COTUTELLE THESIS PRESENTED
TO OBTAIN THE QUALIFICATION OF

DOCTOR OF

THE UNIVERSITY OF BORDEAUX, BORDEAUX - FRANCE
AND THE UNIVERSITY OF THE BASQUE COUNTRY, BILBAO - SPAIN

DOCTORAL SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE
DOCTORAL SCHOOL OF MATHEMATICS AND STATISTICS

SPECIALIZATION : APPLIED MATHEMATICS AND SCIENTIFIC COMPUTATION

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SPECTRAL PROPERTIES OF DIRAC OPERATORS ON CERTAIN DOMAINS

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Will be defended on May 13, 2024 - University of Bordeaux

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TITLE : SPECTRAL PROPERTIES OF DIRAC OPERATORS ON CERTAIN DOMAINS.

Abstract

This thesis mainly focused on the spectral analysis of perturbation models of the free Dirac operator, in 2-D and 3-D space. More precisely, this thesis is divided into two parts : *Dirac operator with MIT bag conditions* (Chapters 2, 3) and *Dirac operator coupled with delta shell interactions* (Chapters 4, 5). Most of these studies are conducted through the analysis of the resolvents of these operators.

The first chapter of this thesis examines perturbation of the Dirac operator by a large mass M, supported on a domain. Our main objective is to establish, under the condition of sufficiently large mass M, the convergence of the perturbed operator, towards the Dirac operator with the MIT bag condition, in the norm resolvent sense. To this end, we introduce what we refer to the Poincaré-Steklov (PS) operators (as an analogue of the Dirichlet-to-Neumann operators for the Laplace operator) and analyze them from the microlocal point of view, in order to understand precisely the convergence rate of the resolvent. On one hand, we show that the PS operators fit into the framework of pseudodifferential operators and we determine their principal symbols. On the other hand, since we are mainly concerned with large masses, we treat our problem from the semiclassical point of view, where the semiclassical parameter is $h = M^{-1}$. Finally, by establishing a Krein formula relating the resolvent of the perturbed operator to that of the MIT bag operator, and using the pseudodifferential properties of the PS operators combined with the matrix structures of the principal symbols, we establish the required convergence with a convergence rate of $\mathcal{O}(M^{-1})$.

In the second chapter, we define a tubular neighborhood of the boundary of a given regular domain. We consider perturbation of the free Dirac operator by a large mass M, within this neighborhood of thickness $\varepsilon := M^{-1}$. Our primary objective is to study the convergence of the perturbed Dirac operator when M tends to $+\infty$. Comparing with the first part, we get here two MIT bag limit operators, which act outside the boundary. It's worth noting that the decoupling of these two MIT bag operators can be considered as the confining version of the Lorentz scalar delta interaction of Dirac operator, supported on a closed surface. The methodology followed, as in the previous problem study the pseudodifferential properties of Poincaré-Steklov operators. However, the novelty in this problem lies in the control of these operators by tracking the dependence on the parameter ε, and consequently, in the convergence as ε goes to 0 and M goes to $+\infty$. With these ingredients, we prove that the perturbed operator converges in the norm resolvent sense to the Dirac operator coupled with Lorentz scalar δ-shell interaction.

In the third chapter, we investigate the generalization of an approximation of the three-dimensional Dirac operator coupled with a singular combination of electrostatic and Lorentz scalar δ-interactions supported on a closed surface, by a Dirac operator with a regular potential localized in a thin layer containing the surface. In the non-critical and non-confining cases, we show that the regular perturbed Dirac operator converges in the strong resolvent sense to the singular δ-interaction of the Dirac operator. Moreover, we deduce that the coupling constants of the limit operator depend nonlinearly on those of the potential under consideration.

In the last chapter, our study focuses on the two-dimensional Dirac operator coupled with the electrostatic and Lorentz scalar δ-interactions. We treat in low regularity Sobolev spaces ($H^{1/2}$) the self-adjointness of certain realizations of these operators in various curve settings. The most important case in this chapter arises when the curves under consideration are curvilinear polygons, with smooth, differentiable edges and without cusps. Under certain conditions on the coupling constants, using the Fredholm property of certain boundary integral operators, and exploiting the explicit form of the Cauchy transform on non-smooth curves, we achieve the self-adjointness of the perturbed operator.

Keywords : Spectral analysis, Dirac operators, self-adjoint extensions, δ-shell interactions, quantum confinement, Poincaré-Steklov operators, the MIT bag model, h-Pseudodifferential operators, large coupling limits.

TITRE : PROPRIÉTÉS SPECTRALES DES OPÉRATEURS DE DIRAC SUR CERTAINS DOMAINES.

Résumé

Cette thèse se focalise sur l'étude spectrale des modèles de perturbations de l'opérateur de Dirac libre en dimensions 2 et 3. Plus précisément, cette thèse est divisée en deux parties : *opérateur de Dirac avec conditions aux bords MIT bag* (Chapitres 2, 3) et *opérateur de Dirac couplé à une delta interaction* (Chapitres 4, 5). La plupart de ces études sont réalisées à travers l'analyse des résolvantes de ces opérateurs.

Le premier chapitre de cette thèse étudie la perturbation de l'opérateur de Dirac par une grande masse M, supportée sur un domaine. Notre objectif principal est d'établir, sous la condition d'une masse M suffisamment grande, la convergence de l'opérateur perturbé vers l'opérateur de Dirac avec la condition au bord MIT bag, au sens de la norme de la résolvante. Pour se faire, nous introduisons ce que nous appelons les opérateurs Poincaré-Steklov (PS) (comme un analogue des opérateurs Dirichlet-to-Neumann pour l'opérateur de Laplace) et les analysons d'un point de vue microlocal, afin de comprendre précisément le taux de convergence de la résolvante. D'une part, nous montrons que les opérateurs PS s'intègrent dans le cadre des opérateurs pseudodifférentiels et nous déterminons leurs symboles principaux. D'autre part, comme nous nous intéressons principalement aux grandes masses, nous traitons notre problème du point de vue semiclassique, où le paramètre semiclassique est $h = M^{-1}$. Enfin, en établissant une formule de Krein reliant la résolvante de l'opérateur perturbé à celle de l'opérateur MIT bag, et en utilisant les propriétés pseudodifférentielles des opérateurs PS combinées aux structures matricielles des symboles principaux, nous établissons la convergence requise avec un taux de convergence de $\mathcal{O}(M^{-1})$.

Dans le deuxième chapitre, nous définissons un voisinage tubulaire de la frontière d'un domaine régulier donné. Nous considérons la perturbation de l'opérateur de Dirac libre par une grande masse M, supportée dans ce voisinage d'épaisseur $\varepsilon := M^{-1}$. Notre objectif principal est d'étudier la convergence de l'opérateur de Dirac perturbé lorsque M tend vers $+\infty$. En comparaison avec la première partie, nous obtenons ici deux opérateurs limites MIT bag, qui agissent en dehors de la frontière. Il est intéressant de noter que le découplage de ces deux opérateurs MIT bag peut être considéré comme la version confinée de δ-interaction scalaire de Lorentz de l'opérateur de Dirac, supportée sur une surface fermée. La méthodologie suivie, comme au problème précédent, porte sur l'étude des propriétés pseudodifférentielles des opérateurs de Poincaré-Steklov. Cependant, la nouveauté de ce problème réside dans le contrôle de ces opérateurs en suivant la dépendance du paramètre ε, et par conséquent, dans la convergence lorsque ε tend vers 0 et M tend vers $+\infty$. Avec ces ingrédients, nous prouvons que l'opérateur perturbé converge au sens de la norme de la résolvante vers l'opérateur de Dirac couplé à une δ-interaction scalaire de Lorentz.

Dans le troisième chapitre, nous généralisation une approximation de l'opérateur de Dirac tridimensionnel couplé à une combinaison singulière de δ-interactions électrostatiques et scalaires de Lorentz supportée sur une surface fermée, par un opérateur de Dirac avec un potentiel régulier localisé dans une couche mince contenant la surface. Dans les cas non-critiques et non-confinants, nous montrons que l'opérateur de Dirac perturbé régulier converge au sens de la résolvante forte vers la δ-interaction singulière de l'opérateur de Dirac. De plus, nous déduisons que les constantes de couplage de l'opérateur limite dépendent de manière non-linéaire de celles du potentiel considéré.

Dans le dernier chapitre de cette thèse, notre étude porte sur l'opérateur de Dirac bidimensionnel couplé à une δ-interaction électrostatique et scalaire de Lorentz. Nous traitons dans des espaces de Sobolev de faible régularité ($H^{1/2}$) l'auto-adjonction de certaines réalisations de ces opérateurs dans divers contextes de courbes. Le cas le plus important dans ce chapitre se présente lorsque les courbes considérées sont des polygones curvilignes, avec des bords lisses et différentiables et sans cuspides. Sous certaines conditions sur les constantes de couplage, en utilisant la propriété de Fredholm de certains opérateurs intégraux de frontière, et en exploitant la forme explicite de la transformée de Cauchy sur des courbes non lisses, nous établissons l'auto-adjonction de l'opérateur perturbé.

Mots-clés : Analyse spectrale, opérateurs de Dirac, extensions auto-adjointes, δ-shell interactions, opérateurs de Poincaré-Steklov, le modèle MIT bag, opérateurs h-Pseudodifferentiel, couplage fort.

TITULO : PROPIEDADES ESPECTRALES DE OPERADORES DE DIRAC EN ALGUNOS DOMINIOS.

Resumen

Esta tesis aborda el análisis espectral de modelos de perturbación del operador libre de Dirac en dimensiones 2 y 3. Más concretamente, esta tesis se divide en dos partes : *Operador de Dirac con condiciones de borde MIT bag* (Capítulo 2, 3) y *Operador de Dirac acoplado a delta interacciones* (Capítulo 4, 5). La mayoría de estos estudios se realizan mediante el análisis de los resolventes de estos operadores.

El primer capítulo de esta tesis estudia la perturbación del operador de Dirac por una masa grande M, soportada en un dominio. Nuestro principal objetivo es establecer, bajo la condición de una masa M suficientemente grande, la convergencia del operador perturbado al operador de Dirac con la condición de borde MIT bag, en el sentido de la norma del resolvente. Para ello, introducimos lo que llamamos los operadores de Poincaré-Steklov (PS) (es decir, un análogo al mapa de Dirichlet-Neumann para el operador de Laplace) y los analizamos desde un punto de vista microlocal, con el fin de comprender con precisión la tasa de convergencia del resolvente. Por un lado, mostramos que los operadores PS encajan en el marco de los operadores pseudodiferenciales y determinamos sus símbolos principales. En segundo lugar, como nos interesan principalmente las masas grandes, tratamos nuestro problema desde el punto de vista semiclásicos, donde el parámetro semiclásicos es $h = M^{-1}$. Finalmente, estableciendo una fórmula de Krein que relaciona el resolvente del operador perturbado con el del operador MIT bag, y utilizando las propiedades pseudodiferenciales de los operadores PS combinadas con las estructuras matriciales de los símbolos principales, establecemos la convergencia requerida con una tasa de convergencia de $\mathcal{O}(M^{-1})$.

En el segundo capítulo, definimos una vecindad tubular de la frontera de un dominio regular dado. Consideramos la perturbación del operador de Dirac por una gran masa M, soportada en esta vecindad de espesor $\varepsilon := M^{-1}$. Nuestro principal objetivo es estudiar la convergencia del operador de Dirac perturbado cuando M tiende a $+\infty$. En comparación con la primera parte, obtenemos aquí dos operadores límite MIT bag, que actúan fuera de la frontera. Curiosamente, el desacoplamiento de estos dos operadores MIT bag puede verse como la versión de δ interacción escalar de Lorentz confinada del operador de Dirac, apoyado en una superficie cerrada. La metodología seguida en este problema en realidad entra en contacto con el problema anterior tratado por analogía con el estudio de las propiedades pseudodiferenciales de los operadores de Poincaré-Steklov. Sin embargo, la novedad de este problema radica en el control de estos operadores siguiendo la dependencia del parámetro ε, y en consecuencia, en la convergencia cuando ε tiende a 0 y M tiende a $+\infty$. Con estos ingredientes, demostramos que el operador perturbado converge en el sentido de la norma del resolvente al operador de Dirac acoplado con la δ-interacción de shell escalar de Lorentz.

En el tercer capítulo, generalizamos una aproximación del operador de Dirac tridimensional acoplado a una combinación singular de δ-interacciones electrostáticas y escalares de Lorentz soportadas sobre una superficie cerrada, por un operador de Dirac con un potencial regular localizado en una capa delgada que contiene la superficie. En los casos no críticos y no finitos, mostramos que el operador de Dirac perturbado regular converge euertemente en el sentido del resolvente a la δ-interacción singular del operador de Dirac. Además, deducimos que las constantes de acoplamiento del operador límite dependen no linealmente de las del potencial considerado.

En el último capítulo de esta tesis, estudiamos el operador de Dirac bidimensional acoplado con las δ-interacciones electrostática y escalar de Lorentz. Tratamos en espacios de Sobolev de baja regularidad ($H^{1/2}$) la autounión de ciertas realizaciones de estos operadores en varios contextos de curvas. El caso más importante de este capítulo surge cuando las curvas consideradas son polígonos curvilíneos, con bordes suaves, diferenciables y sin cúspides. Bajo ciertas condiciones sobre las constantes de acoplamiento, utilizando la propiedad de Fredholm de ciertos operadores integrales de frontera, y explotando la forma explícita de la transformada de Cauchy en curvas no suaves, establecemos la auto-unión del operador perturbado.

Palabras clave : Análisis espectral, operadores de Dirac, extensiones autoadjuntas, δ-shell interacciones, operadores de Poincaré-Steklov, el modelo del MIT bag, acoplamiento fuerte, operadores h-Pseudodiferenciales.

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$\mathbb{I}$	The identity.
$\mathbb{I}_2$ resp. $\mathbb{I}_4$	The 2×2 resp. 4×4 identity matrix.
A^*	The adjoint of the operator A.
$\mathrm{Dom}(A)$	The domain of an operator A.
$\mathrm{Sp}(A)$	Spectrum of the operator A.
$\mathrm{Sp}_{\mathrm{p}}(A)$	Point spectrum of the operator A.
$\mathrm{Sp}_{\mathrm{ess}}(A)$	Essential spectrum of the operator A.
$\mathrm{Sp}_{\mathrm{dis}}(A)$	Discrete spectrum of the operator A.
$\mathrm{Sp}_{\mathrm{cont}}(A)$	Continuous spectrum of the operator A.
$(\cdot,\cdot,\cdot,\cdot)^t$	Transpose of a function vector or a matrix.
$A \lesssim B$	There is a constant $C > 0$ such that $A \leqslant CB$.
$\langle \cdot, \cdot \rangle$	Scalar product on a Hilbert space.
$[\cdot, \cdot]$	The usual commutator bracket.
$\{\cdot, \cdot\}$	The usual anticommutator bracket.
$C_0^\infty(\mathbb{R}^d)$	The space of C^∞-functions with compact support in $\mathbb{R}^d$.
$\mathcal{D}'(\mathbb{R}^d)$	The space of distribution on $\mathbb{R}^d$ which is the dual space of $C_0^\infty(\mathbb{R}^d)$.
$\mathscr{S}(\mathbb{R}^d)$	The Schwartz space.
$\mathscr{S}'(\mathbb{R}^d)$	The space of tempered distribution which is the dual of $\mathscr{S}(\mathbb{R}^d)$.
$B(H)$	The space of bounded linear operators defined everywhere in a Hilbert space H.
$\mathrm{dist}(x, \Sigma)$	The distance between x and a point y belongs to the surface Σ.
$Re(z)$ or $\mathfrak{R}(\rho)$	The real part of a complex number z or a function ρ.
$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$.
mod	Modulo.
$f * g$	The convolution product of f and g.
$f \oplus g$	The direct sum of f and g.
$f \equiv g$	f and g are equivalent.
$f \simeq g$	f is almost equal to g.
$\alpha \cdot x$	$\sum_{j=1}^{j=d} \alpha_j x_j$, for all $x \in \mathbb{R}^d$.
$\mathcal{O}(M)$	There is a constant $C > 0$ such that $\|\|f\|\|_{B(H)} \leqslant C\,M$.
$\mathcal{L}(H)$	The space of bounded linear operators from H to itself.
$\langle \xi \rangle$	Is equal to the quantity $(1 + \|\xi\|^2)^{1/2}$.
$\mathscr{M}_k(\mathbb{C})$	The space of $k \times k$ complex matrices.
$f_{\|\Omega}$	The restriction of a function $f \in \mathbb{R}^3$ on the open set Ω.
r_Ω	The restriction operator on the open set Ω.
e_Ω	The extension operator (by 0) outside of Ω, which is the adjoint of r_Ω.

Chapter 1

General Introduction

1.1 Physical and mathematical motivations

The Dirac equation is of profound and multifaceted importance in both physics and mathematics. Its introduction by Paul Dirac in the 1920s marked a decisive moment in the development of theoretical physics, as it successfully reconciled the principles of quantum mechanics and special relativity. The mathematical formalism of the Dirac's equation, based on the theory of matrices and spinors, not only provided a comprehensive framework for describing the behavior of relativistic electrons, but also stimulated profound mathematical investigations and advances. In many applications in science and technology, it is not possible to solve the underlying mathematical models exactly. Therefore, suitable parameters in these mathematical models are replaced by idealized ones. The parameters should be chosen in such a way that the idealized model is more accessible from a mathematical point of view and still reflects physical reality to a reasonable degree of accuracy. To verify that the idealized models have similar properties as the original ones coming from applications is a difficult mathematical problem which is unsolved in many cases.

The *free Dirac equation*

$$i\partial_t \Psi(t,x) = \mathfrak{D}\Psi(t,x), \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^n, \quad \text{with } \partial_t = \frac{\partial}{\partial t}. \tag{1.1}$$

The equation introduced a set of mathematical operators known as ***free Dirac operators***, which are denoted by $\mathfrak{D}$. The free Dirac operators are represented by matrices acting on wave functions $\Psi(t,x)$, which depending on time t and a position x. The n-dimensional Dirac operator $\mathfrak{D}$ acts on a vector function $f:\Omega\to\mathbb{C}^N$ (where $\Omega\subset\mathbb{R}^n$ is an open set and $N := 2^{[\frac{n+1}{2}]}$) as

$$\mathfrak{D}f = -i\sum_{k=1}^{n}\alpha_k\frac{\partial f}{\partial x_k} + \alpha_{n+1}f,$$

with some special $N\times N$ matrices α_k (the so-called Dirac-Pauli matrices), so that one formally has $\mathfrak{D}^2 = -\Delta + \mathbb{I}_4$, where Δ is the n-dimensional Laplacian, and $\mathbb{I}_4$ is the 4×4 identity matrix. These operators play a crucial role in describing the behavior of spin-$1/2$ particles, such as electrons. In the context of free Dirac operators, the term "free" implies that the particles under consideration are not subject to external forces or interactions. This simplification allows physicists and mathematicians to focus on understanding the intrinsic properties and characteristics of the Dirac operators. Studying free

Dirac operators provides valuable insights into the fundamental nature of relativistic quantum systems and serves as a foundation for more complex quantum field theories.

In mathematical physics, researchers delve into the spectral analysis of free Dirac operators, examining the eigenvalues and eigenvectors associated with these operators. This analysis provides a deeper understanding of the mathematical structure underlying relativistic quantum systems and contributes to the development of mathematical tools applicable in various branches of physics.

The motivation behind the study of free Dirac operators lies at the intersection of quantum mechanics, quantum field theory, and mathematical physics. This motivation can be explored through several key aspects, *e.g., Integration of relativity into quantum mechanics, Introduction of Spinors and Clifford Algebras*, etc. On one hand, the primary motivation for the Dirac equation arises from the need to formulate a relativistically correct description of quantum-mechanical systems, particularly electrons. The Schrödinger equation, which successfully describes non-relativistic quantum mechanics, fails to account for effects associated with high speeds and energies. The Dirac equation, combining quantum mechanics with special relativity, emerged as a solution to this limitation, providing a more accurate description of the behavior of relativistic electrons moving at speeds close to the speed of light. Moreover, free Dirac operators have applications beyond particle physics, extending into areas such as condensed matter physics and materials science. The study of Dirac materials, which exhibit unique electronic properties governed by the principles of relativistic quantum mechanics, has gained significant attention. *Graphene*, for instance, is a well-known example of a material where the behavior of charge carriers can be effectively described by the Dirac equation in the absence of external forces. On the other hand, unlike the Schrödinger equation, the Dirac equation involves spinors, mathematical entities that extend the notion of vectors to include intrinsic angular momentum or spin. The need to account for intrinsic angular momentum and the observed magnetic properties of electrons led to the development of spinors and the utilization of Clifford algebras. The study of Clifford algebras and spinors became a rich area of mathematical investigation with applications in geometry and representation theory.

1.2 Dirac's approach to deriving the equation (1.1) in $\mathbb{R}^3$

Let m be the positive mass of a free particle and denote by $\mathfrak{p}$ the momentum of this particle. We consider the classical relativistic energy-momentum relation $E = \sqrt{c^2\mathfrak{p}^2 + c^4m^2}$ (with c the velocity of light). Then, the operators associated with the energy and the momentum, *i.e.*,

$$E \to i\hbar\partial_t, \quad \mathfrak{p} \to -i\hbar\nabla, \quad \text{with } \hbar = \text{ Planck's constant and } \nabla \text{ the gradient in } \mathbb{R}^3 \tag{1.2}$$

yield the Klein-Gordon equation

$$-\hbar^2\partial_t^2\Psi(t,x) = (-c^2\hbar^2\Delta + c^4m^2)\Psi(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^3,$$

where Δ is the Laplace operator and Ψ is the wave function. However, the Klein-Gordon equation lacks consistency with a quantum mechanical interpretation due to the inclusion of a second order time derivative and the absence of an L^2-conservation law. To do so, it is necessary to establish an equation that conserves the L^2 norm of the solution, ensuring that the wave function at time $t = 0$ determines the wave function at all subsequent times. In response to this challenge, Paul Dirac sought to modify the Klein-Gordon equation to derive an equation that incorporated a first-order time derivative, similar to the Schrödinger equation, while adhering to the principles of covariance in the context of special relativity. The initial step in his approach is to reexamine the energy-momentum relation E and, before to its translation into the language of quantum mechanics using (1.2), he linearized the expression through a written formulation. Taking these factors into account, the resulting equation takes the following form

$$\begin{aligned} i\hbar\partial_t\Psi(t,x) &= -i\hbar c(\alpha_1\partial_{x_1}+\alpha_2\partial_{x_2}+\alpha_3\partial_{x_3})\Psi(t,x)+\beta mc^2\Psi(t,x) \\ &\equiv -i\hbar c\,\alpha\cdot\nabla\Psi(t,x)+\beta mc^2\Psi(t,x), \quad \text{with } \partial_{x_j}=\frac{\partial}{\partial x_j},\ t\in\mathbb{R},\ x\in\mathbb{R}^3, \end{aligned}$$

where ∇ is the gradient in $\mathbb{R}^3$, and β, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ have to be determined from the energy-momentum relation E. The quantities α and β are anticommuting which are most naturally represented by 4×4-Hermitian and unitary matrices, called "Dirac matrices". More precisely, α_j and β are satisfying the following anticommutation relationship

$$\{\alpha_j,\alpha_k\}=2\delta_{jk}\mathbb{I}_4, \quad \{\alpha_j,\beta\}=0_4, \quad \beta^2=\mathbb{I}_4, \quad j,\,k\in\{1,2,3\}, \tag{1.3}$$

where, $\{\cdot,\cdot\}$ denotes the anticommutator bracket, δ_{jk} denotes the Kronecker symbol ($\delta_{jk}=1$ if $j=k$; $\delta_{jk}=0$ if $j\neq k$), and $\mathbb{I}_4$ resp. 0_4 are the 4-dimensional unit and zero matrices.

Paul Dirac introduced the standard representation

$$\beta=\begin{pmatrix}\mathbb{I}_2 & 0_2\\ 0_2 & -\mathbb{I}_2\end{pmatrix}, \quad \alpha_j=\begin{pmatrix}0_2 & \sigma_j\\ \sigma_j & 0_2\end{pmatrix} \quad \text{for } j=1,2,3, \tag{1.4}$$

with $\sigma=(\sigma_1,\,\sigma_2,\sigma_3)$ the 2×2-Hermitian Pauli matrices defined by

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad \mathbb{I}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{1.5}$$

The anticommutation relations

$$\{\sigma_j,\sigma_k\}=\sigma_j\sigma_k+\sigma_k\sigma_j=2\delta_{jk}\mathbb{I}_2 \text{ for all } j,k\in\{1,2,3\} \tag{1.6}$$

are well known. Using the above matrices α_1, α_2, α_3 and β, for m be the mass of a relativistic particle, Dirac proposed the equation known as *the Dirac equation* given by

$$i\partial_t\Psi(t,x)=D_m\Psi(t,x), \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}^3,$$

with D_m (from now on, we use the units $c=\hbar=1$) the three-dimensional free Dirac operators having the following matrix form

$$\begin{aligned} D_m &= -i\boldsymbol{\alpha}\cdot\nabla+m\boldsymbol{\beta} := -i\sum_{j=1}^{3}\alpha_j\partial_j+m\beta \\ &= \begin{pmatrix} m & 0 & -i\partial_3 & -i\partial_1-\partial_2\\ 0 & m & -i\partial_1+\partial_2 & i\partial_3\\ -i\partial_3 & -i\partial_1-\partial_2 & -m & 0\\ -i\partial_1+\partial_2 & i\partial_3 & 0 & -m \end{pmatrix}. \end{aligned} \tag{1.7}$$

D_m acts on $\mathbb{C}^4$-valued functions of $x\in\mathbb{R}^3$, which are denoted by $\psi(x)=(\psi_1(x),\psi_2(x),\psi_3(x),\psi_4(x))^t$, and belong to the following first order Sobolev space

$$H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \equiv H^1(\mathbb{R}^3)^4 = H^1(\mathbb{R}^3, \mathbb{C}^4) = H^1(\mathbb{R}^3) \otimes \mathbb{C}^4.$$

Thus, we define the free Dirac operator by

$$D_m\psi := (-i\alpha \cdot \nabla + m\beta)\psi, \quad \text{for all } \psi \in \mathrm{Dom}(D_m) := H^1(\mathbb{R}^3)^4.$$

The free Dirac operator is essentially self-adjoint on the dense domain $C_0^\infty(\mathbb{R}^3 \backslash \{0\})^4$ and self-adjoint on his domain $\mathrm{Dom}(D_m) = H^1\left(\mathbb{R}^3\right)^4$.

Its spectrum is purely continuous and given by

$$\mathrm{Sp}(D_m) = \mathrm{Sp}_{\mathrm{cont}}(D_m) = (-\infty, -m] \cup [m, +\infty).$$

Since in this thesis we are concerned with analyzing perturbations of Dirac operators, we gather below some spectral contributions regarding of the self-adjointness and the spectrum of a perturbed operator.

We mention that this manuscript is devoted to the analysis of self-adjoint Dirac operators. It is important to point out that several spectral studies have been carried out on non-self-adjoint Dirac operators (including discrete Dirac operators), see *for example* [KND22, DFKS22, CIKS20, FK19].

Definition 1.2.1. Let T be self-adjoint operator. We say that V is T-bounded (or relatively bounded with respect to T), if $\mathrm{Dom}(V) \supset \mathrm{Dom}(T)$ and $\exists$ a, b $\geqslant 0$ such that

$$\|V\xi\| \leqslant a\|T\xi\| + b\|\xi\|, \ \forall \xi \in \mathrm{Dom}(T).$$

We denote by $\mathcal{N}_T(V)$ the infimum of such an a.

Theorem 1.2.2. ***(Kato-Rellich Theorem).*** *Let T be self-adjoint operator, and V is symmetric. If V is T-bounded such that $\mathcal{N}_T(V) < 1$, then the operator $(T+V)u := Tu + Vu$ is self-adjoint in the domain $\mathrm{Dom}(T)$.*

Remark 1.2.1. *If we take a multiplication operator V with a hermitian 4×4 matrix such that each component V_{ik} is a function satisfying the estimate*

$$|V_{ik}(x)| \leqslant a\frac{c}{2|x|} + b, \ \forall x \in \mathbb{R}^3 \backslash \{0\}, \ i, k = 1, 2, 3, 4,$$

for some constant $b > 0$, $a < 1$. Then, for the free Dirac operator, we have the same result as in the previous theorem, see [Tha92, Theorem 4.2].

Theorem 1.2.3. ***(Weyl's essential spectrum theorem).*** *Let A be a self-adjoint operator and let B be a closed operator with $\mathrm{Dom}(A) = \mathrm{Dom}(B)$, so that:*

1) *For some (and hence all) $z \in \rho(A) \cap \rho(B)$, $(A-z)^{-1} - (B-z)^{-1}$ is compact.* *and let*
2) $\mathrm{Sp}(A) \neq \mathbb{R}$ *and* $\rho(B) \neq \emptyset$.

Then $\mathrm{Sp}_{\mathrm{ess}}(A) = \mathrm{Sp}_{\mathrm{ess}}(B)$.

Definition 1.2.4. (Relatively compact.) Let A be self-adjoint. An operator C with $\mathrm{Dom}(A) \subset \mathrm{Dom}(C)$ is called **relatively compact** with respect to A if and only if $C(A+i)^{-1}$ is compact.

Corollary 1.2.5. *Let A be a self-adjoint operator and let C be a relatively compact perturbation of A. Let $B = A + C$, then* $\mathrm{Sp}_{\mathrm{ess}}(B) = \mathrm{Sp}_{\mathrm{ess}}(A)$.

1.3 Sobolev and Dirac-Sobolev spaces

Definition 1.3.1 (Function space)**.** Let Ω be a non-empty open subset of $\mathbb{R}^d$. For an integer $k > 0$, we let

$$C^k(\Omega) := \{u : \Omega \to \mathbb{C} : \partial^\alpha u \text{ exists and is continuous in } \Omega \text{ for } |\alpha| \leqslant k\},$$
$$C_0^k(\Omega) := \{u \in C^k(\Omega) : \mathrm{supp}(u) \subset K \subset \Omega, \text{ for a compact set } K\},$$

where,

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}}, \text{ with } |\alpha| = \sum_{j=1}^{d} \alpha_j \text{ and } x = x_1 \cdots x_d,\ x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d.$$

We denote by

$$C^\infty(\Omega) = \bigcap_{k \geqslant 0} C^k(\Omega) \text{ the usual space of infinitely differentiable functions.}$$

Definition 1.3.2 (Lipschitz domain)**.** A open connected set $\Omega \subset \mathbb{R}^d$ is a κ-Lipschitz domain if for every $x \in \Omega$ there exist $r > 0$ and an isometric coordinate system with origin $x = x_0$ such that

$$C \cap \Omega = C \cap \{(\tilde{y}, t) : \tilde{y} \in \mathbb{R}^{d-1} \text{ and } g(\tilde{y}) < t\},$$

for $C = \{y \in \mathbb{R}^d : |x - y|_\infty < r\}$, and for a Lipschitz continuous function $g : \mathbb{R}^{d-1} \to \mathbb{R}$, with $g(x_0) = x_0$ and $||g'||_\infty \leqslant \kappa$. Then, we say a domain is a Lipschitz domain if it is a κ-Lipschitz domain for some $\kappa \geqslant 0$.

Definition 1.3.3 (Hölder space)**.** Let $\Omega \subset \mathbb{R}^d$ be a open bounded domain, $k \geqslant 0$ be an integer, and $\omega \in (0, 1]$. For $u \in C^k(\Omega)$ define the Hölder norm

$$||u||_{C^{k,\omega}(\Omega)} := \sum_{|\alpha| \leqslant k} ||\partial^\alpha u||_{C^0(\Omega)} + \sum_{|\alpha = k|} [\partial^\alpha u]_\omega, \text{ with } [\partial^\alpha u]_\omega := \sup_{x,y \in \Omega, x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x - y|^\omega}.$$

The function space

$$C^{k,\omega}(\Omega) = \{u \in C^k(\Omega) : ||u||_{C^{k,\omega}(\Omega)} < \infty\}$$

is called the Hölder space with exponent ω.

For a bounded or unbounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, we write $\partial\Omega$ for its boundary and we denote by n and σ the outward pointing normal to Ω and the surface measure on $\partial\Omega$, respectively. By $L^2(\mathbb{R}^3)^4 := L^2(\mathbb{R}^3; \mathbb{C}^4)$ (resp. $L^2(\Omega)^4 := L^2(\Omega, \mathbb{C}^4)$) we denote the usual L^2-space over $\mathbb{R}^3$ (resp. Ω), and we let $r_\Omega : L^2(\mathbb{R}^3)^4 \to L^2(\Omega)^4$ be the restriction operator on Ω and $e_\Omega : L^2(\Omega)^4 \to L^2(\mathbb{R}^3)^4$ its adjoint operator, *i.e.*, the extension by 0 outside of Ω.

We define the unitary Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ as follows:

$$\hat{u}(\xi) := \mathscr{F}[u](\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\mathrm{d}x, \quad \forall \xi \in \mathbb{R}^d,$$

and by $\mathscr{F}^{-1}$ we denote the inverse Fourier transform $\mathscr{F}^{-1} : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$, given by

$$[\hat{u}]^{-1}(x) := \mathscr{F}^{-1}[u](x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} u(\xi)\mathrm{d}\xi, \quad \forall x \in \mathbb{R}^d.$$

The Fourier transform defines a continuous linear operator from the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ into itself. By duality, we can also extend $\mathscr{F}$ to the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$. In particular, the Fourier transform can be extended into an isometry in $L^2(\mathbb{R}^d)$.

For $s \in [0, 1]$, we define the usual ***Sobolev space*** $H^s(\mathbb{R}^d)^4$ as

$$H^s(\mathbb{R}^d)^4 := \{u \in L^2(\mathbb{R}^d)^4 : \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi < \infty\},$$

and we shall designate by $H^s(\Omega)^4$ the standard L^2-based Sobolev space of order s. By $H^0(\mathbb{R}^d)^4 = L^2(\partial\Omega)^4 := L^2(\partial\Omega, \mathrm{d}\sigma)^4$ we denote the usual L^2-space over $\partial\Omega$. If Ω is a C^2-smooth domain with a compact boundary $\partial\Omega$, then the Sobolev space of order $s \in (0, 1]$ along the boundary, $H^s(\partial\Omega)^4$, is defined using local coordinates representation on the surface $\partial\Omega$. As usual, we use the symbol $H^{-s}(\partial\Omega)^4$ to denote the dual space of $H^s(\partial\Omega)^4$. In particular, the first order Sobolev space is

$$H^1(\Omega)^4 = \{\varphi \in L^2(\Omega)^4 : \text{ there exists } \tilde{\varphi} \in H^1(\mathbb{R}^3)^4 \text{ such that } \tilde{\varphi}|_\Omega = \varphi\}.$$

The Sobolev space of order $1/2$ along the boundary, $H^{1/2}(\partial\Omega)^4$, consists of all functions $g \in L^2(\partial\Omega)^4$ for which

$$\|g\|^2_{H^{1/2}(\partial\Omega)^4} := \int_{\partial\Omega} |g(x)|^2 \mathrm{d}\sigma(x) + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|g(x) - g(y)|^2}{|x - y|^3} \mathrm{d}\sigma(y)\mathrm{d}\sigma(x) < \infty,$$

and $H^{-1/2}(\partial\Omega)^4$ is the dual space of $H^{1/2}(\Sigma)^4$. We denote by $t_{\partial\Omega} : H^1(\Omega)^4 \to H^{1/2}(\partial\Omega)^4$ the classical trace operator, and by $\mathcal{E}_\Omega : H^{1/2}(\partial\Omega)^4 \to H^1(\Omega)^4$ the extension operator, that is

$$t_{\partial\Omega}\mathcal{E}_\Omega[f] = f, \quad \forall f \in H^{1/2}(\partial\Omega)^4.$$

Throughout the current manuscript, we denote by $P_\pm$ the orthogonal projections defined by

$$P_\pm := \frac{1}{2} \left(\mathbb{I}_4 \mp i\beta(\alpha \cdot n(x))\right), \quad x \in \partial\Omega. \tag{1.8}$$

We use the symbol $H(\alpha, \Omega)$ for the ***Dirac-Sobolev space*** on a smooth domain Ω defined as

$$H(\alpha, \Omega) = \{\varphi \in L^2(\Omega)^4 : (\alpha \cdot \nabla)\varphi \in L^2(\Omega)^4\}, \tag{1.9}$$

which is a Hilbert space (see [OBV18, Section 2.3]) endowed with the following scalar product

$$\langle \varphi, \psi \rangle_{H(\alpha,\Omega)} = \langle \varphi, \psi \rangle_{L^2(\Omega)^4} + \langle (\alpha \cdot \nabla)\varphi, (\alpha \cdot \nabla)\psi \rangle_{L^2(\Omega)^4}, \quad \varphi, \psi \in H(\alpha, \Omega).$$

We also recall that the trace operator $t_{\partial\Omega}$ extends into a continuous map $t_{\partial\Omega} : H(\alpha,\Omega) \to H^{-1/2}(\partial\Omega)^4$. Moreover, if $v \in H(\alpha,\Omega)$ and $t_{\partial\Omega}v \in H^{1/2}(\partial\Omega)^4$, then $v \in H^1(\Omega)^4$, cf. [OBV18, Proposition 2.1 & Proposition 2.16].

Proposition 1.3.4. *Let Ω be a Lipschitz domain. Then, for all ϕ, $\psi \in H^1(\Omega)^4$, we have the Green's formula*

$$\langle(-i\alpha\cdot\nabla)\varphi,\psi\rangle_{L^2(\Omega)^4} - \langle\varphi,(-i\alpha\cdot\nabla)\psi\rangle_{L^2(\Omega)^4} = \langle(-i\alpha\cdot n)t_{\partial\Omega}\varphi, t_{\partial\Omega}\psi\rangle_{L^2(\partial\Omega)^4}.$$

1.4 The resolvent kernel of the free Dirac operator

Let us denote by $R_m(z)$ the resolvent of the free Dirac operator. From [Tha92, Section 1.E], it is known that

$$\left(R_m(z)f\right)(x) := (D_m - z)^{-1}f(x) := \int_{\mathbb{R}^3}\phi_m^z(x-y)f(y)\,\mathrm{d}y,\ \forall\, z\in\mathbb{C}\setminus\mathrm{Sp}(D_m),$$

where,

$$\phi_m^z(x) = \frac{e^{ik(z)|x|}}{4\pi|x|}\left(z + m\beta + (1 - ik(z)|x|)i\alpha\cdot\frac{x}{|x|^2}\right), \quad \forall x\in\mathbb{R}^3\setminus\{0\}, \tag{1.10}$$

is the kernel of the free Dirac operator.

Let us recall how we obtain this integral kernel. On one hand, we have

$$\begin{aligned}(D_m - z)^{-1} &= (D_m + z)(D_m^2 - z^2)^{-1}\\ &= (D_m + z)(-\Delta + m^2 - z^2)^{-1}\\ &= (D_m + z)\left(p^2 - (z^2 - m^2)\right)^{-1}.\end{aligned}$$

On the other hand, if we let $u(x) = \left(p^2 - k^2(z)\right)^{-1}f(x)$, with $k(z) = \sqrt{z^2 - m^2}$ the branch of the square root fixed by the condition $\mathrm{Im}\sqrt{\lambda} \geqslant 0$, then by Fourier transform, we get that

$$\begin{aligned}&\mathscr{F}(f)(\xi) = \left(\xi^2 - (z^2 - m^2)\right)\mathscr{F}(u)(\xi)\\ &\Rightarrow u(x) = \mathscr{F}^{-1}\left(\frac{1}{\xi^2 - k^2(z)}\right) * f(x).\end{aligned}$$

From the well known formula $\mathscr{F}^{-1}\left(\frac{1}{\xi^2 - k^2(z)}\right)(x) = \frac{e^{ik(z)|x|}}{4\pi|x|}$, we obtain

$$\begin{aligned}(D_m - z)^{-1}f(x) &= (D_m + z)\left(\frac{e^{ik(z)|x|}}{4\pi|x|} * f\right)(x)\\ &= \left((-i\alpha\cdot\nabla + \beta m + z)\left(\frac{e^{ik(z)|x|}}{4\pi|x|}\right)\right) * f(x).\end{aligned}$$

In addition, for $x \neq 0$, we have

$$\begin{aligned}\frac{\partial}{\partial x_1}\left(\frac{e^{ik(z)|x|}}{|x|}\right) &= \frac{ik(z)|x|e^{ik(z)|x|}\cdot x_1 - \frac{x_1}{|x|}e^{ik(z)|x|}}{|x|^2} \\ &= ik(z)e^{ik(z)|x|}\frac{x_1}{|x|^2} - \frac{x_1}{|x|^3}e^{ik(z)|x|} \\ &= \left(ik(z)\frac{x_1}{|x|^2} - \frac{x_1}{|x|^3}\right)e^{ik(z)|x|}.\end{aligned}$$

Consequently,

$$\alpha\cdot\nabla = \alpha\cdot\left(ik(z)\frac{x}{|x|^2} - \frac{x}{|x|^3}\right)e^{ik(z)|x|} = \left(i\frac{\alpha\cdot x}{|x|}k(z) - \frac{\alpha\cdot x}{|x|^2}\right)\frac{e^{ik(z)|x|}}{|x|}$$

yields that

$$\Big((D_m - z)^{-1}f\Big)(x) = \underbrace{\left((-i\alpha\cdot\nabla + \beta m + z)\frac{e^{ik(z)|x|}}{4\pi|x|}\right)}_{\left(i\frac{\alpha\cdot x}{|x|^2}+k(z)\frac{\alpha\cdot x}{|x|}+m\beta+z\right)\frac{e^{ik(z)|x|}}{4\pi|x|}} * f(x) = \underbrace{\int_{\mathbb{R}^3}\phi_m^z(x-y,z)f(y)\,\mathrm{d}y}_{=\phi_m^z(\cdot)*f(x)}.$$

By comparing the two quantities, we deduce (1.10).

1.5 Boundary integral operators associated with the free Dirac operators

The aim of this part is to introduce boundary integral operators associated with the fundamental solution of the free Dirac operator D_m and to summarize some of their well-known properties.

For $z \in \rho(D_m)$ (*i.e.,* the resolvent set of D_m), with the convention that $\mathrm{Im}\sqrt{z^2-m^2} > 0$, the fundamental solution of $(D_m - z)$ is given by

$$\phi_m^z(x) = \frac{e^{i\sqrt{z^2-m^2}|x|}}{4\pi|x|}\left(z + m\beta + (1 - i\sqrt{z^2-m^2}|x|)i\alpha\cdot\frac{x}{|x|^2}\right), \quad \forall x \in \mathbb{R}^3\setminus\{0\}. \tag{1.11}$$

We define the potential operator $\Phi_{z,m}^{\Omega} : L^2(\partial\Omega)^4 \longrightarrow L^2(\Omega)^4$ by

$$\Phi_{z,m}^{\Omega}[g](x) = \int_{\partial\Omega}\phi_m^z(x-y)g(y)\mathrm{d}\sigma(y), \quad \text{for all } x \in \Omega, \tag{1.12}$$

and the Cauchy operators $\mathscr{C}_{z,m} : L^2(\partial\Omega)^4 \longrightarrow L^2(\partial\Omega)^4$ as the singular integral operator acting as

$$\mathscr{C}_{z,m}[f](x) = \lim_{\rho\searrow 0}\int_{|x-y|>\rho}\phi_m^z(x-y)f(y)\mathrm{d}\sigma(y), \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega,\ f \in L^2(\partial\Omega)^4. \tag{1.13}$$

Finally, we define the following operator $C_{\pm,m}^z : L^2(\Sigma)^4 \longrightarrow L^2(\Sigma)^4$ as follows:

$$C_{\pm,m}^z[f](x) := \lim_{\Omega_\pm\ni y \overset{nt}{\to} x}\Phi_m^z[f](y),$$

where $\Omega_\pm \ni y \overset{nt}{\to} x$ means that y tends to x non-tangentially from Ω_+ and Ω_-, respectively, *i.e.*, for $y \in \Omega_\pm$, we get $|x-y| < (1+a)\mathrm{dist}(y,\Sigma)$ for $a>0$ and $x \in \Sigma$. Moreover, the following Plemelj-Sokhotski jump formula holds:

$$C^z_{\pm,m} = \mp\frac{i}{2}(\alpha\cdot n) + \mathscr{C}_{z,m}. \tag{1.14}$$

It is well known that $\Phi^\Omega_{z,m}$, $\mathscr{C}_{z,m}$, and $C^z_{\pm,m}$ are bounded and everywhere defined (see, *for instance*, [AMV14, Section. 2]), and that

$$((\alpha\cdot n)\mathscr{C}_{z,m})^2 = (\mathscr{C}_{z,m}(\alpha\cdot n))^2 = -1/4, \quad \forall z \in \rho(D_m), \tag{1.15}$$

holds in $L^2(\partial\Omega)^4$, cf. [AMV15, Lemma 2.2]. In particular, the inverse $\mathscr{C}^{-1}_{z,m} = -4(\alpha\cdot n)\mathscr{C}_{z,m}(\alpha\cdot n)$ exists and is bounded and everywhere defined. Since we have $\phi^z_m(y-x)^* = \phi^{\overline{z}}_m(x-y)$ for all $z\in\rho(D_m)$, it follows that $\mathscr{C}^*_{z,m} = \mathscr{C}_{\overline{z},m}$ as operators in $L^2(\partial\Omega)^4$. In particular, $\mathscr{C}_{z,m}$ is self-adjoint in $L^2(\partial\Omega)^4$ for all $z\in(-m,m)$.

Next, recall that the trace of the single layer operator (1.12), S_z, associated with the Helmholtz operator $(-\Delta + m^2 - z^2)I_4$ is defined, for every $f\in L^2(\partial\Omega)^4$ and $z\in\rho(D_m)$, by

$$S_z[f](x) := \int_{\partial\Omega} \frac{e^{i\sqrt{z^2-m^2}|x-y|}}{4\pi|x-y|} f(y)\mathrm{d}\sigma(y), \quad \text{for } x\in\partial\Omega.$$

It is well-known that S_z is bounded from $L^2(\partial\Omega)^4$ into $H^{1/2}(\partial\Omega)^4$, and it is a positive operator in $L^2(\partial\Omega)^4$ for all $z\in(-m,m)$, cf. [AMV15, Lemma 4.2]. Now we define the operator Λ^z_m by

$$\Lambda^z_m = \frac{1}{2}\beta + \mathscr{C}_{z,m}, \quad \text{for all } z\in\rho(D_m),$$

which is clearly a bounded operator from $L^2(\partial\Omega)^4$ into itself.

In the next lemma we collect the main properties of the operators $\Phi^\Omega_{z,m}$, $\mathscr{C}_{z,m}$ and Λ^z_m.

Lemma 1.5.1. *Assume that Ω is C^2-smooth. Given $z\in\rho(D_m)$ and let $\Phi^\Omega_{z,m}$, $\mathscr{C}_{z,m}$ and Λ^z_m be as above. Then the following hold true:*

(i) *The operator $\Phi^\Omega_{z,m}$ is bounded from $H^{1/2}(\partial\Omega)^4$ to $H^1(\Omega)^4$, and extends into a bounded operator from $H^{-1/2}(\partial\Omega)^4$ to $H(\alpha,\Omega)$. Moreover, it holds that*

$$t_{\partial\Omega}\Phi^\Omega_{z,m}[f] = \left(-\frac{i}{2}(\alpha\cdot n) + \mathscr{C}_{z,m}\right)[f], \quad \forall f\in H^{1/2}(\partial\Omega)^4. \tag{1.16}$$

(ii) *The operator $\mathscr{C}_{z,m}$ gives rise to a bounded operator $\mathscr{C}_{z,m} : H^{1/2}(\partial\Omega)^4 \longrightarrow H^{1/2}(\partial\Omega)^4$.*

(iii) *The operator $\Lambda^z_m : H^{1/2}(\partial\Omega)^4 \longrightarrow H^{1/2}(\partial\Omega)^4$ is bounded invertible for all $z\in\rho(D_m)$.*

Proof. (i) The proof of the boundedness of $\Phi^\Omega_{z,m}$ from $H^{1/2}(\partial\Omega)^4$ into $H^1(\Omega)^4$ is contained in [BH20, Proposition 4.2], and the jump formula (1.16) is proved in [AMV14, Lemma 3.3] in terms of non-tangential limit which coincides (almost everywhere in $\partial\Omega$) with the trace operator for functions in $H^1(\Omega)^4$. The boundedness of $\Phi^\Omega_{z,m}$ from $H^{-1/2}(\partial\Omega)^4$ to $H(\alpha,\Omega)$ is established in [OBV18, Theorem 2.2].

Since n is smooth, it is clear from (i) that $\mathscr{C}_{z,m}$ is bounded from $H^{1/2}(\partial\Omega)^4$ into itself, which proves (ii). As consequence we also obtain that Λ_m^z is bounded from $H^{1/2}(\partial\Omega)^4$ into itself. Now, the invertibility of Λ_m^z in $H^{1/2}(\partial\Omega)^4$ for $z \in \mathbb{C} \setminus \mathbb{R}$ is shown in [BEHL19, Lemma 3.3 (iii)], see also [BHM20, Lemma 3.12]. To complete the proof of (iii), note that if $f \in L^2(\partial\Omega)^4$ is such that $\Lambda_m^z[f] \in H^{1/2}(\partial\Omega)^4$, then a simple computation shows that

$$H^{1/2}(\partial\Omega)^4 \ni (\Lambda_m^z)^2[f] = \Big(1/4 + (\mathscr{C}_{z,m})^2 + (m + z\beta)S_z\Big)\,[f],$$

which means that $f \in H^{1/2}(\partial\Omega)^4$. From the above computation we see that Λ_m^z is invertible from $H^{1/2}(\partial\Omega)^4$ into itself for all $z \in (-m, m)$, since $((\mathscr{C}_{z,m})^2 + (m + z\beta)S_z)$ is a positive operator. This completes the proof of the lemma. ■

Remark 1.5.1. *Note that if Ω is a Lipschitz domain with a compact boundary, then for all $z \in \rho(D_m)$ the operators $\mathscr{C}_{z,m}$ and Λ_m^z are bounded from $L^2(\partial\Omega)^4$ into itself (see,* e.g, *[AMV14, Lemma 3.3]), and since Λ_m^z is an injective Fredholm operator (see the proof of [Ben22a, Theorem 4.5]) it follows that it is also invertible in $L^2(\partial\Omega)^4$. Note also that, thanks to [BHSS24, Lemma 5.1 and Lemma 5.2], we know that the mapping $\Phi^\Omega_{z,m}$ defined by* (1.12) *is bounded from $L^2(\partial\Omega)^4$ to $H^{1/2}(\Omega)^4$, $t_{\partial\Omega}\Phi^\Omega_{z,m}[g] \in L^2(\partial\Omega)^4$ and the formula* (1.16) *still holds true for all $g \in L^2(\partial\Omega)^4$.*

At the end of this section, we recall a definition of geometrical quantities on the surface $\Sigma := \partial\Omega$, with $\Omega \subset \mathbb{R}^3$ a bounded domain:

Definition 1.5.2. [Weingarten map]. Let Σ be parametrized by the family $\{\phi_j, U_j, V_j,\}_{j\in J}$ wich J is a finite set, $U_j \subset \mathbb{R}^2$, $V_j \subset \mathbb{R}^3$, $\Sigma \subset \bigcup_{j\in J} V_j$ and $\phi_j(U_j) = V_j \cap \Sigma$ for all $j \in J$. For $x = \phi_j(u) \in \Sigma \cap V_j$ with $u \in U_j$, one defines the Weingarten map (arising from the second fundamental form) as the following linear operator

$$\begin{array}{rcl} W_x := W(x) : T_x & \to & T_x \\ \partial_i\phi_j(u) & \mapsto & W(x)[\partial_i\phi_j](u) := -\partial_i n(\phi_j(u)), \end{array} \tag{1.17}$$

where T_x denotes the tangent space of Σ on x and $\{\partial_i\phi_j(u)\}_{i=1,2}$ is a basis vector of T_x.
The eigenvalues $k_1(x), ..., k_n(x)$ of the Weingarten map W_x are called principal curvatures of Σ at x. Then, we have the following proposition:

Proposition 1.5.3. *[[Tho79], Chapter 9 (Theorem 2), 12 (Theorem 2)]. Let Σ be an $n-$surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field n, and let $x \in \Sigma$. The principal curvatures are uniformly bounded on Σ.*

In quantum mechanics, one is usually concerned with the study of operators $D_m + V$ in the Hilbert space $L^2(\mathbb{R}^n, \mathbb{C}^N)$, where V is a potential perturbation, or, more recently, in $L^2(\Omega, \mathbb{C}^N)$ with various boundary conditions. The main attention is paid to the most physically interesting cases $n = 2$ and $n = 3$. In many cases, the potential V depends on some parameters, so one is interested in the study of spectra (in particular, eigenvalues) under various variations of parameters in V as well their dependence on the parameter m (mass) and the underlying geometric object Ω.

In order to provide more context, the paragraphs below contain a brief presentation of my scientific papers. These paragraphs come as a summary of our results, as well as some ingredients of their proofs.

1.6 Chapter summaries

My doctoral research is mainly focused on the spectral analysis of Dirac operators. A large part of my thesis deals with an approximation for the Dirac operator on a domain (with boundary conditions) or coupled with δ-shell interactions. More precisely, the current manuscript studies two types of perturbation for Dirac operators: The three-dimensional Dirac operators with large mass limits and the Dirac operators coupled with singular delta interactions of electrostatic and Lorentz scalar. On one hand, the majority of the studies carried out in this thesis are established through the study of their resolvents. Then, in the first part of this thesis, we introduce the Poincaré-Steklov (PS) operators, which appear naturally in the study of ***Dirac operators with MIT bag boundary conditions*** (Chapters 2 and 3), and analyze them from a microlocal point of view (classical and semiclassical). On the other hand, our study focuses on ***Dirac operators coupled with a singular combination of electrostatic and Lorentz scalar delta interactions*** (Chapters 4 and 5). In three-dimension, we generalize an approximation of this operator with regular local interaction. Besides, in two-dimension, we develop a new technique that allows us to prove, for combinations of interactions, the self-adjointness of the realization of the operator under consideration, in low-regularity Sobolev spaces.

1.6.1 Dirac operator with MIT bag boundary conditions

The MIT bag Dirac operator was introduced by Bogoliubov in the 1970s as a simplified model of confinement of quarks in hadrons. Moreover, as far as we know, the MIT bag Dirac operators are considered as a model of general relativity, see *e.g.,* [BL01, BC05]. For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, the MIT bag operator $H_{\mathrm{MIT}}(m)$ is the realization of D_m in $L^2(\Omega)^4$ corresponding to the boundary conditions $P_- t_{\partial\Omega} v = 0$ on $\partial\Omega$ with some explicit matrices P_- (1.18) depending on the outer unit normal n and $t_{\partial\Omega}$ being the Dirichlet trace operator (restriction to the boundary). Several researchers (*e.g.,* [ALTMR19, MOBP20]), have found that the eigenvalues of $H_{\mathrm{MIT}}(m)$ are the limit (in the sense of resolvent) of the eigenvalues of the Dirac operator in the whole space $\mathbb{R}^n$ when the mass becomes large outside of Ω (so that the MIT bag boundary condition represents a kind of relativistic hard wall at the boundary). Moreover, various resolvent convergence results were established as well see, *for example* [BCLTS19].

For a bounded smooth domain $\Omega \subset \mathbb{R}^3$, the MIT bag operator is defined on the domain

$$\mathrm{Dom}(H_{\mathrm{MIT}}(m)) := \{v \in H^1(\Omega, \mathbb{C}^4) : P_- t_{\partial\Omega} v = 0 \text{ on } \partial\Omega\},$$

by $H_{\mathrm{MIT}}(m)v = D_m v$, for all $v \in \mathrm{Dom}(H_{\mathrm{MIT}}(m))$, and where the boundary condition holds in $H^{1/2}(\partial\Omega, \mathbb{C}^4)$. Here $t_{\partial\Omega} : H^1(\Omega, \mathbb{C}^4) \to H^{1/2}(\partial\Omega, \mathbb{C}^4)$ denotes the classical trace operator.
It is well known [OBV18] that the spectrum of H_{MIT} is purely discrete and is contained in $\mathbb{R} \setminus [-m, m]$. Also, it is known that $H_{\mathrm{MIT}}(m)$ arises as the limit (in the sense of resolvent) of the self-adjoint Dirac operator $D_M := D_m + M\beta \mathbb{1}_{\mathbb{R}^3 \setminus \overline{\Omega}}$ when M tends to ∞.

In the following two parts of this introduction, we will describe our main results from Chapters 2 and 3 on the study of Dirac operators with MIT bag boundary conditions, which correspond to the results obtained in [BBZ37] and [Zre84], respectively.

1.6.1.1 Summary of Chapter 2: A Poincaré-Steklov map for the MIT bag model

Boundary integral operators have played a key role in the study of many boundary value problems for partial differential equations arising in various areas of mathematical physics, such as electromagnetism, elasticity, and potential theory. In particular, they are used as a tool for proving the existence of solutions as well as for their construction by means of integral equation methods, see, *e.g.*, [FJJR78, JK81a, JK81b, Ver84].

The study of boundary integral operators has also been the motivation for the development of various tools and branches of mathematics, *e.g.*, Fredholm theory, Singular integral and Pseudodifferential operators. Moreover, it turned out that functional analytic and spectral properties of some of these operators are strongly related to the regularity and geometric properties of surfaces, see *for example* [HMT10, HMOM+09]. A typical and well-known example which occurs in many applications is the Dirichlet-to-Neumann (DtN) operator. In the classical setting of a bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary, the DtN operator, $\mathcal{N}$, is defined by

$$\mathcal{N} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \quad g \longmapsto \mathcal{N}g = \Gamma_N U(g),$$

where $U(g)$ is the harmonic extension of g (*i.e.*, $\Delta U(g) = 0$ in Ω and $\Gamma_D U = g$ on $\partial\Omega$). Here Γ_D and Γ_N denote the Dirichlet and the Neumann traces, respectively. In this setting, it is well known that the DtN operator fits into the framework of pseudodifferential operators, see *e.g.*, [Tay96]. Moreover, from the viewpoint of the spectral theory, several geometric properties of the eigenvalue problem for the DtN operator (such as isoperimetric inequalities, spectral asymptotics and geometric invariants) are closely related to the theory of minimal surfaces [FS16], as well as the problem of determining a complete Riemannian manifold with boundary from the Cauchy data of harmonic functions, see [LTU03] (see also the survey [GP17] for further details).

A motivation of this chapter is to introduce a Poincaré-Steklov map for the Dirac operator (*i.e.*, an analogue of the DtN map for the Laplace operator) and to study its pseudodifferential properties. Our main motivation for considering this operator is that it arises naturally in the study of the well-known Dirac operator with the MIT bag boundary condition, $H_{\text{MIT}}(m)$, which will be rigorously defined in (1.21). Let $\Omega \subset \mathbb{R}^3$ be a domain with a compact smooth boundary $\partial\Omega$, let n be the outward unit normal to Ω, and let $\Gamma_\pm$ and $P_\pm$ be the trace mappings and the orthogonal projections, respectively, defined by

$$\Gamma_\pm = P_\pm\Gamma_D : H^1(\Omega)^4 \longrightarrow P_\pm H^{1/2}(\partial\Omega)^4 \text{ and } P_\pm := \frac{1}{2}\left(\mathbb{I}_4 \mp i\beta(\alpha \cdot n(x))\right), \quad x \in \partial\Omega. \tag{1.18}$$

In Chapter 2, we investigate the specific case of the Poincaré-Steklov (PS for short) operator, $\mathscr{A}_m$, defined for $z \in \rho(H_{\text{MIT}}(m))$ by

$$\mathscr{A}_m : P_- H^{1/2}(\partial\Omega)^4 \longrightarrow P_+ H^{1/2}(\partial\Omega)^4, \quad g \longmapsto \mathscr{A}_m(g) = \Gamma_+ U_z,$$

where $U_z \in H^1(\Omega)^4$ is the unique solution to the following elliptic boundary problem:

$$\begin{cases} (D_m - z)U_z = 0, & \text{in } \Omega, \\ \Gamma_- U_z = g, & \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

We point out that in the R-matrix theory and the embedding method for the Dirac equation, similar operators linking on $\partial\Omega$ values of the upper and lower components of the spinor wave functions have been studied in [Szm98, Agr01, AR05, BS06]. It corresponds to a different boundary condition (the trace of the upper/lower components) which is not necessarily elliptic. As far as we know, such operators for the MIT bag boundary condition have not been studied yet.

Our results are mainly concerned with the pseudodifferential properties of $\mathscr{A}_m$ and their applications. Thus, our first goal is to show that $\mathscr{A}_m$ fits into the framework of pseudodifferential operators. In Section 2.3, we show that when the mass m is fixed and $z \in \rho(D_m)$, then the Poincaré-Steklov operator $\mathscr{A}_m$ is a classical homogeneous pseudodifferential operator of order 0, and that

$$\mathscr{A}_m = S \cdot \left(\frac{\nabla_{\partial\Omega} \wedge n}{\sqrt{-\Delta_{\partial\Omega}}} \right) P_- \mod Op\mathcal{S}^{-1}(\partial\Omega),$$

where $S = i(\alpha \wedge \alpha)/2$ is the spin angular momentum, $\nabla_{\partial\Omega}$ and $\Delta_{\partial\Omega}$ are, respectively, the surface gradient and the Laplace-Beltrami operator on $\partial\Omega$ (equipped with the Riemann metric induced by the euclidian one in $\mathbb{R}^3$) and $Op\mathcal{S}^{-1}$ is the classical class of pseudodifferential operators of order -1 (see Theorem 2.3.3 for details). For $D_{\partial\Omega}$, the extrinsically defined Dirac operator introduced in Section 2.1.2, we also have:

$$\mathscr{A}_m = D_{\partial\Omega} \, (-\Delta_{\partial\Omega})^{-\frac{1}{2}} P_- \mod Op\mathcal{S}^{-1}(\partial\Omega).$$

The proof of the above result is based on the fact that we have an explicit solution of the system (1.19) for any $z \in \rho(D_m)$, and in this case the PS operator takes the following layer potential form:

$$\mathscr{A}_m = -P_+ \beta \left(\beta/2 + \mathscr{C}_{z,m} \right)^{-1} P_-, \tag{1.20}$$

where $\mathscr{C}_{z,m}$ is the Cauchy operator associated with $(D_m - z)$ defined on $\partial\Omega$ in the principal value sense (see Subsection 1.5 for the precise definition). So the starting point of the proof is to analyze the pseudodifferential properties of the Cauchy operator. In this sense, we show that $2\mathscr{C}_{z,m}$ is equal, modulo $Op\mathcal{S}^{-1}(\partial\Omega)$, to $\alpha \cdot (\nabla_{\partial\Omega}(-\Delta_{\partial\Omega})^{-1/2})$. Using this, the explicit layer potential description of $\mathscr{A}_m$, and the symbol calculus, we then prove that $\mathscr{A}_m$ is a pseudodifferential operator and catch its principal symbol (see Theorem 2.3.3).

While the above strategy allows us to capture the pseudodifferential character of $\mathscr{A}_m$, but unfortunately it does not allow us to trace the dependence on the parameter m, and it also imposes a restriction on the spectral parameter z (*i.e.*, $z \in \rho(D_m)$), whereas $\mathscr{A}_m$ is well-defined for any $z \in \rho(H_{\mathrm{MIT}}(m))$. In Section 2.4, we address the m-dependence of the pseudodifferential properties of $\mathscr{A}_m$ for any $z \in \rho(H_{\mathrm{MIT}}(m))$. Since we are mainly concerned with large masses m in our application, we treat this problem from the semiclassical point of view, where $h = 1/m \in (0, 1]$ is the semiclassical parameter. In fact, we show the following result:

The proof of the following result is presented in Theorem 2.4.1 in Chapter 2.

Theorem 1.6.1. *Let $h \in (0,1]$ and $z \in \rho(H_{MIT}(m))$, and let $\mathscr{A}^h := \mathscr{A}_m$ with $m = h^{-1}$. Then for any $N \in \mathbb{N}$, there exists a h-pseudodifferential operator of order 0, $\mathscr{A}^h_N \in Op^h\mathcal{S}^0(\Sigma)$ such that for h sufficiently small, and any $0 \leqslant l \leqslant N + \frac{1}{2}$*

$$\|\mathscr{A}^h - \mathscr{A}^h_N\|_{H^{\frac{1}{2}}(\Sigma) \to H^{N+\frac{3}{2}-l}(\Sigma)} = O(h^{N+\frac{1}{2}+l}),$$

and

$$\mathscr{A}_N^h = \frac{hD_\Sigma}{\sqrt{-h^2\Delta_\Sigma + \mathbb{I}} + \mathbb{I}} P_- \mod h\,Op^h\mathcal{S}^{-1}(\Sigma).$$

The main idea of the proof is to use the system (1.19) instead of the explicit formula (1.20), and it is based on the following two steps. The first step is to construct a local approximate solution for the pushforward of the system (1.19) of the form

$$U^h(\tilde{x}, x_3) = Op^h(A^h(\cdot,\cdot,x_3))g = \frac{1}{2\pi}\int_{\mathbb{R}^2} A^h(\tilde{x}, h\xi, x_3)e^{iy\cdot\xi}\hat{g}(\xi)\mathrm{d}\xi, \quad (\tilde{x}, x_3) \in \mathbb{R}^2 \times [0,\infty),$$

where A^h belongs to a specific symbol class and has the following asymptotic expansion

$$A^h(\tilde{x}, \xi, x_3) \sim \sum_{j\geqslant 0} h^j A_j(\tilde{x}, \xi, x_3).$$

The second step is to show that when applying the trace mapping Γ_+ to the pull-back of $U^h(\cdot, 0)$ it coincides locally with $\mathscr{A}_{1/h}$ modulo a regularizing and negligible operator. At this point, the properties of the MIT bag operator become crucial, in particular, the regularization property of its resolvent which allows us to achieve this second step, as we will see in Section 2.4. The MIT bag operator on Ω is the Dirac operator on $L^2(\Omega)^4$ defined by

$$H_{\text{MIT}}(m)\psi = D_m\psi, \quad \forall\psi \in \mathrm{Dom}(H_{\text{MIT}}(m)) := \left\{\psi \in H^1(\Omega)^4 : \Gamma_-\psi = 0 \text{ on } \partial\Omega\right\}. \tag{1.21}$$

In Section 2.2, we briefly discuss the basic spectral properties of $H_{\text{MIT}}(m)$ when Ω is a domain with compact Lipschitz boundary (see Theorem 2.2.1). We mention that direct proofs of the self-adjointness of $H_{\text{MIT}}(m)$ have been established in [ALTMR17, ALTR20]. Moreover, in Theorem 2.2.2 we establish regularity results concerning the regularization property of the resolvent and the Sobolev regularity of the eigenfunctions of H_{MIT}. In particular, we prove that $(H_{\text{MIT}}(m) - z)^{-1}$ is bounded from $H^n(\Omega)^4$ into $H^{n+1}(\Omega)^4 \cap \mathrm{Dom}(H_{\text{MIT}}(m))$, for all $n \geqslant 1$. Indeed, we prove the following result:

Theorem 1.6.2. *Let $k \geqslant 1$ be an integer and assume that $\mathcal{U}$ is C^{2+k}-smooth. Then the following statements hold true:*

(i) The mapping $(H_{MIT}(m) - z)^{-1} : H^k(\mathcal{U})^4 \longrightarrow H^{k+1}(\mathcal{U})^4 \cap \mathrm{Dom}(H_{MIT}(m))$ is well-defined and bounded for all $m > 0$ and all $z \in \rho(H_{MIT}(m))$. Moreover, for any compact set $K \subset \mathbb{C}$ there exist $m_0, C > 0$ such that for all $m \geqslant m_0$ and $z \in K$, there holds

$$\|(H_{MIT}(m) - z)^{-1}\|_{H^{k-1}(\mathcal{U})^4 \longrightarrow H^k(\mathcal{U})^4} \leqslant Cm^{k-1}.$$

(ii) If ϕ is an eigenfunction associated with an eigenvalue $z \in \mathrm{Sp}(H_{MIT}(m))$, i.e., $(H_{MIT}(m) - z)\phi = 0$, then $\phi \in H^{1+k}(\mathcal{U})^4$. In particular, if $\mathcal{U}$ is C^∞-smooth, then $\phi \in C^\infty(\mathcal{U})^4$.

Motivated by the natural way in which the PS operator is related to the MIT bag operator, and to illustrate its usefulness, we consider in Section 2.5 the large mass problem for the self-adjoint Dirac operator $H_M = D_m + M\beta\mathbb{1}_{\mathcal{U}}$, where $\mathcal{U} = \mathbb{R}^3 \setminus \overline{\Omega}$. Indeed, it is known that, in the limit $M \to \infty$, every eigenvalue of $H_{\text{MIT}}(m)$ is a limit of eigenvalues of H_M, cf. [ALTMR19, MOBP20] (see also [BCLTS19, Ben19, SV19] for the two-dimensional setting). Moreover, it is shown in [BCLTS19, Ben19] that the two-dimensional analogue of H_M convergences to the two-dimensional analogue of $H_{\text{MIT}}(m)$ in the norm resolvent sense with a convergence rate of $\mathcal{O}(M^{-1/2})$. It is worth noting that we have the following easy consequence contribution of Kato-Rellich theorem and Weyl theorem: By Kato-Rellich theorem, we have that H_M is self-adjoint on $\mathrm{Dom}(H_M) := H^1(\mathbb{R}^3)^4$, and by Weyl theorem we have that

$$\mathrm{Sp}_{\mathrm{ess}}(H_M) = (-\infty, -(m+M)] \cup [m+M, +\infty),$$
$$\mathrm{Sp}(H_M) \cap (-(m+M), m+M) \text{ is purely discrete.}$$

In [ALTMR19, MOBP20], it was shown that any eigenvalue of $H_{\mathrm{MIT}}(m)$ is a limit of eigenvalues of H_M, when $M \to +\infty$. Moreover, in the two-dimensional setting, the authors of [BCLTS19] have shown, in the norm resolvent sense, that the bidimensional analogue of H_M converges to the bidimensional $H_{\mathrm{MIT}}(0)$, with a convergence rate of $\mathcal{O}(M^{-1/2})$.

For the physists, it is well known that for $M = +\infty$, we recover that $H_M = H_{\mathrm{MIT}}(m)$. Then, before $M = +\infty$, it seems reasonable to ask some questions about the intermediate values of M. In Section 2.5 we address the following question:

Let $M_0 > 0$ be large enough and fix $M \geqslant M_0$ and $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$. Given $f \in L^2(\mathbb{R}^3)^4$ such that $f = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, and $U \in H^1(\mathbb{R}^3)^4$, what is the boundary value problem on Ω whose solutions closely approximate those of $(D_m + M\beta \mathbb{1}_{\mathbb{R}^3\setminus\overline{\Omega}} - z)U = f$?

It is worth noting that the answer to this question becomes trivial if one establishes an explicit formula for the resolvent of H_M. Having in mind the connection between the Dirac operators H_M and $H_{\mathrm{MIT}}(m)$, this leads us to address the following question: *for M sufficiently large, is it possible to relate the resolvents of H_M and H_{MIT} via a Krein-type resolvent formula?* In Theorem 2.5.2, which is the main result of Section 2.5, we establish a Krein-type resolvent formula for H_M in terms of the resolvent of $H_{\mathrm{MIT}}(m)$.

The key point to establish this result is to treat the elliptic problem $(H_M - z)U = f \in L^2(\mathbb{R}^3)^4$ as a transmission problem (where $\Gamma_\pm U_{|\Omega} = \Gamma_\pm U_{|\mathbb{R}^3\setminus\Omega}$ are the transmission conditions) and to use the semiclassical properties of the Poincaré-Steklov operators in order to invert an auxiliary operator $\Psi_M(z)$ acting on the boundary $\partial\Omega$ (see Theorem 2.5.2 for the precise definition). In addition, we prove an adapted Birman-Schwinger principle relating the eigenvalues of H_M in the gap $(-(m+M), m+M)$ with a spectral property of $\Psi_M(z)$. With their help, we show in Corollary 2.5.4 that the restriction of U on Ω satisfies the elliptic problem

$$\begin{cases} (D_m - z)U_{|\Omega} = f & \text{in } \Omega, \\ \Gamma_- U_{|\Omega} = \mathscr{B}_M \Gamma_+ R_{MIT}(z) f & \text{on } \partial\Omega, \\ \Gamma_+ U_{|\Omega} = \Gamma_+ R_{MIT}(z) f + \mathscr{A}_m \Gamma_- v & \text{on } \partial\Omega, \end{cases}$$

where $\mathscr{B}_M$ is a semiclassical pseudodifferential operators of order 0. Here, the semiclassical parameter is $1/M$. Moreover, we show that the convergence of H_M to H_{MIT} in the norm-resolvent sense indeed holds with a convergence rate of $\mathcal{O}(M^{-1})$, which improves previous works. Then, the result reads as follows:

The proof of this result is presented in Proposition [Chapter 2, 2.5.6].

Theorem 1.6.3. *Let r_Ω be the restriction operator on Ω and e_Ω be the extension operator by 0 outside of Ω. For any compact set $K \subset \rho(H_{\mathrm{MIT}}(m))$ there is $M_0 > 0$ such that for all $M > M_0$: $K \subset \rho(H_M)$,* *and for all $z \in K$ the resolvent R_M admits an asymptotic expansion in $\mathcal{L}(L^2(\mathbb{R}^3)^4)$ of the form:*

$$(H_M - z)^{-1} = e_\Omega (H_{\mathrm{MIT}}(m) - z)^{-1} r_\Omega + \frac{1}{M}\left(K_M(z) + L_M(z)\right),$$

where $K_M(z),\ L_M(z) : L^2(\mathbb{R}^3)^4 \longrightarrow L^2(\mathbb{R}^3)^4$ are uniformly bounded with respect to M and satisfy

$$r_\Omega L_M(z) e_\Omega = 0 = r_{\mathcal{U}} K_M(z) e_{\mathcal{U}}.$$

In particular, it holds that

$$\left|\left|(H_M - z)^{-1} - e_\Omega (H_{\mathrm{MIT}}(m) - z)^{-1} r_\Omega\right|\right|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}\left(\frac{1}{M}\right).$$

The most important ingredient in proving these results is the use of the Krein formula relating the resolvents of H_M and $H_{\mathrm{MIT}}(m)$, as well as regularity estimates for the PS operators (see Theorem 2.5.3) and layer potential operators (see Lemma 2.5.7 for details).

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1.6.1.2 Summary of Chapter 3: On the approximation of the Dirac operator coupled with confining Lorentz scalar δ-shell interactions

The aim of this chapter is to study the behavior of a perturbed Dirac operator, defined within a tubular neighborhood of thickness $\varepsilon > 0$, as ε and the mass tends to 0 and ∞, respectively. We consider perturbations of the free Dirac operator D_m in the whole space by a large mass M term living in an ε-neighborhood $\mathcal{U}^\varepsilon$ of a surface $\Sigma := \partial\Omega_+$, with Ω_+ a bounded set in $\mathbb{R}^3$. Working with this type of massive potential leads to the appearance of what we've seen in Chapter 2, called Dirac operators with MIT bag boundary conditions, when the mass M becomes large. Indeed, in this chapter we interested in establishing the convergence (for suitable relation between ε and M: $\varepsilon = M^{-1}$) of such perturbations to a direct sum of two MIT bag operators, which we denote by $D^{\Omega_+}_{\text{MIT}}(m)$ and $D^{\Omega_-}_{\text{MIT}}(m)$, acting in Ω_+ and $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega_+}$, respectively. This decoupling of these MIT bag Dirac operators can be linked (as ε goes to 0) to the confining version (*i.e.,* when $\eta = 0$ and $\mu = \pm 2$ in (1.25), below, for μ instead of τ) of the Dirac operator coupled with purely Lorentz scalar delta interaction supported on the surface Σ, which will be discussed briefly in the second part of this introduction, Section 1.6.2.

Our main goal in this part of the thesis is to establish an approximation of the Dirac operator coupled with Lorentz scalar δ-shell interactions by the perturbed Dirac operator, which we denote by $\mathfrak{D}^\varepsilon_M = D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon}$. We would like to point out that the convergence of $\mathfrak{D}^\varepsilon_M$ to the MIT bag operator was established in the previous chapter, in the norm resolvent sense, when M tends to $+\infty$, and ε fixed. However, in that previous chapter, the mass M is supported on an unbounded domain, which has only one boundary. Whereas, in the present chapter, M is supported on a bounded domain with two boundaries, whose distance between them is the thickness ε, as shown in Figure 1.1. Thus, it is then natural to address the following question in this chapter:

Let M be a large mass supported on a tubular vicinity of surface Σ. What happens when the thickness of the tubular tends to zero with M^{-1}?

The methodology followed, as in the previous problem (of Chapter 2) study the pseudodifferential properties of the Poincaré-Steklov (PS) operators. The complexity in the current problem is that these operators take a pair of functions with respect to $\partial\mathcal{U}^\varepsilon := \Sigma \cup \Sigma^\varepsilon$ such that for all $x_\Sigma \in \Sigma$, we have $\Sigma^\varepsilon \ni x = x_\Sigma + \varepsilon n(x_\Sigma)$, where n is the unit normal to the surface Σ pointing outside Ω. So, we will control these operators by tracking the dependence on the parameter ε, and consequently, the convergence when ε goes to 0 and M goes to $+\infty$.

Now, to give a rigorous definition of the operator we are dealing within this chapter and to go into more details, we need to introduce some notations. Let Ω_+ be a bounded smooth domain in $\mathbb{R}^3$. For $(\eta, \mu) \in \mathbb{R}^2$, the three-dimensional Dirac operator coupled with delta interactions is defined formally by (1.25), with μ instead of τ. If $\eta = 0$, physicists in particular have been aware of this phenomenon since the 1970s, when they considered confinement in hadrons with a model (see [CJJ+74] and [Joh75]). The mathematical model describing this using the Dirac operator with MIT boundary conditions has been extensively studied in mathematical papers such as those mentioned in [BHM20]. At the end of this paragraph, the Dirac operator coupled with purely Lorentz scalar delta shell interaction, is the operator $\mathbb{D}_{0,\mu}$ defined in (1.25). Besides, $\mathbb{D}_{0,+2}$ is called the Dirac operator coupled with confining Lorentz scalar δ-shell interactions, and in this chapter, this operator is the direct sum of both MIT bag Dirac operators, *i.e.,* $\mathbb{D}_{0,+2} = D^{\Omega_+}_{\text{MIT}} \oplus D^{\Omega_-}_{\text{MIT}}$.

In this chapter, Ω_+ is a open bounded set in $\mathbb{R}^3$ with a compact smooth boundary $\Sigma := \partial\Omega_+$, and n is the outward unit normal to Ω_+. We denote by $\Omega_-^\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega_+ \cup \mathcal{U}^\varepsilon}$, $\Omega_{+-}^\varepsilon := \Omega_+ \cup \Omega_-^\varepsilon$, and by N^ε the outward unit normal with respect to Ω_-^ε. More precisely, for ε_0 sufficiently small, we assume that Σ, Ω_-^ε, Σ^ε and $\mathcal{U}^\varepsilon$ satisfied

$$\begin{aligned}
&\Sigma^\varepsilon := \{x \in \mathbb{R}^3,\, x = x_\Sigma + \varepsilon n(x_\Sigma) : x_\Sigma \in \Sigma\},\\
&\Omega_-^\varepsilon = \{x \in \mathbb{R}^3,\, \mathrm{dist}(x, \Sigma) > \varepsilon\},\\
&\mathcal{U}^\varepsilon := \{x \in \mathbb{R}^3,\ x = x_\Sigma + t\, n(x_\Sigma) : x_\Sigma \in \Sigma \text{ and } t \in (0, \varepsilon)\}, \quad \text{with } \varepsilon \in (0, \varepsilon_0).
\end{aligned}$$

In other words, the Euclidean space is divided as follows:

$$\mathbb{R}^3 = \Omega_-^\varepsilon \cup \Sigma^\varepsilon \cup \mathcal{U}^\varepsilon \cup \Sigma \cup \Omega_+.$$

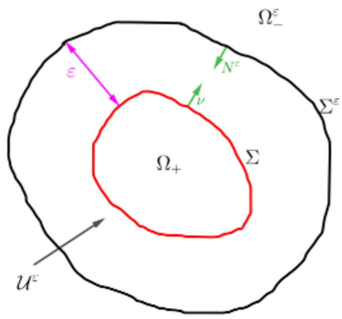

Figure 1.1 – Domain

Description of main results.

The perturbed Dirac operator we are interesting on is $\mathfrak{D}_M^\varepsilon := D_m + M\beta \mathbb{1}_{\mathcal{U}^\varepsilon}$, where $\mathbb{1}_{\mathcal{U}^\varepsilon}$ is the characteristic function of $\mathcal{U}^\varepsilon$. The results of the present chapter are presented as follows:
To establish the main result outlined in Theorem 1.6.6, we will show the following approximations given by Propositions 1.6.4 and 1.6.5:

The following propositions are Propositions 3.1.3 and 3.1.4 of Chapter 3.

Proposition 1.6.4. *We consider the confining version of the Dirac operator coupled with a purely Lorentz scalar δ-shell interaction, denoted by $\mathscr{D}_L := D_m + 2\beta\delta_\Sigma$. Then, for any $z \in \rho(\mathscr{D}_L)$ and ε sufficiently small, the following estimate holds:*

$$\left\| e_{\Omega_{+-}^\varepsilon} (D_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon} - z)^{-1} r_{\Omega_{+-}^\varepsilon} - (\mathscr{D}_L - z)^{-1} \right\|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{1.22}$$

where $D_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}$ is the direct sum of both MIT bag operators, acting in $D_{\mathrm{MIT}}^{\Omega+}$ and $D_{\mathrm{MIT}}^{\Omega_-^\varepsilon}$, refer to $D_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(m) := D_{\mathrm{MIT}}^{\Omega_+}(m) \oplus D_{\mathrm{MIT}}^{\Omega_-^\varepsilon}(m)$, (see Section 3.1.1 for the exact notations).

Proposition 1.6.5. *Let $K \subset \mathbb{C} \setminus \mathbb{R}$ be a compact set. Then, there is $M_0 > 0$ such that for all $M > M_0$ and $\varepsilon = M^{-1}$: $K \subset \rho(\mathfrak{D}^{\varepsilon}_M)$ and for all $z \in K$, the following estimate holds on the whole space*

$$||(\mathfrak{D}^{\varepsilon}_M - z)^{-1} - e_{\Omega^{\varepsilon}_{+-}}(D^{\Omega^{\varepsilon}_{+-}}_{\text{MIT}} - z)^{-1} r_{\Omega^{\varepsilon}_{+-}}||_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(M^{-1}).$$

The latter proposition means that the Dirac operator $\mathfrak{D}^{\varepsilon}_M$ is approximated, in the norm resolvent sense, by both MIT bag Dirac operators, acting in Ω_+ and Ω^{ε}_- with a rate of $\mathcal{O}(M^{-1})$ when M tends to ∞.

By combining Propositions 1.6.4 and 1.6.5, we arrive at the following main result:

Theorem 1.6.6. *Let $z \in \rho(\mathscr{D}_L)$, then for M sufficiently large, $z \in \rho(\mathfrak{D}^{\varepsilon}_M)$, and $\varepsilon = M^{-1}$, the following holds:*

$$\left|\left|(\mathfrak{D}^{\varepsilon}_M - z)^{-1} - (\mathscr{D}_L - z)^{-1}\right|\right|_{L^2(\mathbb{R}^3)^4} = \mathcal{O}\left(M^{-1}\right).$$

Proof ideas

The most important ingredient in proving these results is the use of the Krein formula relating the resolvents of $\mathfrak{D}^{\varepsilon}_M$ and the MIT bag operators (dependent on M or/and ε), examining the convergence of the terms dependent on ε and independent of M, in order to connect them with the fixed boundary surface Σ (namely, Propositions 1.6.4 and 1.6.5). Moreover, the methodology followed in this MIT problem treated in analogy with the study of the pseudodifferential properties (classical and semiclassical) of the Poincaré-Steklov operators. Indeed, we prove in Corollary 3.2.9 that $\mathscr{A}^{\varepsilon}_m$ is a zero-order pseudodifferential operator, and that

$$\mathscr{A}^{\varepsilon}_m = S \cdot \frac{\nabla_{\Sigma^{\varepsilon}} \wedge N^{\varepsilon}(x)}{\sqrt{-\Delta_{\Sigma^{\varepsilon}}}} P^{\varepsilon}_- + \varepsilon\, Op(b^p_0(x_\Sigma, \xi)) + Op(b^p_{-1}(x_\Sigma, \xi)),$$

where $\nabla_{\Sigma^{\varepsilon}}$ is the surface gradient along Σ^{ε}, $-\Delta_{\Sigma^{\varepsilon}}$ is the Laplace-Beltrami operator, and $b^p_0(x_\Sigma, \xi) := b_0\Big(p(x_\Sigma), \big(\nabla p(x_\Sigma)^{-1}\big)^t \xi\Big)$ (similarly for b^p_{-1}), with b^p_0 resp. b^p_{-1} the symbols of order 0 resp. -1, and $\Sigma \ni x_\Sigma \mapsto p(x_\Sigma) = x_\Sigma + \varepsilon n(x_\Sigma) := x \in \Sigma^{\varepsilon}$ a diffeomorphism.

Since, we are interested in large mass coupling, we verify in Proposition 3.2.7 that the PS operator, $\mathcal{A}^h$, fits well within the framework of h-pseudodifferential operators, where $h = \varepsilon = M^{-1}$ is the semiclassical parameter. Moreover, we obtain the following estimate:

$$||\mathcal{A}^h - \mathcal{A}^h_N||_{H^{1/2}(\Sigma^{\varepsilon}) \to H^{\frac{3}{2}-l}(\Sigma^{\varepsilon})} = O(h^{2l+\frac{1}{2}}), \qquad \text{for any } l \in [0, \frac{1}{2}],\ N \in \mathbb{N}.$$

Now, Proposition 1.6.4 can be proved as follows: using the Krein formula of the resolvents of $\mathscr{D}_L$ and both MIT bag operators, $D^{\Omega_+}_{\text{MIT}}$ and $D^{\Omega^{\varepsilon}_-}_{\text{MIT}}$ (see Section 3.3.2), acting in $L^2(\Omega_+)^4$ and $L^2(\Omega^{\varepsilon}_-)^4$, respectively. Besides, we prove in Section 3.4 that following proposition:

Proposition 1.6.7. *Let $\varepsilon_0 > 0$ be small enough, and let $z \in \mathbb{C} \backslash \mathbb{R}$. We set by $\Omega_- := \mathbb{R}^3 \setminus \Omega_+$ the exterior* *fixed domain and by* $\Sigma = \partial\Omega_- = \partial\Omega_+$ *its boundary. Then, for any* $\varepsilon \in (0, \varepsilon_0)$ *the following holds:*

$$\left\| e_{\Omega_-^\varepsilon} \left(D_{\mathrm{MIT}}^{\Omega_-^\varepsilon} - z \right)^{-1} r_{\Omega_-^\varepsilon} - e_{\Omega_-} \left(D_{\mathrm{MIT}}^{\Omega_-} - z \right)^{-1} r_{\Omega_-} \right\|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(\varepsilon), \tag{1.23}$$

where $D_{\mathrm{MIT}}^{\Omega_-}$ *is the* MIT *bag Dirac operator acting in the fixed domain* Ω_-.

By combining the above proposition with Lemma 3.4.2, we then obtain, in the norm resolvent sense, the convergence of $D_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon} := D_{\mathrm{MIT}}^{\Omega_+} \oplus D_{\mathrm{MIT}}^{\Omega_-^\varepsilon}$ to $\mathscr{D}_L = D_{\mathrm{MIT}}^{\Omega_+} \oplus D_{\mathrm{MIT}}^{\Omega_-}$.

In order to prove Proposition 1.6.5, we need to use the following ingredients:

On one hand, the key point to establish this result is to treat the elliptic problem $(\mathfrak{D}_M^\varepsilon - z)\mathfrak{U} = f \in L^2(\mathbb{R}^3)^4$ as a transmission problem with the transmission conditions

$$\begin{cases} (D_m - z)\mathfrak{U}_{|\Omega_+} = f & \text{in } \Omega_+, \\ (D_m - z)\mathfrak{U}_{|\Omega_-^\varepsilon} = f & \text{in } \Omega_-^\varepsilon, \\ (D_{m+M} - z)\mathfrak{U}_{|\mathcal{U}^\varepsilon} = f & \text{in } \mathcal{U}^\varepsilon, \\ P_\pm t_\Sigma \mathfrak{U}_{|\Omega_+} = P_\pm t_\Sigma \mathfrak{U}_{|\mathcal{U}^\varepsilon} & \text{on } \Sigma, \\ P_\mp^\varepsilon t_{\Sigma^\varepsilon} \mathfrak{U}_{|\Omega_-^\varepsilon} = P_\mp^\varepsilon t_{\Sigma^\varepsilon} \mathfrak{U}_{|\mathcal{U}^\varepsilon} & \text{on } \Sigma^\varepsilon. \end{cases}$$

Here $P_\pm$ from (1.18) are the orthogonal projections with respect to n and $P_\pm^\varepsilon$ are the orthogonal projections with respect to N^ε, defined by

$$P_\pm^\varepsilon := (\mathbb{I}_4 \mp i\beta\alpha \cdot N^\varepsilon)/2. \tag{1.24}$$

Then in Section 3.3.2, we establish a Krein resolvent formula that relates the resolvent of the perturbed Dirac operator, $\mathfrak{D}_M^\varepsilon$, in terms of those of the MIT operators bag, $D_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(m)$ and $D_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(m+M)$, acting in $L^2(\Omega_{+-}^\varepsilon)^4$ and $L^2(\mathcal{U}^\varepsilon)^4$, respectively.

On the other hand, we use the semiclassical properties of the Poincaré-Steklov operators in order to invert the auxiliary operator $\Upsilon_M^\varepsilon(z)$ acting on the boundary $\partial\mathcal{U}^\varepsilon = \Sigma \cup \Sigma^\varepsilon$, and which appears in Krein's formula (see (3.57) for the exact notation). Unlike the application of Chapter 2, we remark that in this problem the operator Υ_M^ε (which is constructed by the Poincaré-Steklov operators) takes a pair of functions with respect to $\partial\mathcal{U}^\varepsilon$. With the semiclassical properties verified by the Poincaré-Steklov operators, and subsequently by Υ_M^ε, as well as regularity estimates for the PS operators (see Corollary 3.3.1) and layer potential operators (see Lemma 3.4.3 for more details), we prove the convergence of Proposition 1.6.5. Consequently, using these ingredients, a kind of convergence can be established in Theorem 1.6.6 for $\varepsilon = M^{-1}$.

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1.6.2 Dirac operator coupled with delta shell interactions

A delta shell (δ-shell) interaction, or delta potential, is a mathematical construction used to model a potential energy that acts as an infinitely narrow and infinitely high "spike" at a hypersurface (*e.g.,* curve or surface). The Dirac operators coupled with δ-shell interactions have been studied in detail recently. Mathematically, the Hamiltonian is formally written as

$$\mathbb{D}_{\eta,\tau} := D_m + B_{\eta,\tau}\delta_{\partial\Omega} = D_m + (\eta\mathbb{I}_4 + \tau\beta)\delta_{\partial\Omega}, \tag{1.25}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth open set, β is the Dirac matrix, and $B_{\eta,\tau}$ is a bounded invertible, self-adjoint operator in $L^2(\partial\Omega)^4$. Namely, $\mathbb{D}_{\eta,\tau}$ is called the Dirac operator with a combination of *electrostatic* (strength η) and *Lorentz scalar* (strength τ) δ-shell interactions, with η, $\tau \in \mathbb{R}$.

The initial direct study on the spectral analysis of these Hamiltonian can be traced back to Ref. [DEv89] and [DA90], in which the authors of [DEv89] treated the case that the surface is a sphere, assuming $\eta^2 - \tau^2 = -4$. This specific phenomenon, known as the *confinement case* in physics, signifies the stability of a particle (*for example*, an electron) in its initial region during time evolution. In other words, if the particle is confined to a region $\Omega \subset \mathbb{R}^3$ at time $t = 0$, it remains unable to cross the surface $\partial\Omega$ and enter the region $\mathbb{R}^3 \setminus \overline{\Omega}$ for all subsequent times $t > 0$. Mathematically, this implies that the Dirac operator under consideration can be decomposed into a direct sum of two Dirac operators acting on Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$, respectively, each with appropriate boundary conditions, as we have already seen in the previous chapter. Subsequently, spectral analyses involving Schrödinger operators coupled to δ-shell interactions have developed considerably [KK13, BLL13, AKMN13, EP12, MS12, KM10, EK08, EN03, EI01], while research into the spectral aspects of δ-shell interactions associated with Dirac operators were comparatively inactive. However, Dirac operators with δ-interactions supported on general hypersurfaces have been actively studied since the appearance of the paper [AMV14]. In 2014, a resurgence in the spectral study of these Hamiltonian occurred by N. Arrizabalaga, A. Mas and L. Vega in [AMV14], where the authors developed a new technique to characterize the self-adjointness of the free Dirac operator coupled to a δ-shell potential. In a special case, they treated pure electrostatic δ-shell interactions (*i.e.*, $\tau = 0$) supported on the boundary of a bounded regular domain and proved that the perturbed operator is self-adjoint. The same authors continued their investigation into the spectral analysis of the electrostatic case, exploring the existence of a point spectrum and associated issues in works such as [AMV15] and [AMV16].

Due to the presence of distribution coefficients, the self-adjointness of such operators continues to attract special attention, and it was seen by many authors (primarily for the three-dimensional case) that the self-adjointness domain can be dependent on the coupling constants and the smoothness properties of the hypersurface and that it may lead to unusual spectral properties [BEHL19, BH20, Ben22b, Ben22a, BP24]. Furthermore, J. Behrndt, P. Exner, M. Holzmann and V. Lotoreichik proposed in [BEHL18] the quasi boundary triples theory and their Weyl functions in order to study the spectral properties of the Dirac operators with purely electrostatic δ-shell interactions. Indeed, they were able to confirm the results of [AMV14] about the self-adjointness of $\mathbb{D}_{\eta,0}$ for all $\eta \neq \pm 2$ (called as the non-critical interaction strengths). In the two-dimensional case, the paper [BHOBP20] initiated the study of the Dirac operator coupled with delta shell interactions, and for the case of smooth curves a very complete spectral picture could be found, which was extended in [CLMT23] to a more general class of interactions. Much less attention was given to the case of non-smooth surfaces and curves.

Apart from the self-adjointness of these operators, another area of interest that has been enriched by the contributions of several authors is the approximation of Dirac operators coupled with a singular combination electrostatic and Lorentz scalar, by a Dirac operator coupled with a regular potential. Several progresses have been made in this area, and we present them in the following. The approximation of the Dirac operator $\mathbb{D}_{\eta,\tau}$ by Dirac operators with regular potentials with shrinking support (*i.e.*, of the form (1.26)) provides a justification of the considered idealized model. In the one-dimensional framework, the analysis is carried out in [Š89], where Šeba showed that convergence in the sense of norm resolvent is true. Subsequently, Hughes and Tušek established strong resolvent convergence and norm resolvent convergence for Dirac operators with general point interactions in [Hug97, Hug99] and [Tǔ20], respectively. In the two-dimensional case, [CLMT23, Section 8] considered the approximation of Dirac operators with electrostatic, Lorentz scalar, and anomalous magnetic δ-shell potentials on closed and bounded curves, in the non-critical and non-confinement cases. Additionally, in [BHT23] the authors examined a similar question to [CLMT23], but on a straight line. More precisely, taking parameters $(\eta, \tau) \in \mathbb{R}^2$ in (1.25) and a regular potential $\mathfrak{V}_\Sigma^\varepsilon$ that converges to δ_Σ when ε tends to 0 (in the sense of distributions), then $D_m + B_{\eta,\tau}\mathfrak{V}_\Sigma^\varepsilon$ converges to the Dirac operator $\mathbb{D}_{\hat{\eta},\hat{\tau}}$ with different coupling constants $(\hat{\eta}, \hat{\tau}) \in \mathbb{R}^2$. In particular, these constants depend nonlinearly on the potential $\mathfrak{V}_\Sigma^\varepsilon$.

In the three-dimensional case, the situation seems to be even more complex, as recently shown in [MP18]. There, too, the authors were able to show convergence in the norm resolvent sense in the non-confining case, however, a smallness assumption on the potential $\mathfrak{V}_\Sigma^\varepsilon$ was required to achieve such a result. On the other hand, this assumption unfortunately prevents us from obtaining an approximation of the operator $\mathbb{D}_{\eta,\tau}$ with the physically or mathematically more relevant parameters η and τ. Recognizing this limitation, the authors of the recent paper [BHS23] delved into and verified the approximation problem for two- and three-dimensional Dirac operators with delta-shell potential in the norm resolvent sense. Without the smallness assumption of the potential $\mathfrak{V}_\Sigma^\varepsilon$ no results could be obtained here either.

In the following two parts of this introduction, we will describe our main results from Chapters 4 and 5 on the study of Dirac operators with MIT bag conditions, which correspond to the results obtained in [Zre11] and [BPZ72], respectively.

1.6.2.1 Summary of Chapter 4: On the approximation of the δ-shell interaction for the 3-D Dirac operator

Intuitively, the δ-potential on $\partial\Omega$ corresponds to a potential localized in a small vicinity of $\partial\Omega$. The aim of this chapter is to make this intuition rigorous.

The primary goal of this part is to extend the approximation result explored in [CLMT23, Section 8] to the three-dimensional case. We seek to verify whether the methodologies employed in the two-dimensional context allow us to establish a comparable approximation in terms of strong resolvent. Specifically, we aim to achieve this in the non-critical and non-confinement cases (*i.e.*, when $\eta^2-\tau^2 \neq \pm 4$) without relying on the smallness assumption as stipulated in [MP18]. Mathematically, the Hamiltonian we are interested in is formally written as (1.25). Physically, the Hamiltonian $\mathbb{D}_{\eta,\tau}$ is used as an idealized model for Dirac operators with strongly localized electric and massive potential near the interface Σ (*e.g.*, an annulus), *i.e.*, it replaces a Hamiltonian of the form

$$\mathscr{E}_{\eta,\tau,\varepsilon} = D_m + V_{\eta,\tau,\varepsilon}, \tag{1.26}$$

where $V_{\eta,\tau,\varepsilon}$ are a regular potential localized in a thin layer containing the interface Σ and explicitly defined below.

This chapter answers the following question:

Given the regular approximation achieved in the two-dimensional case in [CLMT23], can we extend this approximation to the three-dimensional case and obtain information on the coupling constant?

In order to answer to this question, we will introduce some additional notations. For a smooth bounded domain $\Omega \subset \mathbb{R}^3$, we consider an interaction supported on the boundary $\Sigma := \partial\Omega$ of Ω. The surface Σ divides the Euclidean space into disjoint union $\mathbb{R}^3 = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ := \Omega$ is a bounded domain and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$. We denote by n the unit outward pointing normal to Ω. We construct a regular symmetric potential $V_{\eta,\tau,\varepsilon} \in L^\infty(\mathbb{R}^3;\mathbb{C}^{4\times 4})$ supported on a tubular ε-neighbourhood of Σ and such that

$$V_{\eta,\tau,\varepsilon} \xrightarrow[\varepsilon\to 0]{} (\eta\,\mathbb{I}_4 + \tau\beta)\delta_\Sigma \quad \text{in the sense of distributions.}$$

Now, for $\gamma > 0$, we define $\Sigma_\gamma := \{x \in \mathbb{R}^3,\ \mathrm{dist}(x,\Sigma) < \gamma\}$ a tubular neighborhood of Σ with width γ. And for $\gamma > 0$ small enough, Σ_γ is parametrized as

$$\Sigma_\gamma = \{x_\Sigma + p n(x_\Sigma),\ x_\Sigma \in \Sigma \quad \text{and} \quad p \in (-\gamma,\gamma)\}.$$

For $0 < \varepsilon < \gamma$, let $h_\varepsilon(p) := \dfrac{1}{\varepsilon} h\left(\dfrac{p}{\varepsilon}\right)$, for all $p \in \mathbb{R}$, with the function h verifying the following

$$h \in L^\infty(\mathbb{R},\mathbb{R}), \quad \mathrm{supp}\, h \subset (-1,1) \text{ and } \int_{-1}^{1} h(t)\,\mathrm{d}t = 1.$$

Thus, we have: $\mathrm{supp}\, h_\varepsilon \subset (-\varepsilon,\varepsilon)$, $\displaystyle\int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t = 1$, and $\displaystyle\lim_{\varepsilon\to 0} h_\varepsilon = \delta_0$ in the sense of the distributions, where δ_0 is the Dirac δ-function supported at the origin. Finally, for any $\varepsilon \in (0,\gamma)$, we define the symmetric approximate potentials $V_{\eta,\tau,\varepsilon} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$, as follows:

$$V_{\eta,\tau,\varepsilon}(x) := \begin{cases} B_{\eta,\tau}\, h_\varepsilon(p), & \text{if} \quad x = x_\Sigma + p n(x_\Sigma) \in \Sigma_\gamma, \\ 0, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\gamma. \end{cases}$$

with $B_{\eta,\tau} = (\eta\, \mathbb{I}_4 + \tau\beta)$. It is easy to see that $\lim_{\varepsilon\to 0} V_{\eta,\tau,\varepsilon} = B_{\eta,\tau}\, \delta_\Sigma$, in $\mathcal{D}'(\mathbb{R}^3)^4$. Then, for $\varepsilon \in (0, \gamma)$, we define the family of Dirac operator $\{\mathscr{E}_{\eta,\tau,\varepsilon}\}_\varepsilon$ as follows:

$$\mathscr{E}_{\eta,\tau,\varepsilon}\psi = D_m\psi + V_{\eta,\tau,\varepsilon}\psi, \quad \text{for all } \psi \in \mathrm{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon}) := \mathrm{Dom}(D_m) = H^1(\mathbb{R}^3)^4. \tag{1.27}$$

Then, in [Chapter 4, Section 4.3] we prove the following result:

Theorem 1.6.8. *Let $(\eta, \tau) \in \mathbb{R}^2$, and denote by $d = \eta^2 - \tau^2$. Let $(\hat{\eta}, \hat{\tau}) \in \mathbb{R}^2$ be defined as follows:*

- *if $d < 0$, then $(\hat{\eta}, \hat{\tau}) = \dfrac{\tanh(\sqrt{-d}/2)}{(\sqrt{-d}/2)}(\eta, \tau)$,*
- *if $d = 0$, then $(\hat{\eta}, \hat{\tau}) = (\eta, \tau)$,*
- *if $d > 0$ such that $d \neq (2k+1)^2\pi^2$, $k \in \mathbb{N} \cup \{0\}$, then $(\hat{\eta}, \hat{\tau}) = \dfrac{\tan(\sqrt{d}/2)}{(\sqrt{d}/2)}(\eta, \tau)$.*

Let $\mathscr{E}_{\eta,\tau,\varepsilon}$ be defined as in (1.27) and $\mathbb{D}_{\hat{\eta},\hat{\tau}}$ as in (1.25) for $(\hat{\eta}, \hat{\tau})$. Then,

$$\mathscr{E}_{\eta,\tau,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathbb{D}_{\hat{\eta},\hat{\tau}} \quad \textit{in the strong resolvent sense.}$$

The proof of this result is to establish the above convergence in the strong graph limit sense. More precisely, the self-adjointness of the limiting operators and the limit operator gives the equivalence between convergence in the strong resolvent and convergence in the strong graph limit sense. The latter means that, for all $\psi \in \mathrm{Dom}(\mathbb{D}_{\hat{\eta},\hat{\tau}})$, there exists a family of vectors $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)} \subset \mathrm{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon})$ such that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathscr{E}_{\eta,\tau,\varepsilon}\psi_\varepsilon = \mathbb{D}_{\hat{\eta},\hat{\tau}}\psi \quad \text{in } L^2(\mathbb{R}^3, \mathbb{C}^4).$$

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1.6.2.2 Summary of Chapter 5: On the self-adjointness of two-dimensional relativistic shell interactions

This chapter is devoted to presenting the results of the article [BPZ72]. First and only in this chapter of the thesis, we mention that D_m is the free Dirac operator in $\mathbb{R}^2$, and that our studies are therefore carried out in 2-D space. Let $m \in \mathbb{R}$. The two-dimensional Dirac operator with mass m is the formally self-adjoint differential expression

$$D_m : \, C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \ni f \, \mapsto \, -i(\sigma_1 \partial_1 f + \sigma_2 \partial_2 f) + m\sigma_3 f \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2),$$

and it naturally extends to a continuous linear map in the space of distributions $\mathcal{D}'(\Omega, \mathbb{C}^2)$ for any open $\Omega \subset \mathbb{R}^2$. Here σ_1, σ_2, and σ_3 are the family of Pauli matrices from (1.5). It is well known that the free two-dimensional Dirac operator

$$A : f \mapsto D_m f, \quad \mathrm{Dom}(A) = H^1(\mathbb{R}^2, \mathbb{C}^2), \tag{1.28}$$

is self-adjoint in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ and has the absolutely continuous spectrum

$$\mathrm{Sp}(A) = \big(-\infty, -|m|\big] \cup \big[|m|, +\infty\big).$$

We will be interested in the study of some special perturbations of A. Namely, let $\Omega_+ \subset \mathbb{R}^2$ be a non-empty bounded open set with Lipschitz boundary. Denote

$$\Sigma := \partial\Omega_+, \quad \Omega_- := \mathbb{R}^2 \setminus \overline{\Omega_+}.$$

For $(\varepsilon, \mu) \in \mathbb{R}^2$, we would like to discuss self-adjoint realizations in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ of operators given formally by

$$f \mapsto D_m f + (\varepsilon \mathbb{I}_2 + \mu\sigma_3)\delta_\Sigma f. \tag{1.29}$$

More specially, we have developed a new technique to prove self-adjointness in low-regularity Sobolev spaces $\big($*i.e.,* domain contained in $H^{1/2}\big)$ namely for general curvilinear polygons Σ. The last sum (1.29) can be considered as an idealized model of a relativistic potential concentrated on Σ, and the constant ε resp. μ measures the strength of the electrostatic resp. Lorentz scalar part of the interaction.

The formal expression (1.29) can be given a more rigorous meaning as follows. First, for any non-empty open set $\Omega \subset \mathbb{R}^2$ consider the two-dimensional Dirac-Sobolev space (*i.e.,* an analogue of the Dirac-Sobolev space in three-dimension (1.9))

$$H(\sigma, \Omega) := \Big\{ f \in L^2(\Omega, \mathbb{C}^2) : \, D_m f \in L^2(\Omega, \mathbb{C}^2) \Big\},$$

which is just the domain of the maximal realization of D_m in $L^2(\Omega, \mathbb{C}^2)$ and becomes a Hilbert space if equipped with the scalar product

$$\langle f, g \rangle_{H(\sigma,\Omega)} := \langle f, g \rangle_{L^2(\Omega,\mathbb{C}^2)} + \langle D_m f, D_m g \rangle_{L^2(\Omega,\mathbb{C}^2)}.$$

For $s > 0$, let $H^s(\Omega, \mathbb{C}^2)$ be the usual fractional Sobolev spaces of order s on Ω (consisting of $\mathbb{C}^2$-valued functions), and we set

$$H^s(\sigma, \Omega) := H(\sigma, \Omega) \cap H^s(\Omega, \mathbb{C}^2),$$

which is a Hilbert space with the scalar product

$$\langle f, g\rangle_{H^s(\sigma,\Omega)} := \langle f, g\rangle_{H(\sigma,\Omega)} + \langle f, g\rangle_{H^s(\Omega,\mathbb{C}^2)}.$$

For what follows it will be convenient to use the identification

$$H(\sigma, \mathbb{R}^2 \setminus \Sigma) \simeq H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-), \quad f \simeq (f_+, f_-),$$

with $f_\pm$ being the restriction of f on $\Omega_\pm$, as well as the analogous identifications for $H^s(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2)$ and $H^s(\sigma, \mathbb{R}^2 \setminus \Sigma)$. We will also use the shorthand notation

$$\sigma \cdot x := x_1\sigma_1 + x_2\sigma_2, \quad x = (x_1, x_2) \in \mathbb{R}^2;$$

from the anticommutation relations (1.6) one easily obtains $(\sigma \cdot x)^2 = |x|^2\mathbb{I}_2$ for all $x \in \mathbb{R}^2$.

It is known that for any $f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma)$ the boundary traces $(\sigma \cdot \nu)f_\pm$ on Σ are well-defined as functions in $H^{-\frac{1}{2}}(\Sigma)$; remark that we keep the same symbols for the boundary traces for better readability. Denote by $\delta_\Sigma f$ the distribution

$$\langle \delta_\Sigma f, \varphi\rangle := \int_\Sigma \frac{f_+ + f_-}{2}\,\varphi\,\mathrm{d}s, \quad \varphi \in C_c^\infty(\mathbb{R}^2),$$

where $\mathrm{d}s$ means the integration with respect to the arclength. An application of the jump formula (distributional derivative for functions with discontinuities along Σ) for a function f shows the identity

$$D_m f = (D_m f_+) \oplus (D_m f_-) + i(\sigma \cdot \nu)(f_+ - f_-)\delta_\Sigma,$$

where $\nu = (\nu_1, \nu_2)$ is the unit normal on Σ pointing to Ω_-. Then it follows that the right-hand side of (1.29) belongs to $L^2(\mathbb{R}^2, \mathbb{C}^2)$ if and only if f satisfies the transmission condition

$$(\varepsilon\mathbb{I}_2 + \mu\sigma_3)\frac{f_+ + f_-}{2} + i(\sigma \cdot \nu)(f_+ - f_-) = 0 \text{ on } \Sigma. \tag{1.30}$$

Therefore, as a first attempt, it is natural to consider the following operator realizations of the expression (1.29) in $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

— the maximal realization $B_{\max}$ with the domain

$$\mathrm{Dom}(B_{\max}) := \{f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma) : f \text{ satisfies (1.30)}\},$$

— the minimal realization $B_{\min}$ with the domain

$$\begin{aligned}\mathrm{Dom}(B_{\min}) &:= \mathrm{Dom}(B_{\max}) \cap H^1(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2)\\ &\equiv \{f \in H^1(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2) : f \text{ satisfies (1.30)}\}.\end{aligned}$$

It is standard to see that $B_{\min}$ is symmetric with $B_{\min}^* = B_{\max}$, therefore, $B_{\min} \subset B \subset B_{\max}$ for any self-adjoint realization B of (1.29). Nevertheless, an explicit description of the self-adjoint realizations turns out to be an involved problem depending on both (ε, μ) and the regularity of Σ.

The most attention was given to the case of C^2-smooth Σ, see [BHSS24] and references therein. Namely, if $\varepsilon^2 - \mu^2 \neq 4$, then $B_{\min} = B_{\max} =: B$, and the spectrum of B consists of the spectrum of the free Dirac operator A and at most finitely many discrete eigenvalues in $(-|m|, |m|)$. For $\varepsilon^2 - \mu^2 = 4$

the operator $B_{\min}$ is not closed, but $\overline{B_{\min}} = B_{\max}$, so $B_{\min}$ is at least essentially self-adjoint (so there is a unique self-adjoint realization), but the loss of regularity leads to peculiar spectral effects (*e.g.*, new pieces of the essential spectrum), see [BHOBP20, BHSS24, BP24]. Remark that [BHSS24, CLMT23] actually consider more general interactions by admitting so-called anomalous magnetic couplings which are not covered by the above framework.

If Σ has corners, one has, in general, $\overline{B_{\min}} \subsetneq B_{\max}$, which means that there are infinitely many self-adjoint realizations [OBP18]. The work [OBP18] suggested that the $H^{\frac{1}{2}}$ regularity should be more natural for the case of non-smooth Σ. Namely, let

$$B \equiv B_{\varepsilon,\mu}$$

be the restriction of $B_{\max}$ to $\mathrm{Dom}(B_{\max}) \cap H^{\frac{1}{2}}(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2)$, *i.e.*,

$$\begin{aligned} B :\ f \simeq (f_+, f_-) &\mapsto (D_m f_+, D_m f_-), \\ \mathrm{Dom}(B) &:= \Big\{ f \in H^{\frac{1}{2}}(\sigma, \mathbb{R}^2 \setminus \Sigma) :\ f \text{ satisfies } (1.30) \Big\}. \end{aligned} \tag{1.31}$$

Due to the standard Sobolev traces theorem, the one-sided traces of functions from $\mathrm{Dom}(B)$ on Σ belong to $L^2(\Sigma, \mathbb{C}^2)$, so the integration by parts shows that B is a symmetric operator. The main result of [PVDB21] reads as follows: if Σ is a curvilinear polygon (a piecewise C^2-smooth closed curve, with finitely many corners and without cusps), $\varepsilon = 0$ and $|\mu| < 2$, then B is self-adjoint. The recent work [BHSS24] presents an extensive study of the case of general compact Lipschitz curves Σ by reducing the self-adjointness to the Fredholmness of some boundary integral operator (see also [AMV14, Ben22a] for the three-dimensional case): we summarize the essential components of the constructions in Section 5.2. Nevertheless, the self-adjoint conditions obtained in [BHSS24] for our case are quite implicit as they depend on the (unknown) spectra of some boundary integral operators.

Presentation of results

Our results in this chapter complement those obtained in the recent papers [BHSS24] and [PVDB21] by providing new very explicit conditions for the self-adjointness of B in terms of the parameters (ε, μ) and the geometry of Σ. The results on the self-adjointness of B are established in several cases, and can be read as follows:

(A) In the case where the curve Σ is Lipschitz, we obtain the following results:

Theorem 1.6.9. *([Chapter 5, Theorems 5.3.1, 5.4.2, and Corollary 5.4.3]). The operator B is self-adjoint for any (ε, μ) with $|\varepsilon| \leq |\mu|$.*

(B) In the case where the curve Σ is C^1-smooth, we have that

Theorem 1.6.10. *If $\varepsilon^2 - \mu^2 \neq 4$, then B is self-adjoint. ([Chapter 5, Theorem 5.4.4])*

(C) In the case where the curve Σ is a curvilinear polygon (with C^1-smooth edges and without cusps), we prove the following:

Theorem 1.6.11. *Denote by ω the smallest angle of Σ, defined by*

$$\omega := \min_{j\in\{1,\dots,n\}} \min\{\theta_j, 2\pi - \theta_j\} \in (0,\pi).$$

If

$$\varepsilon^2 - \mu^2 < \frac{1}{m(\omega)} \quad \text{or} \quad \varepsilon^2 - \mu^2 > 16\, m(\omega), \tag{1.32}$$

then the operator B is self-adjoint. $\big($*[Chapter 5, Theorem 5.5.3]*$\big)$

Here the constant $m(\omega)$ only depends on the sharpest corner ω of Σ. Moreover, the value of $m(\omega)$ is not known explicitly for all ω, but some bounds can be obtained (see Proposition 5.5.2 for more details), and each of the conditions

(i) $\varepsilon^2 - \mu^2 < 2$ or $\varepsilon^2 - \mu^2 > 8$ (without additional geometric assumptions),
(ii) $\varepsilon^2 - \mu^2 \neq 4$ if each angle θ of Σ (measured inside Ω_+) satisfies

$$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2},$$

guarantees the self-adjointness of B $\big($[Chapter 5, Corollary 5.5.4]$\big)$.

Proof ideas

The proofs of these results, all rely on Fredholm properties of boundary integral operators. In fact, we employ two new technical ingredients: The explicit use of the Cauchy transform on non-smooth curves Σ and a characterization of the Fredholmness for boundary integral operators using the approach of [She91].

We would like to describe certain details to demonstrate the results in cases (A), (B) and (C) .

For the results of case (A):

Since the operator B is symmetric, to prove Theorem 1.6.9, it is sufficient to show that $\mathrm{ran}(B-z) = L^2(\mathbb{R}^2,\mathbb{C}^2)$ for any $z \in \mathbb{C}\setminus\mathbb{R}$. To do so, in Section 5.3 we construct an explicit inverse of $(B-z)$ by exploiting the Fredholm property of the boundary integral operator, Λ_z, defined below.

To describe the proof of Theorem 1.6.9, we need to add some notations. Let Λ_z be defined by

$$\Lambda_z := \frac{1}{\varepsilon^2-\mu^2}(\varepsilon\mathbb{I}_2 - \mu\sigma_3) + \mathcal{C}_z,$$

with $\mathcal{C}_z : L^2(\Sigma,\mathbb{C}^2) \longrightarrow L^2(\Sigma,\mathbb{C}^2)$ the singular integral operator given by

$$\mathcal{C}_z g(x) = \mathrm{p.\,v.}\int_\Sigma \phi_z(x-y)g(y)\,\mathrm{d}s(y), \qquad \text{for all} z \in \mathbb{C}\setminus \mathrm{Sp}(A) \text{ and for any } x\in\Sigma,$$

where A is the 2D free Dirac operator defined in (1.28), and $\phi_z : \mathbb{R}^2 \to \mathscr{M}_2(\mathbb{C})$ is the function given by

$$\phi_z(x) := \frac{1}{2\pi}K_0\big(\sqrt{m^2-z^2}|x|\big)(m\sigma_3 + z\mathbb{I}_2) + i\frac{\sqrt{m^2-z^2}}{2\pi|x|}K_1\big(\sqrt{m^2-z^2}|x|\big)(\sigma\cdot x),$$

with K_j the modified Bessel functions of order j.

We also define the layer potentials Φ_z for $D_m - z$ (with $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$)

$$\begin{aligned}\Phi_z : L^2(\Sigma, \mathbb{C}^2) &\longrightarrow L^2(\mathbb{R}^2, \mathbb{C}^2),\\ \Phi_z g(x) &= \int_\Sigma \phi_z(x-y) g(y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \Sigma,\end{aligned}$$

where we recall that $\mathrm{d}s$ means the integration with respect to the arclength.

For $z \in (\mathbb{C} \setminus \mathrm{Sp}(A)) \cup \{m\}$, consider the bounded linear operator

$$\Theta_z := \mathbb{I} + (\varepsilon \mathbb{I}_2 + \mu \sigma_3)\mathcal{C}_z : \ L^2(\Sigma, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2),$$

which is closely related to the operator B from (1.31) as follows:

Lemma 1.6.12. *For any $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$ there holds $\ker(B - z) = \Phi_z \ker \Theta_z$, in particular, $\dim \ker(B - z) = \dim \ker \Theta_z$.*

Thanks to the latter and the following relation between Θ_z and Λ_z

$$\Theta_z \equiv (\varepsilon \mathbb{I}_2 + \mu \sigma_3)\Lambda_z. \tag{1.33}$$

We observe that the self-adjointness of the operator B can be established if the Fredholmness of Λ_z is confirmed.

Since B is symmetric, then we get that $\ker(B - z) = \{0\}$, for any non-real z. Then, for $|\varepsilon| \neq |\mu|$ such that the operator Λ_a is Fredholm for some $a \in (\mathbb{C} \setminus \mathrm{Sp}(A)) \cup \{m\}$, Lemma 1.6.12 and (1.33) imply $\ker \Lambda_z = \{0\}$, and we deduce that Λ_z is surjective and $\operatorname{ran} \Lambda_z = L^2(\Sigma, \mathbb{C}^2)$. Thanks to this surjectivity, we can construct the inverse of $(B - z)$ as follows:

$$(B - z)^{-1} = (A - z)^{-1} - \Phi_z \Lambda_z^{-1} \Phi_{\bar z}^* : \ L^2(\mathbb{R}^2, \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathbb{C}^2)$$

gives the surjectivity of $B - z$ for any non-real z, and then we get $\operatorname{ran}(B - z) = L^2(\mathbb{R}^2, \mathbb{C}^2)$. This complete Theorem 1.6.9.

It is worth noting that if we introduce the tangent vector field

$$\tau = (\tau_1, \tau_2) := (-\nu_2, \nu_1) = \nu$$

on Σ and denote

$$t := \text{the operator of multiplcation by } \tau_1 + i\tau_2 \text{ in } L^2(\Sigma),$$

the operator $(\varepsilon^2 - \mu^2)\Lambda_m$ can then be represented as the following:

$$\begin{aligned}(\varepsilon^2 - \mu^2)\Lambda_m &= (\varepsilon \mathbb{I}_2 - \mu \sigma_3) + (\varepsilon^2 - \mu^2)\mathcal{C}_m\\ &= (\varepsilon \mathbb{I}_2 - \mu \sigma_3) + (\varepsilon^2 - \mu^2)\begin{pmatrix} 0 & C_\Sigma t^* \\ t C_\Sigma^* & 0 \end{pmatrix},\end{aligned}$$

where $C_\Sigma : L^2(\Sigma) \longrightarrow L^2(\Sigma)$ is the Cauchy transform on Σ, defined through the complex line integration

$$C_\Sigma g(x) := \frac{i}{2\pi}\ \mathrm{p.v.}\int_\Sigma \frac{g(y)}{x-y}\,\mathrm{d}y, \quad g \in L^2(\Sigma), \quad x \in \Sigma,$$

and understood in the Cauchy principal value sense.

For the results of case (B):

In the scenario where the curve Σ is C^1-smooth, to do the self-adjointness in the non-critical combinations of coupling constants (*i.e.,* if $\varepsilon^2 - \mu^2 \neq 4$), we adopt for the second time the same strategy as employed in the proof of Theorem 1.6.9. Thus, we prove Theorem 1.6.10.

For the results of case (C):

At the end of this chapter, we prove in Section 5.5 the most important result on the self-adjointness of the operator B when the curve Σ is a curvilinear polygon with C^1-smooth edges and without cusps. Similarly to the preceding cases, under specific assumptions of combination $\varepsilon^2 - \mu^2 \neq 4$, we establish the self-adjointness of B relying on the Fredholmness of the integral operator $(\varepsilon\mathbb{I}_2 + \mu\sigma_3)\Lambda_m$. However, the methodology of the proof is inspired by an algorithm employed by Shelepov [She91] to prove the Fredholmness of a bounded integral operator defined on what is known as a Radon curve. Besides, this methodology requires the introduction of the following concepts:

The bounded integral operator $\Theta_m \equiv (\varepsilon\mathbb{I}_2 + \mu\sigma_3)\Lambda_m \equiv \mathbb{I} + (\varepsilon\mathbb{I}_2 + \mu\sigma_3)\mathcal{C}_m : L^2(\Sigma, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2)$ can be written as follows:

$$\Theta_m g(x) = g - \int_\Sigma \frac{1}{|x-y|} G\Big(x, y, \nu(x), \nu(y), \frac{x-y}{|x-y|}\Big) g(y)\mathrm{d}s(y)$$

with $g \in L^2(\Sigma, \mathbb{C}^2)$ and the 2×2 matrix function G defined by

$$G\Big(x, y, \nu(x), \nu(y), \frac{x-y}{|x-y|}\Big) = -\frac{i}{2\pi}\begin{pmatrix} 0 & (\varepsilon+\mu)\dfrac{\overline{x}-\overline{y}}{|x-y|} \\ (\varepsilon-\mu)\dfrac{x-y}{|x-y|} & 0 \end{pmatrix} \tag{1.34}$$

for $x, y \in \Sigma$, where the integral representations in (5.13) were used. Following [She91], we define a function $\zeta : \mathbb{R} \to \mathbb{R}$ and matrix-valued functions

$$H_a^{(j)} : \mathbb{R} + \frac{i}{2} \to \mathscr{M}_k, \quad j \in \{1, 2\}, \quad \text{with } \mathscr{M}_k \text{ the space of } k \times k \text{ complex matrices.}$$

by

$$\begin{aligned}
\zeta(t) &= \frac{(e^{-\frac{t}{2}}\cos\theta - e^{\frac{t}{2}})\tau - \nu e^{-\frac{t}{2}}\sin\theta}{\sqrt{e^t + e^{-t} - 2\cos\theta}},\\
H_a^{(1)}(\xi) &= \int_{-\infty}^{\infty} \frac{e^{(i\xi+1/2)t}}{\sqrt{e^t + e^{-t} - 2\cos\theta}}\, G\Big(a, a, \nu, -\tau\sin\theta - \nu\cos\theta, \zeta(-t)\Big)\mathrm{d}t,\\
H_a^{(2)}(\xi) &= \int_{-\infty}^{\infty} \frac{e^{(i\xi+1/2)t}}{\sqrt{e^t + e^{-t} - 2\mathrm{cos}(\theta)}}\, G\Big(a, a, -\tau sin\theta - \nu\cos\theta, \nu, -\zeta(t)\Big)\mathrm{d}t.
\end{aligned}$$

Set

$$\Delta_a(\xi) = \det\Big(\mathbb{I}_2 - H_a^{(1)}(\xi)\, H_a^{(2)}(\xi)\Big), \quad \xi \in \mathbb{R} + \frac{i}{2}.$$

The following result was shown in [She91, Theorem 2]:

Proposition 1.6.13. *The operator Θ_m is Fredholm in $L^2(\Sigma, \mathbb{C}^2)$ if and only if*

$$\Delta_{a_j}(\xi) \neq 0 \textit{ for all } \xi \in \mathbb{R} + \frac{i}{2} \textit{ and corners } a_1, ..., a_n \textit{ of } \Sigma.$$

Now, for our matrix function G (1.34), we obtain that

$$\Delta_a(\xi) = \Big(1 - (\varepsilon^2 - \mu^2)M_\theta(2\eta)\Big)^2,$$

where M_θ is the following function

$$M_\theta(x) = \frac{\cosh\left((\pi - \theta)\, x\right)}{2(1 + \cosh(\pi\, x))}, \qquad \text{for all } x \in \mathbb{R}.$$

Applying Proposition 1.6.13, we deduce that the condition $\Delta_a(\xi) \neq 0$ for all ξ is equivalent to

$$M_\theta(x) \neq \frac{1}{\varepsilon^2 - \mu^2} \text{ for all } x \in \mathbb{R}. \tag{1.35}$$

Here θ be the non-oriented interior angle of Σ at the point a measured inside Ω_+. Remark that for any $\theta \in (0, 2\pi)$ one has

$$M_\theta(x) \geq 0 \text{ for all } x \in \mathbb{R}, \qquad \lim_{x\to\pm\infty} M_\theta(x) = 0,$$

then the condition (1.35) is satisfied if any only if

$$\varepsilon^2 - \mu^2 < 0 \quad \text{ or } \quad \frac{1}{\varepsilon^2 - \mu^2} > m(\theta) := \sup_{x\in\mathbb{R}} M_\theta(x),$$

which can be summarized in the single condition $\varepsilon^2 - \mu^2 < \frac{1}{m(\theta)}$.

Under these conditions on $\varepsilon^2 - \mu^2$, we deduce that Θ_z is Fredhom. This means that Λ_z is Fredholm and B therefore is self-adjoint.

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In the following sections, we discuss the different perspectives and questions raised by this thesis.

1.7 Perspectives

1.7.1 Extensions to other dimensions.

The results of Chapters 2–5 are established for specific dimensions only. It is natural to understand the problems for arbitrary dimensions (and at least cover all problems setting for the dimensions two and three). It is clear that a language of Clifford algebras can be used. The analysis of singular integral operators on curves in Chapter 5 could be extended to some rotationally invariant surfaces in $\mathbb{R}^3$ using a separation of variables and the analysis in lower dimensions.

1.7.2 Scattering properties of Dirac operator.

The aim of this section is to study the spectral properties of the Dirac operator in the context of the Spectral Shift Function (SSF). The SSF was introduced by Lifshits and Krein as a generalization of the eigenvalues counting function, and it provides a spectral quantity which makes it possible to compare a self-adjoint perturbed operator to the reference one and it can often be related to physical quantities like the *Scattering Phase* and the *Average Time Delay*. Let (H, H_0) be a pair of two bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $H - H_0$ belongs to the trace class. Then, the general definition of the SSF which I denoted by $\xi(H, H_0; \cdot)$ is the following integral

$$\mathrm{Tr}(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(H, H_0; \cdot) f^{'}(\lambda) \mathrm{d}\lambda, \quad \text{with } f \in C_0^\infty(\mathbb{R}).$$

However, since the perturbations considered in our study are not in trace class, some adaptations are needed, and it reasonable to start with the difference of the resolvents. If one manages to show that

$$(H_M + z)^{-k} - (D_m + z)^{-k} \in \mathcal{S}_1(\mathcal{H})$$

holds for some $k > 0$ and $z \in \rho(H_M) \cap \rho(D_m)$, then there is a SSF, $\xi(H_M, D_m; \cdot)$ of the pair (H_M, D_m). As was observed in the application of the Poincaré-Steklov operators in Chapter 2, we was able to prove the convergence of the perturbed Dirac operator H_M in the norm resolvent sense to the MIT bag operator H_{MIT} when M goes to ∞, and with a convergence rate of $\mathcal{O}(M^{-1})$. The main objective we can study is therefore as follows:

- **Investigate the convergence of $\xi(H_M, D_m; \cdot)$ to $\xi(H_{\mathrm{MIT}}, D_m; \cdot)$ when M tends to ∞.**

We note that it is possible to study the same questions again for the convergence results (in the strong/norm resolvent sense) obtained in Chapters 3 and 4. It is also worth noting that to the best of our knowledge, there is no result dealing with the study of the SSF for Dirac operators in the limit of large coupling constants. In this sense, we have to start from scratch and develop several technical tools to tackle the problem we will consider. Another question that can be studied in the context of the scattering phase is to find a high-energy asymptotic (*i.e.*, $\lambda \to \infty$) of the spectral shift function $\xi(H_{\mathrm{MIT}}, D_m; \lambda)$ where the operator H_{MIT} in this case, acts in an unbounded domain. In our case, we can do the following: For the MIT bag Dirac operator presented in the first part of the summary acting in $\mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is a bounded domain, we can set up an asymptotic expansion for the scattering phase (spectral shift function) $\xi(H_{\mathrm{MIT}}, D_m; \lambda)$ when λ tends to infinity. We note that the geometry of the domain can play a role in the evolution of the asymptotic behavior of the scattering phase. We also mention that, in a tubular neighborhood (as in Chapter 3), it is possible to study other kinds of approximations for any dimension, for example, the approximation of the Dirac operator with a magnetic potential, as has already been done for the magnetic Laplacian in [KRT15].

1.7.3 SSF, Resonances, and non-self-adjoint problems.

When working in the SSF framework, it is important to keep in mind the connection to resonances. In general, a resonance w is a complex number $u+iv$ that describes an unstable quantum state oscillating with a frequency u and a lifetime proportional to $1/v$. In particular, the knowledge that the presence of positive resonances significantly influences the asymptotic completeness of wave operators. The connection between the SSF and the resonance is known as the Breit-Wigner approximation, which states that when λ is close to the real part of resonance w, then an approximation of the derivative of the SSF, $\xi^{'}(\lambda)$ can be found when λ tends to u, the real part of the resonance w. Thanks to the latter, another question can be investigated: Establishing the Breit-Wigner approximation between the SSF and the resonances of the Dirac operators and then to study the existence/presence of the resonances as well as their distribution in the complex plane and the asymptotics in certain regimes, for example, the (semiclassical) Dirac operator with MIT bag boundary conditions on unbounded smooth domains and the Dirac operators with electrostatic and Lorentz scalar δ-shell interactions. We mention that, as resonances are strongly related to non-self-adjoint operators, we find it interesting to study the spectral properties of non-self-adjoint Dirac operators.

1.7.4 Inverse Problems.

The introduction of the Poincaré-Steklov map for the Dirac operator (*i.e.,* an analogue of the Dirichlet-to-Neumann application for the Laplace operator) naturally raises questions about solving Inverse Problems. In the case of Schrödinger, the inverse (boundary value or scattering) problem is whether knowledge of the Dirichlet-to-Neumann map on a particular subset of the boundary determines a potential V uniquely. Indeed, the inverse problems of determining the potential V from the Dirichlet-to-Neumann map have been studied extensively, *e.g.*, for electromagnetic and time-dependent electromagnetic potentials (*see,* [Esk03, Esk08, BJY08]). From a physical point of view, the inverse problem consists in determining the properties, *e.g.*, a dispersion term of an inhomogeneous medium by probing it with perturbations generated on the boundary. Our goal here is to investigate the inverse problem of determining a potential of the Dirac operator from finite measurements on the boundary, via the Poincaré-Steklov map. We remark that some results for analytic domains can be expected from the computation of the complete symbol on the Poincaré-Steklov map similarly to the known studies for the Dirichlet-to-Neumann maps [LU89].

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1.8 How to read this thesis

Chapter 1 contains a complete introduction to the boundary integral operators associated with the free Dirac operator, which are used throughout this thesis.

The body of this thesis is then organised into two parts:

Part 1.6.1 (Chapters 2 and 3) contains our results on three-dimensional Dirac operators with the MIT bag boundary conditions, leading to the introduction of the Poincaré-Steklov (PS) operators. This part corresponds to our papers [BBZ37, Zre84].

Part 1.6.2 (Chapters 4 and 5) deals with Dirac operators coupled with a singular combination of electrostatic and Lorentz scalar delta interactions in three- and two-dimensional setting, respectively, which corresponds to both papers [Zre11] and [BPZ72].

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Chapter 2

A Poincaré-Steklov map for the MIT bag model.

In this chapter, we describe the results obtained in article [BBZ37] in collaboration with Badreddine Benhellal and Vincent Bruneau.

Abstract

The purpose of this chapter is to introduce and study Poincaré-Steklov (PS) operators associated to the Dirac operator D_m with the so-called MIT bag boundary condition. In a domain $\Omega \subset \mathbb{R}^3$, for a complex number z and for U_z a solution of $(D_m - z)U_z = 0$, the associated PS operator maps the value of $\Gamma_- U_z$, the MIT bag boundary value of U_z, to $\Gamma_+ U_z$, where $\Gamma_\pm$ are projections along the boundary $\partial\Omega$ and $(\Gamma_- + \Gamma_+) = t_{\partial\Omega}$ is the trace operator on $\partial\Omega$. Firstly, we show that the PS operator is a zero-order pseudodifferential operator and give its principal symbol. Subsequently, we study the PS operator when the mass m is large, and we prove that it fits into the framework of $1/m$-pseudodifferential operators, and we derive some important properties, especially its semiclassical principal symbol. Then, we apply these results to establish a Krein-type resolvent formula for the Dirac operator $H_M = D_m + M\beta \mathbb{1}_{\mathbb{R}^3 \setminus \overline{\Omega}}$ for large masses $M > 0$, in terms of the resolvent of the MIT bag operator on Ω. With its help, the large coupling convergence with a convergence rate of $\mathcal{O}(M^{-1})$ is shown.

Résumé

Le but de cet chapitre est d'introduire et d'étudier les opérateurs de Poincaré-Steklov (PS) associés à l'opérateur de Dirac D_m avec la condition frontière dite "MIT bag". Dans un domaine $\Omega \subset \mathbb{R}^3$, pour un nombre complexe z et pour U_z une solution de $(D_m - z)U_z = 0$, l'opérateur PS associé fait correspondre la valeur de $\Gamma_- U_z$, la condition au bord MIT de U_z, à $\Gamma_+ U_z$, où $\Gamma_\pm$ sont des projections le long de la frontière $\partial\Omega$ et $(\Gamma_- + \Gamma_+) = t_{\partial\Omega}$ est l'opérateur de trace sur $\partial\Omega$. Premièrement, nous montrons que l'opérateur PS est un opérateur pseudodifférentiel d'ordre zéro et nous donnons son symbole principal. Par la suite, nous étudions l'opérateur PS lorsque la masse m est grande, et nous prouvons qu'il s'intègre dans le cadre des opérateurs $1/m$-pseudodifférentiels, et nous en déduisons quelques propriétés importantes, en particulier son symbole principal semiclassique. Ensuite, nous appliquons ces résultats pour établir une formule de résolvant de type Krein pour l'opérateur de Dirac $H_M = D_m + M\beta \mathbb{1}_{\mathbb{R}^3 \setminus \overline{\Omega}}$ pour les grandes masses $M > 0$, en termes de résolvant de l'opérateur MIT bag sur Ω. Avec son aide, la convergence des grands couplages avec un taux de convergence de $\mathcal{O}(M^{-1})$ est démontrée.

Contents

2.1 Introduction

The main goal of this Chapter is to introduce a Poincaré-Steklov map for the Dirac operator (*i.e.*, an analogue of the Dirichlet-to-Neumann map for the Laplace operator) and to study its pseudodifferential properties. Our main motivation for considering this operator is that it arises naturally in the study of the well-known Dirac operator with the MIT bag boundary condition, $H_{\text{MIT}}(m)$, which will be rigorously defined below.

For a bounded smooth domain $\Omega \subset \mathbb{R}^3$, the MIT bag operator $H_{\text{MIT}}(m)$ is the realization of D_m in $L^2(\Omega, \mathbb{C}^4)$ corresponding to the boundary conditions $P_- t_{\partial\Omega} v = 0$ on $\partial\Omega$ with some explicit matrices P_- depending on the outer unit normal ν and $t_{\partial\Omega}$ being the Dirichlet trace operator (restriction to the boundary). Several researchers, *e.g.,* [MOBP20], have found that the eigenvalues of $H_{\text{MIT}}(m)$ arises as the limit (in the sense of resolvent) of the eigenvalues of the Dirac operator in the whole space $\mathbb{R}^3$ when the mass becomes large outside of Ω (so that the MIT bag boundary condition represents a kind of relativistic hard wall at the boundary). Moreover, various resolvent convergence results were established as well.

The main motivation for the current chapter is to understand the precise rate of the resolvent convergence. For that, we introduce the Poincaré-Steklov operators (PS) $\mathscr{A}_M$ for the Dirac operator with mass M (as an analogue of the Dirichlet-to-Neumann application for the Laplace operator) and studied its microlocal properties. This operator appears naturally in the study of the MIT bag Dirac operator. We show that $\mathscr{A}_M$ fits into the framework of h-pseudodifferential operators (with $h = M^{-1}$) and computed its principal semiclassical symbol.

In the application of this chapter (Section 2.5), based on the pseudodifferential properties of PS, we study the following problem in order to better understand the convergence of H_M to H_{MIT}.

For large M and $z \in \rho(D_{\text{MIT}}(m)) \cap \rho(D_M)$, given $f \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ such that $f = 0$ outside Ω, we ask ourselves, what is the boundary condition on Ω that models the solutions U for $(H_M - z)U = f$ in the whole space? We show that the boundary condition takes the form

$$P_- t_{\partial\Omega}\tilde{v} = \mathcal{B}_M P_+ t_{\partial\Omega} R_{\mathrm{MIT}}(z) f_{|_\Omega} \text{ on } \partial\Omega,$$

where matrix function P_- are explicitly given, $\mathcal{B}_M$ a semiclassical (with respect to $1/M$) pseudodifferential operators of order 0 on $\partial\Omega$, and R_{MIT} is the resolvent of the MIT bag operator $H_{\mathrm{MIT}}(m)$. This implies the resolvent convergence of D_M to $H_{\mathrm{MIT}}(m)$ with the rate $\mathcal{O}(M^{-1})$.

The proof of the above results involves many techniques including the resolvent analysis, pseudodifferential properties of boundary layer potentials, and the construction of a parametrix (*i.e.*, pseudodifferential calculus on $\partial\Omega$) for an inside-outside boundary problem.

In the following section, we recall some properties of symbol classes and their associated pseudodifferential operators.

2.1.1 Symbol classes and Pseudodifferential operators

We recall here the basic facts concerning the classes of pseudodifferential operators that will serve in the rest of the chapter.

Let $\mathscr{M}_4(\mathbb{C})$ be the set of 4×4 matrices over $\mathbb{C}$. For $d \in \mathbb{N}^*$ we let $\mathcal{S}^m(\mathbb{R}^d\times\mathbb{R}^d)$ be the standard symbol class of order $m \in \mathbb{R}$ whose elements are matrix-valued functions a in the space $C^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathscr{M}_4(\mathbb{C}))$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|^2)^{m-|\beta|}, \quad \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,\ \forall \alpha \in \mathbb{N}^d,\ \forall \beta \in \mathbb{N}^d.$$

Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz class of functions. Then, for each $a \in \mathcal{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ and any $h \in (0,1]$, we associate a semiclassical pseudodifferential operator $Op^h(a) : \mathscr{S}(\mathbb{R}^d)^4 \to \mathscr{S}(\mathbb{R}^d)^4$ via the standard formula

$$Op^h(a)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x} a(x, h\xi)\hat{u}(\xi)\mathrm{d}\xi, \quad \forall u \in \mathscr{S}(\mathbb{R}^d)^4.$$

If $a \in \mathcal{S}^0(\mathbb{R}^d\times\mathbb{R}^d)$, then Calderón-Vaillancourt theorem's (see, *e.g.*, [CV72]) yields that $Op^h(a)$ extends to a bounded operator from $L^2(\mathbb{R}^d)^4$ into itself, and there exists $C, N_C > 0$ such that

$$\left|\left|Op^h(a)\right|\right|_{L^2\to L^2} \leqslant C \max_{|\alpha+\beta|\leqslant N_C} \left|\left|\partial_x^\alpha \partial_\xi^\beta a\right|\right|_{L^\infty}. \tag{2.1}$$

By definition, a semiclassical pseudodifferential operator $Op^h(a)$, with $a \in \mathcal{S}^0(\mathbb{R}^d \times \mathbb{R}^d)$, can also be considered as a classical pseudodifferential operator $Op^1(a_h)$ with $a_h = a(x, h\xi)$ which is bounded with respect to $h \in (0, h_0)$, where $h_0 > 0$ is fixed. Thus the Calderón-Vaillancourt theorem also provides the boundedness of these operators in Sobolev spaces $H^s(\mathbb{R}^d)^4 = \langle D_x\rangle^{-s} L^2(\mathbb{R}^d)^4$ where $\langle D_x\rangle = \sqrt{-\Delta + \mathbb{I}}$. Indeed, we have

$$\left|\left|Op^1(a_h)\right|\right|_{H^s\to H^s} = \left|\left|\langle D_x\rangle^s Op^1(a_h)\langle D_x\rangle^{-s}\right|\right|_{L^2\to L^2}, \tag{2.2}$$

and since $\langle D_x\rangle^s Op^1(a_h)\langle D_x\rangle^{-s}$ is a classical pseudodifferential operator with a uniformly bounded symbol in $\mathcal{S}^0$, we deduce that $Op^h(a)$ is uniformly bounded with respect to h from H^s into itself.

Given a C^∞-smooth domain $\Omega \subset \mathbb{R}^3$ with a compact boundary $\Sigma = \partial\Omega$. Then Σ is a 2-dimensional parameterized surface, which in the sense of differential geometry, can also be viewed as a smooth

2-dimensional manifold immersed into $\mathbb{R}^3$. Thus, Σ can be covered by an atlas $\mathbb{A} = \{(U_j, V_j, \varphi_j) | j \in \{1, \cdots, N\}\}$ (*i.e.*, a collection of smooth charts) where $N \in \mathbb{N}^*$. That is

$$\Sigma = \bigcup_{j=1}^{N} U_j,$$

and for each $j \in \{1, \cdots, N\}$, U_j is an open set of Σ, $V_j \subset \mathbb{R}^2$ is an open set of the parametric space $\mathbb{R}^2$, and $\varphi_j : U_j \to V_j$ is a C^∞- diffeormorphism. Moreover, by definition of a smooth manifold, if $U_j \cap U_k \neq \emptyset$ then

$$\varphi_k \circ (\varphi_j)^{-1} \in C^\infty\Big(\varphi_j(U_j \cap U_k);\, \varphi_k(U_j \cap U_k)\Big).$$

As usual, the pull-back $(\varphi_j^{-1})^*$ and the pushforward φ_j^* are defined by

$$(\varphi_j^{-1})^* u = u \circ \varphi_j^{-1} \quad \text{and} \quad \varphi_j^* v = v \circ \varphi_j,$$

for u and v functions on U_j and V_j, respectively. We also recall that a function u on Σ is said to be in the class $C^k(\Sigma)$ if for every chart the pushforward has the property $(\varphi_j^{-1})^* u \in C^k(V_j)$.

Following Zworski [Zwo12, Part 4.], we define pseudodifferential operators on the boundary Σ as follows:

Definition 2.1.1. Let $\mathscr{A} : C^\infty(\Sigma)^4 \to C^\infty(\Sigma)^4$ be a continuous linear operator. Then $\mathscr{A}$ is said to be a h-pseudodifferential operator of order $m \in \mathbb{R}$ on Σ, and we write $\mathscr{A} \in Op^h\mathcal{S}^m(\Sigma)$, if

(1) for every chart (U_j, V_j, φ_j) there exists a symbol $a \in \mathcal{S}^m$ such that

$$\psi_1 \mathscr{A}(\psi_2 u) = \psi_1 \varphi_j^* Op^h(a) (\varphi_j^{-1})^* (\psi_2 u),$$

for any $\psi_1, \psi_2 \in C_0^\infty(U_j)$ and $u \in C^\infty(\Sigma)^4$.

(2) for all $\psi_1, \psi_2 \in C^\infty(\Sigma)$ such that $\operatorname{supp}(\psi_1) \cap \operatorname{supp}(\psi_2) = \emptyset$ and for all $N \in \mathbb{N}$ we have

$$\|\psi_1 \mathscr{A} \psi_2\|_{H^{-N}(\Sigma)^4 \to H^N(\Sigma)^4} = \mathcal{O}(h^\infty).$$

For h fixed (for example $h = 1$), $\mathscr{A}$ is called a pseudodifferential operator.

Since the study of a given pseudodifferential operator on Σ reduces to a local study on local charts, we will recall below the specific local coordinates and surface geometry notations we will use in the rest of the chapter.

We always fix an open set $U \subset \Sigma$, and we let $\chi : V \to \mathbb{R}$ to be a C^∞-function (where $V \subset \mathbb{R}^2$ is open) such that its graph coincides with U. Here and in the following, we omit the possible composition with a rotation that allows this, since changes of variables take h-pseudodifferential operators to h-pseudodifferential operators modulo smoothing operators, and leave the principal symbol invariant. Set $\varphi(\tilde{x}) = (\tilde{x}, \chi(\tilde{x}))$, then for $x \in U$ we write $x = \varphi(\tilde{x})$ with $\tilde{x} \in V$. Here and also in what follows, $\partial_1\chi$ and $\partial_2\chi$ stand for the partial derivatives $\partial_{\tilde{x}_1}\chi$ and $\partial_{\tilde{x}_2}\chi$, respectively. Recall that the first fundamental form, I, and the metric tensor $G(\tilde{x}) = (g_{jk}(\tilde{x}))$, have the following forms:

$$\mathrm{I} = g_{11} d\tilde{x}_1^2 + 2 g_{12} d\tilde{x}_1 d\tilde{x}_2 + g_{22} d\tilde{x}_2^2,$$

$$G(\tilde{x}) = (g_{jk}(\tilde{x})) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(\tilde{x}) := \begin{pmatrix} 1 + |\partial_1\chi|^2 & \partial_1\chi\partial_2\chi \\ \partial_1\chi\partial_2\chi & 1 + |\partial_2\chi|^2 \end{pmatrix}(\tilde{x}).$$

As $G(\tilde{x})$ is symmetric, it follows that it is diagonalizable by an orthogonal matrix. Indeed, let

$$Q(\tilde{x}) := \begin{pmatrix} \frac{|\partial_2\chi|}{|\nabla\chi|} & \frac{\partial_1\chi\partial_2\chi}{|\partial_2\chi||\nabla\chi|} \\ -\frac{\partial_1\chi\partial_2\chi}{|\partial_2\chi||\nabla\chi|} & \frac{|\partial_2\chi|}{|\nabla\chi|}, \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1/2} \end{pmatrix} (\tilde{x}). \tag{2.3}$$

where g stands for the determinant of G. Then, it is straightforward to check that

$$Q^{\mathrm{t}}GQ(\tilde{x}) = \mathbb{I}_2, \quad QQ^{\mathrm{t}}(\tilde{x}) = G(\tilde{x})^{-1} =: \left(g^{jk}(\tilde{x})\right), \quad \det(Q) = \det(Q^{\mathrm{t}}) = g^{-1/2}. \tag{2.4}$$

2.1.2 Operators on the boundary $\Sigma = \partial\Omega$

As above, we consider $\Sigma = \partial\Omega$ the boundary of a smooth bounded domain Ω. On Σ equipped with the Riemann metric induced by the Euclidian one in $\mathbb{R}^3$, we consider the Laplace-Beltrami operator $-\Delta_\Sigma$ and the surface gradient $\nabla_\Sigma = \nabla - n(n\cdot\nabla)$ where n is the unit normal to the surface pointing outside Ω. Note that for (e_1, e_2) an orthonormal basis of the tangent space, $\nabla_\Sigma = e_1\nabla_{e_1} + e_2\nabla_{e_2}$, where ∇_{e_j} stands for the tangential derivative in the direction e_j. With the notation of the previous section, in local coordinates, $-\Delta_\Sigma$ and ∇_Σ are pseudodifferential operators with respective principal symbols

$$p_{-\Delta_\Sigma}(\tilde{x},\xi) = \langle G(\tilde{x})^{-1}\xi,\xi\rangle, \qquad p_{\nabla_\Sigma}(\tilde{x},\xi) = \xi_G := \begin{pmatrix} G(\tilde{x})^{-1}\xi \\ \langle\nabla\chi(\tilde{x}), G(\tilde{x})^{-1}\xi\rangle \end{pmatrix}. \tag{2.5}$$

Let us now introduce D_Σ, the extrinsically defined Dirac operator. To any $x \in \mathbb{R}^3$ we associate the matrix $\alpha(x) = \alpha\cdot x$, where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$. For H_1 the mean curvature of Σ, D_Σ is given by (for more details see Appendix B of [MOBP20]):

$$D_\Sigma = -\alpha(n)\,\alpha(\nabla_\Sigma) + \frac{H_1}{2}.$$

It is a pseudodifferential operator with principal symbol:

$$p_{D_\Sigma}(\tilde{x},\xi) = -i\alpha(n^\varphi(\tilde{x}))\,\alpha(\xi_G),$$

where $n^\varphi = \varphi^* n$. We now define the spin angular momentum S as follows

$$S\cdot X = -\gamma_5(\alpha\cdot X), \quad \forall X\in\mathbb{R}^3, \quad \text{where} \quad \gamma_5 := -i\alpha_1\alpha_2\alpha_3 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}. \tag{2.6}$$

Using the properties (1.3) and (2.74) and the fact that $n\cdot\xi_G = 0$, we then have:

$$p_{D_\Sigma}(\tilde{x},\xi) = -i\alpha\cdot n^\varphi(\tilde{x})\,\alpha\cdot\xi_G = S\cdot(\xi_G\wedge n^\varphi(\tilde{x})).$$

Moreover for $\overline{\xi} := \begin{pmatrix}\xi\\0\end{pmatrix}$, we have: $\overline{\xi} = \xi_G + (\overline{\xi}\cdot n^\varphi)n^\varphi$. Thus, in local coordinates, the principal symbol of D_Σ is also:

$$p_{D_\Sigma}(\tilde{x},\xi) = S\cdot(\overline{\xi}\wedge n^\varphi(\tilde{x})). \tag{2.7}$$

Let us also point out the relationship between the principal symbols of Δ_Σ and D_Σ:

$$|\overline{\xi}\wedge n^\varphi(\tilde{x})|^2 = \langle G(\tilde{x})^{-1}\xi,\xi\rangle. \tag{2.8}$$

2.2 Basic properties of the MIT bag model

In this section, we give a brief review of the basic spectral properties of the Dirac operator with the MIT bag boundary condition on Lipschitz domains. Then, we establish some results concerning the regularization properties of the resolvent and the Sobolev regularity of the eigenfunctions in the case of smooth domains.

Let $\mathcal{U} \subset \mathbb{R}^3$ be a Lipschitz domain with a compact boundary $\partial\mathcal{U}$. Then, for $m > 0$, the Dirac operator with the MIT bag boundary condition on $\mathcal{U}$, $(H_{\text{MIT}}(m), \text{Dom}(H_{\text{MIT}}(m)))$, or simply the MIT bag operator, is defined on the domain

$$\text{Dom}(H_{\text{MIT}}(m)) := \left\{\psi \in H^{1/2}(\mathcal{U})^4 : (\alpha \cdot \nabla)\psi \in L^2(\mathcal{U})^4 \text{ and } P_- t_{\partial\mathcal{U}}\psi = 0 \text{ on } \partial\mathcal{U}\right\},$$

by $H_{\text{MIT}}(m)\psi = D_m\psi$, for all $\psi \in \text{Dom}(H_{\text{MIT}}(m))$, and where the boundary condition holds in $L^2(\partial\mathcal{U})^4$. Here $P_\pm$ are the orthogonal projections defined in (1.8).

The following theorem gathers the basic properties of the MIT bag operator. We mention that some of these properties are well-known in the case of smooth domains, see, *e.g.*, [ALTMR19, ALTMR17, AMSPV23, BHM20, OBV18].

Theorem 2.2.1. *The operator $(H_{MIT}(m), \text{Dom}(H_{MIT}(m)))$ is self-adjoint and we have*

$$(H_{MIT}(m) - z)^{-1} = r_{\mathcal{U}}(D_m - z)^{-1}e_{\mathcal{U}} - \Phi^{\mathcal{U}}_{z,m}(\Lambda^z_m)^{-1}t_{\partial\mathcal{U}}(D_m - z)^{-1}e_{\mathcal{U}}, \quad \forall z \in \rho(D_m). \tag{2.9}$$

Moreover, the following statements hold true:

(i) If $\mathcal{U}$ is bounded, then $\text{Sp}(H_{MIT}(m)) = \text{Sp}_{\text{disc}}(H_{MIT}(m)) \subset \mathbb{R} \setminus [-m, m]$.

(ii) If $\mathcal{U}$ is unbounded, then $\text{Sp}(H_{MIT}(m)) = \text{Sp}_{\text{ess}}(H_{MIT}(m)) = (-\infty, -m] \cup [m, +\infty)$. Moreover, if $\mathcal{U}$ is connected then $\text{Sp}(H_{MIT}(m))$ is purely continuous.

(iii) Let $z \in \rho(H_{MIT}(m))$ be such that $2|z| < m$, then for all $f \in L^2(\mathcal{U})^4$ it holds that

$$\left\|(H_{MIT}(m) - z)^{-1} f\right\|_{L^2(\mathcal{U})^4} \lesssim \frac{1}{m} \|f\|_{L^2(\mathcal{U})^4}.$$

Proof. Let $\varphi, \psi \in \text{Dom}(H_{\text{MIT}}(m))$, then by density arguments we get the Green's formula

$$\langle(-i\alpha \cdot \nabla)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, (-i\alpha \cdot \nabla)\psi\rangle_{L^2(\mathcal{U})^4} = \langle(-i\alpha \cdot n)t_{\partial\mathcal{U}}\varphi, t_{\partial\mathcal{U}}\psi\rangle_{L^2(\partial\mathcal{U})^4}. \tag{2.10}$$

Since $P_- t_{\partial\mathcal{U}}\varphi = P_- t_{\partial\mathcal{U}}\psi = 0$ and $P_\pm(\alpha \cdot n) = (\alpha \cdot n)P_\mp$ (see Lemma 2.6.3), it follows that

$$\langle(-i\alpha \cdot \nabla)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, (-i\alpha \cdot \nabla)\psi\rangle_{L^2(\mathcal{U})^4} = \langle P_+(-i\alpha \cdot n)P_+ t_{\partial\mathcal{U}}\varphi, P_+ t_{\partial\mathcal{U}}\psi\rangle_{L^2(\partial\mathcal{U})^4} = 0.$$

Consequently, we obtain

$$\begin{aligned}\langle H_{\text{MIT}}(m)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, H_{\text{MIT}}(m)\psi\rangle_{L^2(\mathcal{U})^4} &= \langle D_m\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, D_m\psi\rangle_{L^2(\mathcal{U})^4} \\ &= \langle(-i\alpha \cdot \nabla)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, (-i\alpha \cdot \nabla)\psi\rangle_{L^2(\mathcal{U})^4} = 0.\end{aligned}$$

Therefore $(H_{\text{MIT}}(m), \text{Dom}(H_{\text{MIT}}(m)))$ is symmetric. Now, thanks to [Ben22a, Proposition 4.3] we know that the MIT bag operator defined on the domain

$$\mathscr{D} = \left\{ \psi = u + \Phi^{\mathcal{U}}_{0,m}[g], u \in H^1(\mathcal{U})^4, g \in L^2(\partial\mathcal{U})^4 : P_- t_{\partial\mathcal{U}}\psi = 0 \text{ on } \partial\mathcal{U} \right\}, \tag{2.11}$$

by $H_{\text{MIT}}(m)(u + \Phi^{\mathcal{U}}_{0,m}[g]) = D_m u$, for all $(u + \Phi^{\mathcal{U}}_{0,m}[g]) \in \mathscr{D}$, is a self-adjoint operator. As $H_{\text{MIT}}(m)$ is symmetric on $\text{Dom}(H_{\text{MIT}}(m))$ we deduce that $\text{Dom}(H_{\text{MIT}}(m)) \subset \mathscr{D}$. Now, by Remark 1.5.1 we also get that $\mathscr{D} \subset \text{Dom}(H_{\text{MIT}}(m))$ which proves the equality $\mathscr{D} = \text{Dom}(H_{\text{MIT}}(m))$, and thus $(H_{\text{MIT}}(m), \text{Dom}(H_{\text{MIT}}(m)))$ is self-adjoint. Next, we check the resolvent formula (2.9). So let $f \in L^2(\mathcal{U})^4$, $z \in \rho(D_m)$ and set

$$\psi = r_{\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f - \Phi^{\mathcal{U}}_{z,m}(\Lambda^z_m)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f.$$

Since $(D_m - z)^{-1} e_{\mathcal{U}}$ is bounded from $L^2(\mathcal{U})^4$ into $H^1(\mathbb{R}^3)^4$ and $(\Lambda^z_m)^{-1}$ is well-defined by Remark 1.5.1, it follows that

$$u := r_{\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f \in H^1(\mathcal{U})^4 \quad \text{and} \quad g := -(\Lambda^z_m)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f \in L^2(\partial\mathcal{U})^4,$$

which entails that $\psi \in H^{1/2}(\mathcal{U})^4$ and that $(\alpha \cdot \nabla)\psi \in L^2(\mathcal{U})^4$. Next, using Lemma 1.5.1-(i) and Remark 1.5.1 we easily get

$$\begin{aligned} t_{\partial\mathcal{U}}\psi &= t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f + (\frac{i}{2}(\alpha \cdot n) - \mathscr{C}_{z,m})(\frac{1}{2}\beta + \mathscr{C}_{z,m})^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f \\ &= P_+ \beta (\Lambda^z_m)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f, \end{aligned}$$

thus $P_- t_{\partial\mathcal{U}}\psi = 0$ on $\partial\mathcal{U}$, which means that $\psi \in \text{Dom}(H_{\text{MIT}}(m))$. Since $(D_m - z)\Phi^{\mathcal{U}}_{z,m}[g] = 0$ holds in $\mathcal{U}$, it follows that $(H_{\text{MIT}}(m) - z)\psi = f$ and the formula (2.9) is proved.

Now, we are going to prove assertions (i) and (ii). First, note that for $\psi \in \text{Dom}(H_{\text{MIT}}(m))$ a straightforward application of the Green formula (2.10) yields that

$$\|H_{\text{MIT}}(m)\psi\|^2_{L^2(\mathcal{U})^4} = \|(\alpha \cdot \nabla)\psi\|^2_{L^2(\mathcal{U})^4} + m^2 \|\psi\|^2_{L^2(\mathcal{U})^4} + m \|P_+ t_{\partial\mathcal{U}}\psi\|^2_{L^2(\partial\mathcal{U})^4}. \tag{2.12}$$

Thus $\|H_{\text{MIT}}(m)\psi\|^2_{L^2(\mathcal{U})^4} \geqslant m^2 \|\psi\|^2_{L^2(\mathcal{U})^4}$ which yields that $\text{Sp}(H_{\text{MIT}}(m)) \subset (-\infty, -m] \cup [m, +\infty)$. Note that this fact can be seen immediately from the formula (2.9). Next, we show that $\{-m, m\} \notin \text{Sp}_{\text{disc}}(H_{\text{MIT}}(m))$. Assume that there is $0 \neq \psi \in \text{Dom}(H_{\text{MIT}}(m))$ such that $(H_{\text{MIT}}(m) - m)\psi = 0$ in $\mathcal{U}$. Then, from (2.12) we have that

$$\|(-i\alpha \cdot \nabla)\psi\|^2_{L^2(\mathcal{U})^4} + m \|P_+ t_{\partial\mathcal{U}}\psi\|^2_{L^2(\partial\mathcal{U})^4} = 0.$$

Since $m > 0$ it follows that $P_+ t_{\partial\mathcal{U}}\psi = 0$, and thus $t_{\partial\mathcal{U}}\psi = 0$. Using this and the above equation, an integration by parts (using density arguments) gives

$$\|\nabla\psi\|_{L^2(\mathcal{U})^4} = \|(-i\alpha \cdot \nabla)\psi\|_{L^2(\mathcal{U})^4} = 0.$$

From this we conclude that ψ vanishes identically, which contradicts the fact that $\psi \neq 0$, and thus $m \notin \text{Sp}_{\text{disc}}(H_{\text{MIT}}(m))$. Following the same lines as above we also get that $-m \notin \text{Sp}_{\text{disc}}(H_{\text{MIT}}(m))$. Thus, if $\mathcal{U}$ is bounded, then the above considerations and the fact that $\text{Dom}(H_{\text{MIT}}(m)) \subset H^{1/2}(\mathcal{U})^4$ is compactly embedded in $L^2(\mathcal{U})^4$ yield that $\text{Sp}(H_{\text{MIT}}(m)) = \text{Sp}_{\text{disc}}(H_{\text{MIT}}(m)) \subset \mathbb{R} \setminus [-m, m]$, which shows the assertion (i).

Let us now complete the proof of (ii), so suppose that $\mathcal{U}$ is unbounded. We first show that $(-\infty, -m] \cup [m, +\infty) \subset \mathrm{Sp}_{\mathrm{ess}}(H_{\mathrm{MIT}}(m))$ by constructing Weyl sequences as in the case of half-space, see [Ben22b, Theorem 4.1]. As $\mathcal{U}$ is unbounded it follows that there is $R_1 > 0$ such that the half-space $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > R_1\}$ is strictly contained in $\mathcal{U}$ and $\mathbb{R}^3 \setminus \overline{\mathcal{U}} \subset B(0, R_1)$. Fix $\lambda \in (-\infty, -m) \cup (m, +\infty)$ and let $\xi = (\xi_1, \xi_2)$ be such that $|\xi|^2 = \lambda^2 - m^2$. We define the function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{C}^4$ by

$$\varphi(\overline{x}, x_3) = \left(\frac{\xi_1 - i\xi_2}{\lambda - m}, 0, 0, 1\right)^t e^{i\xi\cdot\overline{x}}, \quad \text{with } \overline{x} = (x_1, x_2).$$

Clearly we have $(D_m - \lambda)\varphi = 0$. Now, fix $R_2 > R_1$ and let $\eta \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ and $\chi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ be such that $\mathrm{supp}(\chi) \subset [R_1, R_2]$. For $n \in \mathbb{N}^\star$, we define the sequences of functions

$$\varphi_n(\overline{x}, x_3) = n^{-\frac{3}{2}}\varphi(\overline{x}, x_3)\eta(\overline{x}/n)\chi(x_3/n), \qquad \text{for } (\overline{x}, x_3) \in \mathcal{U}.$$

Then, it is easy to check that $\varphi_n \in H_0^1(\mathcal{U}) \subset \mathrm{Dom}(H_{\mathrm{MIT}}(m))$, $(\varphi_n)_{n\in\mathbb{N}^\star}$ converges weakly to zero, and that

$$\|\varphi_n\|^2_{L^2(\mathcal{U})^4} = \frac{2\lambda}{\lambda - m}\|\eta\|^2_{L^2(\mathbb{R}^2)}\|\chi\|^2_{L^2(\mathbb{R})} > 0, \qquad \frac{\|\left(D_m - \lambda\right)\varphi_n\|_{L^2(\mathcal{U})^4}}{\|\varphi_n\|_{L^2(\mathcal{U})^4}} \underset{n\to\infty}{\longrightarrow} 0,$$

for more details see the proof of [Ben22b, Theorem 4.1]. Therefore, Weyl's criterion yields that

$$(-\infty, -m) \cup (m, +\infty) \subset \mathrm{Sp}_{\mathrm{ess}}(H_{\mathrm{MIT}}(m)).$$

Since the spectrum of a self-adjoint operator is closed, we then get the first statement of (ii). Now, if we assume in addition that $\mathcal{U}$ is connected, then using the same arguments as in the proof of [AMV15, Theorem 3.7] (*i.e.*, using Rellich's lemma and the unique continuation property) one can verify that $H_{\mathrm{MIT}}(m)$ has no eigenvalues in $\mathbb{R} \setminus [-m, m]$. As $\{-m, m\} \notin \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m))$ it follows that $H_{\mathrm{MIT}}(m)$ has a purely continuous spectrum.

Now, we prove (iii). Let $\psi \in \mathrm{Dom}(H_{\mathrm{MIT}}(m))$, then (2.12) yields that $\|H_{\mathrm{MIT}}(m)\psi\|^2_{L^2(\Omega)^4} \geqslant m^2 \|\psi\|^2_{L^2(\Omega)^4}$, and thus

$$m\|\psi\|_{L^2(\mathcal{U})^4} \leqslant \|H_{\mathrm{MIT}}(m)\psi\|_{L^2(\mathcal{U})^4} \leqslant \|(H_{\mathrm{MIT}}(m) - z)\psi\|_{L^2(\mathcal{U})^4} + |z|\,\|\psi\|_{L^2(\mathcal{U})^4}.$$

Therefore, for $2|z| < m$ with $z \in \rho(H_{\mathrm{MIT}}(m))$, we get that $\|\psi\|_{L^2(\mathcal{U})^4} \leqslant 2m^{-1}\|(H_{\mathrm{MIT}}(m) - z)\psi\|_{L^2(\mathcal{U})^4}$. Thus, (iii) follows by taking $\psi = (H_{\mathrm{MIT}}(m) - z)^{-1}f$. ■

Remark 2.2.1. *We mention that the above statement on the self-adjointness can also be deduced from [BHSS24, Theorem 5.4]. We also mention that the MIT bag operator defined on the domain $\mathscr{D}$ given by (2.11) is still self-adjoint for less regular domains, cf. [Ben22a] for more details.*

Remark 2.2.2. *Note that if $\mathcal{U}$ is in the class of Hölder's domains $C^{1,\omega}$, with $\omega \in (1/2, 1)$, then $H_{MIT}(m)$ is self-adjoint and $\mathrm{Dom}(H_{MIT}(m)) := \{\psi \in H^1(\mathcal{U})^4 : P\_t_{\partial\mathcal{U}}\psi = 0 \text{ on } \partial\mathcal{U}\}$, see [Ben22a, Theorem 4.3] for example.*

Now we establish regularity results concerning the regularization property of the resolvent and the Sobolev regularity of the eigenfunctions of $H_{\mathrm{MIT}}(m)$. The first statement of the following theorem will be crucial in Section 2.4 when studying the semiclassical pseudodifferential properties of the Poincaré-Steklov operator.

Theorem 2.2.2. *Let $k \geqslant 1$ be an integer and assume that $\mathcal{U}$ is C^{2+k}-smooth. Then the following statements hold true:*

(i) The mapping $(H_{MIT}(m) - z)^{-1} : H^k(\mathcal{U})^4 \longrightarrow H^{k+1}(\mathcal{U})^4 \cap \mathrm{Dom}(H_{MIT}(m))$ is well-defined and bounded for all $m > 0$ and all $z \in \rho(H_{MIT}(m))$. Moreover, for any compact set $K \subset \mathbb{C}$ there exist $m_0, C > 0$ such that for all $m \geqslant m_0$ and $z \in K$, there holds

$$\|(H_{MIT}(m) - z)^{-1}\|_{H^{k-1}(\mathcal{U})^4 \longrightarrow H^k(\mathcal{U})^4} \leqslant C m^{k-1}.$$

(ii) If ϕ is an eigenfunction associated with an eigenvalue $z \in \mathrm{Sp}(H_{MIT}(m))$, i.e., *$(H_{MIT}(m)-z)\phi = 0$, then $\phi \in H^{1+k}(\mathcal{U})^4$. In particular, if $\mathcal{U}$ is C^∞-smooth, then $\phi \in C^\infty(\mathcal{U})^4$.*

To prove this theorem we need the following classical regularity result.

Proposition 2.2.3. *Let k be a nonnegative integer. Assume that $\mathcal{U}$ is C^{3+k}-smooth and $u \in H^1(\mathcal{U})$. If u solves the Neumann problem*

$$-\Delta u = f \in H^k(\mathcal{U}) \quad \textit{and} \quad \partial_n u = g \in H^{1/2+k}(\partial\mathcal{U}),$$

then $u \in H^{2+k}(\mathcal{U})$.

Proof. First, assume that $k = 0$. As $\mathcal{U}$ is C^3-smooth we know that the Neumann trace $\partial_n : H^2(\mathcal{U}) \to H^{1/2}(\partial\mathcal{U})$ is surjective. Thus, there is $G \in H^2(\mathcal{U})$ such that $\partial_n G = g$ in $\partial\mathcal{U}$. Note that the function $\tilde{u} = u - G$ satisfies the homogeneous Neumann problem

$$-\Delta\tilde{u} = f + \Delta G \ \text{ in } \mathcal{U} \quad \text{and} \quad \partial_n \tilde{u} = 0 \ \text{ on } \partial\mathcal{U}.$$

Therefore, $\tilde{u} \in H^2(\mathcal{U})$ by [Mik78, Theorem 5, p. 217], which implies that $u \in H^2(\mathcal{U})$ and this proves the result for $k = 0$. If $k \geqslant 1$, then the result follows by [Gri85, Theorem 2.5.1.1]. ∎

***Proof* of Theorem 2.2.2.** The theorem will be proved by induction on k. First, we show (i), so fix $z \in \rho(H_{\mathrm{MIT}}(m))$ and assume that $k = 1$. Let $\phi = (\phi_1, \phi_2)^\top \in \mathrm{Dom}(H_{\mathrm{MIT}}(m))$ be such that $(D_m - z)\phi = f$ in $\mathcal{U}$, with $f = (f_1, f_2)^\top \in H^1(\mathcal{U})^4$. By assumption we have $(\Delta + m^2 - z^2)\phi = (D_m + z)f$ in $\mathcal{D}'(\mathcal{U})^4$, and then in $L^2(\mathcal{U})^4$. We next prove that $\partial_n\phi \in H^{1/2}(\partial\mathcal{U})^4$. To this end, consider $\mathcal{U}_\epsilon := \{x \in \mathbb{R}^3 : \mathrm{dist}(x, \partial\mathcal{U}) < \epsilon\}$ for $\epsilon > 0$. Then, for $\delta > 0$ small enough and $0 < \epsilon \leqslant \delta$ the mapping $\Psi : \partial\mathcal{U} \times (-\epsilon, \epsilon) \to \mathcal{U}_\epsilon$, defined by

$$\Psi(x_{\partial\mathcal{U}}, t) = x_{\partial\mathcal{U}} + t n(x_{\partial\mathcal{U}}), \quad x_{\partial\mathcal{U}} \in \partial\mathcal{U},\, t \in (-\epsilon, \epsilon) \tag{2.13}$$

is a C^2-diffeomorphism and $\mathcal{U}_\epsilon := \{x + tn(x) : x \in \partial\mathcal{U},\, t \in (-\epsilon, \epsilon)\}$.

Let $\widetilde{P_-} : L^2(\mathcal{U}_\epsilon \cap \mathcal{U})^4 \to L^2(\mathcal{U}_\epsilon \cap \mathcal{U})^4$ be the bounded operator defined by

$$\widetilde{P_-}\varphi(\Psi(x,t)) = \frac{1}{2}(1 + i\beta(\alpha \cdot n(x)))\varphi(\Psi(x,t)), \quad \Psi(x,t) \in \mathcal{U}_\epsilon \cap \mathcal{U}.$$

Let $x^0_{\partial\mathcal{U}}$ be an arbitrary point on the boundary $\partial\mathcal{U}$, fix $0 < r < \epsilon/2$, and let $\zeta : \mathbb{R}^3 \to [0,1]$ be a C^∞-smooth and compactly supported function such that $\zeta = 1$ on $B(x^0_{\partial\mathcal{U}}, r)$ and $\zeta = 0$ on $\mathbb{R}^3 \setminus B(x^0_{\partial\mathcal{U}}, 2r)$. We claim that $\widetilde{P_-}\zeta\phi$ satisfies the elliptic problem

$$\begin{cases} -\Delta(\widetilde{P_-}\zeta\phi) = g & \text{in } \mathcal{U}, \\ t_{\partial\mathcal{U}}(\widetilde{P_-}\zeta\phi) = 0 & \text{on } \partial\mathcal{U}, \end{cases}$$

with $g \in L^2(\mathcal{U})^4$. Indeed, set $\mathcal{B}(x) = i\beta(\alpha \cdot n(x))$ for $x \in \partial\mathcal{U}$, and observe that

$$(D_m - z)(\widetilde{P_-\zeta\phi}) = \left(\widetilde{P_-\zeta f} + \frac{1}{2}[D_m, \zeta]\phi\right) + \frac{1}{2}[D_m, \zeta\mathcal{B}]\phi =: I(\phi, f) + \frac{1}{2}[D_m, \zeta\mathcal{B}]\phi.$$

Since n is C^2-smooth, ζ is an infinitely differentiable scalar function and $\phi, f \in H^1(\mathcal{U})^4$, it is clear that $I(\phi, f) \in H^1(\mathcal{U})^4$ and $[D_m, \zeta\mathcal{B}]\phi \in L^2(\mathcal{U})^4$. Now, applying $(D_m + z)$ to the above equation yields that $-\Delta(\widetilde{P_-\zeta\phi}) = g$ with

$$g := (z^2 - m^2)\widetilde{P_-\zeta\phi} + (D_m + z)I(\phi, f) + \frac{z}{2}[D_m, \zeta\mathcal{B}]\phi + \frac{1}{2}D_m[D_m, \zeta\mathcal{B}]\phi.$$

As before, it is clear that the first three terms are square integrable. Next, observe that

$$\begin{aligned} D_0[D_0, \zeta\mathcal{B}]\phi &= \{D_0, [D_0, \zeta\mathcal{B}]\}\phi - [D_0, \zeta\mathcal{B}]D_0\phi \\ &= [-\Delta, \zeta\mathcal{B}]\phi - [D_0, \zeta\mathcal{B}]((D_m - z)\phi - (m\beta - z)\phi), \end{aligned}$$

where $\{A, B\} =: AB + BA$ is the anticommutator bracket. Using this, the smoothness assumption on n, the fact that $(D_m - z)\phi = f \in H^1(\mathcal{U})^4$ and that $[D_0, \zeta\mathcal{B}]$ and $[-\Delta, \zeta\mathcal{B}]$ are first order differential operators, we easily see that $D_0[D_0, \zeta\mathcal{B}]\phi \in L^2(\mathcal{U})^4$. Hence, $D_m[D_m, \zeta\mathcal{B}]\phi$ is square integrable, which means that $g \in L^2(\mathcal{U})^4$. As $P_-t_{\partial\mathcal{U}}\phi = 0$ and $t_{\partial\mathcal{U}}(\widetilde{P_-\zeta\phi}) = t_{\partial\mathcal{U}}\zeta P_-t_{\partial\mathcal{U}}\phi = 0$ on $\partial\mathcal{U}$, by [GT01, Theorem 8.12] it follows that $\widetilde{P_-\zeta\phi} \in H^2(\mathcal{U}_\epsilon \cap \mathcal{U})^4$, which implies that

$$\zeta(\phi_1 + i(\sigma \cdot n)\phi_2) \in H^2(B(x^0_{\partial\mathcal{U}}, 2r) \cap \mathcal{U})^2 \quad \text{and} \quad \zeta(-i(\sigma \cdot n)\phi_1 + \phi_2) \in H^2(B(x^0_{\partial\mathcal{U}}, 2r) \cap \mathcal{U})^2.$$

Consequently, we get

$$\phi_1 + i(\sigma \cdot n)\phi_2 \in H^2(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2 \quad \text{and} \quad -i(\sigma \cdot n)\phi_1 + \phi_2 \in H^2(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2. \tag{2.14}$$

Since $-i(\sigma \cdot \nabla)\phi_2 = (z - m)\phi_1 + f_1$ and $-i(\sigma \cdot \nabla)\phi_1 = (z + m)\phi_2 + f_2$ hold in $H^1(\mathcal{U})^2$, it follows from (2.14) that

$$(\sigma \cdot \nabla)\phi_j \in H^1(B(x^0_{\partial\mathcal{U}}, r))^2 \quad \text{and} \quad (\sigma \cdot \nabla)(\sigma \cdot n)\phi_j \in H^1(B(x^0_{\partial\mathcal{U}}, r))^2, \quad j = 1, 2.$$

Using this and the fact that n is C^2-smooth, we easily get that

$$(\sigma \cdot n)(\sigma \cdot \nabla)\phi_j + (\sigma \cdot \nabla)(\sigma \cdot n)\phi_j = (n \cdot \nabla)\phi_j + F_j \in H^1(B(x^0_{\partial\mathcal{U}}, r))^2,$$

with $F_j \in H^1(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2$. As a consequence, we get that $(n \cdot \nabla)\phi_j \in H^1(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2$. Since this holds true for all $x^0_{\partial\mathcal{U}} \in \partial\mathcal{U}$, using the compactness of $\partial\mathcal{U}$ it follows that $\partial_n\phi \in H^{1/2}(\partial\mathcal{U})^4$. Therefore, Propositions 2.2.3 yields that $\phi \in H^2(\mathcal{U})^4$.

Next, assume $k \geqslant 2$, $\mathcal{U}$ is C^{2+k}-smooth and $\phi, f \in H^k(\mathcal{U})^4$. Since n is C^{1+k}-smooth and Ψ defined by (2.13) is a C^{1+k}-diffeomorphism, following the same arguments as above we then conclude that $\partial_n\phi \in H^{k-1/2}(\partial\mathcal{U})^4$. Note also that $-\Delta\phi = (z^2 - m^2)\phi + (D_m - z)f \in H^{k-1}(\mathcal{U})^4$. Therefore, thanks to Proposition 2.2.3, we conclude that $\phi \in H^{k+1}(\mathcal{U})^4$, which proves the first statement of (i).

Now, the second statement of (i) is a consequence of the first one, Theorem 2.2.1-(iii) and the following Gårding-type inequality

$$\|\varphi\|^2_{H^{k+1}(\mathcal{U})^4} \lesssim \|\varphi\|^2_{H^k(\mathcal{U})^4} + \|D_0\varphi\|^2_{H^k(\mathcal{U})^4}, \tag{2.15}$$

which holds for any $\varphi \in \mathrm{Dom}(H_{\mathrm{MIT}}(m)) \cap H^{k+1}(\mathcal{U})^4$, $k \in \mathbb{N}$. Indeed, suppose for instance that (2.15) holds true. Fix a compact set $K \subset \mathbb{C}$ and let $z \in K$. Note that if $z \in \rho(H_{\mathrm{MIT}}(m))$ then for $\psi \in H^k(\mathcal{U})^4$, $k \geqslant 0$, we have

$$\|D_0(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{H^k(\mathcal{U})^4} \leqslant \|\psi\|_{H^k(\mathcal{U})^4} + (m + |z|)\|(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{H^k(\mathcal{U})^4}. \tag{2.16}$$

Let us also remark that Theorem 2.2.1-(iii) entails that there is $m_0 > 0$ such that $z \in \rho(H_{\mathrm{MIT}}(m))$ for any $m \geqslant m_0$, and for any $\psi \in H^k(\mathcal{U})^4$, $k \geqslant 0$, there holds

$$\|D_0(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{L^2(\mathcal{U})^4} \lesssim \|\psi\|_{L^2(\mathcal{U})^4} \leqslant \|\psi\|_{H^k(\mathcal{U})^4}. \tag{2.17}$$

Hence, by iterating the Gårding inequality and taking into account (2.16) and (2.17) we get that

$$\|D_0(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{H^k(\mathcal{U})^4} \lesssim m^k \|\psi\|_{H^k(\mathcal{U})^4},$$

and the conclusion follows by applying again Gårding inequality. We now return to the proof of (2.15).

So let $\varphi \in \mathrm{Dom}(H_{\mathrm{MIT}}(m))$, then [ALTMR17, Theorem 1.5] yields

$$\|D_0\varphi\|^2_{L^2(\mathcal{U})^4} = \|\nabla\varphi\|^2_{L^2(\mathcal{U})^4} + \int_{\partial\mathcal{U}} H_1 |t_{\partial\mathcal{U}}\varphi|^2 \mathrm{d}\sigma, \tag{2.18}$$

where we recall that $H_1(x)$ is the mean curvature at $x \in \partial\mathcal{U}$, and σ is the surface measure on $\partial\mathcal{U}$. Recall that for any $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$\|t_{\partial\mathcal{U}}\varphi\|_{L^2(\partial\mathcal{U})^4} \leqslant \epsilon\|\nabla\varphi\|^2_{L^2(\mathcal{U})^4} + C_\epsilon\|\varphi\|^2_{L^2(\mathcal{U})^4}, \quad \forall\varphi \in H^1(\mathcal{U})^4,$$

see [BCLTS19, Remark 1]. Using this inequality with ϵ sufficiently small and estimating equation (2.18) we get, for all $\varphi \in H^1(\mathcal{U})^4$,

$$\|\varphi\|^2_{H^1(\mathcal{U})^4} = \|\varphi\|^2_{L^2(\mathcal{U})^4} + \|\nabla\varphi\|^2_{L^2(\mathcal{U})^4} \lesssim \|\varphi\|^2_{L^2(\mathcal{U})^4} + \|D_0\varphi\|^2_{L^2(\mathcal{U})^4},$$

which shows (2.15) for $k = 0$. Note that by local arguments one has $\|\varphi\|^2_{H^{k+1}(\mathcal{U})^4} \lesssim \|\varphi\|^2_{L^2(\mathcal{U})^4} + \sum_j \|\partial_j\varphi\|^2_{H^k(\mathcal{U})^4}$, and since $[\partial_j, D_0] = 0$, (2.15) easily follows by induction for any $k \geqslant 1$.

Finally, the proof of the first statement of (ii) follows the same lines as the one of (i). In particular, if $\mathcal{U}$ is C^∞-smooth, we then get $\phi \in H^{k+1}(\mathcal{U})^4$ for any $k \geqslant 0$, which implies that ϕ is infinitely differentiable in $\mathcal{U}$, and the theorem is proved. ■

Remark 2.2.3. *Note that the estimate in Theorem 2.2.2-*(i) *is certainly not sharp but it will be enough for our purposes.*

2.3 Poincaré-Steklov operators as pseudodifferential operators

The main purpose of this section is to introduce the Poincaré-Steklov operator $\mathscr{A}_m$ associated with the MIT bag operator and to prove that it fits into the framework of pseudodifferential operators.

Throughout this section, let Ω be a smooth domain with a compact boundary Σ, and let $P_\pm$ be as in (1.8). Let us start by giving the rigorous definition of the Poincaré-Steklov operator, which is the main subject of this chapter.

Definition 2.3.1. (PS operator) Let $z \in \rho(H_{\text{MIT}}(m))$ and $g \in P_- H^{1/2}(\Sigma)^4$. We denote by $E_m^\Omega(z) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega)^4$ the lifting operator associated with the elliptic problem

$$\begin{cases} (D_m - z)U_z = 0 & \text{in } \Omega, \\ P_- t_\Sigma U_z = g & \text{on } \Sigma. \end{cases} \tag{2.19}$$

That is, $E_m^\Omega(z)g$ is the unique function in $H^1(\Omega)^4$ satisfying $(D_m - z)E_m^\Omega(z)g = 0$ in Ω, and $P_- t_\Sigma E_m^\Omega(z)g = g$ on Σ. Then, the Poincaré-Steklov (PS) operator $\mathscr{A}_m : P_- H^{1/2}(\Sigma)^4 \longrightarrow P_+ H^{1/2}(\Sigma)^4$ associated with the system (2.19) is defined by

$$\mathscr{A}_m(g) = P_+ t_\Sigma E_m^\Omega(z)g,$$

Recall the definitions of $\Phi_{z,m}^\Omega$ and Λ_m^z from Subsection 1.5. Then, the following proposition justifies the existence and the uniqueness of the solution to the elliptic problem (2.19), and gives in particular the explicit formula of the PS operator in terms of the operator $(\Lambda_m^z)^{-1}$ when $z \in \rho(D_m)$. The second assertion of the proposition will be particularly important in Section 2.4 when studying the PS operator from the semiclassical point of view. In the last statement, we use the notations $\mathscr{A}_m(z)$ to highlight the dependence on the parameter $z \in \rho(H_{\text{MIT}}(m))$.

Proposition 2.3.2. *For any $z \in \rho(H_{MIT}(m))$ and $g \in P_- H^{1/2}(\Sigma)^4$, the elliptic problem (2.19) has a unique solution $E_m^\Omega(z)[g] \in H^1(\Omega)^4$. Moreover, the following hold true:*

(i) $\left(E_m^\Omega(z)\right)^* = -\beta P_+ t_\Sigma (H_{MIT}(m) - \overline{z})^{-1}$.

(ii) For any compact set $K \subset \mathbb{C}$, there is $m_0 > 0$ such that for all $m \geqslant m_0$ it holds that $K \subset \rho(H_{MIT}(m))$, and for all $z \in K$ we have

$$\left|\left|E_m^\Omega(z)g\right|\right|_{L^2(\Omega)^4} \lesssim \frac{1}{\sqrt{m}} ||g||_{L^2(\Sigma)^4}, \quad \forall g \in P_- H^{1/2}(\Sigma)^4.$$

(iii) If $z \in \rho(D_m)$, then $E_m^\Omega(z)$ and $\mathscr{A}_m$ are explicitly given by

$$E_m^\Omega(z) = \Phi_{z,m}^\Omega (\Lambda_m^z)^{-1} P_- \quad \textit{and} \quad \mathscr{A}_m = -P_+ \beta (\Lambda_m^z)^{-1} P_-, \tag{2.20}$$

where the boundary integral operators, $\Phi_{z,m}^\Omega$ and Λ_m^z, are introduced in Section 1.5.

(iv) Let $z \in \rho(H_{MIT}(m))$ and let $E_m^\Omega(z)$ be as above. Then, for any $\xi \in \rho(H_{MIT}(m))$, the operator $E_m^\Omega(\xi)$ has the following representation

$$E_m^\Omega(\xi) = (\mathbb{I}_4 + (\xi - z)(H_{MIT}(m) - \xi)^{-1})E_m^\Omega(z). \tag{2.21}$$

In particular, we have

$$\mathscr{A}_m(\xi) - \mathscr{A}_m(z) = (z - \xi)\beta \left(E_m^\Omega(\overline{\xi})\right)^* E_m^\Omega(z). \tag{2.22}$$

(v) For any $z \in \rho(H_{MIT}(m))$ the operator $E_m^\Omega(z)$ extends into a bounded operator from $P_- H^{-1/2}(\Sigma)^4$ to $H(\alpha, \Omega)$.

Proof. We first show that the boundary value problem (2.19) has a unique solution. For this, assume that u_1 and u_2 are both solutions of (2.19), then $(D_m - z)(u_1 - u_2) = 0$ in Ω, and $P_- t_\Sigma (u_1 - u_2) = 0$ on Σ. Thus, $(u_1 - u_2) \in \text{Dom}(H_{\text{MIT}}(m))$ holds by Remark 2.2.2, and since $H_{\text{MIT}}(m)$ is injective by Theorem

2.2.1 it follows that $u_1 = u_2$, which proves the uniqueness. Next, observe that the function

$$v_g = \mathcal{E}_\Omega(P_- g) - (H_{\text{MIT}}(m) - z)^{-1}(D_m - z)\mathcal{E}_\Omega(P_- g)$$

is a solution to (2.19). Indeed, we have $\mathcal{E}_\Omega(P_- g) \in H^1(\Omega)^4$ and thus $v_g \in H^1(\Omega)^4$, moreover, we clearly have that $P_- t_\Sigma v_g = g$ and $(D_m - z)v_g = 0$. Since we already know that the solution to (2.19) is unique, it follows that v_g is independent of the extension operator $\mathcal{E}_\Omega$, and hence there is a unique solution in $H^1(\Omega)^4$ to the elliptic problem (2.19).

Let us show the assertion (i). Let $\psi \in P_- H^{1/2}(\Sigma)^4$ and $f \in L^2(\Omega)^4$, then using the Green's formula and the fact that $P_+(-i\alpha\cdot n) = (-i\alpha\cdot n)P_- = -\beta P_-$ we get that

$$\begin{aligned}
\langle E_m^\Omega(z)\psi, f\rangle_{L^2(\Omega)^4} &= \langle E_m^\Omega(z)\psi, (H_{\text{MIT}}(m) - \overline{z})(H_{\text{MIT}}(m) - \overline{z})^{-1} f\rangle_{L^2(\Omega)^4}\\
&= \langle E_m^\Omega(z)\psi, (D_m - \overline{z})(H_{\text{MIT}}(m) - \overline{z})^{-1} f\rangle_{L^2(\Omega)^4}\\
&= \langle (D_m - z)E_m^\Omega(z)\psi, (H_{\text{MIT}}(m) - \overline{z})^{-1} f\rangle_{L^2(\Omega)^4}\\
&+ \langle (-i\alpha\cdot n)t_\Sigma E_m^\Omega(z)\psi, t_\Sigma(H_{\text{MIT}}(m) - \overline{z})^{-1} f\rangle_{L^2(\Sigma)^4}\\
&= \langle (-i\alpha\cdot n)P_- t_\Sigma E_m^\Omega(z)\psi, P_+ t_\Sigma(H_{\text{MIT}}(m) - \overline{z})^{-1} f\rangle_{L^2(\Sigma)^4}\\
&= \langle \psi, -\beta P_+ t_\Sigma(H_{\text{MIT}}(m) - \overline{z})^{-1} f\rangle_{L^2(\Sigma)^4}
\end{aligned}$$

which entails that $-\beta P_+ t_\Sigma (H_{\text{MIT}}(m) - \overline{z})^{-1}$ is the adjoint of $E_m^\Omega(z)$ and proves (i).

Now we are going to show the assertion (ii). So, let K be a compact set of $\mathbb{C}$, and note that for all $m > \sup\{|\text{Re}(z)| : z \in K\}$ it holds that $K \subset \rho(D_m) \subset \rho(H_{\text{MIT}}(m))$. Hence, $v := E_m^\Omega(z)g$ is well defined for any $z \in K$ and $g \in P_- H^{1/2}(\Sigma)^4$. Then a straightforward application of the Green's formula yields that

$$\begin{aligned}
0 = ||(D_m - z)v||^2_{L^2(\Omega)^4} &= ||(i\alpha\cdot\nabla - z)v||^2_{L^2(\Omega)^4} + m^2\, ||v||^2_{L^2(\Omega)^4}\\
&+ m\Big(\langle -i(\alpha\cdot n)t_\Sigma v, \beta t_\Sigma v\rangle_{L^2(\Sigma)^4} - 2\text{Re}(z)\langle v, \beta v\rangle_{L^2(\Omega)^4}\Big).
\end{aligned} \tag{2.23}$$

Observe that

$$\langle -i(\alpha\cdot n)t_\Sigma v, \beta t_\Sigma v\rangle_{L^2(\Sigma)^4} = \langle (P_+ - P_-)t_\Sigma v, t_\Sigma v\rangle_{L^2(\Sigma)^4} = ||P_+ t_\Sigma v||^2_{L^2(\Sigma)^4} - ||P_- t_\Sigma v||^2_{L^2(\Sigma)^4}.$$

Since $P_- t_\Sigma v = g$ and $P_+ t_\Sigma v = \mathscr{A}_m(g)$ hold by definition, and that

$$-\text{Re}(z)\langle v, \beta v\rangle_{L^2(\Omega)^4} \geqslant -|\text{Re}(z)|\, ||v||^2_{L^2(\Omega)^4}$$

holds by Cauchy-Schwarz inequality, it follows from (2.23) that

$$||g||^2_{L^2(\Sigma)^4} \geqslant m\, ||v||^2_{L^2(\Omega)^4} - 2|\text{Re}(z)|\, ||v||^2_{L^2(\Omega)^4} + ||\mathscr{A}_m(g)||^2_{L^2(\Sigma)^4}.$$

Thus, if we take $m_0 \geqslant 4\sup\{|\text{Re}(z)| : z \in K\}$, then

$$||\mathscr{A}_m(g)||^2_{L^2(\Sigma)^4} + \frac{m}{2}\, ||v||^2_{L^2(\Omega)^4} \leqslant ||g||^2_{L^2(\Sigma)^4}$$

holds for any $m \geqslant m_0$, which proves the desired estimate for $E_m^\Omega(z)$.

Let us now show the assertion (iii), so let $z \in \rho(D_m)$ and recall that $\Phi^{\Omega}_{z,m}(\Lambda^z_m)^{-1} : H^{1/2}(\Sigma)^4 \to H^1(\Omega)^4$ is well defined and bounded by Lemma 1.5.1. Since ϕ^z_m is a fundamental solution of $(D_m - z)$, it holds that

$$(D_m - z)\Phi^{\Omega}_{z,m}(\Lambda^z_m)^{-1}[g] = 0 \text{ in } L^2(\Omega)^4, \quad \forall g \in H^{1/2}(\Sigma)^4.$$

Now, observe that if $g \in P_- H^{1/2}(\Sigma)^4$, then a direct application of the identity (1.16) yields that

$$t_\Sigma \Phi^{\Omega}_{z,m}(\Lambda^z_m)^{-1}[g] = \left(-\frac{i}{2}(\alpha\cdot n) + \mathscr{C}_{z,m}\right)(\Lambda^z_m)^{-1}[g] = g - P_+\beta(\Lambda^z_m)^{-1}[g].$$

Consequently, we get

$$P_- t_\Sigma \Phi^{\Omega}_{z,m}(\Lambda^z_m)^{-1}[g] = g \text{ and } P_+ t_\Sigma \Phi^{\Omega}_{z,m}(\Lambda^z_m)^{-1}[g] = -P_+\beta(\Lambda^z_m)^{-1}[g],$$

which means that $E^{\Omega}_m(z)[g] = \Phi^{\Omega}_{z,m}(\Lambda^z_m)^{-1}[g]$ is the unique solution to the boundary value problem (2.19), and proves the identity $\mathscr{A}_m = -P_+\beta(\Lambda^z_m)^{-1}P_-$.

We are going to prove assertion (iv), so fix $z, \xi \in \rho(H_{\text{MIT}}(m))$ and let $g \in P_- H^{1/2}(\Sigma)^4$. Then, by definition of $E^{\Omega}_m(z)$ we have that

$$\begin{aligned}(D_m - \xi)(1+&(\xi - z)(H_{\text{MIT}}(m) - \xi)^{-1})E^{\Omega}_m(z)g \\ &= (D_m - z)E^{\Omega}_m(z)g - (\xi - z)E^{\Omega}_m(z)g + (\xi - z)(D_m - \xi)(H_{\text{MIT}}(m) - \xi)^{-1}E^{\Omega}_m(z)g, \\ &= (\xi - z)E^{\Omega}_m(z)g - (\xi - z)E^{\Omega}_m(z)g = 0.\end{aligned}$$

Since $(H_{\text{MIT}}(m) - \xi)^{-1}E^{\Omega}_m(z)g \in \text{Dom}(H_{\text{MIT}}(m))$, and hence $P_- t_\Sigma(H_{\text{MIT}}(m) - \xi)^{-1}E^{\Omega}_m(z)g = 0$, it follows that $P_- t_\Sigma(1 + (\xi - z)(H_{\text{MIT}}(m) - \xi)^{-1})E^{\Omega}_m(z)g = P_- t_\Sigma E^{\Omega}_m(z)g = g$, which prove the identity (2.21). Now, (2.22) follows by applying $P_+ t_\Sigma$ to the representation (2.21) and using assertion (i).

It remains to prove item (v). We first consider the case $z \in \rho(D_m)$, then the claim for $z \in \rho(H_{\text{MIT}}(m)) \setminus \rho(D_m)$ follows by the representation formula (2.21). Fix $z \in \rho(D_m)$ and recall that the operators $\mathscr{C}_{z,m}$ and Λ^z_m are bounded invertible in $H^{1/2}(\Sigma)^4$ by Lemma 1.5.1(ii)-(iii) and (1.15). Since $\mathscr{C}^*_{z,m} = \mathscr{C}_{\bar{z},m}$, by duality it follows that Λ^z_m admits a bounded and everywhere defined inverse in $H^{-1/2}(\Sigma)^4$. This together with Lemma 1.5.1(i) and item (iii) of this proposition show that $E^{\Omega}_m(z)$ admits a continuous extension from $P_- H^{-1/2}(\Sigma)^4$ to $H(\alpha,\Omega)$. This completes the proof of the proposition. ■

Remark 2.3.1. *The proof above gives more, namely that for all $m_0 > 0$, $K \subset \rho(D_{m_0})$ a compact set and $z \in K$, there is $m_1 \gg 1$ such that*

$$\sup_{m \geqslant m_1} ||\mathscr{A}_m||_{P_- H^{1/2}(\Sigma)^4 \longrightarrow P_+ L^2(\Sigma)^4} \lesssim 1.$$

Remark 2.3.2. *Thanks to Theorem 2.2.1 and Remark 1.5.1, if Ω is a Lipschitz domain, then $E^{\Omega}_m(z)$ is the unique solution in $H^{1/2}(\Omega)^4$ to the system (2.19) for datum in $L^2(\Sigma)^4$. Moreover, the PS operator $\mathscr{A}_m = -P_+\beta(\Lambda^z_m)^{-1}P_-$ is well-defined and bounded as an operator from $P_- L^2(\Sigma)^4$ to $P_+ L^2(\Sigma)^4$.*

In the rest of this section, we will only address the case $z \in \rho(D_m)$ and we show that the Poincaré-Steklov operator $\mathscr{A}_m$ from Definition 2.3.1 is a homogeneous pseudodifferential operators of order 0 and capture its principal symbol in local coordinates. To this end, we first study the pseudodifferential properties of the Cauchy operator $\mathscr{C}_{z,m}$. Once this is done, we use the explicit formula of $\mathscr{A}_m$ given by (2.20) and the symbol calculus to obtain the principal symbol of $\mathscr{A}_m$.

Recall the definition of ϕ^z_m from (1.11), and observe that

$$\phi^z_m(x-y) = k^z(x-y) + w(x-y),$$

where

$$\begin{aligned}
k^z(x-y) &= \frac{e^{i\sqrt{z^2-m^2}|x-y|}}{4\pi|x-y|}\left(z + m\beta + \sqrt{z^2-m^2}\alpha\cdot\frac{x-y}{|x-y|}\right) + i\frac{e^{i\sqrt{z^2-m^2}|x-y|}-1}{4\pi|x-y|^3}\alpha\cdot(x-y),\\
w(x-y) &= \frac{i}{4\pi|x-y|^3}\alpha\cdot(x-y).
\end{aligned}$$

Using this, it follows that

$$\begin{aligned}
\mathscr{C}_{z,m}[f](x) =& \lim_{\rho\searrow 0}\int_{|x-y|>\rho} w(x-y)f(y)\mathrm{d}\sigma(y) + \int_\Sigma k^z(x-y)f(y)\mathrm{d}\sigma(y)\\
=& W[f](x) + K[f](x).
\end{aligned} \tag{2.24}$$

As $|k^z(x-y)| = \mathcal{O}(|x-y|^{-1})$ when $|x-y|\to 0$, using the standard layer potential techniques (see, *e.g.* [Tay00, Chap. 3, Sec. 4] and [Tay96, Chap. 7, Sec. 11]) it is not hard to prove that the integral operator K gives rise to a pseudodifferential operator of order -1, *i.e.* $K\in Op\mathcal{S}^{-1}(\Sigma)$. Thus, we can (formally) write

$$\mathscr{C}_{z,m} = W \mod Op\mathcal{S}^{-1}(\Sigma), \tag{2.25}$$

which means that the operator W encodes the main contribution in the pseudodifferential character of $\mathscr{C}_{z,m}$. So we only need to focus on the study of the pseudodifferential properties of W. The following theorem makes this heuristic more rigorous. Its proof follows similar arguments as in [AKM17, Miy18, MR20].

Theorem 2.3.3. *Let $\mathscr{C}_{z,m}$ be as (1.13), W as in (2.24) and $\mathscr{A}_m$ as in Definition 2.3.1. Then $\mathscr{C}_{z,m}$, W and $\mathscr{A}_m$ are homogeneous pseudodifferential operators of order 0, and we have*

$$\begin{aligned}
\mathscr{C}_{z,m} &= \frac{1}{2}\alpha\cdot\frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}} \mod Op\mathcal{S}^{-1}(\Sigma),\\
\mathscr{A}_m &= \frac{1}{\sqrt{-\Delta_\Sigma}}S\cdot(\nabla_\Sigma\wedge n)P_- \mod Op\mathcal{S}^{-1}(\Sigma) = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}}P_- \mod Op\mathcal{S}^{-1}(\Sigma),
\end{aligned}$$

where $\mathcal{S}^{-1}$ is the symbol class of order -1 given in Section 2.1.1.

Proof. We first deal with the operator W. So, let $\psi_k : \Sigma\to\mathbb{R}$, $k=1,2$, be a C^∞-smooth function. Clearly, if $\mathrm{supp}(\psi_2)\cap\mathrm{supp}(\psi_1)=\emptyset$, then $\psi_2 W\psi_1$ gives rise to a bounded operator from $H^{-j}(\Sigma)^4$ into $H^j(\Sigma)^4$, for all $j\geqslant 0$.

Now, fix a local chart (U,V,φ) as in Subsection 2.1.1 and recall the definition of the first fundamental form I and the metric tensor $G(\tilde{x})$. That is, up to a rotation, for all $x\in U$ we have $x=\varphi(\tilde{x})=(\tilde{x},\chi(\tilde{x}))$ with $\tilde{x}\in V$, and where the graph of $\chi : V\to\mathbb{R}$ coincides with U. Notice that if we assume that ψ_k is compactly supported with $\mathrm{supp}(\psi_k) \subset U$, then, in this setting, the operator $\psi_2 W \psi_1$ has the form

$$\begin{aligned}\psi_2 W[\psi_1 f](x) =& \psi_2(x)\mathrm{p.v.}\int_V i\alpha \cdot \frac{\varphi(\tilde{x}) - \varphi(\tilde{y})}{4\pi|\varphi(\tilde{x}) - \varphi(\tilde{y})|^3}\psi_1(\varphi(\tilde{y}))f(\varphi(\tilde{y}))\sqrt{g(\tilde{y})}\mathrm{d}\tilde{y} \\ =& \psi_2(x)\sqrt{g(\tilde{x})}\mathrm{p.v.}\int_V i\alpha \cdot \frac{\varphi(\tilde{x}) - \varphi(\tilde{y})}{4\pi|\varphi(\tilde{x}) - \varphi(\tilde{y})|^3}\psi_1(\varphi(\tilde{y}))f(\varphi(\tilde{y}))\mathrm{d}\tilde{y} \\ &+ \psi_2(x)\int_V i\alpha \cdot \frac{\varphi(\tilde{x}) - \varphi(\tilde{y})}{4\pi|\varphi(\tilde{x}) - \varphi(\tilde{y})|^3} f(\varphi(\tilde{y}))\left(\sqrt{g(\tilde{y})} - \sqrt{g(\tilde{x})}\right)\mathrm{d}\tilde{y},\end{aligned} \tag{2.26}$$

where g is the determinant of the metric tensor G. Since $g(\cdot)$ is smooth, it follows that

$$|\sqrt{g(\tilde{y})} - \sqrt{g(\tilde{x})}| \lesssim |\tilde{x} - \tilde{y}|.$$

Therefore, the last integral operator on the right-hand side of (2.26) has a non singular kernel and does not require to write it as an integral operator in the principal value sense. Thus, a simple computation using Taylor's formula shows that

$$|x - y|^2 = |\varphi(\tilde{x}) - \varphi(\tilde{y})|^2 = \langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle(1 + \mathcal{O}|\tilde{x} - \tilde{y}|),$$

where the definition of I was used in the last equality. It follows from the above computations that

$$|x - y|^{-3} = \frac{1}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + k_1(\tilde{x}, \tilde{y}),$$

where the kernel k_1 satisfies $|k_1(\tilde{x}, \tilde{y})| = \mathcal{O}(|\tilde{x} - \tilde{y}|^{-2})$, when $|\tilde{x} - \tilde{y}| \to 0$. Consequently, we get that

$$\frac{x_j - y_j}{|x - y|^3} = \begin{cases} \dfrac{\tilde{x}_j - \tilde{y}_j}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + (\tilde{x}_j - \tilde{y}_j)k_1(\tilde{x}, \tilde{y}), & \text{for } j = 1, 2, \\ \dfrac{\langle \nabla\chi, \tilde{x} - \tilde{y}\rangle}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + k_2(\tilde{x}, \tilde{y}), & \text{for } j = 3, \end{cases}$$

with $|k_2(\tilde{x}, \tilde{y})| = \mathcal{O}(|\tilde{x} - \tilde{y}|^{-1})$, when $|\tilde{x} - \tilde{y}| \to 0$. Note that this implies

$$\alpha \cdot \left(\frac{x - y}{|x - y|^3}\right) = \alpha \cdot \frac{(\tilde{x} - \tilde{y}, \langle \nabla\chi, \tilde{x} - \tilde{y}\rangle)}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + \mathcal{O}(|\tilde{x} - \tilde{y}|^{-1}).$$

Combining the above computations and (2.26), we deduce that

$$\psi_2 W[\psi_1 f](x) = \psi_2(x)\sqrt{g(\tilde{x})}\mathrm{p.v.}\int_V i\alpha \frac{(\tilde{x} - \tilde{y}, \langle \nabla\chi, \tilde{x} - \tilde{y}\rangle)}{4\pi\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} f(\varphi(\tilde{y}))\mathrm{d}\tilde{y} + \psi_2(x)L[\psi_1 f](x), \tag{2.27}$$

where L is an integral operator with a kernel $l(x, y)$ satisfying

$$|l(x, y)| = \mathcal{O}(|x - y|^{-1}) \ \text{ when } \ |x - y| \to 0.$$

Thus, similar arguments as the ones in [Tay96, Chap. 7, Sec. 11] yield that L is a pseudodifferential operator of order -1. Now, for $h \in L^2(\mathbb{R}^2)$ and $k = 1, 2$, observe that if we set

$$R_k[h](\tilde{x}) = \frac{i\sqrt{g(\tilde{x})}}{4\pi} \int_{\mathbb{R}^2} r_k(\tilde{x}, \tilde{x} - \tilde{y}) h(\tilde{y}) \mathrm{d}(\tilde{y}),$$

where for $(\tilde{x}, \tau) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$,

$$r_k(\tilde{x}, \tau) = \frac{\tau_k}{\langle \tau, G(\tilde{x})\tau \rangle^{3/2}}.$$

Then the standard formula connecting a pseudodifferential operator and its symbol yields

$$R_k[h](\tilde{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(\tilde{x}-\tilde{y})\cdot\xi} q_k(\tilde{x}, \xi) h(\tilde{y}) \mathrm{d}\xi \mathrm{d}\tilde{y},$$

where

$$q_k(\tilde{x}, \xi) = \frac{i\sqrt{g(\tilde{x})}}{4\pi} \int_{\mathbb{R}^2} e^{-i\omega\cdot\xi} r_k(\tilde{x}, \omega) \mathrm{d}\omega.$$

Recall the definition of Q from (2.3) and set $\omega = Q(\tilde{x})\tau$. Also recall that

$$\int_{\mathbb{R}^2} e^{-i\omega\cdot\xi} \frac{\omega_k}{|\omega|^3} \mathrm{d}\omega = -2\pi i \frac{\xi_k}{|\xi|}, \quad k = 1, 2. \tag{2.28}$$

Thus, the above change of variables together with the properties (2.4) and (2.28) yield that

$$q_k(\tilde{x}, \xi) = \frac{i}{4\pi} \int_{\mathbb{R}^2} e^{-i(Q(\tilde{x})\tau)\cdot\xi} \frac{(Q(\tilde{x})\tau)_k}{|\tau|^3} \mathrm{d}\tau = \frac{(G^{-1}(\tilde{x})\xi)_k}{2\langle G^{-1}(\tilde{x})\xi, \xi\rangle^{1/2}} = \frac{g_{k1}\xi_1 + g_{k2}\xi_2}{2\langle G^{-1}(\tilde{x})\xi, \xi\rangle^{1/2}},$$

which means that $q_k(\tilde{x}, \xi)$ is homogeneous of degree 0 in ξ. Therefore, R_k is a homogeneous pseudodifferential operators of degree 0. From the above observation and (2.27) if follows that

$$\psi_2 W \psi_1 = \psi_2 \alpha \cdot (R_1, R_2, \partial_1\chi(\tilde{x})R_1 + \partial_2\chi(\tilde{x})R_2)\, \psi_1 + \psi_2 L \psi_1.$$

Since L is a pseudodifferential operator of order -1, we deduce that W is a homogeneous pseudodifferential operators of order 0, and exploiting (2.5), we obtain that

$$W = \frac{1}{2}\alpha \cdot \frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}} \quad \mathrm{mod}\ Op\mathcal{S}^{-1}(\Sigma). \tag{2.29}$$

Thanks to (2.25) and (2.29), we deduce that the Cauchy operator $\mathscr{C}_{z,m}$ has the same principal symbol as the operator W.

Now we are going to deal with the operator $\mathscr{A}_m$. Note that we have

$$\frac{1}{2}\left(\beta + \alpha \cdot \frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}}\right)^2 = \mathbb{I}_4, \tag{2.30}$$

and as $\mathscr{A}_m$ is given by the formula

$$\mathscr{A}_m = -P_+ \beta \left(\frac{1}{2}\beta + \mathscr{C}_{z,m}\right)^{-1} P_-,$$

using (2.30) and the standard mollification arguments, it follows from the product formula for calculus of pseudodifferential operators that, in local coordinates, the symbol of $\mathscr{A}_m$ denoted by $q_{\mathscr{A}_m}$ has the form

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = -P_+\beta\left(\beta + \alpha\cdot\left(\frac{\xi_G}{\langle G^{-1}\xi,\xi\rangle^{1/2}}\right)\right)P_- + p(\tilde{x},\xi),$$

where $p \in \mathcal{S}^{-1}(\Sigma)$ and ξ_G defined in (2.5) is the principal symbol of ∇_Σ. Therefore, we get

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = -P_+\,\beta\,\alpha\cdot\xi_G\,\langle G^{-1}\xi,\xi\rangle^{-1/2}\,P_- + p(\tilde{x},\xi).$$

Hence, using the fact that $P_\pm$ are projectors, and Lemma 2.6.3, we obtain

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = -i\alpha\cdot n^{\varphi}(\tilde{x})\,\alpha\cdot\xi_G\,\langle G^{-1}\xi,\xi\rangle^{-1/2}\,P_- + p(\tilde{x},\xi).$$

Finally, from results of Section 2.1.2 we deduce

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = S\cdot\left(\frac{\xi_G\wedge n^{\varphi}(\tilde{x})}{\langle G^{-1}\xi,\xi\rangle}\right)P_- + p(\tilde{x},\xi),$$

and

$$\mathscr{A}_m = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}}P_- \mod Op\mathcal{S}^{-1}(\Sigma) = \frac{1}{\sqrt{-\Delta_\Sigma}}S\cdot(\nabla_\Sigma\wedge n)P_- \mod Op\mathcal{S}^{-1}(\Sigma).$$

It justifies that $\mathscr{A}_m$ is a homogeneous pseudodifferential operators of order 0 and completes the proof of the theorem. ■

2.4 Approximation of the Poincaré-Steklov operators for large masses

The technique used in the last section allows us to treat the layer potential operator $\mathscr{A}_m$ as pseudodifferential operator and to derive its principal symbol. However, it does not allow us to capture the dependence on m. The main goal of this section is to study the Poincaré-Steklov operator, $\mathscr{A}_m$, as a m-dependent pseudodifferential operator when m is large enough. For this purpose, we consider $h = 1/m$ as a semiclassical parameter (for $m \gg 1$) and use the system (2.19) instead of the layer potential formula of $\mathscr{A}_m$. Roughly speaking, we will look for a local approximate formula for the solution of (2.19). Once this is done, we use the regularization property of the resolvent of the MIT bag operator to catch the semiclassical principal symbol of $\mathscr{A}_m$.

Throughout this section, we assume that $m > 1$, $z \in \rho(H_{\text{MIT}}(m))$ and that Ω is smooth with a compact boundary $\Sigma := \partial\Omega$. Next, we introduce the semiclassical parameter $h = m^{-1} \in (0,1]$, and we set $\mathscr{A}^h := \mathscr{A}_m$. Then, the following theorem is the main result of this section, it ensures that $\mathscr{A}^h$ is a h-pseudodifferential operator of order 0 and gives its semiclassical principal symbol.

Theorem 2.4.1. *Let $h \in (0,1]$ and $z \in \rho(H_{MIT}(m))$, and let $\mathscr{A}^h$ be as above. Then for any $N \in \mathbb{N}$, there exists a h-pseudodifferential operator of order 0, $\mathscr{A}^h_N \in Op^h\mathcal{S}^0(\Sigma)$ such that for h sufficiently small, and any $0 \leqslant l \leqslant N + \frac{1}{2}$*

$$\|\mathscr{A}^h - \mathscr{A}^h_N\|_{H^{\frac{1}{2}}(\Sigma)\to H^{N+\frac{3}{2}-l}(\Sigma)} = O(h^{N+\frac{1}{2}+l}),$$

and
$$\mathscr{A}_N^h = \frac{hD_\Sigma}{\sqrt{-h^2\Delta_\Sigma + \mathbb{I}} + \mathbb{I}} P_- \mod hOp^h\mathcal{S}^{-1}(\Sigma).$$

Let us consider $\mathbb{A} = \{(U_{\varphi_j}, V_{\varphi_j}, \varphi_j) : j \in \{1, \cdots, N\}\}$ an atlas of Σ and $(U_\varphi, V_\varphi, \varphi) \in \mathbb{A}$. As in Section 2.2 we consider the case where U_φ is the graph of a smooth function χ, and we assume that Ω corresponds locally to the side $x_3 > \chi(x_1, x_2)$. Then, for

$$\begin{aligned} U_\varphi =& \{(x_1, x_2, \chi(x_1, x_2));\ (x_1, x_2) \in V_\varphi\}; \quad \varphi((x_1, x_2, \chi(x_1, x_2)) = (x_1, x_2) \\ \mathcal{V}_{\varphi,\varepsilon} :=& \{(y_1, y_2, y_3 + \chi(y_1, y_2));\ (y_1, y_2, y_3) \in V_\varphi \times (0, \varepsilon)\} \subset \Omega, \end{aligned}$$

with ε sufficiently small, we have the following homeomorphism:

$$\begin{aligned} \phi : \mathcal{V}_{\varphi,\varepsilon} &\longrightarrow V_\varphi \times (0, \varepsilon) \\ (x_1, x_2, x_3) &\mapsto (x_1, x_2, x_3 - \chi(x_1, x_2)). \end{aligned}$$

Then the pull-back is

$$\begin{aligned} \phi^* : C^\infty(V_\varphi \times (0, \varepsilon)) &\longrightarrow C^\infty(\mathcal{V}_{\varphi,\varepsilon}) \\ v &\mapsto \phi^* v := v \circ \phi. \end{aligned}$$

We write the change of variables as $y = \phi(x)$ and we assume it is of the form, possibly after a rotation, translation and relabeling:
$$\begin{cases} y_j = x_j, \quad j = 1, 2 \\ y_3 = x_3 - \chi(x_1, x_2). \end{cases} \tag{2.31}$$

Proposition 2.4.2. *By the well-known change of coordinates formula, we can transform the differential operator D_m restricted on $\mathcal{V}_{\varphi,\varepsilon}$ into the following operator on $V_\varphi \times (0, \varepsilon)$:*

$$\begin{aligned} \widetilde{D}_m^\varphi :&= (\phi^{-1})^* D_m (\phi)^* \\ &= -i\,(\alpha_1\partial_{y_1} + \alpha_2\partial_{y_2} - (\alpha_1\partial_{x_1}\chi + \alpha_2\partial_{x_2}\chi - \alpha_3)\partial_{y_3}) + m\beta \\ &= -i(\alpha_1\partial_{y_1} + \alpha_2\partial_{y_2}) + \sqrt{1 + |\nabla\chi|^2}(i\alpha \cdot n^\varphi)(\tilde{y})\partial_{y_3} + m\beta, \end{aligned}$$

where $\tilde{y} = (y_1, y_2)$ and $n^\varphi = (\varphi^{-1})^* n$ is the pull-back of the outward pointing normal to Ω restricted on V_φ:
$$n^\varphi(\tilde{y}) = \frac{1}{\sqrt{1 + |\nabla\chi|^2}} \begin{pmatrix} \partial_{x_1}\chi \\ \partial_{x_2}\chi \\ -1 \end{pmatrix} (y_1, y_2).$$

Proof. Noted by
$$y = \phi(x)$$
$$x = \phi^{-1}(y)$$
$$\tilde{f}(y) = f(\phi^{-1}(y))$$
$$f(x) = \tilde{f}(\phi(x)) = \tilde{f}(y) = \tilde{f}\left(\phi_1(x_1, x_2, x_3), \phi_2(x_1, x_2, x_3), \phi_3(x_1, x_2, x_3)\right).$$

so, we get

$$\begin{aligned} y = (y_1, y_2, y_3) &= \phi(x_1, x_2, x_3) = (x_1, x_2, x_3 - \chi(x_1, x_2)) \\ &= (\phi_1(x_1, x_2, x_3), \phi_2(x_1, x_2, x_3), \phi_3(x_1, x_2, x_3)), \\ \\ &\Leftrightarrow \phi^{-1}(y_1, y_2, y_3) = (y_1, y_2, y_3 - \chi(y_1, y_2)). \end{aligned}$$

Now, we have

$$\begin{cases} \dfrac{\partial \phi_1}{\partial x_1} = 1; \ \dfrac{\partial \phi_1}{\partial x_2} = 0; \ \dfrac{\partial \phi_1}{\partial x_3} = 0; \\ \\ \dfrac{\partial \phi_2}{\partial x_1} = 0; \ \dfrac{\partial \phi_2}{\partial x_2} = 1; \ \dfrac{\partial \phi_3}{\partial x_3} = 0; \\ \\ \dfrac{\partial \phi_3}{\partial x_1} = -\dfrac{\partial \chi(x_1, x_2)}{\partial x_1} = \partial_1 \chi; \ \dfrac{\partial \phi_3}{\partial x_2} = -\dfrac{\partial \chi(x_1, x_2)}{\partial x_2} = \partial_2 \chi; \ \dfrac{\partial \phi_3}{\partial x_3} = 1. \end{cases}$$

then, we obtain

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = \dfrac{\partial \tilde{f}}{\partial y_1} - \partial_1 \chi \dfrac{\partial \tilde{f}}{\partial y_3}, \\ \\ \dfrac{\partial f}{\partial x_2} = \dfrac{\partial \tilde{f}}{\partial y_2} - \partial_2 \chi \dfrac{\partial \tilde{f}}{\partial y_3}, \\ \\ \dfrac{\partial f}{\partial x_1} = \dfrac{\partial \tilde{f}}{\partial y_3}. \end{cases}$$

Hence, for all $f \in L^2(\mathbb{R}^3)^4$

$$\begin{aligned} \widetilde{D}_m^{\varphi} f &= -i\alpha \cdot \nabla f + m\beta f \\ &= -i\alpha_1 \frac{\partial \tilde{f}}{\partial y_1} - i\alpha_2 \frac{\partial \tilde{f}}{\partial y_2} + i\left(\alpha_1 \frac{\partial \chi}{\partial x_1} + \alpha_2 \frac{\partial \chi}{\partial x_2} - \alpha_3\right) \frac{\partial \tilde{f}}{\partial y_3} + m\beta \tilde{f}. \end{aligned}$$

This achieves the proof of the proposition. ■

For the projectors $P_\pm$, we have:

$$P_\pm^{\varphi} := (\varphi^{-1})^* P_\pm (\varphi)^* = \frac{1}{2}\Big(\mathbb{I}_4 \mp i\beta\, \alpha \cdot n^{\varphi}(\tilde{y})\Big).$$

Thus, in the variable $y \in V_\varphi \times (0, \varepsilon)$, the equation (2.19) becomes:

$$\begin{cases} (\widetilde{D}_m^{\varphi} - z)u = 0, & \text{in } V_\varphi \times (0, \varepsilon), \\ \Gamma_-^{\varphi} u = g^{\varphi} = g \circ \varphi^{-1}, & \text{on } V_\varphi \times \{0\}, \end{cases} \tag{2.32}$$

where $\Gamma_\pm^{\varphi} = P_\pm^{\varphi} t_{\{y_3 = 0\}}$.

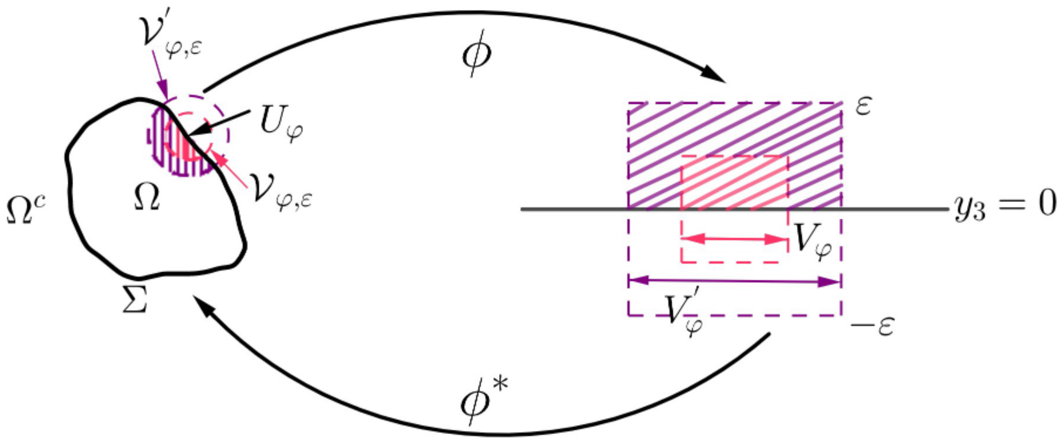

Figure 2.1 – Change of coordinates

By isolating the derivative with respect to y_3, and using that $(i\alpha \cdot n^\varphi)^{-1} = -i\alpha \cdot n^\varphi$, the system (2.32) becomes:

$$\begin{cases} \partial_{y_3} u = \dfrac{i\alpha \cdot n^\varphi(\tilde{y})}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Big(- i\alpha_1 \partial_{y_1} - i\alpha_2 \partial_{y_2} + m\beta - z\Big)u, & \text{in } V_\varphi \times (0,\varepsilon), \\ \Gamma_-^\varphi u = g^\varphi, & \text{on } V_\varphi \times \{0\}. \end{cases} \tag{2.33}$$

Let us now introduce the matrices-valued symbols

$$L_0(\tilde{y},\xi) := \frac{i\alpha \cdot n^\varphi(\tilde{y})}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Big(\alpha\cdot\xi + \beta\Big); \qquad L_1(\tilde{y}) := \frac{-iz\alpha \cdot n^\varphi(\tilde{y})}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}, \tag{2.34}$$

with $\xi = (\xi_1, \xi_2)$ identified with $(\xi_1, \xi_2, 0)$. Then (2.33) is equivalent to

$$\begin{cases} h\partial_{y_3} u = L_0(\tilde{y}, hD_{\tilde{y}})u + hL_1(\tilde{y})u, & \text{in } V_\varphi \times (0,\varepsilon), \\ \Gamma_-^\varphi u = g^\varphi, & \text{on } V_\varphi \times \{0\}. \end{cases} \tag{2.35}$$

Before constructing an approximate solution of the system (2.35), let us give some properties of L_0.

2.4.1 Properties of L_0

The following proposition will be used in the sequel, it gathers some useful spectral properties of the matrix-valued symbol $L_0(\tilde{y},\xi)$ introduced in (2.34). The spectral properties of $l_0(n,\xi) = i(\alpha\cdot n)(\alpha\cdot\xi+\beta)$ given in Proposition 2.6.2 (from Appendix 2.6) provides the following properties for

$$L_0(\tilde{y},\xi) = \frac{1}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}} l_0\big(n^\varphi(\tilde{y}),\xi\big).$$

Proposition 2.4.3. *Let $L_0(\tilde{y},\xi)$ be as in (2.34), then we have*

$$L_0(\tilde{y},\xi) = \frac{1}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Big(i\xi\cdot n^\varphi(\tilde{y}) + S\cdot(n^\varphi(\tilde{y})\wedge\xi) - i\beta(\alpha\cdot n^\varphi(\tilde{y}))\Big),$$
$$= i\xi\cdot\tilde{n}^\varphi(\tilde{y}) + \frac{\lambda(\tilde{y},\xi)}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Pi_+(\tilde{y},\xi) - \frac{\lambda(\tilde{y},\xi)}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Pi_-(\tilde{y},\xi)$$

where

$$\begin{aligned}\lambda(\tilde{y},\xi) :=& \sqrt{|n^\varphi(\tilde{y})\wedge\xi|^2+1} = \sqrt{\langle G(\tilde{y})^{-1}\xi,\xi\rangle+1},\\ \tilde{n}^\varphi(\tilde{y}) :=& \frac{1}{\sqrt{1+|\nabla\chi|^2}}n^\varphi(\tilde{y}),\\ \Pi_\pm(\tilde{y},\xi) :=& \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{S\cdot(n^\varphi(\tilde{y})\wedge\xi) - i\beta(\alpha\cdot n^\varphi(\tilde{y}))}{\lambda(\tilde{y},\xi)}\right),\end{aligned} \tag{2.36}$$

with G the induced metric defined in Section 2.1.1.

In particular, the symbol $L_0(\tilde{y},\xi)$ is elliptic in $\mathcal{S}^1$ and it admits two eigenvalues $\rho_\pm(\cdot,\cdot)\in\mathcal{S}^1$ of multiplicity 2 which are given by

$$\rho_\pm(\tilde{y},\xi) = \frac{in^\varphi(\tilde{y})\cdot\xi \pm \lambda(\tilde{y},\xi)}{\sqrt{1+|\nabla\chi|^2}}, \tag{2.37}$$

and for which there exists $c>0$ such that

$$\frac{(\rho_+-\rho_-)(\tilde{y},\xi)}{2} = \pm\Re\rho_\pm(\tilde{y},\xi) > c\langle\xi\rangle, \tag{2.38}$$

uniformly with respect to $\tilde{y}$. Moreover, $\Pi_\pm(\tilde{y},\xi)$, the projections onto $\mathrm{Kr}(L_0(\tilde{y},\xi)-\rho_\pm(\tilde{y},\xi)\mathbb{I}_4)$, belong to the symbol class $\mathcal{S}^0$ and satisfy:

$$P^\varphi_\pm\,\Pi_\pm(\tilde{y},\xi)\,P^\varphi_\pm = k^\varphi_+(\tilde{y},\xi)P^\varphi_\pm \quad \textit{and} \quad P^\varphi_\pm\,\Pi_\mp(\tilde{y},\xi)\,P^\varphi_\mp = \mp\Theta^\varphi(\tilde{y},\xi)P^\varphi_\mp, \tag{2.39}$$

with

$$k^\varphi_\pm(\tilde{y},\xi) = \frac{1}{2}\left(1\pm\frac{1}{\lambda(\tilde{y},\xi)}\right), \quad \Theta^\varphi(\tilde{y},\xi) = \frac{1}{2\lambda(\tilde{y},\xi)}\left(S\cdot(n^\varphi(\tilde{y})\wedge\xi)\right). \tag{2.40}$$

That is, k^φ_+ is a positive function of $\mathcal{S}^0$, $(k^\varphi_+)^{-1}\in\mathcal{S}^0$ and $\Theta^\varphi\in\mathcal{S}^0$.

Remark 2.4.1. *Thanks to the property (2.39) a 4×4-matrix A is uniquely determined by $P^\varphi_- A$ and $\Pi_+ A$ and we have:*

$$A = P^\varphi_- A + P^\varphi_+ A = P^\varphi_- A + \frac{1}{k^\varphi_+}P^\varphi_+\Pi_+P^\varphi_+ A = \Big(\mathbb{I} - \frac{P^\varphi_+\Pi_+}{k^\varphi_+}\Big)P^\varphi_- A + \frac{P^\varphi_+}{k^\varphi_+}\Pi_+ A.$$

***Proof of Proposition* 2.4.3.** By definition it is clear that $L_0(\tilde{y},\xi)$ belongs to the symbol class $\mathcal{S}^1$, and all the formulas follows for whose for $l_0(n,\xi)$ proved in the Appendix 2.6 (see Proposition 2.6.2 and Lemma 2.6.3), mainly taking $n=n^\varphi(\tilde{y})$ and multiplying by $\frac{1}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}$. Next, using (2.8) we get for some $c > 0$ independent of $\tilde{y}$ that

$$\pm\Re\rho_\pm(\tilde{y},\xi) = \frac{\sqrt{|n^\varphi \wedge \xi|^2 + 1}}{\sqrt{1+|\nabla\chi|^2}} = \frac{\sqrt{\langle G(\tilde{y})^{-1}\xi,\xi\rangle + 1}}{\sqrt{1+|\nabla\chi|^2}} \geqslant c(1+|\xi|),$$

which gives (2.38) and shows that $\rho_\pm$ are elliptic in $\mathcal{S}^1$. Consequently, we also get that $L_0(\tilde{y},\xi)$ is elliptic in $\mathcal{S}^1$ and that the functions $\Pi_\pm$, k_+^φ, $(k_+^\varphi)^{-1}$ and Θ^φ belong to the symbol class $\mathcal{S}^0$. ■

2.4.2 Semiclassical parametrix for the boundary problem

In this section, we construct the approximate solution of the system (1.19) mentioned in the introduction. For simplicity of notation, in the sequel we will use y and $P_\pm$ instead of $\tilde{y}$ and $P_\pm^\varphi$, respectively.

We are going to construct a local approximate solution of the following first order system:

$$\begin{cases} h\partial_\tau u^h = L_0(y,hD_y)u^h + hL_1(y)u^h, & \text{in } \mathbb{R}^2\times(0,+\infty),\\ P_- u^h = f, & \text{on } \mathbb{R}^2\times\{0\}. \end{cases}$$

To be precise, we will look for a solution u^h in the following form:

$$u^h(y,\tau) = Op^h(A^h(\cdot,\cdot,\tau))f = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A^h(y,h\xi,\tau)e^{iy\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi, \tag{2.41}$$

with $A^h(\cdot,\cdot,\tau)\in\mathcal{S}^0$ for any $\tau>0$ constructed inductively in the form:

$$A^h(y,\xi,\tau) \sim \sum_{j\geqslant 0} h^j A_j(y,\xi,\tau).$$

The action of $h\partial_\tau - L_0(y,hD_y) - hL_1(y)$ on $A^h(y,hD_y,\tau)f$ is given by $T^h(y,hD_y,\tau)f$, with

$$T^h(y,\xi,\tau) = h(\partial_\tau A)(y,\xi,\tau) - L_0(y,\xi)A(y,\xi,\tau) - h\Big(L_1(y)A(y,\xi,\tau) - i\partial_\xi L_0(y,\xi)\cdot\partial_y A(y,\xi,\tau)\Big).$$

Here we exploited the particular form of L_1 (independent of ξ) and of L_0 (first order polynomial in ξ).

Then we look for A_0 satisfying:

$$\begin{cases} h\partial_\tau A_0(y,\xi,\tau) = L_0(y,\xi)A_0(y,\xi,\tau),\\ P_-(y)A_0(y,\xi,\tau) = P_-(y), \end{cases} \tag{2.42}$$

and for $j\geqslant 1$,

$$\begin{cases} h\partial_\tau A_j(y,\xi,\tau) = L_0(y,\xi)A_j(y,\xi,\tau) + L_1(y)A_{j-1}(y,\xi,\tau) - i\partial_\xi L_0(y,\xi)\cdot\partial_y A_{j-1}(y,\xi,\tau),\\ P_-(y)A_j(y,\xi,\tau) = 0. \end{cases} \tag{2.43}$$

Let us introduce a class of parametrized symbols, in which we will construct the family A_j:

$$\mathcal{P}_h^m := \{b(\cdot,\cdot,\tau)\in\mathcal{S}^m;\ \forall(k,l)\in\mathbb{N}^2,\ \tau^k\partial_\tau^l b(\cdot,\cdot,\tau)\in h^{k-l}\mathcal{S}^{m-k+l}\};\quad m\in\mathbb{Z}.$$

More precisely, $b\in\mathcal{P}_h^m$ means that for all $(k,l)\in\mathbb{N}^2$, the function $(\tau,h)\mapsto(h^{-1}\tau)^k(h\partial_\tau)^l b(\cdot,\cdot,\tau)$ is uniformly bounded with respect to $(\tau,h)\in(0,+\infty)\times(0,1)$ in $\mathcal{S}^{m-k+l}$.

Proposition 2.4.4. *There exists $A_0 \in \mathcal{P}_h^0$ solution of (2.42) given by:*

$$A_0(y,\xi,\tau) = \frac{\Pi_-(y,\xi)P_-(y)}{k_+^\varphi(y,\xi)}\, e^{h^{-1}\tau\rho_-(y,\xi)}.$$

Proof. The solutions of the differential system $h\partial_\tau A_0 = L_0 A_0$ are $A_0(y,\xi,\tau) = e^{h^{-1}\tau L_0(y,\xi)}A_0(y,\xi,0)$. By definition of $\rho_\pm$ and $\Pi_\pm$, we have:

$$e^{h^{-1}\tau L_0(y,\xi)} = e^{h^{-1}\tau\rho_-(y,\xi)}\Pi_-(y,\xi) + e^{h^{-1}\tau\rho_+(y,\xi)}\Pi_+(y,\xi). \tag{2.44}$$

It follows from (2.38) that A_0 belongs to $\mathcal{S}^0$ for any $\tau > 0$ if and only if $\Pi_+(y,\xi)A_0(y,\xi,0) = 0$. Moreover, the boundary condition $P_-A_0 = P_-$ implies $P_-(y)A_0(y,\xi,0) = P_-(y)$. Thus, thanks to Remark 2.4.1, we deduce that

$$A_0(y,\xi,0) = P_-(y) - \frac{P_+\Pi_+P_-}{k_+^\varphi}(y,\xi) = P_-(y) + \frac{P_+\Pi_-P_-}{k_+^\varphi}(y,\xi) = \frac{\Pi_-P_-}{k_+^\varphi}(y,\xi).$$

The properties of ρ_-, Π_-, P_- and k_+ given in Proposition 2.4.3, imply that $(k_+^\varphi)^{-1}\Pi_-P_- \in \mathcal{S}^0$ and that $e^{h^{-1}\tau\rho_-(y,\xi)} \in \mathcal{P}_h^0$. This concludes the proof of Proposition 2.4.4. ■

For the other terms A_j, $j \geqslant 1$, we have:

Proposition 2.4.5. *Let A_0 be defined by Proposition 2.4.4. Then for any $j \geqslant 1$, there exists $A_j \in h^j\mathcal{P}_h^{-j}$ solution of (2.43) which has the form:*

$$A_j(y,\xi,\tau) = e^{h^{-1}\tau\rho_-(y,\xi)}\sum_{k=0}^{2j}(h^{-1}\tau\langle\xi\rangle)^k B_{j,k}(y,\xi), \tag{2.45}$$

with $B_{j,k} \in h^j\mathcal{S}^{-j}$.

Proof. Let us prove the result by induction. Thanks to Proposition 2.4.4, the claimed property holds for $j = 0$. Now, assume that there exists $A_j \in h^j\mathcal{P}_h^{-j}$ solution of (2.43) satisfying the above property and let us prove that the same holds for A_{j+1}. In order to be a solution of the differential system $h\partial_\tau A_{j+1} = L_0A_{j+1} + L_1A_j - i\partial_\xi L_0\cdot\partial_y A_j$, for A_{j+1} we have:

$$A_{j+1} = e^{h^{-1}\tau L_0}A_{j+1|\tau=0} + e^{h^{-1}\tau L_0}\int_0^\tau e^{-h^{-1}sL_0}(L_1A_j - i\partial_\xi L_0\cdot\partial_y A_j)\mathrm{d}s, \tag{2.46}$$

where L_1A_j has still the form (2.45), and we have

$$\partial_y A_j = e^{h^{-1}\tau\rho_-}\left(h^{-1}\tau\partial_y\rho_- + \partial_y\right)\sum_{k=0}^{2j}(h^{-1}\tau\langle\xi\rangle)^k B_{j,k}.$$

Thus, thanks to the properties ρ_- and $B_{j,k}$, the quantity $(L_1A_j - i\partial_\xi L_0\cdot\partial_y A_j)(y,\xi,s)$ has the form:

$$e^{h^{-1}s\rho_-(y,\xi)}\sum_{k=0}^{2j+1}(h^{-1}s\langle\xi\rangle)^k\tilde{B}_{j,k}(y,\xi), \tag{2.47}$$

with $\tilde{B}_{j,k} \in h^j\mathcal{S}^{-j}$. So, by using the decomposition (2.44), for the second term of the r.h.s. of (2.46) we have:

$$e^{h^{-1}\tau L_0}\int_0^\tau e^{-h^{-1}sL_0}(L_1A_j - i\partial_\xi L_0\cdot\partial_y A_j)\mathrm{d}s = e^{h^{-1}\tau\rho_-}\Pi_- I_-^j(\tau) + e^{h^{-1}\tau\rho_+}\Pi_+ I_+^j(\tau), \quad (2.48)$$

with

$$I_\pm^j(\tau) = \int_0^\tau e^{h^{-1}s(\rho_- - \rho_\pm)}\sum_{k=0}^{2j+1}(h^{-1}s\langle\xi\rangle)^k \tilde{B}_{j,k}\mathrm{d}s.$$

For I_-^j, the exponential term is equal to 1 and by integration of s^k, we obtain:

$$I_-^j(\tau) = \sum_{k=0}^{2j+1}(h^{-1}\tau\langle\xi\rangle)^{k+1}\frac{h\langle\xi\rangle^{-1}}{k+1}\tilde{B}_{j,k}. \quad (2.49)$$

For I_+^j, let us introduce P_k the polynomial of degree k such that

$$\int_0^\tau e^{\lambda s}s^k\mathrm{d}s = \frac{1}{\lambda^{k+1}}(e^{\tau\lambda}P_k(\tau\lambda) - P_k(0)), \quad \text{for any } \lambda\in\mathbb{C}^*.$$

With this notation in hand, we easily see that the term $e^{\tau^h\rho_+}\Pi_+ I_+^j(\tau)$ has the following form:

$$e^{\tau^h\rho_+}\Pi_+ I_+^j(\tau) = \Pi_+\sum_{k=0}^{2j+1}\frac{h\langle\xi\rangle^k}{(\rho_- - \rho_+)^{k+1}}\tilde{B}_{j,k}\Big(e^{\tau^h\rho_-}P_k(\tau^h(\rho_- - \rho_+)) - e^{\tau^h\rho_+}P_k(0)\Big), \quad (2.50)$$

where $\tau^h := h^{-1}\tau$.

Thus, combining (2.49) and (2.50) with (2.46), (2.48) and (2.44), yields that

$$A_{j+1} = e^{h^{-1}\tau\rho_+}\Big(\Pi_+ A_{j+1|\tau=0} - \widetilde{B_{j+1}^+}\Big) + e^{h^{-1}\tau\rho_-}\Big(\Pi_- A_{j+1|\tau=0} + \sum_{k=0}^{2(j+1)}(h^{-1}\tau\langle\xi\rangle)^k\widetilde{B_{j+1,k}^-}\Big),$$

where

$$\widetilde{B_{j+1}^+} = \Pi_+\sum_{k=0}^{2j+1}\frac{h\langle\xi\rangle^k}{(\rho_- - \rho_+)^{k+1}}P_k(0)\tilde{B}_{j,k} \in h^{j+1}\mathcal{S}^{-j-1},$$

and $\widetilde{B_{j+1,k}^-} \in h^{j+1}\mathcal{S}^{-j-1}$ as a linear combination of products of $\Pi_- \in \mathcal{S}^0$, $h\langle\xi\rangle^{-1}$ (or $h\langle\xi\rangle^k(\rho_- - \rho_+)^{-k-1}$) belonging to $h\mathcal{S}^{-1}$, and of $\tilde{B}_{j,k} \in h^j\mathcal{S}^{-j}$.

Now, in order to have $A_{j+1}\in\mathcal{S}^0$, we let the contribution of the exponentially growing term vanish by choosing

$$\Pi_+ A_{j+1}(y,\xi,0) = \widetilde{B_{j+1}^+}(y,\xi).$$

Then, thanks to Remark 2.4.1, the boundary condition $P_-(y)A_{j+1}(y,\xi,0) = 0$ gives

$$A_{j+1}(y,\xi,0) = \frac{P_+\Pi_+}{k_+^\varphi}\widetilde{B_{j+1}^+}(y,\xi).$$

Finally, we have

$$A_{j+1}(y,\xi,\tau) = e^{h^{-1}\tau\rho_-(y,\xi)}\Big(\frac{\Pi_- P_+ \Pi_+}{k_+^\varphi}\widetilde{B^+_{j+1}}(y,\xi) + \sum_{k=0}^{2(j+1)} (h^{-1}\tau\langle\xi\rangle)^k \widetilde{B^-_{j+1,k}}(y,\xi)\Big),$$

and Proposition 2.4.5 is proven with

$$B_{j+1,0} = \frac{\Pi_- P_+ \Pi_+}{k_+^\varphi}\widetilde{B^+_{j+1}} + \widetilde{B^-_{j+1,0}},$$

and for $k \geqslant 1$, $B_{j+1,k} = \widetilde{B^-_{j+1,k}}$. ■

Remark 2.4.2. *The computation of each term $B_{j,0}$ can be done recursively, but this leads to complicated calculations. For example $B_{1,0}$ has the following form*

$$B_{1,0}(y,\xi) = h\left[\Pi_+ a_0 + \frac{\Pi_- P_+ \Pi_+ a_0}{k_+^\varphi}\right]\left(\frac{(z + i\alpha\cdot\partial_y)}{2\lambda} + \frac{i\alpha\cdot\partial_y \rho_-}{4\lambda^2}\right)\Pi_- A_0(y,\xi),$$

with $a_0(\tilde{y}) = i\alpha\cdot\tilde{n}^\varphi(\tilde{y})$.

Thanks to the relation (2.41), to any $A^h \in \mathcal{P}_h^0$ we associate a bounded operator from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2\times(0,+\infty))$. The boundedness in the variable $y \in \mathbb{R}^2$ is a consequence of the Calderon-Vaillancourt theorem (see (2.1)), and in the variable $\tau \in (0,+\infty)$ it is essentially the multiplication by an L^∞-function. Moreover, for A_j of the form (2.45), we have the following mapping property which captures the Sobolev space regularity.

Proposition 2.4.6. *Let A_j, $j \geqslant 0$, be of the form (2.45). Then, for any $s \geqslant -j-\frac{1}{2}$, the operator $\mathcal{A}_j$ defined by*

$$\mathcal{A}_j : f \longmapsto (\mathcal{A}_j f)(y,y_3) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A_j(y,h\xi,y_3)e^{iy\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi$$

gives rise to a bounded operator from $H^s(\mathbb{R}^2)$ into $H^{s+j+\frac{1}{2}}(\mathbb{R}^2\times(0,+\infty))$. Moreover, for any $l \in [0, j+\frac{1}{2}]$ we have:

$$\|\mathcal{A}_j\|_{H^s\to H^{s+j+\frac{1}{2}-l}} = O(h^{l-|s|}). \tag{2.51}$$

Proof. First, let us prove the result for $s = k-j-\frac{1}{2}$, $k\in\mathbb{N}$, between the semiclassical Sobolev spaces

$$H^s_{\text{scl}}(\mathbb{R}^2) := \langle hD_y\rangle^{-s}L^2(\mathbb{R}^2),$$
$$H^k_{\text{scl}}(\mathbb{R}^2\times(0,+\infty)) := \{u\in L^2;\ \langle hD_y\rangle^{k_1}(h\partial_{y_3})^{k_2}u \in L^2 \text{ for } (k_1,k_2)\in\mathbb{N}^2, k_1+k_2 = k\},$$

where $\langle hD_y\rangle = \sqrt{-h^2\Delta_{\mathbb{R}^2} + I}$. Then, for $f\in H^s(\mathbb{R}^2)^4$, we have:

$$\begin{aligned}\|\mathcal{A}_j f\|^2_{H^k_{\text{scl}}(\mathbb{R}^2\times(0,+\infty))} &= \sum_{k_1+k_2=k}\|\langle hD_y\rangle^{k_1}(h\partial_{y_3})^{k_2}\mathcal{A}_j f\|^2_{L^2(\mathbb{R}^2\times(0,+\infty))}\\ &= \sum_{k_1+k_2=k}\int_0^{+\infty}\|\langle hD_y\rangle^{k_1}(h\partial_{y_3})^{k_2}(\mathcal{A}_j f)(\cdot,y_3)\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}y_3.\end{aligned} \tag{2.52}$$

Thanks to the ellipticity property (2.38), for A_j given by Proposition 2.4.5 we have:

$$(h\partial_{y_3})^{k_2} A_j(y,\xi,y_3) = h^j b_j(y,\xi;y_3) e^{-h^{-1}y_3\frac{c}{2}\langle\xi\rangle}\langle\xi\rangle^{k_2-j},$$

with b_j satisfying, for any $(\alpha,\beta)\in\mathbb{N}^2\times\mathbb{N}^2$ there exists $C_{\alpha,\beta}>0$ such that:

$$|\partial_y^\alpha\partial_\xi^\beta b_j(y,\xi;y_3)| \leqslant C_{\alpha,\beta}, \quad \forall (y,\xi;y_3)\in\mathbb{R}^2\times\mathbb{R}^2\times(0,+\infty).$$

Consequently, thanks to the Calderón-Vaillancourt theorem (see (2.1)), we can write:

$$\langle hD_y\rangle^{k_1}(h\partial_{y_3})^{k_2}\mathcal{A}_j = h^j\mathcal{B}_j(y_3)\langle hD_y\rangle^{k_1+k_2-j}e^{-h^{-1}y_3\frac{c}{2}\langle hD_y\rangle},$$

with $(\mathcal{B}_j(y_3))_{y_3>0}$ a family of bounded operators on $L^2(\mathbb{R}^2)$, and uniformly bounded with respect to $y_3>0$. Then, for $f\in H^s(\mathbb{R}^2)^4$, we have:

$$\|\langle hD_y\rangle^{k_1}(h\partial_{y_3})^{k_2}(\mathcal{A}_j f)(\cdot,y_3)\|^2_{L^2(\mathbb{R}^2)} \lesssim h^j\|\langle hD_y\rangle^{k_1+k_2-j}e^{-h^{-1}y_3\frac{c}{2}\langle hD_y\rangle}f\|^2_{L^2(\mathbb{R}^2)},$$

and from (2.52) we deduce that

$$\|\mathcal{A}_j f\|^2_{H^k_{\text{scl}}(\mathbb{R}^2\times(0,+\infty))} \lesssim h^{2j+1}\|\langle hD_y\rangle^{k-j-\frac12}f\|^2_{L^2(\mathbb{R}^2)} = h^{2j+1}\|f\|^2_{H_{\text{scl}}^{k-j-\frac12}(\mathbb{R}^2)},$$

where we used that for any $l\in\mathbb{N}$, $f\in H_{\text{scl}}^{l-\frac12}(\mathbb{R}^2)$,

$$\begin{aligned}\|\langle hD_y\rangle^{l}e^{-h^{-1}y_3\frac{c}{2}\langle hD_y\rangle}f\|^2_{L^2(\mathbb{R}^2)} &= \langle e^{-h^{-1}y_3c\langle hD_y\rangle}\langle hD_y\rangle^l f\,,\,\langle hD_y\rangle^l f\rangle_{L^2}\\ &= -\frac{h}{c}\frac{\partial}{\partial_{y_3}}\langle e^{-h^{-1}y_3c\langle hD_y\rangle}\langle hD_y\rangle^{l-1} f\,,\,\langle hD_y\rangle^l f\rangle_{L^2}.\end{aligned}$$

By interpolation arguments we thus deduce that for any $j\in\mathbb{N}$, $s\geqslant -j-\frac12$, it holds that

$$\|\mathcal{A}_j\|_{H^s_{\text{scl}}\to H_{\text{scl}}^{s+j+\frac12}} = O(h^{j+\frac12}).$$

This means that for $\overline{y}:=(y,y_3)$

$$\|\langle hD_{\overline{y}}\rangle^{s+j+\frac12}\mathcal{A}_j\langle hD_y\rangle^{-s}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2\times(0,+\infty))} = O(h^{j+\frac12}). \tag{2.53}$$

In order to prove (2.51) (in *classical* Sobolev spaces) let us estimate $\langle D_{\overline{y}}\rangle^{s+j+\frac12-l}\mathcal{A}_j\langle D_y\rangle^{-s}$ from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2\times(0,+\infty))$. The inequalities, for all $\xi\in\mathbb{R}^d$, $d=2,3$ and $h\in(0,1)$,

$$1\leqslant\langle\xi\rangle\leqslant h^{-1}\langle h\xi\rangle;\qquad \langle\xi\rangle^{-1}\leqslant\langle h\xi\rangle^{-1};\qquad \langle\xi\rangle^{-1}\leqslant 1;$$

imply for $j+\frac12\geqslant l$, $s_+=\max(s,0)$ and $s_-=s-s_+$, the estimates:

$$\langle\xi\rangle^{s+j+\frac12-l}\leqslant h^{-j-\frac12+l}\,h^{-s_+}\,\langle h\xi\rangle^{s+j+\frac12};\qquad \langle\xi\rangle^{-s}\leqslant h^{s_-}\langle h\xi\rangle^{-s}.$$

We deduce

$$\|\langle D_{\overline{y}}\rangle^{s+j+\frac12-l}\mathcal{A}_j\langle D_y\rangle^{-s}\|_{L^2\to L^2}\lesssim h^{-j-\frac12+l}\,h^{-s_+}h^{s_-}\;\|\langle hD_{\overline{y}}\rangle^{s+j+\frac12}\mathcal{A}_j\langle hD_y\rangle^{-s}\|_{L^2\to L^2}.$$

Then the estimate (2.51) follows from (2.53) using that $s_+ - s_- = |s|$.

Proposition 2.4.7. *Let $f \in H^s(\mathbb{R}^2)$ and A_j, $j \geqslant 0$, be as in Propositions 2.4.4 and 2.4.5. Then for any $N \geqslant -s - \frac{1}{2}$, the function $u_N^h = \sum_{j=0}^N h^j \mathcal{A}_j f$ satisfies:*

$$\begin{cases} h\partial_\tau u_N^h - L_0(y, hD_y)u_N^h - hL_1(y)u_N^h = h^{N+1}\mathcal{R}_N^h f, & \text{in } \mathbb{R}^2 \times (0, +\infty), \\ P_- u_N^h = f, & \text{on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{2.54}$$

with

$$\mathcal{R}_N^h : f \longmapsto \frac{-1}{(2\pi)^2} \int_{\mathbb{R}^2} \Big(L_1 A_N - i\partial_\xi L_0 \cdot \partial_y A_N\Big)(y, h\xi, \tau) e^{iy\cdot\xi} \hat{f}(\xi) \mathrm{d}\xi,$$

a bounded operator from $H^s(\mathbb{R}^2)$ into $H^{s+N+\frac{1}{2}}(\mathbb{R}^2 \times (0, +\infty))$ satisfying for any $l \in [0, N + \frac{1}{2}]$:

$$\|\mathcal{R}_N^h\|_{H^s \to H^{s+N+\frac{1}{2}-l}} = O(h^{l-|s|}). \tag{2.55}$$

Proof. By construction of the sequence $(A_j)_{j\in\{0,\cdots,N-1\}}$ we have the system (2.54) with $\mathcal{R}_N^h = Op^h(r_N^h(\cdot,\cdot,\tau))$,

$$r_N^h(y, \xi, \tau) = -\Big(L_1 A_N - i\partial_\xi L_0 \cdot \partial_y A_N\Big)(y, \xi, \tau),$$

(see the beginning of Section 2.4.2). As in the proof of Proposition 2.4.5, r_N^h has the form (2.47) (with $j = N$). Then, as in the proof of Proposition 2.4.6 we obtain the estimate (2.55). ∎

2.4.3 Proof of Theorem 2.4.1

In this section, we apply the above construction in order to prove Theorem 2.4.1.

Let $g \in P_- H^{1/2}(\partial\Omega)^4$, $(U_\varphi, V_\varphi, \varphi)$ a chart of the atlas $\mathbb{A}$ and $\psi_1, \psi_2 \in C_0^\infty(U_\varphi)$. Then $f := (\varphi^{-1})^*(\psi_2 g)$ is a function of $H^{1/2}(V_\varphi)^4$ which can be extended by 0 to a function of $H^{1/2}(\mathbb{R}^2)^4$. Then for $h = 1/m$ and any $N \in \mathbb{N}$, the previous construction provides a function $u_N^h \in H^1(\mathbb{R}^2 \times (0, +\infty))^4$ satisfying

$$\begin{cases} (\widetilde{D}_m^\varphi - z)u_N^h = h^{N+1}\mathcal{R}_N^h f, & \text{in } \mathbb{R}^2 \times (0, \varepsilon), \\ \Gamma_- u_N^h = f, & \text{on } \mathbb{R}^2 \times \{0\}, \end{cases}$$

with $u_N^h = \sum_{j=0}^N h^j \mathcal{A}_j f$ (see Proposition 2.4.6) and $\mathcal{R}_N^h f \in H^{N+1}(\mathbb{R}^2 \times (0, \varepsilon))$ with norm in H^{N+1-l}, $l \in [0, N + \frac{1}{2}]$, bounded by $O(h^{l-\frac{1}{2}})$. Consequently, $v_N^h := \phi^* u_N^h$, defined on $\mathcal{V}_{\varphi,\varepsilon}$, satisfies:

$$\begin{cases} (D_m - z)v_N^h = h^{N+1}\phi^*(\mathcal{R}_N^h f), & \text{in } \mathcal{V}_{\varphi,\varepsilon}, \\ \Gamma_- v_N^h = \psi_2 g, & \text{on } U_\varphi. \end{cases}$$

Now, let $E_m^\Omega(z)[\psi_2 g] \in H^1(\Omega)^4$ be as in Definition 2.3.1. Since $\Gamma_- v_N^h = \Gamma_- E_m^\Omega(z)[\psi_2 g] = \psi_2 g$, then the following equality holds in $\mathcal{V}_{\varphi,\varepsilon}$:

$$v_N^h - E_m^\Omega(z)[\psi_2 g] = h^{N+1}(H_{\text{MIT}}(m) - z)^{-1}\phi^*\Big(\mathcal{R}_N^h(\varphi^{-1})^*(\psi_2 g)\Big).$$

From this, we deduce that

$$\psi_1 \mathscr{A}_m \psi_2(g) := \psi_1 \Gamma_+ E_m^\Omega(z)[\psi_2 g] = \psi_1 \Gamma_+ v_N^h - h^{N+1}\psi_1 \Gamma_+ (H_{\text{MIT}} - z)^{-1}\phi^*\Big(\mathcal{R}_N^h(\varphi^{-1})^*(\psi_2 g)\Big).$$

Since $\phi\lfloor_{U_\varphi} = \varphi$, for any $u \in H^1(V_\varphi \times (0,\varepsilon))^4$, we have that

$$\Gamma_+\phi^*(u) = \varphi^*(P_+u\lfloor_{V_\varphi\times\{0\}}), \quad \psi_1\Gamma_+v_N^h = \psi_1\varphi^* Op^h(a_N^h)(\varphi^{-1})^*\psi_2 g,$$

with

$$a_N^h(\tilde{y},\xi) = \sum_{j=0}^{N} h^j P_+A_j(y,\xi,0) = \sum_{j=0}^{N} h^j P_+B_{j,0}(y,\xi),$$

where $B_{j,0} \in h^j\mathcal{S}^{-j}$ are introduced in Proposition 2.4.5. Thus, from Proposition 2.4.4, in local coordinates, the principal semiclassical symbol of $\mathscr{A}_m$ is given by

$$P_+B_{0,0}(y,\xi) = P_+A_0(y,\xi,0) = \frac{P_+\Pi_-P_-}{k_+^\varphi}(y,\xi).$$

Thanks to the property (2.39) it is equal to

$$-\Theta^\varphi P_-(y,\xi) = \frac{S\cdot(\xi\wedge n^\varphi(y))}{\sqrt{\langle G(y)^{-1}\xi,\xi\rangle+1}+1}P_-(y,\xi).$$

We conclude the proof of Theorem 2.4.1 from results of Section 2.1.2 and by proving the following Lemma which is a consequence of the above considerations, the regularity estimates from Theorem 2.2.1-(iii), Theorem 2.2.2-(i) and Proposition 2.3.2.

Lemma 2.4.8. *Let $\psi_1,\psi_2 \in C^\infty(\Sigma)$ such that $\operatorname{supp}(\psi_1)\cap\operatorname{supp}(\psi_2) = \emptyset$. Then, for $m_0 > 0$ sufficiently large, $m \geqslant m_0$, and for any $(k,N) \in \mathbb{N}^*\times\mathbb{N}^*$ it holds that*

$$\|\psi_1\mathscr{A}_m\psi_2\|_{P_-H^{1/2}(\Sigma)^4\to P_+H^k(\Sigma)^4} = \mathcal{O}(m^{-N}).$$

Proof. Let $\psi_1,\psi_2 \in C^\infty(\Sigma)$ with disjoint supports. Thanks to Theorem 2.2.1-(iii) and Theorem 2.2.2-(i), to prove the lemma it suffices to show that for any $(N_1,N_2)\in\mathbb{N}^2$, there exists C_{N_1,N_2} such that for $g \in P_-H^{1/2}(\Sigma)^4$,

$$\begin{aligned}\|(\psi_1\mathscr{A}_m\psi_2)g\|_{P_+H^{N_2+\frac{1}{2}}(\Sigma)^4} \leqslant &\frac{C_{N_1,N_2}}{\sqrt{m}}\left(\Pi_{i=0}^{N_2}\|(H_{\text{MIT}}(m)-z)^{-1}\|_{H^i(\Omega)^4\to H^{i+1}(\Omega)^4}\right)\\ &\times\|(H_{\text{MIT}}(m)-z)^{-1}\|^{N_1}_{L^2(\Omega)^4\to L^2(\Omega)^4}\|g\|_{P_-H^{1/2}(\Sigma)^4}.\end{aligned} \tag{2.56}$$

For this, let us introduce $\Phi_1 \in C_0^\infty(\overline{\Omega})$ such that $\Phi_1 = 1$ near $\operatorname{supp}(\psi_1)$ and $\Phi_1 = 0$ near $\operatorname{supp}(\psi_2)$. Thus for $g \in P_-H^{1/2}(\Sigma)^4$ and $E_m^\Omega(z)[\psi_2 g] \in H^1(\Omega)$ as in Definition 2.3.1, the function $u_{1,2} := \Phi_1 E_m^\Omega(z)[\psi_2 g]$ satisfies:

$$\begin{cases}(D_m - z)u_{1,2} = [D_0\,,\,\Phi_1]E_m^\Omega(z)[\psi_2 g], & \text{in } \Omega,\\ \Gamma_-u_{1,2} = \Phi_1{}_{|\Sigma}\psi_2 g = 0, & \text{on } \Sigma.\end{cases}$$

Then, $u_{1,2} = (H_{\text{MIT}}(m)-z)^{-1}[D_0\,,\,\Phi_1]E_m^\Omega(z)[\psi_2 g]$, and for any $\widetilde{\Phi_1} \in C_0^\infty(\overline{\Omega})$ equals to 1 near $\operatorname{supp}(\psi_1)$ we have:

$$\psi_1\mathscr{A}_m\psi_2(g) = \psi_1\Gamma_+\widetilde{\Phi_1}(H_{\text{MIT}}(m)-z)^{-1}[D_0\,,\,\Phi_1]E_m^\Omega(z)[\psi_2 g].$$

Moreover, by choosing $\widetilde{\Phi_1}$ such that $\widetilde{\Phi_1} \prec \Phi_1$, that is $\Phi_1 = 1$ on $\operatorname{supp}(\widetilde{\Phi_1})$, both functions $\widetilde{\Phi_1}$ and

$[D_0\,,\,\Phi_1]$ have disjoint supports, and we can then apply the following telescopic formula:

$$\begin{aligned}\widetilde{\Phi_1}(H_{\text{MIT}}(m)-z)^{-1}(1-\chi_1) =& \widetilde{\Phi_1}(H_{\text{MIT}}(m)-z)^{-1}[D_0,\chi_J]\cdots(H_{\text{MIT}}(m)-z)^{-1}[D_0,\chi_2] \\ &(H_{\text{MIT}}(m)-z)^{-1}(1-\chi_1),\end{aligned}$$

for $(\chi_i)_{1\leqslant i\leqslant J}$ a family of compactly supported smooth functions such that $\widetilde{\Phi_1}\prec\chi_J\prec\chi_{J-1}\prec\cdots\prec\chi_1\prec\Phi_1$, $J=N_1+N_2$. Since $[D_0\,,\,\Phi_1]=(1-\chi_1)[D_0\,,\,\Phi_1]$, the above telescopic formula allows us to write $\psi_1\mathscr{A}_m\psi_2(g)$ as a product of J cutoff resolvents of $H_{\text{MIT}}(m)$. Now, by Proposition 2.3.2 we have

$$\left|\left|E_m^{\Omega}(z)[\psi_2 g]\right|\right|_{L^2(\Omega)^4}\lesssim\frac{1}{\sqrt{m}}\,||g||_{L^2(\Sigma)^4}\,.$$

Thus, using the continuity of Γ_+ from $H^{N_2+1}(\Omega)$ to $H^{N_2+\frac{1}{2}}(\Sigma)$, we then get the estimation (2.56), finishing the proof of the lemma taking $N_2=k$ and N_1 such that for $N_1\geqslant N+N_2(N_2-1)/2$. ■

Remark 2.4.3. *Note that for any $m>0$ and $z\in\rho(H_{MIT}(m))$, the parametrix we have constructed for $\mathscr{A}_m$ is valid from the classical pseudodifferentiel point of view. Actually, Lemma 2.4.8 is the only result where the assumption that m is big enough has been assumed, and it is exclusively required to ensures that away from the diagonal the operator $\mathscr{A}_m$ is negligible in $1/m$. In the same vein, if m is fixed then the proof of Lemma 2.4.8 still ensures that away from the diagonal $\mathscr{A}_m$ is regularizing. Consequently, we deduce that for any $m>0$ and $z\in\rho(H_{MIT}(m))$, the operator $\mathscr{A}_m$ is a homogeneous pseudodifferential operator of order 0, and that*

$$\mathscr{A}_m=\frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}}P_-\quad\mathrm{mod}\ Op\mathcal{S}^{-1}(\Sigma),$$

which is in accordance with Theorem 2.3.3.

$$a_m(\xi)=-\frac{i\alpha_3(\alpha\cdot\xi-z)}{\sqrt{|\xi|^2+m^2}+m}P_-.$$

2.5 Resolvent convergence to the MIT bag model

In the whole section, $\Omega\subset\mathbb{R}^3$ denotes a bounded smooth domain, we set

$$\Omega_i=\Omega,\quad\Omega_e=\mathbb{R}^3\setminus\overline{\Omega}\quad\text{and}\ \ \Sigma=\partial\Omega,$$

and we let n be the outward (with respect to Ω_i) unit normal vector field on Σ.

Fix $m>0$ and let $M>0$. Consider the perturbed Dirac operator

$$H_M\varphi=(D_m+M\beta\mathbb{1}_{\Omega_e})\varphi,\quad\forall\varphi\in\mathrm{Dom}(H_M):=H^1(\mathbb{R}^3)^4,$$

where 1_{Ω_e} is the characteristic function of Ω_e. Using Kato-Rellich theorem and Weyl's theorem, it is easy to see that $(H_M,\mathrm{Dom}(H_M))$ is self-adjoint and that

$$\begin{gathered}\mathrm{Sp}_{\mathrm{ess}}(H_M)=(-\infty,-(m+M)]\cup[m+M,+\infty),\\ \mathrm{Sp}(H_M)\cap(-(m+M),m+M)\ \text{is purely discrete.}\end{gathered}$$

Now, let $H_{\text{MIT}}(m)$ be the MIT bag operator acting on $L^2(\Omega_i)^4$, that is

$$H_{\text{MIT}}(m)v = D_m v, \quad \forall v \in \text{Dom}(H_{\text{MIT}}(m)) := \left\{ v \in H^1(\Omega_i)^4 : P_- t_\Sigma v = 0 \text{ on } \Sigma \right\},$$

where t_Σ and $P_\pm$ are the trace operator and the orthogonal projection from Section 1.3.

The aim of this section is to use the properties of the Poincaré-Steklov operators carried out in the previous sections to study the resolvent of H_M when M is large enough. Namely, we give a Krein-type resolvent formula in terms of the resolvent of $H_{\text{MIT}}(m)$, and we show that the convergence of H_M toward $H_{\text{MIT}}(m)$ holds in the norm resolvent sense with a convergence rate of $\mathcal{O}(1/M)$, which improves the result of [BCLTS19].

Before stating the main results of this section, we need to introduce some notations and definitions. First, we introduce the following Dirac auxiliary operator

$$\widetilde{H}_M u = D_{m+M} u, \quad \forall u \in \text{Dom}(\widetilde{H}_M) := \left\{ u \in H^1(\Omega_e)^4 : P_+ t_\Sigma u = 0 \text{ on } \Sigma \right\}.$$

Notice that $\widetilde{H}_M$ is the MIT bag operator on Ω_e (the boundary condition is with P_+ because the normal n is incoming for Ω_e). Since Ω_e is unbounded, Theorem 2.2.1 together with Remark 2.2.1 imply that $(\widetilde{H}_M, \text{Dom}(\widetilde{H}_M))$ is self-adjoint and that

$$\text{Sp}(\widetilde{H}_M) = \text{Sp}_{\text{ess}}(\widetilde{H}_M) = (-\infty, -(m+M)] \cup [m+M, +\infty).$$

In particular, $\rho(H_M) \subset \rho(\widetilde{H}_M))$. Let $z \in \rho(H_{\text{MIT}}(m)) \cap \rho(\widetilde{H}_M)$, $g \in P_- H^{1/2}(\Sigma)^4$ and $h \in P_+ H^{1/2}(\Sigma)^4$. We denote by $E_m^{\Omega_i}(z) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega_i)^4$ the unique solution of the boundary value problem:

$$\begin{cases} (D_m - z)v = 0, & \text{in } \Omega_i, \\ P_- t_\Sigma v = g, & \text{in } \Sigma. \end{cases} \tag{2.57}$$

Similarly, we denote by $E_{m+M}^{\Omega_e}(z) : P_+ H^{1/2}(\Sigma)^4 \to H^1(\Omega_e)^4$ the unique solution of the boundary value problem:

$$\begin{cases} (D_{m+M} - z)u = 0, & \text{in } \Omega_e, \\ P_+ t_\Sigma u = h, & \text{in } \Sigma. \end{cases} \tag{2.58}$$

Define the Poincaré-Steklov operators associated to the above problems by

$$\mathscr{A}_m^i = P_+ t_\Sigma E_m^{\Omega_i}(z) P_- \quad \text{and} \quad \mathscr{A}_{m+M}^e = P_- t_\Sigma E_{m+M}^{\Omega_e}(z) P_+.$$

Notation 2.5.1. In the sequel we shall denote by $R_M(z)$, $\widetilde{R}_M(z)$ and $R_{\text{MIT}}(z)$ the resolvent of H_M, $\widetilde{H}_M$ and $H_{\text{MIT}}(m)$, respectively. We also use the notations:

- $\Gamma_\pm = P_\pm t_\Sigma$ and $\Gamma = \Gamma_+ r_{\Omega_i} + \Gamma_- r_{\Omega_e}$, with $r_\bullet$ the restriction operator in $\bullet$.
- $E_M(z) = e_{\Omega_i} E_m^{\Omega_i}(z) P_- + e_{\Omega_e} E_{m+M}^{\Omega_e}(z) P_+$, with $e_\bullet$ the extension by 0 outside of $\bullet$.
- $\widetilde{R}_{\text{MIT}}(z) = e_{\Omega_i} R_{\text{MIT}}(z) r_{\Omega_i} + e_{\Omega_e} \widetilde{R}_M(z) r_{\Omega_e}$.

With these notations in hand, we can state the main results of this section. The following theorem is the main tool to show the large coupling convergence with a rate of convergence of $\mathcal{O}(1/M)$.

Theorem 2.5.2. *There is $M_0 > 0$ such that for all $M > M_0$ and all $z \in \rho(H_{MIT}(m)) \cap \rho(H_M)$, the* *operator* $\Psi_M(z) := \left(\mathbb{I} - \mathscr{A}^i_m - \mathscr{A}^e_{m+M}\right)$ *is bounded invertible in* $H^{1/2}(\Sigma)^4$*, and the inverse is given by*

$$\Psi_M^{-1}(z) = \left(\mathbb{I}_4 - \mathscr{A}^i_m \mathscr{A}^e_{m+M} - \mathscr{A}^e_{m+M}\mathscr{A}^i_m\right)^{-1}\left(\mathbb{I} + \mathscr{A}^i_m + \mathscr{A}^e_{m+M}\right),$$

and the following resolvent formula holds:

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\Psi_M^{-1}(z)\Gamma\widetilde{R}_{\mathrm{MIT}}(z). \tag{2.59}$$

Remark 2.5.1. *By Proposition 2.3.2* (i) *we have that*

$$\left(E^{\Omega_i}_m(z)\right)^* = -\beta\Gamma_+ R_{\mathrm{MIT}}(\overline{z}) \quad \text{and} \quad \left(E^{\Omega_e}_{m+M}(z)\right)^* = -\beta\Gamma_-\widetilde{R}_M(\overline{z}),$$

for any $z \in \rho(H_{MIT}(m)) \cap \rho(H_M)$*. Thus, the resolvent formula* (2.59) *can be written in the form*

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) - (\beta\Gamma\widetilde{R}_{\mathrm{MIT}}(\overline{z}))^*\Psi_M^{-1}(z)\Gamma\widetilde{R}_{\mathrm{MIT}}(z).$$

Before going through the proof of Theorem 2.5.2 we first establish a regularity result that will play a crucial role in the rest of this section. It concerns the dependence on the parameter M of the norm of an auxiliary operator which involves the composition of the operators $\mathscr{A}^i_m$ and $\mathscr{A}^e_{m+M}$.

Proposition 2.5.3. *Let* $\mathscr{A}^i_m$ *and* $\mathscr{A}^e_{m+M}$ *be as above. Then, there is* $M_0 > 0$ *such that for every* $M > M_0$ *and all* $z \in \rho(H_{MIT}(m)) \cap \rho(H_M)$ *the following hold true:*

(i) For any $s \in \mathbb{R}$ *the operator* $\Xi_M(z) : H^s(\Sigma)^4 \longrightarrow H^s(\Sigma)^4$ *defined by*

$$\Xi_M(z) = \left(\mathbb{I}_4 - \mathscr{A}^i_m \mathscr{A}^e_{m+M} - \mathscr{A}^e_{m+M}\mathscr{A}^i_m\right)^{-1}, \tag{2.60}$$

is everywhere defined and uniformly bounded with respect to M.

(ii) The Poincaré-Steklov operator, $\mathscr{A}^e_{m+M}$*, satisfies the estimate*

$$||\mathscr{A}^e_{m+M}||_{P_+H^{s+1}(\Sigma)^4 \to P_-H^s(\Sigma)^4} \lesssim M^{-1}, \quad \forall s \in \mathbb{R}.$$

Proof. (i) Set $\tau := (m+M)$, then the result essentially follows from the fact that $\Xi_M(z)$ is a $1/\tau$-pseudodifferential operator of order 0. Indeed, fix $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$ and set $h = \tau^{-1}$. Then, from Theorem 2.3.3 and Remark 2.4.3 we know that $\mathscr{A}^i_m$ is a homogeneous pseudodifferential operator of order 0. Thus $\mathscr{A}^i_m$ can also be viewed as a h-pseudodifferential operators of order 0. That is, $\mathscr{A}^i_m \in Op^h\mathcal{S}^0(\Sigma)$, and in local coordinates, its semiclassical principal symbol is given by

$$p_{h,\mathscr{A}^i_m}(x,\xi) = \frac{S\cdot(\xi \wedge n(x))P_-}{|\xi \wedge n(x)|},$$

where we identify $\xi \in \mathbb{R}^2$ with $\overline{\xi} = (\xi_1, \xi_2, 0)^t \in \mathbb{R}^3$, and for $x = \varphi(\tilde{x}) \in \Sigma$, $n(x)$ stands for $n^\varphi(\tilde{x})$. Similarly, thanks to Theorem 2.4.1, we also know that for h_0 sufficiently small (and hence M_0 big enough) and all $h < h_0$, $\mathscr{A}^e_{m+M}$ is a h-pseudodifferential operator and that

$$\mathscr{A}^e_{m+M} \in Op^h\mathcal{S}^0(\Sigma), \quad p_{h,\mathscr{A}^e_{m+M}}(x,\xi) = -\frac{S\cdot(\xi \wedge n(x))P_+}{\sqrt{|\xi \wedge n(x)|^2+1}+1}.$$

Therefore, the symbol calculus yields for all $h < h_0$ that $\left(\mathbb{I}_4 - \mathscr{A}^i_m \mathscr{A}^e_{m+M} - \mathscr{A}^e_{m+M}\mathscr{A}^i_m\right)$ is a $1/\tau$-

pseudodifferential operator of order 0. Now, Lemmas 2.6.3 and 2.6.1 yield

$$\frac{S\cdot(\xi\wedge n(x))P_{\pm}S\cdot(\xi\wedge n(x))P_{\mp}}{|\xi\wedge n(x)|(\sqrt{|\xi\wedge n(x)|^2+1}+1)}=\frac{|\xi\wedge n(x)|P_{\mp}}{\sqrt{|\xi\wedge n(x)|^2+1}+1}.$$

Thus

$$\begin{aligned}\mathbb{I}_4-p_{h,\mathscr{A}^i_m}(x,\xi)p_{h,\mathscr{A}^e_{m+M}}(x,\xi)-p_{h,\mathscr{A}^e_{m+M}}(x,\xi)p_{h,\mathscr{A}^i_m}(x,\xi)&=\mathbb{I}_4+\frac{|\xi\wedge n(x)|}{\sqrt{|\xi\wedge n(x)|^2+1}+1}\\&=\frac{\sqrt{|\xi\wedge n(x)|^2+1}+1+|\xi\wedge n(x)|}{\sqrt{|\xi\wedge n(x)|^2+1}+1}\gtrsim 1.\end{aligned}$$

From this, we deduce that $\left(\mathbb{I}_4-\mathscr{A}^i_m\mathscr{A}^e_{m+M}-\mathscr{A}^e_{m+M}\mathscr{A}^i_m\right)$ is elliptic in $Op^h\mathcal{S}^0(\Sigma)$. Thus, $\Xi_M(z)\in Op^h\mathcal{S}^0(\Sigma)$, and in local coordinates, its semiclassical principal symbol is given by

$$p_{h,\Xi_M(z)}(x,\xi)=\frac{\sqrt{|\xi\wedge n(x)|^2+1}+1}{\sqrt{|\xi\wedge n(x)|^2+1}+1+|\xi\wedge n(x)|}.$$

As $\Xi_M(z)$ is a h-pseudodifferential operators of order 0, it follows from the Calderón-Vaillancourt theorem (see (2.2)), that $\Xi_M(z):H^s(\Sigma)^4\to H^s(\Sigma)^4$ is well-defined and uniformly bounded with respect to M, for any $s\in\mathbb{R}$, proving the statement (i) of the theorem.

The proof of the statement (ii) exploits also the Calderón-Vaillancourt theorem which shows that for any $s\in\mathbb{R}$, any operator in $hOp^h\mathcal{S}^0(\Sigma)$ is uniformly bounded by $O(h)$, with respect to $h=\tau^{-1}\in(0,1)$, from $H^{s+1}(\Sigma)^4$ into $H^{s+1}(\Sigma)^4\hookrightarrow H^s(\Sigma)^4$ (see (2.2)). Thus for any $s\in\mathbb{R}$,

$$\left|\left|\mathscr{A}^e_\tau-\frac{1}{\tau}D_\Sigma(\sqrt{-\tau^{-2}\Delta_\Sigma+\mathbb{I}}+\mathbb{I})^{-1}P_+\right|\right|_{H^{s+1}(\Sigma)^4\to H^s(\Sigma)^4}\lesssim\tau^{-1},$$

uniformly with respect to τ large enough.

Then we conclude the proof of the statement (ii) by using that $(\sqrt{-\tau^{-2}\Delta_\Sigma+\mathbb{I}}+\mathbb{I})^{-1}$ is uniformly bounded from $H^{s+1}(\Sigma)^4$ into itself and that D_Σ is bounded from $H^{s+1}(\Sigma)^4$ into $H^s(\Sigma)^4$ (as a first order differential operator). ■

We can now give the proof of Theorem 2.5.2.

***Proof of Theorem* 2.5.2.** Let M_0 be as in Proposition 2.5.3 and $M>M_0$, fix $z\in\rho(H_{\text{MIT}}(m))\cap\rho(H_M)$ and let $f\in L^2(\mathbb{R}^3)^4$. We set

$$v=r_{\Omega_i}R_M(z)f\quad\text{ and }\quad u=r_{\Omega_e}R_M(z)f.$$

Then u and v satisfy the following system

$$\begin{cases}(D_m-z)v=f & \text{in } \Omega_i,\\(D_{m+M}-z)u=f & \text{in } \Omega_e,\\P_-t_\Sigma v=P_-t_\Sigma u & \text{on } \Sigma,\\P_+t_\Sigma v=P_+t_\Sigma u & \text{on } \Sigma.\end{cases}$$

Since $E_m^{\Omega_i}(z)$ (resp. $E_{m+M}^{\Omega_e}(z)$) gives the unique solution to the boundary value problem (2.57) (resp. (2.58)), and

$$\Gamma_- R_{\mathrm{MIT}}(z) r_{\Omega_i} f = 0 \quad \text{and} \quad \Gamma_+ \widetilde{R}_M(z) r_{\Omega_e} f = 0,$$

if we let

$$\varphi = \Gamma_- u \quad \text{and} \quad \psi = \Gamma_+ v,$$

then it is easy to check that

$$\begin{cases} v = R_{\mathrm{MIT}}(z) r_{\Omega_i} f + E_m^{\Omega_i}(z)\varphi, \\ u = \widetilde{R}_M(z) r_{\Omega_e} f + E_{m+M}^{\Omega_e}(z)\psi. \end{cases} \tag{2.61}$$

Hence, to get an explicit formula for $R_M(z)$ it remains to find the unknowns φ and ψ. For this, note that from (2.61) we have

$$\begin{cases} \psi = \Gamma_+ r_{\Omega_i} R_M(z) f = \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f + \Gamma_+ E_m^{\Omega_i}(z)[\varphi], \\ \varphi = \Gamma_- r_{\Omega_e} R_M(z) f = \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f + \Gamma_- E_{m+M}^{\Omega_e}(z)[\psi]. \end{cases} \tag{2.62}$$

Substituting the values of ψ and φ (from (2.62)) into the system (2.61), we obtain

$$\begin{aligned} R_M(z) =& e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i} + e_{\Omega_e} \widetilde{R}_M(z) r_{\Omega_e} \\ &+ \left(e_{\Omega_i} E_m^{\Omega_i}(z) \Gamma_- r_{\Omega_e} + e_{\Omega_e} E_{m+M}^{\Omega_e}(z) \Gamma_+ r_{\Omega_i} \right) R_M(z) \\ =& \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z) \Gamma R_M(z). \end{aligned} \tag{2.63}$$

Note that, by definition of the Poincaré-Steklov operators, (2.62) is equivalent to

$$\begin{cases} \psi = \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f + \mathscr{A}_m^i(\varphi), \\ \varphi = \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f + \mathscr{A}_{m+M}^e(\psi). \end{cases} \tag{2.64}$$

Thus, applying Γ to the identity (2.63) yields that

$$\Gamma \widetilde{R}_{\mathrm{MIT}}(z) = \left(\mathbb{I} - \mathscr{A}_m^i - \mathscr{A}_{m+M}^e \right) \Gamma R_M(z) = \Psi_M(z) \Gamma R_M(z).$$

Now, we apply $(I + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e)$ to the last identity and we get

$$\left(\mathbb{I} + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e \right) \Gamma \widetilde{R}_{\mathrm{MIT}}(z) = \left(\mathbb{I}_4 - \mathscr{A}_m^i \mathscr{A}_{m+M}^e - \mathscr{A}_{m+M}^e \mathscr{A}_m^i \right) \Gamma R_M(z) =: (\Xi_M(z))^{-1} \Gamma R_M(z),$$

where $\Xi_M(z)$ is given by (2.60). Then, thanks to Proposition 2.5.3 we know that for $M > M_0$ the operator $(\Xi_M(z))^{-1}$ is bounded invertible from $H^{1/2}(\Sigma)^4$ into itself, which actually means that Ψ_M is bounded invertible from $H^{1/2}(\Sigma)^4$ into itself, and that

$$\Psi_M^{-1} = \Xi_M(z) \left(\mathbb{I} + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e \right).$$

From this, it follows that

$$\Gamma R_M(z) = \Psi_M^{-1}(z) \Gamma \widetilde{R}_{\mathrm{MIT}}(z).$$

Substituting this into formula (2.63) yields that

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\Psi_M^{-1}(z)\Gamma\widetilde{R}_{\mathrm{MIT}}(z),$$

which achieves the proof of the theorem. ■

As an immediate consequence of Theorem 2.5.2 and Proposition 2.5.3 we have:

Corollary 2.5.4. *There is $M_0 > 0$ such that for every $M > M_0$ and all $z \in \rho(H_{MIT}(m)) \cap \rho(H_M)$, the operators $\Xi_M^{\pm}(z) : P_{\pm}H^s(\Sigma)^4 \to P_{\pm}H^s(\Sigma)^4$ defined by*

$$\Xi_M^{+}(z) = \left(\mathbb{I} - \mathscr{A}_m^i \mathscr{A}_{m+M}^e\right)^{-1} \quad \textit{and} \quad \Xi_M^{-}(z) = \left(\mathbb{I} - \mathscr{A}_{m+M}^e \mathscr{A}_m^i\right)^{-1},$$

are everywhere defined and bounded for any $s \in \mathbb{R}$, and it holds that

$$\left|\left|\Xi_M^{\pm}(z)\right|\right|_{P_{\pm}H^s(\Sigma)^4 \to P_{\pm}H^s(\Sigma)^4} \lesssim 1,$$

uniformly with respect to $M > M_0$.

Moreover, if $v \in H^1(\mathbb{R}^3)^4$ solves $(D_m + M\beta\mathbb{1}_{\Omega_e} - z)v = e_{\Omega_i} f$, for some $f \in L^2(\Omega_i)^4$. Then, $r_{\Omega_i} v$ satisfies the following boundary value problem

$$\begin{cases} (D_m - z)r_{\Omega_i} v = f & \textit{in } \Omega_i, \\ \qquad \Gamma_- v = \Xi_M^{-}(z)\mathscr{A}_{m+M}^e \Gamma_+ R_{\mathrm{MIT}}(z) f & \textit{on } \Sigma, \\ \qquad \Gamma_+ v = \Gamma_+ R_{\mathrm{MIT}}(z) f + \mathscr{A}_m^i \Gamma_- v & \textit{on } \Sigma. \end{cases} \tag{2.65}$$

Proof. We first note that $\Xi_M^{\pm}(z) = P_{\pm}\Xi_M(z)P_{\pm}$. Thus, the first statement follows immediately from Proposition 2.5.3 . Now, let $f \in L^2(\Omega_i)^4$, and suppose that $v \in H^1(\mathbb{R}^3)^4$ solves $(D_m + M\beta 1_{\Omega_e} - z)v = e_{\Omega_i} f$. Thus $(D_m - z)r_{\Omega_i} v = f$ in Ω_i, and if we set

$$\varphi = P_- t_\Sigma v \quad \text{and} \quad \psi = P_+ t_\Sigma v,$$

then, from (2.64) we easily get

$$\varphi = \Xi_M^{-}(z)\mathscr{A}_{m+M}^e \Gamma_+ R_{\mathrm{MIT}}(z) f \quad \text{and} \quad \psi = \Gamma_+ R_{\mathrm{MIT}}(z) f + \mathscr{A}_m^i \varphi,$$

which means that $r_{\Omega_i} v$ satisfies (2.65), and this completes the proof of the corollary. ■

Remark 2.5.2. *Notice that from (2.64) and Corollary 2.5.4 we have that*

$$\begin{pmatrix} \Gamma_+ r_{\Omega_i} R_M(z) f \\ \Gamma_- r_{\Omega_e} R_M(z) f \end{pmatrix} = \begin{pmatrix} \Xi_M^{+}(z) & 0 \\ 0 & \Xi_M^{-}(z) \end{pmatrix} \begin{pmatrix} \mathbb{I}_4 & \mathscr{A}_m^i \\ \mathscr{A}_{m+M}^e & \mathbb{I}_4 \end{pmatrix} \begin{pmatrix} \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f \\ \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f \end{pmatrix}.$$

With this observation, we remark that the resolvent formula (2.59) can also be written in the following matrix form

$$\begin{pmatrix} r_{\Omega_i} R_M(z) \\ r_{\Omega_e} R_M(z) \end{pmatrix} = \begin{pmatrix} R_{\mathrm{MIT}}(z) r_{\Omega_i} \\ \widetilde{R}_M(z) r_{\Omega_e} \end{pmatrix} + \begin{pmatrix} E_m^{\Omega_i}(z)\Xi_M^{-}(z)\mathscr{A}_{m+M}^e & E_m^{\Omega_i}(z)\Xi_M^{-}(z) \\ E_{m+M}^{\Omega_e}(z)\Xi_M^{+}(z) & E_{m+M}^{\Omega_e}(z)\Xi_M^{+}(z)\mathscr{A}_m^i \end{pmatrix} \begin{pmatrix} \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} \\ \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} \end{pmatrix}.$$

An inspection of the proof of Theorem 2.5.2 shows that, for any $M > 0$, $z \in \rho(H_{\text{MIT}}(m)) \cap \rho(H_M)$ and $f \in L^2(\mathbb{R}^3)^4$, one has

$$\Gamma \widetilde{R}_{\text{MIT}}(z) f = \Psi_M(z) \Gamma R_M(z) f. \tag{2.66}$$

When f runs through the whole space $L^2(\mathbb{R}^3)^4$, then the values of $\Gamma \widetilde{R}_{\text{MIT}}(z) f$ and $\Gamma R_M(z) f$ cover the whole space $H^{1/2}(\Sigma)^4$, which means that $\text{Rn}(\Psi_M(z)) = H^{1/2}(\Sigma)^4$. Hence, if one proves that $\text{Kr}(\Psi_M(z)) = \{0\}$, then $\Psi_M(z)$ would be boundedly invertible in $H^{1/2}(\Sigma)^4$, and thus (2.59) holds without restriction on $M > 0$. The following theorem provides a Birman-Schwinger-type principle relating $\text{Kr}(H_M - z)$ with $\text{Kr}(\Psi_M(z))$ and allows us to recover the resolvent formula (2.59) for any $M > 0$.

Theorem 2.5.5. *Let $M > 0$ and let Ψ_M be as in Theorem 2.5.2. Then, the following hold:*

(i) For any $a \in (-(m+M), m+M) \cap \rho(H_{MIT}(m))$ we have $a \in \text{Sp}_\text{p}(H_M) \Leftrightarrow 0 \in \text{Sp}_\text{p}(\Psi_M(a))$, and it holds that

$$\text{Kr}(H_M - a) = \{E_M(a) g : g \in \text{Kr}(\Psi_M(a))\}.$$

In particular, $\dim\text{Kr}(H_M - a) = \dim\text{Kr}(\Psi_M(a))$ holds for all $a \in (-(m+M), m+M) \cap \rho(H_{MIT}(m))$.

(ii) The operator $\Psi_M(z)$ is boundedly invertible in $H^{1/2}(\Sigma)^4$ for all $z \in \rho(H_{MIT}(m)) \cap \rho(H_M)$, and the following resolvent formula holds:

$$R_M(z) = \widetilde{R}_{\text{MIT}}(z) + E_M(z) \Psi_M^{-1}(z) \Gamma \widetilde{R}_{\text{MIT}}(z). \tag{2.67}$$

Proof. (i) Let us first prove the implication ($\Longrightarrow$). Let $a \in (-(m+M), m+M) \cap \rho(H_{\text{MIT}}(m))$ be such that $(H_M - a)\varphi = 0$ for some $0 \neq \varphi \in H^1(\mathbb{R}^3)^4$. Set $\varphi_+ = \varphi_{|\Omega_i}$ and $\varphi_- = \varphi_{|\Omega_e}$. Then, it is clear that φ_+ solves the system (2.57) for $z = a$ with $g = \Gamma_- \varphi$, and φ_- solves the system (2.58) with $h = \Gamma_+ \varphi$. Thus, $\varphi_+ = E_m^{\Omega_i}(a) \Gamma_- \varphi$ and $\varphi_- = E_{m+M}^{\Omega_e}(a) \Gamma_+ \varphi$. Hence, $\varphi = E_M(a) t_\Sigma \varphi$ and $\Gamma_\pm \varphi \neq 0$, as otherwise φ would be zero. Using this and the definition of the Poincaré-Steklov operators, we obtain that

$$(\mathbb{I}_4 + \mathscr{A}_m^i) \Gamma_- \varphi =: t_\Sigma \varphi_+ = t_\Sigma \varphi = t_\Sigma \varphi_- := (\mathbb{I}_4 + \mathscr{A}_{m+M}^e) \Gamma_+ \varphi,$$

and since $t_\Sigma \varphi \neq 0$ it follows that

$$\Psi_M(a) t_\Sigma \varphi = (\mathbb{I}_4 - \mathscr{A}_m^i - \mathscr{A}_{m+M}^e) t_\Sigma \varphi = 0,$$

which means that $0 \in \text{Sp}_\text{p}(\Psi_M(a))$ and proves the inclusion $\text{Kr}(H_M - a) \subset \{E_M(a) g : g \in \text{Kr}(\Psi_M(a))\}$.

Now, we turn to the proof of the implication ($\Longleftarrow$). Let $a \in (-(m+M), m+M) \cap \rho(H_{\text{MIT}}(m))$ and assume that 0 is an eigenvalue of $\Psi_M(a)$. Then, there is $g \in H^{1/2}(\Sigma)^4 \setminus \{0\}$ such that $\Psi_M(a) g = 0$ on Σ. Note that this is equivalent to

$$(P_- + \mathscr{A}_m^i) g = (P_+ + \mathscr{A}_{m+M}^e) g. \tag{2.68}$$

Since $a \in (-(m+M), m+M) \cap \rho(H_{\text{MIT}}(m))$, the operators $E_m^{\Omega_i}(a) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega_i)^4$ and $E_{m+M}^{\Omega_e}(a) : P_+ H^{1/2}(\Sigma)^4 \to H^1(\Omega_e)^4$ are well-defined and bounded. Thus, if we let $\varphi = E_M(a) g =$

$(E_m^{\Omega_i}(a)P_-g, E_{m+M}^{\Omega_e}(a)P_+g)$, then $\varphi \neq 0$ and we have that $(D_m - a)\varphi = 0$ in Ω_i, and that $(D_{m+M} - a)\varphi = 0$ in Ω_e. Hence, it remains to show that $\varphi \in H^1(\mathbb{R}^3)^4$. For this, observe that by (2.68) we have

$$t_\Sigma E_m^{\Omega_i}(a)P_-g = (P_- + \mathscr{A}_m^i)g = (P_+ + \mathscr{A}_{m+M}^e)g = t_\Sigma E_{m+M}^{\Omega_e}(a)P_+g.$$

Thanks to the boundedness properties of $E_m^{\Omega_i}(a)$ and $E_{m+M}^{\Omega_e}(a)$, it follows from the above computations that $\varphi = E_M(a)g \in H^1(\mathbb{R}^3)^4\setminus\{0\}$ and satisfies the equation $(H_M - a)\varphi = 0$. Therefore, $a \in \mathrm{Sp_p}(H_M)$ and the inclusion $\{E_M(a)g : g \in \mathrm{Kr}(\Psi_M(a))\} \subset \mathrm{Kr}(H_M - a)$ holds, which completes the proof of (i).

(ii): Let $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$ and note that the self-adjointness of H_M together with assertion (i) imply that $\mathrm{Kr}(\Psi_M(z)) = \{0\}$, as otherwise $\mathrm{Kr}(H_M - z) \neq \{0\}$. Since $\mathrm{Rn}(\Psi_M(z)) = H^{1/2}(\Sigma)^4$ holds for all $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, it follows that $\Psi_M(z)$ admits a bounded and everywhere defined inverse in $H^{1/2}(\Sigma)^4$. Therefore, (2.66) yields that $\Gamma R_M(z) = \Psi_M^{-1}(z)\Gamma\widetilde{R}_{\mathrm{MIT}}(z)$, and the resolvent formula (2.67) follows from this and (2.63). ■

Remark 2.5.3. *Note the different nature of Theorems 2.5.2 and 2.5.5, since the second one ensures the invertibility of Ψ_M and yields the resolvent formula (2.67) without assumption, while the first one is based on a largeness assumption that allows us (thanks to the semiclassical properties of the PS operators) to obtain the explicit formula of the operator $(\Psi_M)^{-1}$. Besides, note that in Theorem 2.5.5 we do not know a priori whether $(\Psi_M)^{-1}$ is uniformly bounded when M is large, and hence (2.67) is not suitable for studying the large coupling convergence.*

In the next proposition we prove the norm convergence of $R_M(z)$ toward $R_{\mathrm{MIT}}(z)$ and estimate the rate of convergence.

Proposition 2.5.6. *For any compact set $K \subset \rho(H_{MIT}(m))$ there is $M_0 > 0$ such that for all $M > M_0$: $K \subset \rho(H_M)$, and for all $z \in K$ the resolvent R_M admits an asymptotic expansion in $\mathcal{L}(L^2(\mathbb{R}^3)^4)$ of the form:*

$$R_M(z) = e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i} + \frac{1}{M}\left(K_M(z) + L_M(z)\right), \tag{2.69}$$

where $K_M(z)$, $L_M(z) : L^2(\mathbb{R}^3)^4 \longrightarrow L^2(\mathbb{R}^3)^4$ are uniformly bounded with respect to M and satisfy

$$r_{\Omega_i} L_M(z) e_{\Omega_i} = 0 = r_{\Omega_e} K_M(z) e_{\Omega_e}.$$

In particular, it holds that

$$||R_M(z) - e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i}||_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}\left(\frac{1}{M}\right). \tag{2.70}$$

Before giving the proof, we need the following estimates.

Lemma 2.5.7. *Let $K \subset \mathbb{C}$ be a compact set. Then, there is $M_0 > 0$ such that for all $M > M_0$:*

$K \subset \rho(\widetilde{H}_M)$ and for every $z \in K$ the following estimates hold:

$$\begin{aligned}\left|\left|\widetilde{R}_M(z)f\right|\right|_{L^2(\Omega_e)^4} + \frac{1}{\sqrt{M}}\left|\left|\Gamma_-\widetilde{R}_M(z)f\right|\right|_{L^2(\Sigma)^4} &\lesssim \frac{1}{M}\,||f||_{L^2(\Omega_e)^4}\,, \quad \forall f \in L^2(\Omega_e)^4,\\ \left|\left|\Gamma_-\widetilde{R}_M(z)f\right|\right|_{H^{-1/2}(\Sigma)^4} &\lesssim \frac{1}{M}\,||f||_{L^2(\Omega_e)^4}\,, \quad \forall f \in L^2(\Omega_e)^4,\\ \left|\left|E^{\Omega_e}_{m+M}(z)\psi\right|\right|_{L^2(\Omega_e)^4} &\lesssim \frac{1}{\sqrt{M}}\,||\psi||_{L^2(\Sigma)^4}\,, \quad \forall \psi \in P_+L^2(\Sigma)^4,\\ \left|\left|E^{\Omega_e}_{m+M}(z)\psi\right|\right|_{L^2(\Omega_e)^4} &\lesssim \frac{1}{M}\,||\psi||_{H^{1/2}(\Sigma)^4}\,, \quad \forall \psi \in P_+H^{1/2}(\Sigma)^4.\end{aligned}$$

Proof. Fix a compact set $K \subset \mathbb{C}$, and note that for $M_1 > \sup_{z\in K}\{|\mathrm{Re}(z)| - m\}$ it holds that $K \subset \rho(D_{m+M_1})$, and hence $K \subset \rho(\widetilde{H}_M)$ for all $M > M_1$.

We next show the claimed estimates for $\widetilde{R}_M(z)$ and $\Gamma_-\widetilde{R}_M(z)$. For this, let $z \in K$ and assume that $M > M_1$. Let $\varphi \in \mathrm{Dom}(\widetilde{H}_M)$, then a straightforward application of the Green's formula yields that

$$\|\widetilde{H}_M\varphi\|^2_{L^2(\Omega_e)^4} = \|(\alpha\cdot\nabla)\varphi\|^2_{L^2(\Omega_e)^4} + (m+M)^2\,||\varphi||^2_{L^2(\Omega_e)^4} + (m+M)\,||P_-t_\Sigma\varphi||^2_{L^2(\Sigma)^4}\,.$$

Using this and the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}\|(\widetilde{H}_M - z)\varphi\|^2_{L^2(\Omega_e)^4} =& \|\widetilde{H}_M\varphi\|^2_{L^2(\Omega_e)^4} + |z|^2\|\varphi\|^2_{L^2(\Omega_e)^4} - 2\mathrm{Re}(z)\langle\widetilde{H}_M\varphi,\varphi\rangle_{L^2(\Omega_e)^4}\\ \geqslant& \|\widetilde{H}_M\varphi\|^2_{L^2(\Omega_e)^4} + |z|^2\|\varphi\|^2_{L^2(\Omega_e)^4} - \frac{1}{2}\|\widetilde{H}_M\varphi\|^2_{L^2(\Omega_e)^4} - 2|\mathrm{Re}(z)|^2\|\varphi\|^2_{L^2(\Omega_e)^4}\\ \geqslant& \left(\frac{(m+M)^2}{2} + |\mathrm{Im}(z)|^2 - |\mathrm{Re}(z)|^2\right)||\varphi||^2_{L^2(\Omega_e)^4} + \frac{M}{2}\,||P_-t_\Sigma\varphi||^2_{L^2(\Sigma)^4}\,.\end{aligned}$$

Therefore, taking $\widetilde{R}_M(z)f = \varphi$ and $M \geqslant M_2 \geqslant \sup_{z\in K}\{\sqrt{|\mathrm{Re}(z)|^2 - |\mathrm{Im}(z)|^2} - m\}$ we obtain the inequality

$$\left|\left|\widetilde{R}_M(z)f\right|\right|_{L^2(\Omega_e)^4} + \frac{1}{\sqrt{M}}\left|\left|\Gamma_-\widetilde{R}_M(z)f\right|\right|_{L^2(\Sigma)^4} \lesssim \frac{1}{M}\,||f||_{L^2(\Omega_e)^4}\,.$$

Since Γ_- is bounded from $L^2(\Omega_e)^4$ into $H^{-1/2}(\Sigma)^4$, it follows from the above inequality that

$$\left|\left|\Gamma_-\widetilde{R}_M(z)f\right|\right|_{H^{-1/2}(\Sigma)^4} \lesssim ||\Gamma_-||_{L^2(\Omega_e)^4\to H^{-1/2}(\Sigma)^4}\left|\left|\widetilde{R}_M(z)f\right|\right|_{L^2(\Omega_e)^4} \lesssim \frac{1}{M}\,||f||_{L^2(\Omega_e)^4}\,,$$

for any $f \in L^2(\Omega_e)^4$, which gives the second inequality.

Let us now turn to the proof of the claimed estimates for $E^{\Omega_e}_{m+M}(z)$. Let $\psi \in P_+L^2(\Sigma)^4$, then from the proof of Proposition 2.3.2 we have

$$||\psi||^2_{L^2(\Sigma)^4} \geqslant (m+M)\left|\left|E^{\Omega_e}_{m+M}(z)\psi\right|\right|^2_{L^2(\Omega_e)^4} - 2|\mathrm{Re}(z)|\left|\left|E^{\Omega_e}_{m+M}(z)\psi\right|\right|^2_{L^2(\Omega_e)^4}\,.$$

Thus, for any $M \geqslant M_3 \geqslant \sup_{z\in K}\{4|\mathrm{Re}(z)| - m\}$, we get that

$$M\left|\left|E^{\Omega_e}_{m+M}(z)\psi\right|\right|^2_{L^2(\Omega_e)^4} \leqslant 2\,||\psi||^2_{L^2(\Sigma)^4}\,,$$

and this proves the first estimate for $E^{\Omega_e}_{m+M}(z)$. Finally, the last inequality is a consequence of the first one and Proposition 2.3.2. Indeed, from Proposition 2.3.2 (ii) we know that $\beta\Gamma_-\widetilde{R}_M(\overline{z})$ is the adjoint of the operator $E^{\Omega_e}_{m+M}(z) : P_+H^{1/2}(\Sigma)^4 \longrightarrow L^2(\Omega_e)^4$. Using this and the estimate fulfilled by $\Gamma_-\widetilde{R}_M(\overline{z})$ we obtain that

$$\begin{aligned}\left|\langle f, E^{\Omega_e}_{m+M}(z)\psi\rangle_{L^2(\Omega_e)^4}\right| &= \left|\langle \Gamma_-\widetilde{R}_M(\overline{z})f, \beta\psi\rangle_{H^{-1/2}(\Sigma)^4,H^{1/2}(\Sigma)^4}\right| \\ &\leqslant \left|\left|\Gamma_-\widetilde{R}_M(z)f\right|\right|_{H^{-1/2}(\Sigma)^4} ||\psi||_{H^{1/2}(\Sigma)^4} \\ &\lesssim \frac{1}{M}\, ||f||_{L^2(\Omega_e)^4}\, ||\psi||_{H^{1/2}(\Sigma)^4}\,.\end{aligned}$$

Since this is true for all $f \in L^2(\Omega_e)^4$, by duality arguments it follows that

$$\left|\left|E^{\Omega_e}_{m+M}(z)\psi\right|\right|_{L^2(\Omega_e)^4} \lesssim \frac{1}{M}\, ||\psi||_{H^{1/2}(\Sigma)^4}\,, \quad \forall\psi \in P_+H^{1/2}(\Sigma)^4,$$

which proves the last inequality. Hence, the lemma follows by taking $M_0 = \max\{M_1, M_2, M_3\}$. ■

***Proof of Proposition* 2.5.6.** We first show (2.70) for some $M_0' > 0$ and any $z \in \mathbb{C} \setminus \mathbb{R}$. So, let us fix such a z and let $f \in L^2(\mathbb{R}^3)^4$. Then, it is clear that $z \in \rho(H_{\text{MIT}}(m)) \cap \rho(H_M)$, and from Theorem 2.5.2 and Remark 2.5.2 we know that there is $M_0' > 0$ such that for all $M > M_0'$ it holds that

$$\begin{aligned}||(R_M(z) - e_{\Omega_i}R_{\text{MIT}}(z)r_{\Omega_i})f||_{L^2(\mathbb{R}^3)^4} \leqslant{}& \left|\left|E^{\Omega_i}_m(z)\Xi^-_M(z)\mathscr{A}^e_{m+M}\Gamma_+R_{\text{MIT}}(z)r_{\Omega_i}f\right|\right|_{L^2(\Omega_i)^4} \\ &+ \left|\left|E^{\Omega_i}_m(z)\Xi^-_M(z)\Gamma_-\widetilde{R}_M(z)r_{\Omega_e}f\right|\right|_{L^2(\Omega_i)^4} \\ &+ \left|\left|E^{\Omega_e}_{m+M}(z)\Xi^+_M(z)\Gamma_+R_{\text{MIT}}(z)r_{\Omega_i}f\right|\right|_{L^2(\Omega_e)^4} \\ &+ \left|\left|E^{\Omega_e}_{m+M}(z)\Xi^+_M(z)\mathscr{A}^i_m\Gamma_-\widetilde{R}_M(z)r_{\Omega_e}f\right|\right|_{L^2(\Omega_e)^4} \\ &+ \left|\left|\widetilde{R}_M(z)r_{\Omega_e}f\right|\right|_{L^2(\Omega_e)^4} =: J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned}$$

From Lemma 2.5.7 we immediately get that $J_5 \lesssim M^{-1}\,||f||$. Next, notice that $\Gamma_+R_{\text{MIT}}(z) : L^2(\Omega_i)^4 \to H^{1/2}(\Sigma)^4$, $\mathscr{A}^i_m : H^{1/2}(\Sigma)^4 \to H^{1/2}(\Sigma)^4$ and $E^{\Omega_i}_m(z) : H^{-1/2}(\Sigma)^4 \to H(\alpha, \Omega_i) \subset L^2(\Omega_i)^4$ (where $H(\alpha, \Omega_i)$ is defined by (1.9)) are bounded operators and do not depend on M. Moreover, thanks to Corollary 2.5.4 we know that for all $s \in \mathbb{R}$ there is $C > 0$ independent of M such that

$$\left|\left|\Xi^\pm_M(z)\right|\right|_{P_\pm H^s(\Sigma)^4 \to P_\pm H^s(\Sigma)^4} \leqslant C.$$

Using this and the above observation, for $j \in \{1, 2, 3, 4\}$, we can estimate J_k as follows

$$\begin{aligned}J_1 &\lesssim \left|\left|E^{\Omega_i}_m(z)\right|\right|_{P_-H^{-1/2}(\Sigma)^4\to L^2(\Omega_i)^4} ||\mathscr{A}^e_{m+M}||_{H^{1/2}(\Sigma)^4\to H^{-1/2}(\Sigma)^4} ||\Gamma_+R_{\text{MIT}}(z)r_{\Omega_i}f||_{H^{1/2}(\Sigma)^4}\,, \\ J_2 &\lesssim \left|\left|E^{\Omega_i}_m(z)\right|\right|_{H^{-1/2}(\Sigma)^4\to L^2(\Omega_i)^4} \left|\left|\Gamma_-\widetilde{R}_M(z)r_{\Omega_e}f\right|\right|_{H^{-1/2}(\Sigma)^4}\,, \\ J_3 &\lesssim \left|\left|E^{\Omega_e}_{m+M}(z)\right|\right|_{H^{1/2}(\Sigma)^4\to L^2(\Omega_e)^4} ||\Gamma_+R_{\text{MIT}}(z)r_{\Omega_i}f||_{H^{1/2}(\Sigma)^4}\,, \\ J_4 &\lesssim \left|\left|E^{\Omega_e}_{m+M}(z)\right|\right|_{L^2(\Sigma)^4\to L^2(\Omega_e)^4} \left|\left|\mathscr{A}^i_m\right|\right|_{L^2(\Sigma)^4\to L^2(\Sigma)^4} \left|\left|\Gamma_-\widetilde{R}_M(z)r_{\Omega_e}f\right|\right|_{L^2(\Sigma)^4}\,.\end{aligned}$$

Therefore, Proposition 2.5.3-(ii) together with Lemma 2.5.7 yield that

$$J_k \lesssim \frac{1}{M} ||f||_{L^2(\mathbb{R}^3)^4} \,, \quad \text{for any } j \in \{1,2,3,4\}.$$

Thus, we obtain the estimate

$$||(R_M(z) - e_{\Omega_i} R_{\text{MIT}}(z) r_{\Omega_i}) f||_{L^2(\mathbb{R}^3)^4} \leqslant \frac{C}{M} ||f||_{L^2(\mathbb{R}^3)^4} \,. \tag{2.71}$$

Moreover, the asymptotic expansion (2.69) holds with

$$\begin{aligned} L_M(z) =& M(e_{\Omega_e} \widetilde{R}_M(z) r_{\Omega_e} + e_{\Omega_i} E_m^{\Omega_i}(z) \Xi_M^-(z) \mathscr{A}^e_{m+M} \Gamma_+ R_{\text{MIT}}(z) r_{\Omega_i} \\ &+ e_{\Omega_e} E_{m+M}^{\Omega_e}(z) \Xi_M^+(z) \mathscr{A}^i_m \Gamma_- \widetilde{R}_M(z) r_{\Omega_e}), \end{aligned}$$

and

$$K_M(z) = M \left(e_{\Omega_i} E_m^{\Omega_i}(z) \Xi_M^-(z) \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} + e_{\Omega_e} E_{m+M}^{\Omega_e}(z) \Xi_M^+(z) \Gamma_+ R_{\text{MIT}}(z) r_{\Omega_i} \right),$$

and we clearly see that $r_{\Omega_i} K_M(z) e_{\Omega_i} = 0 = r_{\Omega_e} K_M(z) e_{\Omega_e}$.

Finally, since (2.71) holds true for every $z \in \mathbb{C} \setminus \mathbb{R}$, for any fixed compact subset $K \subset \rho(H_{\text{MIT}}(m))$, one can show by arguments similar to those in the proof of [BCLTS19, Lemma A.1] that there is $M_0 > M_0'$ such that $K \subset \rho(H_M)$. Therefore, the proposition follows with the same arguments as before. ■

2.5.1 Comments and further remarks at the end of this chapter

In this part we discuss possible generalizations of our results and comment on the usefulness of the pseudodifferential properties of the Poincaré-Steklov operators.

(1) First note that all the results in this article which are proved without the use of the (semi) classical properties of the Poincaré-Steklov operator are valid when Σ is just $C^{1,\omega}$-smooth with $\omega \in (1/2, 1)$, and can also be generalized without difficulty to the case of local deformation of the plane $\mathbb{R}^2 \times \{0\}$ (see [Ben22b] where the self-adjointness of $H_{\text{MIT}}(m)$ and the regularity properties of $\Phi^{\Omega}_{z,m}$, $\mathscr{C}_{z,m}$ and Λ^z_m were shown for this case). We mention, however, that in the latter case the spectrum of the MIT bag operator is equal to that of the free Dirac operator, cf. [Ben22b, Theorem 4.1].

(2) It should also be noted that there are several boundary conditions that lead to self-adjoint realizations of the Dirac operator on domains (see, *e.g.*, [AMSPV23, BHM20, Ben22a]) and for which the associated PS operators can be analyzed in a similar way as for the MIT bag model. In particular, one can consider the PS operator $\mathscr{B}_m(z)$ associated with the self-adjoint Dirac operator

$$\widetilde{H}_{MIT}(m)v = D_m v, \quad \forall v \in \text{Dom}(\widetilde{H}_{MIT}(m)) := \left\{ v \in H^1(\Omega_i)^4 : P_+ t_\Sigma v = 0 \text{ on } \Sigma \right\}.$$

According to the previous considerations, this operator can be viewed as an analogue of the Neumann-to-Dirichlet map for the Dirac operator. Moreover, the same arguments as in the proof of Theorem 2.3.3 show that

$$\mathscr{B}_m(z) = \frac{1}{\sqrt{-\Delta_\Sigma}} S\cdot(\nabla_\Sigma \wedge n)P_+ \mod Op\mathcal{S}^{-1}(\Sigma) = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}} P_+ \mod Op\mathcal{S}^{-1}(\Sigma),$$

for all $z \in \rho(D_m)\cap\rho(\widetilde{H}_{MIT}(m))$.

(3) As already mentioned in the introduction, in [BCLTS19] it was shown that (in the two-dimensional massless case) the norm resolvent convergence of H_M to $H_{\text{MIT}}(m)$ holds with a convergence rate of $M^{-1/2}$. Their proof is based on two main ingredients: The first is a resolvent identity (see [BCLTS19, Lemma 2.2] for the exact formula), and the second is the following inequality

$$||\Gamma_- R_M(z)f||_{L^2(\Sigma)^4} \lesssim \frac{1}{\sqrt{M}}\, ||f||_{L^2(\mathbb{R}^3)^4}\,, \tag{2.72}$$

which is a consequence of the lower bound

$$||\nabla\psi||^2_{L^2(\Omega_e)^4} + M^2\, ||\psi||^2_{L^2(\Omega_e)^4} \geqslant (M-C)\, ||t_\Sigma\psi||^2_{L^2(\Sigma)^4}\,,$$

which holds for all $\psi \in H^1(\mathbb{R}^3)^4$ and M large enough (see [SV19, Lemma 4] for the proof in the 2D-case and [ALTMR19, Proposition 2.1 (i)] for the 3D-case). Note that the resolvent formula (2.63) together with (2.72) yield the same result. Indeed, from (2.62) and (2.72) we easily get the inequality

$$||\Gamma_+ R_M(z)f||_{L^2(\Sigma)^4} \lesssim ||f||_{L^2(\mathbb{R}^3)^4}\,.$$

This together with (2.63) and Lemma 2.5.7 yield

$$\begin{aligned} ||(R_M(z) - e_{\Omega_i}R_{\text{MIT}}(z)r_{\Omega_i})f||_{L^2(\mathbb{R}^3)^4} \leqslant \Big|\Big|E_m^{\Omega_i}(z)\Gamma_- r_{\Omega_e}R_M(z)f\Big|\Big|_{L^2(\Omega_i)^4} \\ + \Big|\Big|\widetilde{R}_M(z)r_{\Omega_e}f\Big|\Big|_{L^2(\Omega_e)^4} + \Big|\Big|E_{m+M}^{\Omega_e}(z)\Gamma_+ r_{\Omega_i}R_M(z)f\Big|\Big|_{L^2(\Omega_e)^4} \\ \lesssim \frac{1}{\sqrt{M}}\, ||f||_{L^2(\mathbb{R}^3)^4}\,. \end{aligned}$$

(4) Finally, let us point out that a first order asymptotic expansion of the eigenvalues of H_M in terms of the eigenvalues of $H_{\text{MIT}}(m)$ was established in [ALTMR19] when $M \to \infty$. In their proof the authors used the min-max characterization and optimization techniques. Note that it is also possible to obtain such a result using the properties of the PS operator, the Krein formula from Theorem 2.5.2 and the finite-dimensional perturbation theory (cf. Kato [Kat66] for example), see, *e.g.*, [Ben19, BC02] for similar arguments.

Note also that the asymptotic expansion of the eigenvalues of H_M depends only on the term $E_m^{\Omega_i}(z)\Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+ R_{\text{MIT}}(z)r_{\Omega_i}$. Indeed, let λ_{MIT} be an eigenvalue of $H_{\text{MIT}}(m)$ with multiplicity l, and let $(f_1,\cdots,f_l)$ be an $L^2(\Omega_i)^4$-orthonormal basis of $\text{Kr}(H_{\text{MIT}}(m) - \lambda_{MIT}\mathbb{I}_4)$.

Then, using the explicit resolvent formula from Remark 2.5.2 we see that

$$\begin{aligned}\langle R_M(z)e_{\Omega_i}f_k, e_{\Omega_i}f_j\rangle_{L^2(\mathbb{R}^3)^4} &= \langle E_m^{\Omega_i}(z)\Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+R_{\mathrm{MIT}}(z)f_k, f_j\rangle_{L^2(\Omega_i)^4}\\ &= \langle \Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+R_{\mathrm{MIT}}(z)f_k, -\beta\Gamma_+R_{\mathrm{MIT}}(\overline{z})f_j\rangle_{L^2(\Sigma)^4}\\ &= \frac{1}{(z-\lambda_{MIT})^2}\langle \Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+f_k, -\beta\Gamma_+f_j\rangle_{L^2(\Sigma)^4},\end{aligned}$$

which means that $E_m^{\Omega_i}(z)\Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+R_{\mathrm{MIT}}(z)r_{\Omega_i}$ is the only term that intervenes in the asymptotic expansion of the eigenvalues of H_M. Besides, recall that the principal symbol of $\Xi_M^-(z)\mathscr{A}_{m+M}^e$ is given by

$$q_M(x,\xi) = -\frac{S\cdot(\xi\wedge n(x))P_+}{\sqrt{|\xi\wedge n(x)|^2+(m+M)^2}+|\xi\wedge n(x)|+(m+M)},$$

and for $M>0$ large enough one has

$$q_M(x,\xi) = -\frac{1}{2M}S\cdot(\xi\wedge n(x))P_+\sum_{l=1}^{\infty}\frac{1}{M^{l+1}}p_l(x,\xi)P_+,\quad p_l\in\mathcal{S}^{-l}.$$

Using this, we formally deduce that for sufficiently large M, H_M has exactly l eigenvalues $(\lambda_k^M)_{1\leqslant k\leqslant l}$ counted according to their multiplicities (in $B(\lambda_{MIT},\eta)$ with $B(\lambda_{MIT},\eta)\cap \mathrm{Sp}(H_{\mathrm{MIT}}(m))=\{\lambda_{MIT}\}$) and these eigenvalues admit an asymptotic expansion of the form

$$\lambda_k^M = \lambda_{MIT} + \frac{1}{M}\mu_k + \sum_{j=2}^{N}\frac{1}{M^j}\mu_k^j + O\left(M^{-(N+1)}\right). \tag{2.73}$$

where $(\mu_k)_{1\leqslant k\leqslant l}$ are the eigenvalues of the matrix $\mathcal{M}$ with coefficients:

$$m_{kj} = \frac{1}{2}\langle\beta Op(S\cdot(\xi\wedge n(x)))\Gamma_+f_k, \Gamma_+f_j\rangle_{L^2(\Sigma)^4}.$$

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2.6 Appendix A: Dirac algebra and applications

In this appendix, we recall the anticommutation relations of Dirac matrices and give formulas used in the current manuscript. Let us consider the 4×4-Hermitian Dirac matrices α_j, $j = 1, 2, 3$, and β, whose possible representation is given at the beginning of this thesis, see (1.4).

Recall the definition of the spin angular momentum S and the matrix γ_5 (see (2.6)), and note that by (1.3) we have $S = (i\alpha_2\alpha_3,\ -i\alpha_1\alpha_3,\ i\alpha_1\alpha_2)$.

Using the anticommutation relations (1.3) we easily get the following identities, for all $X, Y \in \mathbb{R}^3$,

$$\begin{aligned} i(\alpha \cdot X)(\alpha \cdot Y) = iX \cdot Y + S \cdot (X \wedge Y), \qquad [\gamma_5, \alpha \cdot X] = 0, \\ \{S \cdot X, \alpha \cdot Y\} = -2(X \cdot Y)\gamma_5, \qquad [S \cdot X, \beta] = 0. \end{aligned} \tag{2.74}$$

Let us now give some relations we have used for n a normal vector field to a smooth domain $\Omega \subset \mathbb{R}^3$, and for τ, a tangent vector, in particular for $\tau = n \wedge \xi$, where ξ is a Fourier variable.

Lemma 2.6.1. *Let $n \in \mathbb{R}^3$ and let $\tau \in \mathbb{R}^3$ such that $\tau \perp n$. Then the following identity holds:*

$$(S \cdot \tau + i(\alpha \cdot n)\beta)^2 = \left(|\tau|^2 + |n|^2\right) \mathbb{I}_4.$$

Proof. Using the relations (1.3) and (2.74) we get

$$(S \cdot \tau)^2 = \gamma_5(\alpha \cdot \tau)\gamma_5(\alpha \cdot \tau) = (\gamma_5)^2(\alpha \cdot \tau)^2 = |\tau|^2\mathbb{I}_4.$$

Then we have

$$(S \cdot \tau + i(\alpha \cdot n)\beta)^2 = |\tau|^2\mathbb{I}_4 - ((\alpha \cdot n)\beta)^2 + i\{S \cdot \tau, (\alpha \cdot n)\beta\} = (|\tau|^2 + |n|^2)\mathbb{I}_4 + i\{S \cdot \tau, (\alpha \cdot n)\beta\},$$

and since $\tau \cdot n = 0$, by (2.74) we obtain

$$\{S \cdot \tau, (\alpha \cdot n)\beta\} = \{S \cdot \tau, \alpha \cdot n\}\beta + \alpha \cdot n[S \cdot \tau, \beta] = 0,$$

and the conclusion follows. ∎

Proposition 2.6.2. *Given $n \in \mathbb{R}^3$ such that $|n| = 1$, let $\xi \in \mathbb{R}^3$, and define the matrix-valued function*

$$l_0(n, \xi) = i(\alpha \cdot n)(\alpha \cdot \xi + \beta).$$

Then $l_0(n, \xi)$ has two eigenvalues given by

$$\rho_\pm(n, \xi) := i\, n \cdot \xi \pm \lambda(n, \xi), \quad \textit{with} \quad \lambda(n, \xi) = \sqrt{|n \wedge \xi|^2 + 1}.$$

The associated eigenprojections (onto $\mathrm{Kr}(l_0(n, \xi) - \rho_\pm(n, \xi)I_4)$) are given by

$$\Pi_\pm(n, \xi) := \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{S \cdot (n \wedge \xi) + i(\alpha \cdot n)\beta}{\lambda(n, \xi)}\right).$$

Proof. By applying (2.74) for $(X, Y) = (n, \xi)$, we get

$$l_0(n, \xi) = i\, n \cdot \xi\, \mathbb{I}_4 + S \cdot (n \wedge \xi) + i(\alpha \cdot n)\beta.$$

Thanks to Lemma 2.6.1, the Hermitian matrix $h(n, \xi) := S \cdot (n \wedge \xi) + i(\alpha \cdot n)\beta$ satisfies:

$$h(n, \xi)^2 = (|n \wedge \xi|^2 + 1)\mathbb{I}_4 = \lambda(n, \xi)^2 \mathbb{I}_4.$$

Therefore, $h(n, \xi)$ has the eigenvalues $\pm\lambda(n, \xi)$ and the associated eigenprojections are given by

$$\Pi_\pm(n, \xi) = \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{h(n, \xi)}{\lambda(n, \xi)}\right),$$

which proves the claimed results since $l_0(n, \xi) = i\, n \cdot \xi\, \mathbb{I}_4 + h(n, \xi)$. ■

Lemma 2.6.3. *Given $n \in \mathbb{R}^3$ such that $|n| = 1$, let $P_\pm = \Pi_\pm(n, 0) = \frac{1}{2}(\mathbb{I}_4 \pm i(\alpha \cdot n)\beta)$ be the eigenprojections onto $\mathrm{Kr}(i(\alpha \cdot n)\beta \mp \mathbb{I}_4)$. The following properties hold true.*

(i) For any $\tau \in \mathbb{R}^3$ such that $\tau \perp n$, we have

$$P_\pm(S \cdot \tau) = (S \cdot \tau)P_\mp \quad , \quad P_\pm(\alpha \cdot n) = (\alpha \cdot n)P_\mp \quad \textit{and} \quad P_\pm\beta = \beta P_\mp.$$

(ii) For any $\xi \in \mathbb{R}^3$, the projections $\Pi_\pm(n, \xi)$ defined in Proposition 2.6.2 satisfy

$$P_\pm\, \Pi_\pm\, P_\pm = k_+ P_\pm \quad , P_\mp\, \Pi_\pm\, P_\mp = k_- P_\pm \quad \textit{and}\ P_\pm\, \Pi_\mp\, P_\mp = \mp\Theta P_\mp, \tag{2.75}$$

with

$$k_\pm(n, \xi) = \frac{1}{2}\left(1 \pm \frac{1}{\lambda(n, \xi)}\right), \quad \Theta(n, \xi) = \frac{1}{2\lambda(n, \xi)} S \cdot (n \wedge \xi). \tag{2.76}$$

Proof. The relations of (i) follow from (2.74). For the proof of (ii), let us write $\Pi_\pm(n, \xi)$ as

$$\Pi_\pm(n, \xi) = P_\pm \pm \frac{1}{2\lambda(x, \xi)} S \cdot (n \wedge \xi) P_\mp \pm \frac{i}{2}(\alpha \cdot n)\beta\left(\frac{1}{\lambda(n, \xi)} - 1\right).$$

Then, using item (i) if this lemma (with $\tau = n \wedge \xi$) and the fact that $P_\pm\, i(\alpha \cdot n)\beta = \pm P_\pm$, we get

$$P_\pm\Pi_\pm = P_\pm \pm \frac{1}{2\lambda} S \cdot (n \wedge \xi) P_\mp + \frac{1}{2}\left(\frac{1}{\lambda} - 1\right) P_\pm = k_+ P_\pm \pm \Theta\, P_\mp,$$
$$P_\mp\Pi_\pm = \pm\frac{1}{2\lambda} S \cdot (n \wedge \xi) P_\pm - \frac{1}{2}\left(\frac{1}{\lambda} - 1\right) P_\mp = k_- P_\mp \pm \Theta\, P_\pm,$$

with $k_\pm$ and Θ as in (2.76). Hence, (2.75) directly follows from the above formulas and the fact that $P_\pm$ are orthogonal projections. ■

2.7 Appendix B: Poincaré-Steklov operators in half-space

This appendix focuses on the examination of the Poincaré-Steklov operators introduced in this chapter, where we simplify the perturbation domain of the Dirac operator. On a smooth bar C^∞ of infinite size dividing space into two parts Ω_- and Ω_+, we consider the perturbation of the Dirac operator with a potential that affects the lower zone of the bar, $H_M = D_m + M\beta \mathbb{1}_{\Omega_-}$. The simplicity of the domain in this problem allows us to derive an explicit form of the Poincaré-Steklov operators and their symbolic formulas. Moreover, Poincaré-Steklov operators are simply zeroth-order Fourier multipliers. Let N be the outward unit normal with respect to Ω_+.

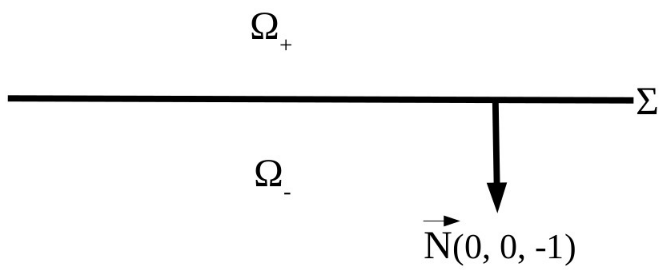

Figure 2.2 – Half-plane domain

Proposition 2.7.1. *Let the projection operators given by*

$$\Pi^{\bullet}_{\pm}(\xi) := \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{d_{\bullet}(\xi)}{\lambda_{\bullet}(\xi)}\right), \quad \textit{with} \quad \bullet = m \textit{ or } \kappa := (m+M). \tag{2.77}$$

Let $\phi \in P_- H^{1/2}(\Sigma)^4$ and $\psi \in P_+ H^{1/2}(\Sigma)^4$. We consider the system (2.57) resp. (2.58). Then, the solution of this system is respectively given by:

$$\begin{cases} \hat{v}\,(\xi_1,\xi_2,x_3) = e^{-\lambda_m(\xi)x_3}\,\Pi^m_-(\xi)\left(\mathbb{I}_4 + \dfrac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m}\right)\hat{\phi}(\xi_1,\xi_2) & \textit{in } \Omega_+ \ (i.e.,\ x_3>0), \\ \\ \hat{u}\,(\xi_1,\xi_2,x_3) = e^{\lambda_\kappa(\xi)x_3}\,\Pi^\kappa_+(\xi)\left(\mathbb{I}_4 - \dfrac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_\kappa(\xi)+\kappa}\right)\hat{\psi}(\xi_1,\xi_2) & \textit{in } \Omega_- \ (i.e.,\ x_3<0), \end{cases} \tag{2.78}$$

where $d_{\bullet}(\xi) = -i\alpha_3(\alpha\cdot\xi + \bullet\beta - z)$ and $\lambda_{\bullet}(\xi) = \sqrt{|\xi|^2 + \bullet^2 - z^2}$, with $\alpha\cdot\xi = \alpha_1\xi_1 + \alpha_2\xi_2$.

Proof. We consider the system

$$\begin{cases} (D_m - z)v\,(x_1,x_2,x_3) = 0, & \text{in } \Omega_+, \\ P_- t_\Sigma v\,(x_1,x_2,x_3) = \phi, & \text{on } \Sigma, \end{cases} \tag{2.79}$$

$$\begin{cases} (D_\kappa - z)u\,(x_1,x_2,x_3) = 0, & \text{in } \Omega_-, \\ P_+ t_\Sigma u\,(x_1,x_2,x_3) = \psi, & \text{on } \Sigma. \end{cases} \tag{2.80}$$

Recall that, the unique solution of the boundary value problem (2.79) resp. (2.80) is $v = E^{\Omega_+}_m \phi$ resp. $u = E^{\Omega_-}_\kappa \psi$, bounded from $P_- H^{1/2}(\Sigma)^4$ resp. $P_+ H^{1/2}(\Sigma)^4$ into $H^1(\Omega_+)^4$ resp. $H^1(\Omega_-)^4$.

By Fourier, $\mathscr{F}_{(x_1,x_2)\longrightarrow(\xi_1,\xi_2)}$ we get

$$\begin{cases} \partial_{x_3}\hat{u}\,(\xi_1,\xi_2,x_3) = -i\alpha_3(\alpha\cdot\xi+\kappa\beta-z)\hat{u}\,(\xi_1,\xi_2,x_3)\,, & \text{in } \Omega_-,\\ \partial_{x_3}\hat{v}\,(\xi_1,\xi_2,x_3) = -i\alpha_3(\alpha\cdot\xi+m\beta-z)\hat{v}\,(\xi_1,\xi_2,x_3)\,, & \text{in } \Omega_+,\\ P_+\hat{u}\,(\xi_1,\xi_2,0) = \hat{\psi}(\xi_1,\xi_2), & \text{on } \Sigma,\\ P_-\hat{v}\,(\xi_1,\xi_2,0) = \hat{\phi}(\xi_1,\xi_2), & \text{on } \Sigma, \end{cases} \tag{2.81}$$

Then, the solution of the system (2.81) is the following:

$$\begin{cases} \hat{u}\,(\xi_1,\xi_2,x_3) = e^{d_\kappa(\xi)x_3}\hat{u}\,(\xi_1,\xi_2,0)\,, & \text{in } \Omega_-,\\ \hat{v}\,(\xi_1,\xi_2,x_3) = e^{d_m(\xi)x_3}\hat{v}\,(\xi_1,\xi_2,0)\,, & \text{in } \Omega_+\\ P_+\hat{u}\,(\xi_1,\xi_2,0) = \hat{\psi}(\xi_1,\xi_2), & \text{on } \Sigma,\\ P_-\hat{v}\,(\xi_1,\xi_2,0) = \hat{\phi}(\xi_1,\xi_2), & \text{on } \Sigma. \end{cases} \tag{2.82}$$

Now, we can write $e^{d_\bullet(\xi)x_3}$ by

$$e^{d_\bullet(\xi)x_3} = e^{\lambda_\bullet(\xi)x_3}\Pi^\bullet_+(\xi) + e^{-\lambda_\bullet(\xi)x_3}\Pi^\bullet_-(\xi),$$

thus, we obtain the following

$$\begin{cases} \hat{u}\,(\xi_1,\xi_2,x_3) = \left(e^{\lambda(\xi)x_3}\Pi^\kappa_+(\xi) + e^{-\lambda_\kappa(\xi)x_3}\Pi^\kappa_-(\xi)\right)(P_+ + P_-)\hat{u}\,(\xi_1,\xi_2,0)\,, & \text{in } \Omega_-,\\ \hat{v}\,(\xi_1,\xi_2,x_3) = \left(e^{\lambda_m(\xi)x_3}\Pi^m_+(\xi) + e^{-\lambda_m(\xi)x_3}\Pi^m_-(\xi)\right)(P_+ + P_-)\hat{v}\,(\xi_1,\xi_2,0)\,, & \text{in } \Omega_+,\\ P_+\hat{u}\,(\xi_1,\xi_2,0) = \hat{\psi}(\xi_1,\xi_2), & \text{on } \Sigma,\\ P_-\hat{v}\,(\xi_1,\xi_2,0) = \hat{\phi}(\xi_1,\xi_2), & \text{on } \Sigma. \end{cases} \tag{2.83}$$

Now, calculate the expression of $P_+\hat{v}\,(\xi_1,\xi_2,0)$ and $P_-\hat{u}\,(\xi_1,\xi_2,0)$.

In Ω_-, $e^{-\lambda_\kappa(\xi)x_3} \notin L^2(\{x_3<0\})$,

$$\begin{aligned} &\Rightarrow \Pi^\kappa_-(\xi)\hat{u}\,(\xi_1,\xi_2,0) = 0 \Leftrightarrow \Pi^\kappa_+(\xi)\hat{v}\,(\xi_1,\xi_2,0) = \hat{u}\,(\xi_1,\xi_2,0)\\ &\Leftrightarrow P_-\hat{u}\,(\xi_1,\xi_2,0) - P_-\Pi^\kappa_+P_-\hat{u}\,(\xi_1,\xi_2,0) = P_-\Pi^\kappa_+P_+\hat{\psi}(\xi_1,\xi_2). \end{aligned} \tag{2.84}$$

In Ω_+, $e^{\lambda_m(\xi)x_3} \notin L^2(\{x_3>0\})$,

$$\begin{aligned} &\Rightarrow \Pi^m_+(\xi)\hat{v}\,(\xi_1,\xi_2,0) = 0 \Leftrightarrow \Pi^m_-(\xi)\hat{v}\,(\xi_1,\xi_2,0) = \hat{v}\,(\xi_1,\xi_2,0)\\ &\Leftrightarrow P_+\hat{v}\,(\xi_1,\xi_2,0) - P_+\Pi^m_-P_+\hat{v}\,(\xi_1,\xi_2,0) = P_+\Pi^m_-P_-\hat{\phi}(\xi_1,\xi_2). \end{aligned} \tag{2.85}$$

Using Lemma 2.6.3-(ii), we express (2.84) and (2.85) in terms of $P_+\Pi^m_-(\xi)$ and $P_-\Pi^\kappa_+(\xi)$ as follows:

From (2.84), we get

$$P_-\hat{u}\,(\xi_1,\xi_2,0) = -\frac{i\alpha_3(\alpha\cdot\xi-z)}{\lambda_\kappa(\xi)+\kappa}\hat{\psi}(\xi_1,\xi_2).$$

From (2.85), we get

$$P_+\hat{v}\,(\xi_1,\xi_2,0) = \frac{i\alpha_3(\alpha\cdot\xi-z)}{\lambda_m(\xi)+m}\hat{\phi}(\xi_1,\xi_2).$$

Combining the above formulas with system (2.83), we deduce that

$$\begin{cases} \hat{v}\,(\xi_1,\xi_2,x_3) = e^{-\lambda_m(\xi)x_3}\Pi_-^m(\xi)\left(\mathbb{I}_4 + \dfrac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m}\right)\hat{\phi}(\xi_1,\xi_2), & \text{in } \Omega_+, \\ \hat{u}\,(\xi_1,\xi_2,x_3) = e^{\lambda_\kappa(\xi)x_3}\Pi_+^\kappa(\xi)\left(\mathbb{I}_4 - \dfrac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_\kappa(\xi)+\kappa}\right)\hat{\psi}(\xi_1,\xi_2), & \text{in } \Omega_-. \end{cases} \tag{2.86}$$

■

Theorem 2.7.2. *We denote by $\mathscr{A}_m$ resp. $\mathscr{A}_\kappa$ the Poincaré-Steklov operators associated with (2.79) resp. (2.80). Then, we can write the operator $\mathscr{A}_m$ resp. $\mathscr{A}_\kappa$ (in which $a_m(\xi)$ resp. $a_\kappa(\xi)$ is the symbols of $\mathscr{A}_m$ resp. $\mathscr{A}_\kappa$) as the following*

$$\mathscr{A}_m P_-\phi = \mathscr{F}^{-1}\underbrace{\left(P_+\frac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m}P_-\right)}_{a_m(\xi)}\mathscr{F}\phi, \tag{2.87}$$

and

$$\mathscr{A}_\kappa P_+\psi = \mathscr{F}^{-1}\underbrace{\left(P_-\frac{-i\alpha_3(\alpha\cdot\xi - z)}{\lambda_\kappa(\xi)+\kappa}P_+\right)}_{a_\kappa(\xi)}\mathscr{F}\psi. \tag{2.88}$$

Proof. Recall that, for all $z \in \rho(D_m)$, we have the following explicit formula for $\mathscr{A}_m(z)$ and $\mathscr{A}_\kappa(z)$:

$$\mathscr{A}_m = P_+ t_\Sigma E_m^{\Omega_+}(z) P_- = -P_+\beta\left(\Lambda_m^z\right)^{-1}P_-,$$

$$\mathscr{A}_\kappa = P_- t_\Sigma E_\kappa^{\Omega_-}(z) P_+ = -P_-\beta\left(\Lambda_\kappa^z\right)^{-1}P_+,$$

Now, we consider the system (2.79) resp. (2.80), we get

$$\mathscr{A}_m P_-\phi = P_+ v_{\lfloor (x_3=0)} \Leftrightarrow \mathscr{F}(\mathscr{A}_m\phi) = \left(P_+\frac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m}P_-\right)\mathscr{F}\phi,$$

$$\Rightarrow \mathscr{A}_m\phi = \mathscr{F}^{-1}\left(P_+\frac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m}P_-\right)\mathscr{F}\phi.$$

$$\mathscr{A}_\kappa\psi = P_- u_{\lfloor (x_3=0)} \Leftrightarrow \mathscr{F}(\mathscr{A}_\kappa\psi) = \left(P_-\frac{-i\alpha_3(\alpha\cdot\xi - z)}{\lambda(\xi)+\kappa}P_+\right)\mathscr{F}\psi.$$

$$\Rightarrow \mathscr{A}_\kappa\psi = \mathscr{F}^{-1}\left(P_-\frac{-i\alpha_3(\alpha\cdot\xi - z)}{\lambda(\xi)+\kappa}P_+\right)\mathscr{F}\psi,$$

■

Corollary 2.7.3. *The operator $\Psi_M(z) = (\mathbb{I} - \mathscr{A}_m - \mathscr{A}_\kappa)$ (associated with the current Appendix) introduced in Theorem 2.5.2 is bounded invertible in $H^{1/2}(\Sigma)$, and has the following inverse:*

$$\Psi_M^{-1}(z) = \left(\mathbb{I}_4 - \mathscr{A}_m\mathscr{A}_\kappa - \mathscr{A}_\kappa\mathscr{A}_m\right)^{-1}\left(\mathbb{I} + \mathscr{A}_m + \mathscr{A}_\kappa\right)$$

$$= \mathscr{F}^{-1}\left[\left(\mathbb{I}_4 + \frac{|\xi|^2 - z^2}{(\lambda_m(\xi)+m)(\lambda_\kappa(\xi)+\kappa)}\right)^{-1}\left(\mathbb{I} + \frac{i\alpha_3 P_-(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m} - \frac{i\alpha_3 P_+(\alpha\cdot\xi - z)}{\lambda_\kappa(\xi)+\kappa}\right)\right]\mathscr{F}.$$

Consequently, the resolvent formula (2.59) *becomes*

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\mathscr{F}^{-1}\left(\mathbb{I}_4 + \frac{|\xi|^2 - z^2}{(\lambda_m(\xi)+m)(\lambda_\kappa(\xi)+\kappa)}\right)^{-1} \times \left(\mathbb{I} + \frac{i\alpha_3 P_-(\alpha\cdot\xi - z)}{\lambda_m(\xi)+m} - \frac{i\alpha_3 P_+(\alpha\cdot\xi - z)}{\lambda_\kappa(\xi)+\kappa}\right)\mathscr{F}\Gamma\widetilde{R}_{\mathrm{MIT}}(z). \tag{2.89}$$

Proof. Firstly, we will calculate the symbols of $\mathscr{A}_m\mathscr{A}_\kappa$ and $\mathscr{A}_\kappa\mathscr{A}_m$. We have

$$\mathscr{A}_m\mathscr{A}_\kappa = \mathscr{F}^{-1}a_m a_\kappa\mathscr{F}, \tag{2.90}$$

$$\mathscr{A}_\kappa\mathscr{A}_m = \mathscr{F}^{-1}a_\kappa a_m\mathscr{F}, \tag{2.91}$$

It is easy to check that, for $a_m(\xi)$ resp. $a_\kappa(\xi)$ as in (2.87) resp. (2.88),

$$\begin{cases} a_m(\xi)a_\kappa(\xi) &= -\dfrac{(|\xi|^2 - z^2)P_+}{(\lambda_m(\xi)+m)(\lambda_\kappa(\xi)+\kappa)}, \\ a_\kappa(\xi)a_m(\xi) &= -\dfrac{(|\xi|^2 - z^2)P_-}{(\lambda_m(\xi)+m)(\lambda_\kappa(\xi)+\kappa)}. \end{cases}$$

Using the above quantities, we get that

$$\Psi_M^{-1}(z) = \mathscr{F}^{-1}\left[\,(\mathbb{I}_4 - a_m(\xi)a_\kappa(\xi) - a_\kappa(\xi)a_m(\xi))^{-1}\,(\mathbb{I} + a_m(\xi) + a_\kappa(\xi))\,\right]\mathscr{F},$$

and then we get the explicit formula of resolvent (2.89). ∎

Proposition 2.7.4. *Let the Poincaré-Steklov operators* $\mathscr{A}_m^{\Omega_+} = \Gamma_+ E_m^{\Omega_+}(z)P_-$ *and* $\mathscr{A}_\kappa^{\Omega_-} = \Gamma_- E_\kappa^{\Omega_-}(z)P_+$. *Then, we easily obtain that* $\mathscr{A}_m^{\Omega_+}$ *and* $\mathscr{A}_\kappa^{\Omega_-}$ *are a Fourier multiplier with symbols*

$$a_m(\xi) = -\frac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_m + m}P_- \quad \text{and} \quad a_\kappa(\xi) = \frac{i\alpha_3(\alpha\cdot\xi - z)}{\lambda_\kappa + \kappa}P_+.$$

Chapter 3

On the approximation of the Dirac operator coupled with confining Lorentz scalar δ-shell interactions.

In this chapter, we present the results obtained in article [Zre84].

Abstract

Let $\Omega_+ \subset \mathbb{R}^3$ be a fixed bounded domain with boundary $\Sigma = \partial\Omega_+$. We consider $\mathcal{U}^\varepsilon$ a tubular neighborhood of the surface Σ with a thickness parameter $\varepsilon > 0$, and we define the perturbed Dirac operator $\mathfrak{D}_M^\varepsilon = D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon}$, with D_m the free Dirac operator, $M > 0$, and $\mathbb{1}_{\mathcal{U}^\varepsilon}$ the characteristic function of $\mathcal{U}^\varepsilon$. Then, in the norm resolvent sense, the Dirac operator $\mathfrak{D}_M^\varepsilon$ converges to the Dirac operator coupled with Lorentz scalar δ-shell interactions as $\varepsilon = M^{-1}$ tends to 0, with a convergence rate of $\mathcal{O}(M^{-1})$.

Résumé

Soit $\Omega_+ \subset \mathbb{R}^3$ un domaine borné fixe, et désignons sa frontière par $\Sigma = \partial\Omega_+$. Nous considérons $\mathcal{U}^\varepsilon$ comme un voisinage tubulaire de la surface Σ avec un paramètre d'épaisseur $\varepsilon > 0$, et nous définissons l'opérateur de Dirac perturbé $\mathfrak{D}_M^\varepsilon = D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon}$, avec D_m l'opérateur de Dirac libre, $M > 0$, et $\mathbb{1}_{\mathcal{U}^\varepsilon}$ la fonction caractéristique de $\mathcal{U}^\varepsilon$. Alors, au sens de la norme de la résolvante, l'opérateur de Dirac $\mathfrak{D}_M^\varepsilon$ converge vers l'opérateur de Dirac couplé aux interactions scalaires de Lorentz δ-shell lorsque $\varepsilon = M^{-1}$ tend vers 0, avec un taux de convergence de $\mathcal{O}(M^{-1})$.

Contents

3.1 Introduction

The aim of this chapter is to approximate the Dirac operator coupled with a singular δ-interactions, supported on a closed surface. More precisely, our main goal in this chapter is to approximate the Dirac operator coupled with confining Lorentz scalar δ-shell interactions (*i.e.,* when $\eta = 0$ and $\mu = \pm 2$ in (3.1)) by a perturbed Dirac operator $\mathfrak{D}^{\varepsilon}_{M} = D_m + M\beta \mathbb{1}_{\mathcal{U}^{\varepsilon}}$, where M is a large mass supported on a tubular neighborhood, $\mathcal{U}^{\varepsilon}$, with thickness $\varepsilon > 0$.

Let Ω_+ be a bounded smooth domain in $\mathbb{R}^3$, and $\Sigma := \partial\Omega_+$ for its boundary. For $(\eta,\mu) \in \mathbb{R}^2$, the three-dimensional Dirac operator coupled with delta interactions is defined formally by

$$\mathbb{D}_{\eta,\mu} : f \mapsto D_m f + V_{\eta,\mu}\delta_\Sigma f := D_m f + (\eta \mathbb{I}_4 + \mu\beta)\delta_\Sigma f, \tag{3.1}$$

where δ_Σ is the Dirac delta distribution supported on Σ, and the constant η (resp. μ) measures the strength of the electrostatic (resp. Lorentz scalar) part of the interaction. In this case, the operator in (3.1) is called by Dirac operator coupled with electrostatic and Lorentz scalar δ-shell interactions.

Definition 3.1.1. Let $\mu \in \mathbb{R} \setminus \{0\}$. The Dirac operator coupled with purely Lorentz scalar δ-shell interaction of strength μ, is the operator $\mathbb{D}_{0,\mu} := D_m + V_L$ (*i.e.,* when $\eta = 0$ in (3.1)), acting in $L^2(\mathbb{R}^3)^4$ and defined on the following domain

$$\mathrm{Dom}(\mathbb{D}_{0,\mu}) := \{\varphi = u + \Phi^z_m[g],\ u \in H^1(\mathbb{R}^3)^4,\ g \in L^2(\Sigma)^4,\ t_\Sigma u = -\Lambda^z_{+,m}[g] \text{ on } \Sigma\}, \tag{3.2}$$

where

$$V_L(\varphi) = \frac{\mu\beta}{2}(\varphi_+ + \varphi_-)\delta_\Sigma, \quad \text{with} \quad \varphi_\pm = t_\Sigma u + C^z_{\pm,m}[g].$$

Hence, $\mathbb{D}_{0,\mu}$ acts in the sense of distributions as $\mathbb{D}_{0,\mu}(\varphi) = D_m u$, for all $\varphi = u + \Phi^z_m[g] \in \mathrm{Dom}(\mathbb{D}_{0,\mu})$. Consequently, we can identify $\mathbb{D}_{0,\mu}$ as

$$\begin{aligned}\mathbb{D}_{0,\mu}\varphi = \mathbb{D}^{\Omega_-}_{0,\mu}\varphi_- \oplus \mathbb{D}^{\Omega_+}_{0,\mu}\varphi_+ = D_m\varphi_- \oplus D_m\varphi_+,&\\ \mathrm{Dom}(\mathbb{D}_{0,\mu}) = \{w_\pm + \Phi^z_{m,\pm}[g],\ w_\pm \in H^1(\Omega_\pm)^4,\ g \in L^2(\Sigma)^4,&\\ P_\pm(t_\Sigma w_\pm + C^z_{\pm,m}[g]) = 0,\ \text{with } t_\Sigma w_\pm = -\Lambda^z_{\pm,m}[g] \text{ on } \Sigma\},&\end{aligned}$$

where $\Phi^z_{m,\pm}[g] : L^2(\Sigma)^4 \longrightarrow L^2(\Omega_\pm)^4$ is the operator defined by $\Phi^z_{m,\pm}[g](x) = \Phi^z_m|_{\Omega_\pm}[g](x)$, for $g \in L^2(\Sigma)^4$ and $x \in \Omega_\pm$. Here Φ^z_m, Λ^z_m, and $C^z_{\pm,m}$ are defined in Section 1.5.

Moreover, recall that $\mathbb{D}_{0,\mu}$ is self-adjoint operator on $H^1(\mathbb{R}^3)^4$ for all $\mu \in \mathbb{R}$ (see, [AMV15, Section 5.1]), and for all $z \in \mathbb{C} \setminus \mathbb{R}$ the following resolvent formula holds [Ben19, Proposition 4.1]

$$(\mathbb{D}_{0,\mu} - z) = (D_m - z)^{-1} - \Phi^z_m(\Lambda^z_{+,m})^{-1} t_\Sigma (D_m - z)^{-1}.$$

Finally, we recall that the version of $\mathbb{D}_{0,\mu}$ for $\mu = \pm 2$ is called by the confining version of the Dirac operator coupled with Lorentz scalar δ-shell interactions. Throughout the current chapter, Ω_+ is a bounded smooth domain in $\mathbb{R}^3$ with a compact smooth boundary $\Sigma := \partial\Omega_+$, and let n resp. $\mathrm{d}\sigma$ is the outward unit normal to Ω_+ resp. the surface measure on Σ. We shall work on the Hilbert space $L^2(\mathbb{R}^3)^4$ (resp. $L^2(\Omega^\varepsilon_\pm)^4$ with $\Omega^\varepsilon_+ = \Omega_+ \cup \mathcal{U}^\varepsilon$ and $\Omega^\varepsilon_- = \mathbb{R}^3 \setminus \overline{\Omega^\varepsilon_+}$, where $\mathcal{U}^\varepsilon$ is an ε-neighborhood of the surface Σ) with respect to the Lebesgue measure, and we will make use of the orthogonal decomposition $L^2(\mathbb{R}^3)^4 = L^2(\Omega^\varepsilon_-)^4 \oplus L^2(\Omega^\varepsilon_+)^4$. We denote by N^ε the outward unit normal with respect to Ω^ε_-. More precisely, for ε_0 sufficiently small, we assume that Σ, Ω^ε_-, Σ^ε and $\mathcal{U}^\varepsilon$ satisfied

$$\begin{aligned}
&\Sigma^\varepsilon := \{x \in \mathbb{R}^3,\ x = x_\Sigma + \varepsilon\, n(x_\Sigma) : x_\Sigma \in \Sigma\},\\
&\Omega^\varepsilon_- = \{x \in \mathbb{R}^3,\ \mathrm{dist}(x, \Sigma) > \varepsilon\},\\
&\mathcal{U}^\varepsilon := \{x \in \mathbb{R}^3,\ x = x_\Sigma + t\ n(x_\Sigma) : x_\Sigma \in \Sigma \text{ and } t \in (0, \varepsilon)\}, \quad \text{with } \varepsilon \in (0, \varepsilon_0).
\end{aligned} \tag{3.3}$$

In other words, the Euclidean space is divided as follows: $\mathbb{R}^3 = \Omega^\varepsilon_- \cup \Sigma^\varepsilon \cup \mathcal{U}^\varepsilon \cup \Sigma \cup \Omega_+$.

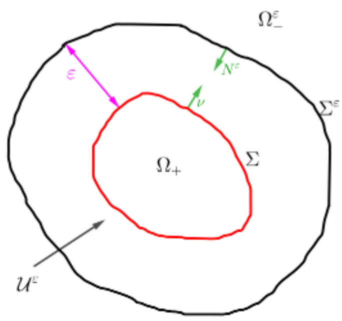

Figure 3.1 – Domain

Definition 3.1.2. [Transformation operator]. Let $\Sigma, \Sigma^\varepsilon \subset \mathbb{R}^3$ be as above. We define the diffeomorphism $p : \Sigma \longrightarrow \Sigma^\varepsilon$ such that for all $x_\Sigma \in \Sigma$, we get $p(x_\Sigma) := x_\Sigma + \varepsilon n(x_\Sigma)$, $\varepsilon \in (0, \varepsilon_0)$. Then for ε_0 sufficiently small, we define the transformation operator as an unitary and invertible operator as follows

$$\begin{aligned}
\mathcal{T}_\varepsilon : L^2(\Sigma)^4 \quad &\to \quad L^2(\Sigma^\varepsilon)^4,\\
\psi \quad &\mapsto \quad \mathcal{T}_\varepsilon[\psi](x) = \frac{1}{\det(1 - \varepsilon W(x_\Sigma))}(\psi \circ p^{-1})(x), \quad x = p(x_\Sigma),
\end{aligned} \tag{3.4}$$

with $W(x_\Sigma)$ the Weingarten defined in Definition 1.5.2. Its inverse, $\mathcal{T}_\varepsilon^{-1}$, is given by

$$\begin{aligned} \mathcal{T}_\varepsilon^{-1} : L^2(\Sigma^\varepsilon)^4 &\to L^2(\Sigma)^4, \\ \varphi &\mapsto \mathcal{T}_\varepsilon^{-1}[\varphi](x_\Sigma) = \det(1-\varepsilon W(x_\Sigma))(\varphi \circ p)(x_\Sigma). \end{aligned}$$

We consider perturbations of the free Dirac operator D_m in the whole space by a large mass M term living in an ε-neighborhood $\mathcal{U}^\varepsilon$ of Σ. The perturbed Dirac operator where interesting on is $\mathfrak{D}_M^\varepsilon := D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon}$, where $\mathbb{1}_{\mathcal{U}^\varepsilon}$ is the characteristic function of $\mathcal{U}^\varepsilon$ and ε is the thickness of the tubular region $\mathcal{U}^\varepsilon$. The results of this chapter are the following:

Proposition 3.1.3. *We consider the confining version of the Dirac operator coupled with a purely Lorentz scalar δ-shell interaction, denoted by $\mathscr{D}_L := \mathbb{D}_{0,+2}$. Then, for any $z \in \rho(\mathscr{D}_L)$ and ε sufficiently small, the following estimate holds:*

$$\left|\left| e_{\Omega^\varepsilon_{+-}} R_{\mathrm{MIT}}^{\Omega^\varepsilon_{+-}}(z) r_{\Omega^\varepsilon_{+-}} - R_L(z) \right|\right|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(\varepsilon), \quad \textit{as } \varepsilon \to 0, \tag{3.5}$$

where $R_{\mathrm{MIT}}^{\Omega^\varepsilon_{+-}}$ is the resolvent of the direct sum of both MIT bag operators, refer to $D_{\mathrm{MIT}}^{\Omega^\varepsilon_{+-}}(m)$ and which will be defined rigorously in Section 3.1.1, R_L is the resolvent of the Dirac operator coupled with purely Lorentz scalar δ-shell interactions, $\mathscr{D}_L$, and $r_{\Omega^\varepsilon_{+-}}$ resp. $e_{\Omega^\varepsilon_{+-}}$ is the restriction operator in Ω^ε_{+-} resp. its adjoint operator, i.e., the extension by 0 outside of Ω^ε_{+-}.

Remark 3.1.1. *We mention that the proof of Proposition 3.1.3 is not difficult to realize. Indeed, we establish the above approximation by tracking the dependence on the thickness ε, when ε goes to 0. However, what is important to achieve is the proof of the following proposition, for which studies and estimates are required by tracking the dependence on the parameters ε and M, in order to establish such a relationship between the parameters, and prove therefore the main result of Theorem 3.1.5.*

Proposition 3.1.4. *Let $K \subset \mathbb{C} \setminus \mathbb{R}$ be a compact set. Then, there is $M_0 > 0$ such that for all $M > M_0$ and $\varepsilon = M^{-1}$: $K \subset \rho(\mathfrak{D}_M^\varepsilon)$ and for all $z \in K$, the following estimate holds on the whole space*

$$||R_M^\varepsilon(z) - e_{\Omega^\varepsilon_{+-}} R_{\mathrm{MIT}}^{\Omega^\varepsilon_{+-}}(z) r_{\Omega^\varepsilon_{+-}}||_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(M^{-1}).$$

The latter proposition means that the Dirac operator $\mathfrak{D}_M^\varepsilon$ is approximated, in the norm resolvent sense, by both MIT Dirac operators, acting in Ω_+ and Ω_-^ε with a rate of $\mathcal{O}(M^{-1})$ when M tends to ∞ and $\varepsilon \in (0, \varepsilon_0)$.

By combining Propositions 3.1.3, 3.1.4, we arrive at the following main result:

Theorem 3.1.5. *Let $z \in \rho(\mathscr{D}_L)$, then for M sufficiently large, $z \in \rho(\mathfrak{D}_M^\varepsilon)$, and $\varepsilon = M^{-1}$, the following holds:*

$$||R_M^\varepsilon(z) - R_L(z)||_{L^2(\mathbb{R}^3)^4} = \mathcal{O}\left(M^{-1}\right).$$

The methodology followed, as in the previous problem of [Chapter 2, Section 2.5] study the pseudodifferential properties of Poincaré-Steklov (PS) operators. The complexity in the current problem is that these operators take a pair of functions with respect to $\partial\mathcal{U}^\varepsilon := \Sigma \cup \Sigma^\varepsilon$ such that for all $x_\Sigma \in \Sigma$, we have $\Sigma^\varepsilon \ni x = x_\Sigma + \varepsilon\, n(x_\Sigma)$, where n is the unit normal to the surface Σ pointing outside Ω_+ (see

Figure 3.1). However, this complication becomes trivial if we fix the parameter ε, and consequently, the results of this chapter become equivalent to those of Chapter 2.

The most important ingredient in proving Proposition 3.1.3 is the use of the Krein formula of the resolvents of $\mathscr{D}_L$ and both MIT bag operators, $D^{\Omega_+}_{\mathrm{MIT}}$ and $D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$ (see Section 3.3.2), acting in $L^2(\Omega_+)^4$ and $L^2(\Omega^\varepsilon_-)^4$, respectively. Then, in Proposition 3.4.1, we establish that the convergence $(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}} - z)^{-1}$ toward $(D^{\Omega_-}_{\mathrm{MIT}} - z)^{-1}$ holds for any non-real z, when ε goes to 0, and we then obtain, in the norm resolvent sense, the convergence of $D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} := D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$ to $\mathscr{D}_L = D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega_-}_{\mathrm{MIT}}$.

The key point to establish the result of Proposition 3.1.4 is to treat the elliptic problem $(\mathfrak{D}^\varepsilon_M - z)\mathfrak{U} = f \in L^2(\mathbb{R}^3)^4$ as a transmission problem (where $P_\pm t_\Sigma \mathfrak{U}_{|\Omega_+} = P_\pm t_\Sigma \mathfrak{U}_{|\mathcal{U}^\varepsilon}$ and $P^\varepsilon_\pm t_{\Sigma^\varepsilon} \mathfrak{U}_{|\Omega^\varepsilon_-} = P^\varepsilon_\pm t_{\Sigma^\varepsilon} \mathfrak{U}_{|\mathcal{U}^\varepsilon}$ are the transmission conditions) and to use the semiclassical properties of the auxiliary operator $\Upsilon^\varepsilon_M(z)$ acting on the boundary $\partial\mathcal{U}^\varepsilon = \Sigma \cup \Sigma^\varepsilon$, which is constructed by the Poincaré-Steklov operators (see (3.57) for the exact notation). Indeed, in Section 3.4, we show convergence of the Dirac operator, $\mathfrak{D}^\varepsilon_M$, to both MIT bag operators, $D^{\Omega_+}_{\mathrm{MIT}}$ and $D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, with a convergence rate of $\mathcal{O}(M^{-1})$ for $M = \varepsilon^{-1}$ sufficiently large. Consequently, using these ingredients, a kind of convergence can be established in Theorem 3.1.5 for $\varepsilon = M^{-1}$.

Unlike the application in [Chapter 2, Theorem 2.5.2], we mention that in this problem the operator Υ^ε_M (which is constructed by the Poincaré-Steklov operators) takes a pair of functions with respect to $\partial\mathcal{U}^\varepsilon$.

We recall, $P^\varepsilon_\pm$ and $P_\pm$ are the orthogonal projections with respect to N^ε and n, respectively, defined by

$$P^\varepsilon_\pm := (\mathbb{I}_4 \mp i\beta\alpha \cdot N^\varepsilon)/2 \quad \text{and} \quad P_\pm := (\mathbb{I}_4 \mp i\beta\alpha \cdot n)/2. \tag{3.6}$$

We end this part with the following remark on the projections $P_\pm$ and $P^\varepsilon_\pm$:

Remark 3.1.2. *We define the diffeomorphism $p : \Sigma \longrightarrow \Sigma^\varepsilon$ such that for all $x_\Sigma \in \Sigma$, we get $p(x_\Sigma) := x_\Sigma + \varepsilon\, n(x_\Sigma) = x$. Then, we have*

$$N^\varepsilon(x) = -(\,n \circ p^{-1})(x) = -\,n(x_\Sigma),$$

with

$$P^\varepsilon_\pm(x) = \frac{1}{2}\left(\mathbb{I}_4 \mp i\beta\alpha \cdot N^\varepsilon_+(x)\right) = \frac{1}{2}\left(\mathbb{I}_4 \pm i\beta\alpha \cdot n(x_\Sigma)\right) := P_\mp \circ p^{-1}(x) = P_\mp(x_\Sigma).$$

3.1.1 Definition and some properties of the MIT bag operator.

Recall the definition of the perturbed Dirac operator $\mathfrak{D}^\varepsilon_M := D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon}$, where $\mathbb{1}_{\mathcal{U}^\varepsilon}$ is the characteristic function of $\mathcal{U}^\varepsilon$. Then, we consider the MIT bag operator, $D^{\Omega_+}_{\mathrm{MIT}}(m)$ and $D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(m)$, acting in Ω_+ and Ω^ε_-, respectively, and defined on the following domains

$$D^{\Omega_+}_{\mathrm{MIT}}(m)v_+ = D_m v_+, \quad \forall v_+ \in \mathrm{Dom}(D^{\Omega_+}_{\mathrm{MIT}}(m)) = \{v_+ \in H^1(\Omega_+)^4, \quad P_- t_\Sigma v_+ = 0 \text{ on } \Sigma\},$$

$$D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(m)v^\varepsilon = D_m v^\varepsilon, \quad \forall v^\varepsilon \in \mathrm{Dom}(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(m)) = \{v^\varepsilon \in H^1(\Omega^\varepsilon_-)^4, \quad P^\varepsilon_- t_{\Sigma^\varepsilon} v^\varepsilon_- = 0 \text{ on } \Sigma^\varepsilon\}.$$

Then, let the MIT Dirac operator, $D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} = D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, acts in $\Omega^\varepsilon_{+-} := \Omega_+ \cup \Omega^\varepsilon_-$, and defined on the following domain

$$\mathrm{Dom}(D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}}) = \{v^\varepsilon = (v^\varepsilon_-, v_+) \in H^1(\Omega^\varepsilon_-)^4 \oplus H^1(\Omega_+)^4, \quad P^\varepsilon_- t_{\Sigma^\varepsilon} v^\varepsilon_- = 0 = P_- t_\Sigma v_+\},$$

with $D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} v^\varepsilon = (D_+ \oplus D_-)v^\varepsilon$; $D_+ = D_- = D_m$ for all $v^\varepsilon \in \mathrm{Dom}(D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}})$, and where the boundary condition holds in $H^{1/2}(\Sigma^\varepsilon)^4$ and $H^{1/2}(\Sigma)^4$, respectively. Here $P_\pm$ and $P^\varepsilon_\pm$ are the projections given in (3.6).

Finally, on $\mathcal{U}^\varepsilon$, we introduce the following Dirac auxiliary operator

$$D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(m+M)u^\varepsilon = D_{m+M}u^\varepsilon,$$
$$u^\varepsilon \in \mathrm{dom}(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(m+M)) = \{u^\varepsilon \in H^1(\mathcal{U}^\varepsilon)^4,\ P^\varepsilon_+ t_{\Sigma^\varepsilon} u^\varepsilon = 0 = P_+ t_\Sigma u^\varepsilon \text{ on } \partial\mathcal{U}^\varepsilon := \Sigma \cup \Sigma^\varepsilon\},$$

with $D_{m+M} = D_m + M\beta = -i\alpha \cdot \nabla + (m+M)\beta$. We note that $D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}$ is the MIT bag operator on $\mathcal{U}^\varepsilon$.

Theorem 3.1.6. *The operators* $(D^{\Omega_+}_{\mathrm{MIT}}, \mathrm{Dom}(D^{\Omega_+}_{\mathrm{MIT}}))$ $\Big($*resp.* $(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}, \mathrm{Dom}(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}))$ *and* $(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}, \mathrm{Dom}(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}))\Big)$ *are self-adjoint and we have*

$$(D^{\Omega_+}_{\mathrm{MIT}} - z)^{-1} = r_{\Omega_+}(D_m - z)^{-1} e_{\Omega_+} - \Phi^z_{m,+}(\Lambda^z_{+,m})^{-1} t_\Sigma (D_m - z)^{-1} e_{\Omega_+}, \quad \forall z \in \rho(D_m).$$

Moreover, the following statements hold true:

(i) $\mathrm{Sp}(D^{\Omega_+}_{\mathrm{MIT}}) = \mathrm{Sp}_{\mathrm{disc}}(D^{\Omega_+}_{\mathrm{MIT}}) \subset \mathbb{R} \setminus [-m, m]$. $\Big($*Similarly for* $D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}$ *for* $(m+M)$ *instead of* $m\Big)$.

(ii) $\mathrm{Sp}(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}) = \mathrm{Sp}_{\mathrm{ess}}(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}) = (-\infty, -m] \cup [m, +\infty)$. *Moreover, if* Ω^ε_- *is connected then* $\mathrm{Sp}(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}})$ *is purely continuous.*

(iii) Let $z \in \rho(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}})$ *be such that* $2|z| < (m+M)$*, then for all* $f \in L^2(\mathcal{U}^\varepsilon)^4$ *it holds that*

$$\left\| (D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} - z)^{-1} f \right\|_{L^2(\mathcal{U}^\varepsilon)^4} \lesssim M^{-1} \|f\|_{L^2(\mathcal{U}^\varepsilon)^4},$$

uniformly with respect to ε.

Proof. The proof of this theorem follows the same arguments as the proof of [Chapter 2 , Theorem 2.2.1], where the estimates are valid uniformly with respect to ε. ■

Definition 3.1.7. Let $z \in \rho(D_m) \cap \rho(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}})$, $g^\varepsilon \in P^\varepsilon_- H^{1/2}(\Sigma^\varepsilon)^4$, $g_+ \in P_- H^{1/2}(\Sigma)^4$ and $(h^\varepsilon, h_+) \in P^\varepsilon_+ H^{1/2}(\Sigma^\varepsilon)^4 \oplus P_+ H^{1/2}(\Sigma)^4$. We denote by $E_m(z) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega^+)^4$, respectively, $E^\varepsilon_m(z) : P^\varepsilon_- H^{1/2}(\Sigma^\varepsilon)^4 \to H^1(\Omega^\varepsilon_-)^4$ the unique solution of the boundary value problem:

$$\begin{cases} (D_m - z)v_+ = 0, & \text{in } \Omega_+, \\ P_- t_\Sigma v_+ = g_+, & \text{on } \Sigma, \end{cases} \tag{3.7}$$

$$\begin{cases} (D_m - z)v^\varepsilon_- = 0, & \text{in } \Omega^\varepsilon_-, \\ P^\varepsilon_- t_{\Sigma^\varepsilon_-} v^\varepsilon_- = g^\varepsilon, & \text{on } \Sigma^\varepsilon. \end{cases} \tag{3.8}$$

Similarly, we denote by $\mathcal{E}^{\varepsilon}_{m+M}(z) : P^{\varepsilon}_{+}H^{1/2}(\Sigma^{\varepsilon})^4 \oplus P_{+}H^{1/2}(\Sigma)^4 \to H^1(\mathcal{U}^{\varepsilon})^4$ the unique solution of the boundary value problem:

$$\begin{cases} (D_{m+M} - z)u^{\varepsilon} = 0, & \text{in } \mathcal{U}^{\varepsilon}, \\ P^{\varepsilon}_{+}t_{\Sigma^{\varepsilon}}u^{\varepsilon} = h^{\varepsilon}, & \text{on } \Sigma^{\varepsilon}, \\ P_{+}t_{\Sigma}u^{\varepsilon} = h_{+}, & \text{on } \Sigma. \end{cases} \tag{3.9}$$

Define the Poincaré-Steklov operators associated with the above problems by

$$\begin{aligned} \mathscr{A}_m(z) : P_{-}H^{1/2}(\Sigma)^4 &\to P_{+}H^{1/2}(\Sigma)^4 \\ g_{+} &\mapsto \mathscr{A}_m(z)g_{+} := P_{+}t_{\Sigma}E_m(z)P_{-}g_{+}, \\ \mathscr{A}^{\varepsilon}_m(z) : P^{\varepsilon}_{-}H^{1/2}(\Sigma^{\varepsilon})^4 &\to P^{\varepsilon}_{+}H^{1/2}(\Sigma^{\varepsilon})^4 \\ g^{\varepsilon}_{-} &\mapsto \mathscr{A}^{\varepsilon}_m(z)g^{\varepsilon} := P^{\varepsilon}_{+}t_{\Sigma}E_m(z)P^{\varepsilon}_{-}g^{\varepsilon}, \end{aligned}$$

$$\mathcal{A}^{\varepsilon}_{m+M}(z) : P_{+}H^{1/2}(\Sigma)^4 \oplus P^{\varepsilon}_{+}H^{1/2}(\Sigma^{\varepsilon})^4 \to P_{-}H^{1/2}(\Sigma)^4 \oplus P^{\varepsilon}_{-}H^{1/2}(\Sigma^{\varepsilon})^4, \quad \text{with}$$

$$\mathcal{A}^{\varepsilon}_{m+M}(h_{+}, h^{\varepsilon}) := \big(P_{-}t_{\Sigma}\mathcal{E}^{\varepsilon}_{m+M}(z)P_{+}, P^{\varepsilon}_{-}t_{\Sigma^{\varepsilon}}\mathcal{E}^{\varepsilon}_{m+M}(z)P^{\varepsilon}_{+}\big).$$

In particular, for $z \in \rho(D_m)$ we have the following explicit formulas

$$\mathscr{A}_m(z) = -P_{+}\beta(\beta/2 + \mathscr{C}_{z,m})^{-1}P_{-}, \quad \mathscr{A}^{\varepsilon}_m(z) = -P^{\varepsilon}_{+}\beta(\beta/2 + \mathscr{C}^{\varepsilon}_{z,m})^{-1}P^{\varepsilon}_{-}.$$

where $\mathscr{C}_{z,m}$ resp. $\mathscr{C}^{\varepsilon}_{z,m}$ are the Cauchy operators associated with Σ resp. Σ^{ε}.

Remark 3.1.3. *We define the Poincaré-Steklov operator, $\mathtt{A}^{\varepsilon}_{m+M}$, as a part of the operator $\mathcal{A}^{\varepsilon}_{m+M}$, which is only associated with Σ^{ε} as follows:*

$$\begin{aligned} \mathtt{A}^{\varepsilon}_{m+M}(z) : P^{\varepsilon}_{+}H^{1/2}(\Sigma^{\varepsilon})^4 &\to P^{\varepsilon}_{-}H^{1/2}(\Sigma^{\varepsilon})^4 \\ h^{\varepsilon} &\mapsto \mathtt{A}^{\varepsilon}_{m+M}(z)h^{\varepsilon} := P^{\varepsilon}_{-}t_{\Sigma^{\varepsilon}}\mathcal{E}^{\varepsilon}_{m+M}(z)P^{\varepsilon}_{+}. \end{aligned}$$

In particular, $\mathtt{A}^{\varepsilon}_{m+M}$ will be used to establish the approximation in Section 3.2.

3.2 Parametrix for the Poincaré-Steklov operators (large mass limit)

Set $\kappa := (M + m)$. This section is devoted to study the (classical and semiclassical) pseudodifferential properties of the Poincaré-Steklov operator, $\mathcal{A}^{\varepsilon}_{\kappa}$, in order to use it in the application of Section 3.3. The main goal of this section is to study the Poincaré-Steklov operator, $\mathcal{A}^{\varepsilon}_{\kappa}$, as a κ-dependent pseudodifferential operator when κ is large enough. Roughly speaking, we will look for a local approximate formula for the solution of (3.9). The approximation in this section follows the steps of the one in [Chapter 2, Section 2.4], but since our elliptic problem (3.9), defined on the domain $\mathcal{U}^{\varepsilon}$, has two different boundary ($\partial\mathcal{U}^{\varepsilon} = \Sigma \cup \Sigma^{\varepsilon}$), and we have to take into account the dependence in ε, so we prefer to study rigorously the construction of the approximation. Once this is done, we use the regularization property of the resolvent of the MIT bag operator to catch the semiclassical principal symbol of $\mathcal{A}^{\varepsilon}_{\kappa}$. Throughout this section, we assume that $z \in \rho(D^{\mathcal{U}^{\varepsilon}}_{\text{MIT}}(\kappa))$.

We see that $\mathcal{U}^{\varepsilon}$ has two boundaries, Σ and Σ^{ε}. Since the approximation with respect to Σ has already been established in [BBZ37, Section 4], and we therefore have this result in the present problem, it is then sufficient to establish the approximation of $\mathcal{A}^{\varepsilon}_{\kappa}$ just with respect to Σ^{ε}. For this purpose, and for simplicity of notation, we set $\mathcal{A}^h := \mathtt{A}^\varepsilon_\kappa$ with $\varepsilon \equiv h := \kappa^{-1} \in (0,1]$ as the semiclassical parameter, where $\mathtt{A}^\varepsilon_\kappa$ is defined in Remark 3.1.3.

3.2.1 Reduction to local coordinates

Let us consider $\mathbb{A} = \{(U_{\varphi_j}, V_{\varphi_j}, \varphi_j) : j \in \{1, \cdots, N\}\}$ an atlas of Σ and $(U_\varphi, V_\varphi, \varphi) \in \mathbb{A}$. We consider also the case where U_φ is the graph of a smooth function χ, and we assume that Ω^ε_- corresponds locally to the side $x_3 > \chi(x_1, x_2)$. Then, for

$$\begin{aligned} U_\varphi =&\{(x^1_\Sigma, x^2_\Sigma, \chi(x^1_\Sigma, x^2_\Sigma));\ (x^1_\Sigma, x^2_\Sigma) \in V_\varphi\}; \quad \varphi((x^1_\Sigma, x^2_\Sigma, \chi(x^1_\Sigma, x^2_\Sigma)) = (x^1_\Sigma, x^2_\Sigma),\\ \mathcal{V}_{\varphi,\eta} :=&\{(y_1, y_2, y_3 + \chi(y_1, y_2));\ (y_1, y_2, y_3) \in V_\varphi \times (0, \eta)\} \subset \Omega_+, \end{aligned}$$

with η sufficiently small, we have the following homeomorphism:

$$\begin{aligned} \phi : \mathcal{V}_{\varphi,\eta} &\longrightarrow V_\varphi \times (\varepsilon, \eta)\\ (x^1_\Sigma, x^2_\Sigma, x^3_\Sigma) &\mapsto (x^1_\Sigma, x^2_\Sigma, x^3_\Sigma - \chi(x^1_\Sigma, x^2_\Sigma)), \end{aligned}$$

and the pull-back

$$\begin{aligned} \phi^* : C^\infty(V_\varphi \times (\varepsilon, \eta)) &\longrightarrow C^\infty(\mathcal{V}_{\varphi,\eta})\\ v &\mapsto \phi^* v := v \circ \phi. \end{aligned}$$

Now, using the coordinates in (3.3), we let the diffeomorphism $\phi_\varepsilon : C^\infty(\mathcal{V}_{\varphi,\eta}) \longrightarrow (\mathcal{V}^\varepsilon_{\varphi,\eta})$ defined by follows:

$$\phi_\varepsilon(x_1, x_2, x_3) := \phi(x^1_\Sigma, x^2_\Sigma, x^3_\Sigma) + \varepsilon\, n(\phi(x_\Sigma)) = (x^1_\Sigma + \varepsilon\, n_1, x^2_\Sigma + \varepsilon\, n_2, x^3_\Sigma + \varepsilon\, n_3 - \chi(x^1_\Sigma, x^2_\Sigma)),$$

with $\tilde{y} = (y_1, y_2)$ and n the outward pointing normal to Ω_+. Now, let $n^\varphi = (\varphi^{-1})^*\, n$ be the pull-back of the outward pointing normal to Ω_+ restricted on V_φ:

$$n^\varphi(\tilde{y}) = \frac{1}{\sqrt{1 + |\nabla\chi|^2}} \begin{pmatrix} -\partial_{x_1}\chi \\ -\partial_{x_2}\chi \\ 1 \end{pmatrix} (y_1, y_2) =: \begin{pmatrix} n^\varphi_1 \\ n^\varphi_2 \\ n^\varphi_3 \end{pmatrix}.$$

Then, the pull-back $(\phi_\varepsilon^{-1})^*$ transforms the differential operator D_m restricted on $\mathcal{V}_{\varphi,\eta}$ into the following operator on $V_\varphi \times (0, \eta)$:

$$\begin{aligned} \widetilde{D}^\varphi_m &:= (\phi_\varepsilon^{-1})^* D_m (\phi_\varepsilon)^*\\ &= -i\,(\alpha_1\partial_{y_1} + \alpha_2\partial_{y_2} - (-\alpha_1\partial_{x_1}\chi - \alpha_2\partial_{x_2}\chi + \alpha_3)\partial_{y_3}) + m\beta - i\varepsilon\,[c_1\partial_{y_1} + c_2\partial_{y_2} + c_3\partial_{y_3}]\\ &= -i(\alpha_1\partial_{y_1} + \alpha_2\partial_{y_2}) + \sqrt{1 + |\nabla\chi|^2}(i\alpha\cdot n^\varphi)(\tilde{y})\partial_{y_3} - i\varepsilon\,[c_1\partial_{y_1} + c_2\partial_{y_2} + c_3\partial_{y_3}] + m\beta, \end{aligned}$$

where $c_\bullet$ are 4×4 matrices having the form $c_\bullet = (\alpha_1\partial_{x_1} + \alpha_2\partial_{x_2})\, n^\varphi_\bullet$, for $\bullet = 1, 2, 3$.

Thus, in the variable $y \in V_\varphi \times (\varepsilon, \eta)$ for $0 < \varepsilon < \eta$, the system (3.9) becomes:

$$\begin{cases} (\widetilde{D}^\varphi_\kappa - z)u = 0, & \text{in } V_\varphi \times (\varepsilon, +\infty),\\ \Gamma^\varphi_- u = g^\varphi = g \circ \varphi^{-1}, & \text{on } V_\varphi \times \{\varepsilon\}, \end{cases} \tag{3.10}$$

where $\Gamma_{\pm}^{\varphi} = P_{\pm}^{\varphi} t_{\{y_3=\varepsilon\}}$.

By isolating the derivative with respect to y_3, and using that $(i\alpha \cdot n^{\varphi})^{-1} = -i\alpha \cdot n^{\varphi}$, we get

$$\partial_{y_3} u = \left(\mathbb{I}_4 - \frac{\varepsilon(\alpha \cdot n^{\varphi} c_3)}{\sqrt{1+|\nabla\chi|^2}}\right)^{-1} \frac{i\alpha \cdot n^{\varphi}(\tilde{y})}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Big(-i\alpha_1\partial_{y_1} - i\alpha_2\partial_{y_2} + m\beta - z - i\varepsilon c_1\partial_{y_1} - i\varepsilon c_2\partial_{y_2}\Big)u.$$

Since, $\dfrac{(\alpha \cdot n^{\varphi} c_3)}{\sqrt{1+|\nabla\chi|^2}}$ is a bounded linear operator, then for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small, the following Neumann series converges

$$\left(\mathbb{I}_4 - \frac{\varepsilon(\alpha \cdot n^{\varphi} c_3)}{\sqrt{1+|\nabla\chi|^2}}\right)^{-1} = \sum_{k=0}^{+\infty} \varepsilon^k \left(\frac{\alpha \cdot n^{\varphi} c_3}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\right)^k,$$

and we obtain

$$\begin{cases} \partial_{y_3} u = \displaystyle\sum_{k=0}^{+\infty} \varepsilon^k \left(\frac{\alpha \cdot n^{\varphi} c_3}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\right)^{k+1} \Big(-i\alpha_1\partial_{y_1} - i\alpha_2\partial_{y_2} + \kappa\beta - i\varepsilon c_1\partial_{y_1} - i\varepsilon c_2\partial_{y_2} - z\Big)u, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } V_{\varphi} \times (\varepsilon, +\infty), \\ \Gamma_{-}^{\varphi} u = g^{\varphi}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } V_{\varphi} \times \{\varepsilon\}. \end{cases}$$

Let us now introduce the matrices-valued symbols

$$L_0(\tilde{y}, \xi) := \frac{i\alpha \cdot n^{\varphi}(\tilde{y})}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Big(\alpha \cdot \xi + \beta\Big), \quad \text{and} \quad L_1(\tilde{y}) := \frac{i\alpha \cdot n^{\varphi}(\tilde{y})}{\sqrt{1+|\nabla\chi(\tilde{y})|^2}}\Big(c \cdot \xi - z\Big), \tag{3.11}$$

with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ identified with $(\xi_1, \xi_2, 0) \in \mathbb{R}^3$ and $c = (c_1, c_2)$. Then for $\varepsilon = h := 1/m$, the system (3.10) becomes:

$$\begin{cases} h\partial_{y_3} u^h = L_0(\tilde{y}, hD_{\tilde{y}})u^h + hL_1(\tilde{y}, hD_{\tilde{y}})u^h \\ \displaystyle + \sum_{k=1}^{+\infty} h^k \frac{(\alpha \cdot n^{\varphi} c_3)^k}{(1+|\nabla\chi|^2)^{k/2}} \Big(L_0(\tilde{y}, hD_{\tilde{y}})u^h + hL_1(\tilde{y}, hD_{\tilde{y}})\Big)u^h, \quad \text{in } \mathbb{R}^2 \times (\varepsilon, +\infty), \\ P_{+}^{\varphi} t_{\{y_3=\varepsilon\}} u^h = g^{\varphi}, \qquad\qquad\qquad\qquad \text{on } \mathbb{R}^2 \times \{\varepsilon\}. \end{cases} \tag{3.12}$$

Remark 3.2.1. *In this remark, we clarify the first difference in the approximation of this section compared to that of [BBZ37, Section 5]. Indeed, according to the formula of L_1 from (3.11), we observe that the term $c \cdot \xi$ appears in our case, whereas it was absent in the case of [BBZ37]. Moreover, we mention that this difference plays an important role in the subsequent progression of this approximation, exerting a significant impact on the symbol class of the solution u^h.*

Before constructing an approximate solution of the system (3.12), let us give some properties of L_0. Besides, we mention that L_1 also verifies these properties.

Lemma 3.2.1. *Recall the projections $P_\pm^\varphi := (\mathbb{I}_4 \mp i\beta\,\alpha\cdot n^\varphi(\tilde y))/2$, and set*

$$\gamma_5 := -i\alpha_1\alpha_2\alpha_3 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \quad \text{and} \quad S\cdot X = -\gamma_5(\alpha\cdot X), \quad \forall X\in\mathbb{R}^3. \tag{3.13}$$

Using the anticommutation relations of the Dirac's matrices we easily get the following identities

$$\begin{aligned} &i(\alpha\cdot X)(\alpha\cdot Y) = iX\cdot Y + S\cdot(X\wedge Y),\\ &\{S\cdot X, \alpha\cdot Y\} = -(X\cdot Y)\gamma_5, \quad [S\cdot X,\beta]=0, \quad \forall X,Y\in\mathbb{R}^3. \end{aligned}$$

Let n^φ and ξ be as above. Then, for any $z\in\mathbb{C}$ and any $\tau\in\mathbb{R}^3$ such that $\tau\perp n^\varphi$, the following identities hold:

$$(S\cdot\tau - im\beta(\alpha\cdot n^\varphi(\tilde y)))^2 = \left(|\tau|^2+m^2\right)\mathbb{I}_4,$$

$$P_\pm^\varphi(S\cdot\tau) = (S\cdot\tau)P_\mp^\varphi \quad \text{and} \quad P_\pm^\varphi(i\alpha\cdot n^\varphi) = (i\alpha\cdot n^\varphi)P_\mp^\varphi.$$

The next proposition gathers the main properties of the operator L_0.

Proposition 3.2.2. *[BBZ37, Proposition 5.1]. Let $L_0(\tilde y,\xi)$ be as in (3.11), then we have*

$$\begin{aligned} L_0(\tilde y,\xi) &= \frac{1}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}\Big(i\xi\cdot n^\varphi(\tilde y) + S\cdot(n^\varphi(\tilde y)\wedge\xi) - i\beta(\alpha\cdot n^\varphi(\tilde y))\Big)\\ &= i\xi\cdot\tilde n^\varphi(\tilde y) + \frac{\lambda(\tilde y,\xi)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}\Pi_+(\tilde y,\xi) - \frac{\lambda(\tilde y,\xi)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}\Pi_-(\tilde y,\xi), \end{aligned}$$

where

$$\begin{aligned} \lambda(\tilde y,\xi) &:= \sqrt{|n^\varphi(\tilde y)\wedge\xi|^2+1},\\ \tilde n^\varphi(\tilde y) &:= \frac{1}{\sqrt{1+|\nabla\chi|^2}}\,n^\varphi(\tilde y),\\ \Pi_\pm(\tilde y,\xi) &:= \frac12\left(\mathbb{I}_4 \pm \frac{S\cdot(n^\varphi(\tilde y)\wedge\xi) - i\beta(\alpha\cdot n^\varphi(\tilde y))}{\lambda(\tilde y,\xi)}\right). \end{aligned} \tag{3.14}$$

In particular, the symbol $L_0(\tilde y,\xi)$ is elliptic in symbol class $\mathcal{S}^1$ (defined in Section 2.1.1) and it admits two eigenvalues $\varrho_\pm(\cdot,\cdot)\in\mathcal{S}^1$ of multiplicity 2 which are given by

$$\varrho_\pm(\tilde y,\xi) = \frac{i\,n^\varphi(\tilde y)\cdot\xi \pm \lambda(\tilde y,\xi)}{\sqrt{1+|\nabla\chi|^2}},$$

and for which there exists $c>0$ such that

$$\pm\Re\varrho_\pm(\tilde y,\xi) > c\langle\xi\rangle, \tag{3.15}$$

uniformly with respect to $\tilde y$. Moreover, $\Pi_\pm(\tilde y,\xi)$ are the projections onto $\mathrm{Kr}(L_0(\tilde y,\xi)-\varrho_\pm(\tilde y,\xi)\mathbb{I}_4)$, belong to the symbol class $\mathcal{S}^0$ and satisfy:

$$P_\pm^\varphi\,\Pi_\pm(\tilde y,\xi)\,P_\pm^\varphi = k_+^\varphi(\tilde y,\xi)P_\pm^\varphi \quad \text{and} \quad P_\pm^\varphi\,\Pi_\mp(\tilde y,\xi)\,P_\mp^\varphi = \mp\Theta^\varphi(\tilde y,\xi)P_\mp^\varphi, \tag{3.16}$$

with

$$k_{\pm}^{\varphi}(\tilde{y},\xi) = \frac{1}{2}\left(1 \pm \frac{1}{\lambda(\tilde{y},\xi)}\right), \quad \Theta^{\varphi}(\tilde{y},\xi) = \frac{1}{2\lambda(\tilde{y},\xi)}\left(S\cdot(n^{\varphi}(\tilde{y})\wedge\xi)\right).$$

Now, using Lemma 3.2.1 and the properties (3.14), a simple computation shows that

$$P_{+}^{\varphi}\Pi_{\pm} = k_{\pm}^{\varphi}P_{+}^{\varphi} \pm \frac{1}{2\lambda}\left(S\cdot(n^{\varphi}(\tilde{y})\wedge\xi)\right)P_{-}^{\varphi},$$
$$P_{-}^{\varphi}\Pi_{\pm} = k_{\mp}^{\varphi}P_{-}^{\varphi} \pm \frac{1}{2\lambda}\left(S\cdot(n^{\varphi}(\tilde{y})\wedge\xi)\right)P_{+}^{\varphi}.$$

That is, k_{+}^{φ} is a positive function of $\mathcal{S}^0$, $(k_{+}^{\varphi})^{-1}\in\mathcal{S}^0$ and $\Theta^{\varphi}\in\mathcal{S}^0$ where $\mathcal{S}^0$ is zero-order symbol class defined in Section 2.1.1.

3.2.2 Semiclassical parametrix for the boundary problem

In this section, we construct the approximate solution of the system (3.12). For simplicity of notation, in the sequel we will use y, τ, and $P_{\pm}$ instead of $\tilde{y}$, y_3, and $P_{\pm}^{\varphi}$, respectively. We are going to construct a local approximate solution of the following first order system:

$$\begin{cases} h\partial_{y_3}u^h = L_0(y,hD_y)u^h + hL_1(y,hD_y)u^h \\ \quad + \displaystyle\sum_{k=1}^{+\infty} h^k \frac{(\alpha\cdot n^{\varphi}c_3)^k}{(1+|\nabla\chi|^2)^{k/2}}\Big(L_0(y,hD_y)u^h + hL_1(y,hD_y)\Big)u^h, & \text{in } \mathbb{R}^2\times(\varepsilon,+\infty), \\ \\ P_+ t_{\{\tau=\varepsilon\}}u^h = g^{\varphi}, & \text{on } \mathbb{R}^2\times\{\varepsilon\}. \end{cases}$$

This system is equivalent to

$$\begin{cases} h\partial_{y_3}u^h = L_0(y,hD_y)u^h + \displaystyle\sum_{k=1}^{+\infty} h^k \frac{(\alpha\cdot n^{\varphi}c_3)^{k-1}}{(1+|\nabla\chi|^2)^{\frac{k-1}{2}}}\widetilde{L}_1(y,hD_y)u^h, & \text{in } \mathbb{R}^2\times(\varepsilon,+\infty), \\ P_+ t_{\{\tau=\varepsilon\}}u^h = g^{\varphi}, & \text{on } \mathbb{R}^2\times\{\varepsilon\}, \end{cases} \tag{3.17}$$

with $\widetilde{L}_1(y,\xi) = L_1(y,\xi) + (\alpha\cdot\tilde{n}^{\varphi}c_3)L_0(y,\xi)$.

To be precise, we will look for a solution u^h in the following form:

$$u^h(y,\tau) = Op^h(A^h(\cdot,\cdot,\tau))f = \int_{\mathbb{R}^2} A^h(y,h\xi,\tau)e^{iy\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi,$$

with $A^h(\cdot,\cdot,\tau)\in\mathcal{S}^0$ for any $\tau>0$ constructed inductively in the form:

$$A^h(y,\xi,\tau) \sim \sum_{j\geqslant 0} h^j A_j(y,\xi,\tau).$$

The action of $h\partial_{y_3} - L_0 - \sum_{k=1}^{+\infty} h^k \dfrac{(\alpha\cdot n^{\varphi}c_3)^{k-1}}{(1+|\nabla\chi|^2)^{\frac{k-1}{2}}}\widetilde{L}_1$ on $A^h(y,hD_y,\tau)f$ is given by $T^h(y,hD_y,\tau)f$,

with

$$\begin{aligned}T^h(y,\xi,\tau) = h(\partial_\tau A^h)(y,\xi,\tau) - L_0(y,\xi)A^h(y,\xi,\tau)\\ - h\Big(\widetilde{L}_1(y,\xi)A^h(y,\xi,\tau) - i\partial_\xi L_0(y,\xi)\cdot\partial_y A^h(y,\xi,\tau)\Big)\\ - h^2\Big(L_0A^h + \widetilde{L}_1(y,\xi)A^h + \partial_\xi L_0\cdot\partial_y A^h - i\partial_\xi\widetilde{L}_1\cdot\partial_y A^h + (\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1(y,\xi)A^h\Big) + \dots.\end{aligned}$$

Then, by identifications of the coefficients of j, $j\geqslant 0$, we look for A_0 satisfying:

$$\begin{cases} h\partial_\tau A_0(y,\xi,\tau) = L_0(y,\xi)A_0(y,\xi,\tau),\\ P_+(y)A_0(y,\xi,\varepsilon) = P_+(y),\end{cases} \tag{3.18}$$

and for $j\geqslant 1$,

$$\begin{cases} h\partial_\tau A_j(y,\xi,\tau) = L_0(y,\xi)A_j(y,\xi,\tau) + \Big(\widetilde{L}_1(y,\xi) - i\partial_\xi L_0(y,\xi)\cdot\partial_y\Big)A_{j-1}(y,\xi,\tau)\\ + \displaystyle\sum_{l\geqslant 2}^{l=j}(\alpha\cdot\tilde{n}^\varphi c_3)^{j-l}\Big((\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1(y,\xi) - i\partial_\xi\widetilde{L}_1(y,\xi)\cdot\partial_y\Big)A_{l-2}(y,\xi,\tau),\\ \\ P_+(y)A_j(y,\xi,\varepsilon) = 0.\end{cases} \tag{3.19}$$

Let us introduce a class of parametrized symbols, in which we will construct the family A_j:

$$\mathcal{P}_h^m := \{b(\cdot,\cdot,\tau)\in\mathcal{S}^m;\ \forall(k,l)\in\mathbb{N}^2,\ \tau^k\partial_\tau^l b(\cdot,\cdot,\tau)\in h^{k-l}\mathcal{S}^{m-k+l}\};\quad m\in\mathbb{Z}.$$

Proposition 3.2.3. *There exists $A_0\in\mathcal{P}_h^0$ solution of (3.18) given by:*

$$\begin{aligned}A_0(y,\xi,\tau) &= e^{h^{-1}(\tau-\varepsilon)\varrho_-(y,\xi)}\ \frac{\Pi_-(y,\xi)P_+(y)A_0(y,\xi,\varepsilon)}{k_-^\varphi(y,\xi)}\\ &= e^{h^{-1}(\tau-\varepsilon)\varrho_-(y,\xi)}\frac{\Pi_-(y,\xi)P_+(y)}{k_-^\varphi(y,\xi)}\\ &= e^{h^{-1}(\tau-\varepsilon)\varrho_-(y,\xi)}\left(\mathbb{I}_4 - \frac{\Theta^\varphi}{k_-^\varphi}\right)P_+.\end{aligned}$$

Proof. The proof follows the same argument as [BBZ37, Proposition 5.2]. The solution of the differential system $h\partial_\tau A_0 = L_0A_0$ is $A_0(y,\xi,\tau) = e^{h^{-1}(\tau-\varepsilon)L_0}A_0(y,\xi,\varepsilon)$. By definition of $\varrho_\pm$ and $\Pi_\pm$, we have:

$$e^{h^{-1}\tau L_0} = e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Pi_-(y,\xi) + e^{h^{-1}(\tau-\varepsilon)\varrho_+}\Pi_+(y,\xi). \tag{3.20}$$

It follows from (3.15) that A_0 belongs to $\mathcal{S}^0$ for any $\tau>\varepsilon$ if and only if $\Pi_+(y,\xi)A_0(y,\xi,\varepsilon) = 0$. Moreover, the boundary condition $P_+A_0 = P_+$ implies $P_+(y)A_0(y,\xi,\varepsilon) = P_+(y)$. Thus, we deduce that

$$A_0(y,\xi,\varepsilon) = P_+(y) - \frac{P_-\Pi_+P_+}{k_-^\varphi}(y,\xi) = P_+(y) + \frac{P_-\Pi_-P_+}{k_-^\varphi}(y,\xi) = \frac{\Pi_-P_+}{k_-^\varphi}(y,\xi).$$

The properties of ϱ_-, Π_-, P_- and k_+ given in Proposition 3.2.2, imply that $(k_+^\varphi)^{-1}\Pi_-P_-\in\mathcal{S}^0$ and that $e^{h^{-1}\tau\varrho_-(y,\xi)}\in\mathcal{P}_h^0$. This concludes the proof of Proposition 3.2.3. ■

Proposition 3.2.4. *Let A_0 be defined by Proposition 3.2.3. Then for any $j \geqslant 1$, there exists A_j solution of (3.19) which has the form:*

$$A_j(y,\xi,\tau) = e^{h^{-1}(\tau-\varepsilon)\varrho_-(y,\xi)} \sum_{k=0}^{2j} (h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k B_{j,k}(y,\xi), \quad \text{with} \quad B_{j,k} \in h\,\mathcal{S}^0. \tag{3.21}$$

Remark 3.2.2. *An important difference in the approximation between the solution A_j resulting from this work and the solution presented in the work [BBZ37, Proposition 5.3] lies in the order of the standard symbol class $\mathcal{S}^m$. Indeed, by referring to the form of A_2 (see (3.76) from Appendix 3.5) one can deduce that the optimal order of the term $\Pi_- a_0\Big(P_+ - \dfrac{P_+\Theta^\varphi}{k_-^\varphi} + \Pi_+ a_0\Big)$ in $B_{2,0}$ is in $h\,\mathcal{S}^0$, and this property is reflected in the construction of A_j for $j \geqslant 3$. However, in [BBZ37, Proposition 5.3], it was possible to obtain all A_j in $h^j\mathcal{S}^{-j}$. This discrepancy leads us to deduce the following propositions concerning the solutions A_j.*

Remark 3.2.3. *We mention that this difference in the symbol class of terms $B_{j,k}$ with that obtained in [BBZ37] is mainly due to the difference discussed in Remark 3.2.1,* i.e., *to the influence of $c\cdot\xi$ as presented in the formula of L_1 in system (3.12), and subsequently to that mentioned in Remark 3.2.2.*

***Proof of Proposition* 3.2.4.** For initialization and calculation of A_1 and A_2, see Appendix 3.5. So, for A_j with $j \geqslant 1$, it is sufficient to prove the induction step. Thus, assume that the A_j solution of (3.19) satisfies the above property and let us prove that the same holds for A_{j+1}. In order to be a solution to the differential system

$$\begin{aligned} h\partial_\tau A_{j+1}(y,\xi,\tau) &= L_0(y,\xi)A_{j+1}(y,\xi,\tau) + \Big(\widetilde{L}_1(y,\xi) - i\partial_\xi L_0(y,\xi)\cdot\partial_y\Big)A_j(y,\xi,\tau) \\ &\quad + \sum_{l=2}^{l=j+1} (\alpha\cdot\tilde{n}^\varphi c_3)^{j+1-l}\Big((\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1(y,\xi) - i\partial_\xi\widetilde{L}_1(y,\xi)\cdot\partial_y\Big)A_{l-2}(y,\xi,\tau), \end{aligned}$$

then, for A_{j+1} we have:

$$\begin{aligned} A_{j+1} &= e^{h^{-1}L_0(\tau-\varepsilon)}A_{j+1|\tau=\varepsilon} + e^{h^{-1}\tau L_0}\int_\varepsilon^\tau e^{-h^{-1}sL_0}\underbrace{\Big(\widetilde{L}_1 - i\partial_\xi L_0\cdot\partial_y\Big)A_j(y,\xi,\tau)}_{(a)}\,\mathrm{d}s \\ &\quad + e^{h^{-1}\tau L_0}\int_\varepsilon^\tau e^{-h^{-1}sL_0}\underbrace{\sum_{l=2}^{l=j+1}(\alpha\cdot\tilde{n}^\varphi c_3)^{j+1-l}\Big((\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1 - i\partial_\xi\widetilde{L}_1\cdot\partial_y\Big)A_{l-2}(y,\xi,\tau))}_{(b)}\,\mathrm{d}s \\ &:= e^{h^{-1}L_0(\tau-\varepsilon)}A_{j+1|\tau=\varepsilon} + e^{h^{-1}\tau L_0}\int_\varepsilon^\tau e^{h^{-1}sL_0}\Big(\boldsymbol{(a)} + \boldsymbol{(b)}\Big)\mathrm{d}s. \end{aligned} \tag{3.22}$$

In order to know the form of $\boldsymbol{(a)}$ and $\boldsymbol{(b)}$, let us consider the formula (3.74). Then for the quantity $\boldsymbol{(a)}$, we have

$$\partial_y A_j = e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Big(h^{-1}(\tau-\varepsilon)\partial_y\varrho_- + \partial_y\Big)\sum_{k=0}^{2j}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{j,k}.$$

Now, applying $(\widetilde{L}_1 - i\partial_\xi L_0 \cdot \partial_y)$ to $A_j(y,\xi,\tau)$:

$$(\widetilde{L}_1 - i\partial_\xi L_0 \cdot \partial_y)A_j = a_0(y)\big(-z + c\cdot\xi - ic_3L_0 - i\alpha\cdot\partial_y\big)e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2j}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{j,k}$$
$$:= \underbrace{e^{h^{-1}(\tau-\varepsilon)\varrho_-}a_0(y)\Big(-z + c_3\alpha\cdot\tilde{n}^\varphi\beta - i\alpha\cdot\partial_y\Big)\sum_{k=0}^{2j}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{j,k}}_{\text{(i)}}$$
$$+\underbrace{e^{h^{-1}(\tau-\varepsilon)\varrho_-}a_0(y)\Big(c + c_3\alpha\cdot\tilde{n}^\varphi\alpha\Big)\cdot\xi\sum_{k=0}^{2j}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{j,k}}_{\text{(ii)}}$$
$$+\underbrace{e^{h^{-1}(\tau-\varepsilon)\varrho_-}a_0(y)\Big(-ih^{-1}(\tau-\varepsilon)\alpha\cdot\partial_y\varrho_-\Big)\sum_{k=0}^{2j}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{j,k}}_{\text{(iii)}}.$$

Thanks to the properties of ϱ_- and $B_{j,k}$, (i), (ii) and (iii) have respectively the form:

$$\text{(i)} = \quad e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2j}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k B'_{j,k}(y,\xi), \tag{3.23}$$

$$\text{(ii)} = \quad e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2j}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\ \langle\xi\rangle\ \overline{B}_{j,k}(y,\xi), \tag{3.24}$$

$$\text{(iii)} = \quad e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2j}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^{k+1} B''_{j,k}(y,\xi), \tag{3.25}$$

with $B'_{j,k}$ and $B''_{j,k}$ verifying the properties of $B_{j,k}$, and $\langle\xi\rangle\ \overline{B}_{j,k} \in h\,\mathcal{S}^1$. Therefore, toghether (3.23), (3.24) and (3.25) give that

$$\textbf{(a)} = \quad e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2j+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k \widetilde{B}_{j,k}(y,\xi), \tag{3.26}$$

where $\widetilde{B}_{j,k}$ verifies

$$\widetilde{B}_{j,k} \in h\,\mathcal{S}^1 \text{ for } k = 0,...,2j, \text{ and } \widetilde{B}_{j,2j+1} \in h\,\mathcal{S}^0.$$

Similarly, to calculate $\boldsymbol{(b)}$, applying $\big(-i\partial_\xi\widetilde{L}_1\cdot\partial_y + (\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1\big)$ (see (3.74)) to the identity (3.21) yields that

$$\big(-i\partial_\xi\widetilde{L}_1\cdot\partial_y + (\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1\big)A_j =$$
$$e^{h^{-1}(\tau-\varepsilon)\varrho_-}a_0(y)\bigg(d + e\cdot\xi - ih^{-1}(y_3-\varepsilon)f\cdot\partial_y\varrho_-\bigg)\sum_{k=0}^{2j}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{j,k}, \tag{3.27}$$

with d, e and f defined in (3.75). Let us decompose $\boldsymbol{(b)}$ as the following

$$\sum_{l=2}^{l=j+1}(\alpha\cdot\tilde{n}^{\varphi}c_3)^{j+1-l}\Big(-i\partial_\xi\widetilde{L}_1\cdot\partial_y+(\alpha\cdot\tilde{n}^{\varphi}c_3)\widetilde{L}_1\Big)A_{l-2}(y,\xi,\tau)):=$$
$$\underbrace{(\alpha\cdot\tilde{n}^{\varphi}c_3)^{j-1}\Big(-i\partial_\xi\widetilde{L}_1\cdot\partial_y+(\alpha\cdot\tilde{n}^{\varphi}c_3)\widetilde{L}_1\Big)A_0(y,\xi))}_{(\mathrm{m1})}$$
$$+\underbrace{\sum_{l\geqslant 3}^{l=j+1}(\alpha\cdot\tilde{n}^{\varphi}c_3)^{j+1-l}\Big(-i\partial_\xi\widetilde{L}_1\cdot\partial_y+(\alpha\cdot\tilde{n}^{\varphi}c_3)\widetilde{L}_1\Big)A_{l-2}(y,\xi,\tau))}_{(\mathrm{m2})}.$$

Since $A_0\in\mathcal{S}^0$, this gives that

$$(\mathrm{m1})\quad=\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}+(-ih^{-1}(\tau-\varepsilon)f\cdot\partial_y\varrho_-)B_{0,0},\tag{3.28}$$

where $\dot{B}_{0,0},\ \widehat{B}_{0,0}\in\mathcal{S}^0$ are respectively the constants obtained by applying d and e to $\dfrac{\Pi_-P_+}{k_-^{\varphi}}$ and $f\cdot\partial_y\varrho_-\in\mathcal{S}^1$. Thus, $(\mathrm{m1})\in\mathcal{S}^1$, $\forall j\geqslant 1$.
In the other hand, and for all $l\geqslant 3$ (*i.e.*, $l-2\geqslant 1$), A_{l-2} has the form

$$A_{l-2}(y,\xi,\tau)=e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2(l-2)}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^kB_{l-2,k}(y,\xi),\tag{3.29}$$

with $B_{l-2,k}\in h\,\mathcal{S}^0$. Applying (3.27) to the identity (3.29) we get

$$(\mathrm{m2})\quad=e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{l\geqslant 3}^{l=j+1}\sum_{k=0}^{2(l-2)+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\ddot{B}_{j,k}(y,\xi),\tag{3.30}$$

with $\ddot{B}_{j,k}\in h\,\mathcal{S}^1$ and $\ddot{B}_{j,2(l-2)+1}\in h\,\mathcal{S}^0$. Therefore, for $i=(l-2)\geqslant 1$ and $j\geqslant 2$, toghether (3.28), (3.30) with (3.26) give that

$$\begin{aligned}\textbf{(a)}+\textbf{(b)}&=e^{h^{-1}(\tau-\varepsilon)\varrho_-}\left(\sum_{k=0}^{2j+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\widetilde{B}_{j,k}+\sum_{l\geqslant 3}^{l=j+1}\sum_{k=0}^{2(l-2)+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\ddot{B}_{l-2,k}+\mathrm{m1}\right)\\&=e^{h^{-1}(\tau-\varepsilon)\varrho_-}\left(\sum_{k=0}^{2j+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\widetilde{B}_{j,k}+\sum_{i\geqslant 1}^{i=j-1}\sum_{k=0}^{2i+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\ddot{B}_{i,k}+\mathrm{m1}\right)\\&=e^{h^{-1}(\tau-\varepsilon)\varrho_-}\left(\sum_{k=0}^{2i+1}(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k\underbrace{\left(\widetilde{B}_{i,k}+\sum_{i\geqslant 1}^{i=j-1}\ddot{B}_{i,k}\right)}_{C_{i,j,k}}+\mathrm{m1}\right),\end{aligned}\tag{3.31}$$

with $C_{i,j,k}\in h\,\mathcal{S}^1$, and $C_{i,j,k}\in h\,\mathcal{S}^0$ for $k=2i+1$.

So, using the decomposition (3.20), for the second term of the r.h.s. of (3.22) we have:

$$e^{h^{-1}\tau L_0}\int_\varepsilon^\tau e^{-h^{-1}sL_0}\Big(\boldsymbol{(a)}+\boldsymbol{(b)}\Big)\mathrm{d}s=e^{h^{-1}\tau\varrho_-}\Pi_-\mathbb{I}_-^j(\tau)+e^{h^{-1}\tau\varrho_+}\Pi_+\mathbb{I}_+^j(\tau),\tag{3.32}$$

with

$$\mathbb{I}^j_{\pm}(\tau) = e^{-h^{-1}\varepsilon\varrho_-} \int_\varepsilon^\tau e^{h^{-1}s(\varrho_- - \varrho_\pm)} \left(\sum_{k=0}^{2i+1} (h^{-1}(s-\varepsilon)\langle\xi\rangle)^k C_{i,j,k} \quad + \quad \mathrm{m}1 \right) \mathrm{d}s.$$

For $\mathbb{I}^j_-$, the exponential term is equal to 1 and by integration of s^k, we obtain:

$$\begin{aligned}
\mathbb{I}^j_-(\tau) = e^{-h^{-1}\varepsilon\varrho_-} \Bigg(\sum_{k=0}^{2i+1} (h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^{k+1} \frac{h\langle\xi\rangle^{-1}}{k+1} C_{i,j,k} \\
+ \Big((\tau-\varepsilon)(\xi\cdot\dot{B}_{0,0} + \widehat{B}_{0,0}\Big) - ih^{-1}(\tau-\varepsilon)^2 \frac{f\cdot\partial_y\varrho_-}{2} B_{0,0} \Bigg) \\
= e^{-h^{-1}\varepsilon\varrho_-} \sum_{k=0}^{2i+1} (h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^{k+1} \frac{h\langle\xi\rangle^{-1}}{k+1} C_{i,j,k} \\
+ e^{-h^{-1}\varepsilon\varrho_-} \Bigg((h^{-1}(\tau-\varepsilon)\langle\xi\rangle)\Big(h\langle\xi\rangle^{-1}\xi\cdot\dot{B}_{0,0} + h\langle\xi\rangle^{-1}\widehat{B}_{0,0}\Big) \\
- i\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^2 \frac{h\langle\xi\rangle^{-1} f\cdot\partial_y\varrho_-}{2} B_{0,0}\Bigg),
\end{aligned}$$

then $e^{h^{-1}\tau\varrho_-}\Pi_-\mathbb{I}^j_-(\tau)$ has the following form:

$$\begin{aligned}
e^{h^{-1}\tau\varrho_-}\Pi_-\mathbb{I}^j_-(\tau) = e^{h^{-1}(\tau-\varepsilon)\varrho_-} Pi_- \sum_{k=0}^{2i+1} \Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^{k+1} \frac{h\langle\xi\rangle^{-1}}{k+1} C_{i,j,k} \quad + \\
e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Pi_- \Bigg((h^{-1}(\tau-\varepsilon)\langle\xi\rangle)\Big(h\langle\xi\rangle^{-1}\xi\cdot\dot{B}_{0,0} + h\langle\xi\rangle^{-1}\widehat{B}_{0,0}\Big) \\
- i\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^2 \frac{h\langle\xi\rangle^{-1} f\cdot\partial_y\varrho_-}{2} B_{0,0}\Bigg).
\end{aligned} \tag{3.33}$$

For $\mathbb{I}^j_+$, let us introduce $\mathbb{P}_k$ the polynomial of degree k such that

$$\int_\varepsilon^\tau e^{\lambda s} s^k \mathrm{d}s = \frac{1}{\lambda^{k+1}}(e^{\tau\lambda}\mathbb{P}_k(\tau\lambda) - e^{\varepsilon\lambda}\mathbb{P}_k(0)), \quad \text{for any } \lambda \in \mathbb{C}^*.$$

Using the above formula, then we obtain:

$$\begin{aligned}
\mathbb{I}^j_+(\tau) = e^{-h^{-1}\varepsilon\varrho_-} \int_\varepsilon^\tau e^{h^{-1}s(\varrho_- - \varrho_+)} \left(\sum_{k=0}^{2i+1} (h^{-1}(s-\varepsilon)\langle\xi\rangle)^k C_{i,j,k} \quad + \quad \mathrm{m}1 \right) \mathrm{d}s \\
= e^{-h^{-1}\varepsilon\varrho_-} \sum_{k=0}^{2j+1} \frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}} \Big(e^{h^{-1}\tau(\varrho_- - \varrho_+)} \mathbb{P}_k(h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+)) \\
- e^{h^{-1}\varepsilon(\varrho_- - \varrho_+)}\mathbb{P}_k(0)\Big) C_{i,j,k}
\end{aligned}$$

$$+ e^{-h^{-1}\varepsilon\varrho_-} e^{h^{-1}(\varrho_- - \varrho_+)\tau}\Big[h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) + i\frac{(\tau-\varepsilon)}{\varrho_- - \varrho_+} f\cdot\partial_y\varrho_- B_{0,0} - i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big]$$
$$- e^{-h^{-1}\varepsilon\varrho_-} e^{h^{-1}(\varrho_- - \varrho_+)\varepsilon}\Big[h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) - i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big].$$

With this notation in hand, we easily see that the term $e^{h^{-1}\tau\varrho_+}\Pi_+\mathbb{I}_+^j(\tau)$ has the following form:

$$\begin{aligned}
&e^{h^{-1}\tau\varrho_+}\Pi_+\mathbb{I}_+^j(\tau) = \\
&\Pi_+\sum_{k=0}^{2i+1}\frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}} C_{i,j,k}\Big(e^{h^{-1}(\tau-\varepsilon)\varrho_-}\mathbb{P}_k(h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+)) - e^{h^{-1}(\tau-\varepsilon)\varrho_+}\mathbb{P}_k(0)\Big)\\
&+ e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Pi_+\Big[h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) + i\frac{(\tau-\varepsilon)}{\varrho_- - \varrho_+} f\cdot\partial_y\varrho_- B_{0,0} - i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big]\\
&- e^{h^{-1}(\tau-\varepsilon)\varrho_+}\Pi_+\Big[h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) - i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big].
\end{aligned} \tag{3.34}$$

Thus, combining (3.33) and (3.34) with (3.22), (3.32) and (3.20), yield that

$$\begin{aligned}
A_{j+1} &= e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Big[\Pi_- A_{j+1|\tau=\varepsilon} + \Pi_-\sum_{k=0}^{2i+1}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^{k+1}\frac{h\langle\xi\rangle^{-1}}{k+1}C_{i,j,k}\\
&+ \Pi_-\Big(\big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\big)\Big(h\langle\xi\rangle^{-1}\xi\cdot\dot{B}_{0,0} + h\langle\xi\rangle^{-1}\widehat{B}_{0,0}\Big) - i\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^2\frac{h\langle\xi\rangle^{-1}f\cdot\partial_y\varrho_-}{2}B_{0,0}\Big)\\
&+ \Pi_+\sum_{k=0}^{2i+1}\frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}}C_{i,j,k}\Big(\mathbb{P}_k(h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+))\Big)\Big]\\
&+ e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Pi_+\Big[h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) - i(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)\frac{h\langle\xi\rangle^{-1}f\cdot\partial_y\varrho_- B_{0,0}}{\varrho_- - \varrho_+}\\
&- i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big]\\
&+ e^{h^{-1}(\tau-\varepsilon)\varrho_+}\Big[\Pi_+ A_{j+1|\tau=\varepsilon} - \Pi_+\sum_{k=0}^{2i+1}\frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}}C_{i,j,k}\Big(\mathbb{P}_k(0)\Big)\Big]\\
&+ e^{h^{-1}(\tau-\varepsilon)\varrho_+}\Pi_+\Big(h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) - i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big).
\end{aligned} \tag{3.35}$$

We set

$$\widehat{B^+_{j+1}} := \Pi_+ \sum_{k=0}^{2i+1} \frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}} C_{i,j,k}\mathbb{P}_k(0) - \Pi_+\Big(h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) - i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2} f\cdot\partial_y\varrho_- B_{0,0}\Big) \tag{3.36}$$

belongs to $h\,\mathcal{S}^0$ as a linear combination of products of $\Pi_+ \in \mathcal{S}^0$, $h\langle\xi\rangle^k(\varrho_- - \varrho_+)^{-k-1} \in h\,\mathcal{S}^{-1}$, and of $C_{i,j,k}$ which verify the properties as in (3.31).
Now, in order to have $A_{j+1} \in \mathcal{S}^0$, we let the contribution of the exponentially growing term vanish by choosing

$$\Pi_+ A_{j+1}(y,\xi,\varepsilon) = \widehat{B^+_{j+1,k}}(y,\xi).$$

Then, we obtain

$$\begin{aligned} A_{j+1} = e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Big[&\Pi_- A_{j+1|\tau=\varepsilon} + \Pi_- \sum_{k=0}^{2i+1}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^{k+1}\frac{h\langle\xi\rangle^{-1}}{k+1}C_{i,j,k} \\ &+\Pi_-\Big((h^{-1}(\tau-\varepsilon)\langle\xi\rangle)\Big(h\langle\xi\rangle^{-1}\xi\cdot\dot{B}_{0,0}+h\langle\xi\rangle^{-1}\widehat{B}_{0,0}\Big) - i\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^2\frac{h\langle\xi\rangle^{-1}f\cdot\partial_y\varrho_-}{2}B_{0,0}\Big) \\ &+\Pi_+\sum_{k=0}^{2i+1}\frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}}C_{i,j,k}\Big(\mathbb{P}_k\big(h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+)\big)\Big)\Big] \\ &+ e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Pi_+\Big[h\Big(\frac{\xi\cdot\dot{B}_{0,0}+\widehat{B}_{0,0}}{\varrho_- - \varrho_+}\Big) - i(h^{-1}(\tau-\varepsilon)\langle\xi\rangle)\frac{h\langle\xi\rangle^{-1}f\cdot\partial_y\varrho_- B_{0,0}}{\varrho_- - \varrho_+} \\ &- i\frac{\varepsilon h}{(\varrho_- - \varrho_+)^2}f\cdot\partial_y\varrho_- B_{0,0}\Big], \end{aligned} \tag{3.37}$$

since the boundary condition $P_+(y)A_{j+1}(y,\xi,\varepsilon)=0$, gives

$$\Pi_- A_{j+1}(y,\xi,\varepsilon) = \Pi_-(P_+ + P_-)A_{j+1}(y,\xi,\varepsilon) = \Pi_- P_- A_{j+1}(y,\xi,\varepsilon),$$

using the formula of $A_{j+1}(y,\xi,\tau)$ above, we get that

$$P_- A_{j+1}(y,\xi,\varepsilon) = \frac{P_-\Pi_+}{k_-^{\varphi}}\widehat{B^+_{j+1,k}},$$

therefore

$$\Pi_- A_{j+1}(y,\xi,\varepsilon) = \frac{\Pi_- P_-\Pi_+}{k_-^{\varphi}}\widehat{B^+_{j+1,k}}. \tag{3.38}$$

In the other hand, regarding the following two series mentioned in (3.35)

$$\Pi_-\sum_{k=0}^{2i+1}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^{k+1}\frac{h\langle\xi\rangle^{-1}}{k+1}C_{i,j,k} + \Pi_+\sum_{k=0}^{2i+1}\frac{h\langle\xi\rangle^k}{(\varrho_- - \varrho_+)^{k+1}}C_{i,j,k}\Big(\mathbb{P}_k\big(h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+)\big)\Big), \tag{3.39}$$

by calculation, it is easy to verify that for all $j \geqslant 2$ (*i.e.*, $i \geqslant 1$), this quantity can be written as follows

$$\sum_{k=0}^{2(j+1)} \left(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\right)^k \widehat{B^-_{j+1,k}}, \tag{3.40}$$

such that $\widehat{B^-_{j+1,k}}$, as a linear combination, belong to $h^2\mathcal{S}^0$ for $k = 0, \ldots, 2j+1$ and $\widehat{B^-_{j+1,2(j+1)}} \in h^2\mathcal{S}^{-1}$.

Finally, the fact that we have the other terms (first and last) of the equality (3.37) of order $h\,\mathcal{S}^0$ and admit the same structure as that of the terms in (3.39), then thanks to (3.38), and (3.36), (3.40), together with (3.37) give that

$$A_{j+1}(y,\xi,\tau) = e^{h^{-1}(\tau-\varepsilon)\varrho_-(y,\xi)}\left(\frac{\Pi_- P_- \Pi_+}{k_-^{\varphi}}\widehat{B^+_{j+1}}(y,\xi) + \sum_{k=0}^{2(j+1)} (h^{-1}(\tau-\varepsilon)\langle\xi\rangle)^k \widehat{B^-_{j+1,k}}(y,\xi)\right),$$

where $\widehat{B^+_{j+1}}(y,\xi)$, $\widehat{B^-_{j+1,k}}(y,\xi)$ belong to $h\,\mathcal{S}^0$, and Proposition 3.2.4 is proven with

$$B_{j+1,0} = \frac{\Pi_- P_+ \Pi_+}{k_+^{\varphi}}\widehat{B^+_{j+1}} + \widehat{B^-_{j+1,0}}, \text{ and for } k \geqslant 1,\ B_{j+1,k} = \widehat{B^-_{j+1,k}}. \qquad \blacksquare$$

Proposition 3.2.5. *Let A_j, $j \geqslant 0$, be of the form (3.21). Then, for any $s \geqslant -\frac{1}{2}$, the operator $\mathcal{A}_j$ defined by*

$$\mathcal{A}_j : f \longmapsto (\mathcal{A}_j f)(y, y_3) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A_j(y, h\xi, y_3) e^{iy\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi$$

gives rise to a bounded operator from $H^s(\mathbb{R}^2)$ into $H^{s+\frac{1}{2}}(\mathbb{R}^2 \times (\varepsilon, +\infty))$. Moreover, for any $l \in [0, \frac{1}{2}]$ we have:

$$\|\mathcal{A}_j\|_{H^s \to H^{s+\frac{1}{2}-l}} = O(h^{l-|s|+1}).$$

Proof. The proof of this proposition follows exactly the arguments of [BBZ37, Proposition 5.4]. However, this difference obtained at the rate level on h is because of the presence of a parameter h in the terms $B_{j,k}$ of the solution A_j. $\blacksquare$

Proposition 3.2.6. *Let $f \in H^s(\mathbb{R}^2)$ and A_j, $j \geqslant 0$, be as in Propositions 3.2.3, 3.2.4. Then for any $N \in \mathbb{N}$, the function $u_N^h = \sum_{j=0}^N h^j \mathcal{A}_j f$ satisfies:*

$$\begin{cases} h^{N+1}\mathcal{R}_N^h f = h\partial_\tau u_N^h - L_0(y, hD_y)u_N^h \\ \qquad - h\displaystyle\sum_{k=1}^{+\infty} h^{k-1}\frac{(\alpha\cdot n^{\varphi}c_3)^{k-1}}{(1+|\nabla\chi|^2)^{\frac{k-1}{2}}}\widetilde{L}_1(y, hD_y)u_N^h, & \textit{in } \mathbb{R}^2\times(\varepsilon,+\infty), \\ P_+ u_N^h = f, & \textit{on } \mathbb{R}^2\times\{\varepsilon\}, \end{cases} \tag{3.41}$$

with

$$\mathcal{R}_N^h f = \frac{-1}{(2\pi)^2}\int_{\mathbb{R}^2}\left(\sum_{k=1}^{+\infty} h^k \frac{(\alpha\cdot n^{\varphi}c_3)^{k-1}}{(1+|\nabla\chi|^2)^{\frac{k-1}{2}}}\left(h^{-1}\widetilde{L}_1 A_N - i\partial_\xi\widetilde{L}_1\cdot\partial_y A_N\right) - i\partial_\xi L_0\cdot\partial_y A_N\right)e^{iy\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi,$$

a bounded operator from $H^s(\mathbb{R}^2)$ into $H^{s+\frac{1}{2}}(\mathbb{R}^2\times(\varepsilon,+\infty))$ satisfying for any $l\in[0,\frac{1}{2}]$:

$$\|\mathcal{R}_N^h\|_{H^s\to H^{s+\frac{1}{2}-l}} = O(h^{l-|s|+1}). \tag{3.42}$$

Proof. By construction of the sequence $(A_j)_{j\in\{0,\cdots,N-1\}}$ as in (3.17), we have the system (3.41) with $\mathcal{R}_N^h = Op^h(r_N^h(\cdot,\cdot,\tau))$, such that

$$r_N^h(y,\xi,\tau) = -\left(\sum_{k=1}^{+\infty} h^k\frac{(\alpha\cdot n^\varphi c_3)^{k-1}}{(1+|\nabla\chi|^2)^{\frac{k-1}{2}}}\left(h^{-1}\widetilde{L}_1A_N - i\partial_\xi\widetilde{L}_1\cdot\partial_yA_N\right) - i\partial_\xi L_0\cdot\partial_yA_N\right)(y,h\xi,\tau).$$

As in the proof of Proposition 3.2.4, $\widetilde{L}_1A_N$ has the form (3.24), and $\partial_\xi\widetilde{L}_1\cdot\partial_yA_N$ and $\partial_\xi L_0\cdot\partial_yA_N$ have the form (3.25). Then, r_N^h has the form (3.26) (with $j=N$). Therefore, as in the proof of Proposition 3.2.5, we obtain the estimate (3.42). ∎

Proposition 3.2.7. *Let us consider the Poincaré-Steklov operator $\mathcal{A}^h$ introduced at the beginning of Section 3.2. For $h=\varepsilon\in(0,1]$ and for all $N\in\mathbb{N}$, there is a h-pseudodifferential operator of order 0, $\mathcal{A}_N^h$ such that for h sufficiently small, we have the following estimate:*

$$||\mathcal{A}^h-\mathcal{A}_N^h||_{H^{1/2}(\Sigma^\varepsilon)\to H^{\frac{3}{2}-l}(\Sigma^\varepsilon)} = O(h^{2l+\frac{1}{2}}), \quad \textit{for any } l\in[0,\frac{1}{2}]. \tag{3.43}$$

Proof. The proof of this proposition follows the same argument of [BBZ37, Theorem 5.1]. That is a consequence of the above Proposition 3.2.5 and 3.2.6, combined with the regularity estimates from Theorem 2.2.1-(iii). More precisely, let $(U_\varphi^\varepsilon, V_\varphi^\varepsilon,\varphi^\varepsilon)$ a chart of an atlas $\mathbb{A}^\varepsilon$ of Σ^ε, and $\psi_1,\psi_2\in C_0^\infty(U_\varphi^\varepsilon)$. Let also $h^\varepsilon\in P_-H^{1/2}(\Sigma^\varepsilon)$ be such that $f^\varepsilon := (\varphi_\varepsilon^{-1})^*[\psi_2h^\varepsilon]\in H^{1/2}(V_\varphi^\varepsilon)^4$, which can be extended by 0 to a function of $H^{1/2}(\mathbb{R}^2)^4$. Then, for $\varepsilon=h=\kappa^{-1}$ and $N\in\mathbb{N}$, the previous construction provides a function $u_N^h\in H^1(\mathbb{R}^2\times(\varepsilon,+\infty))^4$ which verifies the following system

$$\begin{cases}(\widetilde{D}_\kappa^\varphi - z)u_N^h = h^{N+1}\mathcal{R}_N^h f^\varepsilon, & \text{in } \mathbb{R}^2\times(\varepsilon,+\infty),\\ P_-^\varphi t_{\Sigma^\varepsilon}u_N^h = f^\varepsilon, & \text{on } \mathbb{R}^2\times\{\varepsilon\},\end{cases}$$

where u_N^h, $\mathcal{R}_N^h$ are defined in Proposition 3.2.6. Moreover, from the latter, we know that $\mathcal{R}_N^h\in H^{N+1}(\mathbb{R}^2\times(\varepsilon,+\infty))$ with norm in H^{1-l}, $l\in[0,\frac{1}{2}]$, bounded by $O(h^{l+\frac{1}{2}})$. Consequently, $v_N^h := \phi_\varepsilon^* u_N^h$, defined on $\mathcal{V}_{\varphi,\eta}^\varepsilon$ satisfies

$$\begin{cases}(D_\kappa - z)v_N^h = h^{N+1}(\phi_\varepsilon^{-1})^*(\mathcal{R}_N^h f^\varepsilon), & \text{in } \mathcal{V}_{\varphi,\eta}^\varepsilon,\\ P_-t_{\Sigma^\varepsilon}v_N^h = \psi_2h^\varepsilon, & \text{on } U_\varphi^\varepsilon.\end{cases}$$

Recall the definition of the lifting operator $\mathcal{E}_\kappa^\varepsilon$, given in Definition 3.1.7. We have for $h^\varepsilon\in P_-H^{1/2}(\Sigma^\varepsilon)^4$, $\mathcal{E}_\kappa^\varepsilon[\psi_2h^\varepsilon]\in H^1(\mathcal{U}^\varepsilon)^4$. Since $P_-t_{\Sigma^\varepsilon}v_N^h = P_-t_{\Sigma^\varepsilon}\mathcal{E}_\kappa^\varepsilon[\psi_2h^\varepsilon] = \psi_2h^\varepsilon$, it follows that

$$v_N^h - \mathcal{E}_\kappa^\varepsilon[\psi_2h^\varepsilon] = h^{N+1}(D_{\text{MIT}}^\varepsilon(\kappa)-1)^{-1}(\phi_\varepsilon^{-1})^*(\mathcal{R}_N^h(\varphi_\varepsilon^{-1})^*[\psi_2h^\varepsilon]).$$

Thanks to the estimation of [BBZ37, Theorem 3.2-(i)], and also by continuing the steps of the proof of Theorem 5.1 in [BBZ37], we obtain that $\mathcal{A}_N^h\in h\,Op^h\mathcal{S}^0(\Sigma^\varepsilon)$ and the estimate (3.43) holds for any $l\in[0,\frac{1}{2}]$. ∎

At the end of this section, let's give some pseudodifferential properties of the Poincaré-Steklov operators, $\mathscr{A}_m$ and $\mathscr{A}_m^\varepsilon$, introduced in Definition 3.1.7, in order to use it in Section 3.3.

Remark 3.2.4. *We mention that the fixed Poincaré-Steklov operator $\mathscr{A}_m$ have been introduced and studied in details in [Chapter 2, Theorem 2.3.3]. Moreover, it is a pseudodifferential operator of order 0, which can be considered as a h-pseudodifferential operator, and whose semiclassical principal symbol (in local coordinate) is given by*

$$\mathscr{P}_{h,\mathscr{A}_m}(x_\Sigma,\xi) = \frac{S\cdot(\xi\wedge n(x_\Sigma))}{|\xi\wedge n(x_\Sigma)|}P_-, \quad \text{for any } x_\Sigma\in\Sigma.$$

For $\mathscr{A}_m^\varepsilon$, we have the following results:

Theorem 3.2.8. *Let $z\in\rho(D_m)$ and $x_\Sigma\in\Sigma$ and recall the definition of $\mathcal{T}_\varepsilon$ from Definition 3.1.2. We define the Cauchy operator $\mathscr{C}_m^{z,\varepsilon}: L^2(\Sigma^\varepsilon)^4 \longrightarrow L^2(\Sigma^\varepsilon)^4$ as the singular integral operator acting as*

$$\mathscr{C}_m^{z,\varepsilon}[g](x) := \lim_{\rho\searrow 0}\int_{|x-y|>\rho}\phi_m^z(x-y)g(y)\mathrm{d}\sigma(y), \quad \text{for } \mathrm{d}\sigma\text{-a.e.}, x = x_\Sigma+\varepsilon\, n(x_\Sigma)\in\Sigma^\varepsilon,\ g\in L^2(\Sigma^\varepsilon)^4.$$

Also, we consider the Poincaré-Steklov operator $\mathscr{A}_m^\varepsilon$ given in Definition 3.1.7. Then, $\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^\varepsilon\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^{-1}\mathscr{A}_m^\varepsilon\mathcal{T}_\varepsilon$ are homogeneous pseudodifferential operators of order 0, and we have

$$\begin{aligned}\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^\varepsilon\mathcal{T}_\varepsilon &= \det(1-\varepsilon W(x_\Sigma))\Big[\frac{1}{2}\alpha\cdot\frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}} + \varepsilon\, Op(b_0(x_\Sigma,\xi)) + Op(b_{-1}(x_\Sigma,\varepsilon\xi))\Big],\\ \mathcal{T}_\varepsilon^{-1}\mathscr{A}_m^\varepsilon\mathcal{T}_\varepsilon &= \det(1-\varepsilon W(x_\Sigma))\Big[S\cdot\frac{(\nabla_\Sigma\wedge n)}{\sqrt{-\Delta_\Sigma}}P_-^\varepsilon + \varepsilon\, Op(b_0'(x_\Sigma,\xi)) + Op(b_{-1}'(x_\Sigma,\varepsilon\xi))\Big],\end{aligned}$$

where $\nabla_\Sigma = \nabla - n(n\cdot\nabla)$ is the surface gradient along Σ, and $-\Delta_\Sigma$ is the Laplace-Beltrami operator, with b_0, b_0', resp. b_{-1}, b_{-1}' the symbols of order 0, resp. -1.

Proof. The proof follows similar arguments as in [BBZ37, Theorem 4.1]. Let $f\in L^2(\Sigma)^4$ and consider the operator $\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^\varepsilon\mathcal{T}_\varepsilon f$. Using the explicit formula of $\mathscr{A}_m^\varepsilon$, we have the following connection

$$L^2(\Sigma)^4 \ni \mathcal{T}_\varepsilon^{-1}\mathscr{A}_m^\varepsilon\mathcal{T}_\varepsilon f = -P_+^\varepsilon\beta\Big(\beta/2+\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^\varepsilon\mathcal{T}_\varepsilon f\Big)^{-1}P_-^\varepsilon.$$

Now, fix a local chart (U,V,φ) of Σ and let $\psi_k:\Sigma\longrightarrow\mathbb{R}$, $k=1,2$, be a C^∞-smooth function with $\mathrm{supp}(\psi_1)\cap\mathrm{supp}(\psi_2)=\emptyset$. For $x_\Sigma\in\Sigma$,

$$\begin{aligned}\Big(\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^\varepsilon\mathcal{T}_\varepsilon f\Big)(x_\Sigma) &= \det(1-\varepsilon W(x_\Sigma))\,\mathrm{p.v.}\int_{|x_\Sigma+\varepsilon\, n(x_\Sigma)-y|>\rho}\phi_m^z(x_\Sigma+\varepsilon\, n(x_\Sigma)-y)\mathcal{T}_\varepsilon f(y)\mathrm{d}\sigma(y)\\ &= \det(1-\varepsilon W(x_\Sigma))\,\mathrm{p.v.}\int_{|x_\Sigma-y_\Sigma|>\rho'}\phi_m^z(x_\Sigma+\varepsilon\, n(x_\Sigma)-y_\Sigma-\varepsilon\, n(y_\Sigma))f(y_\Sigma)\mathrm{d}\sigma(y_\Sigma)\\ &= \det(1-\varepsilon W(x_\Sigma))\int_V\phi_m^z(x_\Sigma-y_\Sigma+\varepsilon(n(x_\Sigma)-n(y_\Sigma)))f(y_\Sigma)\mathrm{d}\sigma(y_\Sigma).\end{aligned} \tag{3.44}$$

Now, recall the definition of ϕ_m^z from (1.11), and observe that

$$\phi_m^z(x-y) = k(x-y)+a(x-y),$$

where

$$k^z(x-y) = \frac{e^{i\sqrt{z^2-m^2}|x-y|}}{4\pi|x-y|}\Big(z + m\beta + \sqrt{z^2-m^2}\alpha\cdot\frac{x-y}{|x-y|}\Big) + i\frac{e^{i\sqrt{z^2-m^2}|x-y|}-1}{4\pi|x-y|^3}\alpha\cdot(x-y),$$
$$a(x-y) = \frac{i}{4\pi|x-y|^3}\alpha\cdot(x-y).$$

Using this, it follows that

$$\mathscr{C}_m^{z,\varepsilon}[g](x) = \lim_{\rho\searrow 0}\int_{|x-y|>\rho} a(x-y)g(y)\mathrm{d}\sigma(y) + \int_{\Sigma^\varepsilon} k^z(x-y)g(y)\mathrm{d}\sigma(y)$$
$$= A[g](x) + K[g](x).$$

As $|k^z(x-y)| = \mathcal{O}(|x-y|^{-1})$ when $|x-y|\to 0$, using the standard layer potential techniques (see, *e.g.* [Tay00, Chap. 3, Sec. 4] and [Tay96, Chap. 7, Sec. 11]) it is not hard to prove that the integral operator $\mathcal{T}_\varepsilon^{-1}K\mathcal{T}_\varepsilon$ gives rise to a pseudodifferential operator of order -1, *i.e.*, $\mathcal{T}_\varepsilon^{-1}K\mathcal{T}_\varepsilon \in Op\mathcal{S}^{-1}(\Sigma)$. Thus, we can (formally) write

$$\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^{\varepsilon}\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon^{-1}A\mathcal{T}_\varepsilon \mod Op\mathcal{S}^{-1}(\Sigma), \tag{3.45}$$

which means that the operator A encodes the main contribution in the pseudodifferential character of $\mathcal{T}_\varepsilon^{-1}\mathscr{C}_m^{z,\varepsilon}\mathcal{T}_\varepsilon$.

For $\Sigma^\varepsilon \ni x = x_\Sigma + \varepsilon\, n(x_\Sigma)$, $y = y_\Sigma + \varepsilon\, n(y_\Sigma)$,

$$a\big(x_\Sigma - y_\Sigma + \varepsilon(\,n(x_\Sigma) - \,n(y_\Sigma))\big) = i\alpha\cdot\frac{(x_\Sigma - y_\Sigma + \varepsilon(\,n(x_\Sigma) - \,n(y_\Sigma)))}{|x_\Sigma - y_\Sigma + \varepsilon(\,n(x_\Sigma) - \,n(y_\Sigma))|^3}.$$

Set $X = x_\Sigma - y_\Sigma$. Then, $|x_\Sigma - y_\Sigma + \varepsilon(\,n(x_\Sigma) - \,n(y_\Sigma))| = |X + \varepsilon\, nX|$. And $|X+\varepsilon\, nX|^{-3}$ yields

$$|X+\varepsilon\, nX|^{-3} = (1+\varepsilon^2)^{-3/2}|X|^{-3}\Big(1 + 2\varepsilon(1+\varepsilon^2)^{-1}\frac{\langle X,\, nX\rangle}{|X|^2}\Big)^{-3/2}.$$

By a series expansion (first order), we get

$$|X+\varepsilon\, nX|^{-3} = |X|^{-3} + \varepsilon\Big(-3|X|^{-3}\frac{\langle X,\, nX\rangle}{|X|^2}\Big).$$

For any $X\in U$ we have $X = (\widetilde{X}, \chi(\widetilde{X}))$ with $X\in V$ and where the graph of $\chi: V\longrightarrow\mathbb{R}$ coincides with U. With the same argument in [BBZ37, Theorem 4.1] we get that, uniformly with respect to $\varepsilon\in(0,\varepsilon_0)$, with ε_0 sufficiently small

$$|X+\varepsilon\, nX|^{-3} = \frac{1}{\langle \widetilde{X}, G(\widetilde{x}_\Sigma)\widetilde{X}\rangle^{3/2}} + k_1(\widetilde{X}),$$
$$\text{with } |k_1(\widetilde{X})| = \mathcal{O}(|\widetilde{X}|^{-2}) \text{ when } |\widetilde{X}|\longrightarrow 0,$$
$$|X+\varepsilon\, nX|^{-5}\langle X,\, nX\rangle = \frac{\langle \widetilde{X},\, n\widetilde{X}\rangle}{\langle \widetilde{X}, G(\widetilde{x}_\Sigma)\widetilde{X}\rangle^{5/2}} + \langle \widetilde{X},\, n\widetilde{X}\rangle k_2(\widetilde{X}),$$
$$\text{with } |k_2(\widetilde{X})| = \mathcal{O}(|\widetilde{X}|^{-4}) \text{ when } |\widetilde{X}|\longrightarrow 0,$$

where $G(\widetilde{x}_\Sigma)$ is the metric tensor. We deduce that

$$\psi_2(\mathcal{T}_\varepsilon^{-1} A\mathcal{T}_\varepsilon[\psi_1 f])(x_\Sigma) = \psi_2 Op(a_0(x_\Sigma,\xi))\psi_1 f(x_\Sigma) + \varepsilon\,\psi_2 Op(b_0(x_\Sigma,\xi))\psi_1 f(x_\Sigma) + \psi_2 L\psi_1, \tag{3.46}$$

with L a pseudodifferential operator of order -1. Thus, $\mathcal{T}_\varepsilon^{-1} A\mathcal{T}_\varepsilon$ is a zero-order pseudodifferential operator. Furthermore, thanks to (3.45) and (3.46) we get that $\mathcal{T}_\varepsilon^{-1}\mathscr{C}_m^{z,\varepsilon}\mathcal{T}_\varepsilon$ is a homogeneous pseudodifferential operator of order 0, with principal symbol given by

$$\mathcal{T}_\varepsilon^{-1}\mathscr{C}_{z,m}^{\varepsilon}\mathcal{T}_\varepsilon = \det(1-\varepsilon W(x_\Sigma))\Big[\frac{1}{2}\alpha\cdot\frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}} + \varepsilon\, Op(b_0(x_\Sigma,\xi)) + Op(b_{-1}(x_\Sigma,\varepsilon\xi))\Big].$$

Consequently, thanks to the relation between $\mathscr{C}_m^{z,\varepsilon}$ and $\mathscr{A}_m^{\varepsilon}$, we have that $\mathcal{T}_\varepsilon^{-1}\mathscr{A}_m^{\varepsilon}\mathcal{T}_\varepsilon$ is a homogeneous pseudodifferential operators of order 0

$$\mathcal{T}_\varepsilon^{-1}\mathscr{A}_m^{\varepsilon}\mathcal{T}_\varepsilon = \det(1-\varepsilon W(x_\Sigma))\Big[S\cdot\frac{(\nabla_\Sigma\wedge n(x_\Sigma))}{\sqrt{-\Delta_\Sigma}}P_-^{\varepsilon} + \varepsilon\, Op(b_0^{'}(x_\Sigma,\xi)) + Op(b_{-1}^{'}(x_\Sigma,\varepsilon\xi))\Big].$$

■

Corollary 3.2.9. *The Poincaré-Steklov operator* $\mathscr{A}_m^{\varepsilon}$ *is a homogeneous pseudodifferential operator of order* 0*, and we have that*

$$\begin{aligned}\mathscr{A}_m^{\varepsilon} &= S\cdot\frac{(\nabla_{\Sigma^\varepsilon}\wedge N^\varepsilon(p(x_\Sigma)))}{\sqrt{-\Delta_{\Sigma^\varepsilon}}}P_-^{\varepsilon} + \varepsilon\, Op(b_0^p(x_\Sigma,\xi)) + Op(b_{-1}^p(x_\Sigma,\varepsilon\xi))\\ &= -S\cdot\frac{(\nabla_{\Sigma^\varepsilon}\wedge n(x_\Sigma))}{\sqrt{-\Delta_{\Sigma^\varepsilon}}}P_-^{\varepsilon} + \varepsilon\, Op(b_0^p(x_\Sigma,\xi)) + Op(b_{-1}^p(x_\Sigma,\varepsilon\xi)), \quad \textit{with } \varepsilon\in(0,\varepsilon_0),\end{aligned}$$

where $\nabla_{\Sigma^\varepsilon}$ *is the surface gradient along* Σ^ε, $-\Delta_{\Sigma^\varepsilon}$ *is the Laplace-Beltrami operator, and* $b_j^p(x_\Sigma,\xi)$ *has the following form*

$$b_j^p(x_\Sigma,\xi) = b_j\Big(p(x_\Sigma), \big(\nabla p(x_\Sigma)^{-1}\big)^t\xi\Big), \quad \textit{for } j\in\{-1,0\},$$

with $p(x_\Sigma) = x_\Sigma + \varepsilon\, n(x_\Sigma)$ *the diffeomorphism from Definition 3.1.2, and* $\big(\nabla p(x_\Sigma)^{-1}\big)^t$ $= \big((1-\varepsilon W(x_\Sigma))^{-1}\big)^t = (1-\varepsilon W(x_\Sigma))^{-1}$*, where* $W(x_\Sigma)$ *is the Weingarten matrix, symmetric, given in Definition 1.5.2.*

Proof. The proof of this corollary is a consequence of Theorem 3.2.8 and the arguments of [Zwo12, Theorem 9.3]. ■

3.3 Reduction to a MIT bag problem.

Throughout the section, we denote Ω_+, Ω_-^ε and $\mathcal{U}^\varepsilon$ the domains as in Figure 3.1 such that $\Sigma=\partial\Omega_+$, $\Sigma^\varepsilon := \partial\Omega_-^\varepsilon$ and $\partial\mathcal{U}^\varepsilon = \Sigma\cup\Sigma^\varepsilon$, respectively, and we let N^ε be the outward pointing unit normal to Ω_-^ε. We set n the outward unit normal to the fixed domain $\Omega_+\subset\mathbb{R}^3$. Fix $m>0$ and let $M>0$. Remember our perturbed Dirac operator

$$\mathfrak{D}_M^{\varepsilon}\varphi = (D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon})\varphi, \quad \forall\varphi\in\operatorname{dom}(\mathfrak{D}_M^{\varepsilon}) := H^1(\mathbb{R}^3)^4,$$

where $\mathbb{1}_{\mathcal{U}^\varepsilon}$ is the characteristic function of $\mathcal{U}^\varepsilon$.

Let us now recall the definition of the MIT bag operator from Section 3.1.1 by $D^{\Omega_+}_{\mathrm{MIT}}$, $D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, and $D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}$, which act in $L^2(\Omega_+)^4$, $L^2(\Omega^\varepsilon_-)^4$, and $L^2(\mathcal{U}^\varepsilon)^4$ repsectively. The aim of this section is to use the properties of the Poincaré-Steklov operators carried out in the previous sections to study the resolvent of $\mathfrak{D}^\varepsilon_M$ when M is large enough. Namely, we give a Krein-type resolvent formula of the Dirac operator $\mathfrak{D}^\varepsilon_M$ in terms of the resolvent of the MIT bag operator $D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, and we show that the convergence of $\mathfrak{D}^\varepsilon_M$ toward $\mathscr{D}_L$ holds in the norm resolvent sense when M and ε converge to ∞ and 0^+, respectively. To set up Krein's formula between the resolvent of $\mathfrak{D}^\varepsilon_M$ and the resolvent of $D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, we will fix n the only normal acting in our domain. Throughout this section, the projections associated with the surface Σ^ε (*i.e.,* $P^\varepsilon_\pm(x)$, for $x \in \Sigma^\varepsilon$) verify the properties of Remark 3.1.2.

3.3.1 Notations

Let $z \in \rho(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}) \cap \rho(\mathfrak{D}^\varepsilon_M)$. We recall $\Omega^\varepsilon_{+-} := \Omega_+ \cup \Omega^\varepsilon_-$. We define the resolvents associated with the operators $\mathfrak{D}^\varepsilon_M$, $D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}$, and $D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} := D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, respectively, by

- $R^\varepsilon_M(z) := (\mathfrak{D}^\varepsilon_M - z)^{-1} : L^2(\mathbb{R}^3)^4 \to H^1(\mathbb{R}^3)^4$.
- $R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) := (D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} - z)^{-1} : L^2(\mathcal{U}^\varepsilon)^4 \to \mathrm{dom}(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}})$.
- $R^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}}(z) := (D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} - z)^{-1} : L^2(\Omega_+)^4 \oplus L^2(\Omega^\varepsilon_-)^4 \to \mathrm{dom}(D^{\Omega_+}_{\mathrm{MIT}}) \oplus \mathrm{dom}(D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}) \subset L^2(\Omega_+)^4 \oplus L^2(\Omega^\varepsilon_-)^4$

can be read as the following matrix:

$$\begin{aligned} R^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} = \begin{pmatrix} R^{\Omega_+}_{\mathrm{MIT}} r_{\Omega_+} & 0 \\ 0 & R^{\Omega^\varepsilon_-}_{\mathrm{MIT}} r_{\Omega^\varepsilon_-} \end{pmatrix} &\equiv r_{\Omega^\varepsilon_{+-}} e_{\Omega_+} R^{\Omega_+}_{\mathrm{MIT}} r_{\Omega_+} + r_{\Omega^\varepsilon_{+-}} e_{\Omega^\varepsilon_-} R^{\Omega^\varepsilon_-}_{\mathrm{MIT}} r_{\Omega^\varepsilon_-} \\ &= (R^{\Omega_+}_{\mathrm{MIT}} r_{\Omega_+}, R^{\Omega^\varepsilon_-}_{\mathrm{MIT}} r_{\Omega^\varepsilon_-}), \end{aligned} \tag{3.47}$$

where $R^{\Omega_+}_{\mathrm{MIT}}(z)$, $R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(z)$ are the resolvents of $D^{\Omega_+}_{\mathrm{MIT}}$, $D^{\Omega^\varepsilon_-}_{\mathrm{MIT}}$, respectively, and $r_{\Omega^\varepsilon_{+-}}$, $e_{\Omega^\varepsilon_{+-}}$ are defined below.

We define $r_{\Omega^\varepsilon_{+-}}$ and $e_{\Omega^\varepsilon_{+-}}$ as the restriction operator in Ω^ε_{+-} and its adjoint operator, *i.e.*, the extension by 0 outside of Ω^ε_{+-}, respectively, by

$$\begin{aligned} r_{\Omega^\varepsilon_{+-}} : L^2(\mathbb{R}^3)^4 &\to L^2(\Omega_+)^4 \oplus L^2(\Omega^\varepsilon_-)^4 \\ w &\mapsto r_{\Omega^\varepsilon_{+-}} w := (r_{\Omega_+} w \oplus r_{\Omega^\varepsilon_-} w) \equiv (r_{\Omega_+}, r_{\Omega^\varepsilon_-}) w, \\ e_{\Omega^\varepsilon_{+-}} : L^2(\Omega^\varepsilon_-)^4 \oplus L^2(\Omega_+)^4 &\to L^2(\mathbb{R}^3)^4 \\ v = (v^\varepsilon, v_+) &\mapsto e_{\Omega^\varepsilon_{+-}}(v^\varepsilon, v_+) := e_{\Omega^\varepsilon_-} v^\varepsilon + e_{\Omega_+} v_+. \end{aligned} \tag{3.48}$$

Let us recall for $z \in \rho(D_m)$, the lifting operators associated with boundary value problems (3.7), (3.8) and (3.9) are defined respectively, by

$$\begin{aligned} E_m(z) : P_- H^{1/2}(\Sigma)^4 &\to H^1(\Omega_+)^4 \\ g_+ &\mapsto E_m(z) g_+ := \Phi^z_m (\Lambda^z_{+,m})^{-1} P_-, \end{aligned}$$

$$\begin{aligned} E^\varepsilon_m(z) : P_+H^{1/2}(\Sigma^\varepsilon)^4 &\to H^1(\Omega^\varepsilon_-)^4 \\ g^\varepsilon &\mapsto E^\varepsilon_m(z)g^\varepsilon := \Phi^{z,\varepsilon}_m(\Lambda^{z,\varepsilon}_{+,m})^{-1}P_+, \end{aligned}$$

$$\mathcal{E}^\varepsilon_{m+M}(z) : P_+H^{1/2}(\Sigma)^4 \oplus P_-H^{1/2}(\Sigma^\varepsilon)^4 \to H^1(\mathcal{U}^\varepsilon)^4,$$

with $\mathcal{E}^\varepsilon_{m+M}(z)(h_+, h^\varepsilon) := \Phi^z_{m+M}(\Lambda^z_{+,m+M})^{-1}P_+h_+ + \Phi^{z,\varepsilon}_{m+M}(\Lambda^{z,\varepsilon}_{+,m+M})^{-1}P_-h^\varepsilon$.

In addition, we also recall the Poincaré-Steklov operators from Definition 3.1.7

$$\begin{aligned} \mathscr{A}_m(z) : P_-H^{1/2}(\Sigma)^4 &\to P_+H^{1/2}(\Sigma)^4 \\ g_+ &\mapsto \mathscr{A}_m(z)g_+ := -P_+\beta(\Lambda^z_{+,m})^{-1}P_-g_+, \\ \mathscr{A}^\varepsilon_m(z) : P_+H^{1/2}(\Sigma^\varepsilon)^4 &\to P_-H^{1/2}(\Sigma^\varepsilon)^4 \\ g^\varepsilon_- &\mapsto \mathscr{A}^\varepsilon_m(z)g^\varepsilon := -P_-\beta(\Lambda^{z,\varepsilon}_{+,m})^{-1}P_+g^\varepsilon, \end{aligned}$$

$$\mathcal{A}^\varepsilon_{m+M}(z) : P_+H^{1/2}(\Sigma)^4 \oplus P_-H^{1/2}(\Sigma^\varepsilon)^4 \to P_-H^{1/2}(\Sigma)^4 \oplus P_+H^{1/2}(\Sigma^\varepsilon)^4, \quad \text{with}$$

$$\mathcal{A}^\varepsilon_{m+M}(h_+, h^\varepsilon) := \big(-P_-\beta(\Lambda^z_{+,m+M})^{-1}P_+h_+, -P_+\beta(\Lambda^{z,\varepsilon}_{+,m+M})^{-1}P_-h^\varepsilon\big).$$

3.3.2 The Krein resolvent formula of R^ε_M

Let $f \in L^2(\mathbb{R}^3)^4$ and set

$$u^\varepsilon = r_{\mathcal{U}^\varepsilon}R^\varepsilon_M(z)f \quad \text{and} \quad v = r_{\Omega^\varepsilon_{+-}}R^\varepsilon_M(z)f := (v^\varepsilon \oplus v_+).$$

Then u^ε and v satisfy the following system

$$\begin{cases} (D_m - z)v_+ = f & \text{in } \Omega_+, \\ (D_m - z)v^\varepsilon = f & \text{in } \Omega^\varepsilon_-, \\ (D_{m+M} - z)u^\varepsilon = f & \text{in } \mathcal{U}^\varepsilon, \\ P_\pm t_\Sigma v_+ = P_\pm t_\Sigma u^\varepsilon & \text{on } \Sigma, \\ P^\varepsilon_\mp t_{\Sigma^\varepsilon} v^\varepsilon = P^\varepsilon_\mp t_{\Sigma^\varepsilon} u^\varepsilon & \text{on } \Sigma^\varepsilon. \end{cases}$$

Using Lemma 1.5.1, it is straightforward to check that the following resolvent formulas hold:

$$R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(z) = r_{\Omega^\varepsilon_-}(D_m - z)^{-1}e_{\Omega^\varepsilon_-} - \Phi^{z,\varepsilon}_{m,-}(\Lambda^{z,\varepsilon}_{+,m})^{-1}t_{\Sigma^\varepsilon}(D_m - z)^{-1}e_{\Omega^\varepsilon_-}, \tag{3.49}$$

$$\begin{aligned} R^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}}(z) = {} & r_{\Omega^\varepsilon_{+-}}(D_m - z)^{-1}e_{\Omega^\varepsilon_{+-}} - r_{\Omega^\varepsilon_{+-}}e_{\Omega_+}\Phi^z_{m,+}(\Lambda^z_{+,m})^{-1}t_\Sigma(D_m - z)^{-1}r_{\Omega_+}e_{\Omega^\varepsilon_{+-}} \\ & - r_{\Omega^\varepsilon_{+-}}e_{\Omega^\varepsilon_-}\Phi^{z,\varepsilon}_{m,-}(\Lambda^{z,\varepsilon}_{+,m})^{-1}t_{\Sigma^\varepsilon}(D_m - z)^{-1}r_{\Omega^\varepsilon_-}e_{\Omega^\varepsilon_{+-}}, \end{aligned}$$

$$R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) = r_{\mathcal{U}^\varepsilon}(D_m + M\beta - z)^{-1}e_{\mathcal{U}^\varepsilon} - \Phi^{z,\varepsilon}_{m+M}(\Lambda^{z,\varepsilon}_{+,m+M})^{-1}t_{\partial\mathcal{U}^\varepsilon}(D_m + M\beta - z)^{-1}e_{\mathcal{U}^\varepsilon}.$$

In the whole following sections, and for simplicity, we'll use the following notation:

$$(\bullet, \bullet) := \mathrm{diag}(\bullet, \bullet) = \begin{pmatrix} \bullet & 0 \\ 0 & \bullet \end{pmatrix}.$$

Now, we set $\Gamma_{\pm} := P_{\pm}t_{\Sigma}$ and $\Gamma^{\varepsilon}_{\pm} := P_{\pm}t_{\Sigma^{\varepsilon}}$. Since $E_m(z)$, $E^{\varepsilon}_m(z)$ and $\mathcal{E}^{\varepsilon}_{m+M}(z)$ gives the unique solution to the boundary value problem (3.7), (3.8) and (3.9), respectively, and the fact

$$\begin{cases}\Gamma_{-}R^{\Omega_+}_{\mathrm{MIT}}(z)r_{\Omega_+}f = 0, & \Gamma_{+}R^{\mathcal{U}^{\varepsilon}}_{M\,\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}f \quad = 0,\\ \Gamma^{\varepsilon}_{+}R^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}(z)r_{\Omega^{\varepsilon}_-}f = 0, & \Gamma^{\varepsilon}_{-}R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}f \quad = 0.\end{cases}$$

Then, if we let

$$\begin{cases}\varphi = \Gamma_{+}r_{\Omega_+}R^{\varepsilon}_M(z), & \varphi^{\varepsilon} = \Gamma^{\varepsilon}_{-}r_{\Omega^{\varepsilon}_-}R^{\varepsilon}_M(z),\\ \psi = \Gamma_{-}r_{\mathcal{U}^{\varepsilon}}R^{\varepsilon}_M(z), & \psi^{\varepsilon} = \Gamma^{\varepsilon}_{+}r_{\mathcal{U}^{\varepsilon}}R^{\varepsilon}_M(z),\end{cases}$$

it is easy to check that

$$\begin{cases}v_{+} = R^{\Omega_+}_{\mathrm{MIT}}(z)r_{\Omega_+}f + E_m(z)\psi,\\ v^{\varepsilon} = R^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}(z)r_{\Omega^{\varepsilon}_-}f + E^{\varepsilon}_m(z)\psi^{\varepsilon},\\ u^{\varepsilon} = R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}f + \mathcal{E}^{\varepsilon}_{m+M}(z)(\varphi,\varphi^{\varepsilon}).\end{cases} \tag{3.50}$$

Hence, to get an explicit formula for $R^{\varepsilon}_M(z)$ it remains to find the unknowns $(\varphi,\varphi^{\varepsilon},\psi,\psi_{\varepsilon})$. To do this, from (3.50) we get

$$\begin{cases}\varphi = \Gamma_{+}v_{+} = \Gamma_{+}R^{\Omega_+}_{\mathrm{MIT}}r_{\Omega_+}f + \mathscr{A}_m(z)\psi,\\ \varphi^{\varepsilon} = \Gamma^{\varepsilon}_{-}v^{\varepsilon} = \Gamma^{\varepsilon}_{-}R^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}r_{\Omega_{\varepsilon}}f + \mathscr{A}^{\varepsilon}_m(z)\psi^{\varepsilon},\\ \psi = \Gamma_{-}u^{\varepsilon} = \Gamma_{-}R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}f + \Gamma_{-}\mathcal{E}^{\varepsilon}_{m+M}(z)(\varphi,\varphi^{\varepsilon}),\\ \psi^{\varepsilon} = \Gamma^{\varepsilon}_{+}u^{\varepsilon} = \Gamma^{\varepsilon}_{+}R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}f + \Gamma_{+}\mathcal{E}^{\varepsilon}_{m+M}(z)(\varphi,\varphi^{\varepsilon}).\end{cases} \tag{3.51}$$

Using the restriction map $r_{\bullet}$ and the extension map $e_{\bullet}$ given in (3.48), we get

$$\begin{cases}v = e_{\Omega^{\varepsilon}_{+-}}(R^{\Omega_+}_{\mathrm{MIT}}(z), R^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}(z))r_{\Omega^{\varepsilon}_{+-}}f + e_{\Omega^{\varepsilon}_{+-}}\Big(E_m(z)P_{-}, E^{\varepsilon}_m(z)P_{+}\Big)(\Gamma_{-},\Gamma^{\varepsilon}_{+})r_{\mathcal{U}^{\varepsilon}}R^{\varepsilon}_M(z)f,\\ u^{\varepsilon} = R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}f + \mathcal{E}^{\varepsilon}_{m+M}(z)(P_{+},P_{-})(\Gamma_{+}r_{\Omega_+},\Gamma^{\varepsilon}_{-}r_{\Omega^{\varepsilon}_-})R^{\varepsilon}_M(z)f.\end{cases}$$

Thus, we obtain

$$\begin{aligned}R^{\varepsilon}_M(z) &= e_{\Omega_+}R^{\Omega_+}_{\mathrm{MIT}}r_{\Omega_+} + e_{\Omega^{\varepsilon}_-}R^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}r_{\Omega^{\varepsilon}_-} + e_{\mathcal{U}^{\varepsilon}}R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}}\\ &\qquad + \left(e_{\Omega^{\varepsilon}_{+-}}\Big(E_m(z)P_{-}, E^{\varepsilon}_m(z)P_{+}\Big)(\Gamma_{-},\Gamma^{\varepsilon}_{+})r_{\mathcal{U}^{\varepsilon}} + e_{\mathcal{U}^{\varepsilon}}\mathcal{E}^{\varepsilon}_{m+M}(z)\big(\Gamma_{+}r_{\Omega_+},\Gamma^{\varepsilon}_{-}r_{\Omega^{\varepsilon}_-}\big)\right)R^{\varepsilon}_M(z)\\ &= e_{\Omega^{\varepsilon}_{+-}}R^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}}(z)r_{\Omega^{\varepsilon}_{+-}} + e_{\mathcal{U}^{\varepsilon}}R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z)r_{\mathcal{U}^{\varepsilon}} + E^{\varepsilon}_M(z)\Gamma^{\varepsilon}R^{\varepsilon}_M(z),\end{aligned} \tag{3.52}$$

with $R^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}}(z)$ as in (3.47).

Here Γ^{ε} and $E^{\varepsilon}_M(z)$ are defined as follows:

$$\begin{aligned}\Gamma^{\varepsilon}: H^1(\Omega_+)^4 \oplus H^1(\Omega^{\varepsilon}_-)^4 \oplus H^1(\mathcal{U}^{\varepsilon})^4 &\to H^{1/2}(\Sigma)^4 \oplus H^{1/2}(\Sigma^{\varepsilon})^4 \oplus H^{1/2}(\Sigma)^4 \oplus H^{1/2}(\Sigma^{\varepsilon})^4\\ (r_{\Omega_+}, r_{\Omega^{\varepsilon}_-}, r_{\mathcal{U}^{\varepsilon}}) &\mapsto (\Gamma_{+}r_{\Omega_+}\ \Gamma^{\varepsilon}_{-}r_{\Omega^{\varepsilon}_-}\ \Gamma_{-}r_{\mathcal{U}^{\varepsilon}}\ \Gamma^{\varepsilon}_{+}r_{\mathcal{U}^{\varepsilon}})^t,\end{aligned}$$

and

$$E_M^\varepsilon(z) = e_{\Omega_{+-}^\varepsilon} E_m^{\Omega_{+-}^\varepsilon}(z) + e_{\mathcal{U}^\varepsilon}\mathcal{E}_{m+M}^\varepsilon(z)(P_+, P_-),$$

with $E_m^{\Omega_{+-}^\varepsilon}(z) = r_{\Omega_{+-}^\varepsilon} e_{\Omega_+} E_m(z)P_- + r_{\Omega_{+-}^\varepsilon} e_{\Omega_-^\varepsilon} E_m^\varepsilon(z)P_+$ can be read as the following matrix:

$$\begin{aligned} E_m^{\Omega_{+-}^\varepsilon} : P_- H^{1/2}(\Sigma)^4 \oplus P_+ H^{1/2}(\Sigma^\varepsilon)^4 &\rightarrow H^1(\Omega_+)^4 \oplus H^1(\Omega_-^\varepsilon)^4 \\ (\psi,\, \psi_\varepsilon) &\mapsto (E_m P_- \psi,\, E_m^\varepsilon P_+ \psi_\varepsilon) \equiv \begin{pmatrix} E_m P_- & 0 \\ 0 & E_m^\varepsilon P_+ \end{pmatrix}\begin{pmatrix} \psi \\ \psi_\varepsilon \end{pmatrix}. \end{aligned} \tag{3.53}$$

Now, applying Γ^ε to the identity (3.52), it yields

$$\Gamma^\varepsilon R_{\mathrm{MIT}}^\varepsilon(z) = \Big(\mathbb{I} - (\mathscr{A}_m(z)P_-, \mathscr{A}_m^\varepsilon(z)P_+) - \mathcal{A}_{m+M}^\varepsilon(z)(P_+, P_-)\Big)\Gamma^\varepsilon R_M^\varepsilon(z) := \Upsilon_M^\varepsilon(z)\Gamma^\varepsilon R_M^\varepsilon(z), \tag{3.54}$$

with $R_{\mathrm{MIT}}^\varepsilon(z) := e_{\Omega_{+-}^\varepsilon} R_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(z) + e_{\mathcal{U}^\varepsilon} R_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(z)$. Similarly, we mention that $[\mathscr{A}_m(z), \mathscr{A}_m^\varepsilon(z)]$ means the sum of both terms $\mathscr{A}_m$, $\mathscr{A}_m^\varepsilon$ and can be read as the following matrix

$$\begin{aligned} \mathscr{A}_m^{\Omega_{+-}^\varepsilon} &:= \\ (\mathscr{A}_m, \mathscr{A}_m^\varepsilon) : P_- H^{1/2}(\Sigma)^4 \oplus P_+ H^{1/2}(\Sigma^\varepsilon)^4 &\rightarrow P_+ H^{1/2}(\Sigma)^4 \oplus H^{1/2}(\Sigma^\varepsilon)^4 \\ (\psi,\, \psi_\varepsilon) &\mapsto (\mathscr{A}_m, \mathscr{A}_m^\varepsilon)(\psi,\, \psi_\varepsilon) = \begin{pmatrix} \mathscr{A}_m P_- & 0 \\ 0 & \mathscr{A}_m^\varepsilon P_+ \end{pmatrix}\begin{pmatrix} \psi \\ \psi_\varepsilon \end{pmatrix}. \end{aligned} \tag{3.55}$$

Using the formula of $\mathcal{A}_{m+M}^\varepsilon$, the term $(\Gamma_-, \Gamma_+^\varepsilon)\mathcal{E}_{m+M}^\varepsilon(z)$ is identified with $(P_-\mathcal{A}_{m+M}^\varepsilon, P_+\mathcal{A}_{m+M}^\varepsilon) = (P_-, P_+)\mathcal{A}_{m+M}^\varepsilon(z) \equiv (P_-, 0)\mathcal{A}_{m+M}^\varepsilon(z) + (0, P_+)\mathcal{A}_{m+M}^\varepsilon(z)$.

Now, applying also $\Big(\mathbb{I} + \mathscr{A}_m^{\Omega_{+-}^\varepsilon}(z) + (P_-, P_+)\mathcal{A}_{m+M}^\varepsilon(z)\Big)$ to the identity (3.54) we get

$$\Gamma^\varepsilon R_M^\varepsilon(z) = \Xi_M^\varepsilon(z)\Big(\mathbb{I} + \mathscr{A}_m^{\Omega_{+-}^\varepsilon}(z) + (P_-, P_+)\mathcal{A}_{m+M}^\varepsilon(z)\Big)\Gamma^\varepsilon R_{\mathrm{MIT}}^\varepsilon(z),$$

with $\Xi_M^\varepsilon(z) : H^{1/2}(\Sigma)^4 \oplus H^{1/2}(\Sigma^\varepsilon)^4 \rightarrow H^{1/2}(\Sigma)^4 \oplus H^{1/2}(\Sigma^\varepsilon)^4$ the following quantity

$$\Xi_M^\varepsilon(z) := \Big(\mathbb{I}_8 - \mathscr{A}_m^{\Omega_{+-}^\varepsilon}(z)(P_-, P_+)\mathcal{A}_{m+M}^\varepsilon(z) - \mathcal{A}_{m+M}^\varepsilon(z)(P_+, P_-)\mathscr{A}_m^{\Omega_{+-}^\varepsilon}(z)\Big)^{-1}. \tag{3.56}$$

From which it follows that,

$$R_M^\varepsilon(z) = R_{\mathrm{MIT}}^\varepsilon(z) + E_M^\varepsilon(z)[\Upsilon_M^\varepsilon(z)]^{-1}\Gamma^\varepsilon R_{\mathrm{MIT}}^\varepsilon(z), \tag{3.57}$$

with

$$[\Upsilon_M^\varepsilon]^{-1}(z) = \Xi_M^\varepsilon(z)\Big(\mathbb{I} + \mathscr{A}_m^{\Omega_{+-}^\varepsilon}(z) + (P_-, P_+)\mathcal{A}_{m+M}^\varepsilon(z)\Big).$$

Remark 3.3.1. *The identity* (3.54) *has the following matrix form*

$$\begin{pmatrix}\Gamma_+ r_{\Omega_+} R^\varepsilon_M\\ \Gamma_- r_{\Omega^\varepsilon_-} R^\varepsilon_M\\ \Gamma_- r_{\mathcal{U}^\varepsilon} R^\varepsilon_M\\ \Gamma^\varepsilon_+ r_{\mathcal{U}^\varepsilon} R^\varepsilon_M\end{pmatrix} =$$
$$\begin{pmatrix}\Gamma_+ R^{\Omega_+}_{\mathrm{MIT}} r_{\Omega_+}\\ \Gamma^\varepsilon_- R^{\Omega^\varepsilon_-}_{\mathrm{MIT}} r_{\Omega^\varepsilon_-}\\ \Gamma_- R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} r_{\mathcal{U}^\varepsilon}\\ \Gamma^\varepsilon_+ R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} r_{\mathcal{U}^\varepsilon}\end{pmatrix} + \begin{pmatrix}0 & 0 & \mathscr{A}_m P_- & 0\\ 0 & 0 & 0 & \mathscr{A}^\varepsilon_m P_+\\ \mathcal{A}^\varepsilon_{m+M}(P_+,P_-) & \mathcal{A}^\varepsilon_{m+M}(P_+,P_-) & 0 & 0\\ \mathcal{A}^\varepsilon_{m+M}(P_+,P_-) & \mathcal{A}^\varepsilon_{m+M}(P_+,P_-) & 0 & 0\end{pmatrix}\begin{pmatrix}\Gamma_+ r_{\Omega_+} R^\varepsilon_M\\ \Gamma^\varepsilon_- r_{\Omega^\varepsilon_-} R^\varepsilon_M\\ \Gamma_- r_{\mathcal{U}^\varepsilon} R^\varepsilon_M\\ \Gamma^\varepsilon_+ r_{\mathcal{U}^\varepsilon} R^\varepsilon_M\end{pmatrix}.$$

Moreover, if we note by $\Gamma^\varepsilon_{+-} = (\Gamma_+ r_{\Omega_+}\ \Gamma^\varepsilon_- r_{\Omega^\varepsilon_-})^t$ *and* $\Gamma^\varepsilon_{-+} = (\Gamma_- r_{\mathcal{U}^\varepsilon}\ \Gamma^\varepsilon_+ r_{\mathcal{U}^\varepsilon})^t$*. Then, using the quantities of* (3.51)*, we remark that the Krein resolvent formula* 3.57 *can be also written in the following matrix*

$$\begin{pmatrix} r_{\Omega^\varepsilon_{+-}} R^\varepsilon_M\\ r_{\mathcal{U}^\varepsilon} R^\varepsilon_M\end{pmatrix} = \begin{pmatrix} R^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} r_{\Omega^\varepsilon_{+-}}\\ R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} r_{\mathcal{U}^\varepsilon}\end{pmatrix} + \begin{pmatrix} E^{\Omega^\varepsilon_{+-}}_m \Xi^{\varepsilon,-+}_M & 0\\ 0 & E^\varepsilon_{m+M}\Xi^{\varepsilon,+-}_M\end{pmatrix}\begin{pmatrix}\mathcal{A}^\varepsilon_{m+M} & \mathbb{I}_4\\ \mathbb{I}_4 & \mathscr{A}^{\Omega^\varepsilon_{+-}}_m\end{pmatrix}\begin{pmatrix}\Gamma^\varepsilon_{+-} R^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}} r_{\Omega^\varepsilon_{+-}}\\ \Gamma^\varepsilon_{-+} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} r_{\mathcal{U}^\varepsilon}\end{pmatrix},$$

where $\mathscr{A}^{\Omega^\varepsilon_{+-}}_m$ *is the matrix in* (3.55) *and* $\Xi^{\varepsilon,\pm\mp}_M$ *are given in the following corollary.*

Corollary 3.3.1. *Consider the operator* $\Xi^\varepsilon_M(z)$ *given in* (3.56)*. Then, there is* $M_0 > 0$ *such that for every* $M > M_0$*,* $h \equiv \varepsilon = 1/M$ *and for all* $z \in \rho(D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}}) \cap \rho(\mathfrak{D}^\varepsilon_M)$*, the operator* $\Xi^\varepsilon_M(z)$ *is everywhere defined and uniformly bounded with respect to* M*. Moreover, the operators* $\Xi^{\varepsilon,+-}_M(z)$ *and* $\Xi^{\varepsilon,-+}_M(z)$ *defined by*

$$\Xi^{\varepsilon,+-}_M(z) : P_+H^s(\Sigma)^4 \oplus P_-H^s(\Sigma^\varepsilon)^4 \quad\rightarrow\quad P_+H^s(\Sigma)^4 \oplus P_-H^s(\Sigma^\varepsilon)^4,$$

$$\Xi^{\varepsilon,-+}_M(z) : P_-H^s(\Sigma)^4 \oplus P_+H^s(\Sigma^\varepsilon)^4 \quad\rightarrow\quad P_-H^s(\Sigma)^4 \oplus P_+H^s(\Sigma^\varepsilon)^4,$$

which have the following formula

$$\Xi^{\varepsilon,+-}_M(z) = \Big(\mathbb{I} - \mathscr{A}^{\Omega^\varepsilon_{+-}}_m(z)(z)(P_-,P_+)\mathcal{A}^\varepsilon_{m+M}(z)(P_+,P_-)\Big)^{-1},$$
$$\Xi^{\varepsilon,-+}_M(z) = \Big(\mathbb{I} - (P_-,P_+)\mathcal{A}^\varepsilon_{m+M}(z)(P_+,P_-)\mathscr{A}^{\Omega^\varepsilon_{+-}}_m(z)\Big)^{-1}$$

are bounded for any $s \in \mathbb{R}$*, and it holds that*

$$||\Xi^{\varepsilon,\pm\mp}_M(z)||_{P_\pm H^{-1/2}(\Sigma)^4\oplus P_\mp H^{-1/2}(\Sigma^\varepsilon)^4\rightarrow P_\pm H^{-1/2}(\Sigma)^4\oplus P_\mp H^{-1/2}(\Sigma^\varepsilon)^4} \lesssim 1, \tag{3.58}$$

uniformly with respect to $M > M_0$*.*

Moreover, the Poincaré-Steklov $\mathcal{A}^\varepsilon_{m+M}$*, satisfies the following estimate*

$$||\mathcal{A}^\varepsilon_{m+M}||_{P_+H^{1/2}(\Sigma)^4\oplus P_-H^{1/2}(\Sigma^\varepsilon)^4\rightarrow P_-H^{-1/2}(\Sigma)^4\oplus P_+H^{-1/2}(\Sigma^\varepsilon)^4} \lesssim M^{-1}. \tag{3.59}$$

Proof. Set $\kappa = m + M$ and $h = \kappa^{-1}$. The proof of this corollary follows a similar argument as in [BBZ37, Proposition 6.1]. It is based on the pseudodifferential properties of the Poincaré-Steklov operators $\mathscr{A}^\varepsilon_m$ and $\mathcal{A}^\varepsilon_\kappa$. Since $\mathscr{A}_m$ (resp. $\mathscr{A}^\varepsilon_m$) are a pseudodifferential operators of order 0, see Remark 3.2.4 (resp. Corollary 3.2.9), we can consider it as an h-pseudodifferential operator of order 0 whose principal symbol is given by:

$$\mathscr{P}_{h,\mathscr{A}_m}(x_\Sigma,\xi) = \frac{S\cdot(\xi\wedge n(x_\Sigma))}{|\xi\wedge n(x_\Sigma)|}P_-, \quad x_\Sigma\in\Sigma,$$

$$\mathscr{P}_{h,\mathscr{A}^\varepsilon_m}(x,\xi) = -\frac{(1-\varepsilon W(x_\Sigma))^{-1}S\cdot(\xi\wedge n(p^{-1}(x)))}{\left|(1-\varepsilon W(x_\Sigma))^{-1}\right|\left|\xi\wedge n(p^{-1}(x))\right|}P_+,\ \Sigma^\varepsilon\ni x = p(x_\Sigma) = x_\Sigma+\varepsilon\, n(x_\Sigma),$$

where S is the spin angular momentum given in Lemma 3.2.1, $\xi\in\mathbb{R}^2$ can be identify with $\bar{\xi} = (\xi_1,\xi_2,0)^t\in\mathbb{R}^3$, p is the diffeomorphism from Remark 3.1.2, and for $x=\varphi(\tilde{x})$ stands for $n^\varphi(\tilde{x})$.

On the other hand, Proposition 3.2.7 follows that $\mathcal{A}^\varepsilon_\kappa$ is h-pseudodifferential operator of order 0 has the following principal symbol

$$\mathscr{P}_{h,\mathcal{A}^\varepsilon_\kappa}(x,\xi) = (1-\varepsilon W(x_\Sigma))^{-1}\frac{S\cdot(\xi\wedge n(p^{-1}(x)))}{\sqrt{\left((1-\varepsilon W(x_\Sigma))^{-1}\xi\wedge n(p^{-1}(x))\right)^2+1}+1}\begin{pmatrix}-P_+ & 0\\ 0 & P_-\end{pmatrix}.$$

Consequently, the symbol calculus yields for all $h<h_0$ that

$$\left(\mathbb{I}_8 - \mathscr{A}_m^{\Omega^\varepsilon_{+-}}(z)(P_-,P_+)\mathcal{A}^\varepsilon_\kappa(z) - \mathcal{A}^\varepsilon_\kappa(z)(P_+,P_-)\mathscr{A}_m^{\Omega^\varepsilon_{+-}}(z)\right)$$

is a κ^{-1}-pseudodifferential operator of order 0.

Now, using the principal symbols of $\mathscr{A}_m$, $\mathscr{A}^\varepsilon_m$, the principal symbol of $\mathscr{A}_m^{\Omega^\varepsilon_{+-}}$ can be written as the following:

$$\begin{aligned}\mathscr{P}_{h,\mathscr{A}_m^{\Omega^\varepsilon_{+-}}}(x_\Sigma,\xi) &= \begin{pmatrix}\mathscr{P}_{h,\mathscr{A}_m}(x_\Sigma,\xi) & 0\\ 0 & \mathscr{P}_{h,\mathscr{A}^\varepsilon_m}(p(x_\Sigma),\xi)\end{pmatrix}\\ &= \frac{S\cdot(\xi\wedge n(x_\Sigma))}{|\xi\wedge n(x_\Sigma)|}\begin{pmatrix}P_- & 0\\ 0 & -\frac{(1-\varepsilon W(x_\Sigma))^{-1}}{\left|(1-\varepsilon W(x_\Sigma))^{-1}\right|}P_+\end{pmatrix}.\end{aligned}$$

Using Lemma 3.2.1, we obtain

$$\mathscr{P}_{h,\mathscr{A}_m^{\Omega^\varepsilon_{+-}}}(x_\Sigma,\xi)\mathscr{P}_{h,\mathcal{A}^\varepsilon_\kappa}(x,\xi) = \\ -\frac{(1-\varepsilon W(x_\Sigma))^{-1}|\xi\wedge n(x_\Sigma)|}{\sqrt{\left((1-\varepsilon W(x_\Sigma))^{-1}\xi\wedge n(p^{-1}(x))\right)^2+1}+1}\begin{pmatrix}P_+ & 0\\ 0 & \frac{(1-\varepsilon W(x_\Sigma))^{-1}}{\left|(1-\varepsilon W(x_\Sigma))^{-1}\right|}P_-\end{pmatrix}.$$

Then, it yields

$$\mathbb{I}_8 - \mathscr{P}_{h,\mathscr{A}_m^{\Omega^\varepsilon_{+-}}}(x_\Sigma,\xi)\mathscr{P}_{h,\mathcal{A}^\varepsilon_\kappa}(x,\xi) - \mathscr{P}_{h,\mathcal{A}^\varepsilon_\kappa}(x,\xi)\mathscr{P}_{h,\mathscr{A}_m^{\Omega^\varepsilon_{+-}}}(x_\Sigma,\xi) =$$

$$\mathbb{I}_8 + \frac{(1-\varepsilon W(x_\Sigma))^{-1}|\xi\wedge n(x_\Sigma)|}{\sqrt{\left((1-\varepsilon W(x_\Sigma))^{-1}\xi\wedge n(p^{-1}(x))\right)^2+1}+1}\begin{pmatrix}\mathbb{I}_4 & 0\\ 0 & \frac{(1-\varepsilon W(x_\Sigma))^{-1}}{|(1-\varepsilon W(x_\Sigma))^{-1}|}\mathbb{I}_4\end{pmatrix}\gtrsim 1.$$

Thus, Ξ^ε_M is a zero-order pseudoddiferential operator.

Thanks to the following relationship: $\Xi^{\varepsilon,\pm\mp}_M(z) = (P_\pm, P_\mp)\Xi^\varepsilon_M(z)(P_\pm, P_\mp)$, it yields the same properties for $\Xi^{\varepsilon,\pm\mp}_M(z)$ and therefore (3.58) is established.

Regarding the estimate of $\mathcal{A}^\varepsilon_\kappa$, exploits also the Calderón-Vaillancourt theorem which shows that for any operator in $h\,Op^h\mathcal{S}^0(\partial\mathcal{U}^\varepsilon)$ is uniformly bounded by $O(h)$, with respect to $h=\kappa^{-1}\in(0,1)$, from $H^{1/2}(\partial\mathcal{U}^\varepsilon)^4$ into $H^{1/2}(\partial\mathcal{U}^\varepsilon)^4 \hookrightarrow H^{-1/2}(\partial\mathcal{U}^\varepsilon)^4$, see (2.2). Thus,

$$\left|\left|\mathcal{A}^\varepsilon_\kappa - \frac{S\cdot(\nabla_{\partial\mathcal{U}^\varepsilon}\wedge n(p^{-1}(x)))}{\sqrt{-\kappa^{-2}\Delta_{\partial\mathcal{U}^\varepsilon}+\mathbb{I}}+\mathbb{I}}(P_+,P_-)\right|\right|_{H^{1/2}(\partial\mathcal{U}^\varepsilon)^4\to H^{-1/2}(\partial\mathcal{U}^\varepsilon)^4}\lesssim\kappa^{-1},$$

uniformly with respect to κ big enough and $\varepsilon\in(0,\varepsilon_0)$. Then we conclude the proof of the estimate by using that $\left(\sqrt{-\kappa^{-2}\Delta_{\partial\mathcal{U}^\varepsilon}+\mathbb{I}}+\mathbb{I}\right)^{-1}$ is uniformly bounded from $H^{1/2}(\partial\mathcal{U}^\varepsilon)^4$ into itself and $(\nabla_{\partial\mathcal{U}^\varepsilon}\wedge n(p^{-1}(x)))$ is uniformly bounded from $H^{1/2}(\partial\mathcal{U}^\varepsilon)^4$ into $H^{-1/2}(\partial\mathcal{U}^\varepsilon)^4$. ■

Remark 3.3.2. *Let $E^{\Omega^\varepsilon_{+-}}_m$ from (3.53). Thanks to [BBZ37, Proposition 4.1 (ii)], we have that*

$$\left(E^{\Omega^\varepsilon_{+-}}_m(z)\right)^* = -\beta(\Gamma^\varepsilon_{+-})^t R^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}}(\overline{z}) \quad\text{and}\quad (\mathcal{E}^\varepsilon_{m+M}(z))^* = -\beta(\Gamma^\varepsilon_{-+})^t\begin{pmatrix}R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(\overline{z})\\ R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(\overline{z})\end{pmatrix},$$

for any $z\in\rho(D^{\Omega^\varepsilon_{+-}}_{\mathrm{MIT}})\cap\rho(\mathfrak{D}^\varepsilon_M)$.

3.4 Resolvent convergence to the Dirac operator with Lorentz scalar.

In this section, we gather the necessary elements to prove the main result of this work. The components of the proof for the main theorem (*i.e.*, Theorem 3.1.5) are dedicated to examining the convergence of the terms present in the resolvent formula (3.52). It is important to note that this resolvent formula includes certain terms independent of M and ε, namely $E_m, \mathscr{A}_m$, and $R^{\Omega_+}_{\mathrm{MIT}}r_{\Omega_+}$, which remain fixed and act within Ω_+. Consequently, our focus shifts to examining the convergence of terms dependent on ε but independent of M, namely $R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}r_{\Omega^\varepsilon-}$ and E^ε_m (see, Proposition 3.1.3). Subsequently, we will proceed to estimate the remaining terms in relation to M and ε (see, Proposition 3.1.4).

Proposition 3.4.1. *Let $\varepsilon_0>0$ be small enough, and let $z\in\mathbb{C}\backslash\mathbb{R}$. We set $\Omega_- := \mathbb{R}^3\setminus\overline{\Omega_+}$ the exterior fixed domain and by $\Sigma=\partial\Omega_-=\partial\Omega_+$ its boundary. We denote by $R^{\Omega_-}_{\mathrm{MIT}}$ the resolvent of the fixed MIT bag operator, which we denote by $D^{\Omega_-}_{\mathrm{MIT}}$, acts in Ω_-. Then, for any $\varepsilon\in(0,\varepsilon_0)$ the following holds:*

$$\left|\left|e_{\Omega^\varepsilon_-}R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(z)r_{\Omega^\varepsilon_-} - e_{\Omega_-}R^{\Omega_-}_{\mathrm{MIT}}(z)r_{\Omega_-}\right|\right|_{L^2(\mathbb{R}^3)^4\to L^2(\mathbb{R}^3)^4} = \mathcal{O}(\varepsilon). \tag{3.60}$$

Proof. The Krein formula for the resolvent $R_{\text{MIT}}^{\Omega_-^\varepsilon}$ (from equality (3.49))

$$e_{\Omega_-^\varepsilon} R_{\text{MIT}}^{\Omega_-^\varepsilon}(z) r_{\Omega_-^\varepsilon} = (D_m - z)^{-1} - e_{\Omega_-^\varepsilon} \Phi_{m,-}^{z,\varepsilon} (\Lambda_{+,m}^{z,\varepsilon})^{-1} t_{\Sigma^\varepsilon} (D_m - z)^{-1},$$
$$e_{\Omega_-} R_{\text{MIT}}^{\Omega_-}(z) r_{\Omega_-} = (D_m - z)^{-1} - e_{\Omega_-} \Phi_{m,-}^{z} (\Lambda_{m,+}^{z})^{-1} t_{\Sigma} (D_m - z)^{-1}$$

yield that

$$\begin{aligned}
&\left|\left|e_{\Omega_-^\varepsilon} R_{\text{MIT}}^{\Omega_-^\varepsilon}(z) r_{\Omega_-^\varepsilon} - e_{\Omega_-} R_{\text{MIT}}^{\Omega_-}(z) r_{\Omega_-}\right|\right|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} \\
&= \left|\left|e_{\Omega_-} \Phi_{m,-}^{z} (\Lambda_{+,m}^{z})^{-1} t_{\Sigma} (D_m - z)^{-1} - e_{\Omega_-^\varepsilon} \Phi_{m,-}^{z,\varepsilon} (\Lambda_{+,m}^{z,\varepsilon})^{-1} t_{\Sigma^\varepsilon} (D_m - z)^{-1}\right|\right|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} \\
&\leqslant \left|\left|e_{\Omega_-} \Phi_{m,-}^{z} (\Lambda_{+,m}^{z})^{-1} t_{\Sigma} - e_{\Omega_-^\varepsilon} \Phi_{m,-}^{z,\varepsilon} (\Lambda_{+,m}^{z,\varepsilon})^{-1} t_{\Sigma^\varepsilon}\right|\right|_{H^1(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} \left|\left|(D_m - z)^{-1}\right|\right|_{L^2(\mathbb{R}^3)^4 \to H^1(\mathbb{R}^3)^4} \\
&\lesssim \left|\left|e_{\Omega_-} \Phi_{m,-}^{z} (\Lambda_{+,m}^{z})^{-1} t_{\Sigma} - e_{\Omega_-^\varepsilon} \Phi_{m,-}^{z,\varepsilon} (\Lambda_{+,m}^{z,\varepsilon})^{-1} t_{\Sigma^\varepsilon}\right|\right|_{H^1(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4}
\end{aligned} \tag{3.61}$$

since $(D_m - z)^{-1}$ is bounded from $L^2(\mathbb{R}^3)^4$ into $H^1(\mathbb{R}^3)^4$.

To obtain a rigorous estimate of the right-hand side of (3.61), we'll use the unitary transformation $\mathcal{T}_\varepsilon$ from Definition 3.1.2 and the explicit formula for $\Lambda_{+,m}^z$ (resp. $\Lambda_{+,m}^{z,\varepsilon}$). Let $f, g \in L^2(\mathbb{R}^3)^4$. Since $t_\Sigma (D_m - z)^{-1} = (\Phi_m^{\bar{z}})^*$ (resp. $t_{\Sigma^\varepsilon} (D_m - z)^{-1} = (\Phi_m^{\bar{z},\varepsilon})^*$) by duality and interpolation arguments, we get that

$$\begin{aligned}
&\left|\left\langle [e_{\Omega_-} \Phi_{m,-}^{z} \left(\beta/2 + \mathscr{C}_{z,m}\right)^{-1} t_\Sigma - e_{\Omega_-^\varepsilon} \Phi_{m,-}^{z,\varepsilon} \left(\beta/2 + \mathscr{C}_{z,m}^\varepsilon\right)^{-1} t_{\Sigma^\varepsilon}] f, g \right\rangle_{L^2(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4}\right| \\
&= \Big|\left\langle e_{\Omega_-} \Phi_{m,-}^{z} \left(\beta/2 + \mathscr{C}_{z,m}\right)^{-1} t_\Sigma f, g \right\rangle_{L^2(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4} - \\
&\qquad\qquad \left\langle e_{\Omega_-^\varepsilon} \Phi_{m,-}^{z,\varepsilon} \left(\beta/2 + \mathscr{C}_{z,m}^\varepsilon\right)^{-1} t_{\Sigma^\varepsilon} f, g \right\rangle_{L^2(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4}\Big| \\
&= \Big|\left\langle \left(\beta/2 + \mathscr{C}_{z,m}\right)^{-1} t_\Sigma f, t_\Sigma (D_m - z)^{-1} r_{\Omega_-} g \right\rangle_{L^2(\Sigma)^4, L^2(\Sigma)^4} - \\
&\qquad\qquad \left\langle \left(\beta/2 + \mathscr{C}_{z,m}^\varepsilon\right)^{-1} \mathcal{T}_\varepsilon \mathcal{T}_\varepsilon^{-1} t_{\Sigma^\varepsilon} f, t_{\Sigma^\varepsilon} (D_m - z)^{-1} r_{\Omega_-^\varepsilon} g \right\rangle_{L^2(\Sigma^\varepsilon)^4, L^2(\Sigma^\varepsilon)^4}\Big| \\
&= \Big|\left\langle \left(\beta/2 + \mathscr{C}_{z,m}\right)^{-1} t_\Sigma f, t_\Sigma (D_m - z)^{-1} r_{\Omega_-} g \right\rangle_{L^2(\Sigma)^4, L^2(\Sigma)^4} - \\
&\qquad\qquad \left\langle \left(\beta/2 + \mathcal{T}_\varepsilon \mathcal{T}_\varepsilon^{-1} \mathscr{C}_{z,m}^\varepsilon \mathcal{T}_\varepsilon \mathcal{T}_\varepsilon^{-1}\right)^{-1} \mathcal{T}_\varepsilon \mathcal{T}_\varepsilon^{-1} t_{\Sigma^\varepsilon} f, t_{\Sigma^\varepsilon} (D_m - z)^{-1} r_{\Omega_-^\varepsilon} g \right\rangle_{L^2(\Sigma^\varepsilon)^4, L^2(\Sigma^\varepsilon)^4}\Big| \\
&= \Big|\left\langle \left(\beta/2 + \mathscr{C}_{z,m}\right)^{-1} t_\Sigma f, t_\Sigma (D_m - z)^{-1} r_{\Omega_-} g \right\rangle_{L^2(\Sigma)^4, L^2(\Sigma)^4} - \\
&\qquad\qquad \left\langle \left(\beta/2 + \mathcal{T}_\varepsilon^{-1} \mathscr{C}_{z,m}^\varepsilon \mathcal{T}_\varepsilon\right)^{-1} \mathcal{T}_\varepsilon^{-1} t_{\Sigma^\varepsilon} f, \mathcal{T}_\varepsilon^{-1} t_{\Sigma^\varepsilon} (D_m - z)^{-1} r_{\Omega_-^\varepsilon} g \right\rangle_{L^2(\Sigma)^4, L^2(\Sigma)^4}\Big|.
\end{aligned}$$

By adding and subtracting the term $\left\langle \left(\beta/2 + \mathcal{T}_\varepsilon^{-1} \mathscr{C}_m^{z,\varepsilon} \mathcal{T}_\varepsilon\right)^{-1} \mathcal{T}_\varepsilon^{-1} t_{\Sigma^\varepsilon} f, t_\Sigma (D_m - z)^{-1} r_{\Omega_-} g \right\rangle_{L^2(\Sigma)^4, L^2(\Sigma)^4}$ in the last quantity, we obtain that

$$
\begin{aligned}
&\Big|\Big\langle [e_{\Omega_-}\Phi^z_{m,-}\big(\beta/2+\mathscr{C}_{z,m}\big)^{-1}t_\Sigma - e_{\Omega^\varepsilon_-}\Phi^{z,\varepsilon}_{m,-}\big(\beta/2+\mathscr{C}^\varepsilon_{z,m}\big)^{-1}t_{\Sigma^\varepsilon}]f,g\Big\rangle_{L^2(\mathbb{R}^3)^4,L^2(\mathbb{R}^3)^4}\Big|\\
&\quad\leqslant \Big|\Big|\Big[\big(\beta/2+\mathscr{C}_{z,m}\big)^{-1}t_\Sigma - \big(\beta/2+\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\big)^{-1}\mathcal{T}_\varepsilon^{-1}t_{\Sigma^\varepsilon}\Big]f\Big|\Big|_{L^2(\Sigma)^4}\Big|\Big|t_\Sigma(D_m-z)^{-1}r_{\Omega_-}g\Big|\Big|_{L^2(\Sigma)^4}\\
&\quad+\Big|\Big|\big(\beta/2+\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\big)^{-1}\mathcal{T}_\varepsilon^{-1}t_{\Sigma^\varepsilon}f\Big|\Big|_{L^2(\Sigma)^4}\Big|\Big|\Big[t_\Sigma(D_m-z)^{-1}r_{\Omega_-}-\mathcal{T}_\varepsilon^{-1}t_{\Sigma^\varepsilon}(D_m-z)^{-1}r_{\Omega^\varepsilon_-}\Big]g\Big|\Big|_{L^2(\Sigma)^4}\\
&\quad=: r_1+r_2.
\end{aligned}
$$

Now, let $\mathscr{C}_{z,m}$ and $\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon$ from (1.13) and (3.44) respectively. Then, for a fixed $\rho,\rho^{'}>0$ such that $\rho^{''}=\min\{\rho,\rho^{'}\}$, the regularity of Σ and ϕ^z_m, and a combination of the mean value theorem give

$$
|\phi^z_m(x_\Sigma-y_\Sigma+\varepsilon(\,n(x_\Sigma)-\,n(y_\Sigma)))-\phi^z_m(x_\Sigma-y_\Sigma)|\leqslant\varepsilon\,|\partial\phi^z_m|\leqslant\varepsilon\,C,\qquad\text{with } C \text{ only depending on } z.
$$

We set $f_\varepsilon(y_\Sigma):=\det(1-\varepsilon\,n(x_\Sigma))f(y_\Sigma)$. On one hand, using the Cauchy-Schwarz inequality, we obtain that:

$$
\begin{aligned}
&\Big|\mathscr{C}_{z,m}f(x_\Sigma)-(\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon f)(x_\Sigma)\Big|\\
&\qquad\leqslant\int_{|x_\Sigma-y_\Sigma|>\rho^{''}}|\phi^z_m(x_\Sigma-y_\Sigma+\varepsilon(\,n(x_\Sigma)-\,n(y_\Sigma)))f(y_\Sigma)-\phi^z_m(x_\Sigma-y_\Sigma)f_\varepsilon(y_\Sigma)|\mathrm{d}\sigma(y_\Sigma)\\
&\qquad\leqslant\int_\Sigma\Big|\Big(\phi^z_m(x_\Sigma-y_\Sigma+\varepsilon(\,n(x_\Sigma)-\,n(y_\Sigma)))-\phi^z_m(x_\Sigma-y_\Sigma)\Big)f(y_\Sigma)\Big|\mathrm{d}\sigma(y_\Sigma)\\
&\qquad\qquad+\int_\Sigma|\phi^z_m(x_\Sigma-y_\Sigma)(f_\varepsilon(y_\Sigma)-f(y_\Sigma))|\mathrm{d}\sigma(y_\Sigma).
\end{aligned}
$$

On the other hand, Proposition 1.5.3 gives us

$$
\det(1-\varepsilon W(x_\Sigma))=1-\varepsilon\lambda_1(x_\Sigma)-\varepsilon\lambda_2(x_\Sigma)+\varepsilon^2\lambda_1(x_\Sigma)\lambda_2(x_\Sigma),
$$

where $\lambda_1(x_\Sigma)$, $\lambda_2(x_\Sigma)$ are the eigenvalues of the Weingarten map $W(x_\Sigma)$. Then, we get

$$
|f_\varepsilon(y_\Sigma)-f(y_\Sigma)|=|\det(1-\varepsilon W(y_\Sigma))-1||f(y_\Sigma)|\lesssim\varepsilon||f||_{L^2(\Sigma)^4}.
$$

We conclude that

$$
||\left(\mathscr{C}_{z,m}-\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right)||_{L^2(\Sigma)^4\to L^2(\Sigma)^4}=\mathcal{O}(\varepsilon). \tag{3.62}
$$

Now, we are going to establish the estimate r_1. First, we have that $t_\Sigma(D_m-z)^{-1}r_{\Omega_-}$ is bounded from $L^2(\mathbb{R}^3)^4$ into $L^2(\Sigma)^4$. On the other hand, using triangular inequality, we get that

$$
\begin{aligned}
&\Big|\Big|\Big[\Big(\frac{\beta}{2}+\mathscr{C}_{z,m}\Big)^{-1}t_\Sigma-\Big(\frac{\beta}{2}+\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\Big)^{-1}\mathcal{T}_\varepsilon^{-1}t_{\Sigma^\varepsilon}\Big]f\Big|\Big|_{L^2(\Sigma)^4}\\
&\qquad\leqslant\Big|\Big|\Big[\Big(\frac{\beta}{2}+\mathscr{C}_{z,m}\Big)^{-1}-\Big(\frac{\beta}{2}+\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\Big)^{-1}\Big]t_\Sigma f\Big|\Big|_{L^2(\Sigma)^4}\\
&\qquad\qquad+\Big|\Big|\Big(\frac{\beta}{2}+\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\Big)^{-1}\Big[\mathcal{T}_\varepsilon^{-1}t_{\Sigma^\varepsilon}-t_\Sigma\Big]f\Big|\Big|_{L^2(\Sigma)^4}\leqslant q_1+q_2.
\end{aligned}
$$

To prove the estimate q_1, we let $f \in L^2(\Sigma)^4$ and we set $h = \left(\frac{\beta}{2} + \mathscr{C}_{z,m}\right)^{-1} t_\Sigma f$ bounded from $L^2(\Sigma)^4$ into itself. Then, the Cauchy-Schwarz inequality and the following statement

$$\left(\frac{\beta}{2} + \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right)^{-1} - \left(\frac{\beta}{2} + \mathscr{C}_{z,m}\right)^{-1} = \left(\frac{\beta}{2} + \mathscr{C}^{z,\varepsilon}_{m}\right)^{-1} \left(\mathscr{C}_{z,m} - \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right) \left(\frac{\beta}{2} + \mathscr{C}_{z,m}\right)^{-1} \tag{3.63}$$

yields that

$$\begin{aligned} q_1 &= \left|\left|\left(\frac{\beta}{2} + \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right)^{-1} \left(\mathscr{C}_{z,m} - \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right) h\right|\right|_{L^2(\Sigma)^4} \\ &\leqslant \left|\left|\left(\frac{\beta}{2} + \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right)^{-1}\right|\right|_{L^2(\Sigma)^4\to L^2(\Sigma)^4} || \left(\mathscr{C}_{z,m} - \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right) h||_{L^2(\Sigma)^4} \\ &\leqslant || \left(\mathscr{C}_{z,m} - \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right) h||_{L^2(\Sigma)^4} \\ &\lesssim || \left(\mathscr{C}_{z,m} - \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right) ||_{L^2(\Sigma)^4\to L^2(\Sigma)^4} ||h||_{L^2(\Sigma)^4} \lesssim || \left(\mathscr{C}_{z,m} - \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right) ||_{L^2(\Sigma)^4\to L^2(\Sigma)^4} \end{aligned}$$

since $\mathscr{C}_{z,m}$ and $\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon$ are bounded from $L^2(\Sigma)^4$ into itself. Thanks to the estimate (3.62), we get that $q_1 = \mathcal{O}(\varepsilon)$.

To prove the estimate q_2, we have for $x \in \Sigma^\varepsilon$, the following estimate holds in $L^2(\Sigma)^4$

$$\left|\left|t_\Sigma(D_m - z)^{-1} r_{\Omega_-} - \mathcal{T}_\varepsilon^{-1} t_{\Sigma^\varepsilon}(D_m - z)^{-1} r_{\Omega^\varepsilon_-}\right|\right|_{L^2(\mathbb{R}^3)^4\to L^2(\Sigma)^4} = \mathcal{O}(\varepsilon). \tag{3.64}$$

Next, based on (3.44), we immediately get that $\mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon$ is uniformly bounded from $L^2(\Sigma)^4$ into itself. Thus, together with (3.62), (3.64), we deduce that r_2 has a convergence rate of $\mathcal{O}(\varepsilon)$.

Now, for the same reasons as those used to prove the estimate q_2, subsequently, the fact that we have we immediately deduce that $\left(\beta/2 + \mathcal{T}_\varepsilon^{-1}\mathscr{C}^\varepsilon_{z,m}\mathcal{T}_\varepsilon\right)^{-1} = \left(\beta/2 + \mathscr{C}_{z,m}\right)^{-1} + \mathcal{O}(\varepsilon)$ (see the estimate q_1 for more details), we obtain the estimate r_2.

Thus, we conclude that the statement (3.60) is valid in $L^2(\mathbb{R}^3)^4$. The proof of Proposition 3.4.1 is complete. ■

Lemma 3.4.2. *If the Lorentz scalar is $\mu = 2$ (confinement case). We can identify the domain (3.2) by the following form*

$$\mathrm{dom}(\mathscr{D}_L) := \{(\varphi_+, \varphi_-) \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4,\ g \in H^{1/2}(\Sigma)^4,\ P_+\varphi_- = P_-\varphi_+ = 0 \text{ on } \Sigma\},$$

and then, $\mathscr{D}_L = D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega_-}_{\mathrm{MIT}}$, where $D^{\Omega_+}_{\mathrm{MIT}}$ resp. $D^{\Omega_-}_{\mathrm{MIT}}$ is introduced in Section 3.1.1 resp. Proposition 3.4.1.

Proof. Using Plemelj-Sokhotski jump formula from Lemma 1.5.1-(i), and that $\varphi_\pm = t_\Sigma u + C^z_{\pm,m}[g]$, then we get $P_+\varphi_- = -\beta P_- P_+ = 0$ and $P_-\varphi_+ = -\beta P_+ P_- = 0$. Moreover, as $P_+\varphi_- + P_-\varphi_+ =$

$t_\Sigma u + \Lambda^z_{+,m}[g]$, we have that $t_\Sigma u = -\Lambda^z_{+,m}[g]$. ■

***Proof of Proposition* 3.1.3.** For $z \in \rho(\mathscr{D}_L)$, we have the following estimate

$$\begin{aligned}
&\Big|\Big|e_{\Omega^\varepsilon_-} R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(z) r_{\Omega^\varepsilon_-} + e_{\Omega_+} R^{\Omega_+}_{\mathrm{MIT}}(z) r_{\Omega_+} - R_L(z)\Big|\Big|_{L^2(\mathbb{R}^3)^4}\\
&\leqslant \Big|\Big|e_{\Omega^\varepsilon_-} R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(z) r_{\Omega^\varepsilon_-} + e_{\Omega_+} R^{\Omega_+}_{\mathrm{MIT}}(z) r_{\Omega_+} + e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(z) r_{\Omega_-} - e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(z) r_{\Omega_-} - R_L(z)\Big|\Big|_{L^2(\mathbb{R}^3)^4}\\
&\leqslant \Big|\Big|e_{\Omega^\varepsilon_-} R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(z) r_{\Omega^\varepsilon_-} - e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(z) r_{\Omega_-}\Big|\Big|_{L^2(\mathbb{R}^3)^4} +\\
&\qquad\qquad \Big|\Big|e_{\Omega_+} R^{\Omega_+}_{\mathrm{MIT}}(z) r_{\Omega_+} + e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(z) r_{\Omega_-} - R_L(z)\Big|\Big|_{L^2(\mathbb{R}^3)^4}.
\end{aligned}$$

Then, Proposition 3.4.1 and Lemma 3.4.2 yield the statement (1.22). ■

Remark 3.4.1. *For all $f \in L^2(\mathbb{R}^3)^4$, $g \in P_+L^2(\Sigma)^4$ the following convergence holds*

$$||e_{\Omega^\varepsilon_-} E^\varepsilon_m(z)[\mathcal{T}_\varepsilon] - e_{\Omega_-} E^-_m(z)||_{L^2(\Sigma)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(\varepsilon), \tag{3.65}$$

where E^-_m is the lifting operator associated with the boundary value problem $(D_m - z)U = 0$ in Ω_- with $P_+U = 0$ on Σ.

***Proof*.** Now, let me show la convergence considered in (3.65). To this end, let $\tilde g := \mathcal{T}_\varepsilon g \in P_+L^2(\Sigma^\varepsilon)^4$, then we have

$$\begin{aligned}
&\Big|\langle e_{\Omega^\varepsilon_-} E^\varepsilon_m(z)[\mathcal{T}_\varepsilon g], f\rangle_{L^2(\mathbb{R}^3)^4} - \langle e_{\Omega_-} E^-_m(z) g, f\rangle_{L^2(\mathbb{R}^3)^4}\Big|\\
&= \Big|\langle \beta g, \Big(\mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(\bar z) r_{\Omega^\varepsilon_-} - \Gamma_+ R^-_{\mathrm{MIT}}(\bar z) r_{\Omega_-}\Big) f\rangle_{L^2(\Sigma)^4}\Big|\\
&\leqslant ||g||_{L^2(\Sigma)^4}\Big|\Big|(\mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-} e_{\Omega^\varepsilon_-} R^\varepsilon_{\mathrm{MIT}}(\bar z) r_{\Omega^\varepsilon_-} - \Gamma_+ r_{\Omega_-} e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(\bar z) r_{\Omega_-}) f\Big|\Big|_{L^2(\Sigma)^4}\\
&\lesssim \Big|\Big|(\mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-} e_{\Omega^\varepsilon_-} R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(\bar z) r_{\Omega^\varepsilon_-} - \mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-} e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(\bar z) r_{\Omega_-} + \mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-} e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(\bar z) r_{\Omega_-}\\
&\qquad\qquad - \Gamma_+ r_{\Omega_-} e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(\bar z) r_{\Omega_-}) f\Big|\Big|_{L^2(\Sigma)^4}\\
&\lesssim ||\mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-}||_{L^2(\Sigma)^4}\Big|\Big|e_{\Omega^\varepsilon_-} R^{\Omega^\varepsilon_-}_{\mathrm{MIT}}(\bar z) r_{\Omega^\varepsilon_-} f - e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}(\bar z) r_{\Omega_-} f\Big|\Big|_{L^2(\mathbb{R}^3)^4}\\
&\qquad\qquad + \Big|\Big|\mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-} - \Gamma_+ r_{\Omega_-}\Big|\Big|_{L^2(\Sigma)^4} ||e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}} r_{\Omega_-}(\bar z) f||_{L^2(\mathbb{R}^3)^4}.
\end{aligned}$$

Since Γ^ε_+ is bounded form $L^2(\Omega^\varepsilon_-)^4$ to $L^2(\Sigma^\varepsilon)^4$ for ε small enough, then $\mathcal{T}_\varepsilon^{-1}\Gamma^\varepsilon_+ r_{\Omega^\varepsilon_-}$ is bounded in $L^2(\Sigma)^4$. Thus, together with the boundedness of $e_{\Omega_-} R^{\Omega_-}_{\mathrm{MIT}}$ in $L^2(\mathbb{R}^3)^4$ and the convergence established in Proposition 3.1.3, we get

$$\Big|\langle e_{\Omega^\varepsilon_-} E^\varepsilon_m(z)[\mathcal{T}_\varepsilon g], f\rangle_{L^2(\mathbb{R}^3)^4} - \langle e_{\Omega_-} E^-_m(z) g, f\rangle_{L^2(\mathbb{R}^3)^4}\Big| \lesssim \varepsilon, \quad \text{for all } f \in L^2(\mathbb{R}^3)^4.$$

Since this is true for all $g \in L^2(\Sigma)^4$, by duality arguments it follows that

$$||e_{\Omega^\varepsilon_-} E^\varepsilon_m(z)[\mathcal{T}_\varepsilon] - e_{\Omega_-} E^-_m(z)||_{L^2(\Sigma)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}(\varepsilon).$$

■

Lemma 3.4.3. *Let $K \subset \mathbb{C}$ be a compact set. Then, there exists $M_0 > 0$ such that for all $M > M_0$, for $\varepsilon \in (0, \varepsilon_0)$, $K \subset \rho(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(m+M))$, and for $z \in K$ the following estimates hold:*

$$\left|\left|e_{\mathcal{U}^\varepsilon} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{M} \left|\left|f\right|\right|_{L^2(\mathbb{R}^3)^4}, \qquad \forall\ f \in L^2(\mathbb{R}^3)^4,$$

$$\left|\left|\Gamma^\varepsilon_{-+} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\partial\mathcal{U}^\varepsilon)^4} \lesssim \frac{1}{\sqrt{M}} \left|\left|f\right|\right|_{L^2(\mathbb{R}^3)^4}, \qquad \forall\ f \in L^2(\mathbb{R}^3)^4,$$

$$\left|\left|\Gamma^\varepsilon_{-+} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{H^{-1/2}(\partial\mathcal{U}^\varepsilon)^4} \lesssim \frac{1}{M} \left|\left|f\right|\right|_{L^2(\mathbb{R}^3)^4}, \qquad \forall\ f \in L^2(\mathbb{R}^3)^4,$$

$$\left|\left|e_{\mathcal{U}^\varepsilon} \mathcal{E}^\varepsilon_{m+M}(z)(\psi, \mathcal{T}_\varepsilon\varphi)\right|\right|_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{\sqrt{M}} \left|\left|\psi\right|\right|_{L^2(\Sigma)^4} \left|\left|\varphi\right|\right|_{L^2(\Sigma)^4}, \qquad \forall\ (\psi, \mathcal{T}_\varepsilon\varphi) \in P_+ L^2(\Sigma)^4 \oplus P_- L^2(\Sigma^\varepsilon)^4,$$

$$\left|\left|e_{\mathcal{U}^\varepsilon} \mathcal{E}^\varepsilon_{m+M}(z)(\psi, \mathcal{T}_\varepsilon\varphi)\right|\right|_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{M} \left|\left|\psi\right|\right|_{H^{1/2}(\Sigma)^4} \left|\left|\varphi\right|\right|_{H^{1/2}(\Sigma)^4}, \qquad \forall\ (\psi, \mathcal{T}_\varepsilon\varphi) \in P_+ H^{1/2}(\Sigma)^4 \oplus P_- H^{1/2}(\Sigma^\varepsilon)^4.$$

Proof. Using the same arguments as in the proof of [BBZ37, Lemma 6.1], we can show the above estimates with respect to M. First, I want to show the claimed estimates for $e_{\mathcal{U}^\varepsilon} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon}$ and $\Gamma^\varepsilon_{-} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon}$. For this, fix a compact set $K \subset \mathbb{C}$, and note that for $z \in K$ and $M_1 > \sup_{z\in K}\{|\mathrm{Re}(z)| - m\}$ it holds that $K \subset \rho(D_{m+M_1})$, and hence $K \subset \rho(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}})$ for all $M > M_1$. Let $f \in L^2(\mathbb{R}^3)^4$. We have that

$$||e_{\mathcal{U}^\varepsilon} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f||_{L^2(\mathbb{R}^3)^4} = ||R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f||_{L^2(\mathcal{U}^\varepsilon)^4}.$$

Now, for $r_{\mathcal{U}^\varepsilon} f \in L^2(\mathcal{U}^\varepsilon)^4$ and $\varphi \in \mathrm{dom}(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}})$, then a straightforward application of the Green's formula yields that

$$\|D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} = \|(\alpha\cdot\nabla)\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} + (m+M)^2 \left|\left|\varphi\right|\right|^2_{L^2(\mathcal{U}^\varepsilon)^4} + (m+M) \left|\left|P^{\mathcal{U}^\varepsilon}_{-} t_{\partial\mathcal{U}^\varepsilon}\varphi\right|\right|^2_{L^2(\partial\mathcal{U}^\varepsilon)^4},$$

with $P^{\mathcal{U}^\varepsilon}_{-} t_{\partial\mathcal{U}^\varepsilon} = P_- t_\Sigma + P_+ t_{\Sigma^\varepsilon}$. Using this and the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}
\|(D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}} - z)\varphi\|^2{}_{L^2(\mathcal{U}^\varepsilon)^4} &= \|D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} + |z|^2\|\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} - 2\mathrm{Re}(z)\langle D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}\varphi, \varphi\rangle_{L^2(\mathcal{U}^\varepsilon)^4} \\
&\geqslant \|D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} + |z|^2\|\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} - \frac{1}{2}\|D^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} - 2|\mathrm{Re}(z)|^2\|\varphi\|^2_{L^2(\mathcal{U}^\varepsilon)^4} \\
&\geqslant \left(\frac{(m+M)^2}{2} + |\mathrm{Im}(z)|^2 - |\mathrm{Re}(z)|^2\right) ||\varphi||^2_{L^2(\mathcal{U}^\varepsilon)^4} + \frac{M}{2} \left|\left|P^{\mathcal{U}^\varepsilon}_{-} t_{\partial\mathcal{U}^\varepsilon}\varphi\right|\right|^2_{L^2(\partial\mathcal{U}^\varepsilon)^4}.
\end{aligned}$$

Therefore, taking $R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f = \varphi$ and $M \geqslant M_2 \geqslant \sup_{z\in K}\{\sqrt{|\mathrm{Re}(z)|^2 - |\mathrm{Im}(z)|^2} - m\}$ we obtain the inequality

$$\left|\left|R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\mathcal{U}^\varepsilon)^4} + \frac{1}{\sqrt{M}} \left|\left|\Gamma^\varepsilon_{-+} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\partial\mathcal{U}^\varepsilon)^4} \lesssim \frac{1}{M} \left|\left|f\right|\right|_{L^2(\mathbb{R}^3)^4}, \text{ with } \partial\mathcal{U}^\varepsilon = \Sigma \cup \Sigma^\varepsilon.$$

Thus

$$\left|\left|e_{\mathcal{U}^\varepsilon} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{M} \left|\left|f\right|\right|_{L^2(\mathbb{R}^3)^4}, \quad \text{and} \quad \left|\left|\Gamma^\varepsilon_{-+} R^{\mathcal{U}^\varepsilon}_{\mathrm{MIT}}(z) r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\partial\mathcal{U}^\varepsilon)^4} \lesssim \frac{1}{\sqrt{M}} \left|\left|f\right|\right|_{L^2(\mathbb{R}^3)^4}.$$

Since $\Gamma^{\varepsilon}_{-+} := (\Gamma_-, \Gamma^{\varepsilon}_+)$ is bounded from $L^2(\mathcal{U}^{\varepsilon})^4$ into $H^{-1/2}(\partial\mathcal{U}^{\varepsilon})^4$ for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small, it follows from the above inequality that

$$\left|\left|\Gamma^{\varepsilon}_{-+} R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f\right|\right|_{H^{-1/2}(\partial\mathcal{U}^{\varepsilon})^4} \lesssim ||\Gamma^{\varepsilon}_{-+}||_{L^2(\mathcal{U}^{\varepsilon})^4 \to H^{-1/2}(\partial\mathcal{U}^{\varepsilon})^4} \left|\left|R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f\right|\right|_{L^2(\mathcal{U}^{\varepsilon})^4} \lesssim \frac{1}{M} ||f||_{L^2(\mathbb{R}^3)^4},$$

for any $f \in L^2(\mathbb{R}^3)^4$, which gives the last inequality.

Let us now turn to the proof of the claimed estimates for $e_{\mathcal{U}^{\varepsilon}} \mathcal{E}^{\varepsilon}_{m+M}(z)$. Let f, ψ belong to $L^2(\mathbb{R}^3)^4$ and $L^2(\Sigma)^4$, respectively, and consider the transformation operator $\mathcal{T}_{\varepsilon}$ defined in (3.1.2). For $\varphi \in L^2(\Sigma)^4$, we set $\varphi_{\varepsilon} = \mathcal{T}_{\varepsilon}\varphi \in L^2(\Sigma^{\varepsilon})$. We mention that $\beta(\Gamma_-, \Gamma^{\varepsilon}_+) R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(\overline{z})$ is the adjoint of the operator $\mathcal{E}^{\varepsilon}_{m+M}(z) : P_+ L^2(\Sigma)^4 \oplus P_- L^2(\Sigma^{\varepsilon})^4 \longrightarrow L^2(\mathcal{U}^{\varepsilon})^4$. Using this and the estimate fulfilled by $(\Gamma_-, \Gamma^{\varepsilon}_+) R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(\overline{z}) r_{\mathcal{U}^{\varepsilon}}$ we obtain that

$$\begin{aligned}
\left|\langle f, e_{\mathcal{U}^{\varepsilon}} \mathcal{E}^{\varepsilon}_{m+M}(z)(\psi, \varphi_{\varepsilon})\rangle_{L^2(\mathbb{R}^3)^4}\right| &= \left|\langle (\Gamma_-, \mathcal{T}^{-1}_{\varepsilon}\Gamma^{\varepsilon}_+) R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(\overline{z}) r_{\mathcal{U}_{\varepsilon}} f, \beta(\psi, \varphi)\rangle_{L^2(\Sigma)^4}\right| \\
&\leqslant \left|\left|(\Gamma_-, \mathcal{T}^{-1}_{\varepsilon}\Gamma^{\varepsilon}_+) R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f\right|\right|_{L^2(\Sigma)^4} ||\psi||_{L^2(\Sigma)^4} ||\varphi||_{L^2(\Sigma)^4} \\
&\leqslant ||\psi||_{L^2(\Sigma)^4} ||\varphi||_{L^2(\Sigma)^4} \left|\left|\mathcal{T}^{-1}_{\varepsilon}\right|\right|_{L^2(\partial\mathcal{U}^{\varepsilon})^4 \to L^2(\Sigma)^4} \left|\left|\Gamma^{\varepsilon}_{-+} R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f\right|\right|_{L^2(\partial\mathcal{U}^{\varepsilon})^4} \\
&\lesssim \frac{1}{\sqrt{M}} ||f||_{L^2(\mathbb{R}^3)^4} ||\psi||_{L^2(\Sigma)^4} ||\varphi||_{L^2(\Sigma)^4}.
\end{aligned}$$

So, we get

$$||e_{\mathcal{U}^{\varepsilon}} \mathcal{E}^{\varepsilon}_{m+M}(z)(\psi, \mathcal{T}_{\varepsilon}\varphi)||_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{\sqrt{M}} ||\psi||_{L^2(\Sigma)^4} ||\varphi||_{L^2(\Sigma)^4}.$$

Similarly, we established the last inequality of the lemma andthis finishes the proof of the lemma. ■

The last ingredient to prove Theorem 3.1.5 is to show that the second term in the ride hand side of the resolvent formula (3.57) converges to zero when M converges to ∞, (*i.e.*, $h = \varepsilon = M^{-1} \to 0$).

***Proof of Proposition* 3.1.4.** Recall the following notations: $D^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}} = D^{\Omega_+}_{\mathrm{MIT}} \oplus D^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}$ and $R^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}} = R^{\Omega_+}_{\mathrm{MIT}} \oplus R^{\Omega^{\varepsilon}_-}_{\mathrm{MIT}}$, with $\Omega^{\varepsilon}_{+-} = \Omega_+ \cup \Omega^{\varepsilon}_-$. Let $z \in \rho(\mathfrak{D}^{\varepsilon}_M) \cap \rho(D^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}})$ and $f \in L^2(\mathbb{R}^3)^4$. From the resolvent formula (3.57) and Remark 3.3.1, together give us the following

$$\begin{aligned}
||R^{\varepsilon}_M(z) - e_{\Omega^{\varepsilon}_{+-}} R^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}}(z) r_{\Omega^{\varepsilon}_{+-}}||_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} &\leqslant ||e_{\mathcal{U}^{\varepsilon}} R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f||_{L^2(\mathbb{R}^3)^4} \\
&+ ||E^{\Omega^{\varepsilon}_{+-}}_m(z) \Xi^{\varepsilon,-+}_M(z) \mathscr{A}^{\varepsilon}_{m+M} \Gamma^{\varepsilon}_{+-} R^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}}(z) r_{\Omega^{\varepsilon}_{+-}} f||_{L^2(\Omega^{\varepsilon}_{+-})^4} \\
&+ ||E^{\Omega^{\varepsilon}_{+-}}_m(z) \Xi^{\varepsilon,-+}_M(z) \Gamma^{\varepsilon}_{-+} R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f||_{L^2(\Omega^{\varepsilon}_{+-})^4} \\
&+ ||\mathcal{E}^{\varepsilon}_{m+M}(z) \Xi^{\varepsilon,+-}_M(z) \Gamma^{\varepsilon}_{+-} R^{\Omega^{\varepsilon}_{+-}}_{\mathrm{MIT}}(z) r_{\Omega^{\varepsilon}_{+-}} f||_{L^2(\mathcal{U}^{\varepsilon})^4} \\
&+ ||\mathcal{E}^{\varepsilon}_{m+M}(z) \Xi^{\varepsilon,+-}_M(z) \mathscr{A}^{\Omega^{\varepsilon}_{+-}}_m \Gamma^{\varepsilon}_{-+} R^{\mathcal{U}^{\varepsilon}}_{\mathrm{MIT}}(z) r_{\mathcal{U}^{\varepsilon}} f||_{L^2(\mathcal{U}^{\varepsilon})^4} \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

We start with J_1. From the second item of Lemma 3.4.3, we get that $J_1 \lesssim M^{-1}$. Now, thanks to the uniform bound (with respect to M) of $\Xi_M^{\varepsilon,\pm\mp}$, see Corollary 3.3.1, J_2, J_3, J_4, J_5 become as follows

$$\begin{aligned}
J_2 &\lesssim ||E_m^{\Omega_{+-}^\varepsilon}(z)||_{L^2(\Omega_{+-}^\varepsilon)^4}||\mathscr{A}_{m+M}^\varepsilon||_{H^{-1/2}(\Sigma)^4\oplus H^{-1/2}(\Sigma^\varepsilon)^4}||\Gamma_{+-}^\varepsilon R_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(z)r_\Omega f||_{H^{1/2}(\Sigma)^4\oplus H^{1/2}(\Sigma^\varepsilon)^4},\\
J_3 &\lesssim ||E_m^{\Omega_{+-}^\varepsilon}(z)||_{H^{-1/2}(\Sigma)^4\oplus H^{-1/2}(\Sigma^\varepsilon)^4\to L^2(\Omega_{+-}^\varepsilon)^4}||\Gamma_{-+}^\varepsilon R_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(z)r_{\mathcal{U}^\varepsilon} f||_{H^{-1/2}(\Sigma)^4\oplus H^{-1/2}(\Sigma^\varepsilon)^4},\\
J_4 &\lesssim ||\mathcal{E}_{m+M}^\varepsilon(z)||_{H^{1/2}(\Sigma)^4\oplus H^{1/2}(\Sigma^\varepsilon)^4\to L^2(\mathcal{U}^\varepsilon)^4}||\Gamma_{+-}^\varepsilon R_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(z)r_\Omega f||_{H^{1/2}(\Sigma)^4\oplus H^{1/2}(\Sigma^\varepsilon)^4},\\
J_5 &\lesssim ||\mathcal{E}_{m+M}^\varepsilon(z)||_{L^2(\mathcal{U}^\varepsilon)^4}||\mathscr{A}_m^{\Omega_{+-}^\varepsilon}||_{L^2(\Sigma)^4\oplus L^2(\Sigma^\varepsilon)^4}||\Gamma_{-+}^\varepsilon R_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(z)r_{\mathcal{U}^\varepsilon} f||_{L^2(\Sigma)^4\oplus L^2(\Sigma^\varepsilon)^4}.
\end{aligned}$$

Notice that the terms $E_m^{\Omega_{+-}^\varepsilon}$, $\mathscr{A}_m^{\Omega_{+-}^\varepsilon}$, and $\Gamma_{+-}^\varepsilon R_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(z)$ are bounded operators for all $\varepsilon \in (0,\varepsilon_0)$, everywhere defined and do not depend on M. Now, thanks to Lemma 3.4.3, $\Gamma_{-+}^\varepsilon R_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(z)r_{\mathcal{U}^\varepsilon}$ and $e_{\mathcal{U}^\varepsilon}\mathcal{E}_{m+M}^\varepsilon(z)$ hold the following estimate

$$\left|\left|\Gamma_{-+}^\varepsilon R_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(z)r_{\mathcal{U}^\varepsilon} f\right|\right|_{L^2(\partial\mathcal{U}^\varepsilon)^4} \lesssim \frac{1}{\sqrt{M}}\,||f||_{L^2(\mathbb{R}^3)^4}$$

$$\text{and } \left|\left|\Gamma_{-+}^\varepsilon R_{\mathrm{MIT}}^{\mathcal{U}^\varepsilon}(z)r_{\mathcal{U}^\varepsilon} f\right|\right|_{H^{-1/2}(\partial\mathcal{U}^\varepsilon)^4} \lesssim \frac{1}{M}\,||f||_{L^2(\mathbb{R}^3)^4},$$

$$||e_{\mathcal{U}^\varepsilon}\mathcal{E}_{m+M}^\varepsilon(z)(\psi,\mathcal{T}_\varepsilon\varphi)||_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{\sqrt{M}}\,||\psi||_{L^2(\Sigma)^4}\,||\varphi||_{L^2(\Sigma)^4},$$

$$||e_{\mathcal{U}^\varepsilon}\mathcal{E}_{m+M}^\varepsilon(z)(\psi,\mathcal{T}_\varepsilon\varphi)||_{L^2(\mathbb{R}^3)^4} \lesssim \frac{1}{M}\,||\psi||_{H^{1/2}(\Sigma)^4}\,||\varphi||_{H^{1/2}(\Sigma)^4}.$$

Thus, from the above estimates, we deduce that

$$J_k \lesssim M^{-1}||f||_{L^2(\mathbb{R}^3)^4}, \quad \forall\, k \in \{3,4,5\}.$$

Moreover, the following lower bound of $\mathcal{A}_{m+M}^\varepsilon$, see Corollary (3.59),

$$||\mathcal{A}_{m+M}^\varepsilon||_{H^{1/2}(\Sigma)^4\oplus H^{1/2}(\Sigma^\varepsilon)^4\to H^{-1/2}(\Sigma)^4\oplus H^{-1/2}(\Sigma^\varepsilon)^4} \lesssim M^{-1},$$

yields that $J_2 \lesssim M^{-1}||f||_{L^2(\mathbb{R}^3)^4}$. Thus, we obtain the estimate

$$||R_M^\varepsilon(z) - e_{\Omega_{+-}^\varepsilon} R_{\mathrm{MIT}}^{\Omega_{+-}^\varepsilon}(z)r_{\Omega_{+-}^\varepsilon}||_{L^2(\mathbb{R}^3)^4\to L^2(\mathbb{R}^3)^4} \lesssim M^{-1}||f||_{L^2(\mathbb{R}^3)^4}.$$

And this achieves the proof of the proposition. ■

Thus, Theorem 3.1.5 is then obtained by a simple combination of Propositions 3.1.3, 3.1.4.

★ ★ ★ ★ ★ ★

3.5 Appendix

For a better understanding of the construction of the approximation of the solutions $A_j(y,\xi,\tau)$ and the order of the coefficients $B_{j,k}(y,\xi)$ as well as the proof of Proposition 3.2.4, an explicit calculation is presented in this appendix, which aims to obtain an exact form of the solutions $A_j(y,\xi,\tau)$ for $j=1,2$.

For $j=1$, we define $A_1(y,\xi,\tau)$ inductively by

$$\begin{cases} h\partial_\tau A_1(y,\xi,\tau) = L_0(y,\xi)A_1(y,\xi,\tau) + \Big(L_1 + (\alpha\cdot\tilde{n}^\varphi c_3)L_0 - i\partial_\xi L_0\cdot\partial_y\Big)A_0(y,\xi,\tau), \\ P_+A_1(y,\xi,\varepsilon)=0, \end{cases} \tag{3.66}$$

we have $\partial_\xi L_0(y,\xi)\cdot\partial_y = i\alpha\cdot\tilde{n}^\varphi(\alpha\cdot\partial_y) := a_0(y)(\alpha\cdot\partial_y)$, with $a_0(y)=i\alpha\cdot\tilde{n}^\varphi$. The solution of the differential system (3.66) is

$$\begin{aligned} A_1(y,\xi,\tau) &= e^{h^{-1}L_0(\tau-\varepsilon)}A_1(y,\xi,\varepsilon) \\ &\quad + e^{h^{-1}L_0\tau}\int_\varepsilon^\tau e^{-h^{-1}L_0(y,\xi)s}\Big(L_1+(\alpha\cdot\tilde{n}^\varphi c_3)L_0 - i\partial_\xi L_0(y,\xi)\cdot\partial_y\Big)A_0(y,\xi,\tau)\mathrm{d}s \\ &= e^{h^{-1}L_0(\tau-\varepsilon)}A_1(y,\xi,\varepsilon) \\ &\quad + e^{h^{-1}L_0\tau}\int_\varepsilon^\tau e^{-h^{-1}L_0 s}a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)A_0(y,\xi,\tau)\mathrm{d}s \\ &:= I_1+I_2, \end{aligned}$$

where I_1 and I_2 have the following quantity:

$$\begin{aligned} I_1 &= \Big(e^{(\tau-\varepsilon)\varrho_-(y,\xi)}\Pi_- + e^{(\tau-\varepsilon)\varrho_-(y,\xi)}\Pi_+\Big)A_1(y,\xi,\varepsilon), \\ I_2 &= e^{h^{-1}L_0(y,\xi)\tau}\int_\varepsilon^\tau e^{-h^{-1}L_0(y,\xi)s}a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)A_0(y,\xi,s)\mathrm{d}s. \end{aligned}$$

Now, to obtain an explicit form of I_2, let's decompose the quantity $e^{-h^{-1}L_0(y,\xi)s}$. To do this, we have

$$\begin{aligned} &\int_\varepsilon^\tau e^{-h^{-1}L_0(y,\xi)s}a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)A_0(y,\xi,s)\mathrm{d}s \\ &= \int_\varepsilon^\tau\Big(e^{-h^{-1}s\varrho_-(y,\xi)}\Pi_- + e^{-h^{-1}s\varrho_+(y,\xi)}\Pi_+\Big)a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)A_0(y,\xi,s)\mathrm{d}s \\ &= \int_\varepsilon^\tau\Big(e^{-h^{-1}s\varrho_-}\Pi_- + e^{-h^{-1}s\varrho_+}\Pi_+\Big)a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)\left(e^{h^{-1}(s-\varepsilon)\varrho_-}\frac{\Pi_-P_+}{k_-^\varphi}\right)\mathrm{d}s \\ &= \underbrace{\int_\varepsilon^\tau e^{-h^{-1}s\varrho_-}\Pi_-a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)\left(e^{h^{-1}(s-\varepsilon)\varrho_-}\frac{\Pi_-P_+}{k_-^\varphi}\right)\mathrm{d}s}_{(1)} \\ &+ \underbrace{\int_\varepsilon^\tau e^{-h^{-1}s\varrho_+}\Pi_+a_0(y)\Big(-z+c\cdot\xi-ic_3L_0-i\alpha\cdot\partial_y\Big)\left(e^{h^{-1}(s-\varepsilon)\varrho_-}\frac{\Pi_-P_+}{k_-^\varphi}\right)\mathrm{d}s}_{(2)}\,. \end{aligned} \tag{3.67}$$

First of all, note that the quantity

$$(-z + c\cdot\xi - ic_3L_0 - i\alpha\cdot\partial_y)\Big(e^{h^{-1}\varrho_-(\tau-\varepsilon)}\mathfrak{M}\Big) = e^{h^{-1}\varrho_-(\tau-\varepsilon)}(a + b\cdot\xi - ih^{-1}(\tau-\varepsilon)\alpha\cdot\partial_y\varrho_-)\mathfrak{M},$$

with $\mathfrak{M}\in\mathscr{M}_4(\mathbb{C})$ and

$$a = -z + c_3\alpha\cdot\tilde{n}^{\varphi}\beta - i\alpha\cdot\partial_y \quad \text{and} \quad b = c + c_3\alpha\cdot\tilde{n}^{\varphi}\alpha \tag{3.68}$$

belong to $\mathscr{M}_4(\mathbb{C})$. Note also the term $\alpha\cdot\partial_y$ in the quantity a is applies to $\dfrac{\Pi_-P_+}{k_-^{\varphi}}$ in the following calculation. Now, we want to explain the quantities (1) and (2) given in (3.67). Let's start with (1):

$$\begin{aligned}(1) &= \int_{\varepsilon}^{\tau} e^{-h^{-1}s\varrho_-}\Pi_-a_0(y)\Big(-z + c\cdot\xi - ic_3L_0 - i\alpha\cdot\partial_y\Big)\left(e^{h^{-1}(s-\varepsilon)\varrho_-}\frac{\Pi_-P_+}{k_-^{\varphi}}\right)\mathrm{d}s\\ &= \int_{\varepsilon}^{\tau} e^{-\varepsilon h^{-1}\varrho_-}\Pi_-a_0(y)\Big(a + (b\cdot\xi) - ih^{-1}(\tau-\varepsilon)\alpha\cdot\partial_y\varrho_-\Big)\frac{\Pi_-P_+}{k_-^{\varphi}}\mathrm{d}s\\ &= (\tau-\varepsilon)e^{-\varepsilon h^{-1}\varrho_-}\Pi_-a_0(y)\Big(a + b\cdot\xi\Big)B_{0,0} - ih^{-1}(\tau-\varepsilon)^2e^{-\varepsilon h^{-1}\varrho_-}\Pi_-a_0(y)\Big(\frac{\alpha\cdot\partial_y\varrho_-}{2}\Big)B_{0,0},\end{aligned} \tag{3.69}$$

with $B_{0,0}(y,\xi) = \dfrac{\Pi_-P_+}{k_-^{\varphi}}\in\mathcal{S}^0$.

Similarly, for (2) we get

$$\begin{aligned}(2) &= \int_{\varepsilon}^{\tau} e^{-h^{-1}s\varrho_+}\Pi_+a_0(y)\Big(-z + c\cdot\xi - ic_3L_0 - i\alpha\cdot\partial_y\Big)\left(e^{h^{-1}(s-\varepsilon)\varrho_-}B_{0,0}\right)\mathrm{d}s\\ &= e^{-\varepsilon h^{-1}\varrho_-}\int_{\varepsilon}^{\tau} e^{h^{-1}s(\varrho_--\varrho_+)}\Pi_+a_0(y)\Big(a + b\cdot\xi - ih^{-1}(s-\varepsilon)\alpha\cdot\partial_y\varrho_-\Big)B_{0,0}\mathrm{d}s\\ &= e^{-\varepsilon h^{-1}\varrho_-}h(\varrho_- - \varrho_+)^{-1}\Pi_+a_0(y)\left(e^{h^{-1}(\varrho_--\varrho_+)\tau} - e^{h^{-1}(\varrho_--\varrho_+)\varepsilon}\right)\Big(a + b\cdot\xi\Big)B_{0,0}\\ &+ e^{-\varepsilon h^{-1}\varrho_-}e^{h^{-1}(\varrho_--\varrho_+)\tau}\Pi_+a_0(y)\left[\frac{-i(\tau-\varepsilon)\alpha\cdot\partial_y\varrho_-}{\varrho_- - \varrho_+} + \frac{h\,i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0}\\ &+ e^{-\varepsilon h^{-1}\varrho_-}e^{h^{-1}(\varrho_--\varrho_+)\varepsilon}\Pi_+a_0(y)\left[\frac{-h\,i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0}.\end{aligned} \tag{3.70}$$

Putting the formula of (1) and (2) as in (3.69) and (3.70), respectively, in I_2. Together, with I_1, we obtain that

$$\begin{aligned}A_1(y,\xi,\tau) &= \left(e^{h^{-1}(\tau-\varepsilon)\varrho_-}\Pi_- + e^{h^{-1}(\tau-\varepsilon)\varrho_+}\Pi_+\right)A_1(y,\xi,\varepsilon)\\ &+ e^{h^{-1}\varrho_-(\tau-\varepsilon)}\Pi_-a_0(y)\left[(\tau-\varepsilon)\Big(a + (b\cdot\xi)\Big)B_{0,0} - ih^{-1}(\tau-\varepsilon)^2\Big(\frac{\alpha\cdot\partial_y\varrho_-}{2}\Big)B_{0,0}\right]\\ &+ \frac{h}{(\varrho_- - \varrho_+)}\Pi_+a_0(y)e^{h^{-1}\varrho_-(\tau-\varepsilon)}\Big(a + (b\cdot\xi)\Big)B_{0,0}\\ &+ e^{h^{-1}\varrho_-(\tau-\varepsilon)}\Pi_+a_0(y)\left[\frac{-i(\tau-\varepsilon)\alpha\cdot\partial_y\varrho_-}{\varrho_- - \varrho_+} + \frac{h\,i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0}\\ &+ e^{h^{-1}\varrho_+(\tau-\varepsilon)}\Pi_+a_0(y)\left[-i\frac{h\;\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0} - e^{h^{-1}\varrho_+(\tau-\varepsilon)}\frac{h\;\Pi_+a_0(y)}{(\varrho_- - \varrho_+)}\Big(aB_{0,0} + b\cdot\xi B_{0,0}\Big).\end{aligned}$$

Thanks to the properties of ϱ_+ given in (3.15), and the fact that $e^{h^{-1}(\tau-\varepsilon)\varrho_+}\Pi_+ a_0(y)$ is unbounded in $L^2(\{\tau > \varepsilon\})$, then we look $A_1(\tilde{y},\xi,\varepsilon)$ such that

$$\Pi_+ A_1(y,\xi,\varepsilon) = h\ \frac{\Pi_+ a_0}{\varrho_- - \varrho_+}\Big(a + b\cdot\xi + \frac{i\alpha\cdot\partial_y\varrho_-}{\varrho_- - \varrho_+}\Big)B_{0,0}. \tag{3.71}$$

Thus, we obtain

$$\begin{aligned} A_1(y,\xi,\tau) = e^{h^{-1}(\tau-\varepsilon)\varrho_-} \times \Big\{ &\Pi_- A_1(y,\xi,\varepsilon) + h\ \Pi_+ a_0(y)(\varrho_- - \varrho_+)\left[a + b\cdot\xi + \frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)}\right]B_{0,0} \\ &+ (\tau-\varepsilon)\left[\Pi_- a_0(y)(a+b\cdot\xi) - \Pi_+ a_0(y)\frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)}\right]B_{0,0} \\ &+ h^{-1}(\tau-\varepsilon)^2\Pi_- a_0(y)\Big(\frac{-i\alpha\cdot\partial_y\varrho_-}{2}\Big)B_{0,0}\Big\}. \end{aligned} \tag{3.72}$$

Calculate of $\Pi_- A_1(y,\xi,\varepsilon)$. From (3.72), we get that

$$A_1(y,\xi,\varepsilon) = \Pi_-(P_- + P_+)A_1(y,\xi,\varepsilon) + \frac{h\ \Pi_+ a_0(y)}{(\varrho_- - \varrho_+)}\left[a + b\cdot\xi + \frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)}\right]B_{0,0}.$$

From (3.66) we have $P_+ A_1(y,\xi,\varepsilon) = 0$, then

$$P_- A_1(y,\xi,\varepsilon) = P_-\Pi_- P_- A_1(y,\xi,\varepsilon) + \frac{h\ \Pi_+ a_0(y)}{(\varrho_- - \varrho_+)}\left[a + b\cdot\xi + \frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)}\right]B_{0,0}.$$

Thanks to the relations (3.16), we obtain

$$\Pi_- P_- A_1(y,\xi,\varepsilon) = \frac{h\,\Pi_- a_0 P_+}{(\varrho_- - \varrho_+)}\left(\mathbb{I}_4 - \frac{\Theta^\varphi}{k^{\varphi_-}}\right)\left[a + b\cdot\xi + \frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)}\right]B_{0,0},$$

and so (3.72) becomes as follows

$$\begin{aligned} A_1(y,\xi,\tau) = e^{h^{-1}(\tau-\varepsilon)\varrho_-} \times \Big\{ &h\left[\Pi_- a_0\left(P_+ - \frac{P_+\Theta^\varphi}{k^{\varphi_-}}\right) + \Pi_+ a_0\right]\left[\frac{a+b\cdot\xi}{(\varrho_- - \varrho_+)} + \frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0} \\ &+ (\tau-\varepsilon)\left[\Pi_- a_0(y)(a+b\cdot\xi) - \Pi_+ a_0(y)\frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)}\right]B_{0,0} \\ &+ h^{-1}(\tau-\varepsilon)^2\Pi_- a_0(y)\Big(\frac{-i\alpha\cdot\partial_y\varrho_-}{2}\Big)B_{0,0}\Big\}. \end{aligned}$$

Consequently, we get that

$$\begin{aligned} A_1(y,\xi,\tau) &= e^{h^{-1}\varrho_-(\tau-\varepsilon)}\Big\{B_{1,0}(y,\xi) + (h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+))B_{1,1}(y,\xi) \\ &\qquad + (h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+))^2 B_{1,2}(y,\xi)\Big\} \\ &= e^{h^{-1}(\tau-\varepsilon)\varrho_-}\sum_{k=0}^{2}\Big(h^{-1}(\tau-\varepsilon)(\varrho_- - \varrho_+)\Big)^k B_{1,k}(y,\xi), \end{aligned} \tag{3.73}$$

where,

$$B_{1,0}(y,\xi) = h\left[\Pi_- a_0\left(P_+ - \frac{P_+\,\Theta^\varphi}{k_-^\varphi}\right) + \Pi_+ a_0\right]\left[\frac{a+b\cdot\xi}{(\varrho_- - \varrho_+)} + \frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0},$$

$$B_{1,1}(y,\xi) = h\left[\Pi_- a_0(y)\frac{(a+b\cdot\xi)}{(\varrho_- - \varrho_+)} - \Pi_+ a_0(y)\frac{i\alpha\cdot\partial_y\varrho_-}{(\varrho_- - \varrho_+)^2}\right]B_{0,0},$$
$$B_{1,2}(y,\xi) = -h\ \Pi_- a_0(y)\Big(\frac{i\alpha\cdot\partial_y\varrho_-}{2(\varrho_- - \varrho_+)^2}\Big)B_{0,0},$$

with $\varrho_- - \varrho_+ = -2\lambda(y,\xi)\in\mathcal{S}^1$ and $\partial_y\varrho_-\in\mathcal{S}^1$ identified with $\langle\xi\rangle$, then $B_{1,k}\in h\,\mathcal{S}^0$ for $k=0,1$, and $B_{1,2}\in h\,\mathcal{S}^{-1}$.
Let's look at the form of A_j for $j=2$. To do it, we define $A_2(y,\xi,\tau)$ inductively by

$$\begin{cases} h\partial_\tau A_2(y,\xi,\tau) = L_0(y,\xi)A_2(y,\xi,\tau) + \Big(\widetilde{L}_1(y,\xi) - i\partial_\xi L_0(y,\xi)\cdot\partial_y\Big)A_1(y,\xi,\tau) \\ \qquad\qquad + \Big((\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1(y,\xi) - i\partial_\xi\widetilde{L}_1(y,\xi)\cdot\partial_y\Big)A_0(y,\xi,\tau), \\ P_+A_2(y,\xi,\varepsilon) = 0, \end{cases}$$

where,

$$\begin{aligned} &(\widetilde{L}_1 - i\partial_\xi L_0\cdot\partial_y)\Big(e^{h^{-1}\varrho_-(\tau-\varepsilon)}\mathfrak{M}\Big) = e^{h^{-1}\varrho_-(\tau-\varepsilon)}a_0(y)(a+b\cdot\xi - ih^{-1}(\tau-\varepsilon)\alpha\cdot\partial_y\varrho_-)\mathfrak{M}, \\ &(-i\partial_\xi\widetilde{L}_1\cdot\partial_y + (\alpha\cdot\tilde{n}^\varphi c_3)\widetilde{L}_1)\Big(e^{h^{-1}\varrho_-(\tau-\varepsilon)}\mathfrak{M}\Big) = \\ &\qquad\qquad e^{h^{-1}\varrho_-(\tau-\varepsilon)}a_0(y)\Big(d + e\cdot\xi - ih^{-1}(\tau-\varepsilon)f\cdot\partial_y\varrho_-\Big)\mathfrak{M}, \end{aligned} \tag{3.74}$$

with $\mathfrak{M}$, $a,\ b$ were noted in (3.68), $d,\ e,\ f$ belong to $\mathscr{M}_4(\mathbb{C})$, where d, e and f are the following

$$\begin{aligned} &d = (c_3\alpha\cdot\tilde{n}^\varphi)^2\beta - c_3\alpha\cdot\tilde{n}^\varphi z - i(c + c_3\alpha\cdot\tilde{n}^\varphi\alpha)\cdot\partial_y, \\ &e = (c_3\alpha\cdot\tilde{n}^\varphi + (c_3\alpha\cdot\tilde{n}^\varphi)^2\alpha)\cdot\xi \quad \text{and} \quad f = c + c_3\alpha\cdot\tilde{n}^\varphi\alpha. \end{aligned} \tag{3.75}$$

Then, after a many calculation, we arrive at the following formula

$$\begin{aligned} A_2(y,\xi,\tau) &= e^{h^{-1}\varrho_-(\tau-\varepsilon)}\Big\{\big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\big)^0 B_{2,0}(y,\xi) + \big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\big)^1 B_{2,1}(y,\xi) \\ &\quad + \big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\big)^2 B_{2,2}(y,\xi) + \big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\big)^3 B_{2,3}(y,\xi) + \big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\big)^4 B_{2,4}(y,\xi)\Big\} \\ &=: e^{h^{-1}\varrho_-(\tau-\varepsilon)}\sum_{k=0}^{4}\Big(h^{-1}(\tau-\varepsilon)\langle\xi\rangle\Big)^k B_{1,k}(y,\xi), \end{aligned} \tag{3.76}$$

where,

$$B_{2,4}(y,\xi) = h\ \Pi_- a_0(y)\left[\frac{(\alpha\cdot\partial_y\varrho_-)\,B_{1,2}}{4(\varrho_- - \varrho_+)^2}\right],$$

$$B_{2,3}(y,\xi) = h\ \Pi_- a_0(y)\left[\frac{aB_{1,2}}{3\langle\xi\rangle} + \frac{bB_{1,2}}{3} + \frac{(\alpha\cdot\partial_y\varrho_-)B_{1,1}}{3(\varrho_- - \varrho_+)^2}\right] + h\ \Pi_+ a_0(y)\left[\frac{(\alpha\cdot\partial_y\varrho_-)(B_{1,2})}{(\varrho_- - \varrho_+)\langle\xi\rangle}\right],$$

$$\begin{aligned}B_{2,0}(y,\xi) = h\ &\left[\Pi_- a_0\left(P_+ - \frac{P_+\,\Theta^\varphi}{k_-^\varphi}\right) + \Pi_+ a_0\right]\left[\frac{aB_{1,0} + (b\cdot\xi)B_{1,0} + dB_{0,0} + (e\cdot\xi)B_{0,0}}{(\varrho_- - \varrho_+)}\right.\\ &- \frac{(\alpha\cdot\partial_y\varrho_-)B_{1,0} + \langle\xi\rangle aB_{1,1} + \langle\xi\rangle(b\cdot\xi)B_{1,1} + f\cdot\partial_y\varrho_- B_{0,0}}{(\varrho_- - \varrho_+)^2}\\ &\left.+ \frac{2\langle\xi\rangle\alpha\cdot\partial_y\varrho_- B_{1,1} + 2\langle\xi\rangle(b\cdot\xi)B_{1,0} + 2\langle\xi\rangle^2(b\cdot\xi)B_{1,2}}{(\varrho_- - \varrho_+)^3} - \frac{6\langle\xi\rangle^2\alpha\cdot\partial_y\varrho_- B_{1,2}}{(\varrho_- - \varrho_+)^4}\right],\end{aligned}$$

$$\begin{aligned}B_{2,1}(y,\xi) = h\ &\Pi_- a_0(y)\left[\frac{aB_{1,0} + dB_{0,0}}{(\varrho_- - \varrho_+)} + bB_{1,0} + eB_{0,0}\right]\\ &+ h\ \Pi_+ a_0(y)\left[\frac{f\cdot\partial_y\varrho_- B_{0,0} + aB_{1,1} + \langle\xi\rangle B_{1,1}}{(\varrho_- - \varrho_+)} + \frac{\alpha\cdot\partial_y\varrho_- B_{1,0}}{(\varrho_- - \varrho_+)\langle\xi\rangle}\right.\\ &\left.- \frac{2\alpha\cdot\partial_y\varrho_- B_{1,1} + (2\langle\xi\rangle a + 2\langle\xi\rangle^2)B_{1,2}}{(\varrho_- - \varrho_+)^2} + \frac{6\langle\xi\rangle\alpha\cdot\partial_y\varrho_- B_{1,2}}{(\varrho_- - \varrho_+)^3}\right],\end{aligned}$$

$$\begin{aligned}B_{2,2}(y,\xi) = h\ &\Pi_- a_0(y)\left[\frac{aB_{1,1}}{2(\varrho_- - \varrho_+)} + \frac{bB_{1,1}}{2} + \frac{(\alpha\cdot\partial_y\varrho_-)B_{1,0}}{2(\varrho_- - \varrho_+)^2} + \frac{(f\cdot\partial_y\varrho_-)B_{0,0}}{2(\varrho_- - \varrho_+)^2}\right]\\ &+ h\ \Pi_+ a_0(y)\left[\frac{(\alpha\cdot\partial_y\varrho_-)B_{1,1}}{(\varrho_- - \varrho_+)\langle\xi\rangle} - \frac{aB_{1,2}}{(\varrho_- - \varrho_+)} + \frac{(b\cdot\xi)B_{1,2}}{(\varrho_- - \varrho_+)} - \frac{3(\alpha\cdot\partial_y\varrho_-)B_{1,2}}{(\varrho_- - \varrho_+)^2}\right],\end{aligned}$$

with $\varrho_- - \varrho_+ = -2\lambda(y,\xi) \in \mathcal{S}^1$ and $\partial_y\varrho_- \in \mathcal{S}^1$. Then $B_{2,k} \in h\,\mathcal{S}^0$ for $k = 0,1,2$, $B_{2,3} \in h^2\,\mathcal{S}^{-1}$, and $B_{2,4} \in h^2\,\mathcal{S}^{-2}$. ■

Remark 3.5.1. *Using* (3.14) *and* (3.16)*, then the boundary condition associated with* $A_2(y,\xi,\varepsilon)$ *is the following*

$$\begin{aligned}\Pi_+ A_2(y,\xi,\varepsilon) = h\,\Pi_+ a_0&\left[\frac{aB_{1,0} + (\xi\cdot b)B_{1,0} + dB_{0,0} + (e\cdot\xi)B_{0,0}}{(\varrho_- - \varrho_+)}\right.\\ &- \frac{(\alpha\cdot\partial_y\varrho_-)B_{1,0} + \langle\xi\rangle aB_{1,1} + \langle\xi\rangle(b\cdot\xi)B_{1,1} + f\cdot\partial_y\varrho_- B_{0,0}}{(\varrho_- - \varrho_+)^2}\\ &\left.+ \frac{2\langle\xi\rangle\alpha\cdot\partial_y\varrho_- B_{1,1} + 2\langle\xi\rangle(b\cdot\xi)B_{1,0} + 2\langle\xi\rangle^2(b\cdot\xi)B_{1,2}}{(\varrho_- - \varrho_+)^3} - \frac{6\langle\xi\rangle^2\alpha\cdot\partial_y\varrho_- B_{1,2}}{(\varrho_- - \varrho_+)^4}\right].\end{aligned}$$

Chapter 4

On the approximation of the δ-shell interaction for the 3-D Dirac operator.

The results presented in this chapter have been the subject of the paper [Zre11].

Abstract

We consider the three-dimensional Dirac operator coupled with a combination of electrostatic and Lorentz scalar δ-shell interactions. We approximate this operator with general local interactions V. Without any hypotheses of smallness on the potential V, we show convergence in the strong resolvent sense to the Dirac Hamiltonian coupled with a δ-shell potential supported on Σ, a bounded smooth surface. However, the coupling constant depends nonlinearly on the potential V.

Résumé

Nous considérons l'opérateur de Dirac tridimensionnel couplé à une combinaison de δ-shell interactions électrostatiques et scalaires de Lorentz. Nous approximons cet opérateur avec des interactions locales générales V. Sans aucune hypothèse de petitesse sur le potentiel V, nous montrons la convergence dans le sens de la résolvante forte vers l'hamiltonien de Dirac couplé à un potentiel δ-shell supporté sur Σ, une surface lisse bornée. Cependant, la constante de couplage dépend de façon non-linéaire du potentiel V.

Contents

4.1 Introduction

Dirac Hamiltonians of the type D_m+V, where V is a suitable perturbation, are used in many problems where the implications of special relativity play an important role. This is the case, for example, in the description of elementary particles such as quarks, or in the analysis of graphene, which is used in research for batteries, water filters, or photovoltaic cells. For these problems, mathematical investigations are still in their infancy. The present work studies the three-dimensional Dirac operator with a singular interaction on a closed surface Σ. Mathematically, the Hamiltonian we are interested in can be formulated as follows

$$\mathbb{D}_{\eta,\tau} = D_m + B_{\eta,\tau}\delta_\Sigma = D_m + (\eta\,\mathbb{I}_4 + \tau\beta)\delta_\Sigma, \tag{4.1}$$

where $B_{\eta,\tau}$ is the combination of the *electrostatic* and *Lorentz scalar* potentials of strengths η and τ, respectively. Physically, the Hamiltonian $\mathbb{D}_{\eta,\tau}$ is used as an idealized model for Dirac operators with strongly localized electric and massive potential near the interface Σ (*e.g.*, an annulus), *i.e.*, it replaces a Hamiltonian of the form

$$\mathbb{H}_{\tilde\eta,\tilde\tau} = D_m + B_{\tilde\eta,\tilde\tau} = D_m + (\tilde\eta\,\mathbb{I}_4 + \tilde\tau\beta)\mathfrak{P}_\Sigma, \tag{4.2}$$

where $\mathfrak{P}_\Sigma$ is a regular potential localized in a thin layer containing the interface Σ.

In the three-dimensional case, the authors of [MP18] were able to show the convergence in the norm resolvent sense in the non-confining case, however, a smallness assumption for the potential $\mathfrak{P}^\varepsilon_\Sigma$ was required to achieve such a result. On the other hand, this assumption, unfortunately, prevents us from obtaining an approximation of the operator $\mathbb{D}_{\eta,\tau}$ with the parameters η and τ which are more relevant from the physical or mathematical point of view. Believing this to be the case, the authors of the recent paper [BHS23] have studied and confirmed the approximation problem for two- and three-dimensional Dirac operators with delta-shell potential in norm resolvent sense. Without the smallness assumption of the potential $\mathfrak{P}^\varepsilon_\Sigma$, no results could be obtained here either. Finally, we note that in the two- and three-dimensional setting a renormalization of the interaction strength was observed in [CLMT23, MP18, BHS23].

The primary aim of our work is to extend the approximation result explored in [CLMT23, Section 8] to the three-dimensional case. We seek to verify whether the methodologies employed in the two-dimensional context allow us to establish a comparable approximation in terms of strong resolvent. Specifically, we aim to achieve this in the non-critical and non-confinement cases (*i.e.*, when $\eta^2-\tau^2 \neq \pm 4$) without relying on the smallness assumption as stipulated in [MP18]. Finally, we also give the Dirac operator coupled with a combination of electrostatic, Lorentz scalar δ-shell interactions of strength η and τ, respectively, which we will denote $\mathbb{D}_{\eta,\tau}$ in what follows.

Throughout this chapter, for $\Omega \subset \mathbb{R}^3$ a bounded smooth domain with boundary $\Sigma := \partial\Omega$, we refer to $H^1(\Omega, \mathbb{C}^4) := H^1(\Omega)^4$ as the first order Sobolev space

$$H^1(\Omega)^4 = \{\varphi \in L^2(\Omega)^4 : \text{ there exists } \tilde{\varphi} \in H^1(\mathbb{R}^3)^4 \text{ such that } \tilde{\varphi}|_\Omega = \varphi\}.$$

Recall that $H^{1/2}(\Sigma, \mathbb{C}^4) := H^{1/2}(\Sigma)^4$ is the Sobolev space of order $1/2$ along the boundary Σ, and $t_\Sigma : H^1(\Omega)^4 \to H^{1/2}(\Sigma)^4$ is the classical trace operator.

Definition 4.1.1. Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary $\Sigma = \partial\Omega$. Let $(\eta, \tau) \in \mathbb{R}^2$. Then, $\mathbb{D}_{\eta,\tau} = D_m + B_{\eta,\tau}\delta_\Sigma := D_m + (\eta\mathbb{I}_4 + \tau\beta)\delta_\Sigma$ acting in $L^2(\mathbb{R}^3)^4$ and defined as follows:

$$\mathbb{D}_{\eta,\tau} f = D_m f_+ \oplus D_m f_-, \text{ for all } f \in \mathrm{Dom}(\mathbb{D}_{\eta,\tau}) := \{f = f_+ \oplus f_- \in H^1(\Omega)^4 \oplus H^1(\mathbb{R}^3 \setminus \overline{\Omega})^4 :$$
$$\text{the transmission condition (T.C) below holds in } H^{1/2}(\Sigma)^4\}.$$

$$\text{Transmission condition} : i\alpha \cdot n(t_\Sigma f_+ - t_\Sigma f_-) + \frac{1}{2}(\eta\,\mathbb{I}_4 + \tau\beta)(t_\Sigma f_+ + t_\Sigma f_-) = 0, \tag{4.3}$$

where n is the outward pointing normal to Ω. ■

Recall that for $\eta^2 - \tau^2 \neq 0, 4$, the Dirac operator $(\mathbb{D}_{\eta,\tau}, \mathrm{Dom}(\mathbb{D}_{\eta,\tau}))$ is self-adjoint and verifies the following assertions (see, e.g., [BEHL19, Theorem 3.4, 4.1])

(i) $\mathrm{Sp}_{\mathrm{ess}}(\mathbb{D}_{\eta,\tau}) = (-\infty, m] \cup [m, +\infty)$.

(ii) $\mathrm{Sp}_{\mathrm{dis}}(\mathbb{D}_{\eta,\tau})$ is finite.

4.2 Model and Main results

For a smooth bounded domain $\Omega \subset \mathbb{R}^3$, we consider an interaction supported on the boundary $\Sigma := \partial\Omega$ of Ω. The surface Σ divides the Euclidean space into disjoint union $\mathbb{R}^3 = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ := \Omega$ is a bounded domain and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$. We denote by n and $\mathrm{d}\sigma$ the unit outward pointing normal to Ω and the surface measure on Σ, respectively. We also denote by $f_\pm := f \upharpoonright \Omega_\pm$ be the restriction of f in $\Omega_\pm$, for all C^2–valued function f defined on $\mathbb{R}^3$. Then, we define the distribution $\delta_\Sigma f$ by

$$\langle \delta_\Sigma f,\, g\rangle := \frac{1}{2}\int_\Sigma (t_\Sigma f_+ + t_\Sigma f_-)\, g\,\mathrm{d}\sigma, \quad \text{for any test function} \quad g \in C_0^\infty(\mathbb{R}^3)^4,$$

where $t_\Sigma f_\pm$ is the classical trace operator defined below in Definition 4.1.1. Now, we explicitly construct regular symmetric potentials $V_{\eta,\tau,\varepsilon} \in L^\infty(\mathbb{R}^3; \mathbb{C}^{4\times 4})$ supported on a tubular ε-neighbourhood of Σ and such that

$$V_{\eta,\tau,\varepsilon} \xrightarrow[\varepsilon\to 0]{} (\eta\,\mathbb{I}_4 + \tau\beta)\delta_\Sigma \quad \text{in the sense of distributions.}$$

To explicitly describe the approximate potentials $V_{\eta,\tau,\varepsilon}$, we will introduce an additional notation. For $\gamma > 0$, we define $\Sigma_\gamma := \{x \in \mathbb{R}^3,\ \mathrm{dist}(x, \Sigma) < \gamma\}$ a tubular neighborhood of Σ with width γ. For $\gamma > 0$ small enough, Σ_γ is parametrized as

$$\Sigma_\gamma = \{x_\Sigma + p\, n(x_\Sigma),\ x_\Sigma \in \Sigma \quad \text{and} \quad p \in (-\gamma, \gamma)\}. \tag{4.4}$$

For $0 < \varepsilon < \gamma$, let $h_\varepsilon(p) := \dfrac{1}{\varepsilon} h\left(\dfrac{p}{\varepsilon}\right)$, for all $p \in \mathbb{R}$, with the function h verifies the following

$$h \in L^\infty(\mathbb{R}, \mathbb{R}), \quad \operatorname{supp} h \subset (-1,1) \text{ and } \int_{-1}^{1} h(t)\,\mathrm{d}t = 1.$$

Thus, we have:

$$\operatorname{supp} h_\varepsilon \subset (-\varepsilon, \varepsilon), \quad \int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t = 1, \text{ and } \lim_{\varepsilon \to 0} h_\varepsilon = \delta_0 \quad \text{in the sense of the distributions,} \tag{4.5}$$

where δ_0 is the Dirac δ-function supported at the origin. Finally, for any $\varepsilon \in (0, \gamma)$, we define the symmetric approximate potentials $V_{\eta,\tau,\varepsilon} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$, as follows:

$$V_{\eta,\tau,\varepsilon}(x) := \begin{cases} B_{\eta,\tau} h_\varepsilon(p), & \text{if} \quad x = x_\Sigma + p\, n(x_\Sigma) \in \Sigma_\gamma, \\ 0, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\gamma. \end{cases} \tag{4.6}$$

It is easy to see that $\lim_{\varepsilon\to 0} V_{\eta,\tau,\varepsilon} = B_{\eta,\tau}\delta_\Sigma$, in $\mathcal{D}'(\mathbb{R}^3)^4$. For $0 < \varepsilon < \gamma$, we define the family of Dirac operator $\{\mathscr{E}_{\eta,\tau,\varepsilon}\}_\varepsilon$ as follows:

$$\begin{aligned} &\operatorname{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon}) := \operatorname{Dom}(D_m) = H^1(\mathbb{R}^3)^4, \\ &\mathscr{E}_{\eta,\tau,\varepsilon}\psi = D_m\psi + V_{\eta,\tau,\varepsilon}\psi, \quad \text{for all } \psi \in \operatorname{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon}). \end{aligned} \tag{4.7}$$

The main purpose of the present chapter is to study the strong resolvent limit of $\mathscr{E}_{\eta,\tau,\varepsilon}$ at $\varepsilon \to 0$. The following theorem is the main result of this chapter.

Theorem 4.2.1. *Let $(\eta, \tau) \in \mathbb{R}^2$, and denote by $d = \eta^2 - \tau^2$. Let $(\hat\eta, \hat\tau) \in \mathbb{R}^2$ be defined as follows:*

$$\begin{aligned} &\bullet \textit{ if } d < 0, \textit{ then } (\hat\eta, \hat\tau) = \frac{\tanh(\sqrt{-d}/2)}{(\sqrt{-d}/2)}(\eta, \tau), \\ &\bullet \textit{ if } d = 0, \textit{ then } (\hat\eta, \hat\tau) = (\eta, \tau), \\ &\bullet \textit{ if } d > 0 \textit{ such that } d \neq (2k+1)^2\pi^2,\ k \in \mathbb{N}\cup\{0\}, \textit{ then } (\hat\eta, \hat\tau) = \frac{\tan(\sqrt{d}/2)}{(\sqrt{d}/2)}(\eta, \tau). \end{aligned} \tag{4.8}$$

Now, let $\mathscr{E}_{\eta,\tau,\varepsilon}$ be defined as in (4.7) and $\mathbb{D}_{\hat\eta,\hat\tau}$ as in Definition 4.1.1. Then,

$$\mathscr{E}_{\eta,\tau,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathbb{D}_{\hat\eta,\hat\tau} \quad \textit{in the strong resolvent sense.} \tag{4.9}$$

Remark 4.2.1. *We mention that in this work we find approximations by regular potentials in the strong resolvent sense for the Dirac operator with δ-shell potentials $\mathscr{E}_{\eta,\tau,\varepsilon}$ in the non-critical case (*i.e., *when $d \neq 4$) and non-confining case, (*i.e., *when $d \neq -4$) everywhere on Σ. This is what we shall prove in the proof of Theorem 4.2.1.*

4.2.1 Tubular neighborhood of Σ

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Σ parametrized by the family $\{\phi_j, U_j, V_j,\}_{j\in J}$ with J a finite set, $U_j \subset \mathbb{R}^2$, $V_j \subset \mathbb{R}^3$, $\Sigma \subset \bigcup_{j\in J} V_j$ and $\phi_j(U_j) = V_j \cap \Sigma \subset \Sigma \subset \mathbb{R}^3$ for all $j \in J$. We set $s = \phi_j^{-1}(x_\Sigma)$ for any $x_\Sigma \in \Sigma$. We set $n_\phi = n \circ \phi : \Sigma \longrightarrow \mathbb{R}^3$ the unit normal vector field which points outwards of Ω.

For $\gamma > 0$, Σ_γ (4.4) is a tubular neighborhood of Σ of width γ. We define the diffeomorphism Φ_ϕ by:

$$\Phi_\phi : U_{x_\Sigma} \times (-\gamma,\gamma) \longrightarrow \mathbb{R}^3$$
$$(s,p) \longmapsto \Phi_\phi(s,p) = \phi(s) + p\, n(\phi(s)).$$

For γ be small enough, Φ_ϕ is a smooth parametrization of Σ_γ. Moreover, the matrix of the differential $\mathrm{d}\Phi_\phi$ of Φ_ϕ in the canonical basis of $\mathbb{R}^3$ is

$$\mathrm{d}\Phi_\phi(s,p) = \Big(\partial_1\phi(s) + p\,\mathrm{d}n(\partial_1\phi)(s) \quad \partial_2\phi(s) + p\,\mathrm{d}n(\partial_2\phi)(s) \quad n_\phi(s)\Big). \tag{4.10}$$

Thus, the differential on U_{x_Σ} and the differential on $(-\gamma,\gamma)$ of Φ_ϕ are respectively given by

$$\begin{aligned} \mathrm{d}_s\Phi_\phi(s,p) &= \partial_i\phi_j(s) - pW(x_\Sigma)\partial_i\phi_j(s) \quad \text{for } i=1,2 \text{ and } x_\Sigma \in \Sigma, \\ \mathrm{d}_p\Phi_\phi(s,p) &= \; n_\phi(s), \end{aligned} \tag{4.11}$$

where $\partial_i\phi$, n_ϕ should be understood as column vectors, and $W(x_\Sigma)$ is the Weingarten map defined as in Definition 1.5.2. Next, we define

$$\begin{aligned} \mathscr{P}_\phi &:= \left(\Phi_\phi^{-1}\right)_1 : \Sigma_\gamma \longrightarrow U_{x_\Sigma} \subset \mathbb{R}^2; \quad \mathscr{P}_\phi(\phi(s) + p\,n(\phi(s))) = s \in \mathbb{R}^2, \quad x_\Sigma = \phi(s), \\ \mathscr{P}_\perp &:= \left(\Phi_\phi^{-1}\right)_2 : \Sigma_\gamma \longrightarrow (-\gamma,\gamma); \quad \mathscr{P}_\perp(\phi(s) + p\,n(\phi(s))) = p. \end{aligned} \tag{4.12}$$

Using the inverse function theorem and thanks to (4.10), then we have for $x = \phi(s) + p\,n(\phi(s)) \in \Sigma_\gamma$ the following differential

$$\nabla\mathscr{P}_\phi(x) = \left(1 - pW(s)\right)^{-1} t_\phi(s) \quad \text{and} \quad \nabla\mathscr{P}_\perp(x) = n_\phi(s), \tag{4.13}$$

with $t_\phi(s) = \partial_i\phi(s)$, $i = 1,2$.

4.2.2 Preparations for proof

Before presenting the tools for the proof of Theorem 4.2.1, let us state some properties verified by the operator $\mathbb{D}_{\eta,\tau}$.

Lemma 4.2.2. *Let $(\eta,\tau) \in \mathbb{R}^2$, and let $\mathbb{D}_{\eta,\tau}$ be as in Definition 4.1.1. Then, the following hold:*

(i) If $\eta^2 - \tau^2 \neq -4$, then there exists an invertible matrix $R_{\eta,\tau}$ such that a function $f = f_+ \oplus f_- \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4$ belongs to $\mathrm{Dom}(\mathbb{D}_{\eta,\tau})$ if and only if $t_\Sigma f_+ = R_{\eta,\tau} t_\Sigma f_-$, with $R_{\eta,\tau}$ given by

$$R_{\eta,\tau} := \Big(\mathbb{I}_4 - \frac{i\alpha\cdot n}{2}(\eta\,\mathbb{I}_4 + \tau\beta)\Big)^{-1}\Big(\mathbb{I}_4 + \frac{i\alpha\cdot n}{2}(\eta\,\mathbb{I}_4 + \tau\beta)\Big). \tag{4.14}$$

(ii) If $\eta^2 - \tau^2 = -4$, then a function $f = f_+ \oplus f_- \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4$ belongs to $\mathrm{Dom}(\mathbb{D}_{\eta,\tau})$ if and only if

$$\Big(\mathbb{I}_4 - \frac{i\alpha\cdot n}{2}(\eta\,\mathbb{I}_4 + \beta\tau)\Big)t_\Sigma f_+ = 0 \quad \textit{and} \quad \Big(\mathbb{I}_4 + \frac{i\alpha\cdot n}{2}(\eta\,\mathbb{I}_4 + \beta\tau)\Big)t_\Sigma f_- = 0.$$

Proof. Using the transmission condition introduced in (4.3), then for assertion (i): for all $f = f_+ \oplus f_- \in \mathrm{Dom}(\mathbb{D}_{\eta,\tau})$, we have that

$$\Big(i\alpha\cdot n + \frac{1}{2}(\eta\mathbb{I}_4 + \tau\beta)\Big)t_\Sigma f_+ = \Big(i\alpha\cdot n - \frac{1}{2}(\eta\mathbb{I}_4 + \tau\beta)\Big)t_\Sigma f_-.$$

Thanks to properties in (1.3) and the fact that $(i\alpha\cdot n)^{-1} = -i\alpha\cdot n$, we get that

$$\Big(\mathbb{I}_4 - M\Big)t_\Sigma f_+ = \Big(\mathbb{I}_4 + M\Big)t_\Sigma f_-, \tag{4.15}$$

with M a 4×4 matrix has the following form

$$M = \frac{i\alpha\cdot n}{2}(\eta\,\mathbb{I}_4 + \beta\tau),$$

thus (4.14) is established.

Furthermore, as

$$d := \eta^2 - \tau^2 \neq -4, \quad M^2 = -\frac{d}{4}\mathbb{I}_4,$$

$$\text{and } (\mathbb{I}_4 - M)(\mathbb{I}_4 + M) = \frac{4+d}{4}\mathbb{I}_4,$$

then $(\mathbb{I}_4 - M)$ is invertible and $(\mathbb{I}_4 - M)^{-1} = \dfrac{4}{4+d}(\mathbb{I}_4 + M)$.

Consequently, using (4.15) we obtain that $t_\Sigma f_+ = R_{\eta,\tau} t_\Sigma f_-$ which $R_{\eta,\tau}$ has the following explicit form

$$R_{\eta,\tau} = \frac{4}{4+d}\left(\frac{4-d}{4}\mathbb{I}_4 + i\alpha\cdot n(\eta\mathbb{I}_4 + \tau\beta)\right). \tag{4.16}$$

For assertion (ii), one just has to multiply (4.15) by $(\mathbb{I}_4 \pm M)$ we get

$$(\mathbb{I}_4 + M)^2 t_\Sigma f_- = 0 \quad \text{and} \quad (\mathbb{I}_4 - M)^2 t_\Sigma f_+ = 0.$$

This achieves the proof of Lemma 4.2.2. ■

4.3 Proof of Theorem 4.2.1

Let $\{\mathscr{E}_{\eta,\tau,\varepsilon}\}_{\varepsilon\in(0,\gamma)}$ and $\mathbb{D}_{\hat\eta,\hat\tau}$ be as defined in (4.7) and Definition 4.1.1, respectively. Since the singular interaction $V_{\eta,\tau,\varepsilon}$ are bounded and symmetric, then by the Kato-Rellich theorem, the operators $\mathscr{E}_{\eta,\tau,\varepsilon}$ are self-adjoint in $L^2(\mathbb{R}^3)^4$. Moreover, we know that $\mathbb{D}_{\hat\eta,\hat\tau}$ are self-adjoint and $\mathrm{Dom}(\mathbb{D}_{\hat\eta,\hat\tau}) \subset H^1(\mathbb{R}^3\setminus\Sigma)^4$. Although the limiting operators and the limit operator are self-adjoint, it has been shown in [RS78, Theorem VIII.26] that $\{\mathscr{E}_{\eta,\tau,\varepsilon}\}_{\varepsilon\in(0,\gamma)}$ converges in the strong resolvent sense to $\mathbb{D}_{\hat\eta,\hat\tau}$ as $\varepsilon\to 0$ if and only if it converges in the strong graph limit sense. The latter means that, for all $\psi\in\mathrm{Dom}(\mathbb{D}_{\hat\eta,\hat\tau})$, there exists a family of vectors $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)} \subset \mathrm{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon})$ such that

$$\text{(a)} \lim_{\varepsilon\to 0}\psi_\varepsilon = \psi \quad \text{and} \quad \text{(b)} \lim_{\varepsilon\to 0}\mathscr{E}_{\eta,\tau,\varepsilon}\psi_\varepsilon = \mathbb{D}_{\hat\eta,\hat\tau}\psi \quad \text{in } L^2(\mathbb{R}^3)^4. \tag{4.17}$$

Let $\psi \equiv \psi_+ \oplus \psi_- \in \mathrm{Dom}(D_{\hat{\eta},\hat{\tau}})$. From (4.8), we have that

$$\begin{aligned} \hat{d} = \hat{\eta}^2 - \hat{\tau}^2 &= -4\tanh^2(\sqrt{-d}/2), \quad \text{if} \quad d < 0, \\ \hat{d} = \hat{\eta}^2 - \hat{\tau}^2 &= 4\tan^2(\sqrt{d}/2), \quad \text{if} \quad d > 0, \\ \hat{d} = \hat{\eta}^2 - \hat{\tau}^2 &= 0, \quad \text{if} \quad d = 0. \end{aligned}$$

In all cases, we have that $\hat{d} > -4$ (in particular $\hat{d} \neq -4$). Then, by Lemma 4.2.2 (i),

$$t_\Sigma \psi_+ = R_{\hat{\eta},\hat{\tau}} t_\Sigma \psi_-, \tag{4.18}$$

where $R_{\hat{\eta},\hat{\tau}}$ are given in (4.16).

Using the Definition 4.1.1, we get that $t_\Sigma \psi_\pm \in H^{1/2}(\Sigma)^4$.

- ***Show that***

$$e^{i\alpha\cdot nB_{\eta,\tau}} = R_{\hat{\eta},\hat{\tau}}. \tag{4.19}$$

Recall the definition of the family $\mathscr{E}_{\eta,\tau,\varepsilon}$ and $V_{\eta,\tau,\varepsilon}$ defined in (4.7) and (4.6), respectively. We have that

$$(i\alpha \cdot nB_{\eta,\tau})^2 = (i\alpha \cdot n(\eta \mathbb{I}_4 + \tau\beta))^2 = -(\eta^2 - \tau^2) =: D^2, \quad \text{with } D = \sqrt{-(\eta^2 - \tau^2)} = \sqrt{-d}.$$

Using this equality, we can write: $e^{i\alpha\cdot nB_{\eta,\tau}} = e^{-D}\Pi_- + e^{D}\Pi_+$, with $\pm D$ the eigenvalues of $i\alpha \cdot nB_{\eta,\tau}$; and $\Pi_\pm$ the eigenprojections are given by:

$$\Pi_\pm := \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{i\alpha \cdot nB_{\eta,\tau}}{D}\right).$$

Therefore,

$$\begin{aligned} e^{(i\alpha\cdot nB_{\eta,\tau})} &= \left(\frac{e^D + e^{-D}}{2}\right)\mathbb{I}_4 + \frac{i\alpha \cdot nB_{\eta,\tau}}{D}\left(\frac{e^D - e^{-D}}{2}\right) \\ &= \cosh(D)\mathbb{I}_4 + \frac{\sinh(D)}{D}(i\alpha \cdot n(\eta\mathbb{I}_4 + \tau\beta)). \end{aligned}$$

Now, the idea is to show (4.19), *i.e.*, it remains to show

$$\frac{4}{4+\hat{d}}\left(\frac{4-\hat{d}}{4}\mathbb{I}_4 + i\alpha \cdot n(\hat{\eta}\mathbb{I}_4 + \hat{\tau}\beta)\right) - \cosh(D)\mathbb{I}_4 - \frac{\sinh(\mathrm{D})}{D}(i\alpha \cdot n(\eta\mathbb{I}_4 + \tau\beta)) = 0. \tag{4.20}$$

To this end, set $\mathfrak{U} = \dfrac{4-\hat{d}}{4+\hat{d}} - \cosh(\mathrm{D})$ and $\mathfrak{V} = \dfrac{4}{4+\hat{d}} - \dfrac{\sinh(\mathrm{D})}{D}$. If we apply (4.20) to the unit vector $e_1 = (1\ 0\ 0\ 0)^t$, then we get that $\mathfrak{U} = \mathfrak{V} = 0$. Hence, (4.20) makes sense if and only if

$$\cosh(D) = \frac{4-\hat{d}}{4+\hat{d}} \quad \text{and} \quad \frac{\sinh(D)}{D}(\eta,\tau) = \frac{4}{4+\hat{d}}(\hat{\eta},\hat{\tau}).$$

Consequently, we have $R_{\hat{\eta},\hat{\tau}} = e^{i\alpha\cdot nB_{\eta,\tau}}$.

Moreover, dividing $\dfrac{\sinh(D)}{D}$ by $(1+\cosh(D))$ we get that

$$(\hat{\eta},\hat{\tau}) = \frac{\sinh(D)}{1+\cosh(D)}\frac{1}{D/2}(\eta,\tau).$$

Now, applying the elementary identity $\tanh(\dfrac{\theta}{2}) = \dfrac{\sinh(\theta)}{1+\cosh(\theta)}$, for all $\theta \in \mathbb{C}\setminus\{i(2k+1)\pi,\ k\in\mathbb{Z}\}$. We conclude that

$$\frac{\tanh(\sqrt{-d}/2)}{\sqrt{-d}/2}(\eta,\tau) = (\hat{\eta},\hat{\tau}), \quad \text{if } d<0,$$

and so, for $d>0$ we apply the elementary identity $-i\tanh(i\theta) = \tan(\theta)$ for all $\theta \in \mathbb{C}\setminus\{\pi(k+\frac{1}{2}),\ k\in\mathbb{Z}\}$, then we get that

$$\frac{\tanh(\sqrt{-d}/2)}{\sqrt{-d}/2} = \frac{\tan(\sqrt{d}/2)}{\sqrt{d}/2}.$$

Hence, we obtain that $\dfrac{\tan(\sqrt{d}/2)}{\sqrt{d}/2}(\eta,\tau) = (\hat{\eta},\hat{\tau})$ if $d>0$ such that $d\neq(2k+1)^2\pi^2$. Consequently, the equality $e^{i\alpha\cdot nB_{\eta,\tau}} = R_{\hat{\eta},\hat{\tau}}$ is shown such that the following parameters verify:

$$\begin{aligned}
&\bullet\ \frac{\tanh(\sqrt{-d}/2)}{\sqrt{-d}/2}(\eta,\tau) = (\hat{\eta},\hat{\tau}), && \text{if } d<0,\\
&\bullet\ \frac{\tan(\sqrt{d}/2)}{\sqrt{d}/2}(\eta,\tau) = (\hat{\eta},\hat{\tau}), && \text{if } d>0,\\
&\bullet\ (\eta,\tau) = (\hat{\eta},\hat{\tau}), && \text{if } d=0.
\end{aligned} \tag{4.21}$$

Moreover, the fact that $\int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)dt = 1$ (see, (4.5)) with the statement (4.19) make it possible to write

$$\exp\Big[\big(-i\int_{-\varepsilon}^{0} h_\varepsilon(t)\,\mathrm{d}t\big)(\alpha\cdot nB_{\eta,\tau})\Big]t_\Sigma\psi_+ = \exp\Big[\big(i\int_{0}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t\big)(\alpha\cdot nB_{\eta,\tau})\Big]t_\Sigma\psi_-. \tag{4.22}$$

• ***Construction of the family $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)}$.*** For all $0<\varepsilon<\gamma$, we define the function $H_\varepsilon : \mathbb{R}\setminus\{0\}\to\mathbb{R}$ such that

$$H_\varepsilon(p) := \begin{cases} \displaystyle\int_p^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t, & \text{if } \quad 0<p<\varepsilon,\\ \displaystyle -\int_{-\varepsilon}^{p} h_\varepsilon(t)\,\mathrm{d}t, & \text{if } \quad -\varepsilon<p<0,\\ 0, & \text{if } \quad |p|\geqslant\varepsilon. \end{cases} \tag{4.23}$$

Clearly, $H_\varepsilon\in L^\infty(\mathbb{R})$ and supported in $(-\varepsilon,\varepsilon)$. The fact that $||H_\varepsilon||_{L^\infty}\leqslant||h||_{L^1}$, we get $\{H_\varepsilon\}_\varepsilon$ is bounded uniformly in ε. For all $\varepsilon\in(0,\gamma)$, the restrictions of H_ε to $\mathbb{R}_\pm$ are uniformly continuous, so finite limits at $p=0$ exist, and differentiable a.e., with derivative being bounded, since $h_\varepsilon\in L^\infty(\mathbb{R},\mathbb{R})$.

Using these function, we set the matrix functions $\mathbb{U}_\varepsilon : \mathbb{R}^3 \setminus \Sigma \to \mathbb{C}^{4\times 4}$ such that

$$\mathbb{U}_\varepsilon(x) := \begin{cases} e^{(i\alpha\cdot n)B_{\eta,\tau}H_\varepsilon(\mathscr{P}_\perp(x))}, & \text{if} \quad x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_4, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon, \end{cases} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4}), \tag{4.24}$$

where the mappings $\mathscr{P}_\perp$ is defined as in (4.12), and Σ_ε is a tubular neighborhood of Σ of width ε. As the functions $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, and uniformly continuous in $\Omega_\pm$, with a jump discontinuity across Σ, then $\forall x_\Sigma \in \Sigma$ and $y_\pm \in \Omega_\pm$, we get

$$\begin{aligned} \mathbb{U}_\varepsilon(x_\Sigma^-) &:= \lim_{y_- \to x_\Sigma} \mathbb{U}_\varepsilon(y_-) = \exp\Big[i\Big(\int_0^\varepsilon h_\varepsilon(t)\,\mathrm{d}t\Big)(\alpha\cdot n(x_\Sigma))B_{\eta,\tau}\Big], \\ \mathbb{U}_\varepsilon(x_\Sigma^+) &:= \lim_{y_+ \to x_\Sigma} \mathbb{U}_\varepsilon(y_+) = \exp\Big[-i\Big(\int_{-\varepsilon}^0 h_\varepsilon(t)\,\mathrm{d}t\Big)(\alpha\cdot n(x_\Sigma))B_{\eta,\tau}\Big]. \end{aligned} \tag{4.25}$$

Thus, we construct ψ_ε by $\psi_\varepsilon = \psi_{\varepsilon,+} \oplus \psi_{\varepsilon,-} := \mathbb{U}_\varepsilon \psi \in L^2(\mathbb{R}^3)^4$.

Since $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, using the construction of ψ_ε we get that $\psi_\varepsilon - \psi := (\mathbb{U}_\varepsilon - \mathbb{I}_4)\psi$. Then, by the dominated convergence theorem and the fact that $\operatorname{supp}(\mathbb{U}_\varepsilon - \mathbb{I}_4) \subset |\Sigma_\varepsilon|$ with $|\Sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$, it is easy to show that

$$\psi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \psi \quad \text{in } L^2(\mathbb{R}^3)^4. \tag{4.26}$$

This achieves assertion (a).

• ***Show that*** $\boldsymbol{\psi_\varepsilon \in \mathbf{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon}) = H^1(\mathbb{R}^3)^4}$. This means that we must show, for all $0 < \varepsilon < \gamma$,

$$\text{(i) } \psi_{\varepsilon,\pm} \in H^1(\Omega_\pm)^4 \quad \text{and} \quad \text{(ii) } t_\Sigma \psi_{\varepsilon,+} = t_\Sigma \psi_{\varepsilon,-} \in H^{1/2}(\Sigma)^4.$$

Let us show point (i). By construction of ψ_ε, we have $\psi_\varepsilon \in L^2(\mathbb{R}^3)^4$. It remains to have $\partial_j \mathbb{U}_\varepsilon \in L^2(\mathbb{R}^3)^4$, for $j = 1, 2, 3$. To do so, recall the parametrization $\phi : U \to \Sigma \subset \mathbb{R}^3$ of Σ defined at the beginning of Section 4.2.1 and let A a 4×4 matrix such that $A(s) := i\alpha \cdot n(\phi(s))B_{\eta,\tau}$, for $s = (s_1, s_2) \in U \subset \mathbb{R}^2$. Thus, the matrix functions $\mathbb{U}_\varepsilon$ in (4.24) can be written

$$\mathbb{U}_\varepsilon(x) = \begin{cases} e^{A(\mathscr{P}_\phi(x))H_\varepsilon(\mathscr{P}_\perp(x))}, & \text{if} \quad x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_4, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon, \end{cases} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4}), \tag{4.27}$$

where $\mathscr{P}_\phi$ is defined as in (4.12).

For $j = 1, 2, 3$, $\operatorname{supp} \partial_j \mathbb{U}_\varepsilon \subset \Sigma_\varepsilon$. Furthermore, it was mentioned in [WIL67, Eq.(4.1)] that for all $x \in \Sigma_\varepsilon \setminus \Sigma$, $\partial_j \mathbb{U}_\varepsilon$ can be written as follows

$$\begin{aligned} \partial_j \mathbb{U}_\varepsilon(x) = \int_0^1 \Big[&\exp\Big(zA(\mathscr{P}_\phi(x))H_\varepsilon(\mathscr{P}_\perp(x))\Big)\partial_j\Big(A(\mathscr{P}_\phi(x))H_\varepsilon(\mathscr{P}_\perp(x))\Big)\times \\ &\exp\Big((1-z)A(\mathscr{P}_\phi(x))H_\varepsilon(\mathscr{P}_\perp(x))\Big)\Big]\mathrm{d}z. \end{aligned} \tag{4.28}$$

Let $x = \phi(s) + p\, n(\phi(s)) \in \Sigma_\gamma$, and recall the definition of the mappings $\mathscr{P}_\phi(x)$ and $\mathscr{P}_\perp(x)$ introduced in (4.12). Based on the quantities (4.13) (with $s = \mathscr{P}_\phi(x)$ and $p = \mathscr{P}_\perp(x)$), we get that

$$\partial_j\Big(A(\mathscr{P}_\phi(x))H_\varepsilon(\mathscr{P}_\perp(x))\Big) = \partial_s A(s)(1-pW(s))^{-1}(t_\phi(s))_j H_\varepsilon(p) - A(s)h_\varepsilon(p)(n_\phi(s))_j. \tag{4.29}$$

Therefore, $\partial_j \mathbb{U}_\varepsilon$ has the following form

$$\begin{aligned}\partial_j \mathbb{U}_\varepsilon(x) &= -A(s)h_\varepsilon(p)(n_\phi(s))_j \mathbb{U}_\varepsilon(x)\\ &\quad + \int_0^1 e^{zA(s)H_\varepsilon(p)}\big[\partial_s A(s)(1-pW(s))^{-1}(t_\phi(s))_j H_\varepsilon(p)\big] e^{(1-z)A(s)H_\varepsilon(p)}\,\mathrm{d}z.\end{aligned} \tag{4.30}$$

Set by $\mathbb{E}_{\varepsilon,j}$ the second term of the right part of equality (4.30), *i.e.*,

$$\mathbb{E}_{\varepsilon,j} = \int_0^1 e^{zA(s)H_\varepsilon(p)}\big[\partial_s A(s)(1-pW(s))^{-1}(t_\phi(s))_j H_\varepsilon(p)\big] e^{(1-z)A(s)H_\varepsilon(p)}\,\mathrm{d}z. \tag{4.31}$$

Then, thanks to Proposition 1.5.3, the matrix-valued functions $\mathbb{E}_{\varepsilon,j}$ are bounded, uniformly for $0 < \varepsilon < \gamma$, and $\operatorname{supp}\mathbb{E}_{\varepsilon,j} \subset \Sigma_\varepsilon$. Moreover, we have $\mathbb{U}_\varepsilon$ and $\partial_j \mathbb{U}_\varepsilon \in L^\infty(\Omega_\pm, \mathbb{C}^{4\times 4})$. Hence, for all $\psi_\pm \in H^1(\Omega_\pm)^4$ we have that $\psi_{\varepsilon,\pm} = \mathbb{U}_\varepsilon\psi_\pm \in H^1(\Omega_\pm)^4$ and statement (i) is proved.

Now, we show point (ii). As $\psi_{\varepsilon,\pm} \in H^1(\Omega_\pm)^4$, we get that $t_\Sigma\psi_{\varepsilon,\pm} \in H^{1/2}(\Sigma)^4$. On the other hand, it have been showed in [EG15, Chapter 4 (p.133)], for a.e., $x_\Sigma \in \Sigma$ and $r > 0$, that

$$t_\Sigma\psi_{\varepsilon,\pm}(x_\Sigma) = \lim_{r\to 0}\frac{1}{|B(x_\Sigma,r)|}\int_{\Omega_\pm\cap B(x_\Sigma,r)}\psi_\varepsilon(y)\,\mathrm{d}y = \lim_{r\to 0}\frac{1}{|B(x_\Sigma,r)|}\int_{\Omega_\pm\cap B(x_\Sigma,r)}\mathbb{U}_\varepsilon(y)\psi(y)\,\mathrm{d}y,$$

and so, similarly,

$$\mathbb{U}_\varepsilon(x_\Sigma^\pm)t_\Sigma\psi_\pm(x_\Sigma) = \lim_{r\to 0}\frac{1}{|B(x_\Sigma,r)|}\int_{\Omega_\pm\cap B(x_\Sigma,r)}\mathbb{U}_\varepsilon(x_\Sigma^\pm)\psi(y)\,\mathrm{d}y.$$

As $\mathbb{U}_\varepsilon$ is continuous in $\overline{\Omega_\pm}$, we get $t_\Sigma\psi_{\varepsilon,\pm}(x_\Sigma) = \mathbb{U}_\varepsilon(x_\Sigma^\pm)t_\Sigma\psi_\pm(x_\Sigma)$. Consequently, (4.22) with (4.25) give us that $t_\Sigma\psi_{\varepsilon,+} = t_\Sigma\psi_{\varepsilon,-} \in H^{1/2}(\Sigma)^4$. With this, (ii) is valid and $\psi_\varepsilon \in \mathrm{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon})$.

To complete the proof of Theorem 4.2.1, it remains to show the property (b), mentioned in (4.17). Since $(\mathscr{E}_{\eta,\tau,\varepsilon}\psi_\varepsilon - \mathbb{D}_{\hat\eta,\hat\tau}\psi)$ belongs to $L^2(\mathbb{R}^3)^4$, it suffices to prove the following:

$$\mathscr{E}_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} - \mathbb{D}_{\hat\eta,\hat\tau}\psi_\pm \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{in } L^2(\Omega_\pm)^4. \tag{4.32}$$

To do this, let $\psi \equiv \psi_+ \oplus \psi_- \in \mathrm{Dom}(\mathbb{D}_{\hat\eta,\hat\tau})$ and $\psi_\varepsilon \equiv \psi_{\varepsilon,+} \oplus \psi_{\varepsilon,-} \in \mathrm{Dom}(\mathscr{E}_{\eta,\tau,\varepsilon})$. We have

$$\begin{aligned}\mathscr{E}_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} - \mathbb{D}_{\hat\eta,\hat\tau}\psi_\pm &= -i\alpha\cdot\nabla\psi_{\varepsilon,\pm} + m\beta\psi_{\varepsilon,\pm} + V_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} + i\alpha\cdot\nabla\psi_\pm - m\beta\psi_\pm\\ &= -i\alpha\cdot\nabla(\mathbb{U}_\varepsilon\psi_\pm) + i\alpha\cdot\nabla\psi_\pm + m\beta(\mathbb{U}_\varepsilon - \mathbb{I}_4)\psi_\pm + V_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm}\\ &= -i\sum_{j=1}^3\alpha_j\big[(\partial_j\mathbb{U}_\varepsilon)\psi_\pm + (\mathbb{U}_\varepsilon - \mathbb{I}_4)\partial_j\psi_\pm\big] + m\beta(\mathbb{U}_\varepsilon - \mathbb{I}_4)\psi_\pm + V_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm}.\end{aligned} \tag{4.33}$$

Using the form of $\partial_j \mathbb{U}_\varepsilon$ given in (4.30), the quantity $-i\sum_{j=1}^{3}\alpha_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm$ yields

$$\begin{aligned}
-i\sum_{j=1}^{3}\alpha_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm &= -i\sum_{j=1}^{3}\alpha_j\big[-i\alpha\cdot nV_{\eta,\tau,\varepsilon}\,n_j\mathbb{U}_\varepsilon\psi_\pm + \mathbb{E}_{\varepsilon,j}\psi_\pm\big] \\
&= -(\alpha\cdot n)^2 V_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} - i\sum_{j=1}^{3}\alpha_j\mathbb{E}_{\varepsilon,j}\psi_\pm \\
&= -V_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} + \mathbb{R}_\varepsilon\psi_\pm,
\end{aligned}$$

where $\mathbb{E}_{\varepsilon,j}$ is given in (4.31) and $\mathbb{R}_\varepsilon = -i\sum_{j=1}^{3}\alpha_j\mathbb{E}_{\varepsilon,j}$, a matrix-valued functions in $L^\infty(\mathbb{R}^3,\mathbb{C}^{4\times 4})$, verifies the same property of $\mathbb{E}_{\varepsilon,j}$ given in (4.31), for $\varepsilon\in(0,\gamma)$. Thus, (4.33) becomes

$$\mathscr{E}_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} - D_{\hat{\eta},\hat{\tau}}\psi_\pm = -i\sum_{j=1}^{3}\alpha_j\big[(\mathbb{U}_\varepsilon-\mathbb{I}_4)\partial_j\psi_\pm\big] + m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm + \mathbb{R}_\varepsilon\psi.$$

Since $\psi_\pm\in H^1(\Omega_\pm)^4$, $(\mathbb{U}_\varepsilon-\mathbb{I}_4)$ and $\mathbb{R}_\varepsilon$ are bounded, uniformly in $\varepsilon\in(0,\gamma)$ and supported in Σ_ε, and $|\Sigma_\varepsilon|$ tends to 0 as $\varepsilon\to 0$. By the dominated convergence theorem, we conclude that

$$\mathscr{E}_{\eta,\tau,\varepsilon}\psi_{\varepsilon,\pm} - \mathbb{D}_{\hat{\eta},\hat{\tau}}\psi_\pm \xrightarrow[\varepsilon\to 0]{} 0, \quad \text{holds in } L^2(\Omega_\pm)^4, \tag{4.34}$$

and this achieves the assertion (4.32).

Thus, both conditions mentioned in (4.17) (*i.e.*, (a) and (b)) of the convergence in the strong graph limit sense are proved (see, (4.26) and (4.34)). Also, note that the latter remains stable with respect to bounded symmetric perturbations (in our case $m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)$ with $m>0$, so we can assume $m=0$). Hence, the family $\{\mathscr{E}_\varepsilon\}_{\varepsilon\in(0,\gamma)}$ converges in the strong resolvent sense to $\mathbb{D}_{\hat{\eta},\hat{\tau}}$ as $\varepsilon\to 0$. The proof of the Theorem 4.2.1 is complete. ■

Chapter 5

On the self-adjointness of two-dimensional relativistic shell interactions.

In this chapter, we describe the results obtained in article [BPZ72] in collaboration with Badreddine Benhellal and Konstantin Pankrashkin.

Abstract

In this chapter, we discuss the self-adjointness of the two-dimensional Dirac operator with a transmission condition along a closed Lipschitz curve. The main new ingredients are an explicit use of the Cauchy transform on non-smooth curves and a direct link with the Fredholmness of a singular boundary integral operator. This results in a proof of self-adjointness for a new range of coupling constants, which includes and extends all previous results for this class of problems. The study is particularly precise for the case of curvilinear polygons, as the angles can be taken into account in an explicit way. In particular, if the curve is a curvilinear polygon with obtuse angles, then there is a unique self-adjoint realization with domain contained in $H^{\frac{1}{2}}$ for the full range of non-critical coefficients in the transmission condition.

Résumé

Dans ce chapitre, nous discutons de l'auto-adjonction de l'opérateur de Dirac bidimensionnel avec une condition de transmission le long d'une courbe de Lipschitz fermée. Les principaux nouveaux ingrédients sont une utilisation explicite de la transformée de Cauchy sur des courbes non lisses et un lien direct avec le caractère de Fredholm d'un opérateur intégral de frontière singulier. Il en résulte une preuve de l'auto-adjonction pour une nouvelle gamme de constantes de couplage, qui inclut et étend tous les résultats précédents pour cette classe de problèmes. L'étude est particulièrement précise dans le cas des polygones curvilignes, car les angles peuvent être pris en compte de manière explicite. En particulier, si la courbe est un polygone curviligne avec des angles obtus, alors il existe une réalisation unique auto-adjointe avec un domaine contenu dans $H^{\frac{1}{2}}$ pour toute la gamme des coefficients non critiques dans la condition de transmission.

Contents

5.1 Introduction

Dirac operators with δ-interactions supported on general hypersurfaces have been actively studied since the appearance of the paper [AMV14]. Due to the presence of distributional coefficients, the self-adjointness of such operators requires special attention, and it was seen by many authors (primarily for the three-dimensional case) that the self-adjointness domain can be dependent on the coupling constants and the smoothness properties of the hypersurface and that it may lead to unusual spectral properties [BEHL19, BH20, Ben22b, Ben22a, BP24]. The paper [BHOBP20] initiated the study of the two-dimensional case, and for the case of smooth curves a very complete spectral picture could be found, which was extended in [CLMT23] to a more general class of interactions. Much less attention was given to the case of non-smooth surfaces and curves. In the present work, we discuss the self-adjointness of two-dimensional Dirac operators with δ-interactions supported on closed Lipschitz curves (in particular, on curvilinear polygons). Our results complement those obtained in the recent papers [BHSS24, PVDB21] and provide precise ranges of coupling constants and corner openings for which the domain of self-adjointness can be given explicitly. Compared to the preceding works, we employ two new technical ingredients: the explicit use of the Cauchy transform on non-smooth curves and a characterization of the Fredholmness for boundary integral operators using the approach of [She91].

Now let us pass to precise formulations. Through the text we use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and denote by $\mathbb{I}_2$ the 2×2 identity matrix. Let $m \in \mathbb{R}$. The two-dimensional Dirac operator with mass m is the formally self-adjoint differential expression

$$D_m : C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \ni f \mapsto -i(\sigma_1 \partial_1 f + \sigma_2 \partial_2 f) + m\sigma_3 f \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2),$$

and it naturally extends to a continuous linear map in the space of distributions $\mathcal{D}'(\Omega, \mathbb{C}^2)$ for any open set $\Omega \subset \mathbb{R}^2$. It is well known that the operator

$$A : f \mapsto D_m f, \quad \operatorname{Dom}(A) = H^1(\mathbb{R}^2, \mathbb{C}^2), \tag{5.1}$$

(the free two-dimensional Dirac operator), is self-adjoint in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ and has the absolutely continuous spectrum

$$\operatorname{Sp}(A) = \operatorname{Sp}_{\mathrm{cont}}(A) = \big(-\infty, -|m|\big] \cup \big[|m|, +\infty\big),$$

and it occupies a central place in relativistic quantum mechanics [Tha92]. We will be interested in the study of some special perturbations of A.

Namely, let $\Omega_+ \subset \mathbb{R}^2$ be a non-empty bounded open set with Lipschitz boundary. Denote

$$\Sigma := \partial\Omega_+, \quad \Omega_- := \mathbb{R}^2 \setminus \overline{\Omega_+}.$$

For $(\varepsilon, \mu) \in \mathbb{R}^2$ we would like to discuss self-adjoint realizations in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ of operators given formally by

$$f \mapsto D_m f + (\varepsilon \mathbb{I}_2 + \mu\sigma_3)\delta_\Sigma f, \tag{5.2}$$

where δ_Σ is the Dirac δ-distribution supported on Σ. The last summand can be considered as an idealized model of a relativistic potential concentrated on Σ, and the constant ε resp. μ measures the strength of the electrostatic resp. Lorentz scalar part of the interaction. The formal expression (5.2) can be given a more rigorous meaning as follows. First, for any non-empty open set $\Omega \subset \mathbb{R}^2$ consider the space

$$H(\sigma, \Omega) := \Big\{ f \in L^2(\Omega, \mathbb{C}^2) : \ D_m f \in L^2(\Omega, \mathbb{C}^2) \Big\},$$

which is just the domain of the maximal realization of D_m in $L^2(\Omega, \mathbb{C}^2)$ and becomes a Hilbert space if equipped with the scalar product

$$\langle f, g\rangle_{H(\sigma,\Omega)} := \langle f, g\rangle_{L^2(\Omega,\mathbb{C}^2)} + \langle D_m f, D_m g\rangle_{L^2(\Omega,\mathbb{C}^2)}.$$

For $s > 0$ let $H^s(\Omega, \mathbb{C}^2)$ be the usual fractional Sobolev spaces of order s on Ω (consisting of $\mathbb{C}^2$-valued functions), and we set

$$H^s(\sigma, \Omega) := H(\sigma, \Omega) \cap H^s(\Omega, \mathbb{C}^2),$$

which is a Hilbert space with the scalar product

$$\langle f, g\rangle_{H^s(\sigma,\Omega)} := \langle f, g\rangle_{H(\sigma,\Omega)} + \langle f, g\rangle_{H^s(\Omega,\mathbb{C}^2)}.$$

For what follows it will be convenient to use the identification

$$H(\sigma, \mathbb{R}^2 \setminus \Sigma) \simeq H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-), \quad f \simeq (f_+, f_-),$$

with $f_\pm$ being the restriction of f on $\Omega_\pm$, as well as the analogous identifications for $H^s(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2)$ and $H^s(\sigma, \mathbb{R}^2 \setminus \Sigma)$. We will also use the shorthand notation

$$\sigma \cdot x := x_1\sigma_1 + x_2\sigma_2, \quad x = (x_1, x_2) \in \mathbb{R}^2;$$

from the anticommutation relations (1.6) one easily obtains $(\sigma \cdot x)^2 = |x|^2\mathbb{I}_2$ for all $x \in \mathbb{R}^2$.

It is known that for any $f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma)$ the boundary traces $(\sigma \cdot \nu)f_\pm$ on Σ are well-defined as functions in $H^{-\frac{1}{2}}(\Sigma)$; remark that we keep the same symbols for the boundary traces for better readability. Denote by $\delta_\Sigma f$ the distribution

$$\langle \delta_\Sigma f, \varphi\rangle := \int_\Sigma \frac{f_+ + f_-}{2}\,\varphi\,\mathrm{d}s, \quad \varphi \in C_c^\infty(\mathbb{R}^2),$$

where $\mathrm{d}s$ means the integration with respect to the arclength. An application of the jump formula (distributional derivative for functions with discontinuities along Σ) for a function f shows the identity

$$D_m f = (D_m f_+) \oplus (D_m f_-) + i(\sigma \cdot \nu)(f_+ - f_-)\delta_\Sigma, \tag{5.3}$$

where $\nu = (\nu_1, \nu_2)$ is the unit normal on Σ pointing to Ω_-. Then it follows that the right-hand side of (5.2) belongs to $L^2(\mathbb{R}^2, \mathbb{C}^2)$ if and only if f satisfies the transmission condition

$$(\varepsilon \mathbb{I}_2 + \mu\sigma_3)\frac{f_+ + f_-}{2} + i(\sigma \cdot \nu)(f_+ - f_-) = 0 \text{ on } \Sigma. \tag{5.4}$$

Therefore, as a first attempt, it is natural to consider the following operator realizations of the expression (5.2) in $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

- the maximal realization $B_{\max}$ with the domain

$$\mathrm{Dom}(B_{\max}) := \{f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma) : \ f \text{ satisfies (5.4)}\},$$

- the minimal realization $B_{\min}$ with the domain

$$\begin{aligned} \mathrm{Dom}(B_{\min}) &:= \mathrm{Dom}(B_{\max}) \cap H^1(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2) \\ &\equiv \{f \in H^1(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2) : \ f \text{ satisfies (5.4)}\}. \end{aligned}$$

It is standard to see that $B_{\min}$ is symmetric with $B_{\min}^* = B_{\max}$, therefore, $B_{\min} \subset B \subset B_{\max}$ for any self-adjoint realization B of (5.2). Nevertheless, an explicit description of the self-adjoint realizations turns out to be an involved problem depending on both (ε, μ) and the regularity of Σ.

The most attention was given to the case of C^2-smooth Σ, see [BHSS24] and references therein. Namely, if $\varepsilon^2 - \mu^2 \neq 4$, then $B_{\min} = B_{\max} =: B$, and the spectrum of B consists of the spectrum of the free Dirac operator A and at most finitely many discrete eigenvalues in $(-|m|, |m|)$. For $\varepsilon^2 - \mu^2 = 4$ the operator $B_{\min}$ is not closed, but $\overline{B_{\min}} = B_{\max}$, so $B_{\min}$ is at least essentially self-adjoint (so there is a unique self-adjoint realization), but the loss of regularity leads to peculiar spectral effects (e.g. new pieces of the essential spectrum), see [BHOBP20, BHSS24, BP24]. Remark that [BHSS24, CLMT23] actually consider more general interactions by admitting so-called anomalous magnetic couplings which are not covered by the above framework.

If Σ has corners, one has, in general, $\overline{B_{\min}} \subsetneq B_{\max}$, which means that there are infinitely many self-adjoint realizations [OBP18]. The work [OBP18] suggested that the $H^{\frac{1}{2}}$ regularity should be more natural for the case of non-smooth Σ. Namely, let

$$B \equiv B_{\varepsilon,\mu}$$

be the restriction of $B_{\max}$ to $\mathrm{Dom}(B_{\max}) \cap H^{\frac{1}{2}}(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2)$, *i.e.*,

$$\begin{aligned} B :\ & f \simeq (f_+, f_-) \mapsto (D_m f_+, D_m f_-), \\ \mathrm{Dom}(B) &:= \Big\{f \in H^{\frac{1}{2}}(\sigma, \mathbb{R}^2 \setminus \Sigma) : \ f \text{ satisfies (5.4)}\Big\}. \end{aligned} \tag{5.5}$$

Due to the standard Sobolev traces theorem, the one-sided traces of functions from $\mathrm{Dom}(B)$ on Σ belong to $L^2(\Sigma, \mathbb{C}^2)$, so the integration by parts shows that B is a symmetric operator. The main result of [PVDB21] reads as follows: if Σ is a curvilinear polygon (a piecewise C^2-smooth closed curve, with finitely many corners and without cusps), $\varepsilon = 0$ and $|\mu| < 2$, then B is self-adjoint. The recent work [BHSS24] presents an extensive study of the case of general compact Lipschitz curves Σ by reducing the self-adjointness to the Fredholmness of some boundary integral operator (see also [AMV14, Ben22a] for the three-dimensional case): we summarize the essential components of the constructions in Section 5.2. Nevertheless, the self-adjoint conditions obtained in [BHSS24] for our case are quite implicit as they depend on the (unknown) spectra of some boundary integral operators.

In the present work we extend the results of both [BHSS24] and [PVDB21] by providing new very explicit conditions for the self-adjointness of B in terms of the parameters (ε, μ) and the geometry of Σ. Namely, we show that B is self-adjoint in the following cases:

(A) The curve Σ is Lipschitz and $|\varepsilon| \leq |\mu|$ (Corollary 5.4.3),
(B) The curve Σ is C^1-smooth and $\varepsilon^2 - \mu^2 \neq 4$ (Theorem 5.4.4),
(C) The curve Σ is a curvilinear polygon (with C^1-smooth edges and without cusps) and

$$\varepsilon^2 - \mu^2 < \frac{1}{m(\omega)} \quad \text{or} \quad \varepsilon^2 - \mu^2 > 16m(\omega),$$

where the constant $m(\omega)$ only depends on the sharpest corner ω of Σ (Theorem 5.5.3).
The value of $m(\omega)$ is not known explicitly for all ω, but some bounds can be obtained, and each of the conditions
(i) $\varepsilon^2 - \mu^2 < 2$ or $\varepsilon^2 - \mu^2 > 8$ (without additional geometric assumptions),
(ii) $\varepsilon^2 - \mu^2 \neq 4$ if each angle θ of Σ (measured inside Ω_+) satisfies

$$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2},$$

guarantees the self-adjointness of B (Corollary 5.5.4).

The case (B) is formally contained in (C.ii), but the proofs are very different, so we prefer to consider these two situations separately.

Remark 5.1.1. *If the operator B is self-adjoint, a standard analysis shows that its essential spectrum coincides with the spectrum of the free Dirac operator A and that the discrete spectrum is at most finite [BHOBP20, Proposition 3.8]. While all constructions of [BHOBP20] are formally for smooth Σ, the proof of this specific result only uses the compact embedding of $H^s(\Omega)$ to $L^2(\Omega)$ for $s > 0$ and bounded open sets $\Omega \subset \mathbb{R}^2$ with Lipschitz boundaries.*

Remark 5.1.2. *An additional useful property is that for any (ε, μ) with $|\varepsilon| \neq |\mu|$ the operator $B_{\varepsilon,\mu}$ is unitarily equivalent to $B_{-\frac{4\varepsilon}{\varepsilon^2-\mu^2}, -\frac{4\mu}{\varepsilon^2-\mu^2}}$. Namely, a simple direct computation shows that*

$$B_{\varepsilon,\mu} U = U B_{-\frac{4\varepsilon}{\varepsilon^2-\mu^2}, -\frac{4\mu}{\varepsilon^2-\mu^2}}$$

for the unitary linear map $U : L^2(\mathbb{R}^2, \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathbb{C}^2)$ defined by

$$U : (f_+, f_-) \mapsto (f_+, -f_-),$$

see [BHOBP20, Propositon 4.8]. In particular, the self-adjointness of $B_{-\frac{4\varepsilon}{\varepsilon^2-\mu^2}, -\frac{4\mu}{\varepsilon^2-\mu^2}}$ is equivalent to the self-adjoitness of $B_{\varepsilon,\mu}$, which will be used in the last proof steps.

5.2 Preparations for the proof

We will need some constructions related to the free Dirac operator A in (5.1). Most of these required results were already obtained in [BHOBP20, BHSS24] and we simply present them in an adapted form.

First of all, we recall the Cauchy transform on Σ, *i.e.*, the linear operator $C_\Sigma : L^2(\Sigma) \longrightarrow L^2(\Sigma)$ defined through the complex line integration

$$C_\Sigma g(x) := \frac{i}{2\pi}\,\mathrm{p.\,v.}\int_\Sigma \frac{g(y)}{x-y}\,\mathrm{d}y, \quad g \in L^2(\Sigma), \quad x \in \Sigma, \tag{5.6}$$

and understood in the Cauchy principal value sense. It is a classical result that C_Σ is well-defined and bounded [CMM82]. Moreover, if one considers the analytic function

$$F_g : \mathbb{C}\setminus\Sigma \simeq \mathbb{R}^2\setminus\Sigma \ni x \mapsto \frac{i}{2\pi}\,\mathrm{p.\,v.}\int_\Sigma \frac{g(y)}{x-y}\,\mathrm{d}y, \quad g \in L^2(\Sigma),$$

then Plemelj-Sokhotski formulas are valid:

$$F_g(x) = \pm\frac{g(x)}{2} + C_\Sigma g(x) \text{ for a.e. } x \in \Sigma,$$

where the value on the left-hand side is understood as the non-tangential limit [Jou83, p. 108].

Denote by K_j the modified Bessel functions of order j. For $z \in \mathbb{C}\setminus \mathrm{Sp}(A)$ consider the function $\phi_z : \mathbb{R}^2 \to \mathscr{M}_2(\mathbb{C})$ given by

$$\phi_z(x) := \frac{1}{2\pi}K_0\big(\sqrt{m^2-z^2}|x|\big)\big(m\sigma_3 + z\mathbb{I}_2\big) + i\frac{\sqrt{m^2-z^2}}{2\pi|x|}K_1\big(\sqrt{m^2-z^2}|x|\big)(\sigma\cdot x).$$

It will be convenient to admit the additional value $z = m$ by setting

$$\phi_m(x) := \frac{i}{2\pi}\begin{pmatrix} 0 & \dfrac{1}{x_1+ix_2} \\ \dfrac{1}{x_1-ix_2} & 0 \end{pmatrix}.$$

Using the asymptotic expansions of K_j one obtains

$$\phi_z(x) = \phi_m(x) + h_1(x)\log|x| + h_2(x). \tag{5.7}$$

with continuous functions h_j, see [BHOBP20, Lemma 3.3] for details.

For all admissible z the function ϕ_z is a fundamental solution of $D_m - z$, and it gives rise to several (singular) integral operators.

Namely, consider the layer potentials Φ_z for $D_m - z$ (with $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$)

$$\Phi_z : L^2(\Sigma, \mathbb{C}^2) \longrightarrow L^2(\mathbb{R}^2, \mathbb{C}^2),$$
$$\Phi_z g(x) = \int_\Sigma \phi_z(x-y) g(y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \Sigma,$$

where we recall that $\mathrm{d}s$ means the integration with respect to the arclength. Observe that $\phi_z(x)^* = \phi_{\bar z}(-x)$ for all x. Let $\gamma : H^{\frac{1}{2}}(\sigma, \mathbb{R}^2, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2)$ be the Sobolev trace operator (which is a bounded linear operator), then for any $u \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ and $g \in L^2(\Sigma, \mathbb{C}^2)$ one has, using Fubini's theorem,

$$\begin{aligned} \langle \Phi_{\bar z} g, u\rangle_{L^2(\mathbb{R}^2,\mathbb{C}^2)} &= \int_{\mathbb{R}^2} \Big\langle \int_\Sigma \phi_{\bar z}(x-y) g(y)\, \mathrm{d}s(y), u(x)\Big\rangle_{\mathbb{C}^2} \mathrm{d}x \\ &= \int_\Sigma \Big\langle g(y), \int_{\mathbb{R}^2} \phi_{\bar z}^*(x-y) u(x) \mathrm{d}x\Big\rangle_{\mathbb{C}^2} \mathrm{d}s(y), \\ &= \big\langle g, \gamma(A-z)^{-1} u\big\rangle_{L^2(\Sigma,\mathbb{C}^2)}. \end{aligned}$$

This shows that $\Phi_{\bar z} = \big(\gamma(A-z)^{-1}\big)^*$ is bounded, and by replacing z with $\bar z$ one obtains the useful identity

$$\Phi_z^* = \gamma(A-\bar z)^{-1}, \quad z \in \mathbb{C} \setminus \mathrm{Sp}(A). \tag{5.8}$$

Now let $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ and $h \in L^2(\Sigma, \mathbb{C}^2)$, then

$$\begin{aligned} \big\langle \Phi_z h, (D_m - \bar z)\varphi\big\rangle_{L^2(\mathbb{R}^2,\mathbb{C}^2)} &= \big\langle h, \Phi_z^* (D_m - \bar z)\varphi\big\rangle_{L^2(\Sigma,\mathbb{C}^2)} \\ &= \big\langle h, \gamma(D_m - \bar z)^{-1}(D_m - \bar z)\varphi\big\rangle_{L^2(\Sigma,\mathbb{C}^2)} \\ &= \big\langle h, \gamma\varphi\big\rangle_{L^2(\Sigma,\mathbb{C}^2)}, \end{aligned}$$

and it follows that $(D_m - z)\Phi_z h = 0$ in $\mathcal{D}'(\mathbb{R}^2 \setminus \Sigma)$. In particular,

$$\operatorname{ran} \Phi_z \subset \ker(B_{\max} - z) \subset \mathrm{Dom}(B_{\max}).$$

In fact, for any $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$ one has the stronger property [BHSS24, Lemma 4.2]:

$$\Phi_z : \; L^2(\Sigma, \mathbb{C}^2) \to H^{\frac{1}{2}}(\sigma, \mathbb{R}^2 \setminus \Sigma) \text{ is bounded.} \tag{5.9}$$

For all admissible z consider the singular integral operator

$$\mathcal{C}_z : L^2(\Sigma, \mathbb{C}^2) \longrightarrow L^2(\Sigma, \mathbb{C}^2)$$

given by

$$\mathcal{C}_z g(x) = \mathrm{p.\,v.} \int_\Sigma \phi_z(x-y) g(y)\, \mathrm{d}s(y), \quad x \in \Sigma.$$

To summarize its properties we introduce the tangent vector field

$$\tau = (\tau_1, \tau_2) := (-\nu_2, \nu_1)$$

on Σ and denote

$$t := \text{the operator of multiplcation by } \tau_1 + i\tau_2 \text{ in } L^2(\Sigma).$$

With an arc-length parametrization γ of Σ and $x = \gamma(r), y = \gamma(s)$ it follows that the Cauchy transform C_Σ from (5.6) acts as

$$C_\Sigma g(\gamma(r)) = \frac{\mathrm{i}}{2\pi}\text{p.v}\int_0^\ell \frac{(\gamma_1'(s) + \mathrm{i}\gamma_2'(s))u(\gamma(s))}{(\gamma_1(r) + \mathrm{i}\gamma_2(r)) - (\gamma_1(s) + \mathrm{i}\gamma_2(s))}\mathrm{d}s.$$

Using the notation $t(y) := t_1(y) + \mathrm{i}t_2(y) = \gamma_1'(s) + \mathrm{i}\gamma_2'(s)$. We shall also view $y \mapsto t(y)$ as a function on Σ or $s \mapsto t(\gamma(s))$ as a function on $[0, \ell]$. The same holds for the function $t^*(y) := t_1(y) - \mathrm{i}t_2(y) = \gamma_1'(s) - \mathrm{i}\gamma_2'(s)$, and we will also denote the corresponding multiplication operators by t and t^*. With this we see for $g \in C^\infty(\Sigma)$ and $x = \gamma(r) \in \Sigma$ that

$$\begin{aligned}(C_\Sigma t^* g)(x) &= \frac{\mathrm{i}}{2\pi}\text{p.v}\int_0^\ell \frac{(\gamma_1'(s) + \mathrm{i}\gamma_2'(s))(\gamma_1'(s) - \mathrm{i}\gamma_2'(s))g(\gamma(s))}{(\gamma_1(r) + \mathrm{i}\gamma_2(r)) - (\gamma_1(s) + \mathrm{i}\gamma_2(s))}\mathrm{d}s \\ &= \frac{\mathrm{i}}{2\pi}\text{p.v}\int_\Sigma \frac{g(y)}{(x_1 + \mathrm{i}x_2) - (y_1 + \mathrm{i}y_2)}\mathrm{d}s(y).\end{aligned} \tag{5.10}$$

In our considerations also the formal dual C_Σ^* of C_Σ in $L^2(\Sigma)$, which acts as

$$C_\Sigma^* g(\gamma(r)) = \frac{\mathrm{i}}{2\pi}\text{p.v}\int_0^\ell \frac{(\gamma_1'(r) - \mathrm{i}\gamma_2'(r))g(\gamma(s))}{(\gamma_1(r) - \mathrm{i}\gamma_2(r)) - (\gamma_1(s) - \mathrm{i}\gamma_2(s))}\mathrm{d}s \tag{5.11}$$

for $g \in C^\infty(\Sigma)$ and $x = \gamma(r) \in \Sigma$ will play an important role. Note that C_Σ^* is the operator which satisfies $(C_\Sigma g, f)_{L^2(\Sigma)} = (g, C_\Sigma^* f)_{L^2(\Sigma)}$ for all $g, f \in C^\infty(\Sigma)$. Similarly as in (5.10) we have

$$\begin{aligned}(tC_\Sigma^* g)(x) &= \frac{\mathrm{i}}{2\pi}\text{p.v}\int_0^\ell \frac{(\gamma_1'(r) + \mathrm{i}\gamma_2'(r))(\gamma_1'(r) - \mathrm{i}\gamma_2'(r))g(\gamma(s))}{(\gamma_1(r) - \mathrm{i}\gamma_2(r)) - (\gamma_1(s) - \mathrm{i}\gamma_2(s))}\mathrm{d}s \\ &= \frac{\mathrm{i}}{2\pi}\text{p.v}\int_\Sigma \frac{g(y)}{(x_1 - \mathrm{i}x_2) - (y_1 - \mathrm{i}y_2)}\mathrm{d}s(y).\end{aligned} \tag{5.12}$$

Then

$$\begin{aligned}C_\Sigma t^* g(x) &= \frac{i}{2\pi}\text{ p.v}\int_\Sigma \frac{g(y)}{(x_1 - y_1) - i(x_2 - y_2)}\,\mathrm{d}s(y), \\ tC_\Sigma^* g(x) &= \frac{i}{2\pi}\text{ p.v}\int_\Sigma \frac{g(y)}{(x_1 - y_1) + i(x_2 - y_2)}\,\mathrm{d}s(y), \quad x \in \Sigma,\end{aligned} \tag{5.13}$$

and

$$\mathcal{C}_m = \begin{pmatrix} 0 & C_\Sigma t^* \\ tC_\Sigma^* & 0 \end{pmatrix}. \tag{5.14}$$

Therefore, the boundedness of C_Σ implies the boundedness of $\mathcal{C}_m$. In addition, the expansion (5.7) shows that $\mathcal{C}_z - \mathcal{C}_m$ is an integral operator with a Hilbert-Schmidt kernel, in particular,

$$\mathcal{C}_z - \mathcal{C}_m : \ L^2(\Sigma, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2) \text{ is compact for any } z \in \mathbb{C} \setminus \mathrm{Sp}(A),$$

which also shows the well-definedness and boundedness of $\mathcal{C}_z$ for all admissible z.

Let $\gamma_\pm : H^{\frac{1}{2}}(\sigma, \Omega_\pm) \to L^2(\Sigma)$ be the Sobolev trace operators, and for any $f \in H^{\frac{1}{2}}(\mathbb{R}^2 \setminus \Sigma)$ we set

$$\gamma_\pm f := \gamma_\pm f_\pm,$$

then one has the so-called jump formula

$$\gamma_\pm \Phi_z g = \Big(\mp \frac{i}{2} \sigma \cdot \nu + \mathcal{C}_z \Big) g, \quad g \in L^2(\Sigma, \mathbb{C}^2). \tag{5.15}$$

In [BHOBP20, Proposition 3.5] the jump formula was proved under the formal assumption that Σ is C^∞ smooth, but the same proof applies to our case as well, as the Plemelj-Sokhotski formula used in the proof also holds for closed Lipschitz curves. From the jump formula (5.15) one obtains

$$g = i(\sigma \cdot \nu)\Big[\gamma_+ \Phi_z g - \gamma_- \Phi_z g\Big], \quad g \in L^2(\Sigma, \mathbb{C}^2),$$

which shows the injectivity of Φ_z. Further direct consequences of the jump formula are the identities

$$\begin{aligned} &\gamma_+ \Phi_z g - \gamma_- \Phi_z g = -i(\sigma \cdot \nu) g, \\ &\frac{\gamma_+ \Phi_z g + \gamma_- \Phi_z g}{2} = \mathcal{C}_z g, \quad g \in L^2(\Sigma, \mathbb{C}^2). \end{aligned} \tag{5.16}$$

For $z \in (\mathbb{C} \setminus \mathrm{Sp}(A)) \cup \{m\}$ consider the bounded linear operator

$$\Theta_z := \mathbb{I} + (\varepsilon \mathbb{I}_2 + \mu \sigma_3)\mathcal{C}_z : \ L^2(\Sigma, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2),$$

which is closely related to the operator B from (5.5) as follows:

Lemma 5.2.1. *For any $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$ there holds $\ker(B - z) = \Phi_z \ker \Theta_z$, in particular, $\dim \ker(B - z) = \dim \ker \Theta_z$.*

Proof. Remark that the last assertion follows from the injectivity of Φ_z.

Let $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$ and $g \in \ker \Theta_z$. Denote $f := \Phi_z g$, then $f \in \ker(B_{\max} - z)$ due to the above properties of Φ_z. We need to show $f \in \mathrm{Dom}(B)$. By (5.9) we have already $f \in H^{\frac{1}{2}}(\sigma, \mathbb{R}^2 \setminus \Sigma)$. By (5.16) we have

$$\begin{aligned} (\varepsilon \mathbb{I}_2 + \mu \sigma_3) &\frac{\gamma_+ \Phi_z g + \gamma_- \Phi_z g}{2} + i(\sigma \cdot \nu)(\gamma_+ \Phi_z g - \gamma_- \Phi_z g) \\ &= (\varepsilon \mathbb{I}_2 + \mu \sigma_3)\mathcal{C}_z g + i(\sigma \cdot \nu)\big(-i(\sigma \cdot \nu)\big) g \\ &= (\varepsilon \mathbb{I}_2 + \mu \sigma_3)\mathcal{C}_z g + g = \Theta_z g = 0. \end{aligned}$$

Hence, $f \in \ker(B - z)$. This shows the inclusion $\Phi_z \ker \Theta_z \subset \ker(B - z)$.

Now let $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$ and $f \in \ker(B - z)$. Due to (5.3) we have

$$(D_m - z) f = (B - z) f + i(\sigma \cdot \nu)(f_+ - f_-)\delta_\Sigma. \tag{5.17}$$

Let $\mathcal{F} : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ be the Fourier transform. For any $\psi \in \mathcal{S}'(\mathbb{R}^2)$ we have

$$\mathcal{F}(D_m - z)\psi = (\sigma \cdot \xi + m\sigma_3 - z\mathbb{I}_2)\mathcal{F}\psi.$$

The matrix $\sigma \cdot \xi + m\sigma_3 - z\mathbb{I}_2$ is invertible for any $\xi \in \mathbb{R}^2$ and has polynomial entries, which shows that $D_m - z : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ is injective. As the function $\phi_z \in \mathcal{S}'(\mathbb{R}^2)$ is a fundamental solution of

$D_m - z$, from (5.17) one obtains

$$f = \phi_z * \big[i(\sigma\cdot\nu)(f_+ - f_-)\delta_\Sigma\big].$$

Due to $f \in \mathrm{Dom}(B)$ we have $f_\pm \in H^{\frac{1}{2}}(\Omega_\pm, \mathbb{C}^2)$, and, hence

$$g := i(\sigma\cdot\nu)(\gamma_+ f - \gamma_- f) \in L^2(\Sigma, \mathbb{C}^2).$$

Then

$$f = \phi_z * g = \int_\Sigma \phi_z(\cdot - y) g(y)\,\mathrm{d}s(y) \equiv \Phi_z g.$$

With the help of (5.16) we obtain

$$\begin{aligned} 0 &= (\varepsilon\mathbb{I}_2 + \mu\sigma_3)\frac{\gamma_+ f + \gamma_- f}{2} + i(\sigma\cdot\nu)(\gamma_+ f - \gamma_- f) \\ &= (\varepsilon\mathbb{I}_2 + \mu\sigma_3)\mathcal{C}_z g + g = \Theta_z g, \end{aligned}$$

which implies $g \in \ker\Theta_z$. Hence, $\ker(B - z) \subset \Phi_z \ker\Theta_z$. ■

For the sake of completeness, we include the proof of the following important statement (which is based on similar ideas):

Lemma 5.2.2. *The operator* $C_\Sigma^2 - \frac{1}{4}$ *is compact in* $L^2(\Sigma, \mathbb{C}^2)$.

Proof. Let $h \in L^2(\Sigma, \mathbb{C}^2)$ and $z \in \mathbb{C} \setminus \mathrm{Sp}(A)$. Consider $f := \Phi_z h$, then $(D_m - z)f = 0$ in $\Omega_\pm$. Consider further the function

$$\widetilde{f} : \mathbb{R}^2 \ni x \mapsto \begin{cases} f(x), & x \in \Omega_+, \\ 0, & \text{otherwise.} \end{cases}$$

One has $\gamma_+\widetilde{f} = \gamma_+ f$ and $\gamma_-\widetilde{f} = 0$, with $(D_m - z)\widetilde{f} = 0$ in $\Omega_\pm$, and (5.3) gives

$$(D_m - z)\widetilde{f} = i(\sigma\cdot\nu)(\gamma_+\widetilde{f} - \gamma_-\widetilde{f})\delta_\Sigma \equiv i(\sigma\cdot\nu)\gamma_+ f\,\delta_\Sigma \text{ in } \mathcal{D}'(\mathbb{R}^2),$$

which implies $\widetilde{f} = \phi_z * \big[i(\sigma\cdot\nu)\gamma_+ f\,\delta_\Sigma\big] \equiv \Phi_z i(\sigma\cdot\nu)\gamma_+ f$. In particular,

$$\Phi_z i(\sigma\cdot\nu)\gamma_+ f = f = \Phi_z h \text{ in } \Omega_+. \tag{5.18}$$

Remark that by the construction of f we have

$$\gamma_+ f = \Big(-\frac{i(\sigma\cdot\nu)}{2} + \mathcal{C}_z\Big)h.$$

Use this last equality in (5.18) and then apply γ_+ on the both parts, then one arrives at

$$\Big(-\frac{i(\sigma\cdot\nu)}{2} + \mathcal{C}_z\Big)i(\sigma\cdot\nu)\Big(-\frac{i(\sigma\cdot\nu)}{2} + \mathcal{C}_z\Big)h = \Big(-\frac{i(\sigma\cdot\nu)}{2} + \mathcal{C}_z\Big)h,$$

which after a simple algebra takes the form

$$\mathcal{C}_z i(\sigma\cdot\nu)\mathcal{C}_z h = -\frac{i(\sigma\cdot\nu)}{4}\,h,$$

and results in the identity

$$\Big(\mathcal{C}_z(\sigma\cdot\nu)\Big)^2 = -\frac{1}{4}\,\mathbb{I}. \tag{5.19}$$

The identities are well-known for the three-dimensional case [AMV14, Lemma 3.3], but we gave a complete argument to stay self-contained. Further remark that

$$\sigma\cdot\nu = \begin{pmatrix} 0 & n^* \\ n & 0 \end{pmatrix},$$

where n is the operator of multiplication by $\nu_1 + i\nu_2$. Using (5.14) we write

$$\mathcal{C}_z = \begin{pmatrix} 0 & C_\Sigma t^* \\ tC_\Sigma^* & 0 \end{pmatrix} + M_0$$

with a compact operator M_0. We have $t^*n = -i\mathbb{I}$, so the substitution into (5.19) gives, with some compact operators M_j,

$$-\frac{1}{4}\,\mathbb{I} = \left[\begin{pmatrix} -iC_\Sigma & 0 \\ 0 & tC_\Sigma^* n^* \end{pmatrix} + M_1\right]^2 = \begin{pmatrix} -C_\Sigma^2 & 0 \\ 0 & (tC_\Sigma^* n^*)^2 \end{pmatrix} + M_2,$$

and the upper left block gives the sought result. ■

5.3 Case $|\varepsilon| = |\mu|$

We first consider the self-adjointness of B for $|\varepsilon| = |\mu|$.

Theorem 5.3.1. *The operator B in (5.5) is self-adjoint for $|\varepsilon| = |\mu|$.*

Proof. In the case $\varepsilon = \mu = 0$ we obviously have $B = A$. From now on let

$$\mu = \pm\varepsilon \text{ with } \varepsilon \neq 0.$$

Consider the following maps

$$P_+ : L^2(\Sigma) \ni f \mapsto \begin{pmatrix} f \\ 0 \end{pmatrix} \in L^2(\Sigma, \mathbb{C}^2),$$
$$P_- : L^2(\Sigma) \ni f \mapsto \begin{pmatrix} 0 \\ f \end{pmatrix} \in L^2(\Sigma, \mathbb{C}^2),$$

and their adjoints

$$P_+^* : L^2(\Sigma, \mathbb{C}^2) \ni \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto f_1 \in L^2(\Sigma),$$
$$P_-^* : L^2(\Sigma, \mathbb{C}^2) \ni \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto f_2 \in L^2(\Sigma).$$

We set

$$P := P_\pm \text{ for } \varepsilon = \pm\mu.$$

As the operator B is symmetric, it is sufficient to show that $\mathrm{ran}(B-z) = L^2(\mathbb{R}^2,\mathbb{C}^2)$ for any $z\in\mathbb{C}\setminus\mathbb{R}$. For that, we will explicitly construct the inverse $(B-z)^{-1}$.

Let $z\in\mathbb{C}\setminus\mathbb{R}$. As B is symmetric, $\ker(B-z)=\{0\}$, and Lemma 5.2.1 implies $\ker\Theta_z=\{0\}$. Remark that in the present case, we have

$$\Theta_z = \mathbb{I} + 2\varepsilon PP^*\mathcal{C}_z,\quad \Theta_z P = P + 2\varepsilon PP^*\mathcal{C}_z P \equiv 2\varepsilon P\lambda_z$$
$$\text{for } \lambda_z := \frac{1}{2\varepsilon}\mathbb{I} + P^*\mathcal{C}_z P:\ L^2(\Sigma,\mathbb{C}^2)\to L^2(\Sigma,\mathbb{C}^2) \equiv \frac{1}{2\varepsilon}\mathbb{I} + (z\pm m)S_z$$

with the operator $S_z : L^2(\Sigma)\to L^2(\Sigma)$ given by

$$(S_z g)(x) := \frac{1}{2\pi}\int_\Sigma K_0\big(\sqrt{m^2-z^2}|x-y|\big)g(y)\,\mathrm{d}s(y),\quad x\in\Sigma,\quad g\in L^2(\Sigma).$$

The integral kernel of S_z has a logarithmic singularity on the diagonal, therefore, S_z is Hilbert-Schmidt (in particular, compact). It follows that λ_z is a Fredholm operator of index zero. From the injectivity of Θ_z and P one obtains the injectivity of λ_z, and it follows that $\lambda_z : L^2(\Sigma)\to L^2(\Sigma)$ is bijective.

Now we are going to show that the operator

$$R(z) := (A-z)^{-1} - \Phi_z P\lambda_z^{-1}P^*\Phi_{\bar z}^*,$$

is the inverse of $B-z$. Let $v\in L^2(\mathbb{R}^2,\mathbb{C}^2)$. Due to (5.8) one has

$$f := R(z)v \in H^{\frac{1}{2}}(\mathbb{R}^2\setminus\Sigma,\mathbb{C}^2).$$

Using the jump formulas (5.16) we obtain

$$\frac{\gamma_+ f + \gamma_- f}{2} = \gamma(A-z)^{-1}v - \mathcal{C}_z P\lambda_z^{-1}P^*\Phi_{\bar z}^* v \equiv \Phi_{\bar z}^* v - \mathcal{C}_z P\lambda_z^{-1}P^*\Phi_{\bar z}^* v,$$
$$\gamma_+ f - \gamma_- f = i(\sigma\cdot\nu)P\lambda_z^{-1}P^*\Phi_{\bar z}^* v.$$

We have then

$$\begin{aligned}
(\varepsilon\mathbb{I}_2 + \mu\sigma_3)&\frac{\gamma_+ f + \gamma_- f}{2} + i(\sigma\cdot\nu)(\gamma_+ - \gamma_- f)\\
&\equiv 2\varepsilon PP^*\frac{\gamma_+ f + \gamma_- f}{2} + i(\sigma\cdot\nu)(\gamma_+ - \gamma_- f)\\
&= 2\varepsilon PP^*\big(\Phi_{\bar z}^* v - \mathcal{C}_z P\lambda_z^{-1}P^*\Phi_{\bar z}^* v\big) + i(\sigma\cdot\nu)i(\sigma\cdot\nu)P\lambda_z^{-1}P^*\Phi_{\bar z}^* v\\
&= 2\varepsilon PP^*\big(\Phi_{\bar z}^* v - \mathcal{C}_z P\lambda_z^{-1}P^*\Phi_{\bar z}^* v\big) - P\lambda_z^{-1}P^*\Phi_{\bar z}^* v\\
&= P\big(2\varepsilon\mathbb{I} - 2\varepsilon P^*\mathcal{C}_z P\lambda_z^{-1} - \lambda_z^{-1}\big)P^*\Phi_{\bar z}^* v,
\end{aligned}$$

while

$$\begin{aligned}
2\varepsilon - 2\varepsilon P^*\mathcal{C}_z P\lambda_z^{-1} - \lambda_z^{-1} &= 2\varepsilon\mathbb{I} - 2\varepsilon\Big(P^*\mathcal{C}_z P + \frac{1}{2\varepsilon}\mathbb{I}\Big)\lambda_z^{-1}\\
&= 2\varepsilon\mathbb{I} - 2\varepsilon\lambda_z\lambda_z^{-1} = 0.
\end{aligned}$$

This shows that f satisfies the transmission condition (5.4) and, therefore, $f\in\mathrm{Dom}(B)$.

Further, in $\mathcal{D}'(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2)$ we have $(D_m - z)\Phi_z P \lambda_z^{-1} P^* \Phi_{\bar{z}}^* v = 0$, therefore,

$$(B - z)f = (D_m - z)f = (D_m - z)(A - z)^{-1}v = (A - z)(A - z)^{-1}v = v,$$

which shows $R(z) = (B - z)^{-1}$. ■

5.4 Case $|\varepsilon| \neq |\mu|$

For $|\varepsilon| \neq |\mu|$ the matrix $\varepsilon\mathbb{I}_2 + \mu\sigma_3$ is invertible, with

$$(\varepsilon\mathbb{I}_2 + \mu\sigma_3)^{-1} = \frac{1}{\varepsilon^2 - \mu^2}(\varepsilon\mathbb{I}_2 - \mu\sigma_3),$$

and it will be more convenient to consider the auxiliary bounded linear operators

$$\Lambda_z := \frac{1}{\varepsilon^2 - \mu^2}(\varepsilon\mathbb{I}_2 - \mu\sigma_3) + \mathcal{C}_z \equiv (\varepsilon\mathbb{I}_2 + \mu\sigma_3)^{-1}\Theta_z$$

for $z \in (\mathbb{C} \setminus \operatorname{Sp}(A)) \cup \{m\}$. The symmetry property $\phi_z(y - x)^* = \phi_{\bar{z}}(x - y)$ entails that both $\mathcal{C}_z$ and Λ_z are self-adjoint for real admissible z.

The following assertion can be viewed as a simplified version of the results of [BHSS24], and this is the entry point for the subsequent analysis:

Theorem 5.4.1. *Let $|\varepsilon| \neq |\mu|$ such that the operator Λ_a is Fredholm for some $a \in (\mathbb{C} \setminus \operatorname{Sp}(A)) \cup \{m\}$, then the operator B in (5.5) is self-adjoint.*

Proof. Let Λ_a be Fredholm. As noted above, for any $z \in \mathbb{C} \setminus \operatorname{Sp}(A)$ the difference $\Lambda_z - \Lambda_a \equiv \mathcal{C}_z - \mathcal{C}_a$ is a compact operator, and it follows that Λ_z is also Fredholm and has the same index as Λ_a.

Now let $z \in \big(-|m|, |m|\big) \cup \{m\}$, then Λ_z is self-adjoint. From the Fredholmness and the self-adjointness, it follows that the index of Λ_z is zero. We have just seen above that the index is independent of z, so Λ_z is Fredholm of index zero for all $z \in \mathbb{C} \setminus \operatorname{Sp}(A)$.
As B is symmetric, and in order to show its self-adjointness it is sufficient to show that $\operatorname{ran}(B - z) = L^2(\mathbb{R}^2, \mathbb{C}^2)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. We will do it by constructing explicitly the inverse $(B - z)^{-1}$ defined on $L^2(\mathbb{R}^2, \mathbb{C}^2)$.

Let $z \in \mathbb{C} \setminus \mathbb{R}$. As B is symmetric, there holds $\ker(B - z) = \{0\}$. By Lemma 5.2.1 one obtains $\ker \Lambda_z = \{0\}$. As Λ_z is Fredholm of index zero, one has $\operatorname{ran} \Lambda_z = L^2(\Sigma, \mathbb{C}^2)$, so $\Lambda_z : L^2(\Sigma, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2)$ is bijective with a bounded inverse. Consider the bounded linear operator

$$R(z) = (A - z)^{-1} - \Phi_z \Lambda_z^{-1} \Phi_{\bar{z}}^* : \; L^2(\mathbb{R}^2, \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathbb{C}^2).$$

We are going to show that $R(z) = (B - z)^{-1}$.

Let $v \in L^2(\mathbb{R}^2, \mathbb{C}^2)$. Due to (5.8) one has

$$f := R(z)v \in H^{\frac{1}{2}}(\mathbb{R}^2 \setminus \Sigma, \mathbb{C}^2).$$

Using (5.16) we obtain

$$\begin{aligned}\frac{\gamma_+ f + \gamma_- f}{2} &= \gamma(A-z)^{-1}v - \mathcal{C}_z\Lambda_z^{-1}\Phi_{\bar z}^* v = \Phi_{\bar z}^* v - \mathcal{C}_z\Lambda_z^{-1}\Phi_{\bar z}^* v,\\ \gamma_+ f - \gamma_- f &= i\,(\sigma\cdot\nu)(\Lambda_z)^{-1}\Phi_{\bar z}^* v.\end{aligned}$$

Then

$$\begin{aligned}(\varepsilon\mathbb{I}_2 + \mu\sigma_3)&\frac{\gamma_+ f + \gamma_- f}{2} + i(\sigma\cdot\nu)(\gamma_+ f - \gamma_- f)\\ &= \Big[(\varepsilon\mathbb{I}_2 + \mu\sigma_3)(\mathbb{I} - \mathcal{C}_z\Lambda_z^{-1}) - \Lambda_z^{-1}\Big]\Phi_{\bar z}^* v,\end{aligned}$$

while

$$\begin{aligned}(\varepsilon\mathbb{I}_2 + \mu\sigma_3)(\mathbb{I} - \mathcal{C}_z\Lambda_z^{-1}) - \Lambda_z^{-1} &= \Big[(\varepsilon\mathbb{I}_2 + \mu\sigma_3)(\Lambda_z - \mathcal{C}_z) - \mathbb{I}\Big]\Lambda_z^{-1}\\ &= \Big[(\varepsilon\mathbb{I}_2 + \mu\sigma_3)\frac{1}{\varepsilon^2-\mu^2}(\varepsilon\mathbb{I}_2 - \mu\sigma_3) - \mathbb{I}\Big]\Lambda_z^{-1}\\ &= (\mathbb{I} - \mathbb{I})\Lambda_z^{-1} = 0.\end{aligned}$$

This shows that f satisfies the transmission condition (5.4), *i.e.*, $f \in \mathrm{Dom}(B)$. In addition, in $\mathcal{D}'(\mathbb{R}^2\setminus\Sigma)$ we have $(D_m - z)\Phi_z\Lambda_z^{-1}\Phi_{\bar z}^* = 0$, therefore,

$$\begin{aligned}(B-z)f &= (D_m - z)f = (D_m - z)R(z)v\\ &= (D_m - z)(A-z)^{-1} = (A-z)(A-z)^{-1}v = v,\end{aligned}$$

which shows the required identity $R(z) = (B-z)^{-1}$. ■

The following lemma gives a precise range of (ε,μ) for which B is self-adjoint without additional assumptions on Σ.

Theorem 5.4.2. *Assume that* $|\varepsilon| < |\mu|$, *then* B *is self-adjoint.*

Proof. By Theorem 5.4.1 it is sufficient to show that $(\varepsilon^2-\mu^2)\Lambda_m$ is Fredholm. Using (5.14) we represent

$$\begin{aligned}(\varepsilon^2-\mu^2)\Lambda_m &= (\varepsilon\mathbb{I}_2 - \mu\sigma_3) + (\varepsilon^2-\mu^2)\mathcal{C}_m\\ &= (\varepsilon\mathbb{I}_2 - \mu\sigma_3) + (\varepsilon^2-\mu^2)\begin{pmatrix}0 & C_\Sigma t^*\\ tC_\Sigma^* & 0\end{pmatrix} = \varepsilon\mathbb{I}_2 + \Gamma,\\ \text{with } \Gamma &:= \begin{pmatrix}-\mu & (\varepsilon^2-\mu^2)C_\Sigma t^*\\ (\varepsilon^2-\mu^2)tC_\Sigma^* & \mu\end{pmatrix}.\end{aligned}$$

Remark that Γ is self-adjoint and

$$\Gamma^2 = \mu^2 + (\varepsilon^2-\mu^2)^2\begin{pmatrix}C_\Sigma C_\Sigma^* & 0\\ 0 & tC_\Sigma^* C_\Sigma t^*\end{pmatrix}.$$

The last term is a non-negative operator, which shows

$$\Gamma^2) \subset [\mu^2,\infty), \quad \cap(-|\mu|,|\mu|) = \emptyset.$$

Therefore, if $|\varepsilon| < |\mu|$, then the operator

$$(\varepsilon^2 - \mu^2)\Lambda_m \equiv \varepsilon + \Gamma : \; L^2(\Sigma, \mathbb{C}^2) \to L^2(\Sigma, \mathbb{C}^2)$$

is an isomorphism and, in particular, Fredholm. ■

By summarizing Theorems 5.3.1 and 5.4.2 we arrive at

Corollary 5.4.3. *The operator B is self-adjoint for any (ε, μ) with $|\varepsilon| \leq |\mu|$.*

Remark that the preceding discussion is valid without any additional assumptions on Σ (*i.e.*, only assumes that Σ is Lipschitz). Under stronger geometric assumptions one can indeed enlarge the range of parameters for which the self-adjointness is guaranteed. The following result follows implicitly from the machinery of [BHSS24], but we prefer to give an explicit formulation with a direct argument.

Theorem 5.4.4. *If Σ is C^1-smooth and $\varepsilon^2 - \mu^2 \neq 4$, then B is self-adjoint.*

Proof. The case $|\varepsilon| = |\mu|$ is already covered by Theorem 5.3.1, so from now on assume $|\varepsilon| \neq |\mu|$. By Theorem 5.4.1 it is sufficient to show that Λ_m is Fredholm. Due to the self-adjointness of Λ_m this is equivalent to

$$0 \notin \mathrm{Sp}_{\mathrm{ess}}(\varepsilon^2 - \mu^2)\Lambda_m. \tag{5.20}$$

Using (5.14) we represent

$$\begin{aligned}
(\varepsilon^2 - \mu^2)\Lambda_m &= (\varepsilon\mathbb{I}_2 - \mu\sigma_3) + (\varepsilon^2 - \mu^2)\mathcal{C}_m \\
&= (\varepsilon\mathbb{I}_2 - \mu\sigma_3) + (\varepsilon^2 - \mu^2)\begin{pmatrix} 0 & C_\Sigma t^* \\ tC_\Sigma^* & 0 \end{pmatrix} = \varepsilon\mathbb{I}_2 + \Gamma, \\
\text{with } \Gamma &:= \begin{pmatrix} -\mu\mathbb{I} & (\varepsilon^2 - \mu^2)C_\Sigma t^* \\ (\varepsilon^2 - \mu^2)tC_\Sigma^* & \mu\mathbb{I} \end{pmatrix}.
\end{aligned}$$

By [Lan99, Theorem 3.2] the operator $C_\Sigma - C_\Sigma^*$ is compact, therefore,

$$\Gamma = \begin{pmatrix} -\mu & (\varepsilon^2 - \mu^2)C_\Sigma t^* \\ (\varepsilon^2 - \mu^2)tC_\Sigma & \mu \end{pmatrix} + M_0$$

with some compact operator M_0. Using Lemma 5.2.2 we obtain, with some compact operators M_1 and M_2,

$$\begin{aligned}
\Gamma^2 &= \mu^2 + (\varepsilon^2 - \mu^2)^2 \begin{pmatrix} C_\Sigma^2 & 0 \\ 0 & tC_\Sigma^2 t^* \end{pmatrix} + M_1 \\
&\equiv \mu^2 + \frac{(\varepsilon^2 - \mu^2)^2}{4} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} + M_2.
\end{aligned}$$

It follows that

$$\mathrm{Sp}_{\mathrm{ess}}(\Gamma^2) = \mu^2 + \frac{(\varepsilon^2 - \mu^2)^2}{4},$$

and the self-adjointness of Γ implies

$$\mathrm{Sp}_{\mathrm{ess}}\Gamma \in \Big\{ -\sqrt{\mu^2 + \frac{(\varepsilon^2 - \mu^2)^2}{4}}, \sqrt{\mu^2 + \frac{(\varepsilon^2 - \mu^2)^2}{4}} \Big\}.$$

Due to the above identity $(\varepsilon^2-\mu^2)\Lambda_m=\varepsilon+\Gamma$ the condition (5.20) is equivalent to

$$|\varepsilon|\neq\sqrt{\mu^2+\frac{(\varepsilon^2-\mu^2)^2}{4}},\quad \textit{i.e.},\ \varepsilon^2-\mu^2\neq\frac{(\varepsilon^2-\mu^2)^2}{4}$$

which reduces to $\varepsilon^2-\mu^2\neq 4$. ■

5.5 Fredholmness for curvilinear polygons

From now assume that Σ is a piecewise C^1-smooth Lipschitz curve, with finitely many corner points $a_1,\dots,a_n$. For each corner a_j, let

$$\theta_j\in(0,2\pi)\setminus\{\pi\}$$

be the non-oriented interior angle of Σ at the point a_j measured inside Ω_+. Our main goal is to give a complete characterization of the values of ε and μ for which the operators Λ_z are Fredholm in $L^2(\Sigma,\mathbb{C}^2)$. To do so, we are going to implement the technique proposed by Shelepov [She91]. Remark that some components of the approach implicitly appear in other works [BR15, CS85].

Actually the work [She91] also applies to the so-called Radon curves, which are more general than curvilinear polygons, but we prefer to restrict our attention to the case of piecewise C^1-smooth curves in order to avoid a series of involved definitions. Let us describe the general scheme of [She91].

Denote

$$\mathbb{S}:=\{x\in\mathbb{R}^2:\ |x|=1\}$$

and let $\mathscr{M}_k(\mathbb{C})$ be the space of $k\times k$ complex matrices. Let

$$G:\ \mathbb{R}\times\mathbb{R}\times\mathbb{S}\times\mathbb{S}\times\mathbb{S}\to\mathscr{M}_k(\mathbb{C})$$

be a matrix-valued function whose entries $G_{i,j}$ are Lipschitz (with respect to all variables) and such that for some $C>0$ one has

$$|G_{ij}(x,y,\xi,\eta,\zeta)|\leqslant C\Big(|\langle\xi,\zeta\rangle|+|\langle\eta,\zeta\rangle|\Big)\tag{5.21}$$

for all (x,y,ξ,η,ζ).

Consider the bounded integral operator $T:L^2(\Sigma,\mathbb{C}^k)\to L^2(\Sigma,\mathbb{C}^k)$,

$$Tg(x)=\int_\Sigma\frac{1}{|x-y|}\,G\Big(x,y,\nu(x),\nu(y),\frac{x-y}{|x-y|}\Big)g(y)\,\mathrm{ds}(y),$$
$$x,y\in\Sigma,\quad g\in L^2(\Sigma,\mathbb{C}^k).$$

We assume without loss of generality that each connected component of Σ is oriented in the anticlockwise sense. Fix a corner point a on Σ with an interior angle θ. A small arc of Σ around a is separated by a into two nonempty parts Γ_+ and Γ_- that project in one-to-one fashion on the one-sided tangents to Σ at a, and denote the projections by $\overline{\Gamma_+}$ and $\overline{\Gamma_-}$ respectively. Let τ_+ and τ_- be the unit vectors along $\overline{\Gamma_+}$ and $\overline{\Gamma_-}$ directed away from the corner a, and let $\nu_+(a)$ and $\nu_-(a)$ be the corresponding one-sided limits of the inner normal to Σ at a. We then denote by $\tau=-\tau_-$ the unit vector of the left positive tangent to Σ

at a and by $\nu(a) = \nu_-(a)$ the vector obtained from τ by a counterclockwise rotation through the angle $\pi/2$, see Figure 5.1. Finally, we will use the parameters

$$\xi := \eta + \frac{i}{2}, \quad \eta \in \mathbb{R}.$$

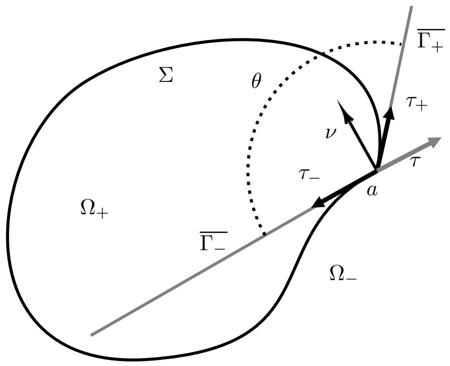

Figure 5.1 – Construction near a corner a.

Following [She91], we define a function $\zeta : \mathbb{R} \to \mathbb{R}$ and matrix-valued functions

$$H_a^{(j)} : \mathbb{R} + \frac{i}{2} \to \mathscr{M}_2(\mathbb{C}), \quad j \in \{1, 2\},$$

by

$$\zeta(t) = \frac{(e^{-\frac{t}{2}} \cos\theta - e^{\frac{t}{2}})\tau - \nu e^{-\frac{t}{2}} \sin\theta}{\sqrt{e^t + e^{-t} - 2\cos\theta}},$$

$$H_a^{(1)}(\xi) = \int_{-\infty}^{\infty} \frac{e^{(i\xi+1/2)t}}{\sqrt{e^t + e^{-t} - 2\cos\theta}} G\Big(a, a, \nu, -\tau\sin\theta - \nu\cos\theta, \zeta(-t)\Big)\mathrm{d}t,$$

$$H_a^{(2)}(\xi) = \int_{-\infty}^{\infty} \frac{e^{(i\xi+1/2)t}}{\sqrt{e^t + e^{-t} - 2\mathrm{cos}(\theta)}} G\Big(a, a, -\tau sin\theta - \nu\cos\theta, \nu, -\zeta(t)\Big)\mathrm{d}t,$$

and set

$$\Delta_a(\xi) = \det\Big(\mathbb{I}_2 - H_a^{(1)}(\xi)\, H_a^{(2)}(\xi)\Big), \quad \xi \in \mathbb{R} + \frac{i}{2}. \tag{5.22}$$

The following result was shown in [She91, Theorem 2]:

Proposition 5.5.1. *The operator $\mathbb{I} - T$ is Fredholm in $L^2(\Sigma, \mathbb{C}^2)$ if and only if*

$$\Delta_{a_j}(\xi) \neq 0 \textit{ for all } \xi \in \mathbb{R} + \frac{i}{2} \textit{ and all corners } a_1, ..., a_n \textit{ of } \Sigma.$$

We are now going to apply this machinery to our particular situation. For $\theta \in (0, 2\pi)$ consider the function

$$M_\theta : \mathbb{R} \ni x \mapsto \frac{\cosh\big((\pi - \theta)x\big)}{2\big(1 + \cosh(\pi x)\big)} \in \mathbb{R},$$

and denote

$$m(\theta) := \sup_{x\in\mathbb{R}} M_\theta(x).$$

We have the obvious symmetry

$$m(\theta) = m(2\pi-\theta) \text{ for any } \theta \in (0,2\pi). \tag{5.23}$$

The following elementary properties of m will be needed as well:

Proposition 5.5.2. *For any $\omega \in (0,\pi)$ there holds*

$$\frac{1}{4} \le m(\omega) \le \frac{1}{2}. \tag{5.24}$$

Moreover, the function $\omega \mapsto m(\omega)$ is non-increasing, with

$$\lim_{\omega\to 0^+} m(\omega) = \frac{1}{2} \tag{5.25}$$

and

$$m(\omega) = \frac{1}{4} \textit{ for all } \omega \in \Big[\frac{\pi}{2},\pi\Big). \tag{5.26}$$

Proof. For any $|a| \le |b|$ we have $\cosh a \le \cosh b$. It follows that for any $x \in \mathbb{R}$ there holds

$$\frac{1}{4} = M_\omega(0) \le M_\omega(x) = \frac{\cosh\big((\pi-\omega)x\big)}{2(1+\cosh(\pi x))} \le \frac{\cosh(\pi x)}{2(1+\cosh(\pi x))} \le \frac{1}{2},$$

which gives (5.24). For $0<\omega\le\omega'<\pi$ and any $x\in\mathbb{R}$ one has

$$M_{\omega'}(x) = \frac{\cosh\big((\pi-\omega')x\big)}{2(1+\cosh(\pi x))} \le \frac{\cosh\big((\pi-\omega)x\big)}{2(1+\cosh(\pi x))} = M_\omega(x),$$

so taking the supremum over all x one shows $m(\omega') \le m(\omega)$, *i.e.*, m is non-increasing. In addition, for any fixed x the function $\theta \mapsto M_\theta(x)$ is non-increasing too. It follows

$$\begin{aligned}
\lim_{\omega\to 0^+} m(\omega) &= \sup_{\omega\in(0,\pi)} m(\omega) = \sup_{\omega\in(0,\pi)} \sup_{x\in\mathbb{R}} M_\omega(x) \\
&= \sup_{x\in\mathbb{R}} \sup_{\omega\in(0,\pi)} M_\omega(x) = \sup_{x\in\mathbb{R}} \lim_{\omega\to 0^+} M_\omega(x) \\
&= \sup_{x\in\mathbb{R}} \lim_{\omega\to 0^+} \frac{\cosh\big((\pi-\omega)x\big)}{2(1+\cosh(\pi x))} = \sup_{x\in\mathbb{R}} \frac{\cosh(\pi x)}{2(1+\cosh(\pi x))} = \frac{1}{2}.
\end{aligned}$$

We further remark that for any $\omega\in(0,\pi)$ the function M_θ is even, and for any $x \ge 0$ one has

$$\begin{aligned}
M'_\omega(x) &= \frac{1}{2\big(1+\cosh(\pi x)\big)^2}\Big[(\pi-\omega)\sinh\big((\pi-\omega)x\big)\big(1+\cosh(\pi x)\big) \\
&\qquad - \pi\cosh\big((\pi-\omega)x\big)\sinh(\pi x)\Big] \\
&\equiv \frac{\pi\big(1+\cosh(\pi x)\big)\cosh\big((\pi-\omega)x\big)}{2\big(1+\cosh(\pi x)\big)^2} N_\omega(x)
\end{aligned}$$

with

$$N_\omega(x) := \frac{\pi-\omega}{\pi}\,\frac{\sinh\big((\pi-\omega)x\big)}{\cosh\big((\pi-\omega)x\big)} - \frac{\sinh(\pi x)}{1+\cosh(\pi x)}$$

$$\equiv \frac{\pi-\omega}{\pi}\,\frac{\sinh\big((\pi-\omega)x\big)}{\cosh\big((\pi-\omega)x\big)} - \frac{\sinh\dfrac{\pi x}{2}}{\cosh\dfrac{\pi x}{2}}$$

$$\equiv \frac{\pi-\omega}{\pi}\tanh\big((\pi-\omega)x\big) - \tanh\frac{\pi x}{2}.$$

The function $[0,\infty)\ni a\mapsto \tanh a$ in increasing, therefore, $N_\omega(x)<0$ for all $x>0$ and $\omega\in\big[\frac{\pi}{2},\pi\big)$, and then $M'_\omega(x)<0$ for the same x and ω. Then for each $\omega\in\big[\frac{\pi}{2},\pi\big)$ the function M_ω is decreasing on $(0,+\infty)$, and by parity its maximum is located at the origin, *i.e.*,

$$m(\omega)=\sup_{x\in\mathbb{R}} M_\omega(x) = M_\omega(0) = \frac{1}{4} \text{ for all } \omega\in\Big[\frac{\pi}{2},\pi\Big).$$

■

Remark 5.5.1. *The condition for ω in (5.26) is not expected to be optimal. A rough numerical simulation indicates that*

$$\min\Big\{\omega\in(0,\pi):\ m(\omega)=\frac{1}{4}\Big\}\simeq 0.3\,\pi.$$

Using the above preparations we arrive at the main result:

Theorem 5.5.3. *Denote by ω the smallest angle of Σ, defined by*

$$\omega := \min_{j\in\{1,\dots,n\}} \min\{\theta_j, 2\pi-\theta_j\}\in(0,\pi).$$

If

$$\varepsilon^2-\mu^2<\frac{1}{m(\omega)} \quad \textit{or} \quad \varepsilon^2-\mu^2>16m(\omega), \tag{5.27}$$

then the operator B is self-adjoint.

Proof. As the case $|\varepsilon|\le|\mu|$ is already covered by Corollary 5.4.3, for the rest of the proof we assume

$$|\varepsilon|>|\mu|.$$

By Theorem 5.4.1 it is sufficient to show that Λ_m is Fredholm, which is in turn equivalent to the Fredholmness of the operator

$$\Theta_m \equiv (\varepsilon\mathbb{I}_2+\mu\sigma_3)\Lambda_m \equiv \mathbb{I}+(\varepsilon\mathbb{I}_2+\mu\sigma_3)\mathcal{C}_m:\ L^2(\Sigma,\mathbb{C}^2)\to L^2(\Sigma,\mathbb{C}^2).$$

Eq. (5.14) for $\mathcal{C}_m$ gives the representation

$$\Theta_m g(x) = g - \int_\Sigma \frac{1}{|x-y|} G\Big(x,y,\nu(x),\nu(y),\frac{x-y}{|x-y|}\Big) g(y)\mathrm{d}s(y)$$

with $g \in L^2(\Sigma, \mathbb{C}^2)$ and the 2×2 matrix function G defined by

$$G\Big(x, y, \nu(x), \nu(y), \frac{x-y}{|x-y|}\Big) = -\frac{i}{2\pi}\begin{pmatrix} 0 & (\varepsilon+\mu)\dfrac{\overline{x}-\overline{y}}{|x-y|} \\ (\varepsilon-\mu)\dfrac{x-y}{|x-y|} & 0 \end{pmatrix}$$

for $x, y \in \Sigma$, where the integral representations in (5.13) were used. The entries of G are obviously Lipschitz and satisfy (5.21), so the above machinery is applicable to the analysis of Θ_m.

Let a be a corner point of Σ with an interior angle θ, then

$$G(a, a, \nu, -\tau\sin\theta - \nu\cos\theta, \zeta(-t)) = -\frac{i}{2\pi}\begin{pmatrix} 0 & (\varepsilon+\mu)\overline{\zeta(-t)} \\ (\varepsilon-\mu)\zeta(-t) & 0 \end{pmatrix},$$

$$G(a, a, -\tau\sin\theta - \nu\cos\theta, \nu, -\zeta(t)) = \frac{i}{2\pi}\begin{pmatrix} 0 & (\varepsilon+\mu)\overline{\zeta(t)} \\ (\varepsilon-\mu)\zeta(t) & 0 \end{pmatrix},$$

where one uses the usual identification $\mathbb{R}^2 \ni (x_1, x_2) = x \simeq x = x_1 + ix_2 \in \mathbb{C}$.

We have

$$i\xi + 1 = i\bar{\xi} \text{ for all } \xi \in \mathbb{R} + \frac{i}{2},$$

and one easily sees that the matrices $H_a^{(1)}$ and $H_a^{(2)}$ for this specific case have the form

$$H_a^{(1)}(\xi) = \begin{pmatrix} 0 & (\varepsilon+\mu)A_{\overline{\tau},\overline{\nu}} \\ (\varepsilon-\mu)A_{\tau,\nu} & 0 \end{pmatrix},$$

$$H_a^{(2)}(\xi) = \begin{pmatrix} 0 & (\varepsilon+\mu)B_{\overline{\tau},\overline{\nu}} \\ (\varepsilon-\mu)B_{\tau,\nu} & 0 \end{pmatrix},$$

where $A_{\tau,\nu}$ and $B_{\tau,\nu}$ are given by

$$A_{\tau,\nu} = \int_{-\infty}^{+\infty} \frac{(e^{i\bar{\xi}t}\cos(\theta) - e^{i\xi t})\tau - e^{i\bar{\xi}t}\sin(\theta)\,\nu}{e^t + e^{-t} - 2\cos(\theta)}\,\mathrm{d}t,$$

$$B_{\tau,\nu} = \int_{-\infty}^{+\infty} \frac{(e^{i\xi t}\cos(\theta) - e^{i\bar{\xi}t})\tau - e^{i\xi t}\sin(\theta)\,\nu}{e^t + e^{-t} - 2\cos(\theta)}\,\mathrm{d}t.$$

Hence, applying the change of variable $x = e^t$, we can rewrite $A_{\tau,\nu}$ and $B_{\tau,\nu}$ as follows

$$A_{\tau,\nu} = \int_0^{+\infty} \frac{(x^{i\bar{\xi}}\cos(\theta) - x^{i\xi})\tau - x^{i\bar{\xi}}\sin(\theta)\,\nu}{x^2 + 2x\cos(\pi-\theta) + 1}\,\mathrm{d}x,$$

$$B_{\tau,\nu} = \int_0^{+\infty} \frac{(x^{i\xi}\cos(\theta) - x^{i\bar{\xi}})\tau - x^{i\xi}\sin(\theta)\,\nu}{x^2 + 2x\cos(\pi-\theta)+1}\,\mathrm{d}x.$$

Now recall that for all $b > 0$, $0 < |\omega| < \pi$ and $0 < \mathrm{Re}(\alpha) < 2$ one has

$$\int_0^{+\infty} \frac{x^{\alpha-1}}{x^2 + 2bx\,\cos(\omega) + b^2}\,\mathrm{d}x = -\pi b^{\alpha-2}\frac{1}{\sin(\omega)}\frac{1}{\sin(\alpha\pi)}\sin\left((\alpha-1)\omega\right),$$

see the formula (12) in [GR65, p. 327]. Applying this formula with $b = 1$ and $\omega = \pi - \theta$, one obtains that

$$\int_0^{\infty} \frac{x^{i\bar{\xi}}\cos(\theta)\tau}{x^2+1+2\cos(\pi-\theta)}\,\mathrm{d}x = -\frac{\pi\cos(\theta)}{\sin(\theta)}\frac{\sin(i\bar{\xi}(\pi-\theta))}{\sin(i\xi\theta)}\tau,$$
$$\int_0^{\infty} \frac{x^{i\xi}\tau}{x^2+1+2\cos(\pi-\theta)}\,\mathrm{d}x = -\frac{\pi}{\sin(\theta)}\frac{1}{\sin(i\bar{\xi}\pi)}\sin(i\xi(\pi-\theta))\tau,$$
$$\int_0^{\infty} \frac{x^{i\bar{\xi}}\sin(\theta)\nu}{x^2+1+2\cos(\pi-\theta)}\,\mathrm{d}x = -\pi\frac{\sin(i\bar{\xi}(\pi-\theta))}{\sin(i\xi\theta)}\nu.$$

Thus, $A_{\tau,\nu}$ and $B_{\tau,\nu}$ become as follows

$$A_{\tau,\nu} = \frac{i}{2\sin(\theta)}\Bigg[\Big(\cos(\theta)\frac{\sinh\left(\bar{\xi}(\pi-\theta)\right)}{\sinh(\xi\pi)} - \frac{\sinh\left(\xi(\pi-\theta)\right)}{\sinh(\bar{\xi}\pi)}\Big)\tau - \sin(\theta)\frac{\sinh\left(\bar{\xi}(\pi-\theta)\right)}{\sinh(\xi\pi)}\nu\Bigg],$$

$$B_{\tau,\nu} = \frac{-i}{2\sin(\theta)}\Bigg[\Big(\cos(\theta)\frac{\sinh\left(\xi(\pi-\theta)\right)}{\sinh(\bar{\xi}\pi)} - \frac{\sinh\left(\bar{\xi}(\pi-\theta)\right)}{\sinh(\xi\pi)}\Big)\tau - \sin(\theta)\frac{\sinh(\xi(\pi-\theta))}{\sinh(\bar{\xi}\pi)}\nu\Bigg].$$

Consequently, the product $H_a^{(1)}(\xi)H_a^{(2)}(\xi)$ yields

$$H_a^{(1)}(\xi)H_a^{(2)}(\xi) = \frac{\varepsilon^2-\mu^2}{4\sin^2(\theta)}\begin{pmatrix}\mathfrak{A}(\xi)+\mathfrak{B}(\xi) & 0\\ 0 & \mathfrak{A}(\xi)+\mathfrak{B}(\xi)\end{pmatrix},$$

with

$$\mathfrak{A}(\xi) = \Bigg(-\cos(\theta)\frac{\sin(i\bar{\xi}(\pi-\theta))}{\sin(i\xi\pi)} + \frac{\sin(i\xi(\pi-\theta))}{\sin(i\bar{\xi}\pi)}\Bigg)\Bigg(-\cos(\theta)\frac{\sin(i\xi(\pi-\theta))}{\sin(i\bar{\xi}\pi)} + \frac{\sin(i\bar{\xi}(\pi-\theta))}{\sin(i\xi\pi)}\Bigg),$$

$$\mathfrak{B}(\xi) = \Bigg(\sin(\theta)\frac{\sin(i\bar{\xi}(\pi-\theta))}{\sin(i\xi\pi)}\Bigg)\Bigg(\sin(\theta)\frac{\sin(i\xi(\pi-\theta))}{\sin(i\bar{\xi}\pi)}\Bigg).$$

The trigonometric identity

$$\sin(z) = -i\sinh(iz) \quad \text{and} \quad \sinh(-z) = -\sinh(\mathrm{z}), \qquad \text{for all } z \in \mathbb{C}$$

yield that

$$\mathfrak{A}(\xi) + \mathfrak{B}(\xi) := \frac{\varepsilon^2 - \mu^2}{4\sin^2(\theta)} \times S(\xi)\, \mathbb{I}_2,$$

where $S(\xi)$ is given by

$$\begin{aligned} S(\xi) = 2\, &\frac{\sinh\left(\bar{\xi}(\pi-\theta)\right)}{\sinh(\xi\pi)} \frac{\sinh\left(\xi(\pi-\theta)\right)}{\sinh(\bar{\xi}\pi)} \\ &- \cos(\theta)\left(\frac{\sinh^2\left(\xi(\pi-\theta)\right)}{\sinh^2(\bar{\xi}\pi)} + \frac{\sinh^2\left(\bar{\xi}(\pi-\theta)\right)}{\sinh^2(\xi\pi)}\right). \end{aligned} \tag{5.28}$$

We want to simplify the first term on the right-hand side of the previous identity. To do this, we use the exponential form of the function sinh

$$\sinh(\bar{\xi}(\pi-\theta)) = \frac{e^{\bar{\xi}(\pi-\theta)} - e^{-\bar{\xi}(\pi-\theta)}}{2}, \text{ for } \xi = \eta + \frac{i}{2},\ \bar{\xi} = \eta - \frac{i}{2}.$$

Then, we can write

$$\begin{aligned} \sinh\left(\bar{\xi}(\pi-\theta)\right)\sinh\left(\xi(\pi-\theta)\right) &= \frac{1}{2}\Big(\cos(\theta) + \cosh(2\eta(\pi-\theta))\Big), \\ \sinh(\xi\pi)\sinh(\bar{\xi}\pi) &= \frac{1}{2}(\cosh(2\eta\pi)+1). \end{aligned}$$

Thus, we deduce

$$2\,\frac{\sinh\left(\bar{\xi}(\pi-\theta)\right)}{\sinh(\xi\pi)}\frac{\sinh\left(\xi(\pi-\theta)\right)}{\sinh(\bar{\xi}\pi)} = 2\frac{\cos(\theta)+\cosh(2\eta(\pi-\theta))}{1+\cosh(2\eta\pi)}$$

Now, we want to simplify the second term on the right-hand side of the identity (5.28). Using the exponential formula of sinh and the trigonometric identity

$$\cosh(x \pm iy) = \cosh(x)\cos(y) \pm i\,\sinh(x)\sin(y), \qquad \text{for all } x, y \in \mathbb{R},$$

we obtain the following computation quantities

$$\begin{aligned} \sinh^2(\xi(\pi-\theta)) &= -\frac{1}{2}\Big(1+\cosh(2\eta(\pi-\theta))\cos(\theta)\Big) + i\,\frac{\sinh(2\eta(\pi-\theta))}{2}, \\ \sinh^2(\bar{\xi}(\pi-\theta)) &= -\frac{1}{2}\Big(1+\cosh(2\eta(\pi-\theta))\cos(\theta)\Big) - i\,\frac{\sinh(2\eta(\pi-\theta))}{2}, \\ \sinh^2(\xi\pi) = \sinh^2(\bar{\xi}\pi) &= \frac{1}{2}\Big(1+\cosh(2\eta(\pi-\theta))\Big). \end{aligned}$$

Hence, with a straightforward computation we transform the expression for $S(\xi)$ (5.28) to

$$S(\xi) = \frac{2\,\sin^2(\theta)\cosh\left(2\eta(\pi-\theta)\right)}{(1+\cosh(2\pi\eta))} \text{ with } \xi = \eta + \frac{i}{2}.$$

Thus,

$$\Delta_a(\xi) = \left(1 - (\varepsilon^2-\mu^2)\frac{\cosh\left(2\eta(\pi-\theta)\right)}{2(1+\cosh(2\pi\eta))}\right)^2 = \Big(1-(\varepsilon^2-\mu^2)M_\theta(2\eta)\Big)^2,$$

and the condition $\Delta_a(\xi) \neq 0$ for all ξ is equivalent to

$$M_\theta(x) \neq \frac{1}{\varepsilon^2-\mu^2} \text{ for all } x \in \mathbb{R}. \tag{5.29}$$

Remark that for any $\theta \in (0, 2\pi)$ one has

$$M_\theta(x) \geq 0 \text{ for all } x \in \mathbb{R}, \qquad \lim_{x\to\pm\infty} M_\theta(x) = 0,$$

then the condition (5.29) is satisfied if any only if (recall that $|\varepsilon| > |\mu|$ by assumption)

$$\frac{1}{\varepsilon^2-\mu^2} > m(\theta) := \sup_{x\in\mathbb{R}} M_\theta(x), \qquad \textit{i.e.,} \quad \varepsilon^2-\mu^2 < \frac{1}{m(\theta)}.$$

Thus, for each corner point a_j we have shown the equivalence

$$\Delta_{a_j}(\xi) \neq 0 \text{ for all } \xi \in \mathbb{R} + \frac{i}{2} \quad \text{if and only if} \quad \varepsilon^2-\mu^2 < \frac{1}{m(\theta_j)}. \tag{5.30}$$

Using the symmetry and monotonicity properties of m, see (5.23) and Proposition 5.5.2, we conclude that that Θ_m is Fredholm if and only if

$$\varepsilon^2-\mu^2 < \min_{j\in\{1,\dots,n\}} \frac{1}{m(\theta_j)} = \frac{1}{\max_{j\in\{1,\dots,n\}} m(\theta_j)} = \frac{1}{m(\omega)},$$

which is a sufficient condition for the self-adjointness of $B \equiv B_{\varepsilon,\mu}$ and gives the first half of (5.27).

By applying the above result to $\widetilde{B} := B_{-\frac{4\varepsilon}{\varepsilon^2-\mu^2}, -\frac{4\mu}{\varepsilon^2-\mu^2}}$ we see that $\widetilde{B}$ is self-adjoint for

$$\Big(-\frac{4\varepsilon}{\varepsilon^2-\mu^2}\Big)^2 - \Big(-\frac{4\mu}{\varepsilon^2-\mu^2}\Big)^2 < \frac{1}{m(\omega)},$$

which holds for $\varepsilon^2-\mu^2 > 16m(\omega)$. As the self-adjointness of $\widetilde{B}$ is equivalent to the self-adjointness of B (see Remark 5.1.2), we obtain the second half of (5.27). ∎

By combining Theorem 5.5.3 with Proposition 5.5.2 we obtain:

Corollary 5.5.4. *Let Σ be a curvilinear polygon (with C^1-smooth edges and without cusps). Assume that one of the following three conditions holds:*

(a) $\varepsilon^2-\mu^2 < 2$,
(b) $\varepsilon^2-\mu^2 > 8$,

(c) $\varepsilon^2 - \mu^2 \neq 4$ *and the interior angles* θ_j *of* Σ *satisfy*

$$\frac{\pi}{2} \leq \theta_j \leq \frac{3\pi}{2} \text{ for all } j \in \{1, \ldots, n\},$$

then B *is self-adjoint.*

We finish this chapter by pointing out the following remark.

Remark 5.5.2. *In the proof of Theorem 5.5.3 one sees that for* $\varepsilon^2 - \mu^2 > 16m(\omega)$ *the operator* B *is self-adjoint but the operators* Λ_z *are not Fredholm. This shows that the converse of Theorem 5.4.1 does not hold.*

It would be interesting to understand if the quantity $m(\omega)$ *has any geometric meaning: the preceding analysis does not give any indication in this direction.*

Publication

Title: "*A Poincaré-Steklov map for the MIT bag model*". To appear in Analysis & PDE.

Abstract: The purpose of this paper is to introduce and study Poincaré-Steklov (PS) operators associated to the Dirac operator D_m with the so-called MIT bag boundary condition. In a domain $\Omega \subset \mathbb{R}^3$, for a complex number z and for U_z a solution of $(D_m - z)U_z = 0$, the associated PS operator maps the value of $\Gamma_- U_z$, the MIT bag boundary value of U_z, to $\Gamma_+ U_z$, where $\Gamma_\pm$ are projections along the boundary $\partial\Omega$ and $(\Gamma_- + \Gamma_+) = t_{\partial\Omega}$ is the trace operator on $\partial\Omega$.
In the first part of this paper, we show that the PS operator is a zero-order pseudodifferential operator and give its principal symbol. In the second part, we study the PS operator when the mass m is large, and we prove that it fits into the framework of $1/m$-pseudodifferential operators, and we derive some important properties, especially its semiclassical principal symbol. Subsequently, we apply these results to establish a Krein-type resolvent formula for the Dirac operator $H_M = D_m + M\beta\mathbb{1}_{\mathbb{R}^3\setminus\overline{\Omega}}$ for large masses $M > 0$, in terms of the resolvent of the MIT bag operator on Ω. With its help, the large coupling convergence with a convergence rate of $\mathcal{O}(M^{-1})$ is shown.

Lien: https://doi.org/10.48550/arXiv.2206.13337.

Title: "*On the approximation of the Dirac operator coupled with confining Lorentz scalar δ-shell interactions*". To appear soon on arXiv, (2024).

Abstract: Let $\Omega_+ \subset \mathbb{R}^3$ be a fixed bounded domain, and denote its boundary as $\Sigma = \partial\Omega_+$. Consider $\mathcal{U}^\varepsilon$ a tubular neighborhood of the surface Σ with a thickness parameter $\varepsilon > 0$. Define the perturbed Dirac operator by $\mathfrak{D}_M^\varepsilon = D_m + M\beta\mathbb{1}_{\mathcal{U}^\varepsilon}$, with D_m the free Dirac operator, $M > 0$ and $\mathbb{1}_{\mathcal{U}^\varepsilon}$ the characteristic function of $\mathcal{U}^\varepsilon$. Then, in the norm resolvent sense, the Dirac operator $\mathfrak{D}_M^\varepsilon$ converges to the Dirac operator coupled with Lorentz scalar δ-shell interactions as $\varepsilon = M^{-1}$ tends to 0, with a convergence rate of $\mathcal{O}(M^{-1})$.

Title: "*On the approximation of the δ-shell interaction for the 3-D Dirac operator*". Submitted, (2023).

Abstract: We consider the three-dimensional Dirac operator coupled with a combination of electrostatic and Lorentz scalar δ-shell interactions. We approximate this operator with general local interactions V. Without any hypotheses of smallness on the potential V, converges in the strong resolvent sense to the Dirac Hamiltonian coupled with a δ-shell potential supported on Σ, a bounded smooth surface. However,

the coupling constant depends nonlinearly on the potential V.

Lien: https://doi.org/10.48550/arXiv.2309.12911.

Title: "*On the self-adjointness of two-dimensional relativistic shell interactions*". Journal of Operator Theory (JOT), in press, (2024).

Abstract: In this paper, we discuss the self-adjointness of the two-dimensional Dirac operator with a transmission condition along a closed Lipschitz curve. The main new ingredients are an explicit use of the Cauchy transform on non-smooth curves and a direct link with the Fredholmness of a singular boundary integral operator. This results in a proof of self-adjointness for a new range of coupling constants, which includes and extends all previous results for this class of problems. The study is particularly precise for the case of curvilinear polygons, as the angles can be taken into account in an explicit way. In particular, if the curve is a curvilinear polygon with obtuse angles, then there is a unique self-adjoint realization with domain contained in $H^{1/2}$ for the full range of non-critical coefficients in the transmission condition.

Lien: http://arxiv.org/abs/2307.12772.

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Bibliography

[Agr01] M.S. Agranovich. *Spectral problems for the Dirac system with a spectral parameter in the local boundary conditions* (russian) funktsional. *Anal. i Prilozhen. 35 (2001)*, no. 3, 1-18, 95; translation in Funct. Anal. Appl. **35** (2001)(1):161–175, 2001.

[Aie07] P. Aiena. *Semi-Fredholm operators, perturbation theory and localized SVEP*. *Ediciones IVIC, Instituto Venezolano de Investigaciones Cientificas, Venezuela*, (2007).

[AKM17] K. Ando, H. Kang, and Y. Miyanishi. *Elastic Neumann-Poincaré operators on three dimensional smooth domains: Polynomial compactness ans spectral structure*. *Int. Math. Res. Notices, rnx258*, 2017.

[AKMN13] S. Albeverio, A. Kostenko, M. M. Malamud, and H. Neidhardt. *Spherical Schrödinger operators with δ-type interactions*. *J. Math. Phys.*, 54 (052103), (2013).

[ALTMR17] N. Arrizabalaga, L. Le Treust, A. Mas, and N. Raymond. *On the MIT bag model in the non-relativistic limit*. *Comm. Math. Phys.*, (354):641–669, 2017.

[ALTMR19] N. Arrizabalaga, L. Le Treust, A. Mas, and N. Raymond. *The MIT Bag Model as an infinite mass limit*. *J. Éc. polytech. Math.*, pages 329–365, 2019.

[ALTR20] N. Arrizabalaga, L. Le Treust, and N. Raymond. *Extension operator for the MIT bag model*. *Ann. Fac. Sci. Toulouse Math.*, (6) **29**(no. 1):135–147, (2020).

[AMSPV23] N. Arrizabalaga, A. Mas, T. Sanz-Perela, and L. Vega. *Eigenvalue curves for generalized MIT bag models*. *Communications in Mathematical Physics*, **397**(1):337–392, 2023.

[AMV14] N. Arrizabalaga, A. Mas, and L. Vega. *Shell interactions for Dirac operators*. *J. Math. Pures Appl.*, (9)(102(4)):617–639, 2014.

[AMV15] N. Arrizabalaga, A. Mas, and L. Vega. *Shell interactions for Dirac operators: on the point spectrum and the confinement*. *SIAM J. Math. Anal*, **47**(2):1044–1069, 2015.

[AMV16] N. Arrizabalaga, A. Mas, and L. Vega. An isoperimetric-type inequality for electrostatic shell interactions for Dirac operators. *Commun. Math. Phys.*, **344**(2):483–505, 2016.

[AR05] M.S. Agranovich and G.V. Rozenblum. *Spectral boundary value problems for a Dirac system with singular potential. (Russian) Algebra i Analiz 16 (2004), no. 1, 33-69; translation in St. Petersburg Math. J. **16** (2005)*, **1**(no. 1):25–57, 2005.

[BBZ37] B. Benhellal, V. Bruneau, and M. Zreik. *A Poincaré-Steklov map for the MIT bag model*. *Analysis & PDE (in press).*, (2023). https://doi.org/10.48550/arXiv.2206.13337.

[BC02] V. Bruneau and G. Carbou. *Spectral asymptotic in the large coupling limit*. *Asymp. Anal*, **29** (2):91–113, 2002.

[BC05] R. A. Bartnik and P. T. Chruściel. *Boundary value problems for Dirac equations with applications*. *J. Reine Angew. Math.*, (579):13–73, (2005).

[BCLTS19] J. M. Barbaroux, H. Cornean, L. Le Treust, and E. Stockmeyer. *Resolvent convergence to Dirac operators on planar domains*. *Ann. Henri Poincaré*, **20**:1877–1891, 2019.

[BEHL18] J. Behrndt, P. Exner, M. Holzmann, and V. Lotoreichik. *On the spectral properties of Dirac operators with electrostatic δ-shell interaction*. *J. Math. Pures Appl.*, **(9)**(111):47–78, 2018.

[BEHL19] J. Behrndt, P. Exner, M. Holzmann, and V. Lotoreichik. *On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar δ-shell interactions*. *Quantum Stud. Math. Found.*, **6**:295–314, 2019.

[Ben19] B. Benhellal. *Spectral asymptotic for the infinite mass Dirac operator in bounded domain*. *ArXiv preprint (2019)*, 2019.

[Ben22a] B. Benhellal. *Spectral Analysis of Dirac operators with singular interactions supported on the boundaries of rough domains*. *J. Math. Phys.*, **63**(011507), 2022.

[Ben22b] B. Benhellal. *Spectral properties of the Dirac operator coupled with δ-shell interactions*. *Lett Math Phys 112*, **52**(112), 2022.

[BH20] J. Behrndt and M. Holzmann. *On Dirac operators with electrostatic δ-shell interactions of critical strength*. *J. Spectr. Theory*, **10**(147), 2020.

[BHM20] J. Behrndt, M. Holzmann, and A. Mas. *Self-Adjoint Dirac Operators on Domains in $\mathbb{R}^3$*. *Ann. Henri Poincaré*, **21**:2681–2735, 2020.

[BHOBP20] J. Behrndt, M. Holzmann, T. Ourmières-Bonafos, and K. Pankrashkin. *Two-dimensional Dirac operators with singular interactions supported on closed curves*. *J. Funct. Anal.*, **279**, (2020).

[BHS23] J. Behrndt, M. Holzmann, and C. Stelzer. *Approximation of Dirac operators with δ-shell potentials in the norm resolvent sense*. *Preprint: arXiv:2308.13344*, 2023.

[BHSS24] J. Behrndt, M. Holzmann, C. Stelzer, and G. Stenzel. *Boundary triples and Weyl functions for Dirac operators with singular interactions*. *Rev. Math. Phys.*, **Vol. 36**(No. 2):2350036, (2024).

[BHT23] J. Behrndt, M. Holzmann, and M. Tůsek. *Two-dimensional Dirac operators with general δ-shell interactions supported on a straight line*. *J. Phys.*, **A 56**:(045201):(29 pages), 2023.

[BJY08] M. Bellassoued, D. Jellali, and M. Yamamoto. *Stability estimate for the hyperbolic inverse boundary value problem by local Dirichlet-to-Neumann map*. *J. Math. Anal. Appl.*, (343):1036–1046, (2008).

[BL01] J. Brüning and M. Lesch. *On boundary value problems for Dirac type operators I. Regularity and self-adjointness*. *J. Func. Anal.*, (185):1–62, (2001).

[BLL13] J. Behrndt, M. Langer, and V. Lotoreichik. *Schrödinger operators with δ and δ'' - potentials supported on hypersurfaces*. *Ann. Henri Poincaré*, **14** :385–423, (2013).

[BP24] Badredine Benhellal and Konstantin Pankrashkin. *Curvature contribution to the essential spectrum of Dirac operators with critical shell interactions*. *Pure Appl. Anal.*, **Vol. 6**(No. 1):237–252, (2024).

[BPZ72] B. Benhellal, K. Pankrashkin, and M. Zreik. *On the self-adjointness of two- dimensional relativistic shell interactions*. *Journal of Operator Theory (in press)*, (2024). https://doi.org/10.48550/arXiv.2307.12772.

[BR15] M. Bolt and A. Raich. *The Kerzman–Stein operator for piecewise continuously differentiable regions*. *Compl. Var. Elliptic. Equ.*, **60**:478–492, (2015).

[BS06] S. Bielski and R. Szmytkowski. *Dirichlet-to-Neumann and Neumann-to-Dirichlet embedding methods for bound states of the Dirac equation*. *J. Phys. A*, **39**(no. 23):7359–7381, 2006.

[CIKS20] B. Cassano, O.O. Ibrogimov, D. Krejčiřík, and F. Stampach. *Location of Eigenvalues of Non-self-adjoint Discrete Dirac Operators*. *Ann. Henri Poincaré*, **21**:2193–2217, (2020).

[CJJ$^+$74] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, and V.F. Weisskopf. *New extended model of hadrons*. *Phys. Rev. D*, **9**(12):3471–3495, (1974).

[CLMT23] B. Cassano, V. Lotoreichik, A. Mas, and M. Tůsek. *General δ-Shell Interactions for the two-dimensional Dirac Operator: Self-adjointness and Approximation*. *Rev. Mat. Iberoam.*, **39**(no. 4):pp. 1443–1492, (2023).

[CMM82] R. R. Coifman, A. McIntosh, and Y. Meyer. *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*. *J. Funct. Anal. Ann. of Math.*, **(2)**(116):no. 2 361–387, (1982).

[CS85] M. Costabel and E. Stephan. *A direct boundary integral equation method for transmission problems*. *J. Math. Anal. Appl.*, **106**:367–413, (1985).

[CV72] A. P. Calderón and R. Vaillancourt. *A class of bounded pseudodifferential operators*. *Proc. Nat. Acad. Sci. U.S.A.*, pages 69:1185–1187, 1972.

[DA90] F. Dominguez-Adame. *Exact solutions of the Dirac equation with surface delta interactions*. *J. Phys. A: Math.*, **Gen. 23**, 1990.

[DEv89] J. Dittrich, P. Exner, and P. Šeba. *Dirac operators with a spherically δ-shell interactions*. *J.Math. Phys.*, **30**:2875–2882, 1989.

[DFKS22] P. D'Ancona, L. Fanelli, D. Krejčiřík, and N. M. Schiavone. *Localization of eigenvalues for non-self-adjoint Dirac and Klein–Gordon operators*. *Nonlinear Analysis*, **214**(112565), (2022).

[EG15] L. C. Evans and R. F. Gariepy. Measure theory and fine properties of functions. CRC Press, 2015.

[EI01] P. Exner and I. Ichinose. *Geometrically induced spectrum in curved leaky wires*. *J. Phys. A*, **34** : 1439–1450, (2001).

[EK08] P. Exner and S. Kondej. *Hiatus perturbation for a singular Schrödinger operator with an interaction supported by a curve in $\mathbb{R}^3$*. *J. Math. Phys.*, **49** (032111), (2008).

[EN03] P. Exner and K. Němcová. *Leaky quantum graphs: approximations by point-interaction Hamiltonians*. *J. Phys. A*, **36** :10173–10193, (2003).

[EP12] P. Exner and K. Pankrashkin. *Strong Coupling Asymptotics for a Singular Schrödinger Operator with an Interaction Supported by an Open Arc*. *Communications in Partial Differential Equations*, **39** :193–212, 2012.

[Esk03] G. Eskin. *Inverse problems for the Schrödinger operators with electromagnetic potentials in domains with obstacles*. *Inverse Problems*, **19**:985–998, (2003).

[Esk08] G. Eskin. *Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov–Bohm effect*. *J. Math. Phys.*, **49**:022105, (2008).

[FJJR78] E. B. Fabes, M. Jodeit Jr., and N. M. Rivière. *Potential techniques for boundary value problems on C^1-domains*. *Acta Math.*, **141**(1):pp. 165–186, 1978.

[FK19] L. Fanelli and D. Krejčiřík. *Location of eigenvalues of three-dimensional non-self-adjoint Dirac operators*. *Lett Math Phys*, **109**:1473–1485, (2019).

[Fol95] G. B. Folland. Introduction to partial differential equations. 2nd ed. Princeton, NJ: Princeton University Press, 2nd ed. edition,, 1995.

[FS16] A. Fraser and R. Schoen. *Sharp eigenvalue bounds and minimal surfaces in the ball*. *Invent. Math.*, **203**(no. 3):823–890, 2016.

[GP17] A. Girouard and L. Polterovich. *Spectral geometry of the Steklov problem (Survey article)*. *J. Spectr. Theory*, **7**(no. 2):pp. 321–359, 2017.

[GR65] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. 4th ed., Fizmatgiz, Moscow, 1963; English Transl., Academic Press, 1965.

[Gri85] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. *Monographs and Studies in Mathematics*, vol. **24**(Pitman (Advanced Publishing Program), Boston, MA):MR 775683, 1985.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. *Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin*, 2001.

[HMOM$^+$09] S. Hofmann, E. Marmolejo-Olea, M. Mitrea, S. Pérez-Esteva, and M. Taylor. *Hardy spaces, singular integrals and the geometry of Euclidean domains of locally finite perimeter*. *Geom. Funct. Anal.*, **19**(no. 3):842–882, 2009.

[HMT10] S. Hofmann, M. Mitrea, and M. Taylor. *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains*. *International Mathematics Research Notices*, Volume **2010**(Issue 14):Pages 2567–2865, 2010.

[HMT15] S. Hofmann, M. Mitrea, and M. Taylor. *Symbol calculus for operators of layer potential type on Lipschitz surfaces with VMO normals, and related pseudodifferential operator calculus*. *Anal. PDE*, **8**(no. 1):115–181, 2015.

[Hug97] R. J. Hughes. *Relativistic point interactions: approximation by smooth potentials*. *Rep. Mathematical Phys*, **39**(3):425–432, 1997.

[Hug99] R. J. Hughes. *Finite-rank perturbations of the Dirac operator*. *J. Math. Anal. Appl.*, **238**:67–81, 1999.

[JK81a] D. Jerison and C. Kenig. *The Dirichlet problem on nonsmooth domains*. *Ann. of Math.*, **113**(no. 2):367–382, 1981.

[JK81b] D. Jerison and C. Kenig. *The Neumann problem on Lipschitz domains*. *Bull. Amer. Math. Soc.*, **4**:203–207, 1981.

[Joh75] K. Johnson. *The MIT bag model*. *Acta Phys. Pol. B*, **9** 12(8):865–892, (1975).

[Jou83] J.-L. Journé. Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón. 2Lecture Notes in Math., vol. 994, Springer, Berlin-New York, 1983.

[Kat66] T. Kato. Perturbation Theory for Linear Operators. Springer Verlag, Berlin/Heidelberg/New York, 1966.

[KK13] Sylwia Kondej and David Krejčiřík. *Spectral Analysis of a Quantum System with a Double Line Singular Interaction*. *Publ. Res. Inst. Math. Sci.*, **49** (no. 4): pp. 831–859, (2013).

[KM10] A. Kostenko and M. M. Malamud. *1-D Schrödinger operators with local point interactions on a discrete set*. *J. Differential Equations*, **249** : 253–304, (2010).

[KND22] D. Krejčiřík and T. Nguyen Duc. *Pseudomodes for non-self-adjoint Dirac operators*. *J. Funct. Anal.*, **282** (12) (Article 109440), (2022).

[KRT15] D. Krejčiřík, N. Raymond, and M. Tušek. *The Magnetic Laplacian in Shrinking Tubular Neighborhoods of Hypersurfaces*. *Journal of Geometric Analysis*, **25** (4): 2546–2564, (2015).

[Lan99] L. Lanzani. *Szego projection versus potential theory for non-smooth planar domains*. *Indiana Univ. Math. J.*, **48**(No. 2):537–556, (1999).

[LTU03] M. Lassas, M. Taylor, and G. Uhlmann. *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary*. *Comm. Anal. Geom. 11*, (no. 2):207–221, 2003.

[LU89] J. M. Lee and G. Uhlmann. *Determining anisotropic real-analytic conductivities by boundary measurements*. *Comm. Pure Appl. Math*, **42**:pp. 1097–1112, (1989).

[Mik78] V. P. Mikhaĭlov. Partial differential equations. “Mir”, distributed by Imported Publications, Inc., Chicago, Ill. Translated from the Russian by P. C. Sinha. MR601389, Moscow, Chicago, 1978.

[Miy18] Y. Miyanishi. *Weyl's law for the eigenvalues of the Neumann-Poincaré operators in three dimensions: Willmore energy and surface geometry*. *ArXiv:1806.03657*, 2018.

[MOBP20] A. Moroianu, T. Ourmières-Bonafos, and K. Pankrashkin. *Dirac Operators on Hypersurfaces as Large Mass Limits*. *Commun. Math. Phys.*, **374**:1963–2013, 2020.

[MP18] A. Mas and F. Pizzichillo. *Klein's Paradox and the Relativistic δ-shell Interaction in $\mathbb{R}^3$*. *Analysis & PDE*, **11**:pp. 705–744, 2018.

[MR20] Y. Miyanishi and G. Rozenblum. *Eigenvalues of the Neumann-Poincaré operators in dimension 3: Weyl's Law and geometry*. *St. Petersburg Math. J.*, **31**:371–386, 2020.

[MS12] M. M. Malamud and K. Schmüdgen. *Spectral theory of Schrödinger operators with infinitely many point interactions and radial positive definite functions*. *J. Funct. Anal*, 263 : 3144–3194, (2012).

[OBP18] T. Ourmières-Bonafos and F. Pizzichillo. *Self-adjointness of Dirac operators with infinite mass boundary conditions in sectors*. *Ann. Henri Poincaré*, **19**(no. 5):1465–1487, (2018).

[OBV18] T. Ourmières-Bonafos and L. Vega. *A strategy for self-adjointness of Dirac operators: applications to the MIT bag model and δ-shell interactions*. *Publicacions matemàtiques*, vol. **62**:pp. 397–437, 2018.

[PVDB21] F. Pizzichillo and H. Van Den Bosch. *Self-adjointness of two dimensional Dirac operators on corner domains*. *J. Spectr. Theory*, **11**:1043–1079, (2021).

[RS78] M. Reed and B. Simon. Methods of Modern Mathematical Physics IV, Analysis of Operators. Academic Press, New York, San Francisco, London 1978 - Notes 4.7., London, 1978.

[RS80] M. Reed and B. Simon. A Methods of Modern Mathematical Physics. vol. I. Functional analysis, Academic Press, New York, 1980.

[She91] V. Yu. Shelepov. *On the Index and spectrum of integral operators of potential type along Radon curves*. *Math. USSR Sb.*, **70**(175), (1991).

[SV19] E. Stockmeyer and S. Vugalter. *Infinite mass boundary conditions for Dirac operators*. *Spectr. Theory*, **9**:pp 569–600, 2019.

[Szm98] R. Szmytkowski. *Operator formulation of Wigner's R-matrix theories for the Schrödinger and Dirac equations*. *J. Math. Phys.*, **39**(5231), 1998.

[Tay96] M. Taylor. Partial differential equations. II. Qualitative studies of linear equations. Applied Mathematical Sciences, 116. Springer-Verlag, New York, 1996.

[Tay00] M. Taylor. Tools for PDE. Pseudo Differential Operators, Paradifferential Operators, and Layer Potentials. Mathematical Surveys and Monographs, vol. **81**, AMS, Providence, 2000.

[Tes14] G. Teschl. Mathematical methods in quantum mechanics. With applications to Schrodinger operators. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2014.

[Tha92] B. Thaller. The Dirac equation. Text and Monographs in Physics, Springer-Verlag, Berlin, 1992.

[Tho79] J. A. Thorpe. Elementary Topics in Differential Geometry. Undergraduate Texts in Mathematics, Springer-Verlag, New York Inc., 1979.

[Tŭ20] M. Tŭsek. *Approximation of one-dimensional relativistic point interactions by regular potentials revised*. *Lett. Math. Phys.*, **110**::2585–2601, 2020.

[Ver84] G. Verchota. *Layer potentials and regularity for the Dirichlet problem for Laplace's equation*. *J. of Funct. Anal.*, **59**:572–611, 1984.

[Š89] P. Šeba. *Klein's paradox and the relativistic point interaction*. *Lett. Math. Phys.*, **18**:77–86, 1989.

[WIL67] R. M. WILCOX. Exponential operators and parameter differentiation in quantum physics. Journal of Mathematical Physics, **8**, pp. 962–982, 1967.

[Zre11] M. Zreik. *On the approximation of the δ-shell interaction for the 3-D Dirac operator*. Submitted, (2023). https://doi.org/10.48550/arXiv.2309.12911.

[Zre84] M. Zreik. *On the approximation of the Dirac operator coupled with confining Lorentz scalar δ-shell interactions*. (2024). https://doi.org/10.48550/arXiv.2404.07784.

[Zwo12] M. Zworski. Semiclassical Analysis. Graduate Studies in Mathematics 138, AMS, 2012.

Title : Spectral Properties of Dirac Operators on Certain Domains.

Abstract : We are interested in spectral study of perturbations of Dirac operators on certain domains. The majority of the studies carried out in this thesis are established through the study of the resolvents of these operators. On one hand, we introduce Poincaré-Steklov (PS) operators, which appear naturally in the study of Dirac operators with MIT bag boundary conditions, and analyze them from a microlocal point of view (Chapter 2). On the other hand, our study focuses on the three-dimensional Dirac operator coupled with a singular delta interactions: Chapter 3 is devoted to an approximation of the confining version of Dirac operator coupled with purely Lorentz scalar delta shell interactions. Chapters 2 and 3 deal with the large mass limit (supported on a fixed domain and a domain whose thickness tends to zero). Chapter 4 also generalizes an approximation of the non-confining version of Dirac operator coupled with a singular combination of electrostatic and Lorentz scalar delta interactions by a Dirac operator with regular local interaction. Finally, in two-dimension, we develop a new technique that allows us to prove, for combinations of delta interactions supported on non-smooth curves, the self-adjointness of the realization of the Dirac operator under consideration, in Sobolev space of order one-half (Chapter 5).

Keywords : Spectral analysis, Dirac operators, self-adjoint extensions, shell interactions, quantum confinement, Poincaré-Steklov operators, the MIT bag model, h-Pseudodifferential operators, large coupling limits.

Titre : Propriétés Spectrales des Opérateurs de Dirac sur Certains Domaines.

Résumé : Nous nous intéressons à l'études spectrale des perturbations d'opérateurs de Dirac sur certains domaines. La majorité des études effectuées dans cette thèse est établie à travers l'étude de la résolvante de ces opérateurs. D'une part, nous introduisons les opérateurs de Poincaré-Steklov (PS), qui apparaissent naturellement dans l'étude des opérateurs de Dirac avec les conditions aux bords MIT bag, et nous les analysons d'un point de vue microlocal (Chapitre 2). D'autre part, notre étude porte sur les opérateurs de Dirac couplés à une combinaison singulière de delta interactions : le Chapitre 3 se consacre à l'approximation de la version confinée de l'opérateur de Dirac couplé avec delta interaction scalaire de Lorentz. Les Chapitres 2 et 3 traitent de la limite de grande masse (supportée sur un domaine fixe et un domaine dont l'épaisseur tend vers zéro). Le Chapitre 4 généralise une approximation de la version non-confinée de l'opérateur de Dirac couplé avec une combinaison singulière de delta interactions électrostatique et scalaire de Lorentz par un opérateur de Dirac avec une interaction locale régulière. Enfin, nous développons, en dimension deux, une nouvelle technique qui nous permet de prouver, pour des combinaisons de delta interactions supportées sur des courbes non-régulières, l'auto-adjonction de la réalisation de l'opérateur considéré, et ce, dans l'espace de Sobolev d'ordre un-demi (Chapitre 5).

Mot clés : Analyse spectrale, opérateurs de Dirac, extensions auto-adjointes, δ-interactions, opérateurs de Poincaré-Steklov, le modèle MIT bag, opérateurs h-Pseudodifferentiel, couplage fort.

Research units

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