-module V then $\mathrm{YExt}^\bullet_{RG}(V, V)$ has the structure of a graded ring with unity when equipped with the Yoneda product.

Remark 2.9. It is easy to see that any Yoneda n-fold extension can be decomposed in a unique way as a Yoneda product of n Yoneda 1-fold extensions.

We thus have two different products defined on $\mathrm{H}^\bullet(G, R)$, the Yoneda product and the cup product. These two products are actually the same, see [Ben91, Proposition 3.2.1].

Theorem 2.10. *Let G be a finite group and $n, m \geq 1$. Then, the Yoneda product*

$$\mathrm{YExt}^n_{RG}(R, R) \otimes \mathrm{YExt}^m_{RG}(R, R) \longrightarrow \mathrm{YExt}^{n+m}_{RG}(R, R)$$

coincides with the cup product

$$\mathrm{H}^n(G, R) \otimes \mathrm{H}^m(G, R) \longrightarrow \mathrm{H}^{n+m}(G, R).$$

Moreover, for a fixed RG-module W, the Yoneda product turns $\mathrm{YExt}^\bullet_{RG}(W, -)$ into a δ-functor, where the connecting homomorphism associated to a short exact sequence is nothing more than multiplying by said short exact sequence on the left [Mac63, Theorem 9.1].

2.2 Crossed extensions

In Section 1.2.1, we saw how to identify the elements of the second cohomology group as group extensions. In this section, we will see how to generalize this characterization for higher degree cohomology groups, using the concept of crossed extensions. We remark that most of the properties and constructions of crossed extensions are analogues to those of Yoneda extensions.

Definition 2.11. Let M_1 and M_2 be groups with M_1 acting on M_2. A *crossed module* is an M_1-invariant group homomorphism $\rho\colon M_2 \longrightarrow M_1$ satisfying the following properties:

(i) $\rho(y_2)x_2 = x_2^{y_2^{-1}}$ for all $x_2, y_2 \in M_2$, and

(ii) $\rho(x_1x_2) = \rho(x_2)^{x_1^{-1}}$ for all $x_1 \in M_1$ and $x_2 \in M_2$.

Definition 2.12. Let G be a finite group, V be an RG-module and $n \geq 2$ be an integer. A *crossed n-fold extension of G by V* is an exact sequence of groups ψ of the form

$$\psi\colon 0 \longrightarrow V \xrightarrow{\rho_n} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\rho_1} M_1 \longrightarrow G \longrightarrow 1,$$

satisfying the following conditions:

(i) $\rho_1\colon M_2 \longrightarrow M_1$ is a crossed module,

(ii) M_i is an RG-module for every $i = 3, \ldots, n$, and

(iii) ρ_i is an RG-module homomorphism for every $i = 2, \ldots, n$.

A *crossed* 1*-fold extension* of G by V is a group extension

$$\psi\colon 0 \longrightarrow V \longrightarrow M \longrightarrow G \longrightarrow 1.$$

Example 2.13. Let G be a finite group with center $\mathrm{Z}(G)$ and group of automorphisms $\mathrm{Aut}(G)$. If we denote by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the subgroup $\mathrm{Inn}(G)$ of inner automorphisms, i.e. the automorphisms induced by conjugation by elements of G. Then, we have a crossed 2-fold extension of $\mathrm{Out}(G)$ by $\mathrm{Z}(G)$ of the form

$$1 \longrightarrow \mathrm{Z}(G) \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

We can define an equivalence relation between crossed extensions as for Yoneda extensions. Firstly, a *morphism of crossed n-fold extensions* is a morphism of exact sequences of groups

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "f_{n+1}"] & M_n \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & M_2 \arrow[r] \arrow[d, "f_2"] & M_1 \arrow[r] \arrow[d, "f_1"] & G \arrow[r] \arrow[d, "f_0"] & 1 \\
0 \arrow[r] & V' \arrow[r] & M'_n \arrow[r] & \cdots \arrow[r] & M'_2 \arrow[r] & M'_1 \arrow[r] & G' \arrow[r] & 1
\end{tikzcd}$$

such that f_i is an RG-module homomorphism for every $i = 3, \ldots, n+1$, and f_1 and f_2 are compatible with the actions of M_1 and M'_1 on M_2 and M'_2, respectively. Now, let ψ and ψ' be crossed n-fold extensions of G by V. If we have a morphism of crossed extensions $\psi \longrightarrow \psi'$ of the form

$$\begin{tikzcd}
\psi\colon 0 \arrow[r] & V \arrow[r] \arrow[d, equal] & M_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
\psi'\colon 0 \arrow[r] & V \arrow[r] & M_n' \arrow[r] & \cdots \arrow[r] & M_1' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

we write $\psi' \implies \psi'$. In general, we say that ψ is *equivalent* to ψ', and write $\psi \equiv \psi'$, if there is a chain of crossed n-fold extensions of G by V and morphisms of the form

$$\psi \implies \psi_1 \impliedby \psi_2 \implies \cdots \impliedby \psi_{r-1} \implies \psi_r \impliedby \psi'.$$

This defines an equivalence relation. We will denote by $\mathrm{XExt}_R^n(G, V)$ the set of all crossed n-fold extensions of G by V up to equivalence.

For the case $n = 2$, we can use the following characterization of equivalent crossed extensions.

Proposition 2.14 ([Hol79, Lemma 2.5])**.** *Let G be a finite group and V be an RG-module. Then, two crossed 2-fold extensions of G by V of the form*

$$\psi\colon \quad 0 \longrightarrow V \xrightarrow{\rho_2} M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} G \longrightarrow 1$$
$$\psi'\colon \quad 0 \longrightarrow V \xrightarrow{\tau_2} N_2 \xrightarrow{\tau_1} N_1 \xrightarrow{\tau_0} G \longrightarrow 1$$

are equivalent if and only if there exists a group X fitting into a commutative diagram of the form

$$\begin{tikzcd}
 & 1 \arrow[dr] & & & & 1 & \\
 & & M_2 \arrow[rr, "\rho_1"] \arrow[dr, "\mu_1"'] & & M_1 \arrow[ur] \arrow[dr, "\rho_0"] & & \\
0 \arrow[r] & V \arrow[ur, "\rho_2"] \arrow[dr, "-\tau_2"'] & & X \arrow[ur, "\nu_1"'] \arrow[dr, "\nu_2"] & & G \arrow[r] & 1 \\
 & & N_2 \arrow[ur, "\mu_2"] \arrow[rr, "\tau_1"'] & & N_1 \arrow[ur, "\tau_0"'] \arrow[dr] & & \\
 & 1 \arrow[ur] & & & & 1 &
\end{tikzcd} \tag{2.4}$$

satisfying the following properties:

(a) $-\tau_2 \colon V \longrightarrow N_2$ *is given by* $(-\tau_2)(a) = \tau_2(-a)$ *for* $a \in V$*,*

(b) the diagonals are short exact sequences,

(c) $\mu_1 \circ \rho_2(V) = \mu_1(M_2) \cap \mu_2(N_2)$*, and*

(d) conjugation in X *coincides with the actions of both* M_1 *on* M_2 *and* N_1 *on* N_2*.*

Remark 2.15. The previous theorem can be generalized for crossed n-fold extensions with $n \geq 2$, see [Hol79, Lemma 3.3]. The proof of these results can be used to prove an analogue of Proposition 2.2 for crossed extensions.

Proposition 2.16. *Let* G *be a finite group,* V *be an* RG*-module and* $n \geq 2$ *be an integer. Then, every element in* $\mathrm{XExt}_R^n(G, V)$ *can be represented by a crossed extension of the form*

$$\psi \colon 0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow G \longrightarrow 1,$$

with M_2 *abelian.*

Analogous to the case of Yoneda extensions, $\mathrm{XExt}_R(-,-)$ is functorial on both components. Let G and G' be finite groups, V and V' be RG-modules and $n \geq 1$ be an integer. Consider the n-fold crossed extension $\psi \in \mathrm{XExt}_R^n(G, V)$ represented by

$$0 \longrightarrow V \xrightarrow{\rho_n} M_n \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\rho_0} G \longrightarrow 1.$$

On the one hand, given an RG-module homomorphism $\alpha \colon V \longrightarrow V'$, we say that $\psi' \in \mathrm{XExt}_R^n(G, V')$ is a *pushout* of ψ via α if there is a morphism $\psi \longrightarrow \psi'$ of the following form:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "\alpha"] & M_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & V' \arrow[r] & M_n' \arrow[r] & \cdots \arrow[r] & M_1' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

We can always construct the pushout $\alpha_*\psi \in \mathrm{XExt}_R^n(G, V')$ represented by the extension

$$0 \longrightarrow V' \longrightarrow N \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\rho_0} G \longrightarrow 1,$$

where

$$N = \frac{M_n \ltimes V'}{\big\{\big(\rho_n(v), -\alpha(v)\big) \mid v \in V\big\}}$$

with the action of M_n on V' being the one induced by ρ_0 if $n = 1$, or trivial otherwise.

On the other hand, given a group homomorphism $\zeta\colon G' \longrightarrow G$, we say that $\psi' \in \mathrm{XExt}^n_R(G', V)$ is a *pullback* of ψ via ζ if there is a morphism $\psi \longrightarrow \psi'$ of the following form:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & M'_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M'_1 \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d, "\zeta"] & 1 \\
0 \arrow[r] & V \arrow[r] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

We can always take the pullback $\zeta^*\psi \in \mathrm{XExt}^n_R(G', V)$ represented by the extension

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow N \longrightarrow G' \longrightarrow 1,$$

where

$$N = \big\{(x, g') \in M_1 \times G' \mid \rho_0(x) = \zeta(g')\big\}.$$

As for Yoneda extensions, pushouts and pullbacks of crossed extensions are unique, see [Hol79, Proposition 4.1]. Therefore, given $\alpha\colon V \longrightarrow V'$, we can define a morphism $\alpha_*\colon \mathrm{XExt}^\bullet_R(G, V) \longrightarrow \mathrm{XExt}^\bullet_R(G, V')$ via the pushout construction. Analogously, given $\zeta\colon G' \longrightarrow G$, there is a morphism $\zeta^*\colon \mathrm{XExt}^\bullet_R(G, V) \longrightarrow \mathrm{XExt}^\bullet_R(G', V)$ defined via the pullback construction.

Given an RG-module V, the set $\mathrm{XExt}^n_{RG}(G, V)$ can be given the structure of an abelian group for every $n \geq 1$.

Suppose that we have two crossed extensions $\psi, \psi' \in \mathrm{XExt}^n_R(G, V)$. Then, we can construct its direct product $\psi \times \psi' \in \mathrm{XExt}^n_R(G \times G, V \times V)$ in the obvious way. Now, consider the diagonal homomorphism $\Delta_G\colon G \longrightarrow G \times G$ defined by

$$\Delta_G(g) = (g, g)$$

for $g \in G$, and the codiagonal homomorphism $\nabla_V\colon V \times V \longrightarrow V$ defined by

$$\nabla_V(v_1, v_2) = v_1 + v_2$$

for $(v_1, v_2) \in V \times V$. We define the *Baer sum* of ψ and ψ' to be the crossed extension

$$\psi + \psi' = (\nabla_V)_*(\Delta_G)^*(\psi \times \psi') \in \mathrm{XExt}^n_R(G, V).$$

Equipped with the Baer sum, the set $\mathrm{XExt}^n_R(G, V)$ acquires the structure of an abelian group for every $n \geq 1$ [Hol79]. The zero element $0 \in \mathrm{XExt}^n_R(G, V)$ is the crossed extension

$$0 \longrightarrow V \longrightarrow G \ltimes V \longrightarrow G \longrightarrow 1$$

for $n = 1$, and

$$0 \longrightarrow V \longrightarrow V \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow G \longrightarrow G \longrightarrow 1$$

for $n > 1$. If $\psi \in \mathrm{XExt}^n_R(G, V)$ is the crossed extension

$$0 \longrightarrow V \xrightarrow{\rho_n} M_n \xrightarrow{\rho_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\rho_0} G \longrightarrow 1,$$

then its opposite $-\psi \in \mathrm{XExt}^n_R(G, V)$ is the extension

$$0 \longrightarrow V \xrightarrow{-\rho_n} M_n \xrightarrow{\rho_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\rho_0} G \longrightarrow 1.$$

Combining Theorem 2.4 with [Hol79, Theorem 4.5], we finally obtain that the cohomology of a finite group can be expressed in terms of both Yoneda and crossed extensions.

Theorem 2.17. *Let G be a finite group. For every RG-module V and every integer $n \geq 1$, there are group isomorphisms*

$$\mathrm{H}^{n+1}(G, V) \cong \mathrm{YExt}^{n+1}_{RG}(R, V) \cong \mathrm{XExt}^n_R(G, V)$$

that are natural in both G and V.

2.3 Product of Yoneda extensions and crossed extensions

We proceed now to define the analogous to the Yoneda product of two Yoneda extensions, this time for a Yoneda extension with a crossed extension.

Definition 2.18. Let G be a finite group, V and W be RG-modules and $n, m \geq 1$ be integers. Given a Yoneda n-fold extension $\varphi \in \mathrm{YExt}^n_{RG}(V, W)$ represented by

$$0 \longrightarrow W \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow V \longrightarrow 0,$$

and a crossed m-fold extension $\psi \in \mathrm{XExt}^m_R(G, V)$ represented by

$$0 \longrightarrow V \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1,$$

we define their *Yoneda product* $\varphi \smile \psi$ as the extension

$$0 \longrightarrow W \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

Remark 2.19. It can be readily checked that the Yoneda product gives a well defined bilinear pairing

$$\smile\colon \operatorname{YExt}^n_{RG}(V,W)\otimes \operatorname{XExt}^m_R(G,V) \longrightarrow \operatorname{XExt}^{n+m}_R(G,W)$$

by following the analogous proofs for the Yoneda product of two Yoneda extensions, compare [Mac63, Section III.5] and [Hol79].

In order to show that the Yoneda product of Yoneda extensions with crossed extensions coincides with the usual cup product, we will use the universality of Ext as a δ-functor.

For a fixed finite group G, we have that the functor $\operatorname{XExt}^\bullet_R(G,-)$ is a universal δ-functor with connecting homomorphism given by the Yoneda product, and starting with $\operatorname{XExt}^1_R(G,-)=\operatorname{Ext}^2_{RG}(R,-)$ [Hol79, Section 4]. Then, we apply Theorem 1.12 to obtain that

$$\big(\operatorname{XExt}^\bullet_R(G,-),\smile\big)\cong\big(\operatorname{Ext}^{\bullet+1}_{RG}(R,-),\delta\big)\cong\big(\operatorname{YExt}^{\bullet+1}_{RG}(R,-),\smile\big).$$

Because both Yoneda products coincide with the connecting homomorphism, they must be the same, see also Remark 2.9.

Theorem 2.20. *Let G be a finite group, V and W be RG-modules, and $n,m\geq 1$. Then, the Yoneda product*

$$\operatorname{YExt}^n_{RG}(V,W)\otimes \operatorname{XExt}^m_R(G,V) \longrightarrow \operatorname{XExt}^{n+m}_R(G,W)$$

coincides with the Yoneda product

$$\operatorname{YExt}^n_{RG}(V,W)\otimes \operatorname{YExt}^m_{RG}(R,V) \longrightarrow \operatorname{YExt}^{n+m}_{RG}(R,W).$$

Remark 2.21. In [Con85], Conrad defines an explicit correspondence between crossed extensions and Yoneda extensions. In fact, starting with a crossed n-fold extension $\psi\in\operatorname{XExt}^n_R(G,V)$ written in the form of Proposition 2.16, he is able to construct a $(n+1)$-fold Yoneda extension $\Upsilon(\psi)\in\operatorname{YExt}^{n+1}_{RG}(R,V)$ in such a way that the map

$$\Upsilon\colon \operatorname{XExt}^n_R(G,V)\longrightarrow \operatorname{YExt}^{n+1}_{RG}(R,V)$$

thus defined is an isomorphism for all $n\geq 2$. It is possible to use this isomorphism to give an alternative proof of Theorem 2.20, sidestepping the use of the machinery of δ-functors. Indeed, this is the approach taken in [GGG22, Section 3.2].

3

Carlson's depth conjecture

In order to better understand the mod-p cohomology ring of a finite group G, we can study its algebraic invariants, such as the Krull dimension or its more exotic cousin, the depth. It is well known, from a result by Quillen, that the Krull dimension of the mod-p cohomology ring of G is simply the largest rank of an elementary abelian p-subgroup of G. As such, the Krull dimension is completely determined by the subgroup structure of the group G, and its computation does not require any prior knowledge of the cohomology ring. The same cannot be said of the depth, the computation of which usually requires determining the structure of the cohomology ring beforehand. The lack of examples of computations of such rings in the literature has thus proved a huge hurdle to the study of this invariant. While it is true that some partial results exist bounding its value, so far no complete characterization of the depth has been found. In [Car95], Carlson stated a conjecture that links the computation of the depth of the cohomology ring of G to its detection by certain families of subgroups of G. So far, the conjecture remains open. In this chapter, we will introduce the concept of depth for the mod-p cohomology ring of a finite group and state Carlson's depth conjecture. Afterwards, we will compute the depth of the mod-p cohomology rings of certain quotients of the pro-p group of maximal class that, moreover, satisfy Carlson's depth conjecture, using techniques from Chapter 2 and without first computing the cohomology rings. This chapter is based on [GGG22].

3.1 Depth detection in cohomology

In this section, we will give some background on the depth of mod-p cohomology rings of finite groups and introduce Carlson's depth conjecture. For a more thorough introduction to the topic, see [CTVZ03, Chapter 12]. Let G be a finite group, $n \geq 1$ be an integer and $x_1, \ldots, x_n \in \mathrm{H}^\bullet(G, \mathbb{F}_p) \setminus \mathrm{H}^0(G, \mathbb{F}_p)$. We say that $x_1, \ldots, x_n$ is a *regular sequence* if the element x_1 is not a zero divisor in $\mathrm{H}^\bullet(G, \mathbb{F}_p)$ and, for every $i = 2, \ldots, n$, the element x_i is not a zero divisor in the quotient $\mathrm{H}^\bullet(G, \mathbb{F}_p)/(x_1, \ldots, x_{i-1})\,\mathrm{H}^\bullet(G, \mathbb{F}_p)$.

Definition 3.1. The *depth* of $\mathrm{H}^\bullet(G, \mathbb{F}_p)$, denoted by $\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p)$, is the maximal length of a regular sequence of elements of $\mathrm{H}^\bullet(G, \mathbb{F}_p)$.

The computation of the depth of the graded-commutative ring can be reduced to the computation of the depth of a (classical) commutative ring, so that all the classical results about depth from commutative algebra hold (see for example [Eis95, Chapters 17 and 18]) and, in particular, every regular sequence can be extended to a regular sequence of maximal length and the depth is well defined. Indeed, for $p = 2$ the ring $\mathrm{H}^\bullet(G, \mathbb{F}_2)$ is commutative, so we can assume that $p > 2$. It is not difficult to see that, in Definition 3.1, we can consider only sequences of homogeneous elements. Now, since every element of odd degree is nilpotent, our regular sequences will consist only of elements of even degree, and so

$$\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p) = \operatorname{depth} \bigoplus_{r=0}^{\infty} \mathrm{H}^{2r}(G, \mathbb{F}_p).$$

As we have already mentioned, the depth is related to the Krull dimension, but while the Krull dimension can be computed simply by studying the structure of the group, the same cannot be said about the depth, which is considerably more difficult to compute. In general, this can only be achieved by first computing the cohomology ring explicitly. Let us state Quillen's classical result giving us the value of the Krull dimension of the mod-p cohomology of a finite group, the proof of which can be found in [CTVZ03, Corollary 8.4.7].

Theorem 3.2 (Quillen)**.** *Let G be a finite group and denote by $r = \operatorname{rk}_p G$ the largest integer such that G has an elementary abelian p-subgroup of rank r. Then, we have that*

$$\dim \mathrm{H}^\bullet(G, \mathbb{F}_p) = r.$$

We will now see some examples of the value of the depth of the mod-p cohomology rings of the p-groups in Example 1.21, and compare it to the Krull dimension.

Examples 3.3. (i) Let G be an abelian p-group with $\operatorname{rk}_p G = r$. Then, its mod-$p$ cohomology is given by

$$\mathrm{H}^\bullet(G, \mathbb{F}_p) = \begin{cases} \Lambda(x_1, \ldots, x_s) \otimes \mathbb{F}_2[y_1, \ldots, y_r], & \text{if } p = 2, \\ \Lambda(x_1, \ldots, x_r) \otimes \mathbb{F}_p[y_1, \ldots, y_r], & \text{if } p > 2, \end{cases}$$

where $s = \operatorname{rk}_2 G^2$. It is clear that $y_1, \ldots, y_r$ is a maximal regular sequence in $\mathrm{H}^\bullet(G, \mathbb{F}_p)$, and so $\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p) = r$, which is equal to $\dim \mathrm{H}^\bullet(G, \mathbb{F}_p) = r$.

(ii) Let D_{2^n} be the dihedral group of order 2^n with $n \geq 3$. Then, its mod-2 cohomology ring is given by

$$\mathrm{H}^\bullet(D_{2^n}, \mathbb{F}_2) = \frac{\mathbb{F}_2[x, y, z]}{(xy)}$$

with $\deg x = \deg y = 1$ and $\deg z = 2$. We can see that z constitutes a maximal regular sequence, and so $\operatorname{depth} \mathrm{H}^\bullet(D_{2^n}, \mathbb{F}_2) = 1$, whereas $\dim \mathrm{H}^\bullet(D_{2^n}, \mathbb{F}_2) = 2$.

(iii) Let Q_{2^n} be the generalized quaternion group of order 2^n with $n \geq 4$. Then, its mod-2 cohomology ring is given by

$$\mathrm{H}^\bullet(Q_{2^n}, \mathbb{F}_2) = \frac{\mathbb{F}_2[x, y, z]}{(xy, x^3 + y^3)}$$

with $\deg x = \deg y = 1$ and $\deg z = 4$. We can see that z constitutes a maximal regular sequence, and so $\operatorname{depth} \mathrm{H}^\bullet(Q_{2^n}, \mathbb{F}_2) = 1$, which is equal to $\dim \mathrm{H}^\bullet(Q_{2^n}, \mathbb{F}_2) = 1$.

(iv) Let $G = C_3 \ltimes (C_3 \times C_3)$ be the extraspecial 3-group of order 27 and exponent 3. It is known (compare [Lea92] and [Min01]) that $\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_3) = 2$, which is equal to $\dim \mathrm{H}^\bullet(G, \mathbb{F}_3) = 2$.

We will now state some results that give upper and lower bounds on the value of the depth of a mod-p cohomology ring of a finite group G. First, recall that a prime ideal $\mathfrak{p} \subseteq \mathrm{H}^\bullet(G, \mathbb{F}_p)$ is an *associated prime* of $\mathrm{H}^\bullet(G, \mathbb{F}_p)$ if it is the annihilator of some $\varphi \in \mathrm{H}^\bullet(G, \mathbb{F}_p)$, i.e. it is of the form

$$\mathfrak{p} = \left\{\psi \in \mathrm{H}^\bullet(G, \mathbb{F}_p) \mid \varphi \smile \psi = 0\right\}.$$

The set of all associated primes of $\mathrm{H}^\bullet(G, \mathbb{F}_p)$ is denoted by $\operatorname{Ass} \mathrm{H}^\bullet(G, \mathbb{F}_p)$. We can obtain an upper bound on the depth as follows (see [CTVZ03, Proposition 12.2.5]).

Proposition 3.4. *Let G be a finite group and $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^\bullet(G, \mathbb{F}_p)$. Then, the following inequality holds:*

$$\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p) \leq \dim \mathrm{H}^\bullet(G, \mathbb{F}_p)/\mathfrak{p}.$$

In particular, we have that

$$\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p) \leq \dim \mathrm{H}^\bullet(G, \mathbb{F}_p).$$

There is also a lower bound on the depth, obtained by Duflot in [Duf81].

Proposition 3.5. *Let G be a finite group such that p divides $|G|$. Given a Sylow p-subgroup $S \leq G$ with center $\mathrm{Z}(S) \leq S$, we have that*

$$1 \leq \operatorname{rk}_p \mathrm{Z}(S) \leq \operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p).$$

In order to state Carlson's depth conjecture, we need to introduce the concept of detection in cohomology.

Definition 3.6. Let G be a finite group and let $\mathcal{H}$ be a collection of subgroups of G. We say that $\mathrm{H}^\bullet(G, \mathbb{F}_p)$ is *detected* by $\mathcal{H}$ if

$$\bigcap_{H \in \mathcal{H}} \operatorname{Ker} \operatorname{res}_{G \to H} = 0.$$

The computation of the depth is intimately linked to detection on certain families of subgroups. Given a finite group G and a subgroup $E \leq G$, let $\mathrm{C}_G(E)$ denote the centralizer of E in G. For $s \geq 1$, define the following:

$$\begin{aligned}
\mathcal{H}_s(G) &= \big\{ \mathrm{C}_G(E) \mid E \text{ is an elementary abelian } p\text{-subgroup of } G,\ \operatorname{rk}_p E = s \big\}, \\
\omega_a(G) &= \min \big\{ \dim \mathrm{H}^\bullet(G, \mathbb{F}_p)/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} \mathrm{H}^\bullet(G, \mathbb{F}_p) \big\}, \\
\omega_d(G) &= \max \big\{ s \geq 1 \mid \mathrm{H}^\bullet(G, \mathbb{F}_p) \text{ is detected by } \mathcal{H}_s(G) \big\}.
\end{aligned}$$

Theorem 3.7 ([Car95, Theorem 2.3])**.** *Let G be a finite group. Then, the following inequalities hold:*

$$\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p) \leq \omega_a(G) \leq \omega_d(G).$$

In fact, in the same article, Carlson conjectured that the previous inequalities are actually equalities.

Conjecture 3.8 (Carlson)**.** Let G be a finite group. Then, we have that

$$\operatorname{depth} \mathrm{H}^\bullet(G, \mathbb{F}_p) = \omega_a(G) = \omega_d(G).$$

Remark 3.9. The equality $\omega_a(G) = \omega_d(G)$ always holds. This follows from combining Theorem 3.7 with [Car95, Remark 3.5(a)].

A particular case of Conjecture 3.8 was proven by Green in [Gre03, Theorem 0.1], and Theorem 3.7 was generalized in the context of compact Lie groups (see [Kuh13, Theorem 2.30] and [Kuh07, Theorem 2.13]) and saturated fusion systems (see [Hea20, Theorem 4.16]).

3.2 Finite p-groups of mod-p cohomology depth at most 2

In this section, we will introduce the family of p-groups under study and provide an initial bound on the depth of their mod-p cohomology. Let p be an odd prime number, let $\mathbb{Z}_p$ denote the ring of p-adic integers and let ζ be a primitive p-th root of unity. Consider the cyclotomic extension $\mathbb{Z}_p[\zeta]$ of degree $p-1$ and note that its additive group is isomorphic to $\mathbb{Z}_p^{p-1}$. The cyclic group $C_p = \langle \sigma \rangle$ acts (on the right) on $\mathbb{Z}_p[\zeta]$ via multiplication by ζ, i.e., for any $x \in \mathbb{Z}_p$, the action is given as $x^\sigma = \zeta x$. Using the ordered basis $1, \zeta, \dots, \zeta^{p-2}$ in $\mathbb{Z}_p[\zeta] \cong \mathbb{Z}_p^{p-1}$, this action is given by multiplication by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & -1 & -1 & \dots & -1 \end{pmatrix}.$$

We form the semidirect product $S = C_p \ltimes \mathbb{Z}_p^{p-1}$, which is the unique pro-p group of maximal nilpotency class. Note that this is the analogue of the infinite dihedral pro-2 group for the p odd case. Moreover, S is a uniserial p-adic space group with cyclic point group C_p (compare [LM02, Section 7.4]). We write $[x,_k \sigma] = [x, \sigma, \overset{k}{\dots}, \sigma]$ for the iterated group commutator and γ_k for the k-th term of the lower central series. Set $N_0 = \mathbb{Z}_p[\zeta]$ and define, for each integer $i \geq 1$, the subgroup

$$N_i = (\zeta - 1)^i \mathbb{Z}_p[\zeta] = [N_0,_i \sigma] = \gamma_{i+1}(S).$$

These are all the C_p-invariant subgroups of N_0, and all successive quotients satisfy that

$$\frac{N_i}{N_{i+1}} \cong \frac{\mathbb{Z}_p[\zeta]}{(\zeta - 1)\mathbb{Z}_p[\zeta]} \cong C_p.$$

Hence, $|N_0 : N_i| = p^i$ for every $i \geq 0$. For each integer $r > 0$, consider the quotient $N_0/N_r = \mathbb{Z}_p[\zeta]/(\zeta - 1)^r \mathbb{Z}_p[\zeta]$. Since the subgroups N_r are C_p-invariant, we can form the semidirect product

$$G_r = C_p \ltimes \frac{N_0}{N_r}. \tag{3.1}$$

For each integer r with $1 < r < p$, we can choose a minimal generating set for N_0/N_r as follows. Consider the elements

$$a_1 = 1 + N_r, \quad a_2 = (\zeta - 1) + N_r, \quad \ldots, \quad a_r = (\zeta - 1)^{r-1} + N_r.$$

Using multiplicative notation, we obtain that

$$\frac{N_0}{N_r} = \langle a_1, \ldots, a_r \rangle \cong C_p \times \overset{r}{\cdots} \times C_p,$$

and thus,

$$G_r = C_p \ltimes \frac{N_0}{N_r} \cong C_p \ltimes (C_p \times \overset{r}{\cdots} \times C_p),$$

generated by the elements $\sigma, a_1, \ldots, a_r$ satisfying the following relations:

- $\sigma^p = a_i^p = [a_i, a_j] = [a_r, \sigma] = 1$, for all $i = 1, \ldots, r$ and $j = 1, \ldots, r-1$,
- $[a_j, \sigma] = a_{j+1}$ for all $j = 1, \ldots, r-1$.

The finite p-group G_r has size p^{r+1} and exponent p. Note that in particular, G_2 is the extraspecial group of size p^3 and exponent p. In general, if we write $r = (p-1) \cdot n + m$ with $n, m \geq 0$ integers such that $m < p-1$, then G_r can be described as a semidirect product

$$G_r = C_p \ltimes (C_{p^{n+1}} \times \overset{m}{\cdots} \times C_{p^{n+1}} \times C_{p^n} \times \overset{p-m-1}{\cdots} \times C_{p^n}) = C_p \ltimes \frac{N_0}{N_r},$$

where N_0/N_r is the maximal abelian p-subgroup of G_r. Hence, if $r > p-1$, then G_r has size p^{r+1} and exponent bigger than p.

We can easily bound the depth of the mod-p cohomology of G_r using the results from Section 3.1.

Proposition 3.10. *For every integer $r > 1$, the following inequalities hold:*

$$1 \leq \operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) \leq 2.$$

Proof. The inequality $1 \leq \operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ holds by Proposition 3.5. Let us prove that $\mathrm{H}^\bullet(G_r, \mathbb{F}_p) \leq 2$. First, suppose that $p = 3$. Then, for every $r > 1$, we have that

$$\mathrm{rk}_p(G_r) = 2 = \dim \mathrm{H}^\bullet(G_r, \mathbb{F}_p),$$

and by Proposition 3.4, we conclude that $\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) \leq 2$.

Now, suppose that $p \geq 5$. It can be readily checked that, for any $r > 1$, every elementary abelian p-subgroup E of G_r with $\mathrm{rk}_p(E) = 3$ satisfies that $E \leq N_0/N_r$, and the centralizer is $\mathrm{C}_{G_r}(E) = N_0/N_r$. Therefore, for every such E, its centralizer $\mathrm{C}_{G_r}(E)$ is contained in the maximal subgroup N_0/N_r of G_r and the cohomology class $\sigma^* \in \mathrm{H}^1(G_r, \mathbb{F}_p)$ restricts to zero on every $\mathrm{C}_{G_r}(E)$. Hence, by Theorem 3.4, we conclude that $\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) < 3$. $\square$

3.3 Finite p-groups of depth one mod-p cohomology

Until the end of Section 3.3, we assume that $p > 3$ is a prime number, that r is an arbitrary but fixed integer satisfying $1 < r < p-1$, and that G_r is the finite p-group described in (3.1). The aim of this section is to prove that $\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) = 1$. To show the result, we construct a non-trivial mod-p cohomology class in $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ that restricts trivially to the mod-p cohomologies of the centralizers of all rank 2 elementary abelian subgroups of G_r. Then, $\omega_d(G_r) = 1$ and Theorem 3.7 yields that $\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) = 1$.

We begin by constructing a cohomology class $\theta_r \in \mathrm{H}^3(G_r, \mathbb{F}_p)$ that is a cup product of a Yoneda 1-fold extension and a crossed 1-fold extension. We will use multiplicative notation and write C_p instead of $\mathbb{F}_p$ when seen as a subgroup rather than as a module.

We start by defining a cohomology class in $\mathrm{H}^1(G_r, \mathbb{F}_p) = \mathrm{Hom}(G_r, \mathbb{F}_p)$. To that aim, consider the homomorphism $\sigma^* \colon G_r \longrightarrow \mathbb{F}_p$. Following Example 2.7, the class $\sigma^* \in \mathrm{H}^1(G_r, \mathbb{F}_p)$ can be represented by the Yoneda extension

$$1 \longrightarrow C_p = \langle a_{r+2} \rangle \longrightarrow C_p \times C_p \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow 1,$$

where the action of G_r on $C_p \times C_p = \langle a_{r+1}, a_{r+2} \rangle$ is described by

$$\text{for } g \in G_r, \text{ set } a_{r+1}^g = a_{r+1} a_{r+2}^{\sigma^*(g)}, \quad a_{r+2}^g = a_{r+2}.$$

We continue by defining a crossed 1-fold extension $\eta_r \in \mathrm{H}^2(G_r, \mathbb{F}_p)$ as follows. Let

$$\lambda_r \colon \frac{N_0}{N_{r+1}} \times \frac{N_0}{N_{r+1}} \longrightarrow \frac{N_0}{N_{r+1}}$$

be the alternating bilinear map satisfying

$$\lambda_r(a_{r-1}, a_r) = a_{r+1} \text{ and } \lambda_r(a_i, a_j) = 0, \text{ for all } i < j \text{ with } (i,j) \neq (r-1, r).$$

Now, define $(N_0/N_{r+1})_{\lambda_r}$ to be the group with underlying set N_0/N_{r+1} and with group operation given by

$$x \cdot_{\lambda_r} y = xy\lambda_r(x,y)^{1/2}$$

for $x, y \in N_0/N_{r+1}$.

Remark 3.11. The group $(N_0/N_{r+1})_{\lambda_r}$ is an extension of the elementary abelian group N_0/N_r of rank r by C_p that, by following Section 1.2.4, can be identified with the extension class $a_{r-1}^* \smile a_r^* \in \mathrm{H}^2(N_0/N_r, \mathbb{F}_p)$. Consequently, we can write

$$(N_0/N_{r+1})_{\lambda_r} = \langle a_1, \dots, a_{r+1} \rangle$$

with the generators subject to the following relations:

- $a_i^p = [a_j, a_k] = [a_i, a_{r+1}] = 1$, for every $i = 1, \dots, r+1$, $j = 1, \dots, r$ and $k = 1, \dots, r-2$, and
- $[a_{r-1}, a_r] = a_{r+1}$.

Finally, define the p-group $\widehat{G}_r = C_p \ltimes (N_0/N_{r+1})_{\lambda_r}$ of size $|\widehat{G}_r| = p^{r+2}$ and exponent p. Let $\eta_r \in \mathrm{H}^2(G_r, \mathbb{F}_p)$ be the cohomology class represented by the crossed 1-fold extension

$$1 \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow \widehat{G}_r \longrightarrow G_r \longrightarrow 1. \tag{3.2}$$

Then, we define the cohomology class $\theta_r = \sigma^* \smile \eta_r \in \mathrm{H}^3(G_r, \mathbb{F}_p)$, which is represented by the crossed 2-fold extension

$$1 \longrightarrow C_p \longrightarrow C_p \times C_p \longrightarrow \widehat{G}_r \longrightarrow G_r \longrightarrow 1. \tag{3.3}$$

After constructing θ_r, we need to show that it is not trivial.

Proposition 3.12. *The cohomology class θ_r constructed in* (3.3) *is non-trivial.*

Proof. Assume by contradiction that $\theta_r = 0$. Then, by Proposition 2.14 there exists a group X such that the following diagram commutes:

$$\begin{tikzcd}
 & 1 \arrow[dr] & & & & 1 & \\
 & & C_p \times C_p \arrow[rr] \arrow[dr, "\mu"'] & & \widehat{G}_r \arrow[ur] \arrow[dr] & & \\
1 \arrow[r] & C_p \arrow[ur] \arrow[dr, equal] & & X \arrow[ur, "\nu"'] \arrow[dr] & & G_r \arrow[r] & 1 \\
 & & C_p \arrow[ur] \arrow[rr] & & G_r \arrow[ur, equal] \arrow[dr] & & \\
 & 1 \arrow[ur] & & & & 1. &
\end{tikzcd}$$

We have that $X = \langle \bar{\sigma}, \bar{a}_1, \ldots, \bar{a}_{r+2} \rangle$ with elements $\bar{\sigma}, \bar{a}_1, \ldots, \bar{a}_{r+1}, \bar{a}_{r+2} \in X$ that satisfy

$$\bar{a}_{r+2} = \mu(a_{r+2}), \quad \nu(\bar{\sigma}) = \sigma \quad \text{and} \quad \nu(\bar{a}_i) = a_i \text{ for all } i = 1, \ldots, r+1,$$

and we have that $\mathrm{Z}(X) = \langle \bar{a}_{r+2} \rangle$ and $\gamma_r(X) = \langle \bar{a}_r, \bar{a}_{r+1}, \bar{a}_{r+2} \rangle$. Consider the normal subgroup

$$Y = \langle \bar{a}_{r-1}, \bar{a}_r, \bar{a}_{r+1}, \bar{a}_{r+2} \rangle \trianglelefteq X,$$

which fits into the following commutative diagram:

$$\begin{tikzcd}
 & 1 \arrow[dr] & & & & 1 & \\
 & & \langle a_{r+1}, a_{r+2} \rangle \arrow[rr] \arrow[dr] & & \langle a_{r-1}, a_r, a_{r+1} \rangle \arrow[ur] \arrow[dr] & & \\
1 \arrow[r] & \langle a_{r+2} \rangle \arrow[ur] \arrow[dr, equal] & & Y \arrow[ur] \arrow[dr] & & \langle a_{r-1}, a_r \rangle \arrow[r] & 1 \\
 & & \langle a_{r+2} \rangle \arrow[ur] \arrow[rr] & & \langle a_{r-1}, a_r \rangle \arrow[ur, equal] \arrow[dr] & & \\
 & 1 \arrow[ur] & & & & 1 &
\end{tikzcd}$$

Then, we have that $\mathrm{Z}(Y) = \langle \bar{a}_{r+1}, \bar{a}_{r+2} \rangle$, and moreover,

$$\big[\bar{\sigma}, Y, \gamma_r(X)\big] = \big[\gamma_r(X), \gamma_r(X)\big] = 1 \quad \text{and} \quad \big[\gamma_r(X), \bar{\sigma}, Y\big] = \big[\,\mathrm{Z}(Y), Y\big] = 1.$$

Therefore, the Three Subgroup Lemma (see [Rob96, 5.1.10]) leads us to the conclusion that $\big[Y, \gamma_r(X), \bar{\sigma}\big] = 1$. Nevertheless, using that $[\bar{a}_{r-1}, \bar{a}_r] = \bar{a}_{r+1}$, we can show that

$$\big[Y, \gamma_r(X), \bar{\sigma}\big] = \big[\,\mathrm{Z}(Y), \bar{\sigma}\big] = \mathrm{Z}(X) \neq 1,$$

which gives a contradiction. Hence, we have that $\theta_r \neq 0$. $\square$

Now that we know that $\theta_r \neq 0$, we show that for every elementary abelian subgroup E of G_r of p-rank $\mathrm{rk}_p\, E = 2$, the image of θ_r via the restriction map,

$$\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \colon\ \mathrm{H}^3(G_r, \mathbb{F}_p) \longrightarrow \mathrm{H}^3\left(\,\mathrm{C}_{G_r}(E), \mathbb{F}_p\right),$$

is trivial, i.e., $\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \theta_r = 0$. This will imply that the cohomology class θ_r is not detected by $\mathcal{H}_2(G_r)$.

Proposition 3.13. *Let $E \leq G_r$ be an elementary abelian subgroup with $\mathrm{rk}_p\, E = 2$. Then, $\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \theta_r = 0$. Consequently, $\omega_d(G) = 1$.*

Proof. We can distinguish two types of elementary abelian subgroups $E \leq G_r$, either $E \leq \langle a_1, \ldots, a_r \rangle$ or $E \not\leq \langle a_1, \ldots, a_r \rangle$. Assume first that $E \leq \langle a_1, \ldots, a_r \rangle$. Then, we have that $\mathrm{C}_{G_r}(E) = \langle a_1, \ldots, a_r \rangle$ and $\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \sigma^* = 0$. Therefore, we obtain that

$$\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \theta_r = (\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \sigma^*) \smile (\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \eta_r) = 0.$$

Assume now that $E \not\leq \langle a_1, \ldots, a_r \rangle$. Then, $E = \langle b, a_r \rangle$ with $b = \sigma x$ for some $x \in \langle a_1, \ldots, a_{r-1} \rangle$, and $\mathrm{C}_{G_r}(E) = E$. Moreover, $\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \eta_r$ is represented by the extension that is obtained by taking the pullback of η_r via the inclusion $E \longrightarrow G_r$, as illustrated in the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \langle a_{r+1} \rangle \arrow[r] \arrow[d, equal] & \widehat{E} = \langle b, a_r, a_{r+1} \rangle \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \langle a_{r+1} \rangle \arrow[r] & \widehat{G}_r \arrow[r] & G_r \arrow[r] & 1
\end{tikzcd}$$

Observe that $\widehat{E} \cong C_p \ltimes (C_p \times C_p)$ is the extraspecial group of order p^3 and exponent p. Hence, $\mathrm{res}_{\mathrm{C}_{G_r}(E) \to G_r} \eta_r$ is represented by the extension

$$1 \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow \widehat{E} = C_p \ltimes (C_p \times C_p) \longrightarrow C_p \times C_p = \langle b, a_r \rangle \longrightarrow 1. \tag{3.4}$$

Consider $a_r^*, b^* \in \operatorname{Hom}(G_r, \mathbb{F}_p)$. As discussed in Section 1.2.4, the extension class of (3.4) coincides with the cup product $b^* \smile a_r^*$, and so $\operatorname{res}_{\mathrm{C}_{G_r}(E)\to G_r} \eta_r = b^* \smile a_r^*$. Consequently, we obtain that

$$\operatorname{res}_{\mathrm{C}_{G_r}(E)\to G_r} \theta_r = (\operatorname{res}_{\mathrm{C}_{G_r}(E)\to G_r} \sigma^*) \smile b^* \smile a_r^* = 0,$$

as the product of any three elements of degree one is trivial in $\mathrm{H}^3(E, \mathbb{F}_p)$. In particular, this means that $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ is not detected by $\mathcal{H}_2(G_r)$, and so $\omega_d(G_r) = 1$. □

We are finally ready to compute the depth of the mod-p cohomology ring of G_r for $1 \leq r \leq p-1$.

Theorem 3.14. *Let $p > 3$ be a prime number, let r be an integer with $1 < r < p-1$ and let G_r be given as in* (3.1)*. Then,* $\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) = \omega_d(G_r) = 1$.

Proof. By Proposition 3.10, we know that $1 \leq \operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p)$, and Proposition 3.13 yields that $\omega_d(G) = 1$. Then, by Theorem 3.7, we conclude that

$$\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) = \omega_d(G_r) = 1.$$

□

3.4 Conclusion and further work

We have managed to compute the depth of the cohomology rings $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ for $1 \leq r \leq p-2$ without first computing the rings themselves.

In particular, by Proposition 3.10 we obtain that, for $p \geq 3$ and $r \geq p-1$, the inequality $\operatorname{depth} \mathrm{H}^\bullet(G_r, \mathbb{F}_p) \leq 2$ holds. We observed that if we mimic the construction of the mod-p cohomology class θ_r in (3.3) for such p-groups, it is no longer true that its restriction in the mod-p cohomology of the centralizer of all elementary abelian subgroups of G_r of rank 2 vanishes. Moreover, for the $p = 3$ and $r = 2$ case, G_2 is the extraspecial 3-group of order 27 and exponent 3, and it is known that the depth of its mod-3 cohomology ring is 2 (compare [Lea92] and [Min01]). We believe that this phenomena will occur with more generality and we propose the following conjecture.

Conjecture 3.15. Let p be an odd prime, let $r \geq p-1$ be an integer, and let

$$G_r = C_p \ltimes \frac{N_0}{N_r}$$

be as in (3.1). Then $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ has depth 2.

The above conjecture is known to be true for the particular cases where $p = 3$ and $r = 2$ or $r = 3$. In these two cases the mod-p cohomology rings have been calculated using computational tools (see [Kin15]). Another argument supporting the validity of the conjecture is that, for a fixed prime p and $r \geq p - 1$, the groups G_r have isomorphic mod-p cohomology groups; not as rings, but as $\mathbb{F}_p$-modules (see [Gar18]).

4

Computing differentials in spectral sequences

One of the most powerful tools for computing the cohomology of a group is the Lyndon-Hochschild-Serre spectral sequence, which allows us to compute the cohomology of a group in terms of the cohomologies of its normal subgroups and quotients. The main difficulty in using spectral sequences lies in the fact that there are no general methods for computing the differentials that appear. In this chapter, we will state a theorem by Charlap and Vasquez [CV69] regarding the computation of the second differential of the Lyndon-Hochschild-Serre (LHS) spectral sequence associated to a split extension of finite groups. Afterwards, we will introduce a generalization of a result by Siegel [Sie96, Corollary 2] that can be used to compute the differentials appearing in the spectral sequence associated to a split extension of finite groups with cyclic quotient of prime power order. This chapter is based on [GG23].

4.1 A formula for the second differential of a split extension

Consider the extension of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

and its LHS spectral sequence $E = E(G)$. From the properties in Section 1.3.1, for each $k \geq 2$ it is only necessary to compute the differential of a finite number of elements of E_k, namely the generators of E_k as an R-algebra. However, there are no general methods for computing these differentials. In this section, we will state a theorem by Charlap and Vasquez [CV69] regarding the computation of the second differential of the LHS spectral sequence associated to a split extension of finite groups and then provide a generalization of [Sie96, Corollary 2] for split extensions of cyclic p-groups of any order.

We start by introducing the necessary definitions and notation to state the aforementioned result by Charlap and Vasquez. Let p denote an odd prime number and assume that $G = Q \ltimes N$ is a split extension of Q by the finite group N and let V be an $\mathbb{F}_pG$-module with trivial N-action.

Let $X_\bullet \longrightarrow \mathbb{F}_p$ be a projective $\mathbb{F}_pG$-resolution, let $\mathcal{B}_\bullet Q \longrightarrow \mathbb{F}_p$ be the $\mathbb{F}_pQ$-bar resolution from Section 1.2.1 and let $P_\bullet \longrightarrow \mathbb{F}_p$ be the minimal $\mathbb{F}_pN$-resolution. Note that $\mathrm{H}^\bullet(N, V) \cong \mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, V)$ due to the resolution being minimal, as noted in Section 1.1.3. Then, we can identify the zeroth and first pages of the LHS spectral sequence $E = E(G)$ associated to the split extension of Q by N following (1.8) as

$$\begin{aligned} E_0 &= \mathrm{Hom}_{\mathbb{F}_pQ}\big(\mathcal{B}_\bullet Q, \mathrm{Hom}_{\mathbb{F}_pN}(X_\bullet, V)\big), \\ E_1 &= \mathrm{Hom}_{\mathbb{F}_pQ}\big(\mathcal{B}_\bullet Q, \mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, V)\big). \end{aligned} \tag{4.1}$$

For each $g \in Q$, we write $P_\bullet^g$ for the $\mathbb{F}_pN$-complex with underlying $\mathbb{F}_p$-complex $P_\bullet$ and N-action given by

$$\text{for } \ h \in N \ \text{ and } \ x \in P_\bullet, \ \text{ set } \quad h \cdot x = h^{g^{-1}} x.$$

It is clear that $P_\bullet^g \longrightarrow \mathbb{F}_p$ is a projective $\mathbb{F}_pN$-resolution for every $g \in Q$. Also, recall that, for $i \geq 0$, we write $\mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, P_\bullet^g)_i$ to denote $\prod_{k=0}^{\infty} \mathrm{Hom}_{\mathbb{F}_pN}(P_k, P_{k+i}^g)$. Then, for each $g, g' \in Q$ the Comparison Theorem guarantees (see [Ben91, Theorem 2.4.2] and subsequent remark) the existence of maps $A(g) \in \mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, P_\bullet^g)_0$ and $U(g, g') \in \mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, P_\bullet^{gg'})_1$ satisfying the following conditions:

(i) $\partial \circ A(g) - A(g) \circ \partial = 0$ and $\varepsilon \circ A(g) - \varepsilon = 0$, and

(ii) $\partial \circ U(g, g') + U(g, g') \circ \partial = A(gg') - A(g) \circ A(g')$.

Theorem 4.1 ([Sie96, Theorem 1]). *Let A and U be as above. Let $r \geq 0, s \geq 1$ and suppose that $\zeta \in E_2^{r,s}$ is represented by $f \in \mathrm{Hom}_{\mathbb{F}_pQ}(\mathcal{B}_rQ, \mathrm{Hom}_{\mathbb{F}_pN}(P_s, V))$. Then, $\mathrm{d}_2(\zeta)$ is represented by $(-1)^r D_2(f)$, where*

$$D_2(f)[g_1|\cdots|g_{r+2}] = g_1g_2 \circ f[g_3|\cdots|g_{r+2}] \circ U(g_2^{-1}, g_1^{-1}).$$

4.2 Comparing resolutions for cyclic groups

Although the previous result is for a split extension of a general finite group Q, it requires the use of the $\mathbb{F}_pQ$-bar resolution of $\mathbb{F}_p$. In [Sie96], the previous result was extended for the minimal resolution of a cyclic group Q of size p. We generalise Siegel's result to the case where $Q = C_{p^n}$ is a cyclic p-group of size p^n, with $n \geq 1$.

In order to use Theorem 4.1 for this situation, we need explicit chain maps between the bar resolution $\mathcal{B}_\bullet Q \longrightarrow \mathbb{F}_p$ and the special resolution $\mathcal{S}_\bullet Q \longrightarrow \mathbb{F}_p$ that we defined in Example 1.17. We will abuse notation and denote the differentials of both resolutions in the same way, as the notation of the elements to which we are applying them makes it clear which differential we are using at each moment. Let us recall the definitions of the differentials of both resolutions. For $k \geq 0$, the differential of the special resolution $\partial_k \colon \mathcal{S}_kQ \longrightarrow \mathcal{S}_{k-1}Q$ is given by

$$\partial_k(e_k) = \begin{cases} (\sigma - 1)e_{k-1}, & \text{if } k \text{ is odd,} \\ T(\sigma)e_{k-1}, & \text{if } k \text{ is even,} \end{cases}$$

and the differential of the bar resolution $\partial_k \colon \mathcal{B}_kQ \longrightarrow \mathcal{B}_{k-1}Q$ is given by

$$\begin{aligned} \partial_k[\sigma^{i_1}|\cdots|\sigma^{i_k}] =& \sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_k}] + \sum_{j=1}^{k-1}(-1)^j[\sigma^{i_1}|\cdots|\sigma^{i_{j-1}}|\sigma^{i_j+i_{j+1}}|\sigma^{i_{j+2}}|\cdots|\sigma^{i_k}] \\ &+ (-1)^k[\sigma^{i_1}|\cdots|\sigma^{i_{k-1}}]. \end{aligned}$$

We will now define a pair of chain maps between the special and bar resolutions resolutions. These maps were originally defined by Siegel in [Sie96] for $n = 1$.

Proposition 4.2. *The $\mathbb{F}_pQ$-module homomorphism $\theta \colon \mathcal{B}_\bullet Q \longrightarrow \mathcal{S}_\bullet Q$ defined by*

$$\begin{aligned} \theta[] &= e_0, \\ \theta[\sigma^{i_1}] &= T_{i_1}(\sigma)e_1, \\ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \begin{cases} e_{2k}, & \text{if } i_{2j-1} + i_{2j} \geq p^n \text{ for all } 1 \leq j \leq k, \\ 0, & \text{otherwise,} \end{cases} \\ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] &= \begin{cases} \sum_{i=0}^{i_1-1} \sigma^i e_{2k+1} = T_{i_1}(\sigma)e_{2k+1}, & \text{if } i_{2j} + i_{2j+1} \geq p^n \text{ for all } 1 \leq j \leq k, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

for $k \geq 1$ and $0 \leq i_1, \ldots, i_{2k+1} < p^n$, is a chain map.

Proof. Let us show that θ is a chain map. To that aim, we need to prove, for all $k \geq 1$, the equalities

$$(\partial \circ \theta - \theta \circ \partial)(\mathcal{B}_{2k}Q) = (\partial \circ \theta - \theta \circ \partial)(\mathcal{B}_{2k+1}Q) = 0.$$

We will begin by showing that $(\partial \circ \theta - \theta \circ \partial)(\mathcal{B}_{2k}Q) = 0$. Observe that, for every $1 \leq j \leq k-1$ such that $i_{2j} + i_{2j+1} \geq p^n$ and $i_{2j+1} + i_{2j+2} \geq p^n$, we have that

$$(i_{2j} + i_{2j+1} \mod p^n) + i_{2j+2} = i_{2j} + i_{2j+1} - p^n + i_{2j+2} = i_{2j} + (i_{2j+1} + i_{2j+2} \mod p^n),$$

and thus

$$\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2j}+i_{2j+1}}|\sigma^{i_{2j+2}}|\cdots|\sigma^{i_{2k}}] = \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2j}}|\sigma^{i_{2j+1}+i_{2j+2}}|\cdots|\sigma^{i_{2k}}]. \tag{4.2}$$

Also note that, by definition, if there is some $1 \leq l \leq k-1$ such that $i_{2l} + i_{2l+1} < p^n$, then

$$\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l}}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k}}] = 0, \tag{4.3}$$

for every $2l+2 \leq j \leq 2k-1$. Likewise, the existence of some $1 \leq m \leq k$ such that $i_{2m-1} + i_{2m} < p^n$ implies that

$$\theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2m}}|\cdots|\sigma^{i_{2k}}] = 0, \tag{4.4}$$

for every $1 \leq j \leq 2m-3$.

In order to show that $\partial \circ \theta(\mathcal{B}_{2k}Q) = \theta \circ \partial(\mathcal{B}_{2k}Q)$, we need to distinguish four different cases:

(i) There are a smallest $1 \leq l \leq k-1$ such that $i_{2l} + i_{2l+1} < p^n$, and a largest $1 \leq m \leq k$ such that $i_{2m-1} + i_{2m} < p^n$.

On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] = 0.$$

On the other hand, for $m > l+1$, it is clear that

$$\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] = 0.$$

Furthermore, (4.3) and (4.4) yield that, for $2 \leq m \leq l+1$,

$$\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] = \sum_{j=2m-2}^{2l+1} (-1)^j \theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k}}]. \tag{4.5}$$

If $2 \leq m \leq l$, using (4.2), the expression (4.5) is reduced to

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \sum_{t=m-1}^{l} \Big(\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2t}+i_{2t+1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2t+1}+i_{2t+2}}|\cdots|\sigma^{i_{2k}}]\Big) \\
&= \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-2}+i_{2m-1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-1}+i_{2m}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l}+i_{2l+1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l+1}+i_{2l+2}}|\cdots|\sigma^{i_{2k}}] \\
&= 0 - T_{i_1}(\sigma)e_{2k-1} + T_{i_1}(\sigma)e_{2k-1} - 0 \\
&= 0.
\end{aligned}$$

Likewise, if $m = l + 1$ we obtain that

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-2}+i_{2m-1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-1}+i_{2m}}|\cdots|\sigma^{i_{2k}}] \\
&= T_{i_1}(\sigma)e_{2k-1} - T_{i_1}(\sigma)e_{2k-1} \\
&= 0.
\end{aligned}$$

Finally, if $m = 1$ then

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k}}]\big) + \sum_{j=1}^{2l+1}(-1)^j\theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k}}] \\
&= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k}}]\big) - \theta[\sigma^{i_1+i_2}|\cdots|\sigma^{i_{2k}}] \\
&\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l}+i_{2l+1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l+1}+i_{2l+2}}|\cdots|\sigma^{i_{2k}}] \\
&= \sigma^{i_1}T_{i_2}(\sigma)e_{2k-1} - T_{i_1+i_2}(\sigma)e_{2k-1} + T_{i_1}(\sigma)e_{2k-1} - 0 \\
&= 0.
\end{aligned}$$

(ii) There is a largest $1 \leq m \leq k$ such that $i_{2m-1} + i_{2m} < p^n$, but $i_{2j} + i_{2j+1} \geq p^n$ for every $1 \leq j \leq k-1$. On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] = 0.$$

On the other hand, for $m > 1$, using (4.2) and (4.4) we obtain that

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \sum_{j=2m-2}^{2k-1} (-1)^j \theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k}}] + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k-1}}] \\
&= \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-2}+i_{2m-1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-1}+i_{2m}}|\cdots|\sigma^{i_{2k}}] + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k-1}}] \\
&= 0 - T_{i_1}(\sigma)e_{2k-1} + T_{i_1}(\sigma)e_{2k-1} \\
&= 0.
\end{aligned}$$

Likewise, for $m = 1$, (4.2) and (4.4) yield that

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k}}]\big) - \theta[\sigma^{i_1+i_2}|\cdots|\sigma^{i_{2k}}] + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k-1}}] \\
&= \sigma^{i_1}T_{i_2}(\sigma)e_{2k-1} - T_{i_1+i_2}(\sigma)e_{2k-1} + T_{i_1}(\sigma)e_{2k-1} - 0 \\
&= 0.
\end{aligned}$$

(iii) There is a smallest $1 \le l \le k-1$ such that $i_{2l} + i_{2l+1} < p^n$, but $i_{2j-1} + i_{2j} \ge p^n$ for every $1 \le j \le k$. On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] = T(\sigma)e_{2k-1}.$$

On the other hand, (4.2) and (4.3) yield that

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k}}]\big) + \sum_{j=1}^{2l+1}(-1)^j \theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k}}] \\
&= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k}}]\big) - \theta[\sigma^{i_1+i_2}|\cdots|\sigma^{i_{2k}}] \\
&\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l}+i_{2l+1}}|\cdots|\sigma^{i_{2k}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l+1}+i_{2l+2}}|\cdots|\sigma^{i_{2k}}] \\
&= \sigma^{i_1}T_{i_2}(\sigma)e_{2k-1} - T_{i_1+i_2-p^n}(\sigma)e_{2k-1} + T_{i_1}(\sigma)e_{2k-1} - 0 \\
&= T(\sigma)e_{2k-1}.
\end{aligned}$$

(iv) For every $1 \le j \le k$, we have that $i_{2j-1} + i_{2j} \ge p^n$, and for every $1 \le j' \le k-1$, we have that $i_{2j'} + i_{2j'+1} \ge p^n$. On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] = T(\sigma)e_{2k-1}.$$

On the other hand, (4.2) yields that

$$\begin{aligned}\theta\circ\partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] &= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k}}]\big)\\ &\qquad+\sum_{j=1}^{2k}(-1)^j\theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k}}]\\ &\qquad+\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k-1}}]\\ &= \sigma^{i_1}T_{i_2}(\sigma)e_{2k-1}-T_{i_1+i_2-p^n}(\sigma)e_{2k-1}+0+T_{i_1}(\sigma)e_{2k-1}\\ &= T(\sigma)e_{2k-1}.\end{aligned}$$

Now, we will see that $(\partial\circ\theta-\theta\circ\partial)(\mathcal{B}_{2k+1}Q)=0$. Observe that, for every $1\le j\le k$ such that $i_{2j}+i_{2j+1}\ge p^n$ and $i_{2j-1}+i_{2j}\ge p^n$, we have that

$$(i_{2j-1}+i_{2j}\mod p^n)+i_{2j+1}=i_{2j-1}+i_{2j}-p^n+i_{2j+1}=i_{2j-1}+(i_{2j}+i_{2j+1}\mod p^n),$$

and thus

$$\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2j-1}+i_{2j}}|\sigma^{i_{2j+1}}|\cdots|\sigma^{i_{2k+1}}]=\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2j-1}}|\sigma^{i_{2j}+i_{2j+1}}|\cdots|\sigma^{i_{2k+1}}]. \tag{4.6}$$

Also note that, if there is some $1\le l\le k$ such that $i_{2l}+i_{2l+1}<p^n$, then

$$\theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2l}}|\cdots|\sigma^{i_{2k+1}}]=0, \tag{4.7}$$

for every $1\le j\le 2l-2$. Likewise, the existence of some $1\le m\le k$ such that $i_{2m-1}+i_{2m}<p^n$ implies that

$$\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m}}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k+1}}]=0, \tag{4.8}$$

for every $2m+1\le j\le 2k$.

In order to show that $\partial\circ\theta(\mathcal{B}_{2k+1}Q)=\theta\circ\partial(\mathcal{B}_{2k+1}Q)$, we need to distinguish four different cases:

(i) There are a largest $1\le l\le k$ such that $i_{2l}+i_{2l+1}<p^n$, and a smallest $1\le m\le k$ such that $i_{2m-1}+i_{2m}<p^n$. On the one hand, we can easily see that

$$\partial\circ\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}]=0.$$

On the other hand, for $m<l$ it is clear that

$$\theta\circ\partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}]=0.$$

Furthermore, (4.7) and (4.8) yield that, for $m \geq l$,

$$\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] = \sum_{j=2l-1}^{2m} (-1)^j \theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k+1}}]. \tag{4.9}$$

If $m > l$, using using (4.6), the expression (4.9) is reduced to

$$\begin{aligned}\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] &= -\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l-1}+i_{2l}}|\cdots|\sigma^{i_{2k+1}}] \\ &\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l}+i_{2l+1}}|\cdots|\sigma^{i_{2k+1}}] \\ &\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-1}+i_{2m}}|\cdots|\sigma^{i_{2k+1}}] \\ &\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m}+i_{2m+1}}|\cdots|\sigma^{i_{2k+1}}] \\ &= 0 + e_{2k} - e_{2k} + 0 \\ &= 0.\end{aligned}$$

Likewise, if $m = l$ we obtain that

$$\begin{aligned}\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] &= -\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-1}+i_{2m}}|\cdots|\sigma^{i_{2k+1}}] \\ &\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m}+i_{2m+1}}|\cdots|\sigma^{i_{2k+1}}] \\ &= 0.\end{aligned}$$

(ii) There is a largest $1 \leq l \leq k$ such that $i_{2l} + i_{2l+1} < p^n$, but $i_{2j-1} + i_{2j} \geq p^n$ for every $1 \leq j \leq k$. On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] = 0.$$

On the other hand, (4.6) and (4.7) yield that

$$\begin{aligned}\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] &= \sum_{j=2l-1}^{2k} (-1)^j \theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k+1}}] - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] \\ &= -\theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l-1}+i_{2l}}|\cdots|\sigma^{i_{2k+1}}] \\ &\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2l}+i_{2l+1}}|\cdots|\sigma^{i_{2k+1}}] - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] \\ &= 0 + e_{2k} - e_{2k} \\ &= 0.\end{aligned}$$

(iii) There is a smallest $1 \leq m \leq k$ such that $i_{2m-1} + i_{2m} < p^n$, but $i_{2j} + i_{2j+1} \geq p^n$ for every $1 \leq j \leq k$. On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] = (\sigma^{i_1} - 1)e_{2k}.$$

On the other hand, (4.6) and (4.8) yield that

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] &= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k+1}}]\big) \\
&\qquad + \sum_{j=1}^{2m}(-1)^j\theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k+1}}] \\
&= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k+1}}]\big) - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m-1}+i_{2m}}|\cdots|\sigma^{i_{2k+1}}] \\
&\qquad + \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2m}+i_{2m+1}}|\cdots|\sigma^{i_{2k+1}}] \\
&= \sigma^{i_1}e_{2k} - e_{2k} + 0 \\
&= (\sigma^{i_1}-1)e_{2k}.
\end{aligned}$$

(iv) For every $1 \leq j \leq k$, we have that $i_{2j-1}+i_{2j} \geq p^n$ and $i_{2j}+i_{2j+1} \geq p^n$. On the one hand, we can easily see that

$$\partial \circ \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] = (\sigma^{i_1}-1)e_{2k}.$$

On the other hand, (4.6) yields that

$$\begin{aligned}
\theta \circ \partial[\sigma^{i_1}|\cdots|\sigma^{i_{2k+1}}] &= \theta\big(\sigma^{i_1}[\sigma^{i_2}|\cdots|\sigma^{i_{2k+1}}]\big) \\
&\qquad + \sum_{j=1}^{2k}(-1)^j\theta[\sigma^{i_1}|\cdots|\sigma^{i_j+i_{j+1}}|\cdots|\sigma^{i_{2k+1}}] \\
&\qquad - \theta[\sigma^{i_1}|\cdots|\sigma^{i_{2k}}] \\
&= \sigma^{i_1}e_{2k} + 0 - e_{2k} \\
&= (\sigma^{i_1}-1)e_{2k}.
\end{aligned}$$

Finally, we can easily check that $(\varepsilon \circ \theta - \varepsilon)(e_0) = 0$ and $(\partial \circ \theta - \theta \circ \partial)(e_1) = 0$, allowing us to conclude that θ is a chain map. □

Proposition 4.3. *The $\mathbb{F}_pQ$-module homomorphism $\eta\colon \mathcal{S}_\bullet Q \longrightarrow \mathcal{B}_\bullet Q$ defined by*

$$\begin{aligned}
\eta(e_0) &= [], \\
\eta(e_1) &= [\sigma], \\
\eta(e_{2k}) &= \sum_{0\leq i_1,\dots,i_k<p^n}[\sigma^{i_1}|\sigma|\cdots|\sigma^{i_k}|\sigma], \\
\eta(e_{2k+1}) &= \sum_{0\leq i_1,\dots,i_k<p^n}[\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_k}|\sigma],
\end{aligned}$$

for $k \geq 1$, is a chain map.

Proof. Let us show that η is a chain map. To that aim, we need to prove, for all $k \geq 1$, the equalities

$$(\partial \circ \eta - \eta \circ \partial)(\mathcal{S}_{2k}Q) = (\partial \circ \eta - \eta \circ \partial)(\mathcal{S}_{2k+1}Q) = 0.$$

We will first show that $(\partial \circ \eta - \eta \circ \partial)(e_{2k}) = 0$ for $k \geq 1$. On the one hand, because the initial sum covers all possible exponents $0 \leq i_1, \ldots, i_k < p^n$, it is easy to see that

$$\sum_{0 \leq i_1, \ldots, i_k < p^n} \sum_{j=1}^{k-1} [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_j}|\sigma^{i_{j+1}+1}|\sigma| \cdots |\sigma^{i_k}|\sigma] = \sum_{0 \leq i_1, \ldots, i_k < p^n} \sum_{j=1}^{k-1} [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_j+1}|\sigma^{i_{j+1}}|\sigma| \cdots |\sigma^{i_k}|\sigma],$$
$$\sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_k}] = \sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_k+1}],$$

and so we have that

$$\begin{aligned}
\partial \circ \eta(e_{2k}) &= \partial\Bigg(\sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma^{i_1}|\sigma| \cdots |\sigma^{i_k}|\sigma] \Bigg) \\
&= \sum_{0 \leq i_1, \ldots, i_k < p^n} \Bigg(\sigma^{i_1}[\sigma|\sigma^{i_2}| \cdots |\sigma^{i_k}|\sigma] \\
&\qquad - \sum_{j=1}^{k-1} [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_j+1}|\sigma^{i_{j+1}}|\sigma| \cdots |\sigma^{i_k}|\sigma] - [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_k+1}] \\
&\qquad + \sum_{j=1}^{k-1} [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_j}|\sigma^{i_{j+1}+1}|\sigma| \cdots |\sigma^{i_k}|\sigma] + [\sigma^{i_1}| \cdots |\sigma|\sigma^{i_k}] \Bigg) \\
&= \sum_{0 \leq i_1, \ldots, i_k < p^n} \sigma^{i_1}[\sigma|\sigma^{i_2}| \cdots |\sigma^{i_k}|\sigma].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\eta \circ \partial(e_{2k}) &= \eta\Bigg(\sum_{i=0}^{p^n-1} \sigma^i e_{2k-1} \Bigg) \\
&= \sum_{0 \leq i, i_1, \ldots, i_{k-1} < p^n} \sigma^i [\sigma|\sigma^{i_1}| \cdots |\sigma|\sigma^{i_{k-1}}|\sigma].
\end{aligned}$$

Therefore, $(\partial \circ \eta - \eta \circ \partial)(e_{2k}) = 0$.

Now, we will show that $(\partial \circ \eta - \eta \circ \partial)(e_{2k+1}) = 0$ for $k \geq 1$. On the one hand, because the initial sum covers all possible exponents $0 \leq i_1, \ldots, i_k < p^n$, it is easy to see that

$$\sum_{0\le i_1,\dots,i_k<p^n}\sum_{j=1}^{k-1}[\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_j+1}|\sigma^{i_{j+1}}|\sigma|\cdots|\sigma^{i_k}] = \sum_{0\le i_1,\dots,i_k<p^n}\sum_{j=1}^{k-1}[\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_j}|\sigma^{i_{j+1}+1}|\sigma|\cdots|\sigma^{i_k}],$$

$$\sum_{0\le i_1,\dots,i_k<p^n}[\sigma^{i_1}|\cdots|\sigma|\sigma^{i_k}|\sigma] = \sum_{0\le i_1,\dots,i_k<p^n}[\sigma^{i_1}|\cdots|\sigma|\sigma^{i_k+1}|\sigma],$$

and so we have that

$$\begin{aligned}
\partial\circ\eta(e_{2k+1}) &= \partial\Bigg(\sum_{0\le i_1,\dots,i_k<p^n}[\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_k}|\sigma]\Bigg)\\
&= \sum_{0\le i_1,\dots,i_k<p^n}\Bigg(\sigma[\sigma^{i_1}|\sigma|\sigma^{i_2}|\cdots|\sigma^{i_k}|\sigma] - [\sigma^{i_1+1}|\sigma|\sigma^{i_2}|\cdots|\sigma^{i_k}|\sigma]\\
&\qquad - \sum_{j=2}^{k}[\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_{j-1}}|\sigma^{i_j+1}|\sigma|\cdots|\sigma|\sigma^{i_k}]\\
&\qquad + \sum_{j=1}^{k-1}[\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_j+1}|\sigma^{i_{j+1}}|\sigma|\cdots|\sigma|\sigma^{i_k}]\\
&\qquad + [\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_k+1}] - [\sigma|\sigma^{i_1}|\cdots|\sigma|\sigma^{i_k}]\Bigg)\\
&= \sum_{0\le i_1,\dots,i_k<p^n}\big(\sigma[\sigma^{i_1}|\sigma|\sigma^{i_2}|\cdots|\sigma^{i_k}|\sigma] - [\sigma^{i_1+1}|\sigma|\sigma^{i_2}|\cdots|\sigma^{i_k}|\sigma]\big)\\
&= \sum_{0\le i_1,\dots,i_k<p^n}(\sigma-1)[\sigma^{i_1}|\sigma|\sigma^{i_2}|\cdots|\sigma^{i_k}|\sigma].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\eta\circ\partial(e_{2k+1}) &= \eta\big((\sigma-1)e_{2k}\big)\\
&= (\sigma-1)\sum_{0\le i_1,\dots,i_k<p^n}\sigma^{i}[\sigma^{i_1}|\cdots|\sigma|\sigma^{i_{k-1}}|\sigma].
\end{aligned}$$

Therefore, $(\partial\circ\eta-\eta\circ\partial)(e_{2k+1})=0$.

Finally, we can easily check that $(\varepsilon\circ\eta-\varepsilon)(e_0)=0$ and $(\partial\circ\eta-\eta\circ\partial)(e_1)=0$, allowing us to conclude that η is a chain map. □

4.3 Application to split extensions with cyclic quotient

Let $Q=C_{p^n}$ with $n\ge 1$ an integer, and $G=Q\ltimes N$ be a semidirect product of Q by a finite group N. Under these hypotheses, if we consider the special $\mathbb{F}_pQ$-resolution

$\mathcal{S}_\bullet Q \longrightarrow \mathbb{F}_p$, the first page of the LHS spectral sequence of the extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

can be identified with

$$E_1 = \mathrm{Hom}_{\mathbb{F}_p Q}\big(\mathcal{S}_\bullet Q, \mathrm{Hom}_{\mathbb{F}_p N}(P_\bullet, V)\big).$$

Furthermore, since $\mathcal{S}_r Q \cong \mathbb{F}_p Q$, we have, for $r, s \geq 0$, that

$$E_1^{r,s} \cong \mathrm{Hom}_{\mathbb{F}_p N}(P_s, V).$$

Using the chain maps θ and η in Proposition 4.2 and Proposition 4.3, we can prove the following generalization of a theorem by Siegel [Sie96, Corollary 2].

Theorem 4.4. *Let $\alpha\colon P_\bullet \longrightarrow P_\bullet^{\sigma^{-1}}$ be an $\mathbb{F}_p N$-chain map commuting with the augmentation, and $\tau \in \mathrm{Hom}_{\mathbb{F}_p N}(P_\bullet, P_\bullet)_1$ such that $\partial\tau + \tau\partial = 1 - \alpha^{p^n}$. Suppose that $\zeta \in E_2^{r,s}$ with $r \geq 0, s \geq 1$ is represented by $f \in \mathrm{Hom}_{\mathbb{F}_p N}(P_s, V)$. Then, $\mathrm{d}_2(\zeta)$ is represented by $(-1)^r f \circ \tau$.*

Proof. The proof of this result can be done by following that of [Sie96, Corollary 2], using the chain maps from Proposition 4.2 and Proposition 4.3, and writing p^n instead of p where appropriate. $\square$

5

A spectral sequence for Heisenberg groups

Computing the cohomology ring of a group explicitly is generally a daunting task, and it is for this reason that not many examples of computations of such rings for individual groups, let alone families of groups, exist in the literature. It is interesting to know which types of graded rings can occur as cohomology rings of finite groups and how many of them are distinct (compare [Car05], [DGG17], [Sym21]). In this chapter, we will compute the Lyndon-Hochschild-Serre (LHS) spectral sequence of a family of finite Heisenberg groups of prime power order, up to the infinity page. We will begin by computing the second page of the spectral sequence and its structure as an algebra, before putting the results from the last chapter to use in the computation of the second differential. Afterwards, we will determine the third page and show that it is at this point that the spectral sequence collapses. In so doing, we provide one of the first infinite families of groups of prime power order whose associated LHS spectral sequences collapse in the same page and are isomorphic. Finally, we will compute the Poincaré series of the cohomology rings. This chapter is based on [GG23].

5.1 Heisenberg groups of prime power order

Throughout, let $n \geq 1$ be an integer. We write

$$G = \mathrm{Heis}(p^n) = C_{p^n} \ltimes (C_{p^n} \times C_{p^n})$$

for the Heisenberg group modulo p^n. Note that G is just a finite quotient of the infinite Heisenberg group $\widehat{G} = \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z})$. We set $N = C_{p^n} \times C_{p^n} = \langle a, b\rangle$ and $Q = C_{p^n} = \langle \sigma \rangle$. Then, we have that

$$G = Q \ltimes N = \langle \sigma, a, b \mid \sigma^{p^n} = a^{p^n} = b^{p^n} = [\sigma, b] = [a, b] = 1,\ [\sigma, a] = b\rangle.$$

Note that the element $\sigma \in Q$ acts (on the right) on N via $a^\sigma = ab$ and $b^\sigma = b$.

The cohomology ring of N with coefficients in $\mathbb{F}_p$ is

$$\mathrm{H}^\bullet(N, \mathbb{F}_p) = \Lambda(x_1, y_1) \otimes_{\mathbb{F}_p} \mathbb{F}_p[x_2, y_2] = \Lambda(x_1, y_1) \otimes \mathbb{F}_p[x_2, y_2],$$

with $|x_i| = |y_i| = i$, for $i = 1, 2$. We can take

$$\begin{aligned} x_1 &= a^*, & y_1 &= b^*, \\ x_2 &= \beta_n(x_1), & y_2 &= \beta_n(y_1). \end{aligned}$$

The (left) action of σ on $\mathrm{H}^\bullet(N, \mathbb{F}_p)$ can be shown to be given by

$$\begin{aligned} \sigma \cdot x_1 &= x_1, & \sigma \cdot y_1 &= x_1 + y_1, \\ \sigma \cdot x_2 &= x_2, & \sigma \cdot y_2 &= x_2 + y_2. \end{aligned} \tag{5.1}$$

We will denote by E the LHS spectral sequence associated to the split extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1, \tag{5.2}$$

which takes the form

$$E_2^{r,s} = \mathrm{H}^r\left(Q, \mathrm{H}^s(N, \mathbb{F}_p)\right) \Longrightarrow \mathrm{H}^{r+s}(G, \mathbb{F}_p),$$

for $r, s \geq 0$.

5.2 Description of the second page of the spectral sequence

We follow the notation in the previous section and unless otherwise stated, we additionally assume until the end of the chapter that $n \geq 2$. We compute the cohomology groups $E_2^{r,s}$ as in (1.2). Take $T(\sigma) = \sum_{i=0}^{p^n-1} \sigma^i \in \mathbb{F}_p N$ and, as $n \geq 2$, it can be readily checked that, for all $\varphi \in \mathrm{H}^\bullet(N, \mathbb{F}_p)$, the identities $\sigma^p \cdot \varphi = \varphi$ and $T(\sigma) \cdot \varphi = 0$ hold. Indeed, that $\sigma^p \cdot \varphi = \varphi$ follows from the description of the action of σ in (5.1), and this can in turn be used to compute

$$T(\sigma) \cdot \varphi = \sum_{k=0}^{p^{n-1}-1} \sum_{i=0}^{p-1} \sigma^{i+kp} \cdot \varphi = \sum_{k=0}^{p^{n-1}-1} \sum_{i=0}^{p-1} \sigma^i \cdot \varphi = p^{n-1}\frac{(p^{n-1}-1)}{2} \sum_{i=0}^{p-1} \sigma^i \cdot \varphi = 0.$$

The second page of the spectral sequence then takes the following form:

$$E_2^{r,s} = \mathrm{H}^r\left(Q, \mathrm{H}^s(N, \mathbb{F}_p)\right) \cong \begin{cases} \mathrm{H}^s(N, \mathbb{F}_p)^Q, & \text{if } r \text{ is even,} \\ \dfrac{\mathrm{H}^s(N, \mathbb{F}_p)}{(\sigma - 1)\cdot \mathrm{H}^s(N, \mathbb{F}_p)}, & \text{if } r \text{ is odd.} \end{cases}$$

Let now

$$z_{2p} = \prod_{i=0}^{p-1} \sigma^i \cdot y_2 = \prod_{i=0}^{p-1} (ix_2 + y_2) \in \mathrm{H}^{2p}(N, \mathbb{F}_p),$$

and observe that the element z_{2p} is invariant under the action of σ. Furthermore, if we write

$$W = \Lambda[x_1, y_1] \otimes \langle x_2^i y_2^j \mid i \geq 0, \quad 0 \leq j < p\rangle,$$

$$D_2^{r,\bullet} = \begin{cases} W^Q, & \text{if } r \text{ is even,} \\ \dfrac{W}{(\sigma - 1)\cdot W}, & \text{if } r \text{ is odd,} \end{cases}$$

we have that $\mathrm{H}^\bullet(N, \mathbb{F}_p) = \mathbb{F}_p[z_{2p}] \otimes W$, and so

$$E_2^{r,\bullet} = \mathbb{F}_p[z_{2p}] \otimes D_2^{r,\bullet} = \begin{cases} \mathbb{F}_p[z_{2p}] \otimes W^Q, & \text{if } r \text{ is even,} \\ \mathbb{F}_p[z_{2p}] \otimes \dfrac{W}{(\sigma - 1)\cdot W}, & \text{if } r \text{ is odd.} \end{cases} \tag{5.3}$$

Consequently, it suffices to study the structure of D_2 so that the structure of E_2 may be determined.

5.2.1 Structure as a vector space

The first step will be determining a basis of the $\mathbb{F}_p$-vector space $D_2^{r,s}$ for each $r, s \geq 0$.

Proposition 5.1.

(i) For $s \geq 0$, the basis elements of $(W^s)^Q$ are the following:

s	$(W^s)^Q$
0	1
1	x_1
$2i \geq 2$	$x_2^i, \quad x_1 y_1 x_2^{i-1}$
$2i+1 \geq 3$	$x_1 x_2^i, \quad (x_1 y_2 - y_1 x_2) x_2^{i-1}$

(ii) For $s \geq 1$, the basis elements of $(\sigma - 1)W^s$ are the following:

s	$(\sigma - 1)W^s$
1	x_1
$2i \geq 2$	$x_2^j y_2^k$, with $j \geq 1$, $\quad 0 \leq k \leq p-2$, $\quad j+k=i$
	$x_1 y_1 x_2^j y_2^k$, with $j \geq 1$, $\quad 0 \leq k \leq p-2$, $\quad j+k+1=i$
$2i+1 \geq 3$	$x_1 x_2^j y_2^k$, with $j \geq 1$, $\quad 0 \leq k \leq p-2$, $\quad j+k=i$
	$y_1 x_2^j y_2^k$, with $j \geq 2$, $\quad 0 \leq k \leq p-3$, $\quad j+k=i$
	$x_1 y_2^k + k y_1 x_2 y_2^{k-1}$, with $1 \leq k \leq p-1$, $\quad k=i$
	$x_1 x_2^j y_2^{p-1} - y_1 x_2^{j+1} y_2^{p-2}$, with $j \geq 0$, $\quad j+p-1=i$

(iii) For $s \geq 0$, the basis elements of $W^s/(\sigma - 1)W^s$ are the following:

s	$W^s/(\sigma - 1)W^s$
0	$\bar{1}$
1	$\bar{y}_1$
$2i \geq 2$	$\overline{(x_1 y_1)^\varepsilon y_2^k}$, with $\varepsilon = 0,1$, $\quad 0 \leq k \leq p-1$, $\quad \varepsilon + k = i$
	$\overline{(x_1 y_1)^\varepsilon x_2^j y_2^{p-1}}$, with $\varepsilon = 0,1$, $\quad j \geq 1$, $\quad \varepsilon + j + p - 1 = i$
$2i+1 \geq 3$	$\overline{x_1^\varepsilon y_1^{1-\varepsilon} y_2^k}$, with $\varepsilon = 0,1$, $\quad 0 \leq k \leq p-1$, $\quad k = i$
	$\overline{x_1^\varepsilon y_1^{1-\varepsilon} x_2^j y_2^{p-1}}$, with $\varepsilon = 0,1$, $\quad j \geq 1$, $\quad j+p-1=i$

Proof. The proof follows verbatim that of [Sie96, Proposition 3]. $\square$

Using this result, we can write a table with the basis elements of $D_2^{r,s}$:

$2i+1 \geq 2p+1$	$x_1x_2^i,\ (x_1y_2 - y_1x_2)x_2^{i-1}$	$\overline{x_1x_2^{i-p+1}y_2^{p-1}},\ \overline{y_1x_2^{i-p+1}y_2^{p-1}}$
$2i \geq 2p$	$x_2^i,\ x_1y_1x_2^{i-1}$	$\overline{x_1y_1x_2^{i-p}y_2^{p-1}},\ \overline{x_2^{i-p+1}y_2^{p-1}}$
$2i+1 \leq 2p-1$	$x_1x_2^i,\ (x_1y_2 - y_1x_2)x_2^{i-1}$	$\overline{x_1y_2^i},\ \overline{y_1y_2^i}$
$2i \leq 2p-2$	$x_2^i,\ x_1y_1x_2^{i-1}$	$\overline{x_1y_1y_2^{i-1}},\ \overline{y_2^i}$
1	x_1	$\overline{y_1}$
0	1	$\bar{1}$
s	$D_2^{2j,s} = (W^s)^Q$	$D_2^{2j+1,s} = W^s/(\sigma-1)W^s$

Figure 5.1: Basis of $D_2^{r,s}$ for $r, s \geq 0$, with $j \geq 0$.

5.2.2 Structure as an algebra

Following Example 1.17 and Section 1.3.1 (see also [Sie96, Section 4]), we describe the structure of E_2 as a bigraded $\mathbb{F}_p$-algebra. For $r, s, r', s' \geq 0$, let $\varphi \in \mathrm{H}^s(N, \mathbb{F}_p)$ and $\varphi' \in \mathrm{H}^{s'}(N, \mathbb{F}_p)$ represent the elements $\bar{\varphi} \in E_2^{r,s}$ and $\bar{\varphi}' \in E_2^{r',s'}$, respectively. Then, their product in E_2 is the element $\bar{\varphi}\bar{\varphi}' \in E_2^{r+r',s+s'}$ with

$$(-1)^{r's}\bar{\varphi}\bar{\varphi}' = \begin{cases} \overline{\varphi \smile \varphi'}, & \text{if } r \text{ or } r' \text{ is even,} \\ \displaystyle\sum_{0 \leq i < j < p^n} \overline{\sigma^i \cdot \varphi \smile \sigma^j \cdot \varphi'}, & \text{if } r \text{ and } r' \text{ are odd.} \end{cases}$$

Lemma 5.2. *Let $\bar{\varphi} \in E_2^{r,s}$ and $\bar{\varphi}' \in E_2^{r',s'}$ be as above with r and r' odd. Then, $\bar{\varphi}\bar{\varphi}' = 0$.*

Proof. For simplicity, write

$$T_0(\sigma) = 0, \quad \text{and for } k \geq 1,\ T_k(\sigma) = \sum_{i=0}^{k-1} \sigma^i.$$

In particular, we have that $T(\sigma) = T_{p^n}(\sigma)$. Furthermore, note that, for $0 \leq i \leq p-1$ and $k \geq 1$, using that $\sigma^p \cdot \varphi = \varphi$ we obtain that

$$\sigma^{i+kp} \cdot \varphi = \sigma^i \cdot \varphi \ \text{ and } \ T_{i+kp}(\sigma) \cdot \varphi = T_i(\sigma) \cdot \varphi + kT_p(\sigma) \cdot \varphi.$$

Then, we compute

$$
\begin{aligned}
\sum_{0\le i<j<p^n} \sigma^i\cdot\varphi\smile\sigma^j\cdot\varphi' &= \sum_{j=0}^{p^n-1} T_j(\sigma)\cdot\varphi\smile\sigma^j\cdot\varphi' \\
&= \sum_{j=0}^{p-1}\left(\sum_{k=0}^{p^{n-1}-1} T_{j+kp}(\sigma)\right)\cdot\varphi\smile\sigma^j\cdot\varphi' \\
&= \sum_{j=0}^{p-1}\left(\sum_{k=0}^{p^{n-1}-1} T_j(\sigma)+kT_p(\sigma)\right)\cdot\varphi\smile\sigma^j\cdot\varphi' \\
&= \sum_{j=0}^{p-1}\left(p^{n-1}T_j(\sigma)+p^{n-1}\frac{(p^{n-1}-1)}{2}T_p(\sigma)\right)\cdot\varphi\smile\sigma^j\cdot\varphi' = 0.
\end{aligned}
$$

As a consequence, $\bar{\varphi}\bar{\varphi}' = 0$ in E_2. □

In what follows, we fix the following notation:

$$
\begin{gathered}
\lambda_1 = x_1 \in E_2^{0,1}, \qquad \lambda_2 = x_2 \in E_2^{0,2}, \\
\nu_2 = x_1y_1 \in E_2^{0,2}, \qquad \nu_3 = x_1y_2 - y_1x_2 \in E_2^{0,3}, \qquad \nu_{2p} = z_{2p} \in E_2^{0,2p}, \\
\gamma_1 = \bar{1} \in E_2^{1,0}, \qquad \gamma_2 = \bar{1} \in E_2^{2,0}, \\
\text{for } 1\le i\le p, \qquad \mu_{2i} = \overline{y_1y_2^{i-1}} \in E_2^{1,2i-1}, \\
\text{for } 1\le i\le p-1, \qquad \mu_{2i+1} = \overline{y_2^i} \in E_2^{1,2i}.
\end{gathered}
\tag{5.4}
$$

Proposition 5.3. *Multiplication by the elements $\nu_{2p}, \gamma_2, \lambda_2$ induces vector space homomorphisms as follows:*

(i) Multiplication $\cdot\nu_{2p}\colon E_2^{r,s} \longrightarrow E_2^{r,s+2p}$ is injective for all $r, s \ge 0$.

(ii) Multiplication $\cdot\gamma_2\colon E_2^{r,s} \longrightarrow E_2^{r+2,s}$ is an isomorphism for all $r, s \ge 0$.

(iii) Multiplication $\cdot\lambda_2\colon D_2^{r,s} \longrightarrow D_2^{r,s+2}$ is an isomorphism for all $s \ge 2p-1$.

Proof. The first claim follows from Equation (5.3). Using the identifications in Proposition 5.1, note that multiplication by $\gamma_2 = \bar{1}$ is simply the identity homomorphism and so, the second item holds. The last statement is clear by the description of the bases in Proposition 5.1 □

Using the previous results, we can deduce the structure of the second page E_2 as a graded-commutative $\mathbb{F}_p$-algebra.

Theorem 5.4.

(i) The graded-commutative $\mathbb{F}_p$-algebra structure of the zeroth column is given by the following tensor product:

$$E_2^{0,\bullet} = \mathbb{F}_p[\nu_{2p}] \otimes \frac{\mathbb{F}_p[\lambda_1, \lambda_2, \nu_2, \nu_3]}{(\nu_2^2, \lambda_1\nu_2, \nu_2\nu_3, \lambda_1\nu_3 + \lambda_2\nu_2)}.$$

(ii) For $r = 0, 1$ and $s \geq 0$, the basis elements of $D_2^{r,s}$ are the following:

$2i+1 \geq 2p+1$	$\lambda_1\lambda_2^i, \quad \lambda_2^{i-1}\nu_3$	$\lambda_2^{i-p+1}\mu_{2p}, \quad \lambda_1\lambda_2^{i-p+1}\mu_{2p-1}$
$2i \geq 2p$	$\lambda_2^i, \quad \lambda_2^{i-1}\nu_2$	$\lambda_1\lambda_2^{i-p}\mu_{2p}, \quad \lambda_2^{i-p+1}\mu_{2p-1}$
$3 \leq 2i+1 < 2p$	$\lambda_1\lambda_2^i, \quad \lambda_2^{i-1}\nu_3$	$\mu_{s+1}, \quad \lambda_1\mu_s$
$3 \leq 2i < 2p$	$\lambda_2^i, \quad \lambda_2^{i-1}\nu_2$	$\mu_{s+1}, \quad \lambda_1\mu_s$
1	λ_1	μ_2
0	1	γ_1
s	$D_2^{0,s}$	$D_2^{1,s}$

For $r \geq 2$ and $s \geq 0$, we have that $D_2^{r,s} = D_2^{r-2,s}\gamma_2$.

(iii) We can write $E_2 = \mathbb{F}_p[\nu_{2p}] \otimes D_2$. Furthermore, E_2 is generated by the elements

$$\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_{2p}, \gamma_1, \gamma_2, \mu_2, \ldots, \mu_{2p}.$$

Proof. The first statement can be obtained as in [Sie96, Corollary 4 (iii)] and the remaining assertions follow from Propositions 5.1 and 5.3. □

We encapsulate the previous result in the following table:

$2p$	$\boxed{\nu_{2p}}$ λ_2^p $\lambda_2^{p-1}\nu_2$	$\nu_{2p}\gamma_1$ $\lambda_1\mu_{2p}$ $\lambda_2\mu_{2p-1}$	$\nu_{2p}\gamma_2$ $\lambda_2^p\gamma_2$ $\lambda_2^{p-1}\nu_2\gamma_2$
$2p-1$	$\lambda_1\lambda_2^{p-1}$ $\lambda_2^{p-2}\nu_3$	$\boxed{\mu_{2p}}$ $\lambda_1\mu_{2p-1}$	$\lambda_1\lambda_2^{p-1}\gamma_2$ $\lambda_2^{p-2}\nu_3\gamma_2$
$2p-2$	λ_2^{p-1} $\lambda_2^{p-2}\nu_2$	$\boxed{\mu_{2p-1}}$ $\lambda_1\mu_{2p-2}$	$\lambda_2^{p-1}\gamma_2$ $\lambda_2^{p-2}\nu_2\gamma_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	λ_2^2 $\lambda_2\nu_2$	$\boxed{\mu_5}$ $\lambda_1\mu_4$	$\lambda_2^2\gamma_2$ $\lambda_2\nu_2\gamma_2$
3	$\boxed{\nu_3}$ $\lambda_1\lambda_2$	$\boxed{\mu_4}$ $\lambda_1\mu_3$	$\nu_3\gamma_2$ $\lambda_1\lambda_2\gamma_2$
2	$\boxed{\lambda_2}$ $\boxed{\nu_2}$	$\boxed{\mu_3}$ $\lambda_1\mu_2$	$\lambda_2\gamma_2$ $\nu_2\gamma_2$
1	$\boxed{\lambda_1}$	$\boxed{\mu_2}$	$\lambda_1\gamma_2$
0	1	$\boxed{\gamma_1}$	$\boxed{\gamma_2}$
	0	1	2

Figure 5.2: Basis elements of $E_2^{r,s}$ for $0 \leq r \leq 2$ and $0 \leq s \leq 2p$, with the $\mathbb{F}_p$-algebra generators highlighted.

Remark 5.5. In [Sie96, Corollary 4], using notation analogous to ours, Siegel obtains that the $\mathbb{F}_p$-algebra generators of $E_2(\mathrm{Heis}(p))$ are

$$\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_{2p}, \gamma_1, \gamma_2, \mu_2, \ldots, \mu_{2p-3}.$$

5.3 Indirect second differential computations

In this section, we use restriction, inflation and the norm maps to determine some of the bigraded $\mathbb{F}_p$-algebra generators of E_2 that survive to the infinity page E_∞.

Proposition 5.6. *The elements* $\lambda_1, \lambda_2, \nu_2, \nu_{2p}, \gamma_1, \gamma_2, \mu_2, \mu_3$ *survive to* E_∞.

Proof. It is clear that $\gamma_1, \gamma_2 \in E_\infty$. Since the extension (5.2) splits, the image of the second differential on $E_2^{\bullet,0}$ is trivial. Consequently, $\lambda_1, \mu_2 \in E_\infty$.

For $\lambda_2 \in E_2^{0,2} = \mathrm{H}^2(N, \mathbb{F}_p)^Q$, consider the map $\pi\colon \mathrm{H}^1(G, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathrm{H}^1(G, \mathbb{F}_p)$ and let $\tilde{\lambda}_1 \in \mathrm{H}^1(G, \mathbb{Z}/p^n\mathbb{Z})$ be such that $\pi(\tilde{\lambda}_1) = \lambda_1$. It can be readily checked that $\lambda_1 = \mathrm{res}_{G\to N} \circ \pi(\tilde{\lambda}_1)$. Furthermore, let $\beta_n^N\colon \mathrm{H}^1(N, \mathbb{F}_p) \longrightarrow \mathrm{H}^2(N, \mathbb{F}_p)$ denotes the n-th Bockstein homomorphism on the cohomology of N. Then, we have that $\lambda_2 = \beta_n^N(\lambda_1)$ and thus,

$$\lambda_2 = (\beta_n^N \circ \mathrm{res}_{G\to N} \circ \pi)(\tilde{\lambda}_1) = \mathrm{res}_{G\to N} \circ \beta_n^G(\tilde{\lambda}_1).$$

This yields that $\lambda_2 \in \mathrm{Im}(\mathrm{res}_{G\to N}) = E_\infty^{0,2}$.

For ν_2, consider the inflation homomorphism $\mathrm{inf}_{\mathrm{Heis}(p)\to G}\colon E_2(\mathrm{Heis}(p)) \longrightarrow E_2$. In particular, for $\tilde{\nu}_2 \in E_2^{0,2}(\mathrm{Heis}(p))$ defined analogously to ν_2 (see [Sie96, Corollary 4], where Siegel uses y_2), we have that $\nu_2 = \mathrm{inf}_{\mathrm{Heis}(p)\to G}(\tilde{\nu}_2)$. By [Sie96, Theorem 5], we have that $\tilde{\nu}_2 \in E_\infty(\mathrm{Heis}(p))$, and since the inflation map commutes with differentials, we conclude that $\nu_2 \in E_\infty$.

For μ_3, consider the subgroup $H = Q \ltimes (C_{p^n}^p \times C_{p^n})$ of G. The action of σ on

$$\mathrm{H}^\bullet(C_{p^n}^p \times C_{p^n}, \mathbb{F}_p) = \Lambda(w_1, \tilde{y}_1) \otimes \mathbb{F}_p[w_2, \tilde{y}_2]$$

can be computed in the same way as the action on $\mathrm{H}^\bullet(N, \mathbb{F}_p)$ to obtain that it is trivial, and so

$$E_2(H) = \mathrm{H}^\bullet(Q, \mathbb{F}_p) \otimes \mathrm{H}^\bullet(C_{p^n}^p \times C_{p^n}, \mathbb{F}_p) = E_\infty(H).$$

The restriction homomorphism $\mathrm{res}_{G\to H}\colon E_2 \longrightarrow E_2(H)$ then sends $\mu_3 = \bar{y}_2$ to

$$\mathrm{res}_{G\to H}(\mu_3) = \tilde{y}_2\gamma_1 \neq 0.$$

Furthermore, $\mathrm{d}_2(\mu_3) \in \langle \mu_2\gamma_2\rangle$ and $\mathrm{res}_{G\to H}(\mu_2\gamma_2) = \tilde{y}_1\gamma_1\gamma_2 \neq 0$. Nevertheless, we have that $\mathrm{d}_2(\tilde{y}_2\gamma_1) = 0$ and, as a consequence, $\mathrm{d}_2(\mu_3) = 0$. Hence, $\mu_3 \in E_\infty$.

Finally, we will study the generator ν_{2p}. The subgroup $M = C_{p^n}^p \ltimes N$ of G is normal, and so we have that $M\backslash G/N = G/MN = G/M$. For $k \geq 0$ even, let $\mathcal{N}_{M\to G}\colon \mathrm{H}^k(M, \mathbb{F}_p) \longrightarrow \mathrm{H}^{kp}(G, \mathbb{F}_p)$ be the Evens norm map as defined in [Eve91, Section 6.1]. Applying the properties in [Eve91, Theorem 6.1.1], we obtain, for any $\varphi \in \mathrm{H}^k(N, \mathbb{F}_p)$ with $k \geq 0$ even, that

$$\begin{aligned}\mathrm{res}_{G\to N}\big(\mathcal{N}_{M\to G}(\varphi)\big) &= \prod_{g\in G/M} \mathcal{N}_{N\to N}\big(\mathrm{res}_{M\to N}(g\cdot\varphi)\big) = \prod_{g\in G/M} \mathrm{res}_{M\to N}(g\cdot\varphi)\\ &= \prod_{g\in G/M} g\cdot \mathrm{res}_{M\to N}(\varphi).\end{aligned}$$

Moreover, since the action of σ^p on $\mathrm{H}^\bullet(N, \mathbb{F}_p) = \Lambda(\tilde{x}_1, \tilde{y}_1) \otimes \mathbb{F}_p[\tilde{x}_2, \tilde{y}_2]$ is once again trivial, we have that

$$E_2(M) = \mathrm{H}^\bullet(C_{p^n}^p, \mathbb{F}_p) \otimes \mathrm{H}^\bullet(N, \mathbb{F}_p) = E_\infty(M),$$

and we can write $y_2 = \mathrm{res}_{M\to N}(\tilde{y}_2)$. Therefore,

$$\nu_{2p} = z_{2p} = \prod_{g\in C_p} g\cdot y_2 = \mathrm{res}_{G\to N}\left(\mathcal{N}_{M\to G}(\tilde{y}_2)\right)$$

and we deduce that $\nu_{2p} \in E_\infty$.

□

5.4 Direct second differential computations

In this section, we explicitly compute the image of the second differential on the remaining bigraded $\mathbb{F}_p$-algebra generators of E_2. To that aim, we employ Theorem 4.4 following the same strategy as Siegel in the proof of [Sie96, Theorem 5].

The problem of computing d_2 is reduced to finding appropriate maps α and τ satisfying the hypotheses in Theorem 4.4. We start by defining such maps.

Denote by $P'_\bullet \longrightarrow \mathbb{F}_p$ and $P''_\bullet \longrightarrow \mathbb{F}_p$ the special resolutions of $\mathbb{F}_p$ as a module over $\mathbb{F}_p\langle a\rangle$ and $\mathbb{F}_p\langle b\rangle$, respectively. Both of these resolutions are minimal. For each $k\geq 0$, let e'_k and e''_k be the basis elements of P'_k and P''_k, respectively. We can then write $P'_k = \mathbb{F}_p\langle a\rangle e'_k$ and $P''_k = \mathbb{F}_p\langle b\rangle e''_k$, and so $P_\bullet = P'_\bullet \otimes P''_\bullet \longrightarrow \mathbb{F}_p$ is the minimal projective $\mathbb{F}_pN$-resolution of $\mathbb{F}_p$. If we set

$$e^i_j = \begin{cases} e'_{i-j}\otimes e''_j, & \text{if } 0\leq j\leq i,\\ 0, & \text{otherwise,}\end{cases} \tag{5.5}$$

then, for each $k\geq 0$, the elements $e^k_0,\dots,e^k_k$ constitute a basis of P_k as an $\mathbb{F}_pN$-module. Using the duality $\mathrm{H}^\bullet(N,\mathbb{F}_p)\cong \mathrm{Hom}_{\mathbb{F}_p}(\mathrm{H}_\bullet(N,\mathbb{F}_p))$ from Theorem 1.16 and the fact that $\mathrm{H}_\bullet(N,\mathbb{F}_p) = P_\bullet\otimes_{\mathbb{F}_pN}\mathbb{F}_p$ is a quotient of $P_\bullet$ via the canonical map $P_\bullet \longrightarrow P_\bullet\otimes_{\mathbb{F}_pN}\mathbb{F}_p$, with a slight abuse of notation we can identify the elements of $\mathrm{H}^\bullet(N,\mathbb{F}_p)$ as follows:

$$\text{for } i_1,i_2,j_1,j_2\geq 0, \qquad x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2} = (e^{i_1+j_1+2i_2+2j_2}_{j_1+2j_2})^*. \tag{5.6}$$

Consider the elements $\rho,\kappa\in\mathbb{F}_pN$ given by

$$\rho = \sum_{0\leq j\leq i<p^n} a^i b^j, \quad \kappa = \sum_{i=0}^{p^n-1}(i+1)a^i,$$

and define the maps $\alpha \in \mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, P_\bullet^{\sigma^{-1}})_0$ and $\tau \in \mathrm{Hom}_{\mathbb{F}_pN}(P_\bullet, P_\bullet)_1$ as the homomorphisms that for $0 \le j \le i < p^n$ satisfy the following equalities:

$$
\begin{aligned}
\alpha(e^{2i}_{2j}) &= \sum_{j\le k\le i} \binom{k}{j} (e^{2i}_{2k} - \rho e^{2i}_{2k+1}), & \tau(e^{2i}_{2j}) &= -(j+1)\kappa e^{2i+1}_{2j+2},\\
\alpha(e^{2i}_{2j+1}) &= \sum_{j\le k\le i} \binom{k}{j} b e^{2i}_{2k+1}, & \tau(e^{2i}_{2j+1}) &= -(j+1) e^{2i+1}_{2j+3},\\
\alpha(e^{2i+1}_{2j}) &= \sum_{j\le k\le i} \binom{k}{j} (b e^{2i+1}_{2k} + e^{2i+1}_{2k+1}), & \tau(e^{2i+1}_{2j}) &= -(j+1) e^{2i+2}_{2j+2},\\
\alpha(e^{2i+1}_{2j+1}) &= \sum_{j\le k\le i} \binom{k}{j} e^{2i+1}_{2k+1}, & \tau(e^{2i+1}_{2j+1}) &= -(j+1)\kappa e^{2i+2}_{2j+3}.
\end{aligned}
$$

We will now prove that these maps satisfy the hypotheses in Theorem 4.4.

The first step is proving that α is a chain map, for which we will make use of the following result.

Lemma 5.7. *Let e^i_j be as in* (5.5).

(i) The following identities hold:

$$
\begin{aligned}
\rho(b-1) &= bT(ab) - T(a), & \rho(a-1) &= T(b) - T(ab),\\
\rho(ab-1) &= T(b) - T(a), & \kappa(a-1) &= -T(a).
\end{aligned}
$$

(ii) The differential of the element e^i_j is as follows:

$$
\begin{aligned}
\partial(e^{2i}_{2j}) &= T(a)e^{2i-1}_{2j} + T(b)e^{2i-1}_{2j-1}, & \partial(e^{2i}_{2j+1}) &= (a-1)e^{2i-1}_{2j+1} - (b-1)e^{2i-1}_{2j},\\
\partial(e^{2i+1}_{2j}) &= (a-1)e^{2i}_{2j} - T(b)e^{2i}_{2j-1}, & \partial(e^{2i+1}_{2j+1}) &= T(a)e^{2i}_{2j+1} + (b-1)e^{2i}_{2j}.
\end{aligned}
$$

Proof. (i) We give the proof of the first identity, as all the others follow similarly. We can compute

$$
\rho(b-1) = \sum_{i=0}^{p^n-1} \sum_{j=0}^{i} a^i(b^{j+1} - b^j) = \sum_{i=0}^{p^n-1} a^i(b^{i+1} - 1) = bT(ab) - T(a).
$$

(ii) Once again, we prove the first identity. Using the definition of the differential of a tensor product in (1.1) and the differential in Examples 1.17(i), we can compute

$$\begin{aligned}\partial(e^{2i}_{2j}) &= \partial(e'_{2i-2j}) \otimes e''_{2j} + e'_{2i-2j} \otimes \partial(e''_{2j})\\ &= T(a)e'_{2i-2j-1} \otimes e''_{2j} + T(b)e'_{2i-2j} \otimes e''_{2j-1}\\ &= T(a)e^{2i-1}_{2j} + T(b)e^{2i-1}_{2j-1}.\end{aligned}$$

□

Proposition 5.8. *The map α is a chain map, i.e. $\partial \circ \alpha - \alpha \circ \partial = 0$.*

Proof. We will use Lemma 5.7 during the computations. We need to consider four cases:

(i) Let us show that $(\partial \circ \alpha - \alpha \circ \partial)(e^{2i}_{2j}) = 0$. On the one hand,

$$\begin{aligned}\partial \circ \alpha(e^{2i}_{2j}) &= \sum_{j \le k \le i} \binom{k}{j} \partial(e^{2i}_{2k} - \rho e^{2i}_{2k+1})\\ &= \sum_{j \le k \le i} \binom{k}{j} \Big(\big(T(a) + \rho(b-1)\big)e^{2i-1}_{2k} + T(b)e^{2i-1}_{2k-1} - \rho(a-1)e^{2i-1}_{2k+1}\Big)\\ &= \sum_{j \le k \le i} \binom{k}{j} \Big(bT(ab)e^{2i-1}_{2k} + T(b)e^{2i-1}_{2k-1} + \big(T(ab) - T(b)\big)e^{2i-1}_{2k+1}\Big)\\ &= \sum_{j \le k \le i} \binom{k}{j} \big(bT(ab)e^{2i-1}_{2k} + T(ab)e^{2i-1}_{2k+1}\big)\\ &\qquad + \sum_{j \le k+1 \le i} \left[\binom{k+1}{j} - \binom{k}{j}\right] T(b)e^{2i-1}_{2k+1}\\ &= \sum_{j \le k \le i} \binom{k}{j} \big(bT(ab)e^{2i-1}_{2k} + T(ab)e^{2i-1}_{2k+1}\big) + \sum_{j \le k+1 \le i} \binom{k}{j-1} T(b)e^{2i-1}_{2k+1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha \circ \partial(e^{2i}_{2j}) &= \alpha\big(T(a)e^{2i-1}_{2j} + T(b)e^{2i-1}_{2j-1}\big)\\ &= \sum_{j \le k \le i} \binom{k}{j} T(ab)(be^{2i-1}_{2k} + e^{2i-1}_{2k+1}) + \sum_{j \le k \le i} \binom{k}{j-1} T(b)e^{2i-1}_{2k+1}\\ &= \sum_{j \le k \le i} \binom{k}{j} T(ab)(be^{2i-1}_{2k} + e^{2i-1}_{2k+1}) + \sum_{j \le k+1 \le i} \binom{k}{j-1} T(b)e^{2i-1}_{2k+1}.\end{aligned}$$

Therefore, $(\partial \circ \alpha - \alpha \circ \partial)(e^{2i}_{2j}) = 0$.

(ii) Let us show that $(\partial \circ \alpha - \alpha \circ \partial)(e_{2j+1}^{2i}) = 0$. On the one hand,

$$\begin{aligned}
\partial \circ \alpha(e_{2j+1}^{2i}) &= \sum_{j \le k \le i} \binom{k}{j} \partial(b e_{2k+1}^{2i}) \\
&= \sum_{j \le k \le i} \binom{k}{j} \big(b(a-1)e_{2k+1}^{2i-1} - b(b-1)e_{2k}^{2i-1}\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha \circ \partial(e_{2j+1}^{2i}) &= \alpha\big((a-1)e_{2j+1}^{2i-1} - (b-1)e_{2j}^{2i-1}\big) \\
&= \sum_{j \le k \le i} \binom{k}{j} \big((ab-1)e_{2k+1}^{2i-1} - (b-1)(b e_{2k}^{2i-1} + e_{2k+1}^{2i-1})\big) \\
&= \sum_{j \le k \le i} \binom{k}{j} \big(b(a-1)e_{2k+1}^{2i-1} - b(b-1)e_{2k}^{2i-1}\big).
\end{aligned}$$

Therefore, $(\partial \circ \alpha - \alpha \circ \partial)(e_{2j+1}^{2i}) = 0$.

(iii) Let us show that $(\partial \circ \alpha - \alpha \circ \partial)(e_{2j}^{2i+1}) = 0$. On the one hand,

$$\begin{aligned}
\partial \circ \alpha(e_{2j}^{2i+1}) &= \sum_{j \le k \le i} \binom{k}{j} \partial(b e_{2k}^{2i+1} + e_{2k+1}^{2i+1}) \\
&= \sum_{j \le k \le i} \binom{k}{j} \Big(\big(b(a-1) + (b-1)\big)e_{2k}^{2i} - bT(b)e_{2k-1}^{2i} + T(a)e_{2k+1}^{2i}\Big) \\
&= \sum_{j \le k \le i} \binom{k}{j} \big((ab-1)e_{2k}^{2i} + T(a)e_{2k+1}^{2i}\big) \\
&\quad - \sum_{j \le k+1 \le i} \binom{k+1}{j} T(b)e_{2k+1}^{2i}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\alpha \circ \partial(e^{2i+1}_{2j}) &= \alpha\big((a-1)e^{2i}_{2j} - T(b)e^{2i}_{2j-1}\big) \\
&= \sum_{j \le k \le i} \binom{k}{j}(ab-1)(e^{2i}_{2k} - \rho e^{2i}_{2k+1}) - \sum_{j \le k+1 \le i} \binom{k}{j-1} bT(b)e^{2i}_{2k+1} \\
&= \sum_{j \le k \le i} \binom{k}{j}\big((ab-1)e^{2i}_{2k} + T(a)e^{2i}_{2k+1}\big) \\
&\qquad - \sum_{j \le k \le i} \left[\binom{k}{j} + \binom{k}{j-1}\right] T(b)e^{2i}_{2k+1} \\
&= \sum_{j \le k \le i} \binom{k}{j}\big((ab-1)e^{2i}_{2k} + T(a)e^{2i}_{2k+1}\big) \\
&\qquad - \sum_{j \le k+1 \le i} \binom{k+1}{j} T(b)e^{2i}_{2k+1}.
\end{aligned}
$$

Therefore, $(\partial \circ \alpha - \alpha \circ \partial)(e^{2i+1}_{2j}) = 0$.

(iv) Let us show that $(\partial \circ \alpha - \alpha \circ \partial)(e^{2i+1}_{2j+1}) = 0$. On the one hand,

$$
\begin{aligned}
\partial \circ \alpha(e^{2i+1}_{2j+1}) &= \sum_{j \le k \le i} \binom{k}{j} \partial(e^{2i+1}_{2k+1}) \\
&= \sum_{j \le k \le i} \binom{k}{j} \big(T(a)e^{2i}_{2k+1} + (b-1)e^{2i}_{2k}\big) \\
&= \sum_{j \le k \le i} \binom{k}{j} \Big(\big(bT(ab) - \rho(b-1)\big)e^{2i}_{2k+1} + (b-1)e^{2i}_{2k}\Big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\alpha \circ \partial(e^{2i+1}_{2j+1}) &= \alpha\big(T(a)e^{2i}_{2j+1} + (b-1)e^{2i}_{2j}\big) \\
&= \sum_{j \le k \le i} \binom{k}{j} \big(T(ab)be^{2i}_{2k+1} + (b-1)(e^{2i}_{2k} - \rho e^{2i}_{2k+1})\big).
\end{aligned}
$$

Therefore, $(\partial \circ \alpha - \alpha \circ \partial)(e^{2i+1}_{2j+1}) = 0$.

$\square$

We are left to prove that the identity $\partial \circ \tau + \tau \circ \partial = 1 - \alpha^{p^n}$ holds. In order to do that, we first list the identities that will be used throughout the proof.

Lemma 5.9.

(i) We have that

$$\sum_{r=0}^{p^n-1} \rho^{\sigma^r} b^r = \kappa T(b).$$

(ii) For any $i, j \geq 0$ and $m \geq 1$, we have that

$$\sum_{j \leq k \leq l \leq i} m^{k-j} \binom{l}{k} \binom{k}{j} = \sum_{l=j}^{i} (m+1)^{l-j} \binom{l}{j}.$$

Proof. (i) If p does not divide $i+1$, then $T(b^{i+1}) = T(b)$. Otherwise, $i+1 = 0$ in $\mathbb{F}_p$. Note that

$$\rho^{\sigma^r} = \sum_{0 \leq j \leq i < p^n} a^i b^{ri+j}$$

for every $r \geq 0$. Hence,

$$\begin{aligned}
\sum_{r=0}^{p^n-1} \alpha^r(\rho) b^r &= \sum_{i=0}^{p^n-1} \sum_{j=0}^{i} a^i b^j \sum_{r=0}^{p^n-1} b^{r(i+1)} = \sum_{i=0}^{p^n-1} \sum_{j=0}^{i} a^i b^j T(b^{i+1}) \\
&= \sum_{i=0}^{p^n-1} (i+1) a^i T(b) = \kappa T(b).
\end{aligned}$$

(ii) Using standard identities of binomial coefficients, we compute

$$\begin{aligned}
\sum_{j \leq k \leq l \leq i} m^{k-j} \binom{l}{k} \binom{k}{j} &= \sum_{l=j}^{i} \sum_{k=j}^{l} m^{k-j} \binom{l}{j} \binom{l-j}{k-j} \\
&= \sum_{l=j}^{i} \binom{l}{j} \sum_{k=0}^{l-j} m^{k-j} \binom{l-j}{k-j} = \sum_{l=j}^{i} (m+1)^{l-j} \binom{l}{j}.
\end{aligned}$$

$\square$

Proposition 5.10. *The maps α and τ satisfy the identity $\partial \circ \tau + \tau \circ \partial = 1 - \alpha^{p^n}$.*

Proof. We will use Lemma 5.7 during the computations. We need to consider four cases:

(i) Let us show that $(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j})=(1-\alpha^{p^n})(e^{2i}_{2j})$. First, we will compute $(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j})$. On the one hand,

$$\begin{aligned}\partial\circ\tau(e^{2i}_{2j})&=\partial\big(-(j+1)\kappa e^{2i+1}_{2j+2}\big)\\&=-(j+1)\kappa(a-1)e^{2i}_{2j+2}+(j+1)\kappa T(b)e^{2i+1}_{2j+1}\\&=(j+1)T(a)e^{2i}_{2j+2}+(j+1)\kappa T(b)e^{2i+1}_{2j+1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau\circ\partial(e^{2i}_{2j})&=\tau\big(T(a)e^{2i-1}_{2j}+T(b)e^{2i-1}_{2j-1}\big)\\&=-(j+1)T(a)e^{2i}_{2j+2}-jT(b)\kappa e^{2i}_{2j+1}.\end{aligned}$$

As a consequence,

$$(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j})=\kappa T(b)e^{2i}_{2j+1}.$$

Now, we compute $(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j})$. Applying α repeatedly to e^{2i}_{2j}, we obtain that

$$\alpha^m(e^{2i}_{2j})=\sum_{j\le k\le i}m^{k-j}\binom{k}{j}\left(e^{2i}_{2k}-\sum_{r=0}^{m-1}\rho^{\sigma^r}b^re^{2i}_{2k+1}\right)$$

for any $1\le m\le p^n$. Therefore,

$$\alpha^{p^n}(e^{2i}_{2j})=\sum_{j\le k\le i}p^{n(k-j)}\binom{k}{j}\big(e^{2i}_{2k}-\kappa T(b)e^{2i}_{2k+1}\big)=e^{2i}_{2j}-\kappa T(b)e^{2i}_{2j+1},$$

and thus $(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j})=(1-\alpha^{p^n})(e^{2i}_{2j})$.

(ii) Let us show that $(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j+1})=(1-\alpha^{p^n})(e^{2i}_{2j+1})$. First, we will compute $(\partial\circ\tau+\tau\circ\partial)(e^{2i}_{2j+1})$. On the one hand,

$$\begin{aligned}\partial\circ\tau(e^{2i}_{2j+1})&=\partial\big(-(j+1)e^{2i+1}_{2j+3}\big)\\&=-(j+1)T(a)e^{2i}_{2j+3}-(j+1)(b-1)e^{2i}_{2j+2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau\circ\partial(e^{2i}_{2j+1})&=\tau\big((a-1)e^{2i-1}_{2j+1}-(b-1)e^{2i-1}_{2j}\big)\\&=-(j+1)(a-1)\kappa e^{2i}_{2j+3}+(j+1)(b-1)e^{2i}_{2j+2}\\&=(j+1)T(a)e^{2i}_{2j+3}+(j+1)(b-1)e^{2i}_{2j+2}.\end{aligned}$$

As a consequence,

$$(\partial \circ \tau + \tau \circ \partial)(e^{2i}_{2j+1}) = 0.$$

Now, applying α repeatedly to e^{2i}_{2j+1}, we obtain that

$$\alpha^m(e^{2i}_{2j+1}) = \sum_{j \leq k \leq i} m^{k-j} \binom{k}{j} b^m e^{2i}_{2k+1}.$$

Therefore,

$$\alpha^{p^n}(e^{2i}_{2j+1}) = \sum_{j \leq k \leq i} p^{n(k-j)} \binom{k}{j} b^{p^n} e^{2i}_{2k+1} = e^{2i}_{2j+1},$$

and thus $(\partial \circ \tau + \tau \circ \partial)(e^{2i}_{2j+1}) = (1 - \alpha^{p^n})(e^{2i}_{2j+1})$.

(iii) Let us show that $(\partial \circ \tau + \tau \circ \partial)(e^{2i+1}_{2j}) = (1 - \alpha^{p^n})(e^{2i+1}_{2j})$. First, we will compute $(\partial \circ \tau + \tau \circ \partial)(e^{2i+1}_{2j})$. On the one hand,

$$\begin{aligned}\partial \circ \tau(e^{2i+1}_{2j}) &= \partial\big(-(j+1)e^{2i+2}_{2j+2}\big) \\ &= -(j+1)T(a)e^{2i+1}_{2j+2} - (j+1)T(b)e^{2i+1}_{2j+1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau \circ \partial(e^{2i+1}_{2j}) &= \tau\big((a-1)e^{2i}_{2j} - T(b)e^{2i}_{2j-1}\big) \\ &= -(j+1)(a-1)\kappa e^{2i+1}_{2j+2} + jT(b)e^{2i+1}_{2j+1} \\ &= (j+1)T(a)e^{2i+1}_{2j+2} + jT(b)e^{2i+1}_{2j+1}.\end{aligned}$$

As a consequence,

$$(\partial \circ \tau + \tau \circ \partial)(e^{2i+1}_{2j}) = -T(b)e^{2i+1}_{2j+1}.$$

Now, applying α repeatedly to e^{2i+1}_{2j}, we obtain that

$$\alpha^m(e^{2i+1}_{2j}) = \sum_{j \leq k \leq i} m^{k-j} \binom{k}{j} \left(b^m e^{2i+1}_{2k} + \sum_{r=0}^{m-1} b^r e^{2i+1}_{2k+1} \right).$$

Therefore,

$$\alpha^{p^n}(e^{2i+1}_{2j}) = \sum_{j \leq k \leq i} p^{n(k-j)} \binom{k}{j} \big(b^{p^n} e^{2i+1}_{2k} + T(b)e^{2i+1}_{2k+1} \big) = e^{2i+1}_{2j} + T(b)e^{2i+1}_{2j+1},$$

and thus $(\partial \circ \tau + \tau \circ \partial)(e^{2i+1}_{2j}) = (1 - \alpha^{p^n})(e^{2i+1}_{2j})$.

(iv) Let us show that $(\partial\circ\tau+\tau\circ\partial)(e^{2i+1}_{2j+1}) = (1-\alpha^{p^n})(e^{2i+1}_{2j+1})$. First, we will compute $(\partial\circ\tau+\tau\circ\partial)(e^{2i+1}_{2j+1})$. On the one hand,

$$\begin{aligned}\partial\circ\tau(e^{2i+1}_{2j+1}) &= \partial\big(-(j+1)\kappa e^{2i+2}_{2j+3}\big)\\ &= -(j+1)(a-1)\kappa e^{2i+1}_{2j+3} + (j+1)(b-1)\kappa e^{2i+1}_{2j+2}\\ &= (j+1)T(a)e^{2i+1}_{2j+3} + (j+1)(b-1)\kappa e^{2i+1}_{2j+2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau\circ\partial(e^{2i+1}_{2j+1}) &= \tau\big(T(a)e^{2i}_{2j+1} + (b-1)e^{2i}_{2j}\big)\\ &= -(j+1)T(a)e^{2i+1}_{2j+3} - (j+1)(b-1)\kappa e^{2i+1}_{2j+2}.\end{aligned}$$

As a consequence,

$$(\partial\circ\tau+\tau\circ\partial)(e^{2i+1}_{2j+1}) = 0.$$

Now, applying α repeatedly to e^{2i+1}_{2j+1}, we obtain that

$$\alpha^m(e^{2i+1}_{2j+1}) = \sum_{j\le k\le i} m^{k-j}\binom{k}{j} e^{2i+1}_{2k+1}.$$

Therefore,

$$\alpha^{p^n}(e^{2i+1}_{2j+1}) = \sum_{j\le k\le i} p^{n(k-j)}\binom{k}{j} e^{2i+1}_{2k+1} = e^{2i+1}_{2j+1},$$

and thus $(\partial\circ\tau+\tau\circ\partial)(e^{2i+1}_{2j+1}) = (1-\alpha^{p^n})(e^{2i+1}_{2j+1})$.

□

We can now combine Proposition 5.8 and Proposition 5.10 into the following result.

Lemma 5.11. *The maps α and τ defined as before satisfy $\partial\circ\alpha-\alpha\circ\partial = 0$ and $\partial\circ\tau-\tau\circ\partial = 1-\alpha^{p^n}$.*

Using Theorem 4.4 and the maps α and τ in Lemma 5.11, we can now compute the second differential of the remaining bigraded $\mathbb{F}_p$-algebra generators. We will need to use the explicit definition of the generators in (5.2.2) and the duality between $\mathrm{H}^\bullet(N,\mathbb{F}_p)$ and $\mathrm{H}_\bullet(N,\mathbb{F}_p)$ in (5.6).

Proposition 5.12. *The second differential of the elements $\mu_4,\dots,\mu_{2p}$ is as follows:*

(i) For $2 \leq i \leq p$, we have that

$$\mathrm{d}_2(\mu_{2i}) = -(i-1)\lambda_1\mu_{2i-2}\gamma_2.$$

(ii) For $2 \leq i \leq p-1$, we have that

$$\mathrm{d}_2(\mu_{2i+1}) = -i\lambda_1\mu_{2i-1}\gamma_2.$$

Proof. Consider $\mu_{2i+2} = \overline{y_1 y_2^i} \in E_2^{1,2i+1}$ with $1 \leq i \leq p-1$, which, by (5.6), is represented by the map $f\colon P_{2i+1} \longrightarrow \mathbb{F}_p$ with $f = (e_{2i+1}^{2i+1})^*$. We can easily compute $f \circ \tau$ to obtain that, for $0 \leq j \leq k < p^n$, we have that

$$(f\circ\tau)(e_{2j}^{2k+1}) = 0, \quad (f\circ\tau)(e_{2j+1}^{2k+1}) = 0, \quad (f\circ\tau)(e_{2j}^{2k}) = 0,$$
$$(f\circ\tau)(e_{2j+1}^{2k}) = \begin{cases} -i, & \text{if } k = i \text{ and } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $-(f\circ\tau) = i(e_{2i-1}^{2i})^*$, which represents $-i\lambda_1\mu_{2i}\gamma_2 = i\overline{x_1 y_1 y_2^{i-1}}$. Consequently,

$$\mathrm{d}_2(\mu_{2i+2}) = -i\lambda_1\mu_{2i}\gamma_2.$$

Take now $\mu_{2i+1} = \overline{y_2^i} \in E_2^{1,2i}$ with $2 \leq i \leq p-1$, which is represented by the map $f\colon P_{2i} \longrightarrow \mathbb{F}_p$ with $f = (e_{2i}^{2i})^*$. We compute $f \circ \tau$ to obtain that

$$(f\circ\tau)(e_{2j}^{2k}) = 0, \quad (f\circ\tau)(e_{2j+1}^{2k}) = 0, \quad (f\circ\tau)(e_{2j+1}^{2k+1}) = 0,$$
$$(f\circ\tau)(e_{2j}^{2k+1}) = \begin{cases} -i, & \text{if } k = i-1 \text{ and } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $-(f\circ\tau) = i(e_{2i-2}^{2i-1})^*$, which represents $-i\lambda_1\mu_{2i-1}\gamma_2 = i\overline{x_1 y_2^{i-1}}$. Consequently,

$$\mathrm{d}_2(\mu_{2i+1}) = -i\lambda_1\mu_{2i-1}\gamma_2.$$

□

The proof of the next result is verbatim to the previous one.

Proposition 5.13. *The second differential of the element ν_3 is trivial.*

Proof. Consider $\nu_3 = x_1 y_2 - y_1 x_2 \in E_2^{0,3}$ which, by (5.6), is represented by the map $f\colon P_3 \longrightarrow \mathbb{F}_p$ with $f = (e_2^3)^* - (e_1^3)^*$. We can easily compute $f \circ \tau = 0$, and so $\mathrm{d}_2(\nu_3) = 0$. □

Using the results in Sections 5.3 and 5.4, we can now determine the bigraded $\mathbb{F}_p$-vector space E_3. First, write $D_3 = E_3/\langle \nu_{2p} \rangle$, and define the elements

$$\begin{aligned} \text{for } 4 \leq i \leq 2p+1, \quad \omega_i &= -\lambda_1 \mu_{i-1} \in E_2^{1,i-1}, \\ \omega_{2p+2} &= \lambda_2 \mu_{2p} \in E_2^{1,2p+1}, \\ \xi_{2p+1} &= \lambda_2 \mu_{2p-1} \in E_2^{1,2p}. \end{aligned}$$

One can easily verify that these elements have trivial second differential, and so they are in fact elements of E_3. For example, we can compute

$$\mathrm{d}_2(\omega_{2p+2}) = \mathrm{d}_2(\lambda_2\mu_{2p}) = \mathrm{d}_2(\lambda_2)\mu_{2p} + \lambda_2\,\mathrm{d}_2(\mu_{2p}) = \lambda_1\lambda_2\mu_{2p-2} = \overline{x_1 y_1 x_2 y_2^{p-2}} = 0.$$

Proposition 5.14. *Multiplication by the elements $\nu_{2p}, \gamma_2, \lambda_2$ induces vector space homomorphisms as follows:*

(i) Multiplication $\cdot\nu_{2p}\colon E_3^{r,s} \longrightarrow E_3^{r,s+2p}$ is injective for all $r, s \geq 0$. As a consequence, $E_3 = \mathbb{F}_p[\nu_{2p}] \otimes D_3$.

(ii) Multiplication $\cdot\gamma_2\colon E_3^{r,s} \longrightarrow E_3^{r+2,s}$ is surjective for all $r, s \geq 0$, and an isomorphism for all $r \neq 1$, as is $\cdot\gamma_2\colon D_3^{1,s} \longrightarrow D_3^{3,s}$ for $s \geq 2p-1$.

(iii) Multiplication $\cdot\lambda_2\colon D_3^{r,s} \longrightarrow D_3^{r,s+2}$ is an isomorphism for all $s \geq 2p$.

Proof. The proofs of (i) and (ii) are based on the proof of [Sie96, Corollary 6].

We start with the first statement. For $r, s \geq 0$, let $\varphi \in E_2^{r,s}$ be such that $\mathrm{d}_2(\varphi) = 0$, and suppose that $\varphi\nu_{2p}$ is a trivial element in E_3, i.e. there exists $\psi \in E_2^{r-2,s+2p+1}$ such that $\varphi\nu_{2p} = \mathrm{d}_2(\psi)$. Then, since $\langle \nu_{2p} \rangle \cap \mathrm{d}_2(E_2 \backslash \langle \nu_{2p} \rangle) = 0$, there exists $\upsilon \in E_2^{r-2,s+1}$ such that $\psi = \upsilon\nu_{2p}$. Consequently, $\varphi\nu_{2p} = \mathrm{d}_2(\upsilon)\nu_{2p}$ and, because $\cdot\nu_{2p}\colon E_2^{r,s} \longrightarrow E_2^{r,s+2p}$ is injective (see Proposition 5.3(i)), we have that $\varphi = \mathrm{d}_2(\upsilon)$, i.e. $\varphi = 0$ in E_3.

For the next claim, we first show that multiplication by γ_2 is surjective. Take $\varphi \in E_2^{r+2,s}$ with $r, s \geq 0$ such that $\mathrm{d}_2(\varphi) = 0$. By Proposition 5.3(ii), there is some $\psi \in E_2^{r,s}$ such that $\varphi = \psi\gamma_2$ in E_2. Then, we have that $\mathrm{d}_2(\psi)\gamma_2 = \mathrm{d}_2(\varphi) = 0$ and, because the product $\cdot\gamma_2\colon E_2^{r+2,s-1} \longrightarrow E_2^{r+4,s-1}$ is injective, we deduce that $\mathrm{d}_2(\psi) = 0$, i.e. ψ survives to E_3 and $\varphi = \psi\gamma_2$ in E_3.

We will now study the injectivity of the multiplication by γ_2. Let $\varphi \in E_2^{r,s}$ with $r \neq 1$ or $s \geq 2p-1$ such that $\mathrm{d}_2(\varphi) = 0$. Suppose that there exists $\psi \in E_2^{r,s+1}$ such that $\varphi\gamma_2 = \mathrm{d}_2(\psi)$ and we want to deduce that $\varphi = 0$. If $r = 0$, or if $\varphi \in D_2^{r,s}$ with $s \geq 2p-1$, then $\mathrm{d}_2(\psi) = 0$, and by the injectivity of $\cdot\gamma_2\colon E_2^{r,s} \longrightarrow E_2^{r+2,s}$ we obtain that $\varphi = 0$. Otherwise, if $r \geq 2$ we have that $\psi = \upsilon\gamma_2$ with $\upsilon \in E_2^{r-2,s+1}$. Hence, $\varphi\gamma_2 = \mathrm{d}_2(\upsilon)\gamma_2$ and, because $\cdot\gamma_2\colon E_2^{r,s} \longrightarrow E_2^{r+2,s}$ is injective, we have that $\varphi = \mathrm{d}_2(\upsilon)$, i.e. $\varphi = 0$ in E_3.

Now, let us show that multiplication by λ_2 is surjective for $s \geq 2p$. Take $\varphi \in E_2^{r,s+2}$ with $s \geq 2p$ such that $\mathrm{d}_2(\varphi) = 0$. By Proposition 5.3(iii), there is some $\psi \in E_2^{r,s}$ such that $\varphi = \psi\lambda_2$ in E_2. Then, we have that $\mathrm{d}_2(\psi)\lambda_2 = \mathrm{d}_2(\varphi) = 0$ and, because the product $\cdot\lambda_2\colon E_2^{r+2,s-1} \longrightarrow E_2^{r+2,s+1}$ is injective, we deduce that $\mathrm{d}_2(\psi) = 0$, i.e. ψ survives to E_3 and $\varphi = \psi\lambda_2$ in E_3.

Finally, we show that multiplication by λ_2 is injective for $s \geq 2p$. Let $\varphi \in E_2^{r,s}$ with $g \geq 2p$ such that $\mathrm{d}_2(\varphi) = 0$. Suppose that $\varphi\lambda_2 = \mathrm{d}_2(\psi)$ for some $\psi \in E_2^{r-2,s+3}$. Then, as $\langle\lambda_2\rangle \cap \mathrm{d}_2(E_2 \setminus \langle\lambda_2\rangle) = 0$, there exists $\upsilon \in E_2^{r-2,s+1}$ such that $\psi = \upsilon\lambda_2$. Therefore, $\varphi\lambda_2 = \mathrm{d}_2(\upsilon)\lambda_2$ and, because $\cdot\lambda_2\colon E_2^{r,s} \longrightarrow E_2^{r,s+2}$ is injective, we have that $\varphi = \mathrm{d}_2(\upsilon)$, i.e. $\varphi = 0$ in E_3.

$\square$

We can now fully determine the structure of the bigraded $\mathbb{F}_p$-vector space E_3.

Theorem 5.15.

(i) For $r \geq 0$ even, we have that $E_3^{r,\bullet} = E_2^{r,\bullet}$. For $r \geq 5$ odd, we have that $D_3^{r,s} = D_3^{r-2,s}\gamma_2$. For $r = 1, 3$, the basis elements of $D_3^{r,s}$ are the following:

$2i+1 \geq 2p+1$	$\lambda_2^{i-p+1}\omega_{2p},\quad \lambda_2^{i-p}\omega_{2p+2}$	$\lambda_2^{i-p+1}\omega_{2p}\gamma_2,\quad \lambda_2^{i-p}\omega_{2p+2}\gamma_2$
$2i \geq 2p$	$\lambda_2^{i-p}\omega_{2p+1},\quad \lambda_2^{i-p}\xi_{2p+1}$	$\lambda_2^{i-p}\omega_{2p+1}\gamma_2,\quad \lambda_2^{i-p}\xi_{2p+1}\gamma_2$
$2p-1$	ω_{2p}	$\omega_{2p}\gamma_2$
$6 \leq s < 2p-2$	ω_{s+1}	$\emptyset$
5	ω_6	$\emptyset$
4	$\mu_2\nu_3$	$\emptyset$
3	$\lambda_1\mu_3$	$\emptyset$
2	μ_3	$\mu_3\gamma_2$
1	μ_2	$\mu_2\gamma_2$
0	γ_1	$\gamma_1\gamma_2$
s	$D_3^{1,s}$	$D_3^{3,s}$

Additionally, if $p \geq 5$ we have that $\omega_6 = \frac{2}{3}\mu_3\nu_3$, and so $D_3^{1,5} = \langle\mu_3\nu_3\rangle$.

(ii) We can write $E_3 = \mathbb{F}_p[\nu_{2p}] \otimes D_3$. Furthermore, for $p = 3$, the third page E_3 is generated by the elements

$$\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_6, \gamma_1, \gamma_2, \mu_2, \mu_3, \omega_6, \omega_7, \omega_8, \xi_7,$$

and for $p \geq 5$, by the elements

$$\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_{2p}, \gamma_1, \gamma_2, \mu_2, \mu_3, \omega_7, \dots, \omega_{2p+2}, \xi_{2p+1}.$$

Proof. We can deduce from Theorem 5.4 that $E_3^{1,s} = \langle \omega_{s+1} \rangle$ for $3 \leq s \leq 2p-1$. Nevertheless, we can easily compute

$$-\omega_4 = \lambda_1\mu_3, \quad -\omega_5 = \mu_2\nu_3, \quad \frac{3}{2}\omega_6 = \mu_3\nu_3.$$

For example, using Proposition 5.1(ii) we obtain that $\overline{x_1y_2^2} = -2\overline{y_1x_2y_2}$, and so

$$\mu_3\nu_3 = \bar{y_2} \cdot (x_1y_2 - y_1x_2) = \overline{y_1x_2y_2 - x_1y_2^2} = \frac{3}{2}\overline{x_1y_2^2} = -\frac{3}{2}\lambda_1\mu_5 = \frac{3}{2}\omega_6.$$

Everything else follows from Propositions 5.12 and 5.14. $\square$

5.6 To infinity and beyond

Our objective in this section is to show that if $p \geq 5$ the spectral sequence E collapses at the third page, i.e. $E_3 = E_\infty$. In order to achieve our goal, we will define two group automorphisms that will help us show that all the differentials starting with d_3 are trivial. Let $u \in \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$ be a generator, i.e. $u^{p^{n-1}(p-1)} = 1$ but $u^i \neq 1$ for any $1 \leq i < p^{n-1}(p-1)$. For $0 \leq i, j, k \leq p^n - 1$, we define the group automorphisms $\Phi \colon G \longrightarrow G$ and $\Psi \colon G \longrightarrow G$ by

$$\Phi(\sigma^k a^i b^j) = \sigma^{uk} a^i b^{uj}, \quad \text{and} \quad \Psi(\sigma^k a^i b^j) = \sigma^k a^{ui} b^{uj}.$$

Because $\Phi(N), \Psi(N) \leq N$, for every $m \geq 2$, there are induced automorphisms $\Phi^* \colon E_m \longrightarrow E_m$ and $\Psi^* \colon E_m \longrightarrow E_m$. These automorphisms act on the generators of D_3 by multiplying each of them by a power of u as described in the following table:

	λ_i	γ_i	ν_2	ν_3	μ_{2i}	μ_{2i+1}	ω_{2i}	ω_{2i+1}	ξ_{2p+1}
Φ	1	u	u	u	u^{i+1}	u^{i+1}	u^i	u^{i+1}	u^p
Ψ	u	1	u^2	u^2	u^i	u^i	u^i	u^{i+1}	u^p

Proposition 5.16. *For $p \geq 5$, the element $\xi_{2p+1} \in E_3$ survives to E_∞.*

Proof. Assume by induction that, for $m \geq 3$, $\xi_{2p+1} \in E_m$, and we will show that $\xi_{2p+1} \in E_{m+1}$. Consider first the case $m = 2j+1$ with $j \geq 1$. We have that

$$\mathrm{d}_{2j+1}(\xi_{2p+1}) = t_1 \lambda_2^{p-j} \gamma_2^{j+1} + t_2 \lambda_2^{p-j-1} \nu_2 \gamma_2^{j+1} \tag{5.7}$$

with $t_1, t_2 \in \mathbb{F}_p$. Applying Ψ, we obtain that

$$u^p \,\mathrm{d}_{2j+1}(\xi_{2p+1}) = t_1 u^{p-j} \lambda_2^{p-j} \gamma_2^{j+1} + t_2 u^{p-j+1} \lambda_2^{p-j-1} \nu_2 \gamma_2^{j+1}$$

and, equating coefficients with those in (5.7), we get the conditions

$$\begin{cases} t_1(1-u^j) = 0, \\ t_2(1-u^{j-1}) = 0. \end{cases}$$

From these, we deduce that $t_1 = 0$ for all $j \geq 1$, and $t_2 = 0$ for all $j > 1$. If $j = 1$, applying Φ to (5.7) we deduce that $t_2(1-u^{p-3}) = 0$ and $t_2 = 0$ for $p \geq 5$. Therefore, $\xi_{2p+1} \in E_{2j+1}$ survives to E_{2j+2}.

If $m = 2j$ with $j \geq 2$, the only case in which the differential might be non-trivial is $j = p$. We have that

$$\mathrm{d}_{2p}(\xi_{2p+1}) = t \mu_2 \gamma_2^p$$

with $t \in \mathbb{F}_p$. Applying Φ, we obtain that

$$u^p \,\mathrm{d}_{2p}(\xi_{2p+1}) = t u^{p+2} \mu_2 \gamma_2^p,$$

which implies that $t(1-u^2) = 0$, and so $t = 0$. Therefore, $\xi_{2p+1} \in E_{2j}$ survives to E_{2j+1}. $\square$

Remark 5.17. For $p = 3$, following the proof of Proposition 5.16, we are only able to show that $\mathrm{d}_3(\xi_7) = t \lambda_2 \nu_2 \gamma_2^2$ for some $t \in \mathbb{F}_p$. If $t = 0$, then $E_3 = E_\infty$. Otherwise, the spectral sequence does not collapse until at least the fourth page. This stands in contrast with [Sie96, Theorem 5], where it is shown that $E_2(\mathrm{Heis}(3)) = E_\infty(\mathrm{Heis}(3))$.

Proposition 5.18. *For $p \geq 3$, the element $\nu_3 \in E_3$ survives to E_∞.*

Proof. Observe that, for some $t \in \mathbb{F}_p$, we have that $\mathrm{d}_3(\nu_3) = t \mu_2 \gamma_2 \in \langle \mu_2 \gamma_2 \rangle$. Applying Φ we obtain that $\Phi(\mathrm{d}_3(\nu_3)) = t u^2 \mu_2 \gamma_2$. Then, $t(u^2-1)\mu_2\gamma_2 = 0$ implies that $t = 0$, as desired. $\square$

Proposition 5.19. *For $p \geq 3$, the elements $\omega_6, \omega_7, \dots, \omega_{2p+2} \in E_3$ survive to E_∞.*

Proof. The proof for the elements $\omega_7, \dots, \omega_{2p+2} \in E_3$ with any $p \geq 3$, and for ω_6 with $p = 3$, is analogous to the proof of Proposition 5.16, and can be done following the proof of [Sie96, Theorem 7]. For $p \geq 5$, the element $\omega_6 = \frac{2}{3}\nu_3\mu_3$ also survives to E_∞, for degree reasons. □

Therefore, Propositions 5.16, 5.18 and 5.19 prove the following result.

Theorem 5.20. *Let $n \geq 2$ and let $p \geq 5$. Then, the LHS spectral sequence E associated to G collapses in the third page, i.e. $E_3 = E_\infty$.*

Remark 5.21. For $p \geq 5$, combining our result with [Sie96, Theorem 7], we obtain that $E_3(\mathrm{Heis}(p^n)) = E_\infty(\mathrm{Heis}(p^n))$ for all $n \geq 1$.

5.7 Poincaré series

In this section, we will compute the Poincaré series of $\mathrm{H}^\bullet(G, \mathbb{F}_p)$, i.e. the power series

$$P(t) = \sum_{k=0}^{\infty} \big(\dim \mathrm{H}^k(G)\big)t^k = \sum_{k=0}^{\infty}\sum_{r=0}^{k}(\dim E_\infty^{r,k-r})t^k.$$

Let $D_\infty = E_\infty/\langle \nu_{2p}\rangle = D_3$, which is the subring of E_∞ generated by all the generators except for ν_{2p}. Given that $E_\infty = \mathbb{F}_p[\nu_{2p}] \otimes D_\infty$, in order to obtain the Poincaré series of E_∞ we only need to compute the Poincaré series of D_∞ and multiply it by the Poincaré series of $\mathbb{F}_p[\nu_{2p}]$. For $k \geq 0$, write

$$D_\infty^k = \bigoplus_{r+s=k} D_\infty^{r,s}, \quad \text{so that} \quad \dim D_\infty^k = \sum_{r=0}^{k} \dim D_\infty^{r,k-r}.$$

Then, the Poincaré series of D_∞ is given by the power series

$$P_D(t) = \sum_{k=0}^{\infty}(\dim D_\infty^k)t^k,$$

and so we first need to obtain the values $\dim D_\infty^k$ for each $k \geq 0$. Note that, for every $r, s \geq 0$, the number $\dim D_\infty^{r,s}$ is computed in Theorem 5.15. Indeed, for $i \geq 0$, we have that

$$\dim D_\infty^{1,s} = \begin{cases} 1, & \text{if } 0 \leq s \leq 2p-1,\\ 2, & \text{if } s \geq 2p, \end{cases} \qquad \dim D_\infty^{2i,s} = \begin{cases} 1, & \text{if } s = 0,1,\\ 2, & \text{if } s \geq 2, \end{cases}$$

$$\dim D_\infty^{2i+3,s} = \begin{cases} 1, & \text{if } s = 0,1,2,2p-1,\\ 0, & \text{if } 3 \leq s \leq 2p-2,\\ 2, & \text{if } s \geq 2p. \end{cases}$$

This information can be showcased in the following table:

$2p+1$	2	2	2	2	2	2	2
$2p$	2	2	2	2	2	2	2
$2p-1$	2	1	2	1	2	1	2
$2p-2$	2	1	2	0	2	0	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	2	1	2	0	2	0	2
2	2	1	2	1	2	1	2
1	1	1	1	1	1	1	1
0	1	1	1	1	1	1	1
	0	1	2	3	4	5	6

Figure 5.3: Dimension of $D_\infty^{r,s}$ for $0 \leq r \leq 6$ and $0 \leq s \leq 2p+1$.

Lemma 5.22. *For $k \geq 0$, we have that*

$$\dim D_\infty^k = \begin{cases} k+1, & \text{if } k = 0, 1, \\ k+2, & \text{if } k = 2, 3, \\ k+3, & \text{if } 4 \leq k \leq 2p, \\ 2k-2p+3, & \text{if } k \geq 2p+1. \end{cases}$$

Proof. The values $\dim D_\infty^k$ for $0 \leq k \leq 3$ can be easily computed from the table in Figure 5.3. Let $4 \leq k \leq 2p$ and write $k = 2i + \varepsilon$ with $\varepsilon = 0, 1$. Then, we can compute

$$\begin{aligned} \sum_{r=2}^{k-3} \dim D_\infty^{r,k-r} &= 2(i-2+\varepsilon) = k-4+\varepsilon, \\ \dim D_\infty^{k-2,2} &= 2-\varepsilon. \end{aligned}$$

Therefore, we obtain that

$$\dim D_\infty^k = \sum_{r=2}^{k-3} \dim D_\infty^{r,k-r} + \dim D_\infty^{k-2,2} + 5 = (k-4+\varepsilon) + (2-\varepsilon) + 5 = k+3.$$

Let now $k \geq 2p+1$ and write $k = 2i + \varepsilon$ with $\varepsilon = 0, 1$. Then, we can compute the following values:

$$\sum_{r=0}^{k-2p} \dim D_\infty^{r,k-r} = 2(k-2p+1) = 2k-4p+2, \qquad \dim D_\infty^{k-2p+1,2p-1} = 1+\varepsilon,$$

$$\sum_{r=k-2p+2}^{k-3} \dim D_\infty^{r,k-r} = 2(p-2) = 2p-4, \qquad \dim D_\infty^{k-2,2} = 2-\varepsilon.$$

Therefore, we obtain that

$$\begin{aligned}
\dim D_\infty^k &= \sum_{r=0}^{k-2p} \dim D_\infty^{r,k-r} + \dim D_\infty^{k-2p+1,2p-1} \\
&\quad + \sum_{r=k-2p+2}^{k-3} \dim D_\infty^{r,k-r} + \dim D_\infty^{k-2,2} + 2 \\
&= (2k-4p+2) + (2p-4) + (1+\varepsilon) + (2-\varepsilon) + 2 \\
&= 2k-2p+3.
\end{aligned}$$

□

As a result, we can compute the Poincaré series of $\mathrm{H}^\bullet(G, \mathbb{F}_p)$.

Theorem 5.23. *The Poincaré series of* $\mathrm{H}^\bullet(G, \mathbb{F}_p)$ *is*

$$P(t) = \frac{1+t^2-t^3+t^4-t^5+t^{2p+1}}{(1-t)^2(1-t^{2p})}.$$

Proof. Using Lemma 5.22, we can compute the Poincaré series for D_∞ as follows:

$$\begin{aligned}
P_D(t) &= \sum_{k=0}^{\infty} (\dim D_\infty^k) t^k \\
&= 1 + 2t + 4t^2 + 5t^3 + \sum_{k=4}^{2p} (k+3)t^k + \sum_{k=2p+1}^{\infty} (2k-2p+3)t^k \\
&= \frac{1+t^2-t^3+t^4-t^5+t^{2p+1}}{(1-t)^2}.
\end{aligned}$$

Therefore, because $E_\infty = \mathbb{F}_p[\nu_{2p}] \otimes D_\infty$, we have that

$$P(t) = \frac{P_D(t)}{(1-t^{2p})} = \frac{1+t^2-t^3+t^4-t^5+t^{2p+1}}{(1-t)^2(1-t^{2p})}.$$

□

As a consequence of Theorem 5.20, we obtain that, for a prime number $p \geq 5$, the LHS spectral sequences E of G are isomorphic from the second page on as bigraded $\mathbb{F}_p$-algebras. We have not, however, determined the ring structure of $\mathrm{H}^\bullet(\mathrm{Heis}(p^n), \mathbb{F}_p)$. Now, as we discussed in Section 1.3.1, by [Car05, Theorem 2.1], there are finitely many liftings of $E_\infty(\mathrm{Heis}(p^n))$ to the cohomology ring $\mathrm{H}^\bullet(\mathrm{Heis}(p^n), \mathbb{F}_p)$. This in particular yields the following result.

Corollary 5.24. *Let $p \geq 5$ be a prime number. Then, there are only finitely many isomorphism types of graded-commutative $\mathbb{F}_p$-algebras in the infinite collection $\{\mathrm{H}^\bullet(\mathrm{Heis}(p^n), \mathbb{F}_p) \mid n \geq 1\}$.*

The above result is in slight analogy with the previously obtained results in the area [Car05], [DGG17], [DGG18], [GG19], [Sym21]. Let $\mathbb{G}(-)$ denote an affine group scheme over a ring. For example, the Heisenberg group $\widehat{G} = \mathbb{Z} \ltimes (Z \times \mathbb{Z})$ and the group G are obtained by applying such a functor $\mathbb{G}(-)$ to $\mathbb{Z}$ and to $\mathbb{Z}/p^n\mathbb{Z}$, respectively. The presentations of the cohomology rings of such groups are intrinsically hard to obtain. For instance, in [Qui72], Quillen described the cohomology rings of the general linear groups $\mathrm{GL}_n(L)$ over a field L of characteristic p with coefficients in a field K of characteristic coprime to p and with trivial $\mathrm{GL}_n(L)$-action. However, the case where L and K have the same characteristic is widely open. Based on Corollary 5.24, we state the following conjecture.

Conjecture 5.25. Let p be a prime number and let $\mathbb{G}(-)$ be an affine group scheme over the p-adic integers $\mathbb{Z}_p$. Then, there exists a natural number $f = f(p, \mathbb{G})$ that depends only on p and on $\mathbb{G}$ such that, for each p and for all $n \geq f$, the graded-commutative $\mathbb{F}_p$-algebras $\mathrm{H}^\bullet(\mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p), \mathbb{F}_p)$ are isomorphic, where $\mathbb{F}_p$ has trivial $\mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p)$-action.

The first reason to support the previous conjecture is that the Quillen categories of the groups $\mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ are isomorphic, equivalently $\mathrm{H}^\bullet(\mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p), \mathbb{F}_p)$ are F-isomorphic (see [Qui71a], [Qui71b]). Secondly, observe that for each $n \geq 2$, there is an extension

$$\mathbb{G}^1(\mathbb{Z}_p/p^n\mathbb{Z}_p) \to \mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p) \to \mathbb{G}(\mathbb{Z}_p/p\mathbb{Z}_p),$$

where $\mathbb{G}^1(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ denotes the first congruence subgroup of $\mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p)$. It is known that $\mathbb{G}^1(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ is a powerful p-central group with the Ω-extension property and thus, for every $n \geq 2$, the graded-commutative $\mathbb{F}_p$-algebras $\mathrm{H}^\bullet(\mathbb{G}^1(\mathbb{Z}_p/p^n\mathbb{Z}_p), \mathbb{F}_p)$ are isomorphic [Wei00]. Moreover, the actions of $\mathbb{G}(\mathbb{Z}_p/p\mathbb{Z}_p)$ on $\mathrm{H}^\bullet(\mathbb{G}^1(\mathbb{Z}_p/p^n\mathbb{Z}_p), \mathbb{F}_p)$

are isomorphic, in the sense of [DGG17, Definition 5.5]. In turn, the second pages $E_2(\mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p))$ are isomorphic as bigraded $\mathbb{F}_p$-algebras. Therefore, based on [DGG17, Conjecture 6.1], we would expect that the above conjecture holds by taking f to be equal to 2.

Resumen en castellano

La cohomología de grupos surge, junto con el álgebra homológica, de la topología algebraica y el estudio de los grupos de cohomología de ciertos espacios topológicos. Dado un grupo finito G, podemos asociarle un espacio clasificador BG, el cual satisface que su primer grupo de homotopía es isomorfo a G, y sus grupos de homotopía de órdenes superiores son triviales. Podemos definir los grupos de cohomología $\mathrm{H}^n(G,V)$ de G con coeficientes en un RG-módulo V, donde R es un anillo conmutativo, como los grupos de cohomología de su espacio clasificador BG, véase [Hat02]. También es posible dar una definición puramente algebraica de los grupos de cohomología de G en términos de functores derivados, y de hecho este es enfoque que seguiremos a lo largo de esta tesis.

Posteriormente, el estudio de la cohomología de grupos se ha convertido en un área de investigación importante por derecho propio. Si tomamos la suma directa $\mathrm{H}^\bullet(G,R)$ de todos los grupos de cohomología del grupo finito G con coeficientes en el módulo trivial R equipada con el así llamado producto cup, obtenemos un anillo conmutativo-graduado finitamente generado [Eve91, Capítulo 3]. Esto demuestra que la cohomología de grupos posee una rica estructura algebraica que puede ser explotada para obtener una gran cantidad de información, como el número mínimo de generadores y relaciones en una presentación del grupo. También tiene innumerables aplicaciones fuera de la teoría de grupos, en áreas como la teoría de números y la geometría algebraica, véanse [Gui18] y [Sil13].

Una característica importante de la cohomología de grupos es que admite múltiples caracterizaciones, las cuales contribuyen a darnos una visión más completa de la materia. De hecho, la cohomología puede describirse en términos de extensiones tanto de módulos como de grupos. Por un lado, la caracterización del functor Ext en términos de extensiones de módulos, debida originalmente a Yoneda [Yon92], nos permite describir los elementos de $\mathrm{H}^n(G, V)$ como clases de equivalencia de extensiones de RG-módulos de la forma

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow R \longrightarrow 0.$$

Por otra parte, es bien sabido que el segundo grupo de cohomología $\mathrm{H}^2(G, V)$ clasifica, salvo equivalencia, las extensiones de grupos de la forma

$$0 \longrightarrow V \longrightarrow E \longrightarrow G \longrightarrow 1.$$

Esto puede generalizarse, tal y como se hace en [Hol79], a grupos de cohomología de grado superior, de modo que los elementos de $\mathrm{H}^n(G, V)$ puedan verse como clases de equivalencia de extensiones cruzadas de la forma

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

Estas dos caracterizaciones de la cohomología de grupos comparten muchas de sus características fundamentales. Cuando se estudian una junto a la otra, proporcionan una descripción mucho más natural, aunque menos computacional, no sólo de los propios grupos de cohomología, sino también de sus propiedades functoriales, homomorfismos conectores y el producto cup. Además, estas caracterizaciones también nos ayudan a construir explícitamente clases de cohomología con propiedades específicas que, de otro modo, serían muy difíciles de encontrar.

Nos interesarán sobre todo los anillos de cohomología módulo p de p-grupos finitos, ya que el cálculo de los anillos de cohomología de grupos finitos puede reducirse a ese caso. De hecho, la cohomología es en general más fácil de calcular cuando se trabaja con coeficientes sobre un cuerpo K debido a resultados como la fórmula de Künneth y el teorema de coeficientes universales, véase [Eve91, Sección 2.5]. Además, podemos utilizar el teorema de Maschke [Ben91, Corolario 3.6.12] para demostrar que la cohomología de G es trivial siempre que la característica de K no divida el orden de G. Entonces, es posible suponer que $K = \mathbb{F}_p$, siendo p un factor primo del orden de G, utilizando el teorema de coeficientes universales. Por último, puede demostrarse que la cohomología módulo p de G se embebe en la cohomología módulo p de cualquiera de sus p-subgrupos de Sylow [Bro82, Sección III.10].

En general, no es posible determinar si dos grupos finitos dados son isomorfos simplemente mirando a sus anillos de cohomología, es decir, la familia de grupos finitos no posee la propiedad de unicidad cohomológica. En efecto, si p es un primo impar y G es un p-grupo abeliano finito, el tipo de isomorfismo del anillo de cohomología módulo p de G depende únicamente del número mínimo de generadores de G, véase

[Eve91, Sección 3.5]. En particular, dos p-grupos cíclicos no isomorfos cualesquiera tienen anillos de cohomología módulo p isomorfos.

No obstante, podemos restringir nuestra atención a familias específicas de p-grupos finitos y estudiar si los grupos de estas familias pueden ser distinguidos por sus cohomologías módulo p. Nos planteamos entonces las siguientes preguntas: ¿Es una familia dada de grupos cohomológicamente única? Y por el contrario, ¿qué familias de grupos no pueden distinguirse cohomológicamente, porque sólo tienen un número finito de anillos de cohomología salvo isomorfismo? Como ya hemos explicado, la respuesta a la primera pregunta es afirmativa para la familia de p-grupos abelianos elementales finitos. Por otra parte, si tomamos todos los p-grupos abelianos finitos de un cierto rango, entonces todos los grupos de esta familia tienen el mismo anillo de cohomología módulo p.

En nuestros intentos de determinar la unicidad cohomológica de ciertas familias de grupos, nos encontramos con un reto importante. En general, es extremadamente difícil calcular el anillo de cohomología de un grupo dado, no hablemos ya de una familia de grupos. De hecho, hay muy pocos ejemplos, aparte de los ya mencionados, de cálculos explícitos de anillos de cohomología en la literatura. Es por esta razón que podemos optar por centrar nuestra atención en el cálculo de ciertos invariantes de la cohomología, en lugar de tratar de determinar la estructura completa del anillo.

Así, es interesante estudiar los invariantes algebraicos del anillo de cohomología de G, y cómo se relacionan con la estructura de grupo de G. Por ejemplo, la dimensión de Krull de la cohomología módulo p de G se puede calcular fácilmente, gracias a un resultado de Quillen [CTVZ03, Corolario 8.4.7], como el rango de cualquier p-subgrupo abeliano elemental maximal de G. Este importante resultado determina completamente la dimensión de Krull de la cohomología en términos de la estructura de los subgrupos del grupo.

Los intentos de hacer lo mismo con otro invariante algebraico, la profundidad, han sido hasta ahora mucho menos exitosos. A pesar de estar estrechamente relacionada con la dimensión de Krull, la profundidad es considerablemente más difícil de calcular. Hasta ahora, sólo se han encontrado cotas superiores [CTVZ03, Proposición 12.2.5] e inferiores [Duf81]. No obstante, Carlson enunció en [Car95] una conjetura que caracteriza la profundidad de la cohomología módulo p de G en términos de lo bien que puede detectarse dicha cohomología al restringir a las cohomologías de ciertos subgrupos de G.

El principal obstáculo en el estudio de esta conjetura es que, normalmente, necesitaríamos calcular primero el anillo de cohomología y luego utilizar métodos computacionales para determinar la profundidad. La falta de ejemplos en la literatura, sin embargo, hace que este enfoque sea inútil para el estudio de la profundidad en familias infinitas de p-grupos. Para p impar, consideramos el grupo pro-p de clase de nilpotencia máxima G, que tiene un único cociente finito G_r de orden p^{r+1} para cada entero $r \geq 2$. Utilizando las cotas antes mencionados para la profundidad de $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$, somos capaces de determinar que su valor es 1 o 2, para cada $r \geq 2$. Es ahora cuando podemos emplear la caracterización de la cohomología de grupos en términos de extensiones. Para calcular la profundidad cuando $r \leq p-2$, construimos una clase de cohomología no trivial en $\mathrm{H}^3(G_r, \mathbb{F}_p)$ como producto de una extensión de Yoneda en $\mathrm{H}^1(G_r, \mathbb{F}_p)$ y una extensión cruzada en $\mathrm{H}^2(G_r, \mathbb{F}_p)$, y demostramos que su restricción es trivial en cada subgrupo de una cierta familia de subgrupos de G_r. Esto nos permite utilizar los resultados de Carlson en [Car95] para demostrar que la profundidad de $\mathrm{H}^\bullet(G_r, \mathbb{F}_p)$ es 1 para $2 \leq r \leq p-2$, lo cual significa que estos grupos satisfacen la conjetura de profundidad de Carlson. Y lo que es más importante, hemos sido capaces de calcular el valor de la profundidad sin tener que calcular primero los anillos de cohomología. Estos resultados publicados en [GGG22], junto con otros de Garaialde Ocaña [Gar18], sugieren que los anillos de cohomología de los cocientes finitos de G son idénticos o extremadamente similares.

Pasamos ahora a centrarnos en el cálculo de los anillos de cohomología en sí. Hacerlo para familias infinitas de grupos sólo es posible a través de un estudio detallado de los grupos específicos considerados. Las sucesiones espectrales han demostrado ser una herramienta extraordinariamente poderosa en el cálculo de la cohomología de grupos finitos, y se han convertido en una de las principales técnicas empleadas con este objetivo. Aparecen principalmente en la forma de la sucesión espectral de Lyndon-Hochschild-Serre (LHS) [Eve91, Sección 7.2], que nos permite en cierto modo aproximar la cohomología de un grupo G que puede obtenerse como una extensión de grupos

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

partiendo de las cohomologías del cociente Q y del subgrupo normal N, y calculando repetidamente grupos de cohomología. El principal inconveniente de este enfoque es que depende del cálculo de ciertas diferenciales, de las cuales normalmente poseemos poca o ninguna información. Este problema puede resolverse en determinadas circunstancias, en las cuales es posible encontrar fórmulas explícitas que nos ayudan a calcular dichas diferenciales. Tal es el caso cuando G es una extensión escindida de Q por N, como demostraron Charlap y Vasquez [CV69] y posteriormente adaptó Siegel en [Sie96] al caso concreto cuando Q es un p-grupo cíclico de orden p. En [GG23], generalizamos este resultado de Siegel para cuando Q es un p-grupo cíclico de cualquier orden.

También en [Sie96], Siegel utiliza sus resultados sobre diferenciales en el cálculo de la sucesión espectral LHS del grupo de Heisenberg $\mathrm{Heis}(p)$ módulo p para $p \geq 3$.

Podemos seguir el mismo argumento que Siegel para calcular las sucesiones espectrales LHS de los grupos de Heisenberg $\mathrm{Heis}(p^n)$ módulo p^n para $p \geq 5$ y $n \geq 2$, y demostrar que son todas isomorfas empezando por la segunda página. Esto implica que, en esta familia infinita de grupos, sólo aparece un número finito de clases de isomorfismo de anillos de cohomología. Los resultados anteriores han sido publicados en [GG23].

La estructura de la tesis es la siguiente:

En el capítulo 1, introducimos los conceptos y resultados principales de álgebra homológica que usaremos a lo largo de esta tesis. Comenzamos introduciendo complejos de cadenas y cocadenas de módulos, así como módulos proyectivos y resoluciones proyectivas, las cuales usamos para definir los functores Tor y Ext. A continuación, definimos los grupos de cohomología de grupos finitos y revisamos algunas de sus propiedades básicas. Construimos la resolución barra de un grupo finito y calculamos con ella los grupos de cohomología de bajo grado. Después, recordamos la estructura de la cohomología como anillo conmutativo-graduado dada por el producto cup. Además, damos una descripción detallada de los homomorfismos de Bockstein, los cuales empleamos a continuación para clasificar las extensiones centrales de p-grupos abelianos elementales de rango dos con núcleo cíclico de orden p. Concluímos el capítulo con una introducción a las sucesiones espectrales, centrándonos en particular en la sucesión espectral de Lyndon-Hochschild-Serre y sus propiedades principales.

En el capítulo 2, describimos la cohomología de un grupo finito en términos de extensiones. En primer lugar, damos la descripción clásica de Ext usando extensiones de Yoneda. Seguidamente, introducimos las extensiones cruzadas para describir grupos de cohomología. Después, definimos un producto de extensiones de Yoneda con extensiones cruzadas que coincide con el producto cup usual en cohomología.

En el capítulo 3, introducimos el concepto de profundidad para el anillo de cohomología módulo p de un grupo finito y formulamos la conjetura de profundidad de Carlson. A continuación, calculamos la profundidad de los anillos de cohomología módulo p de ciertos cocientes del grupo pro-p de clase maximal que, además, satisfacen la conjetura de profundidad de Carlson.

En el capítulo 4, formulamos un teorema de Charlap y Vasquez sobre el cálculo de la segunda diferencial de la sucesión espectral de Lyndon-Hochschild-Serre asociada a una extension escindida de grupos finitos. Después, introducimos una generalización de un resultado de Siegel que puede ser utilizado para calcular las diferenciales que aparecen en la sucesión espectral asociada a una sucesión escindida de grupos finitos con cociente cíclico de orden potencia prima.

En el capítulo 5, calculamos la sucesión espectral de Lyndon-Hochschild-Serre de una familia de grupos de Heisenberg finitos de orden potencia prima, hasta obtener la página del infinito. Comenzamos calculando la segunda página de la sucesión espectral y su estructura como álgebra, antes de poner en uso los resultados del capítulo anterior para calcular la segunda diferencial. Después, determinamos la tercera página y mostramos que es en este punto cuando la sucesión espectral colapsa. De este modo, proporcionamos una de las primeras familias infinitas de p-grupos cuyas sucesiones espectrales de Lyndon-Hochschild-Serre colapsan en la misma página y son isomorfas. Finalmente, calculamos la serie de Poincaré de los anillos de cohomología.

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