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On minimal bases and indices of rational matrices and their linearizations

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ABSTRACT

A complete theory of the relationship between the minimal bases and indices of rational matrices and those of their strong linearizations is presented. Such theory is based on establishing first the relationships between the minimal bases and indices of rational matrices and those of their polynomial system matrices under the classical minimality condition and certain additional conditions of properness. This is related to pioneering results obtained by Verghese, Van Dooren and Kailath in 1979-1980, which were the first proving results of this type. It is shown that the definitions of linearizations and strong linearizations do not guarantee any relationship between the minimal bases and indices of the linearizations and the rational matrices in general. In contrast, simple relationships are obtained for the family of strong block minimal bases linearizations, which can be used to compute minimal bases and indices of any rational matrix, including rectangular ones, via algorithms for pencils. These results extend the cor-

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responding ones for other families of linearizations available in recent literature for square rational matrices.

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1. Introduction

Rational matrices, i.e., matrices whose entries are rational functions have been studied intensively since the 1950s in connection with linear systems and control theory [27,32]. The pioneering work of Rosenbrock [32] showed that some properties of a rational matrix $G(\lambda)$ can be studied from its polynomial system matrices, that is, polynomial matrices of the form

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{1}$$

with $A(\lambda)$ regular, i.e., $\det A(\lambda) \not\equiv 0$, and with transfer function matrix $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$. A classical result of Rosenbrock states that if $P(\lambda)$ has least order (equivalently, is minimal or irreducible), i.e., $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix}$ have no *finite* zeros, then the *finite pole* structure of $G(\lambda)$ is given by the finite zero structure of $A(\lambda)$ and the *finite zero* structure of $G(\lambda)$ is given by the finite zero structure of $P(\lambda)$. Thus, the finite pole and zero structures of a rational matrix can be obtained from any of its minimal polynomial system matrices. However, rational matrices have other structural data that are interesting in applications and that cannot be obtained, in general, from its minimal polynomial system matrices. Actually, rational matrices have *poles and zeros at infinity* [27, pp. 449-450] (see also [33] for applications and some classical algorithms for computing the structure at infinity). Moreover, singular, i.e., nonregular, rational matrices have *minimal bases and indices* [26], [27, Section 6.5.4], which form their null-space or singular structure. Minimal bases of rational matrices have a number of important applications, as, for instance, in the solution of minimal design problems, originally proposed in [41], and further studied in [26, Section 8] and [28]. In this type of problems one must solve $G_1(\lambda)X(\lambda) = G_2(\lambda)$ for a rational unknown $X(\lambda)$ with special properties, where $G_1(\lambda)$ and $G_2(\lambda)$ are given rational matrices. When the solution exists, it can be constructed from a minimal basis of $[G_1(\lambda),\ -G_2(\lambda)]$.

Motivated by the previous discussion, the pioneering work of Verghese, Van Dooren and Kailath established in [39,38] sufficient conditions on a polynomial system matrix as in (1) that allow to obtain from $P(\lambda)$ the pole and zero structures at infinity and the minimal bases and indices of its transfer function matrix $G(\lambda)$, in addition to the finite pole and zero structures. In [39], the authors considered polynomial system matrices in generalized state-space form (GSSF), that is, $A(\lambda) = \lambda E - A$ and $B(\lambda) = B$, $C(\lambda) = C$

and $D(\lambda) = D$ are constant matrices. These polynomial system matrices were defined to be strongly irreducible if $\left[\begin{smallmatrix} \lambda E - A & B \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \lambda E - A \\ -C \end{smallmatrix}\right]$ have no finite or infinite zeros. Under these conditions, it was shown in [39] that the pole and zero structures at infinity and the minimal bases and indices of $P(\lambda)$ and $G(\lambda)$ are very easily related to each other. One year later, Verghese extended in [38] the results of [39] to general polynomial system matrices. However, the definition of strong irreducibility in this case is more complicated since it involves checking not only the finite zeros of $\left[\begin{smallmatrix} A(\lambda) & B(\lambda) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A(\lambda) \\ -C(\lambda) \end{smallmatrix}\right]$ but also the infinite zeros of two extensions of $P(\lambda)$.

The results in [39] were fundamental for the development of efficient and stable numerical algorithms for computing the complete list of structural data of any rational matrix $G(\lambda)$, i.e., its finite and infinite pole and zero structures and its minimal indices. The reason is that Van Dooren presented in [36, Section V] stable algorithms for constructing strongly irreducible polynomial system matrices in GSSF, starting from an easily constructible one in GSSF that is not strongly irreducible. Since a polynomial system matrix in GSSF is a pencil, the staircase algorithm in [35] can be applied to compute its structural data, which are related to those of $G(\lambda)$ as established in [39]. The minimal bases of this pencil can be computed by combining the output of [35] with the method in [9, Sections 3 and 4] and, finally, one can recover the minimal bases of $G(\lambda)$ as explained in [39].

Despite of its importance and major impact, the results in [36,39] are not always the most convenient tools for computing the complete list of structural data of a rational matrix $G(\lambda)$ and its minimal bases. One reason of this is that in recent years a number of new *linearizations* of rational matrices arising in applications have been developed for computing their structural data via algorithms for pencils and that these linearizations are neither in GSSF nor are they necessarily strongly irreducible in the sense of [38]. In plain words, these new linearizations are minimal linear polynomial system matrices of a rational matrix $\widehat{G}(\lambda)$, not necessarily equal to $G(\lambda)$, but related to $G(\lambda)$. This motivates to look for some conditions that can replace Verghese's strong irreducibility conditions at infinity in [38], that are useful in this new setting, and that guarantee that a minimal polynomial system matrix as in (1) allows to obtain the pole and zero structures at infinity and the minimal bases and indices of its transfer function matrix, in addition to the finite pole and zero structures. As a first contribution, we will prove in this paper that $C(\lambda)A(\lambda)^{-1}$ and $A(\lambda)^{-1}B(\lambda)$ being proper rational matrices are such conditions, which have the advantage of being directly checked on the polynomial system matrix itself. Moreover, we will show that these conditions do not imply Verghese's conditions and vice versa. These properness conditions were in fact introduced in [6, Lemma 2.4 and Corollary 2.5], where it is proved that they allow to recover the pole and zero structures at infinity of a rational matrix from its polynomial system matrices satisfying these conditions. We will prove in Section 3 that they also allow to recover the minimal indices and bases and will use this fact for proving other results in this paper.

In recent years, rational matrices have received considerable attention in the context of nonlinear eigenvalue problems (NLEPs), either because they arise directly in some of these problems or because they are used for approximating matrices of nonlinear

functions in order to solve numerically NLEPs [24,31]. Thus, the numerical solution of a NLEP reduces in practice to computing the finite zeros that are not poles of a rational matrix $G(\lambda)$. An effective approach for doing so is to apply an eigenvalue algorithm to a linearization of $G(\lambda)$, i.e., a pencil that allows to recover the information of interest of $G(\lambda)$, in the spirit of Van Doreen's approach [36]. In NLEPs, this approach started in [34], continued in [25], and influenced the search for rigorous definitions of linearizations of rational matrices and the development of new families of linearizations, as it has been done before for polynomial matrices (see the references in [11,30]).

The first formal definition of linearization of a rational matrix was given in [1]. A different definition of linearization was introduced in [6], together with the first definition of strong linearization of a rational matrix (i.e., a linearization that allows to recover the infinite pole and zero structures, in addition to the finite ones). The definition of linearization in [6] includes the one in [1]. A comparison between both definitions can be found in [5, Section 5]. One of the differences between both definitions is that both are linear polynomial system matrices, but the off-diagonal blocks in [1] are forced to be constant matrices, while this is not the case in [6]. Another definition of strong linearization was introduced in [13], that is equivalent to the one in [6], except for the fact that the off-diagonal matrices are constant. Since the definitions mentioned above do not capture some of the pencils that have been used recently in the numerical solution of NLEPs [25,29], the concept of local linearizations of rational matrices was introduced in [20].

Simultaneously to the definitions of linearizations of rational matrices, different families of linearizations that can be constructed without any numerical operation have been developed. Some of them are restricted to square rational matrices, as the different Fiedler-like linearizations presented in [1,3,12,14], and the affine-spaces and $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations introduced in [13] and in [19], respectively, which are closely connected to each other. In contrast, the family of strong block minimal bases linearizations introduced in [6] is valid for general rational matrices, i.e., rectangular or square. An important property of this wide family is that it contains modulo permutations the Fiedler-like linearizations mentioned above. This is proved in Section 8 and extends to rational matrices a well-known result for polynomial matrices [11]. Moreover, it is known that the $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations are very simply connected to strong block minimal bases linearizations [19]. Thus, the properties of strong block minimal bases linearizations imply results for other families of linearizations and for more general rational matrices.

The main focus of this paper is on the relationship of the minimal bases and indices of a rational matrix and its linearizations. Previous works on this problem deal with strongly irreducible pencils in GSSF [39], strongly irreducible polynomial system matrices when they are linear [38] (the pencils in [39,38] are particular cases of linearizations in the sense of [6], though not strong) and, recently, Fiedler-like linearizations [12,14] and affine spaces of linearizations [13]. The results in [12–14] have the advantage with respect to those in [39,38] of dealing with linearizations that can be constructed without computational cost from certain representations of the rational matrix. However, the results in [12–14]

have a disadvantage for computing minimal indices and bases, since they only work for square rational matrices while the applications where minimal indices and bases are of interest very often include rectangular matrices. This is the case, for example, for transfer function matrices of systems with different numbers of inputs and outputs and in the minimal design problems mentioned at the beginning of this section. In this setting, a second contribution of this paper is to establish for the first time simple relationships between the minimal bases and indices of a rational matrix and its strong block minimal bases linearizations. The proof of these results is presented in Section 6 based on those in Section 3 and allow to use strong block minimal bases linearizations for computing minimal bases and indices via algorithms for pencils [35,9]. Since many strong block minimal bases linearizations can be constructed without computational cost and are valid for rectangular matrices, this approach is advantageous with respect to the previous ones. In addition, due to the relation of this family of linearizations with other families of linearizations, the results in Section 6 allow to obtain as corollaries the corresponding ones for Fiedler-like pencils, already presented in [12,14], in Subsection 8.4 and the ones for $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations in Section 7. This approach constitutes a unified treatment of the relationships between minimal bases and indices of rational matrices and those of most classes of linearizations developed in the recent literature.

The paper is completed with the study in Sections 4 and 5 of a question of a fundamental nature: what is the relationship between the minimal bases and indices of a rational matrix and those of any of its linearizations or strong linearizations? Here and in what follows by linearizations and strong linearizations we will mean those defined in [6]. For linearizations, our conclusions are that we can recover polynomial bases, but not minimal ones, and that the dimension of the left (right, respectively) nullspace of any rational matrix and that of its linearizations coincide. Using Van Dooren's index sum theorem [39], we can obtain the sums of the right and left minimal indices of any rational matrix from any of its strong linearizations. However, as far as the minimal indices themselves are concerned, we will show that the minimal indices of a rational matrix may differ arbitrarily from those of its strong linearizations.

The paper is organized as follows: Section 2 contains the notation and some preliminary results. In Section 3 the relationship between the minimal bases and indices of a rational matrix and its minimal polynomial system matrices satisfying some conditions of properness is given. Section 4 relates polynomial bases of rational matrices and their linearizations in general. Section 5 shows that the minimal indices of a rational matrix and of its strong linearizations may differ arbitrarily, but that there is a connection between the sums of their left and right minimal indices. Section 6 is devoted to obtain minimal bases and indices of any rational matrix from its strong block minimal bases linearizations and vice versa. The same goal is pursued in Section 7 for $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations. Connections between strong block minimal bases linearizations and Fiedler-like linearizations are investigated in Section 8. Some brief remarks on eigenvectors are included for completeness in Section 9. Finally, some conclusions are discussed in Section 10.

2. Preliminaries

Most of the results included in this section are classic and can be found in standard references as [27,32,37], together with much more information on rational matrices.

Let $\mathbb{F}$ be an arbitrary field and $\overline{\mathbb{F}}$ its algebraic closure. Let $\mathbb{F}[\lambda]$ be the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ the field of fractions of $\mathbb{F}[\lambda]$, i.e., the field of rational functions over $\mathbb{F}$. The elements of $\mathbb{F}(\lambda)$ with the degree of the numerator at most the degree of the denominator are called proper rational functions. The set of proper rational functions over $\mathbb{F}$ form a ring denoted $\mathbb{F}_{pr}(\lambda)$. If the degree of the denominator of a rational function is strictly larger than the degree of its numerator then the rational function is called strictly proper.

Vectors with entries in $\mathbb{F}[\lambda]$ are called vector polynomials. $\mathbb{F}(\lambda)^p$ stands for the vector space of p-tuples of rational functions. We denote by $\mathbb{F}[\lambda]^{p\times m}$ (resp., $\mathbb{F}(\lambda)^{p\times m}$, $\mathbb{F}_{pr}(\lambda)^{p\times m}$) the set of $p\times m$ matrices with entries in $\mathbb{F}[\lambda]$ (resp., $\mathbb{F}(\lambda)$, $\mathbb{F}_{pr}(\lambda)$). Matrices in $\mathbb{F}[\lambda]^{p\times m}$ are called polynomial matrices or matrix polynomials indistinctly. The degree of a polynomial matrix is the highest degree of all its entries. The square polynomial matrices whose inverses are polynomial matrices are called unimodular matrices. Matrices in $\mathbb{F}(\lambda)^{p\times m}$ are known as rational matrices and matrices with entries in $\mathbb{F}_{pr}(\lambda)$ are termed as proper rational matrices. In particular, if the entries are all strictly proper then they are called strictly proper rational matrices. Invertible matrices in $\mathbb{F}_{pr}(\lambda)^{p\times p}$, that is, square proper rational matrices whose inverses are also proper, are called biproper. Equivalently, biproper matrices are square proper rational matrices whose determinants are biproper rational functions. I_n denotes the $n\times n$ identity matrix and $0_{p\times m}$ the $p\times m$ zero matrix, which reduces to 0_p when $p=m$. We will write just I or 0 when the size is clear from the context.

2.1. Spectral structure of rational matrices

We introduce now the spectral structure (both finite and infinite) of rational matrices. Recall that two rational matrices $G_1(\lambda), G_2(\lambda)\in\mathbb{F}(\lambda)^{p\times m}$ are unimodularly equivalent if there exist unimodular matrices $U_1(\lambda)\in\mathbb{F}[\lambda]^{p\times p}$ and $U_2(\lambda)\in\mathbb{F}[\lambda]^{m\times m}$ such that $G_2(\lambda)=U_1(\lambda)G_1(\lambda)U_2(\lambda)$. Any rational matrix is unimodularly equivalent to its finite Smith–McMillan form (see, for example, [32, Chapter 3, Section 4] or [27, Section 6.5.2]). That is to say, if $G(\lambda)\in\mathbb{F}(\lambda)^{p\times m}$ then there are unimodular matrices $U_1(\lambda)\in\mathbb{F}[\lambda]^{p\times p}$ and $U_2(\lambda)\in\mathbb{F}[\lambda]^{m\times m}$ such that

$$U_1(\lambda)G(\lambda)U_2(\lambda)=\begin{bmatrix}\operatorname{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)},\dots,\frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}\right) & 0\\ 0 & 0\end{bmatrix} \tag{2}$$

where $\epsilon_1(\lambda),\dots,\epsilon_r(\lambda),\psi_1(\lambda),\dots,\psi_r(\lambda)$ are nonzero monic (leading coefficient equal to 1) polynomials, $\epsilon_i(\lambda),\psi_i(\lambda)$ are pairwise coprime for all $i=1,\dots,r$, and $\epsilon_1(\lambda)\mid\cdots\mid\epsilon_r(\lambda)$ while $\psi_r(\lambda)\mid\cdots\mid\psi_1(\lambda)$, where $\mid$ stands for divisibility. The index r is the normal rank

of $G(\lambda)$, which we denote by $\operatorname{rank} G(\lambda)$. The finite zeros of $G(\lambda)$ are the roots in $\overline{\mathbb{F}}$ of $\epsilon_r(\lambda)$ and its finite poles are the roots in $\overline{\mathbb{F}}$ of $\psi_1(\lambda)$. If $\lambda_0 \in \overline{\mathbb{F}}$ is a zero of $G(\lambda)$ then, for $i = 1, \ldots, r$, we can write $\epsilon_i(\lambda) = (\lambda - \lambda_0)^{m_i}\widehat{\epsilon}_i(\lambda)$ with $\widehat{\epsilon}_i(\lambda_0) \neq 0$ and $m_i \geq 0$. The nonzero elements in $(m_1, \ldots, m_r)$ are called the partial multiplicities of λ_0 as a zero of $G(\lambda)$. In the same way, if $\lambda_0 \in \overline{\mathbb{F}}$ is a pole of $G(\lambda)$ then, for $i = 1, \ldots, r$, we can write $\psi_i(\lambda) = (\lambda - \lambda_0)^{n_i}\widehat{\psi}_i(\lambda)$ with $\widehat{\psi}_i(\lambda_0) \neq 0$ and $n_i \geq 0$. The nonzero elements in $(n_1, \ldots, n_r)$ are called the partial multiplicities of λ_0 as a pole of $G(\lambda)$. We understand by finite zero structure of $G(\lambda)$ its finite zeros together with their respective partial multiplicities. Analogously, the finite pole structure of $G(\lambda)$ consists of its finite poles each with its partial multiplicities.

Rational matrices may have structure at infinity as well. Recall (see, for example, [37]) that two rational matrices of the same size $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are equivalent at infinity if there exist biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that $G_2(\lambda) = B_1(\lambda)G_1(\lambda)B_2(\lambda)$. Any rational matrix is equivalent at infinity to its Smith–McMillan form at infinity. That is to say, if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ then there are biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that

$$B_1(\lambda)G(\lambda)B_2(\lambda) = \begin{bmatrix} \operatorname{Diag}\left((\frac{1}{\lambda})^{q_1}, \ldots, (\frac{1}{\lambda})^{q_r}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} G(\lambda)$ and $q_1 \leq \cdots \leq q_r$ are integers. These are called the invariant orders at infinity of $G(\lambda)$. They determine the zeros and poles at infinity of $G(\lambda)$, also called infinite zeros and poles. Namely, if $q_1 \leq \cdots \leq q_k < 0 = q_{k+1} = \cdots = q_{u-1} < q_u \leq \cdots \leq q_r$ are the invariant orders at infinity of $G(\lambda)$ then $G(\lambda)$ has $r - u + 1$ zeros at infinity each one of order $q_u, \ldots, q_r$ and k poles at infinity each one of order $-q_k, \ldots, -q_1$. Notice that proper rational matrices have all nonnegative invariant orders at infinity, that is, they do not have poles at infinity. Moreover, all the invariant orders at infinity of strictly proper rational matrices are positive.

Note that any rational matrix can be decomposed uniquely as $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ with $D(\lambda)$ a polynomial matrix and $G_{sp}(\lambda)$ a strictly proper matrix. When $G(\lambda)$ is not strictly proper, that is, when $D(\lambda) \neq 0$, the first invariant order at infinity of $G(\lambda)$, q_1, turns out to be minus the degree of the polynomial part of $G(\lambda)$, i.e., $q_1 = -\deg(D(\lambda))$ (see [6, Section 2]), where $\deg(\cdot)$ stands for "degree of".

2.2. Polynomial system matrices

Any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be written as $G(\lambda)=D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$ where $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is regular, i.e., $\det A(\lambda) \not\equiv 0$, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. The polynomial matrix formed with these matrices

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{3}$$

is called a polynomial system matrix of (or giving rise to) $G(\lambda)$ (see [32]). The rational matrix $G(\lambda)$ is called the transfer function matrix of $P(\lambda)$ and $\deg(\det A(\lambda))$ is known as the order of $P(\lambda)$. We allow n to be equal to 0 in the definition of polynomial system matrix. In this case we say that $P(\lambda) = D(\lambda)$ is a polynomial system matrix giving rise to $G(\lambda) = D(\lambda)$, that is, $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are empty matrices. Besides, when $A(\lambda)$ is a monic linear matrix polynomial, say $A(\lambda) = \lambda I_n - A$, and $B(\lambda) = B$ and $C(\lambda) = C$ are constant matrices, $P(\lambda)$ is said to be a polynomial system matrix of $G(\lambda)$ in state-space form.

Different polynomial system matrices may exist with different orders giving rise to the same transfer function matrix. A polynomial system matrix of $G(\lambda)$ is said to have least order, or to be minimal, if its order is the smallest integer for which matrix polynomials $A(\lambda)$ (regular), $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ giving rise to $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ exist ([32, Chapter 3, Section 5.1] or [37, Section 1.10]). In consequence, associated with any rational matrix $G(\lambda)$ there is a unique least order, which is the order of any minimal polynomial system matrix giving rise to $G(\lambda)$, and is denoted by $\nu(G(\lambda))$. Interested readers can find three algorithms in [32, Chapter 3, Section 5.1] to compute $\nu(G(\lambda))$ without going to the length of finding a least order polynomial system matrix giving rise to $G(\lambda)$.

One of the many characterizations of when a polynomial system matrix has least order is given in terms of coprimeness. Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{q\times m}$ with $p + q \geq m$ are called right coprime if their only right common divisors are unimodular matrices. That is to say, if there exist $\widehat{A}(\lambda) \in \mathbb{F}[\lambda]^{p\times m}, \widehat{C}(\lambda) \in \mathbb{F}[\lambda]^{q\times m}, X(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that $A(\lambda) = \widehat{A}(\lambda)X(\lambda)$ and $C(\lambda) = \widehat{C}(\lambda)X(\lambda)$, then $X(\lambda)$ is unimodular. Let us recall some equivalent conditions that characterize when two polynomial matrices are right coprime (see, for example, [32, Chapter 2, Section 6], [37, Chapter 1], [4]):

Proposition 2.1. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{q\times m}$ with $p+q \geq m$. The following conditions are equivalent:*

(i) *$A(\lambda)$ and $C(\lambda)$ are right coprime.*
(ii) *There exist matrices $X(\lambda) \in \mathbb{F}[\lambda]^{m\times p}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{m\times q}$ such that $X(\lambda)A(\lambda) + Y(\lambda)C(\lambda) = I_m$.*
(iii) $\operatorname{rank}\begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = m$ *for all $\lambda_0 \in \overline{\mathbb{F}}$.*

On the other hand, $A(\lambda) \in \mathbb{F}[\lambda]^{m\times p}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{m\times q}$, $p+q \geq m$, are left coprime if their transposes $A(\lambda)^T$ and $C(\lambda)^T$ are right coprime.

It turns out that the polynomial system matrix in (3) has least order if and only if $A(\lambda)$ and $B(\lambda)$ are left coprime and $A(\lambda)$ and $C(\lambda)$ are right coprime ([32, Chapter 3]).

A celebrated result by Rosenbrock [32, Chapter 3, Theorem 4.1] relates the finite structure (zero and pole structure) of a rational matrix with the finite structure of its minimal polynomial system matrices. Namely, when the polynomial system matrix in

(3) giving rise to $G(\lambda)$ has least order, the finite zero structure of $G(\lambda)$ is the finite zero structure of $P(\lambda)$ and the finite pole structure of $G(\lambda)$ is the finite zero structure of $A(\lambda)$. A consequence of this fact is that the least order of $G(\lambda)$, $\nu(G(\lambda))$, which is the degree of the determinant of $A(\lambda)$, is equal to the sum of the partial multiplicities of the finite poles of $G(\lambda)$. In other words, $\nu(G(\lambda))$ is the sum of the degrees of the denominators in the finite Smith–McMillan form of $G(\lambda)$.

2.3. Singular structure of rational matrices

Let us introduce now the singular structure of a rational matrix. Denote by $\mathcal{N}_\ell(G(\lambda))$ and $\mathcal{N}_r(G(\lambda))$ the left and right null-spaces over $\mathbb{F}(\lambda)$ of $G(\lambda)$, respectively, i.e., if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$,

$$\mathcal{N}_\ell(G(\lambda)) = \{x(\lambda) \in \mathbb{F}(\lambda)^p : x(\lambda)^T G(\lambda) = 0\},$$
$$\mathcal{N}_r(G(\lambda)) = \{x(\lambda) \in \mathbb{F}(\lambda)^m : G(\lambda)x(\lambda) = 0\}.$$

These sets are vector subspaces of $\mathbb{F}(\lambda)^p$ and $\mathbb{F}(\lambda)^m$, respectively. For any subspace of $\mathbb{F}(\lambda)^p$, it is always possible to find a basis consisting of vector polynomials; simply take an arbitrary basis and multiply each vector by the least common multiple of the denominators of its entries. The order of a polynomial basis is defined as the sum of the degrees of its vectors (see [26]). If $\mathcal{V}$ is a subspace of $\mathbb{F}(\lambda)^p$, a minimal basis of $\mathcal{V}$ is a polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$. The fundamental result in this setting is that the non-decreasing ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same (see [26]). These degrees are called the minimal indices of $\mathcal{V}$.

We refer to a polynomial matrix $N(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$ itself as a right polynomial basis of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ if the columns of $N(\lambda)$ form a basis of $\mathcal{N}_r(G(\lambda))$. If the columns of $N(\lambda)$ form a minimal basis of $\mathcal{N}_r(G(\lambda))$ then $N(\lambda)$ is referred to as a right minimal basis of $G(\lambda)$. Notice that $l = \dim \mathcal{N}_r(G(\lambda)) \leq m$. Moreover, $l = m$ if and only if $G(\lambda) = 0$.

Analogously, a polynomial matrix $N(\lambda) \in \mathbb{F}[\lambda]^{p\times q}$ is a left polynomial (resp., minimal) basis of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ if the columns of $N(\lambda)$ form a polynomial (resp., minimal) basis of $\mathcal{N}_\ell(G(\lambda))$. As above, $q = \dim \mathcal{N}_\ell(G(\lambda)) \leq p$, and $q = p$ if and only if $G(\lambda) = 0$.

The right (resp., left) minimal indices of a rational matrix $G(\lambda)$ are the minimal indices of $\mathcal{N}_r(G(\lambda))$ (resp., $\mathcal{N}_\ell(G(\lambda))$). If $N(\lambda)$ is a right (resp., left) minimal basis of $G(\lambda)$ then the right (resp., left) minimal indices of $G(\lambda)$ are the degrees of the columns of $N(\lambda)$ when ordered non-decreasingly.

One of the most usual characterizations of minimal bases is a slightly modified version of the Main Theorem given in [26], which can be also found in [16, Theorem 2.14]. Before presenting this theorem let us recall what a column proper or column reduced matrix is. Let $N(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$. We denote by $\deg(\mathrm{col}_j(N(\lambda)))$ the degree of the j-

th column of $N(\lambda)$, that is, the degree of the highest degree entry in column j. Put $d_j = \deg(\mathrm{col}_j(N(\lambda)))$. The matrix $N(\lambda)$ can always be written (see [27, Section 6.3.2]) as

$$N(\lambda) = N_h \,\mathrm{Diag}(\lambda^{d_1}, \ldots, \lambda^{d_l}) + L(\lambda) \tag{4}$$

where N_h is the highest column degree coefficient matrix of $N(\lambda)$, and $L(\lambda)$ is a polynomial matrix collecting the remaining terms, which has lower column degrees than the corresponding ones of $N(\lambda)$. The polynomial matrix $N(\lambda)$ is called column proper or column reduced if $\operatorname{rank} N_h = l$.

Theorem 2.2. *([16, Theorem 2.14]) The columns of a matrix polynomial $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if $N(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $N(\lambda)$ is column reduced.*

3. Minimal bases and indices of polynomial system matrices

The goal of this section is to show that if the blocks of a minimal polynomial system matrix $P(\lambda)$ of a rational matrix $G(\lambda)$ satisfy certain properness conditions, then the right (resp. left) minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same and the right (resp. left) minimal bases of $P(\lambda)$ and $G(\lambda)$ are easily related to each other through a simple bijection. This is stated in Corollary 3.9, which is the final result in this section and is obtained as a consequence of a number of intermediate results that deal independently with the left and the right minimal indices and bases. The properness conditions mentioned above are satisfied, in particular, by many classes of linearizations of rational matrices introduced recently [1,3,6,12–14,19], which are minimal linear polynomial system matrices of some rational matrices $\widehat{G}(\lambda)$ different from $G(\lambda)$, in general, but related to $G(\lambda)$. The results of this section are used in Section 6 to establish the relationships between the minimal indices and bases of $G(\lambda)$ and those of its strong block minimal bases linearizations, which in turn imply the corresponding results for $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations and Fiedler-like linearizations of $G(\lambda)$ in Sections 7 and 8. This provides a unified treatment of all these results and alternative proofs to the results in [12,14].

The properness conditions used in this section have been introduced in [6, Lemma 2.4 and Corollary 2.5] with the purpose of obtaining the invariant orders at infinity of $G(\lambda)$ from those of $P(\lambda)$. The new result is that they also allow to recover the minimal indices and bases of $G(\lambda)$ from those of $P(\lambda)$. Thus the coprimeness conditions of the blocks of $P(\lambda)$ guaranteeing its minimality and the properness conditions together allow to recover the complete finite eigenstructure of $G(\lambda)$ from $P(\lambda)$, as a consequence of the classical result of Rosenbrock [32] (see the comments following Proposition 2.1), the complete infinite eigenstructure of $G(\lambda)$ from $P(\lambda)$, as a consequence of the results in [6], and the minimal bases and indices of $G(\lambda)$ from $P(\lambda)$, as a consequence of the results in this section.

As outlined in the introduction, another set of conditions on a polynomial system matrix $P(\lambda)$ that allow to recover the complete finite and infinite spectral structures and the minimal indices and bases of its transfer function matrix $G(\lambda)$ from $P(\lambda)$ is the strong irreducibility introduced in [38]. The polynomial system matrix $P(\lambda)$ in (3) is strongly irreducible if $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix}$ have no finite zeros, and $\begin{bmatrix} A(\lambda) & B(\lambda) & 0 \\ -C(\lambda) & D(\lambda) & -I_p \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \\ 0 & I_m \end{bmatrix}$ have no infinite zeros.[3] The former two conditions are equivalent to $A(\lambda)$ and $B(\lambda)$ be left coprime and $A(\lambda)$ and $C(\lambda)$ be right coprime, i.e., the classical Rosenbrock's minimality of $P(\lambda)$ (see Proposition 2.1). And the latter two conditions are used in [38] to relate the invariant orders at infinity and the minimal bases and indices of $P(\lambda)$ and $G(\lambda)$. We substitute in this section the conditions on the infinite zeros by the easier to check conditions that $C(\lambda)A(\lambda)^{-1}$ and $A(\lambda)^{-1}B(\lambda)$ are proper rational functions. However, we remark that these pairs of conditions, the one on the infinite zeros and the one on properness, are not equivalent, and, even more, that none of them implies the other one, as we show at the end of this section via two examples. This is consistent with the fact that the rules for recovering the invariant orders at infinity are different under the properness conditions [6, Corollary 2.5] that under the strong irreducibility conditions [38, Result 1].

The proofs of the following two lemmas follow the same pattern as the first part of the proofs of Theorem 2 in [39] and Result 2 in [38] and they are omitted.

Lemma 3.1. *Let $P(\lambda)$ of (3) be a polynomial system matrix of a rational matrix $G(\lambda)$. Then* $\operatorname{rank} P(\lambda) = n + \operatorname{rank} G(\lambda)$, $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(P(\lambda))$ *and* $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(P(\lambda))$.

Lemma 3.2. *Under the assumptions of Lemma 3.1, if $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right polynomial basis of $P(\lambda)$ then $H_2(\lambda)$ is a right polynomial basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$.*

Lemma 3.1 means that any polynomial system matrix and its transfer function have the same number of right minimal indices and the same number of left minimal indices. In turns, Lemma 3.2 shows how to obtain a right polynomial basis of a rational matrix from a right polynomial basis of any of its polynomial system matrices. The extension of this result to right minimal bases under coprimeness and properness conditions requires some technical preliminary lemmas.

Lemma 3.3. *Let $N_1(\lambda) \in \mathbb{F}[\lambda]^{n\times l}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$.*

(i) *If $N_1(\lambda) = R(\lambda)N_2(\lambda)$ with $R(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times m}$ then* $\deg(\operatorname{col}_j(N_1(\lambda))) \leq \deg(\operatorname{col}_j(N_2(\lambda)))$ *for $j = 1, \ldots, l$.*

[3] In the case $B(\lambda), C(\lambda)$ and $D(\lambda)$ are constant matrices these conditions are equivalent to the strong irreducibility originally introduced in [39].

(ii) *If* $N_1(\lambda) = R(\lambda)N_2(\lambda)$ *with* $R(\lambda)$ *strictly proper then* $\deg(\mathrm{col}_j(N_1(\lambda))) < \deg(\mathrm{col}_j(N_2(\lambda)))$ *for* $j = 1, \ldots, l$.

Proof. Let $\frac{p(\lambda)}{q(\lambda)}$ be a proper rational function and let $n(\lambda)$ be a polynomial. Notice that

$$\deg(p(\lambda)) + \deg(n(\lambda)) - \deg(q(\lambda)) \leq \deg(n(\lambda)). \tag{5}$$

Let $n_{ij}^{(1)}(\lambda)$ be an arbitrary element of the j-th column of $N_1(\lambda)$. As $N_1(\lambda) = R(\lambda)N_2(\lambda)$, we can write $n_{ij}^{(1)}(\lambda) = \sum_{k=1}^{m} \frac{p_{ik}(\lambda)}{q_{ik}(\lambda)} n_{kj}^{(2)}(\lambda)$ where $\frac{p_{ik}(\lambda)}{q_{ik}(\lambda)}$ is the element in position (i,k) of $R(\lambda)$ and $n_{kj}^{(2)}(\lambda)$ is the element in position (k,j) of $N_2(\lambda)$. It follows from (5) that for each element of the j-th column of $N_1(\lambda)$

$$\begin{aligned}
&\deg\left(n_{ij}^{(1)}(\lambda)\right) = \deg\left(\textstyle\sum_{k=1}^{m} \frac{p_{ik}(\lambda)}{q_{ik}(\lambda)} n_{kj}^{(2)}(\lambda)\right) \\
&\leq \max_k\{\deg(p_{ik}(\lambda)) + \deg(n_{kj}^{(2)}(\lambda)) - \deg(q_{ik}(\lambda))\} \\
&\leq \max_k\{\deg(n_{kj}^{(2)}(\lambda))\} = \deg(\mathrm{col}_j(N_2(\lambda)), \qquad \text{for all } i = 1, \ldots, n.
\end{aligned}$$

Then $\deg(\mathrm{col}_j(N_1(\lambda))) = \max_i\{\deg(n_{ij}^{(1)}(\lambda))\} \leq \deg(\mathrm{col}_j(N_2(\lambda)))$ and (i) follows. If $R(\lambda)$ is strictly proper the previous inequality is strict. □

The following corollary is an immediate consequence of Lemma 3.3.

Corollary 3.4. *With the same assumptions and notation of Lemma 3.2, if* $A(\lambda)^{-1}B(\lambda)$ *is proper then* $\deg(\mathrm{col}_j(H_1(\lambda))) \leq \deg(\mathrm{col}_j(H_2(\lambda)))$ *for all* j. *The inequality is strict if* $A(\lambda)^{-1}B(\lambda)$ *is strictly proper.*

The following lemma relates the minimal bases of a rational matrix and its transpose as well as their minimal indices. It also states that the transpose of a polynomial system matrix gives rise to the transpose of its transfer function. It can be proved straightforwardly and, therefore, the proof is omitted.

Lemma 3.5.

(a) *For any rational matrix* $G(\lambda)$, $\mathcal{N}_\ell(G(\lambda)) = \mathcal{N}_r(G(\lambda)^T)$ *and* $\mathcal{N}_r(G(\lambda)) = \mathcal{N}_\ell(G(\lambda)^T)$. *Moreover,* $H(\lambda)$ *is a left minimal basis of* $G(\lambda)$ *if and only if it is a right minimal basis of* $G(\lambda)^T$. *Also the left minimal indices of* $G(\lambda)$ *and the right minimal indices of* $G(\lambda)^T$ *coincide.*

(b) *If* $P(\lambda)$ *is a (minimal) polynomial system matrix giving rise to* $G(\lambda)$ *then* $P(\lambda)^T$ *is a (minimal) polynomial system matrix giving rise to* $G(\lambda)^T$.

As announced, the following result shows how to obtain a minimal basis of a rational matrix from a minimal basis of those of its polynomial system matrices that satisfy coprimeness and properness conditions, and relates their minimal indices.

Theorem 3.6. *Let $P(\lambda)$ of (3) be a polynomial system matrix of a rational matrix $G(\lambda)$.*

(a) *If $A(\lambda)$ and $C(\lambda)$ are right coprime, $A(\lambda)^{-1}B(\lambda)$ is proper and $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m)\times l}$ is a right minimal basis of $P(\lambda)$, then $H_2(\lambda)$ is a right minimal basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$. Moreover, the right minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

(b) *If $A(\lambda)$ and $B(\lambda)$ are left coprime, $C(\lambda)A(\lambda)^{-1}$ is proper and $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times q}$ is a left minimal basis of $P(\lambda)$, then $H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda)A(\lambda)^{-1})^T H_2(\lambda)$. Moreover, the left minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

Proof. We prove part (a). By Lemma 3.2, $H_2(\lambda)$ is a right polynomial basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$. We show that $H_2(\lambda)$ is a minimal basis of $G(\lambda)$ by applying Theorem 2.2. Let us prove first that $H_2(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$. If this were not true, there would exist $\lambda_1 \in \overline{\mathbb{F}}$ and a vector, $v \neq 0$, such that $H_2(\lambda_1)v = 0$. But since $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right minimal basis for $P(\lambda)$, $\begin{bmatrix} H_1(\lambda_1) \\ H_2(\lambda_1) \end{bmatrix} v = \begin{bmatrix} w \\ 0 \end{bmatrix}$ with $w \neq 0$ and

$$P(\lambda_1)\begin{bmatrix} H_1(\lambda_1) \\ H_2(\lambda_1) \end{bmatrix} v = \begin{bmatrix} A(\lambda_1) & B(\lambda_1) \\ -C(\lambda_1) & D(\lambda_1) \end{bmatrix}\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} A(\lambda_1) \\ -C(\lambda_1) \end{bmatrix} w = 0.$$

This would be a contradiction because $A(\lambda)$ and $C(\lambda)$ are right coprime, i.e., $\begin{bmatrix} A(\lambda_1) \\ -C(\lambda_1) \end{bmatrix}$ has full column rank (see Proposition 2.1).

Next, let us see that $H_2(\lambda)$ is column reduced. By hypothesis and Theorem 2.2, we know that $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is column reduced. Our goal is to express the highest column degree coefficient matrix of $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ in terms of the highest column degree coefficient matrix of $H_2(\lambda)$, which is denoted by H_{2h}. For this purpose, note that the assumption that $A(\lambda)^{-1}B(\lambda)$ is proper implies that $-A(\lambda)^{-1}B(\lambda) = J + R(\lambda)$, where J is a constant matrix and $R(\lambda)$ is strictly proper. Thus, $H_1(\lambda) = JH_2(\lambda) + R(\lambda)H_2(\lambda)$ and

$$\operatorname{col}_j\left(\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}\right) = \begin{bmatrix} J\operatorname{col}_j(H_2(\lambda)) + R(\lambda)\operatorname{col}_j(H_2(\lambda)) \\ \operatorname{col}_j(H_2(\lambda)) \end{bmatrix}. \tag{6}$$

Bear in mind that $\operatorname{col}_j(H_2(\lambda)) \neq 0$ since $H_2(\lambda)$ is a right polynomial basis of $G(\lambda)$. Moreover, $R(\lambda)\operatorname{col}_j(H_2(\lambda))$ is a vector polynomial, because $\operatorname{col}_j(H_1(\lambda))$ and $J\operatorname{col}_j(H_2(\lambda))$ are both vector polynomials. Then, Lemma 3.3 (ii) guarantees that $\deg(R(\lambda)\operatorname{col}_j(H_2(\lambda))) < \deg(\operatorname{col}_j(H_2(\lambda)))$. Therefore, the highest degree coefficient of (6) is $\begin{bmatrix} J\operatorname{col}_j(H_{2h}) \\ \operatorname{col}_j(H_{2h}) \end{bmatrix}$, the degree d_j of (6) is $d_j = \deg(\operatorname{col}_j(H_2(\lambda)))$ and the highest column degree coefficient matrix of $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is $\begin{bmatrix} JH_{2h} \\ H_{2h} \end{bmatrix}$. This latter matrix has full column rank, which implies that H_{2h} has also full column rank, since otherwise there would exist a nonzero constant vector v such that $H_{2h}v = 0$ and $\begin{bmatrix} JH_{2h} \\ H_{2h} \end{bmatrix} v = 0$, which is a contradiction. This proves that

$H_2(\lambda)$ is column reduced and, so, a right minimal basis of $G(\lambda)$. Since the degrees of the corresponding columns of $H_2(\lambda)$ and $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ coincide, the right minimal indices of $G(\lambda)$ and $P(\lambda)$ are the same.

Part (b) is a consequence of part (a) and Lemma 3.5. □

We have shown so far how to obtain a minimal basis of the transfer function matrix of a polynomial system matrix out of a minimal basis of the latter, which is the most interesting scenario in applications. For completeness, we consider now the reciprocal problem. In this respect, Lemma 3.2 motivates the following result.

Lemma 3.7. *Let $P(\lambda)$ of (3) be a polynomial system matrix of a rational matrix $G(\lambda)$ where $A(\lambda)$ and $C(\lambda)$ are right coprime. Let $H_2(\lambda)$ be a right polynomial basis of $G(\lambda)$ and let $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$. Then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right polynomial basis of $P(\lambda)$.*

Proof. Note that

$$P(\lambda)\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}\begin{bmatrix} -A(\lambda)^{-1}B(\lambda)H_2(\lambda) \\ H_2(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ G(\lambda)H_2(\lambda) \end{bmatrix} = 0.$$

Let us see first that $H_1(\lambda)$ is polynomial. As $A(\lambda)$ and $C(\lambda)$ are right coprime, by Bezout's identity (see Proposition 2.1), there exist polynomial matrices $X(\lambda)$ and $Y(\lambda)$ of appropriate sizes such that

$$\begin{bmatrix} X(\lambda) & -Y(\lambda) \end{bmatrix}\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix} = I_n.$$

Put $H(\lambda) = X(\lambda)B(\lambda) - Y(\lambda)D(\lambda)$. Then,

$$\begin{bmatrix} X(\lambda) & -Y(\lambda) \end{bmatrix}\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} = \begin{bmatrix} I_n & H(\lambda) \end{bmatrix}\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} = 0.$$

Hence $H_1(\lambda) = -H(\lambda)H_2(\lambda)$ is a matrix polynomial. Moreover, $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right polynomial basis of $P(\lambda)$, because its columns belong to $\mathcal{N}_r(P(\lambda))$, its columns are linearly independent, since $H_2(\lambda)$ is a basis of $\mathcal{N}_r(G(\lambda))$, and $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(P(\lambda))$. □

We can now prove the reciprocal of Theorem 3.6, which shows that, under certain assumptions, minimal bases of polynomial system matrices can be obtained from minimal bases of their transfer functions.

Theorem 3.8. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix and let $P(\lambda)$ of (3) be a polynomial system matrix of $G(\lambda)$.*

(a) *If $A(\lambda)$ and $C(\lambda)$ are right coprime, $A(\lambda)^{-1}B(\lambda)$ is proper, $H_2(\lambda)$ is a right minimal basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$ then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right minimal basis of $P(\lambda)$. Moreover, the right minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

(b) *If $A(\lambda)$ and $B(\lambda)$ are left coprime, $C(\lambda)A(\lambda)^{-1}$ is proper, $H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda)A(\lambda)^{-1})^T H_2(\lambda)$ then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a left minimal basis of $P(\lambda)$. Moreover, the left minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

Proof. We prove part (a). By Lemma 3.7 and Theorem 2.2, we just need to prove that $\begin{bmatrix} H_1(\lambda_0) \\ H_2(\lambda_0) \end{bmatrix}$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is column reduced. As $H_2(\lambda)$ is a right minimal basis of $G(\lambda)$, $H_2(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$, which implies that the matrix $\begin{bmatrix} H_1(\lambda_0) \\ H_2(\lambda_0) \end{bmatrix}$ has full column rank as well. Moreover, $H_2(\lambda)$ is column reduced. Write $H_2(\lambda) = H_{2h}\,\mathrm{Diag}(\lambda^{d_1}, \ldots, \lambda^{d_l}) + L_2(\lambda)$ with H_{2h} of full column rank, $d_1, \ldots, d_l$ the right minimal indices of $G(\lambda)$ and the degree of the j-th column of $L_2(\lambda)$ less than d_j for each j. Since $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$, with $A(\lambda)^{-1}B(\lambda)$ proper, it follows from Corollary 3.4 that each column of $H_1(\lambda)$ has degree less than or equal to the degree of the same column of $H_2(\lambda)$. Therefore, there is a matrix H_{1h} such that the highest column degree coefficient matrix of $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is $\begin{bmatrix} H_{1h} \\ H_{2h} \end{bmatrix}$, a full column rank matrix. Moreover, its column degrees are those of $H_2(\lambda)$.

Part (b) follows from (a) and Lemma 3.5. □

Theorems 3.6 and 3.8 together provide our next result.

Corollary 3.9. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix and let $P(\lambda)$ of (3) be a minimal polynomial system matrix of $G(\lambda)$. If both $A(\lambda)^{-1}B(\lambda)$ and $C(\lambda)A(\lambda)^{-1}$ are proper matrices then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right (resp., left) minimal basis of $P(\lambda)$ if and only if $H_2(\lambda)$ is a right (resp., left) minimal basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$ (resp., $H_1(\lambda) = (C(\lambda)A(\lambda)^{-1})^T H_2(\lambda)$). Moreover, the right (resp., left) minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

Remark 3.10. As announced at the beginning of this section, we show that the conditions on properness in Theorems 3.6 and 3.8 and Verghese's conditions on infinite zeros in [38] are not equivalent. More precisely, we prove by means of two examples that for a polynomial system matrix $P(\lambda)$ as in (3), $C(\lambda)A(\lambda)^{-1}$ being proper does not imply $\begin{bmatrix} A(\lambda) & B(\lambda) & 0 \\ -C(\lambda) & D(\lambda) & -I \end{bmatrix}$ having no infinite zeros and vice versa, even when $A(\lambda)$ and $B(\lambda)$ are left coprime. The same two examples prove that $A(\lambda)^{-1}B(\lambda)$ being proper does not imply $\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \\ 0 & I \end{bmatrix}$ having no infinite zeros and vice versa, even when $A(\lambda)$ and $C(\lambda)$ are right coprime.

Example 3.11. Let $A(\lambda) = \begin{bmatrix} \lambda+1 & \lambda^2 \\ 1 & \lambda \end{bmatrix}, B(\lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C(\lambda) = \begin{bmatrix} 0 & 1 \end{bmatrix}, D(\lambda) = 0$. It is easy to check that $G(\lambda) = -\frac{1}{\lambda}$, $A(\lambda)$ and $B(\lambda)$ are left coprime and $C(\lambda)A(\lambda)^{-1} = \begin{bmatrix} -\frac{1}{\lambda}, & 1+\frac{1}{\lambda} \end{bmatrix}$

is proper. However, $\begin{bmatrix} A(\lambda) & B(\lambda) & 0 \\ -C(\lambda) & D(\lambda) & -1 \end{bmatrix}$ has an infinite zero, because its invariant orders at infinity are $q_1 = -2$, $q_2 = 0$ and $q_3 = 1$. Moreover, $A(\lambda)$ and $C(\lambda)$ are right coprime, $A(\lambda)^{-1}B(\lambda)$ is proper, but $\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \\ 0 & 1 \end{bmatrix}$ has again an infinite zero, because its invariant orders at infinity are also $q_1 = -2$, $q_2 = 0$ and $q_3 = 1$. Thus, $P(\lambda)$ of (3) is not strongly irreducible, but allows to recover the complete finite and infinite eigenstructures of $G(\lambda)$ as well as its minimal indices and bases, since it satisfies the coprimeness and properness conditions.

Example 3.12. Let $A(\lambda) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ 1 \end{bmatrix}, C(\lambda) = \begin{bmatrix} 1 & 0 & \lambda \end{bmatrix}, D(\lambda) = 0$. It is easy to check that $G(\lambda) = \frac{1}{\lambda} - \lambda^2 + \lambda$, and $A(\lambda)$ and $B(\lambda)$ are left coprime. The matrix $\begin{bmatrix} A(\lambda) & B(\lambda) & 0 \\ -C(\lambda) & D(\lambda) & -1 \end{bmatrix}$ has no infinite zeros since its invariant orders at infinity are $q_1 = -1$, $q_2 = -1$, $q_3 = -1$ and $q_4 = 0$. Nevertheless, $C(\lambda)A(\lambda)^{-1} = \begin{bmatrix} \frac{1}{\lambda}, & -\lambda, & \lambda \end{bmatrix}$ is not proper. Moreover, $A(\lambda)$ and $C(\lambda)$ are right coprime, but the matrix $\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \\ 0 & 1 \end{bmatrix}$ has no infinite zeros, since its invariant orders at infinity are again $q_1 = -1$, $q_2 = -1$, $q_3 = -1$ and $q_4 = 0$. However, the matrix $A(\lambda)^{-1}B(\lambda) = \begin{bmatrix} \frac{1}{\lambda}, & \lambda, & -\lambda+1 \end{bmatrix}$ is not proper. Thus, $P(\lambda)$ of (3) is strongly irreducible and, so, allows to recover the complete finite and infinite eigenstructures of $G(\lambda)$ as well as its minimal indices and bases, but none of the properness conditions hold.

4. Polynomial bases of linearizations of rational matrices

The aim of this section is to study the relationship between the polynomial bases of a rational matrix and the polynomial bases of its linearizations. It is not possible to extend this relationship to minimal bases because it was already proved in [15, Theorem 4.10 (b)] that the minimal bases and indices of a polynomial matrix can not be obtained from the minimal bases and indices of its linearizations in general, and polynomial matrices are particular cases of rational matrices.

A linear pencil

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \tag{7}$$

is said to be a linearization of a rational matrix $G(\lambda)$ (see [6, Definition 3.2]) if it is a minimal polynomial system matrix of a rational matrix $\widehat{G}(\lambda)$ such that, for some non-negative integers s_1, s_2, $\mathrm{Diag}(\widehat{G}(\lambda), I_{s_2})$ and $\mathrm{Diag}(G(\lambda), I_{s_1})$ are unimodularly equivalent. Without loss of generality we can assume that $s_1 = s$ and $s_2 = 0$. This assumption will be adopted in the rest of the paper every time we deal with linearizations.

A first consequence of this definition is that, by the rank-nullity theorem, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$ and $\dim \mathcal{N}_\ell(\widehat{G}(\lambda)) = \dim \mathcal{N}_\ell(G(\lambda))$. Therefore, $G(\lambda)$ and $\widehat{G}(\lambda)$ have the same number of right minimal indices and the same number of left minimal indices. Furthermore, by Lemma 3.1, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$ and $\dim \mathcal{N}_\ell(\widehat{G}(\lambda)) =$

$\dim \mathcal{N}_\ell(L(\lambda))$. Thus, a rational matrix and any of its linearizations have the same number of right minimal indices and the same number of left minimal indices.

Proposition 4.1 relates right polynomial bases of $G(\lambda)$ and $\widehat{G}(\lambda)$. An analogous result holds for left polynomial bases of $G(\lambda)$ and $\widehat{G}(\lambda)$ as a consequence of Lemma 3.5. Such "left" result is omitted for brevity.

Proposition 4.1. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $\widehat{G}(\lambda) \in \mathbb{F}(\lambda)^{(p+s)\times(m+s)}$, $s \geq 0$. Let $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s)\times(p+s)}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s)\times(m+s)}$ be unimodular matrices such that $U(\lambda)\widehat{G}(\lambda)V(\lambda) = \operatorname{Diag}(G(\lambda), I_s)$.*

(a) *If $H(\lambda)$ is a right polynomial basis of $G(\lambda)$ then $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ is a right polynomial basis of $\widehat{G}(\lambda)$.*
(b) *If $\widehat{H}(\lambda)$ is a right polynomial basis of $\widehat{G}(\lambda)$ then $V(\lambda)^{-1}\widehat{H}(\lambda) = \left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ and $H(\lambda)$ is a right polynomial basis of $G(\lambda)$.*

Proof. In order to prove (a) assume that $G(\lambda)H(\lambda) = 0$. We obtain, via a direct multiplication, that

$$\widehat{G}(\lambda)V(\lambda)\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix} = U(\lambda)^{-1}\begin{bmatrix} G(\lambda) & 0 \\ 0 & I_s \end{bmatrix}\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix} = 0.$$

So, $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ is a right polynomial basis of $\widehat{G}(\lambda)$, because its columns are linearly independent and $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$.

For proving (b) assume that $\widehat{G}(\lambda)\widehat{H}(\lambda) = 0$. Therefore,

$$\begin{bmatrix} G(\lambda) & 0 \\ 0 & I_s \end{bmatrix} V(\lambda)^{-1}\widehat{H}(\lambda) = 0.$$

Write $V(\lambda)^{-1} = \left[\begin{smallmatrix} V_1(\lambda) \\ V_2(\lambda) \end{smallmatrix}\right]$, where $V_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+s)}$ and $V_2(\lambda) \in \mathbb{F}[\lambda]^{s\times(m+s)}$. Thus, $G(\lambda)V_1(\lambda)\widehat{H}(\lambda) = 0$ and $V_2(\lambda)\widehat{H}(\lambda) = 0$. Set $H(\lambda) = V_1(\lambda)\widehat{H}(\lambda)$. It follows that $V(\lambda)^{-1}\widehat{H}(\lambda) = \left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ and $G(\lambda)H(\lambda) = 0$. Thus, the columns of $H(\lambda)$ form a right polynomial basis of $G(\lambda)$. □

Remark 4.2. Proposition 4.1 cannot be extended to right minimal bases, i.e., if $H(\lambda)$ is a right minimal basis of $G(\lambda)$, $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ may not be a right minimal basis of $\widehat{G}(\lambda)$, and if $\widehat{H}(\lambda)$ is a right minimal basis of $\widehat{G}(\lambda)$, $V(\lambda)^{-1}\widehat{H}(\lambda)$ may not contain a minimal basis of $G(\lambda)$ in its first m rows. Otherwise, if Proposition 4.1 could be extended to right minimal bases, taking $G(\lambda)$ polynomial and $\widehat{G}(\lambda)$ a linearization of $G(\lambda)$, then their right minimal bases and indices would be always related, which is in contradiction with [15, Theorem 4.10 (b)].

The next result relates the polynomial bases of a rational matrix and its linearizations through the unimodular matrices that connect the rational matrix and the transfer function matrix of the linearizations.

Theorem 4.3. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ of (7) be a linearization of $G(\lambda)$ with transfer function matrix $\widehat{G}(\lambda)$. Let $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s)\times(p+s)}$, $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s)\times(m+s)}$ be unimodular matrices such that $U(\lambda)\widehat{G}(\lambda)V(\lambda) = \mathrm{Diag}(G(\lambda), I_s)$.*

(a) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ *is a right polynomial basis of $L(\lambda)$ if and only if $H_2(\lambda) = V(\lambda)\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix}$ for some right polynomial basis $H(\lambda)$ of $G(\lambda)$ and $H_1(\lambda) = -(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)H_2(\lambda)$.*

(b) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ *is a left polynomial basis of $L(\lambda)$ if and only if $H_2(\lambda) = U(\lambda)^T\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix}$ for some left polynomial basis $H(\lambda)$ of $G(\lambda)$ and $H_1(\lambda) = ((C_1\lambda + C_0)(A_1\lambda + A_0)^{-1})^T H_2(\lambda)$.*

Proof. As $L(\lambda)$ is a linearization of $G(\lambda)$, $L(\lambda)$ is a minimal polynomial system matrix and, therefore, $A_1\lambda + A_0$ and $C_1\lambda + C_0$ are right coprime and $A_1\lambda + A_0$ and $B_1\lambda + B_0$ are left coprime. Thus we can apply Lemmas 3.2 and 3.7 and Proposition 4.1 to prove part (a). To prove part (b), use Lemma 3.5 and part (a). □

5. Minimal indices of strong linearizations of rational matrices

In this section we begin to study the relationship between the minimal indices of a rational matrix and the minimal indices of its strong linearizations. As discussed in [6, Remark 3.5], strong linearizations are particular cases of linearizations and, therefore, we know that the number of right (resp., left) minimal indices of a rational matrix and of its strong linearizations coincide. However, we will show in this section that it is not possible to obtain the right (resp., left) minimal indices of a rational matrix from those of its strong linearizations in general. Nevertheless, we will prove in Theorem 5.9 that the total sum of the right and left minimal indices of a rational matrix can be easily obtained from the total sum of the right and left minimal indices of any of its strong linearizations. It has been recently shown in [12–14] that for the families of Fiedler-like and affine spaces of strong linearizations of *square* rational matrices it is possible to recover easily the minimal bases and indices of the rational matrix from these linearizations. We prove in Section 6 that the same is possible for any rational matrix, i.e., possibly rectangular, from its strong block minimal bases linearizations. As corollaries, analogous results are proved in Section 7 for $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations.

We start by recalling the definition of strong linearization of a rational matrix.

Definition 5.1. ([6, Definition 3.4]) Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. Let q_1 be its first invariant order at infinity and $g = \min(0, q_1)$. Let $n = \nu(G(\lambda))$. A strong linearization of $G(\lambda)$ is a linear polynomial matrix

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}$$

such that the following conditions hold:

(a) if $n > 0$ then $\det(A_1\lambda + A_0) \neq 0$, and
(b) if $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$, $\widehat{q}_1$ is its first invariant order at infinity and $\widehat{g} = \min(0, \widehat{q}_1)$ then:
   (i) there are integers $s_1, s_2 \geq 0$ and unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ so that $s_1 - s_2 = q - p = r - m$ and

$$U_1(\lambda)\,\mathrm{Diag}(G(\lambda), I_{s_1})U_2(\lambda) = \mathrm{Diag}(\widehat{G}(\lambda), I_{s_2}), \text{ and}$$

   (ii) there are biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+s_1)\times(p+s_1)}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+s_1)\times(m+s_1)}$ such that

$$B_1(\lambda)\,\mathrm{Diag}(\lambda^g G(\lambda), I_{s_1})B_2(\lambda) = \mathrm{Diag}(\lambda^{\widehat{g}}\widehat{G}(\lambda), I_{s_2}).$$

As in the case of linearizations, we can also assume without loss of generality that $s_1 = s$ and $s_2 = 0$ in the definition of strong linearizations. We will adopt such assumption in the rest of the paper.

Remark 5.2. As commented in [6, Remark 3.5], the requirement $n = \nu(G(\lambda))$ in Definition 5.1 might seem very restrictive. Thus, it is worth to emphasize that such requirement may be replaced by the assumptions that $L(\lambda)$ is a minimal polynomial system matrix and A_1 is invertible when $n > 0$, as a consequence of the discussion in [6, Remark 3.5], which are more direct requirements. We have decided to state Definition 5.1 exactly as in [6] in order to avoid confusions.

Recall that any rational matrix can be written uniquely as $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ with $D(\lambda)$ a polynomial matrix and $G_{sp}(\lambda)$ a strictly proper matrix. Moreover, if $D(\lambda) \neq 0$ then the first invariant order at infinity of $G(\lambda)$, q_1, is equal to $-\deg(D(\lambda))$; otherwise, if $G(\lambda)$ is strictly proper, $q_1 > 0$. We define

$$d = -\min(0, q_1) = \begin{cases} \deg(D(\lambda)) & \text{if } D(\lambda) \neq 0 \\ 0 & \text{if } D(\lambda) = 0 \end{cases}. \tag{8}$$

Notice that g in Definition 5.1 is equal to $-d$.

We now show with Example 5.4 that the minimal indices of a strong linearization of a rational matrix may be arbitrarily different than the minimal indices of the rational matrix in general. In order to develop Example 5.4, we present the following lemma first.

Lemma 5.3. *Let*

$$K_u(\lambda) = \begin{bmatrix} 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{u\times(u+1)}$$

for any positive integer u and let $0_{u\times 1}$ be the $u \times 1$ zero matrix. Then,

(i) $K_u(\lambda)$ *is unimodularly equivalent to* $[I_u \quad 0_{u\times 1}]$.
(ii) $\lambda^{-1}K_u(\lambda)$ *is equivalent at infinity to* $[I_u \quad 0_{u\times 1}]$.

Proof. In order to prove (i), multiply $K_u(\lambda)$ on the right by the unimodular matrix

$$\begin{bmatrix} 1 & -\lambda & \lambda^2 & (-\lambda)^3 & \cdots & (-\lambda)^u \\ & 1 & -\lambda & \lambda^2 & \cdots & (-\lambda)^{u-1} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & -\lambda & \lambda^2 \\ & & & & 1 & -\lambda \\ & & & & & 1 \end{bmatrix}.$$

To prove (ii), multiply $\lambda^{-1}K_u(\lambda)$ on the right by the biproper matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -1/\lambda \\ -1/\lambda & 1 & 0 & \cdots & 0 & (-1/\lambda)^2 \\ (-1/\lambda)^2 & -1/\lambda & 1 & \cdots & 0 & (-1/\lambda)^3 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ (-1/\lambda)^{u-1} & (-1/\lambda)^{u-2} & & \cdots & 1 & (-1/\lambda)^u \end{bmatrix}. \quad \square$$

Example 5.4. Let $G(\lambda) = \left[\begin{smallmatrix} \lambda+\lambda^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{F}(\lambda)^{2\times 2}$. We may consider infinitely many strong linearizations of $G(\lambda)$. Let

$$L_{\epsilon,\eta}(\lambda) = \left[\begin{array}{c|cc} \lambda & 1 & \\ \hline -1 & \lambda & \\ & & K_\epsilon(\lambda) \\ & & \\ & & \\ & & K_\eta(\lambda)^T \end{array}\right] \in \mathbb{F}[\lambda]^{(1+(2+\epsilon+\eta))\times(1+(2+\epsilon+\eta))}.$$

We prove now that for each pair of positive integers ϵ and η, $L_{\epsilon,\eta}(\lambda)$ is a strong linearization of $G(\lambda)$. First, notice that $L_{\epsilon,\eta}(\lambda)$ is a minimal polynomial system matrix with transfer function matrix

$$\widehat{G}_{\epsilon,\eta}(\lambda) = \begin{bmatrix} \lambda+\lambda^{-1} & & \\ & K_\epsilon(\lambda) & \\ & & K_\eta(\lambda)^T \end{bmatrix}.$$

Using Lemma 5.3, it is easy to prove that $\widehat{G}_{\epsilon,\eta}(\lambda)$ is unimodularly equivalent to

$$\begin{bmatrix} \lambda+\lambda^{-1} & & & & \\ & I_\epsilon & 0_{\epsilon\times 1} & & \\ & & & I_\eta & \\ & & & 0_{1\times\eta} & \end{bmatrix},$$

which is unimodularly equivalent to $\begin{bmatrix} G(\lambda) & 0 \\ 0 & I_{\epsilon+\eta} \end{bmatrix}$. Thus, $L_{\epsilon,\eta}(\lambda)$ is a linearization of $G(\lambda)$. Furthermore, $G(\lambda)$ can be written as

$$G(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and $\widehat{G}_{\epsilon,\eta}(\lambda)$ can be written as

$$\widehat{G}_{\epsilon,\eta}(\lambda) = \begin{bmatrix} \lambda & & \\ & K_\epsilon(\lambda) & \\ & & K_\eta(\lambda)^T \end{bmatrix} + \begin{bmatrix} \lambda^{-1} & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

Therefore, with the notation of Definition 5.1, $g = \widehat{g} = -1$. The matrix $\lambda^{-1}\widehat{G}_{\epsilon,\eta}(\lambda)$ is

$$\begin{bmatrix} 1+\lambda^{-2} & & \\ & \lambda^{-1}K_\epsilon(\lambda) & \\ & & \lambda^{-1}K_\eta(\lambda)^T \end{bmatrix},$$

which, by Lemma 5.3, is equivalent at infinity to

$$\begin{bmatrix} 1+\lambda^{-2} & & & & \\ & I_\epsilon & 0_{\epsilon\times 1} & & \\ & & & I_\eta & \\ & & & 0_{1\times\eta} & \end{bmatrix} \text{ and to } \begin{bmatrix} \lambda^{-1}G(\lambda) & 0 \\ 0 & I_{\epsilon+\eta} \end{bmatrix}.$$

Hence, $L_{\epsilon,\eta}(\lambda)$ is a strong linearization of $G(\lambda)$. Notice that the *unique right minimal index of $G(\lambda)$ is 0 and the unique left minimal index of $G(\lambda)$ is 0 as well, while the unique right minimal index of $L_{\epsilon,\eta}(\lambda)$ is ϵ and the unique left minimal index of $L_{\epsilon,\eta}(\lambda)$ is η. Thus, strong linearizations do not preserve minimal indices.*

Denote by $\mu(G(\lambda))$ the sum of the right and left minimal indices of a rational matrix $G(\lambda)$. Our next goal is to analyze how this is related with the sum of the right and left minimal indices of any of its strong linearizations. In order to study this relationship, we will make use of Van Dooren's index sum theorem, proved for the first time in [39, Theorem 3], and that we rewrite in a way convenient for our purposes in Lemma 5.5. Interested readers are referred to the recent paper [7] for more information on this fundamental result.

Lemma 5.5. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix with finite Smith–McMillan form* $\operatorname{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{(p-r)\times(m-r)}\right)$. *Let $q_1 \leq \cdots \leq q_r$ be its invariant orders at infinity. Then*

$$\mu(G(\lambda)) = \sum_{i=1}^{r} \deg(\psi_i(\lambda)) - \sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - \sum_{i=1}^{r} q_i. \tag{9}$$

Proof. By the index sum theorem (see [39, Theorem 3] or [27, Theorem 6.5-11]) $\mu(G(\lambda))$ is equal to the total number of poles (finite and at infinity) of $G(\lambda)$ minus the total number of zeros (finite and at infinity) of $G(\lambda)$. The total number of finite zeros of $G(\lambda)$ is the sum of all partial multiplicities of all finite zeros of $G(\lambda)$, that is, $\sum_{i=1}^{r} \deg(\epsilon_i(\lambda))$. In the same way, the total number of finite poles of $G(\lambda)$ is the sum of all partial multiplicities of all finite poles of $G(\lambda)$, i.e., $\sum_{i=1}^{r} \deg(\psi_i(\lambda))$. Therefore, the total number of finite poles minus the total number of finite zeros is $\sum_{i=1}^{r} \deg(\psi_i(\lambda)) - \sum_{i=1}^{r} \deg(\epsilon_i(\lambda))$. On the other hand, the total number of infinite poles minus the total number of infinite zeros is $-\sum_{i=1}^{r} q_i$ since the positive q_i are the orders of the infinite zeros while minus the negative q_i are the orders of the infinite poles. Thus, equation (9) is obtained. □

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix, let d be defined as in (8) and let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \tag{10}$$

be a linear minimal polynomial system matrix with A_1 invertible if $n > 0$. We say that $L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$ if the following conditions simultaneously hold:

(i) the finite poles of $G(\lambda)$ are the finite zeros of $A_1\lambda + A_0$, with the same partial multiplicities in both matrices,
(ii) the finite zeros of $G(\lambda)$ are the finite zeros of $L(\lambda)$, with the same partial multiplicities, and
(iii) the number and orders of the infinite zeros of $\lambda^{-1}L(\lambda)$ are the same as the number and orders of the infinite zeros of $\lambda^{-d}G(\lambda)$ if $D_1 + C_1A_1^{-1}B_1 \neq 0$ or of $\operatorname{Diag}(\lambda^{-1}I_s, \lambda^{-d-1}G(\lambda))$ otherwise.

Theorem 5.6. *([6, Theorem 3.10]) Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $n = \nu(G(\lambda))$. Let $L(\lambda)$ be the pencil of (10). Then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if the following two conditions hold:*

(I) $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$ *(which is equivalent to* $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(L(\lambda))$*), and*
(II) *$L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$.*

The following result relates the invariant orders at infinity of a rational matrix and its a strong linearization. We remark that although Lemma 5.7 was not explicitly stated in [6], it is related to discussions in [6, pp. 1682–1683].

Lemma 5.7. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix with invariant orders at infinity $q_1 \leq \cdots \leq q_r$ and d be defined as in (8). Let $L(\lambda)$ of (10) be any strong linearization of $G(\lambda)$ and $q_1^L \leq \ldots \leq q_\ell^L$ be the invariant orders at infinity of $L(\lambda)$. Then $\ell = n + s + r$ and*

(i) *If $D_1 + C_1A_1^{-1}B_1 \neq 0$ then $q_i^L = -1$ for $i = 1, \ldots, n + s$, and $q_{n+s+i}^L = q_i + d - 1$ for $i = 1, \ldots, r$.*
(ii) *If $n > 0$ and $D_1 + C_1A_1^{-1}B_1 = 0$ then $q_i^L = -1$ for $i = 1, \ldots, n$, $q_{n+i}^L = 0$ for $i = 1, \ldots, s$, and $q_{n+s+i}^L = q_i + d$ for $i = 1, \ldots, r$.*
(iii) *If $n = 0$ and $D_1 = 0$ then $L(\lambda) = D_0$, $q_i^L = 0$ for $i = 1, \ldots, s + r$, and $q_i = -d$ for $i = 1, \ldots, r$.*

Proof. By Theorem 5.6 (I) and the rank-nullity theorem, $\ell = n+s+r$ is the rank of $L(\lambda)$. As $q_1 \leq \cdots \leq q_r$ are the invariant orders at infinity of $G(\lambda)$, there exist two biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that

$$G(\lambda) = B_1(\lambda)\,\mathrm{Diag}\left(\left(\frac{1}{\lambda}\right)^{q_1}, \ldots, \left(\frac{1}{\lambda}\right)^{q_r}, 0_{(p-r)\times(m-r)}\right) B_2(\lambda). \tag{11}$$

We distinguish two cases:

Suppose first that $D_1 + C_1A_1^{-1}B_1 \neq 0$. By Theorem 5.6 again, the number and orders of the infinite zeros of $\lambda^{-1}L(\lambda)$ are the same as the number and orders of the infinite zeros of $\lambda^{-d}G(\lambda)$. Since $\lambda^{-1}L(\lambda)$ and $\lambda^{-d}G(\lambda)$ are both proper rational matrices and $\operatorname{rank} L(\lambda) - \operatorname{rank} G(\lambda) = n+s$, $\lambda^{-1}L(\lambda)$ must be equivalent at infinity to $\left[\begin{smallmatrix} \lambda^{-d}G(\lambda) & 0 \\ 0 & I_{n+s} \end{smallmatrix}\right]$. Thus $L(\lambda)$ is equivalent at infinity to $\begin{bmatrix} \lambda^{-d+1}G(\lambda) & 0 \\ 0 & \lambda I_{n+s} \end{bmatrix}$, that is, there exist two biproper matrices $B_3(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+n+s)\times(p+n+s)}$ and $B_4(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+n+s)\times(m+n+s)}$ such that

$$L(\lambda) = B_3(\lambda)\begin{bmatrix} \lambda^{-d+1}G(\lambda) & 0 \\ 0 & \lambda I_{n+s} \end{bmatrix} B_4(\lambda) = B_3(\lambda)\lambda^{-d+1}\begin{bmatrix} G(\lambda) & 0 \\ 0 & \lambda^d I_{n+s} \end{bmatrix} B_4(\lambda).$$

Put $\overline{B}_1(\lambda) = B_3(\lambda)\left[\begin{smallmatrix} B_1(\lambda) & 0 \\ 0 & I_{n+s} \end{smallmatrix}\right]$ and $\overline{B}_2(\lambda) = \left[\begin{smallmatrix} B_2(\lambda) & 0 \\ 0 & I_{n+s} \end{smallmatrix}\right] B_4(\lambda)$, which are biproper matrices. Using (11),

$$\begin{aligned} L(\lambda) &= \overline{B}_1(\lambda)\lambda^{-d+1}\begin{bmatrix} \mathrm{Diag}\left((\frac{1}{\lambda})^{q_1}, \ldots, (\frac{1}{\lambda})^{q_r}, 0\right) & 0 \\ 0 & \lambda^d I_{n+s} \end{bmatrix} \overline{B}_2(\lambda) \\ &= \overline{B}_1(\lambda)\begin{bmatrix} \mathrm{Diag}\left((\frac{1}{\lambda})^{q_1+d-1}, \ldots, (\frac{1}{\lambda})^{q_r+d-1}, 0\right) & 0 \\ 0 & (\frac{1}{\lambda})^{-1} I_{n+s} \end{bmatrix} \overline{B}_2(\lambda). \end{aligned}$$

Notice, by (8), that $q_1 + d \geq 0$. Therefore $-1 \leq q_1 + d - 1 \leq \cdots \leq q_r + d - 1$. Thus, $q_i^L = -1$ for $i = 1, \ldots, n+s$, and $q_{n+s+i}^L = q_i + d - 1$ for $i = 1, \ldots, r$.

Suppose now that $D_1 + C_1 A_1^{-1} B_1 = 0$. By Theorem 5.6, the number and orders of the infinite zeros of $\lambda^{-1} L(\lambda)$ are the same as those of $\mathrm{Diag}(\lambda^{-1} I_s, \lambda^{-d-1} G(\lambda))$. As both matrices are proper and their rank difference is n, $\lambda^{-1} L(\lambda)$ must be equivalent at infinity to $\begin{bmatrix} \lambda^{-d-1} G(\lambda) & 0 & 0 \\ 0 & \lambda^{-1} I_s & 0 \\ 0 & 0 & I_n \end{bmatrix}$. Thus $L(\lambda)$ is equivalent at infinity to $\begin{bmatrix} \lambda^{-d} G(\lambda) & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & \lambda I_n \end{bmatrix}$, that is, there exist two biproper matrices $B_5(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+n+s)\times(p+n+s)}$ and $B_6(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+n+s)\times(m+n+s)}$ such that

$$L(\lambda) = B_5(\lambda) \lambda^{-d} \begin{bmatrix} G(\lambda) & 0 & 0 \\ 0 & \lambda^d I_s & 0 \\ 0 & 0 & \lambda^{d+1} I_n \end{bmatrix} B_6(\lambda).$$

By using (11) and proceeding as in the previous case, if $n > 0$ then the invariant orders at infinity of $L(\lambda)$ are $q_i^L = -1$ for $i = 1, \ldots, n$, $q_{n+i}^L = 0$ for $i = 1, \ldots, s$, and $q_{n+s+i}^L = q_i + d$ for $i = 1, \ldots, r$. Otherwise, if $n = 0$ then $D_1 = 0$, $L(\lambda) = D_0$ and, therefore, $q_i^L = 0$ for $i = 1, \ldots, s + r$. Moreover, since $D_0 = B_5(\lambda) \begin{bmatrix} \lambda^{-d} G(\lambda) & 0 \\ 0 & I_s \end{bmatrix} B_6(\lambda)$, the invariant orders at infinity of $\lambda^{-d} G(\lambda)$ must be 0 and, in consequence, $q_i = -d$ for $i = 1, \ldots, r$. □

The following lemma gives $\mu(L(\lambda))$, the sum of the right and left minimal indices of a strong linearization $L(\lambda)$ of a rational matrix $G(\lambda)$, in terms of the spectral invariants of $G(\lambda)$.

Lemma 5.8. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix with $\epsilon_1(\lambda), \ldots, \epsilon_r(\lambda)$ as numerators in its finite Smith–McMillan form and with $q_1 \leq \cdots \leq q_r$ as invariant orders at infinity. Let d be defined as in (8). Let $L(\lambda)$ of (10) be any strong linearization of $G(\lambda)$.*

(i) *If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then*

$$\mu(L(\lambda)) = s + r(1-d) + n - \textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - \sum_{i=1}^r q_i.$$

(ii) *If $n > 0$ and $D_1 + C_1 A_1^{-1} B_1 = 0$ then*

$$\mu(L(\lambda)) = -dr + n - \textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - \sum_{i=1}^r q_i.$$

(iii) *If $n = 0$ and $D_1 = 0$ then $L(\lambda) = D_0$, $\mu(L(\lambda)) = 0$, and $\epsilon_i(\lambda) = 1$ for $i = 1, \ldots, r$.*

Proof. We aim to apply Lemma 5.5 to $L(\lambda)$. As seen in Lemma 5.7, $\mathrm{rank}\, L(\lambda) = n+s+r$. Since $L(\lambda)$ is a polynomial matrix it has no finite poles. Moreover, by Theorem 5.6, its total number of finite zeros is $\sum_{i=1}^r \deg(\epsilon_i(\lambda))$. Denote by q_i^L, $i = 1, \ldots, n+s+r$, the invariant orders at infinity of $L(\lambda)$. By Lemma 5.5,

$$\mu(L(\lambda)) = -\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - \sum_{i=1}^{n+s+r} q_i^L.$$

By Lemma 5.7:

(i) If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then

$$\begin{aligned} \mu(L(\lambda)) &= -\textstyle\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - (\sum_{i=1}^{n+s}(-1) + \sum_{i=1}^{r}(q_i + d - 1)) \\ &= -\textstyle\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) + n + s + r - dr - \sum_{i=1}^{r} q_i. \end{aligned}$$

(ii) If $n > 0$ and $D_1 + C_1 A_1^{-1} B_1 = 0$ then

$$\begin{aligned} \mu(L(\lambda)) &= -\textstyle\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - (\sum_{i=1}^{n}(-1) + \sum_{i=1}^{r}(q_i + d)) \\ &= -\textstyle\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) + n - dr - \sum_{i=1}^{r} q_i. \end{aligned}$$

(iii) If $n = 0$ and $D_1 = 0$ then $L(\lambda) = D_0$ and $\mu(L(\lambda)) = -\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - 0$. But since $L(\lambda)$ is constant its total number of finite zeros is 0 and, therefore, $\epsilon_i(\lambda) = 1$ for $i = 1, \ldots, r$. □

Finally, the following result shows the relationship between the sum of the right and left minimal indices of a rational matrix and of its strong linearizations.

Theorem 5.9. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix of rank r. Let d be defined as in (8). Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be any strong linearization of $G(\lambda)$. Then

$$\mu(G(\lambda)) = \begin{cases} \mu(L(\lambda)) + dr - (r+s), & \text{if } D_1 + C_1 A_1^{-1} B_1 \neq 0 \\ \mu(L(\lambda)) + dr, & \text{if } n > 0 \text{ and } D_1 + C_1 A_1^{-1} B_1 = 0 \\ dr, & \text{if } n = 0 \text{ and } D_1 = 0 \end{cases}.$$

Proof. Let $\mathrm{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{(p-r)\times(m-r)}\right)$ be the finite Smith–McMillan form of $G(\lambda)$ and $q_1 \leq \cdots \leq q_r$ be its invariant orders at infinity. By definition of strong linearization, $n = \nu(G(\lambda))$. Moreover, $\nu(G(\lambda)) = \sum_{i=1}^{r} \deg(\psi_i(\lambda))$ and, therefore, $n = \sum_{i=1}^{r} \deg(\psi_i(\lambda))$. By using Lemma 5.5, $\mu(G(\lambda)) = n - \sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - \sum_{i=1}^{r} q_i$. Now, by Lemma 5.8:

(i) If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then $\mu(L(\lambda)) = s + r(1-d) + \mu(G(\lambda))$.
(ii) If $n > 0$ and $D_1 + C_1 A_1^{-1} B_1 = 0$ then $\mu(L(\lambda)) = -dr + \mu(G(\lambda))$.

(iii) If $n = 0$ and $D_1 = 0$ then $\mu(L(\lambda)) = 0$ and, by Lemmas 5.7 and 5.8, $\mu(G(\lambda)) = dr$. □

Example 5.10. We show that, certainly, the previous result is satisfied for the matrices in Example 5.4. It was proved that the matrices $L_{\epsilon,\eta}(\lambda)$ are strong linearizations of $G(\lambda) = \left[\begin{smallmatrix} \lambda+\lambda^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Notice that, under the same notation as above, $r = 1$, $d = 1$, $n = 1$, $s = \epsilon + \eta$, $A_1 = 1, B_1 = 0, C_1 = 0$ and $D_1 + C_1 A_1^{-1} B_1 \neq 0$. As we proved $\mu(G(\lambda)) = 0$ and $\mu(L_{\epsilon,\eta}(\lambda)) = \epsilon + \eta$. Thus, $\mu(G(\lambda)) = \mu(L_{\epsilon,\eta}(\lambda)) + dr - (r + s)$, as claimed.

6. Minimal bases and indices of strong block minimal bases linearizations of rational matrices

The aim of this section is to study the relationship between the minimal bases and indices of a rational matrix and the minimal bases and indices of its strong block minimal bases linearizations. This family of strong linearizations is a rather general family introduced in [6, Theorem 5.11]. It will be shown in Section 8 that the families of Fiedler-like linearizations of rational matrices introduced in [1,3,12,14] are, modulo permutations, particular instances of strong block minimal bases linearizations. Actually, this is a consequence of the corresponding results for polynomial matrices in [11] and [19, Lemma 2.7]. Moreover, the strong block minimal bases linearizations are closely connected to those introduced in [19]. In contrast to Fiedler-like linearizations, affine spaces of linearizations [13] and the linearizations in [19], which are only defined for square rational matrices, strong block minimal bases linearizations are valid for general rectangular rational matrices. Strong block minimal bases linearizations of rational matrices are built on strong block minimal bases linearizations of polynomial matrices, presented previously in [18, Definition 3.1] (see [17] for an expanded version of this latter reference). In order to introduce these families of linearizations and prove the results in this section, we need to recall first a number of concepts in the next paragraphs.

A matrix polynomial $N(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$ with $m < l$ is a minimal basis if the columns of $N(\lambda)^T$ form a minimal basis of the subspace they span. Moreover, two matrix polynomials $K(\lambda) \in \mathbb{F}[\lambda]^{m_1\times l}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{m_2\times l}$ are dual minimal bases if they are both minimal bases satisfying $m_1 + m_2 = l$ and $K(\lambda)N(\lambda)^T = 0$ (see [18,26]).

Let us recall the definition of strong block minimal bases pencils associated to a polynomial matrix (see [18, Definition 3.1 and Theorem 3.3] or [6, Definition 5.2]). Let $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be a polynomial matrix. A strong block minimal bases pencil associated to $P(\lambda)$ is a linear polynomial matrix with the following structure

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{matrix} \} \, p+\widehat{p} \\ \} \, \widehat{m} \end{matrix}, \qquad (12)$$

$$\underbrace{}_{m+\widehat{m}} \quad \underbrace{}_{\widehat{p}}$$

where $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ (respectively $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$) is a minimal basis with all its row degrees equal to 1 and with the row degrees of a minimal basis $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ (respectively $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$) dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$) all equal, and such that

$$P(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T. \tag{13}$$

If, in addition, $\deg(P(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ then $\mathcal{L}(\lambda)$ is said to be a strong block minimal bases pencil associated to $P(\lambda)$ with sharp degree. The key property is that any strong block minimal bases pencil associated to $P(\lambda)$ is a strong linearization of $P(\lambda)$ [18, Theorem 3.3].

Let $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ be the unique decomposition of $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ and its strictly proper part $G_{sp}(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times m}$, and let $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal order state-space realization of $G_{sp}(\lambda)$ with $n = \nu(G(\lambda))$. Assume[4] that $\deg(D(\lambda)) > 1$ and let (12) be a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree, with $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$ minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively, such that $D(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$. Let $\widehat{K}_1 \in \mathbb{F}^{m\times(m+\widehat{m})}$, $\widehat{N}_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$, $\widehat{K}_2 \in \mathbb{F}^{p\times(p+\widehat{p})}$ and $\widehat{N}_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$ be matrices such that for $i = 1, 2$

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix} \tag{14}$$

are unimodular (see in [6, Lemma 5.5] the result that guarantees that all these matrices exist and are well-defined). Let $T, S \in \mathbb{F}^{n\times n}$ be any nonsingular constant matrices. By [6, Theorem 5.11] the linear polynomial matrix

$$L(\lambda) = \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CS & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \tag{15}$$

is a strong linearization of $G(\lambda)$ and is called strong block minimal bases linearization of $G(\lambda)$.

Furthermore, by [6, Theorem 5.7], there are matrices $X(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times m}$ ($X(\lambda) = \widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T$), $Y(\lambda) \in \mathbb{F}[\lambda]^{p\times\widehat{m}}$ ($Y(\lambda) = N_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T$), and $Z(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times\widehat{m}}$ ($Z(\lambda) = \widehat{N}_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T$) such that

$$V(\lambda) = \begin{bmatrix} N_1(\lambda)^T & \widehat{N}_1(\lambda)^T & 0 \\ -X(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix} \quad \text{and} \quad U(\lambda) = \begin{bmatrix} N_2(\lambda) & -Y(\lambda) \\ 0 & I_{\widehat{m}} \\ \widehat{N}_2(\lambda) & -Z(\lambda) \end{bmatrix} \tag{16}$$

[4] If $\deg(D(\lambda)) \leq 1$, then the polynomial system matrix $\left[\begin{smallmatrix} \lambda I_n - A & B \\ -C & D(\lambda) \end{smallmatrix}\right]$ with transfer function matrix $G(\lambda)$ gives directly a strong linearization of $G(\lambda)$, as discussed in [6], and the idea of strong block minimal bases linearizations is of no interest.

are unimodular matrices and

$$U(\lambda)\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} V(\lambda) = \mathrm{Diag}(D(\lambda), I_{\widehat{m}+\widehat{p}}),$$

as can be easily checked through a direct matrix multiplication. Moreover, $U(\lambda)\begin{bmatrix} -\widehat{K}_2^T CS \\ 0 \end{bmatrix} = \begin{bmatrix} -CS \\ 0 \end{bmatrix}$ and $\begin{bmatrix} TB\widehat{K}_1 & 0 \end{bmatrix} V(\lambda) = \begin{bmatrix} TB & 0 \end{bmatrix}$. Thus,

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & U(\lambda) \end{bmatrix} L(\lambda) \begin{bmatrix} S^{-1} & 0 \\ 0 & V(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda I_n - A & B & 0 \\ -C & D(\lambda) & 0 \\ 0 & 0 & I_{\widehat{m}+\widehat{p}} \end{bmatrix}.$$

Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$, i.e.,

$$\widehat{G}(\lambda) = \begin{bmatrix} M(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B\widehat{K}_1 & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}. \tag{17}$$

Taking into account the developments above, a straightforward computation yields

$$U(\lambda)\widehat{G}(\lambda)V(\lambda) = \mathrm{Diag}(G(\lambda), I_{\widehat{m}+\widehat{p}}), \tag{18}$$

which implies, among other properties, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$ and $\dim \mathcal{N}_\ell(\widehat{G}(\lambda)) = \dim \mathcal{N}_\ell(G(\lambda))$, in agreement with the properties of any (strong) linearization of $G(\lambda)$.

In order to investigate the relationship between the minimal bases and indices of a rational matrix and those of its strong block minimal bases linearizations, we prove Lemma 6.1. This lemma first establishes the relationship between vectors in the right null-space of the rational matrix and in the right null-spaces of the transfer functions of any of its strong block minimal bases linearizations. Secondly, it relates the right minimal bases of the rational matrix and those of the transfer functions of its strong block minimal bases linearizations. Lemma 6.1 is based on [18, Lemma A.1], which is a similar result corresponding to strong block minimal bases pencils of polynomial matrices.

Lemma 6.1. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $\widehat{G}(\lambda)$ be its transfer function matrix, as in (17). Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$ and let $\widehat{N}_2(\lambda)$ be the matrix in (14).*

(a) *If $h(\lambda) \in \mathcal{N}_r(G(\lambda))$ then*

$$z(\lambda) = \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h(\lambda) \in \mathcal{N}_r(\widehat{G}(\lambda)).$$

Moreover, if $0 \neq h(\lambda) \in \mathcal{N}_r(G(\lambda))$ is a vector polynomial then $z(\lambda)$ is also a vector polynomial and

$$\deg(z(\lambda)) = \deg(N_1(\lambda)^T h(\lambda)) = \deg(N_1(\lambda)) + \deg(h(\lambda)). \tag{19}$$

(b) *If* $\{h_1(\lambda), \ldots, h_l(\lambda)\}$ *is a right minimal basis of* $G(\lambda)$ *then*

$$\left\{ \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_1(\lambda), \ldots, \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_l(\lambda) \right\}$$

is a right minimal basis of $\widehat{G}(\lambda)$.

Proof. By Proposition 4.1, equation (18) and using the structure of $V(\lambda)$ in (16) (recall that $X(\lambda) = \widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T$) we obtain the first part of (a). Now, we are going to prove (19) following the ideas of [18, Lemma A.1]. It can be seen as in the proof of [18, Lemma A.1] that for any vector polynomial $g(\lambda) \neq 0$

$$\deg(N_1(\lambda)^T g(\lambda)) = \deg(N_1(\lambda)) + \deg(g(\lambda)), \tag{20}$$

for any vector polynomial $y(\lambda) \neq 0$

$$\deg(K_2(\lambda)^T y(\lambda)) = \deg(K_2(\lambda)) + \deg(y(\lambda)) = 1 + \deg(y(\lambda)), \tag{21}$$

and

$$\deg(z(\lambda)) = \max\{\deg(N_1(\lambda)^T h(\lambda)), \deg(X(\lambda)h(\lambda))\}. \tag{22}$$

If $X(\lambda)h(\lambda) = 0$ then (19) follows. Otherwise, use $0 = \widehat{G}(\lambda)z(\lambda)$ and consider the expression of $\widehat{G}(\lambda)$ in (17)

$$\begin{aligned} 0 &= \begin{bmatrix} M(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B \widehat{K}_1 & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} N_1(\lambda)^T \\ -X(\lambda) \end{bmatrix} h(\lambda) \\ &= \begin{bmatrix} M(\lambda)N_1(\lambda)^T + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B - K_2(\lambda)^T X(\lambda) \\ 0 \end{bmatrix} h(\lambda). \end{aligned}$$

Therefore, $M(\lambda)N_1(\lambda)^T h(\lambda) - K_2(\lambda)^T X(\lambda)h(\lambda) = -\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)$. Since the expression on the left hand side of this equality is polynomial, the expression on the right hand side must be polynomial. Moreover, by Lemma 3.3, $\deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) < \deg(h(\lambda))$ since $\widehat{K}_2^T C(\lambda I_n - A)^{-1} B$ is strictly proper. Write the previous expression as

$$K_2(\lambda)^T X(\lambda)h(\lambda) = M(\lambda)N_1(\lambda)^T h(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda).$$

Notice that (21) implies that

$$1 + \deg(X(\lambda)h(\lambda)) = \deg(M(\lambda)N_1(\lambda)^T h(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)).$$

By using the previous expression, we now show

$$\deg(X(\lambda)h(\lambda)) \leq \deg(N_1(\lambda)^T h(\lambda)). \tag{23}$$

If $\deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) \leq \deg(M(\lambda) N_1(\lambda)^T h(\lambda))$ then

$$1 + \deg(X(\lambda)h(\lambda)) \leq \deg(M(\lambda)N_1(\lambda)^T h(\lambda)) \leq 1 + \deg(N_1(\lambda)^T h(\lambda)).$$

Otherwise, if $\deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) > \deg(M(\lambda) N_1(\lambda)^T h(\lambda))$ then

$$1 + \deg(X(\lambda)h(\lambda)) = \deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) < \deg(h(\lambda)) \text{ and}$$
$$\deg(X(\lambda)h(\lambda)) < \deg(h(\lambda)) - 1 < \deg(h(\lambda)) + \deg(N_1(\lambda)) = \deg(N_1(\lambda)^T h(\lambda)).$$

Therefore, (20), (22) and (23) prove that $\deg(z(\lambda)) = \deg(N_1(\lambda)) + \deg(h(\lambda))$.

The proof of part (b) is similar to the proof of [18, Lemma A.1] taking into account that $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$. Therefore, the details are omitted. □

As a corollary of Lemma 6.1 we get the following result on the relationship between the minimal indices of a rational matrix and of the transfer function of any of its strong block minimal bases linearizations.

Corollary 6.2. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $\widehat{G}(\lambda)$ be its transfer function matrix, as in (17). Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$ and $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$.*

(a) *If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_l + \deg(N_1(\lambda))$ are the right minimal indices of $\widehat{G}(\lambda)$.*
(b) *If $\eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $G(\lambda)$ then $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the left minimal indices of $\widehat{G}(\lambda)$.*

Proof. Part (a) follows from part (b) of Lemma 6.1 and (19). Suppose now that $\eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $G(\lambda)$. By Lemma 3.5, $\eta_1 \leq \cdots \leq \eta_q$ are the right minimal indices of $G(\lambda)^T$. Notice that $L(\lambda)^T$ is a strong block minimal bases linearization of $G(\lambda)^T$ with transfer function matrix $\widehat{G}(\lambda)^T$. Observe that S^T, A^T, T^T, B^T, C^T, $\widehat{K}_1$, $\widehat{K}_2$, $M(\lambda)^T$, $K_1(\lambda)$, $K_2(\lambda)$ in $L(\lambda)^T$ play the role of T, A, S, $-C$, $-B$, $\widehat{K}_2$, $\widehat{K}_1$, $M(\lambda)$, $K_2(\lambda)$, $K_1(\lambda)$ in $L(\lambda)$ respectively. In particular, $K_2(\lambda)$ in $L(\lambda)^T$ plays the role of $K_1(\lambda)$ in $L(\lambda)$. Thus, by part (a), $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the right minimal indices of $\widehat{G}(\lambda)^T$. By Lemma 3.5 again, $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the left minimal indices of $\widehat{G}(\lambda)$. □

Now, we provide a recovery result for the minimal bases of a rational matrix from the minimal bases of the transfer functions of any of its strong block minimal bases linearizations, i.e., the converse of Lemma 6.1-(b).

Lemma 6.3. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $\widehat{G}(\lambda)$ be its transfer function matrix, as in (17). Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$, $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$ and $\widehat{N}_1(\lambda)$ and $\widehat{N}_2(\lambda)$ be the matrices appearing in (14).*

(a) *Any right minimal basis of $\widehat{G}(\lambda)$ has the form*

$$\left\{ \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_1(\lambda), \ldots, \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_l(\lambda) \right\}$$

where $\{h_1(\lambda), \ldots, h_l(\lambda)\}$ is some right minimal basis of $G(\lambda)$.

(b) *Any left minimal basis of $\widehat{G}(\lambda)$ has the form*

$$\left\{ \begin{bmatrix} N_2(\lambda)^T \\ -\widehat{N}_1(\lambda)M(\lambda)^T N_2(\lambda)^T \end{bmatrix} j_1(\lambda), \ldots, \begin{bmatrix} N_2(\lambda)^T \\ -\widehat{N}_1(\lambda)M(\lambda)^T N_2(\lambda)^T \end{bmatrix} j_q(\lambda) \right\}$$

where $\{j_1(\lambda), \ldots, j_q(\lambda)\}$ is some left minimal basis of $G(\lambda)$.

Proof. The proof is like the one of [17, Lemma 7.1]. Therefore, it is omitted. □

Remark 6.4. Lemma 6.3 implies that a right (resp., left) minimal basis of $G(\lambda)$ can be obtained, or recovered, from any right (resp., left) minimal basis of $\widehat{G}(\lambda)$, as it is described in this remark. Let us focus for brevity only on right minimal bases, since the procedure for left minimal bases is completely analogous. Note first that the vectors $\{\widehat{h}_1(\lambda), \ldots, \widehat{h}_l(\lambda)\}$ obtained by taking the top $m + \widehat{m}$ entries of the vectors of any right minimal basis of $\widehat{G}(\lambda)$ are always of the form

$$\{\widehat{h}_1(\lambda), \ldots, \widehat{h}_l(\lambda)\} = \{N_1(\lambda)^T h_1(\lambda), \ldots, N_1(\lambda)^T h_l(\lambda)\}, \tag{24}$$

with $\{h_1(\lambda), \ldots, h_l(\lambda)\}$ a right minimal basis of $G(\lambda)$. Then, it is enough to multiply each $\widehat{h}_j(\lambda)$ by a left inverse of $N_1(\lambda)^T$ in order to get the right minimal basis $\{h_1(\lambda), \ldots, h_l(\lambda)\}$ of $G(\lambda)$. Such left inverse may be, for instance, the matrix $\widehat{K}_1$ in (14). Moreover, in some cases important in applications, the matrices $N_1(\lambda)$ and $\widehat{K}_1$ are very simple and allow us to recover a right minimal basis of $G(\lambda)$ without the need of performing any matrix multiplication. This happens, for instance, if $K_1(\lambda) = L_\varepsilon(\lambda) \otimes I_m$ (and $K_2(\lambda) = L_\eta(\lambda) \otimes I_p$) in (15), where

$$L_k(\lambda) = \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)}, \tag{25}$$

which corresponds to the well-known block Kronecker linearizations of the polynomial part of $G(\lambda)$ [18, Section 4] (see also [6, Examples 5.3 and 5.6]). In this case,

$$N_1(\lambda)^T = \begin{bmatrix} \lambda^\varepsilon \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \otimes I_m \quad \text{and} \quad \widehat{K}_1 = [0 \quad \cdots \quad 0 \quad 1] \otimes I_m.$$

Thus a minimal bases of $G(\lambda)$ can be obtained just by taking the last m entries of the vectors $\{\widehat{h}_1(\lambda), \ldots, \widehat{h}_l(\lambda)\}$ in (24).

The next Theorem 6.5 is the main result in this section, together with Theorem 6.7, and one of the most relevant results in this paper. Theorem 6.5 describes the complete relationship between the minimal bases of a rational matrix and the minimal bases of its strong block minimal bases linearizations in both directions. It follows from combining results in Section 3 with results previously obtained in this section.

Theorem 6.5. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$, $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$ and $\widehat{N}_1(\lambda)$ and $\widehat{N}_2(\lambda)$ be the matrices appearing in (14).*

(a) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ *is a right minimal basis of $L(\lambda)$ if and only if*

$$\begin{aligned} H_1(\lambda) &= -S^{-1}(\lambda I_n - A)^{-1}BH(\lambda), \\ H_2(\lambda) &= N_1(\lambda)^T H(\lambda), \\ H_3(\lambda) &= -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T H(\lambda) \end{aligned}$$

for some right minimal basis $H(\lambda)$ of $G(\lambda)$.

(b) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ *is a left minimal basis of $L(\lambda)$ if and only if*

$$\begin{aligned} H_1(\lambda) &= (C(\lambda I_n - A)^{-1}T^{-1})^T H(\lambda), \\ H_2(\lambda) &= N_2(\lambda)^T H(\lambda), \\ H_3(\lambda) &= -\widehat{N}_1(\lambda)M(\lambda)^T N_2(\lambda)^T H(\lambda) \end{aligned}$$

for some left minimal basis $H(\lambda)$ of $G(\lambda)$.

Proof. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. Notice that both $(T(\lambda I_n - A)S)^{-1}\begin{bmatrix} TB\widehat{K}_1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -\widehat{K}_2^T CS \\ 0 \end{bmatrix}(T(\lambda I_n - A)S)^{-1}$ are strictly proper matrices. By Corollary 3.9, $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ is a right minimal basis of $L(\lambda)$ if and only if $\begin{bmatrix} H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ is a right minimal basis of $\widehat{G}(\lambda)$ and $H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}B\widehat{K}_1 H_2(\lambda)$. Now, by Lemma 6.3, $H_2(\lambda) = N_1(\lambda)^T H(\lambda)$ and $H_3(\lambda) = -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T H(\lambda)$ for some $H(\lambda)$ right minimal basis of $G(\lambda)$.

Part (b) is proved similarly. □

Remark 6.6. Theorem 6.5 implies that a right (resp., left) minimal basis of $G(\lambda)$ can be recovered from any right (resp., left) minimal basis of any of its strong block minimal bases linearizations. Such recovery procedure is completely analogous to the one described in Remark 6.4 except for the following minor variation: in the case of Theorem 6.5 the right (resp., left) minimal bases of $G(\lambda)$ have to be recovered from the entries $n+1, n+2, \ldots, n+m+\widehat{m}$ (resp., $n+1, n+2, \ldots, n+p+\widehat{p}$) of the vectors of the right (resp., left) minimal bases of its strong block minimal bases linearizations. As in Remark 6.4, the recovery is extremely simple for strong block minimal bases linearizations of $G(\lambda)$ constructed from a block Kronecker linearization of its polynomial part.

In the last result of this section, the relationship between the minimal indices of a rational matrix and those of its strong block minimal bases linearizations is established.

Theorem 6.7. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$ and $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$.*

(a) *If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_l + \deg(N_1(\lambda))$ are the right minimal indices of $L(\lambda)$.*
(b) *If $\eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $G(\lambda)$ then $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the left minimal indices of $L(\lambda)$.*

Proof. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then, by Corollary 6.2, $\varepsilon_1 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_l + \deg(N_1(\lambda))$ are the right minimal indices of $\widehat{G}(\lambda)$. Now, by Theorem 3.6, these are the right minimal indices of $L(\lambda)$.

A similar proof can be done in order to prove (b). □

7. Minimal bases and indices of $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices

$\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of square rational matrices have been recently introduced in [19] by combining results from [6] with the $\mathbb{M}_1$ and $\mathbb{M}_2$ ansatz spaces of linearizations of a polynomial matrix developed in [21], which in turn are inspired by the pioneer $\mathbb{L}_1$ and $\mathbb{L}_2$ vector spaces of linearizations of matrix polynomials introduced in [30]. Among other properties, $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices allow us to deal very easily with rational matrices whose polynomial part is expressed in any orthogonal basis. In this section, we study the minimal bases and indices of $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices. Since these families of linearizations are closely connected to strong block minimal bases linearizations, it is not surprising that the results of this section are easily obtained from combining those in Section 6 with

specific properties of $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations. Besides, we extend the results on minimal indices and bases to the strong linearizations of rational matrices whose polynomial part is expressed in other polynomial bases, not only orthogonal. These were developed in [19, Section 9]. In order to proceed, we need to recap first some results and notations taken from [19].

The following lemma establishes a general result about the relationship between the minimal bases and indices of two rational matrices connected by a nonsingular constant matrix on the left. We will see that this simple result will allow us to obtain the relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_1$, $\mathbb{M}_2$-strong linearizations. The reason is that an $\mathbb{M}_1$-strong linearization is a strong block minimal bases linearization premultiplied by a nonsingular constant matrix, and an $\mathbb{M}_2$-strong linearization is a strong block minimal bases linearization postmultiplied by a nonsingular constant matrix.

Lemma 7.1. *Let $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $X \in \mathbb{F}^{p\times p}$ be nonsingular such that $G_2(\lambda) = XG_1(\lambda)$. Then, $H(\lambda)$ is a right minimal basis of $G_1(\lambda)$ if and only if $H(\lambda)$ is a right minimal basis of $G_2(\lambda)$ and $\overline{H}(\lambda)$ is a left minimal basis of $G_1(\lambda)$ if and only if $X^{-T}\overline{H}(\lambda)$ is a left minimal basis of $G_2(\lambda)$. Moreover, $G_1(\lambda)$ and $G_2(\lambda)$ have the same right minimal indices and the same left minimal indices.*

Proof. Notice that $G_1(\lambda)H(\lambda) = 0$ if and only if $G_2(\lambda)H(\lambda) = 0$. Moreover, by Lemma 3.5, $\overline{H}(\lambda)$ is a left minimal basis of $G_1(\lambda)$ if and only if $\overline{H}(\lambda)$ is a right minimal basis of $G_1(\lambda)^T$. Furthermore, $G_1(\lambda)^T\overline{H}(\lambda) = 0$ if and only if $G_2(\lambda)^T X^{-T}\overline{H}(\lambda) = 0$ and, by [16, Lemma 2.16], $X^{-T}\overline{H}(\lambda)$ is a minimal basis with the same column degrees as $\overline{H}(\lambda)$. Therefore, $\overline{H}(\lambda)$ is a right minimal basis of $G_1(\lambda)^T$ if and only if $X^{-T}\overline{H}(\lambda)$ is a right minimal basis of $G_2(\lambda)^T$ and, by Lemma 3.5 again, $X^{-T}\overline{H}(\lambda)$ is a left minimal basis of $G_2(\lambda)$. □

The definitions of the $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations introduced in Subsections 7.1 and 7.3 are based on the matrices and vectors presented in the next paragraphs. Consider a polynomial basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ of $\mathbb{F}[\lambda]$, viewed as an $\mathbb{F}$-vector space, with $\phi_j(\lambda)$ a polynomial of degree j, that satisfies the following three-term recurrence relation:

$$\alpha_j\phi_{j+1}(\lambda) = (\lambda - \beta_j)\phi_j(\lambda) - \gamma_j\phi_{j-1}(\lambda) \quad j \geq 0 \tag{26}$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}$, $\alpha_j \neq 0$, $\phi_{-1}(\lambda) = 0$, and $\phi_0(\lambda) = 1$.

Let $G(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix, let $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and its strictly proper part $G_{sp}(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$, and let $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal order state-space realization of $G_{sp}(\lambda)$, where $n = \nu(G(\lambda))$. Assume that $\deg(D(\lambda)) \geq 2$. Write $D(\lambda)$ in terms of the polynomial basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$, as

$$D(\lambda) = D_k\phi_k(\lambda) + D_{k-1}\phi_{k-1}(\lambda) + \cdots + D_1\phi_1(\lambda) + D_0\phi_0(\lambda) \tag{27}$$

with $D_k \neq 0$. Let

$$\Phi_k(\lambda) = [\phi_{k-1}(\lambda) \cdots \phi_1(\lambda)\ \phi_0(\lambda)]^T, \tag{28}$$

$$m_\Phi^D(\lambda) = \begin{bmatrix} \dfrac{(\lambda-\beta_{k-1})}{\alpha_{k-1}} D_k + D_{k-1} & D_{k-2} - \dfrac{\gamma_{k-1}}{\alpha_{k-1}} D_k & D_{k-3} & \cdots & D_1 & D_0 \end{bmatrix},$$

$$M_\Phi(\lambda) = \begin{bmatrix} -\alpha_{k-2} & (\lambda-\beta_{k-2}) & -\gamma_{k-2} & & & \\ & -\alpha_{k-3} & (\lambda-\beta_{k-3}) & -\gamma_{k-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 & (\lambda-\beta_1) & -\gamma_1 \\ & & & & -\alpha_0 & (\lambda-\beta_0) \end{bmatrix},$$

$$F_\Phi^D(\lambda) = \begin{bmatrix} m_\Phi^D(\lambda) \\ M_\Phi(\lambda) \otimes I_m \end{bmatrix}. \tag{29}$$

7.1. Minimal bases and indices of $\mathbb{M}_1$-strong linearizations of rational matrices

We investigate first the relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_1$-strong linearizations.

For any nonsingular constant matrices $T, S \in \mathbb{F}^{n\times n}$ the linear polynomial matrix

$$\begin{aligned} L(\lambda) &= \left[\begin{array}{c|c} T(\lambda I_n - A)S & 0_{n\times(k-1)m} \quad TB \\ \hline -CS & m_\Phi^D(\lambda) \\ 0_{(k-1)m\times n} & M_\Phi(\lambda)\otimes I_m \end{array}\right] \\ &= \left[\begin{array}{c|c} T(\lambda I_n - A)S & 0_{n\times(k-1)m} \quad TB \\ \hline -CS & \\ 0_{(k-1)m\times n} & F_\Phi^D(\lambda) \end{array}\right] \end{aligned} \tag{30}$$

is a strong linearization of $G(\lambda)$ (see [19, Theorem 3.8]). Taking into account also [19, Lemmas 3.3 and 3.7] and their proofs, note that $L(\lambda)$ is a strong block minimal bases linearization of $G(\lambda)$ as in (15) with $M(\lambda) = m_\Phi^D(\lambda)$, $K_1(\lambda) = M_\Phi(\lambda)\otimes I_m$, $K_2(\lambda)$ empty, $N_1(\lambda) = (\Phi_k(\lambda)\otimes I_m)^T = \Phi_k(\lambda)^T \otimes I_m$, $N_2(\lambda) = I_m$, $\widehat{K}_1 = e_k^T \otimes I_m$, $\widehat{K}_2 = I_m$ and $\widehat{N}_2(\lambda)$ empty, where e_k is the kth canonical vector of size $k\times 1$. Moreover, $\widehat{N}_1(\lambda)$ is of the form $\widehat{N}_1(\lambda) = Q(\lambda)^T \otimes I_m$ with $Q(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$.

Furthermore, let $v \in \mathbb{F}^k$, $J \in \mathbb{F}^{km\times(k-1)m}$ with $[v\otimes I_m \quad J]$ nonsingular and let $\mathcal{L}(\lambda) = [v\otimes I_m \quad J]F_\Phi^D(\lambda)$. Then, the linear polynomial matrix

$$\begin{aligned} L_1(\lambda) &= \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & v\otimes I_m \quad J \end{array}\right] \left[\begin{array}{c|c} T(\lambda I_n - A)S & 0_{n\times(k-1)m} \quad TB \\ \hline -CS & m_\Phi^D(\lambda) \\ 0_{(k-1)m\times n} & M_\Phi(\lambda)\otimes I_m \end{array}\right] \\ &= \left[\begin{array}{c|c} T(\lambda I_n - A)S & 0_{n\times(k-1)m} \quad TB \\ \hline -(v\otimes I_m)CS & \mathcal{L}(\lambda) \end{array}\right] \end{aligned} \tag{31}$$

is a strong linearization of $G(\lambda)$, which is called $\mathbb{M}_1$-strong linearization of $G(\lambda)$ (see [19, Theorem 3.9]). Put $X = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & v \otimes I_m \quad J \end{array}\right]$, which is nonsingular. Thus, $L_1(\lambda) = XL(\lambda)$.

With all these results at hand, Theorem 7.2 establishes the relationships between the minimal bases and indices of a rational matrix and its $\mathbb{M}_1$-strong linearizations.

Theorem 7.2. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ and let $L_1(\lambda)$ as in (31) be an $\mathbb{M}_1$-strong linearization of $G(\lambda)$. Let $\Phi_k(\lambda)$ be as in (28).*

(a) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix}$ *is a right minimal basis of $L_1(\lambda)$ if and only if $H_{k+1}(\lambda)$ is a right minimal basis of $G(\lambda)$ and*

$$H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}BH_{k+1}(\lambda),$$
$$H_i(\lambda) = \phi_{k-i+1}(\lambda)H_{k+1}(\lambda),\ i = 2, \ldots, k.$$

(b) *If $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a left minimal basis of $L_1(\lambda)$ then $(v^T \otimes I_m)H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda I_n - A)^{-1}T^{-1})^T(v^T \otimes I_m)H_2(\lambda)$.*

(c) *If $H(\lambda)$ is a left minimal basis of $G(\lambda)$ then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a left minimal basis of $L_1(\lambda)$ where*

$$H_1(\lambda) = (C(\lambda I_n - A)^{-1}T^{-1})^T H(\lambda),$$
$$H_2(\lambda) = [v \otimes I_m \quad J]^{-T} \begin{bmatrix} H(\lambda) \\ -\widehat{N}_1(\lambda)m_\Phi^D(\lambda)^T H(\lambda) \end{bmatrix}$$

with $\widehat{N}_1(\lambda) = Q(\lambda)^T \otimes I_m$ such that $Q(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$.

(d) *If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 + k - 1 \leq \cdots \leq \varepsilon_l + k - 1$ are the right minimal indices of $L_1(\lambda)$.*

(e) *If $\eta_1 \leq \cdots \leq \eta_l$ are the left minimal indices of $G(\lambda)$ then $\eta_1 \leq \cdots \leq \eta_l$ are the left minimal indices of $L_1(\lambda)$.*

Proof. To prove (a), by using Lemma 7.1, we get that $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix}$ is a right minimal basis of $L_1(\lambda)$ if and only if it is a right minimal basis of $L(\lambda)$ in (30). By the fact that $L(\lambda)$ is a strong block minimal basis linearization of $G(\lambda)$ and Theorem 6.5 (with $\widehat{N}_2(\lambda)$ empty), this occurs if and only if $H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}BH(\lambda)$ and $\begin{bmatrix} H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix} = (\Phi_k(\lambda) \otimes$

$I_m)H(\lambda)$ for some right minimal basis $H(\lambda)$ of $G(\lambda)$. But, since $\phi_0(\lambda) = 1$, $H(\lambda)$ is uniquely determined as $H(\lambda) = H_{k+1}(\lambda)$, and $H_i(\lambda) = \phi_{k-i+1}(\lambda)H_{k+1}(\lambda)$, $i = 2, \ldots, k$.

The proof of the other parts can be done similarly by using Lemma 7.1 and Theorems 6.5 or 6.7. Observe that in this case $G(\lambda)$ is square and, therefore, it has a number of left minimal indices equal to the number of right minimal indices. □

Remark 7.3. Part (a) of Theorem 7.2, together with the fact that $\phi_0(\lambda) = 1$, provides a very simple recovery rule of a right minimal basis of $G(\lambda)$ from any right minimal basis of any of its $\mathbb{M}_1$-strong linearizations: simply take the last m rows of the right minimal basis of the $\mathbb{M}_1$-strong linearization. Part (b) of Theorem 7.2 also provides a simple recovery rule of a left minimal basis of $G(\lambda)$ from any left minimal basis of any of its $\mathbb{M}_1$-strong linearizations, though in this case some arithmetic operations are required unless v is one of the canonical vectors of $\mathbb{F}^k$.

7.2. Minimal bases and indices of extended $\mathbb{M}_1$-strong linearizations of rational matrices

Similar ideas and techniques to those in Subsection 7.1 can be used to obtain a result similar to Theorem 7.2 for the strong linearizations of rational matrices with polynomial part expressed in any degree-graded polynomial basis, i.e., one whose jth element has degree j. These strong linearizations were introduced in [19, Section 9]. We call such linearizations extended $\mathbb{M}_1$-strong linearizations. More precisely, suppose as in [19, Section 9] that $G(\lambda) = D(\lambda)+C(\lambda I_n - A)^{-1}B$ with $\deg(D(\lambda)) = k \geq 2$ and consider a polynomial basis $\{\psi_j(\lambda)\}_{j=0}^{\infty}$ of $\mathbb{F}[\lambda]$, with $\psi_j(\lambda)$ a polynomial of degree j, that satisfies the linear relation $M_\Psi(\lambda)\Psi_k(\lambda) = 0$, where $M_\Psi(\lambda) \in \mathbb{F}[\lambda]^{(k-1)\times k}$ is a minimal basis with all its row degrees equal to 1, and

$$\Psi_k(\lambda) = [\psi_{k-1}(\lambda) \ \cdots \ \psi_1(\lambda) \ \psi_0(\lambda)]^T. \tag{32}$$

Note that $\Psi_k(\lambda_0) \neq 0$ for all $\lambda_0 \in \overline{\mathbb{F}}$ because $\psi_0(\lambda)$ has degree 0. Example 9.2 in [19] shows how to construct $M_\Psi(\lambda)$ from a recurrence relation that holds under the assumption that the polynomials in the basis are monic. Let $m_\Psi^D(\lambda)$ be a pencil such that $m_\Psi^D(\lambda)(\Psi_k(\lambda)\otimes I_m) = D(\lambda)$ (see again [19, Example 9.2] for an explicit construction). By [6, Lemma 5.5], there exists a vector $w \in \mathbb{F}^k$ such that $U(\lambda) = \begin{bmatrix} M_\Psi(\lambda) \\ w^T \end{bmatrix}$ is unimodular, and its inverse has the form $U(\lambda)^{-1} = [R(\lambda) \quad \Psi_k(\lambda)]$ with $R(\lambda) = \begin{bmatrix} M_\Psi(\lambda) \\ w^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$. Note that in the construction of [19, Example 9.2], one can take simply $w^T = e_k^T = [0 \ \cdots \ 0 \ 1]^T$. Let $v \in \mathbb{F}^k$, $[v\otimes I_m \quad J]$ nonsingular, and $\mathcal{L}(\lambda) = [v\otimes I_m \quad J] \begin{bmatrix} m_\Psi^D(\lambda) \\ M_\Psi(\lambda) \otimes I_m \end{bmatrix}$. By [19, Theorem 9.1], for any T and S nonsingular,

$$
\begin{aligned}
L_1(\lambda) &= \left[\begin{array}{c|cc} I_n & 0 & \\ \hline 0 & v\otimes I_m & J \end{array}\right] \left[\begin{array}{c|c} T(\lambda I_n - A)S & TB(w^T\otimes I_m) \\ \hline -CS & m_\Psi^D(\lambda) \\ 0_{(k-1)m\times n} & M_\Psi(\lambda)\otimes I_m \end{array}\right] \\
&= \left[\begin{array}{c|c} T(\lambda I_n - A)S & TB(w^T\otimes I_m) \\ \hline -(v\otimes I_m)CS & \mathcal{L}(\lambda) \end{array}\right]
\end{aligned} \tag{33}
$$

is a strong linearization of $G(\lambda)$, that we call extended $\mathbb{M}_1$-strong linearization. Moreover, the matrix on the right of the first equality is a strong block minimal bases linearization of $G(\lambda)$ as in (15) with $M(\lambda) = m_\Psi^D(\lambda)$, $K_1(\lambda) = M_\Psi(\lambda)\otimes I_m$, $K_2(\lambda)$ empty, $\widehat{K}_1 = w^T\otimes I_m$, and $\widehat{K}_2 = I_m$. Furthermore, $N_1(\lambda) = \Psi_k(\lambda)^T\otimes I_m$, $N_2(\lambda) = I_m$, $\widehat{N}_1(\lambda) = R(\lambda)^T\otimes I_m$, and $\widehat{N}_2(\lambda)$ is empty. With all this in mind, by using the same techniques as for the proof of Theorem 7.2, the following result is obtained.

Theorem 7.4. *Let $G(\lambda)\in\mathbb{F}(\lambda)^{m\times m}$ and let $L_1(\lambda)$ as in (33) be an extended $\mathbb{M}_1$-strong linearization of $G(\lambda)$. Let $\Psi_k(\lambda)$ be as in (32).*

(a) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix}$ *is a right minimal basis of $L_1(\lambda)$ if and only if*

$$
\begin{aligned}
H_1(\lambda) &= -S^{-1}(\lambda I_n - A)^{-1}BH(\lambda), \\
H_i(\lambda) &= \psi_{k-i+1}(\lambda)H(\lambda),\ \ i = 2,\dots,k+1
\end{aligned}
$$

for some right minimal basis $H(\lambda)$ of $G(\lambda)$.

(b) *If $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a left minimal basis of $L_1(\lambda)$ then $(v^T\otimes I_m)H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda I_n - A)^{-1}T^{-1})^T(v^T\otimes I_m)H_2(\lambda)$.*

(c) *If $H(\lambda)$ is a left minimal basis of $G(\lambda)$ then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a left minimal basis of $L_1(\lambda)$ where*

$$
\begin{aligned}
H_1(\lambda) &= (C(\lambda I_n - A)^{-1}T^{-1})^T H(\lambda), \\
H_2(\lambda) &= \begin{bmatrix} v\otimes I_m & J \end{bmatrix}^{-T} \begin{bmatrix} H(\lambda) \\ -\widehat{N}_1(\lambda)m_\Psi^D(\lambda)^T H(\lambda) \end{bmatrix}
\end{aligned}
$$

with $\widehat{N}_1(\lambda) = R(\lambda)^T\otimes I_m$ such that $R(\lambda) = \begin{bmatrix} M_\psi(\lambda) \\ w^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$.

(d) *If $\varepsilon_1\le\cdots\le\varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1+k-1\le\cdots\le\varepsilon_l+k-1$ are the right minimal indices of $L_1(\lambda)$.*

(e) *If $\eta_1\le\cdots\le\eta_l$ are the left minimal indices of $G(\lambda)$ then $\eta_1\le\cdots\le\eta_l$ are the left minimal indices of $L_1(\lambda)$.*

Note that in Theorem 7.4(a), $H_{k+1}(\lambda)$ itself is a right minimal basis of $G(\lambda)$ since $\psi_0(\lambda)$ has degree zero and is a constant.

7.3. Minimal bases and indices of $\mathbb{M}_2$-strong linearizations of rational matrices

We investigate now the relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_2$-strong linearizations. The developments and results are very similar to those in Subsection 7.1 and, therefore, are described briefly.

Let $Q(\lambda)$ be a $km\times lm$ pencil of the form $Q(\lambda) = \sum_{i=1}^{k}\sum_{j=1}^{l} e_i e_j^T \otimes Q_{ij}(\lambda)$ for certain $m\times m$ pencils $Q_{ij}(\lambda)$, and where e_i (resp., e_j) is the ith (resp., jth) canonical vector in $\mathbb{F}^k$ (resp., $\mathbb{F}^l$). The $lm \times km$ pencil $Q(\lambda)^{\mathcal{B}} = \sum_{i=1}^{k}\sum_{j=1}^{l} e_j e_i^T \otimes Q_{ij}(\lambda)$ is the block-transpose of $Q(\lambda)$. Notice that the block-transpose of $F_\Phi^D(\lambda)$ in (29) is $F_\Phi^D(\lambda)^{\mathcal{B}} = [m_\Phi^D(\lambda)^{\mathcal{B}} \quad M_\Phi(\lambda)^T \otimes I_m]$.

For any nonsingular constant matrices $T, S \in \mathbb{F}^{n\times n}$ the linear polynomial matrix

$$\begin{aligned}\mathbb{L}(\lambda) &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB & 0_{n\times(k-1)m} \\ \hline \begin{array}{c} 0_{(k-1)m\times n} \\ -CS \end{array} & m_\Phi^D(\lambda)^{\mathcal{B}} & M_\Phi(\lambda)^T \otimes I_m \end{array}\right] \\ &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB & 0_{n\times(k-1)m} \\ \hline \begin{array}{c} 0_{(k-1)m\times n} \\ -CS \end{array} & \multicolumn{2}{c}{F_\Phi^D(\lambda)^{\mathcal{B}}} \end{array}\right]\end{aligned} \tag{34}$$

is a strong linearization of $G(\lambda)$ (see [19, Theorem 4.3]). Notice that $\mathbb{L}(\lambda)$ is a strong block minimal bases linearization of $G(\lambda)$ as in (15) with $M(\lambda) = m_\Phi^D(\lambda)^{\mathcal{B}}$, $K_1(\lambda)$ empty, $K_2(\lambda) = M_\Phi(\lambda)\otimes I_m$, $N_1(\lambda) = I_m$, $N_2(\lambda) = (\Phi_k(\lambda)\otimes I_m)^T = \Phi_k(\lambda)^T \otimes I_m$, $\widehat{K}_1 = I_m$, $\widehat{K}_2 = e_k^T \otimes I_m$ and $\widehat{N}_1(\lambda)$ empty. Moreover, $\widehat{N}_2(\lambda) = Q(\lambda)^T \otimes I_m$ such that $Q(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$.

Furthermore, let $w \in \mathbb{F}^k$, $J \in \mathbb{F}^{km\times(k-1)m}$ with $\begin{bmatrix} w^T\otimes I_m \\ J^{\mathcal{B}} \end{bmatrix}$ nonsingular and $\mathcal{L}(\lambda) = F_\Phi^D(\lambda)^{\mathcal{B}}\begin{bmatrix} w^T\otimes I_m \\ J^{\mathcal{B}} \end{bmatrix}$. Then, the linear polynomial matrix

$$\begin{aligned}\mathbb{L}_2(\lambda) &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB & 0_{n\times(k-1)m} \\ \hline \begin{array}{c} 0_{(k-1)m\times n} \\ -CS \end{array} & m_\Phi^D(\lambda)^{\mathcal{B}} & M_\Phi(\lambda)^T \otimes I_m \end{array}\right] \left[\begin{array}{c|c} I_n & 0 \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} w^T\otimes I_m \\ J^{\mathcal{B}} \end{array} \end{array}\right] \\ &= \left[\begin{array}{c|c} T(\lambda I_n - A)S & TB(w^T\otimes I_m) \\ \hline \begin{array}{c} 0_{(k-1)m\times n} \\ -CS \end{array} & \mathcal{L}(\lambda) \end{array}\right]\end{aligned} \tag{35}$$

is a strong linearization of $G(\lambda)$, which is called $\mathbb{M}_2$-strong linearization of $G(\lambda)$ (see [19, Theorem 4.4]). Put $Y = \left[\begin{array}{c|c} I_n & 0 \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} w^T\otimes I_m \\ J^{\mathcal{B}} \end{array} \end{array}\right]$, which is nonsingular. Thus, $\mathbb{L}_2(\lambda) = \mathbb{L}(\lambda)Y$.

The relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_2$-strong linearizations is given in Theorem 7.5.

Theorem 7.5. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ and let $\mathbb{L}_2(\lambda)$ as in (35) be an $\mathbb{M}_2$-strong linearization of $G(\lambda)$. Let $\Phi_k(\lambda)$ be as in (28).*

(a) *If $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right minimal basis of $\mathbb{L}_2(\lambda)$ then $(w^T \otimes I_m)H_2(\lambda)$ is a right minimal basis of $G(\lambda)$ and $H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}B(w^T \otimes I_m)H_2(\lambda)$.*

(b) *If $H(\lambda)$ is a right minimal basis of $G(\lambda)$ then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}$ is a right minimal basis of $\mathbb{L}_2(\lambda)$ where*

$$H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}BH(\lambda),$$
$$H_2(\lambda) = \begin{bmatrix} w^T \otimes I_m \\ J^{\mathcal{B}} \end{bmatrix}^{-1} \begin{bmatrix} H(\lambda) \\ -\widehat{N}_2(\lambda) m_\Phi^D(\lambda)^{\mathcal{B}} H(\lambda) \end{bmatrix}$$

with $\widehat{N}_2(\lambda) = Q(\lambda)^T \otimes I_m$ such that $Q(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix}$ *is a left minimal basis of $\mathbb{L}_2(\lambda)$ if and only if $H_{k+1}(\lambda)$ is a left minimal basis of $G(\lambda)$ and*

$$H_1(\lambda) = (C(\lambda I_n - A)^{-1}T^{-1})^T H_{k+1}(\lambda),$$
$$H_i(\lambda) = \phi_{k-i+1}(\lambda) H_{k+1}(\lambda),\ i = 2, \ldots, k.$$

(d) *If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $\mathbb{L}_2(\lambda)$.*

(e) *If $\eta_1 \leq \cdots \leq \eta_l$ are the left minimal indices of $G(\lambda)$ then $\eta_1 + k - 1 \leq \cdots \leq \eta_l + k - 1$ are the left minimal indices of $\mathbb{L}_2(\lambda)$.*

Proof. The proof can be done by using Lemmas 3.5(a) and 7.1, and Theorems 6.5 and 6.7, and by following the same pattern as in the proof of Theorem 7.2. □

Remark 7.6. Comments similar to those in Remark 7.3 can be done in order to apply Theorem 7.5 to recover minimal bases of $G(\lambda)$ from those of any of its $\mathbb{M}_2$-strong linearizations. The only difference to be emphasized is that the roles of left and right minimal bases are interchanged in Theorems 7.2 and 7.5.

Results completely analogous after obvious modifications to those in Subsection 7.2 can be obtained in the $\mathbb{M}_2$-framework. They are omitted for brevity.

8. Fiedler-like linearizations of rational matrices are block permutations of strong block minimal bases linearizations. Consequences

Fiedler pencils (FP), generalized Fiedler pencils (GFP), Fiedler pencils with repetition (FPR) and generalized Fiedler pencils with repetition (GFPR) of square rational matrices $G(\lambda)$ have been introduced in the references [1,3,12,14], respectively. In simple words, the main idea in those definitions is to combine the corresponding class of Fiedler-like pencils of the polynomial part of $G(\lambda)$ (introduced originally in [23,8,40,10]) with a minimal order state-space realization of its strictly proper part in order to construct minimal order linear polynomial system matrices, of other rational matrices $\widehat{G}(\lambda)$, which are strong linearizations of $G(\lambda)$. We emphasize that FPRs and GFPRs are strong linearizations of $G(\lambda)$ under certain non-singularity generic hypotheses that, for simplicity, are assumed to hold in this section. We will recall that such assumptions hold by using the wordings "FPR linearization" and "GFPR linearization". Moreover, among the GFPs we only consider the *proper* GFPs (see [3] and the references therein), since nonproper ones involve the inversion of some coefficients of the polynomial part of $G(\lambda)$ and are less interesting in applications.

It was proved in [17, Section 4] that every FP of a square polynomial matrix $D(\lambda)$ is a (block) permutation of a (particular) strong block minimal bases linearization of $D(\lambda)$. This was extended to proper GFPs, FPR linearizations and GFPR linearizations of $D(\lambda)$ in [11]. This result allows to unify many different classes of strong linearizations of polynomial matrices and derive many properties of Fiedler-like linearizations from strong block minimal bases linearizations.

We prove in this section that, not surprisingly, FPs, proper GFPs, FPR linearizations and GFPR linearizations of a square rational matrix $G(\lambda)$ are also (block) permutations of particular strong block minimal bases linearizations of $G(\lambda)$. Moreover, the involved permutations are direct sums of an identity matrix plus the permutations corresponding to the Fiedler-like pencils of the polynomial part of $G(\lambda)$. We will discuss how the structure of the permutations allows us to prove immediately that FPs, proper GFPs, FPR linearizations and GFPR linearizations of rational matrices are strong linearizations as a consequence of [6, Theorem 5.11] and [19, Lemma 2.7]. In addition, we will obtain as corollaries of Theorems 6.5 and 6.7, the recovery rules of the minimal bases and indices from FPs, proper GFPs, FPR linearizations and GFPR linearizations of square rational matrices previously deduced in [12,14] (see [2] for a related result for eigenvectors).

We remark that the permutation results in this section were mentioned very briefly (without proof) in [6,19] and that they are simple consequences of results available in the literature for linearizations of polynomial matrices. They set most Fiedler-like linearizations of rational matrices into the unified framework of strong block minimal bases linearizations. For proving the results, we need to recall some definitions on strong block minimal bases linearizations and Fiedler-like linearizations. In this section we consider only square polynomial and rational matrices, since most Fiedler-like pencils have been defined only in the square case.

8.1. Extended block Kronecker linearizations and the antidiagonal sum condition

Let $L_k(\lambda)$ be the pencil defined in (25), and

$$\Lambda_k(\lambda) := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix}. \tag{36}$$

It is well-known and easy to see that $L_k(\lambda)$ and $\Lambda_k(\lambda)$ are dual minimal bases, as well as $L_k(\lambda) \otimes I_p$ and $\Lambda_k(\lambda) \otimes I_p$ for any integer $p > 0$ [18]. Observe that, for any nonsingular constant matrix Y, also $Y(L_k(\lambda) \otimes I_p)$ and $\Lambda_k(\lambda) \otimes I_p$ are dual minimal bases. Moreover the row degrees of $Y(L_k(\lambda) \otimes I_p)$ are all equal to 1 and the ones of $\Lambda_k(\lambda) \otimes I_p$ are all equal to k.

Given $D(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_1\lambda + D_0 \in \mathbb{F}[\lambda]^{p\times p}$, we consider the family of strong block minimal bases linearizations associated to $D(\lambda)$ defined by setting in $\mathcal{L}(\lambda)$ in (12)

$$K_1(\lambda) = Y(L_\varepsilon(\lambda) \otimes I_p), \qquad K_2(\lambda) = Z(L_\eta(\lambda) \otimes I_p), \tag{37}$$

where Y and Z are nonsingular matrices and $q = \varepsilon + \eta + 1$, and $M(\lambda)$ such that

$$D(\lambda) = (\Lambda_\eta(\lambda) \otimes I_p)M(\lambda)(\Lambda_\varepsilon(\lambda)^T \otimes I_p). \tag{38}$$

For simplicity, we assume that $D_q \neq 0$, which implies that $\mathcal{L}(\lambda)$ has sharp degree. Moreover, if $M(\lambda) = M_1\lambda + M_0$ is partitioned into $(\eta+1) \times (\varepsilon+1)$ blocks each of size $p \times p$ and we denote such blocks by $[M(\lambda)]_{ij} = [M_1]_{ij}\lambda + [M_0]_{ij}$, $1 \leq i \leq \eta + 1$ and $1 \leq j \leq \varepsilon + 1$, then (38) is equivalent to

$$\sum_{i+j=q+2-k} [M_1]_{ij} + \sum_{i+j=q+1-k} [M_0]_{ij} = D_k, \quad \text{for } k = 0, 1, \ldots, q. \tag{39}$$

This condition follows from [18, Theorem 4.4] and, if (39) is satisfied, it is said in [11, Definition 3.9] that "$M(\lambda)$ satisfies the antidiagonal sum (AS) condition for $D(\lambda)$". Thus, using the terminology in [11, Section 3], we call the strong block minimal bases linearizations defined by (37) and (38) *extended* $(\varepsilon, p, \eta, p)$*-block Kronecker linearizations satisfying the AS condition for* $D(\lambda)$. All of them are strong linearizations of $D(\lambda)$ and, for brevity, we often refer to them simply as extended block Kronecker linearizations[5] for $D(\lambda)$. Obviously, they include the well known block Kronecker linearizations introduced in [18, Section 4] (see also Remark 6.4) just by taking Y and Z identity matrices.

[5] We remark that we are simplifying a bit the terminology used in [11, Section 3]. First, in [11] the term "extended block Kronecker *pencil*" is used since Y and Z in (37) are allowed to be singular. However, see [11, Theorem 3.8], such pencils are linearizations only when Y and Z are nonsingular. In addition, in [11] expressions like "extended block Kronecker pencil with body $M(\lambda)$ satisfying the AS condition for $D(\lambda)$" are used instead of "extended block Kronecker linearization satisfying the AS condition for $D(\lambda)$". The reason of this is that other AS conditions are investigated in [11].

Extended block Kronecker linearizations of polynomial matrices have also been analyzed (with other name) in [22].

The following very simple corollary of the AS condition (39) is key for obtaining the main results of this section. For block Kronecker pencils, it was described in [18, eq. (4.5)].

Corollary 8.1. *Let* $D(\lambda) = D_q\lambda^q + \cdots + D_1\lambda + D_0 \in \mathbb{F}[\lambda]^{p\times p}$ *and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & (Z(L_\eta(\lambda)\otimes I_p))^T \\ Y(L_\varepsilon(\lambda)\otimes I_p) & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{pq\times pq}$$

be an extended $(\varepsilon, p, \eta, p)$*-block Kronecker linearization satisfying the AS condition for* $D(\lambda)$*. If* $M(\lambda) = M_1\lambda + M_0$ *is partitioned into* $(\eta+1)\times(\varepsilon+1)$ *blocks each of size* $p\times p$*, then*

$$[M_0]_{\eta+1,\varepsilon+1} = D_0.$$

The AS condition (39) and Corollary 8.1 make it convenient to view any extended $(\varepsilon, p, \eta, p)$-block Kronecker linearization satisfying the AS condition for any polynomial matrix $D(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ of degree q partitioned into $q \times q$ blocks each of size $p \times p$. For brevity we will refer to this partition as the *natural partition* of a extended block Kronecker linearization.

Based on the definitions above, we define extended $(\varepsilon, p, \eta, p)$-block Kronecker linearizations satisfying the AS condition for a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$, which are particular cases of strong block minimal bases linearizations of $G(\lambda)$. For this purpose, we first express $G(\lambda) = D(\lambda) + G_{sp}(\lambda) = D_q\lambda^q + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B$ in terms of the coefficients in the monomial basis of the polynomial part $D(\lambda)$ of $G(\lambda)$ and of a minimal order state-space realization $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ of the strictly proper part of $G(\lambda)$. Then we use (15) with $K_1(\lambda)$ and $K_2(\lambda)$ as in (37), with $M(\lambda)$ satisfying (38), and $\widehat{K}_1 = e_{\varepsilon+1}^T \otimes I_p$ and $\widehat{K}_2 = e_{\eta+1}^T \otimes I_p$, where e_k is the last column of I_k. The expressions for $\widehat{K}_1$ and $\widehat{K}_2$ follow from [6, Example 5.6] (see also Remark 6.4). This leads (for any T, S nonsingular matrices) to

$$L(\lambda) = \left[\begin{array}{c|c|c} T(\lambda I_n - A)S & e_{\varepsilon+1}^T \otimes (TB) & 0 \\ \hline -e_{\eta+1}\otimes(CS) & M(\lambda) & (Z(L_\eta(\lambda)\otimes I_p))^T \\ 0 & Y(L_\varepsilon(\lambda)\otimes I_p) & 0 \end{array}\right], \tag{40}$$

which are the desired extended $(\varepsilon, p, \eta, p)$-block Kronecker linearizations satisfying the AS condition for a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$. The natural partition of $\mathcal{L}(\lambda)$ in Corollary 8.1 induces the *natural partition* of $L(\lambda)$ in (40), which has the $(1,1)$-block of size $n \times n$, the other blocks in the first block row (resp. column) of size $n \times p$ (resp. $p \times n$), and the remaining blocks of size $p \times p$. In the case Y and Z are identity matrices,

we simply say that (40) is an $(\varepsilon, p, \eta, p)$-block Kronecker linearization satisfying the AS condition for $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$.

Remark 8.2. The simple facts that in the natural partition of $L(\lambda)$ in (40) the blocks $-CS$ and TB are, together with the $(1,1)$-block, the only non-identically zero blocks in the first block column and row, respectively, and that $-CS$ and TB are located, respectively, in the rows and columns corresponding to the block $[M_0]_{\eta+1,\varepsilon+1} = D_0$ in Corollary 8.1 will be important for obtaining the results in this section.

8.2. Fiedler-like linearizations

The definitions of the families of Fiedler-like linearizations of polynomial and rational matrices require a good number of concepts and notations. Several of them are omitted for brevity. For polynomial matrices, we refer to the summary in [11, Section 4], since we follow similar notations and definitions.

Given a polynomial matrix $D(\lambda) = D_q\lambda^q + \cdots + D_1\lambda + D_0 \in \mathbb{F}[\lambda]^{p\times p}$, its associated Fiedler-like pencils are defined in terms of products of the $pq \times pq$ elementary matrices introduced in [11, pp. 66-67], which are constructed from a $p \times p$ matrix X and are denoted by $M_i(X)$, where $i \in \{-q, \ldots, 0, \ldots, q\}$ is an index that determines $M_i(X)$. If the matrix X is the coefficient of $D(\lambda)$ in [11, p. 67], then $M_i(X)$ is denoted simply by M_i^D. Index tuples, i.e., finite ordered sequences of integers belonging to $\{-q, \ldots, 0, \ldots, q\}$, and matrix assignments are used to denote in concise way products of elementary matrices. For instance, let $\mathbf{t} = (t_1, t_2, t_3)$ and $\mathcal{X} = (X_1, X_2, X_3)$ be an index tuple and a matrix assignment for $\mathbf{t}$, respectively, then $M_{\mathbf{t}}(\mathcal{X}) := M_{t_1}(X_1)M_{t_2}(X_2)M_{t_3}(X_3)$. Moreover, if $\mathbf{t}$ is the empty tuple, then we define $M_{\mathbf{t}}(\mathcal{X}) := I_{pq}$. As in the case of extended block Kronecker linearizations, it is very convenient to view these elementary matrices and their products partitioned into $q \times q$ blocks each of size $p \times p$. We will call this partition again the *natural partition* of $M_{\mathbf{t}}(\mathcal{X})$ and $[M_{\mathbf{t}}(\mathcal{X})]_{jk}$, $1 \le j, k \le q$, will denote the block of $M_{\mathbf{t}}(\mathcal{X})$ in the block position (j, k). A fundamental property is that the products $M_{\mathbf{t}}(\mathcal{X})$ of elementary matrices appearing in the definitions of Fiedler-like pencils are *operation-free* [40] (see also [10, Definition 4.5]), which means that their blocks in the natural partition are either the matrices in the matrix assignments, or I_p or 0_p and that the positions of such blocks only depend on the index tuple $\mathbf{t}$, i.e., do not depend on the particular matrix assignment $\mathcal{X}$. In order to guarantee the operation-free property, we will require that some of the index tuples involved in the definitions of Fiedler-like pencils satisfy the Successor Infix Property (SIP) introduced in [40, Definition 7] (see also [11, Definition 4.6]).

Next, we define the FPs, proper GFPs, FPR linearizations and GFPR linearizations of the polynomial matrix $D(\lambda)$ in terms of the following generic pencil (see [11, Definition 4.29]),

$$\mathcal{L}_D(\lambda) = M_{\boldsymbol{\ell}_t,\boldsymbol{\ell}_z}(\mathcal{X}, \mathcal{Z})(\lambda M_{\mathbf{z}}^D - M_{\mathbf{t}}^D)M_{\mathbf{r}_z,\mathbf{r}_t}(\mathcal{W}, \mathcal{Y}). \tag{41}$$

• If in (41) $\boldsymbol{\ell}_t, \boldsymbol{\ell}_z, \mathbf{r}_z, \mathbf{r}_t$ are empty tuples, $\mathbf{z} = -q$ and $\mathbf{t}$ is a permutation of $\{0, 1, \ldots, q-1\}$, then $\mathcal{L}_D(\lambda)$ is a FP associated to $D(\lambda)$.

• If in (41) $\boldsymbol{\ell}_t, \boldsymbol{\ell}_z, \mathbf{r}_z, \mathbf{r}_t$ are empty tuples, $\mathbf{t}$ and $\mathbf{z}$ are permutations, respectively, of C_0 and $-C_1$, where $\{C_0, C_1\}$ is a partition of $\{0, 1, \ldots, q\}$ such that $0 \in C_0$ and $q \in C_1$, then $\mathcal{L}_D(\lambda)$ is a proper GFP associated to $D(\lambda)$.

• Let $h \in \{0, 1, \ldots, q-1\}$. If in (41) $\mathbf{t}$ and $\mathbf{z}$ are permutations of $\{0, \ldots, h\}$ and $\{-q, -q+1, \ldots, -h-1\}$, respectively, $\boldsymbol{\ell}_t$ and $\mathbf{r}_t$ are tuples with indices from $\{0, \ldots, h-1\}$ such that $(\boldsymbol{\ell}_t, \mathbf{t}, \mathbf{r}_t)$ satisfies the SIP, and $\boldsymbol{\ell}_z$ and $\mathbf{r}_z$ are tuples with indices from $\{-q, -q+1, \ldots, -h-2\}$ such that $(\boldsymbol{\ell}_z, \mathbf{z}, \mathbf{r}_z)$ satisfies the SIP, then $\mathcal{L}_D(\lambda)$ is a GFPR associated to $D(\lambda)$. This pencil is a GFPR linearization whenever the matrices $M_{\boldsymbol{\ell}_t, \boldsymbol{\ell}_z}(\mathcal{X}, \mathcal{Z})$ and $M_{\mathbf{r}_z, \mathbf{r}_t}(\mathcal{W}, \mathcal{Y})$ are nonsingular, or, equivalently, the matrix assignments $\mathcal{X}, \mathcal{Z}, \mathcal{W}, \mathcal{Y}$ are nonsingular according to [10, Definition 4.1]. The FPRs associated to $D(\lambda)$ are those GFPRs such that $\mathcal{X}, \mathcal{Z}, \mathcal{W}, \mathcal{Y}$ are the trivial matrix assignments for $\boldsymbol{\ell}_t, \boldsymbol{\ell}_z, \mathbf{r}_z, \mathbf{r}_t$, respectively (see [11, p. 67]).

The operation-free property of the products of elementary matrices in $\mathcal{L}_D(\lambda) := \lambda\mathcal{L}_1 + \mathcal{L}_0$ in (41) for FPs, proper GFPs and GFPRs and the definition of the involved index tuples imply that for every FP, for every proper GFP, and for every GFPR for which the matrix assignments $\mathcal{X}, \mathcal{Z}, \mathcal{W}, \mathcal{Y}$ do not contain $-D_0$ (the matrix coefficient of degree zero of $D(\lambda)$), there is exactly one block position in the natural partition of $\mathcal{L}_0$ whose entry is identically equal[6] to D_0. This property is very easy to prove for FPs and proper GFPs; for GFPRs, it can be found in [10, Theorem 5.3]. Moreover, this block position in the natural partition of $\mathcal{L}_0$ is uniquely determined by the index tuples in (41), i.e., does not depend on the particular matrix assignment in the case of GFPRs, and we call it the *intrinsic block position of* D_0 *in* $\mathcal{L}_0$. We use the word "intrinsic" because for other GFPRs with the same index tuples and with some of the matrix assignments $\mathcal{X}, \mathcal{Z}, \mathcal{W}, \mathcal{Y}$ containing $-D_0$, D_0 is the entry of the "intrinsic block position" and of other block positions of $\mathcal{L}_0$. However, under arbitrary changes of the matrix assignments (including symbolic changes of variables), D_0 only remains in the "intrinsic position". This allows us to define the intrinsic block position of D_0 in $\mathcal{L}_0$ for the family of all GFPRs sharing the same index tuples in (41).

The intrinsic block position of D_0 in $\mathcal{L}_0$ can be easily determined by using the consecutive consecutions and the consecutive inversions at 0 of two index tuples (see [12, Definition 5.7] or [14, Definition 2.8] for the definition of consecutive consecutions and inversions). This is stated in Lemma 8.3, which is the only new result so far in this section.

Lemma 8.3. *Let* $D(\lambda) = D_q\lambda^q + \cdots + D_1\lambda + D_0 \in \mathbb{F}[\lambda]^{p\times p}$ *be a polynomial matrix and let* $\mathcal{L}_D(\lambda) := \lambda\mathcal{L}_1 + \mathcal{L}_0$ *as in* (41) *be a FP or a proper GFP or a GFPR associated to* $D(\lambda)$. *Let* $i_0(\boldsymbol{\ell}_t, \mathbf{t})$ *be the number of consecutive inversions at* 0 *of the index tuple* $(\boldsymbol{\ell}_t, \mathbf{t})$

[6] The expression "identically equal" means that this happens always in the same block entry for any value of D_0.

and $c_0(\mathbf{t}, \mathbf{r}_t)$ *be the number of consecutive consecutions at* 0 *of* $(\mathbf{t}, \mathbf{r}_t)$. *Then the intrinsic block position of* D_0 *in* $\mathcal{L}_0$ *is* $(q - i_0(\boldsymbol{\ell}_t, \mathbf{t}), q - c_0(\mathbf{t}, \mathbf{r}_t))$.

Proof. Once the (highly non-trivial) operation-free property is taken into account, the proof is a simple exercise of block matrix multiplication. For brevity, we only sketch the proof for GFPRs with matrix assignments $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{Y}$ not containing $-D_0$, since the proofs of the other cases are similar. Note first that the commutativity relations of the elementary matrices [11, Remark 4.3] imply that the elementary matrices in $M_{\boldsymbol{\ell}_z}(\mathcal{Z})$ and $M_{\mathbf{r}_z}(\mathcal{W})$ commute with the other elementary matrices in $\mathcal{L}_0$ and, so, they do not affect the position of D_0 (see also [11, Lemma 4.33 and p. 76]). Next, note that D_0 is in the block position (q, q) if $i_0(\boldsymbol{\ell}_t, \mathbf{t}) = c_0(\mathbf{t}, \mathbf{r}_t) = 0$, as a consequence of the commutativity relations of the elementary matrices. Otherwise, note that each elementary matrix different from M_0^D corresponding to the subtuple of $(\boldsymbol{\ell}_t, \mathbf{t})$ (resp. $(\mathbf{t}, \mathbf{r}_t)$) containing the index $0 \in \mathbf{t}$ and defining $i_0(\boldsymbol{\ell}_t, \mathbf{t})$ (resp. $c_0(\mathbf{t}, \mathbf{r}_t)$) moves D_0 one position up (resp. left), while the rest of elementary matrices in $(\boldsymbol{\ell}_t, \mathbf{t}, \mathbf{r}_t)$ do not move D_0. □

Remark 8.4. We emphasize that $(i_0(\boldsymbol{\ell}_t, \mathbf{t}), c_0(\mathbf{t}, \mathbf{r}_t)) = (i_0(\mathbf{t}), c_0(\mathbf{t}))$ for FPs and proper GFPs. For FPs, one of the elements in this pair is necessarily different from zero and the other one is necessarily zero. For proper GFPs, at most one is different from zero, but it may happen that both are zero.

Based on the definitions of FPs, proper GFPs, FPRs and GFPRs of polynomial matrices, FPs, proper GFPs, FPRs and GFPRs of square rational matrices have been defined in [1, Definition 3.2], [3, Definition 2.2], [12, Definition 5.4] and [14, Definition 3.2], respectively. As in the case of polynomial matrices, for rational matrices GFPRs include FPRs, and GFPRs are strong linearizations if the involved matrix assignments are nonsingular. Essentially, the strategy in [1,3,12,14] is to consider the rational matrix $G(\lambda) = D(\lambda) + G_{sp}(\lambda) = D_q\lambda^q + \cdots + D_1\lambda + D_0 + C(\lambda E - A)^{-1}B \in \mathbb{F}(\lambda)^{p\times p}$ expressed in terms of the coefficients in the monomial basis of the polynomial part $D(\lambda)$ of $G(\lambda)$ and of a minimal order state-space realization $C(\lambda E - A)^{-1}B$ of the strictly proper part, with $E \in \mathbb{F}^{n\times n}$ nonsingular. Then, the elementary matrices $M_i(X)$ and M_i^D of the polynomial part $D(\lambda)$ are carefully embedded into augmented elementary matrices $\mathbb{M}_i(X)$ and $\mathbb{M}_i^G$ of the rational matrix that incorporate the information of the state-space realization of the strictly proper part. Finally, the corresponding families of Fiedler-like pencils of $G(\lambda)$ are defined through products of these augmented elementary matrices using matrix assignments and the same index tuples as in the corresponding families of Fiedler-like pencils of $D(\lambda)$. Theorem 8.5 is a very important result in this context that summarizes in a concise way [1, Theorem 3.6], [3, Theorem 2.7], [12, Theorem 5.12] and [14, Theorem 3.6].

Theorem 8.5. *Let* $G(\lambda) = D(\lambda) + G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ *be the unique decomposition of* $G(\lambda)$ *into its polynomial part* $D(\lambda) = D_q\lambda^q + \cdots + D_0$ *and its strictly proper part* $G_{sp}(\lambda)$,

and let $C(\lambda E - A)^{-1}B$ *be a minimal order state-space realization of* $G_{sp}(\lambda)$ *(with* $E \in \mathbb{F}^{n\times n}$ *nonsingular). Let* $L_G(\lambda) = \mathbb{M}_{\boldsymbol{\ell}_t,\boldsymbol{\ell}_z}(\mathcal{X},\mathcal{Z})(\lambda\mathbb{M}^G_{\mathbf{z}} - \mathbb{M}^G_{\mathbf{t}})\mathbb{M}_{\mathbf{r}_z,\mathbf{r}_t}(\mathcal{W},\mathcal{Y})$ *and* $\mathcal{L}_D(\lambda) = M_{\boldsymbol{\ell}_t,\boldsymbol{\ell}_z}(\mathcal{X},\mathcal{Z})(\lambda M^D_{\mathbf{z}} - M^D_{\mathbf{t}})M_{\mathbf{r}_z,\mathbf{r}_t}(\mathcal{W},\mathcal{Y})$ *be FPs, or proper GFPs or GFPRs associated to* $G(\lambda)$ *and to* $D(\lambda)$*, respectively. Then*

$$L_G(\lambda) = \left[\begin{array}{c|c} A - \lambda E & e^T_{q-c_0(\mathbf{t},\mathbf{r}_t)} \otimes B \\ \hline e_{q-i_0(\boldsymbol{\ell}_t,\mathbf{t})} \otimes C & \mathcal{L}_D(\lambda) \end{array}\right], \tag{42}$$

where e_j *is the* j*th column of* I_q*.*

Remark 8.6. We have followed in (42) the classical notation in [32] for polynomial system matrices and we have set $A-\lambda E$ in the upper-left corner. In contrast in [1,3,12,14], $A-\lambda E$ is set in the lower-right corner and the two block rows and the two block columns in (42) appear interchanged. Note also that in previous sections of this paper we have written the strictly proper part of $G(\lambda)$ as $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$, which is the same as $G_{sp}(\lambda) = (CS)(\lambda TS - TAS)^{-1}(TB)$ for any nonsingular T and S matrices. With an obvious change of notation, it can be written as $G_{sp}(\lambda) = C(\lambda E - A)^{-1}B$, with E nonsingular. This allows to compare (42) and (40) in Subsection 8.3.

Remark 8.7. If we consider for (42) the same natural partition as in (40) and we take into account Lemma 8.3, then we obtain from Theorem 8.5 a simple recipe to construct the FPs, proper GFPs, FPRs and GFPRs of a rational matrix when the explicit expression of the corresponding pencil of its polynomial part is known: (1) construct any FP, proper GFP, FPR or GFPR of the polynomial part, $\mathcal{L}_D(\lambda) = \lambda\mathcal{L}_1 + \mathcal{L}_0$; (2) identify in $\mathcal{L}_0$ the intrinsic block position of D_0 (this is particularly simple in FPs, proper GFPs, and GFPRs with matrix assignments not containing $-D_0$, since there is only one block entry identically equal to D_0); (3) augment $\mathcal{L}_D(\lambda)$ to a larger matrix partitioned in the natural way by adding one block column and one block row as follows: the $(1,1)$-block is $(A-\lambda E)$ and C (resp. B) is the only remaining nonzero block in the first block column (resp. row) and is located in the block row (resp. column) of the intrinsic block position of D_0 in $\mathcal{L}_0$. This remark is related to Remark 8.2 for extended block Kronecker linearizations of rational matrices and stresses the importance of the intrinsic position of D_0 in these families of linearizations.

8.3. Block permutations of Fiedler-like pencils

Given two positive integers q and p, we say that a matrix Π is a (q,p)-block permutation matrix if $\Pi = \Sigma \otimes I_p$, where Σ is a $q \times q$ permutation matrix. The following result follows from Theorems 6.3, 7.1 and 8.1 of [11].

Theorem 8.8. *Let* $D(\lambda) = D_q\lambda^q + \cdots + D_1\lambda + D_0 \in \mathbb{F}[\lambda]^{p\times p}$ *be a polynomial matrix and* $\mathcal{L}_D(\lambda)$ *be a FP, or a proper GFP or a GFPR with nonsingular matrix assignments*

associated to $D(\lambda)$. Then, there exist two (q,p)-block permutation matrices Π_1 and Π_2 such that

$$\Pi_1 \mathcal{L}_D(\lambda)\Pi_2 = \begin{bmatrix} M(\lambda) & (Z(L_\eta(\lambda)\otimes I_p))^T \\ Y(L_\varepsilon(\lambda)\otimes I_p) & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{pq\times pq} \tag{43}$$

is an extended (ε,p,η,p)-block Kronecker linearization satisfying the AS condition for $D(\lambda)$. Moreover the parameters ε and η are determined uniquely from the index tuples defining $\mathcal{L}_D(\lambda)$. If $\mathcal{L}_D(\lambda)$ is a FP or a proper GFP, then Y and Z in (43) are identity matrices and $\Pi_1\mathcal{L}_D(\lambda)\Pi_2$ is a (ε,p,η,p)-block Kronecker linearization.

Next, we extend Theorem 8.8 to Fiedler-like and extended block Kronecker linearizations of rational matrices. We warn the reader that for writing (44) in a compact way, the vectors $e_{\varepsilon+1}$ and $e_{\eta+1}$ in (44) are different from those in (40): in (44) they are the corresponding columns of I_q, while in (40) they are the last columns of $I_{\varepsilon+1}$ and $I_{\eta+1}$, respectively.

Theorem 8.9. *Let $G(\lambda) = D(\lambda) + G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ be the unique decomposition of $G(\lambda)$ into its polynomial part $D(\lambda) = D_q\lambda^q + \cdots + D_0$ and its strictly proper part $G_{sp}(\lambda)$, and let $C(\lambda E - A)^{-1}B$ be a minimal order state-space realization of $G_{sp}(\lambda)$ (with $E \in \mathbb{F}^{n\times n}$ nonsingular). Let $L_G(\lambda)$ be a FP, or a proper GFP or a GFPR with nonsingular matrix assignments associated to $G(\lambda)$ as in (42), where $\mathcal{L}_D(\lambda)$ is the corresponding FP, or proper GFP or GFPR associated to $D(\lambda)$. Then, there exist two (q,p)-block permutation matrices Π_1 and Π_2 such that*

$$\begin{bmatrix} I_n & \\ & \Pi_1 \end{bmatrix} L_G(\lambda) \begin{bmatrix} I_n & \\ & \Pi_2 \end{bmatrix} = \left[\begin{array}{c|c} A-\lambda E & e_{\varepsilon+1}^T\otimes B \\ \hline e_{\eta+1}\otimes C & \Pi_1\mathcal{L}_D(\lambda)\Pi_2 \end{array}\right], \tag{44}$$

is an extended (ε,p,η,p)-block Kronecker linearization satisfying the AS condition for $G(\lambda)$. Moreover the parameters ε and η are determined uniquely from the index tuples defining $L_G(\lambda)$. If $L_G(\lambda)$ is a FP or a proper GFP, then the pencil in the right-hand side of (44) is an (ε,p,η,p)-block Kronecker linearization of $G(\lambda)$.

Proof. Let Π_1 and Π_2 be the two block permutation matrices in Theorem 8.8. Then, from (42), we get

$$\begin{bmatrix} I_n & \\ & \Pi_1 \end{bmatrix} L_G(\lambda) \begin{bmatrix} I_n & \\ & \Pi_2 \end{bmatrix} = \left[\begin{array}{c|c} A-\lambda E & (e_{q-c_0(\mathbf{t},\mathbf{r}_t)}^T\otimes B)\Pi_2 \\ \hline \Pi_1(e_{q-i_0(\boldsymbol{\ell}_t,\mathbf{t})}\otimes C) & \Pi_1\mathcal{L}_D(\lambda)\Pi_2 \end{array}\right], \tag{45}$$

where $\Pi_1\mathcal{L}_D(\lambda)\Pi_2$ is the extended (ε,p,η,p)-block Kronecker linearization satisfying the AS condition for $D(\lambda)$ in Theorem 8.8. According to (40), it only remains to prove that

$$(e_{q-c_0(\mathbf{t},\mathbf{r}_t)}^T\otimes B)\Pi_2 = e_{\varepsilon+1}^T\otimes B \quad \text{and} \quad \Pi_1(e_{q-i_0(\boldsymbol{\ell}_t,\mathbf{t})}\otimes C) = e_{\eta+1}\otimes C. \tag{46}$$

The proof of these two equalities is simple in the case $L_G(\lambda)$ is a FP, a proper GFP, or a GFPR whose matrix assignments do not contain $-D_0$. The key fact in these cases is that if $\mathcal{L}_D(\lambda) := \lambda\mathcal{L}_1 + \mathcal{L}_0$ then there is exactly one block identically equal to D_0 in the natural partition of $\mathcal{L}_0$ located in the block position $(q - i_0(\boldsymbol{\ell}_t, \mathbf{t}), q - c_0(\mathbf{t}, \mathbf{r}_t))$, according to Lemma 8.3. Therefore, $\Pi_1\mathcal{L}_D(\lambda)\Pi_2 := \lambda\mathcal{T}_1 + \mathcal{T}_0$ has also exactly one block identically equal to D_0 in the natural partition of $\mathcal{T}_0$, since Π_1 and Π_2 are (q, p)-block permutations. Moreover, Corollary 8.1 implies that $(\eta + 1, \varepsilon + 1)$ is the block position of D_0 in $\mathcal{T}_0$. This implies that Π_1 moves the block row $q - i_0(\boldsymbol{\ell}_t, \mathbf{t})$ to the block row $\eta + 1$ and that Π_2 moves the block column $q - c_0(\mathbf{t}, \mathbf{r}_t)$ to the block column $\varepsilon + 1$, which imply (46).

In the case of GFPRs whose matrix assignments contain $-D_0$, there are more than one blocks equal to D_0 in $\mathcal{L}_0$ (and, so, in $\mathcal{T}_0$), and the proof of (46) requires to use the concept of the intrinsic block position of D_0 in $\mathcal{L}_0$, introduced before Lemma 8.3. Note that Theorem 8.8 implies that the parameters ε and η are determined uniquely by the index tuples defining $L_G(\lambda)$, which are the same as those defining $\mathcal{L}_D(\lambda)$, and, so, are the same for all the GFPRs with the same index tuples (independently of the matrix assignments). Thus, the intrinsic block position of D_0 in $\mathcal{L}_0$ in $(q - i_0(\boldsymbol{\ell}_t, \mathbf{t}), q - c_0(\mathbf{t}, \mathbf{r}_t))$ is mapped by Π_1 and Π_2 to $(\eta + 1, \varepsilon + 1)$ in $\mathcal{T}_0$, since by Corollary 8.1 is the only block entry of $\mathcal{T}_0$ that contains D_0 when the matrix assignments change arbitrarily but the defining tuples do not change. This implies (46). □

8.4. Some consequences of Theorem 8.9: Fiedler-like pencils are strong linearizations and recovery of minimal bases from Fiedler-like pencils

We study three easy consequences of Theorem 8.9 and its proof. More precisely, (1) we provide alternative proofs to those in [1,3,12,14] for the facts that FPs, proper GFPs, FPRs and GFPRs (with nonsingular matrix assignments) of rational matrices are strong linearizations of rational matrices; (2) we provide alternative proofs to those in [12,14] for the recovery rules of the minimal bases of a rational matrix from those of its Fiedler-like pencils; (3) we discuss briefly how to recover minimal indices of a rational matrix from those of its Fiedler-like pencils, a problem that has been solved in [12,14].

Corollary 8.10. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ be a rational matrix. If $L_G(\lambda)$ is a FP, or a proper GFP or a GFPR with nonsingular matrix assignments associated with $G(\lambda)$, then $L_G(\lambda)$ is a strong linearization of $G(\lambda)$.*

Proof. $L_G(\lambda)$ satisfies (44) and the right-hand side of (44) is a strong linearization of $G(\lambda)$ by [6, Theorem 5.11]. Then, [19, Lemma 2.7] implies that $L_G(\lambda)$ is also a strong linearization of $G(\lambda)$. □

Corollary 8.11 covers the recovery of minimal bases. In order to check that the results in Corollary 8.11 are the same as those in [12,14] recall that for FPs and proper GFPs $\boldsymbol{\ell}_t, \boldsymbol{\ell}_z, \mathbf{r}_z, \mathbf{r}_t$ are empty tuples and also Remark 8.6.

Corollary 8.11. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ be as in Theorem 8.9. Let*

$$L_G(\lambda) = \mathbb{M}_{\boldsymbol{\ell}_t,\boldsymbol{\ell}_z}(\mathcal{X},\mathcal{Z})(\lambda\mathbb{M}^G_{\mathbf{z}} - \mathbb{M}^G_{\mathbf{t}})\mathbb{M}_{\mathbf{r}_z,\mathbf{r}_t}(\mathcal{W},\mathcal{Y})$$

be a FP, or a proper GFP or a GFPR linearization associated to $G(\lambda)$, let $i_0(\boldsymbol{\ell}_t,\mathbf{t})$ be the number of consecutive inversions at 0 of the index tuple $(\boldsymbol{\ell}_t,\mathbf{t})$, $c_0(\mathbf{t},\mathbf{r}_t)$ be the number of consecutive consecutions at 0 of $(\mathbf{t},\mathbf{r}_t)$ and e_j be the jth column of I_q.

(a) *If $Q(\lambda) \in \mathbb{F}[\lambda]^{(n+pq)\times l}$ is a right minimal basis of $L_G(\lambda)$ then*

$$\begin{bmatrix} 0_{p\times n} & e^T_{q-c_0(\mathbf{t},\mathbf{r}_t)} \otimes I_p \end{bmatrix} Q(\lambda)$$

is a right minimal basis of $G(\lambda)$.

(b) *If $Q(\lambda) \in \mathbb{F}[\lambda]^{(n+pq)\times l}$ is a left minimal basis of $L_G(\lambda)$ then*

$$\begin{bmatrix} 0_{p\times n} & e^T_{q-i_0(\boldsymbol{\ell}_t,\mathbf{t})} \otimes I_p \end{bmatrix} Q(\lambda)$$

is a left minimal basis of $G(\lambda)$.

Proof. We prove part (a). Let $L(\lambda)$ be the extended (ε,p,η,p)-block Kronecker linearization satisfying the AS condition for $G(\lambda)$ in the right-hand side of (44). Note that $Q(\lambda) = \operatorname{diag}(I_n,\Pi_2)\,\widetilde{Q}(\lambda)$, where $\widetilde{Q}(\lambda) \in \mathbb{F}[\lambda]^{(n+pq)\times l}$ is a right minimal basis of $L(\lambda)$. Next, we apply Theorem 6.5(a) to $L(\lambda)$ and $\widetilde{Q}(\lambda)$, taking into account that in this case $N_1(\lambda)^T = \Lambda_\varepsilon(\lambda)^T \otimes I_p$, with $\Lambda_\varepsilon(\lambda)$ as in (36). Therefore, Theorem 6.5 implies that

$$\begin{bmatrix} 0_{p\times n} & e^T_{\varepsilon+1} \otimes I_p \end{bmatrix} \widetilde{Q}(\lambda) = \begin{bmatrix} 0_{p\times n} & (e^T_{\varepsilon+1} \otimes I_p)\Pi_2^T \end{bmatrix} Q(\lambda)$$

is a right minimal basis of $G(\lambda)$. Finally, note that $(e^T_{\varepsilon+1} \otimes I_p)\Pi_2^T = e^T_{q-c_0(\mathbf{t},\mathbf{r}_t)} \otimes I_p$, since we have seen in the proof of Theorem 8.9 that the block permutation Π_2 moves the block column $q - c_0(\mathbf{t},\mathbf{r}_t)$ to the block column $\varepsilon+1$. Part (b) is proved analogously via Theorem 6.5(b) and $N_2(\lambda)^T = \Lambda_\eta(\lambda)^T \otimes I_p$. □

Remark 8.12. In the same spirit of Remark 8.7 on Theorem 8.5, Lemma 8.3 allows to express the recovery rules in Corollary 8.11 in simple words as follows: (1) identify in the explicit expression of $L_G(\lambda) = \lambda L_1 + L_0$ the intrinsic block position of D_0 in the natural partition of L_0 (which is immediate for FPs, proper GFPs and GFPRs with matrix assignments not containing $-D_0$); (2) the rows of any right (resp. left) minimal basis of $L_G(\lambda)$ corresponding to the columns (resp. rows) of the intrinsic block position of D_0 in L_0 are a right (resp. left) minimal basis of $G(\lambda)$.

Though Corollary 8.11 is enough for recovering the minimal bases of $G(\lambda)$ from those of its Fiedler-like pencils in applications, we remark that it is weaker than Theorem 6.5

for the minimal bases of strong block minimal bases linearizations. In contrast to Theorem 6.5, Corollary 8.11 does not allow to construct the minimal bases of $L_G(\lambda)$ from those of $G(\lambda)$. In this context, note that for block Kronecker linearizations the matrices $\widehat{N}_1(\lambda)$ and $\widehat{N}_2(\lambda)$ in Theorem 6.5 are known and have simple expressions [17, Remark 7.5]. In the case of extended block Kronecker linearizations, expressions of $\widehat{N}_1(\lambda)$ and $\widehat{N}_2(\lambda)$ involving the inverses of the matrices Y and Z in (40) can also be obtained.

Finally, note that (44) implies that the minimal indices of $L_G(\lambda)$ are those of the extended $(\varepsilon, p, \eta, p)$-block Kronecker linearization of $G(\lambda)$ in the right hand-side. Then, Theorem 6.7 with $N_1(\lambda) = \Lambda_\varepsilon(\lambda) \otimes I_p$ and $N_2(\lambda) = \Lambda_\eta(\lambda) \otimes I_p$ imply that the right (resp. left) minimal indices of $G(\lambda)$ are those of $L_G(\lambda)$ minus ε (resp. η). The parameters ε and η can be obtained from the index tuples defining $L_G(\lambda)$ as explained in Theorems 6.3, 7.1 and 8.1 of [11], which requires to use a number of definitions related to index tuples that we omit for brevity.

9. Some remarks on eigenvectors

If a rational matrix $G(\lambda)$ is regular, i.e., it is square and $\det G(\lambda)$ is not identically zero, then it does not have minimal bases nor minimal indices. In this case, $\lambda_0 \in \overline{\mathbb{F}}$ is an eigenvalue of $G(\lambda)$ if λ_0 is a finite zero but not a pole of $G(\lambda)$. Then, there exist nonzero vectors x, y such that $y^T G(\lambda_0) = 0$ and $G(\lambda_0)x = 0$. Such vectors are called, respectively, left and right eigenvectors of $G(\lambda)$ associated to λ_0 and are very interesting magnitudes in rational and nonlinear eigenvalue problems [24]. Equivalently, the left (resp. right) eigenvectors associated to λ_0 are the nonzero vectors of the left (resp. right) null-space of the constant matrix $G(\lambda_0)$. We denote such null-spaces by $\mathcal{N}_\ell(G(\lambda_0))$ and $\mathcal{N}_r(G(\lambda_0))$, respectively. A standard method for computing eigenvectors of $G(\lambda)$, or more precisely bases of $\mathcal{N}_\ell(G(\lambda_0))$ and $\mathcal{N}_r(G(\lambda_0))$, is to compute those of one of its linearizations and to recover from them the eigenvectors of $G(\lambda)$. This has led to recovery procedures for eigenvectors from Fiedler-like linearizations [2,12,14], from $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations [19], and from strong linearizations in the affine spaces defined in [13]. For completeness, we describe in this section very briefly how to recover eigenvectors from strong block minimal bases linearizations and how such recovery method allows to obtain the ones for Fiedler-like linearizations.

It can be shown that if λ_0 is an eigenvalue of a regular rational matrix $G(\lambda)$ and $L(\lambda)$ is a strong block minimal bases linearization of $G(\lambda)$, then the bases of $\mathcal{N}_r(G(\lambda_0))$ and $\mathcal{N}_r(L(\lambda_0))$ are related as the right minimal bases in Theorem 6.5(a) with the only change of replacing λ by λ_0. Similarly, the bases of $\mathcal{N}_\ell(G(\lambda_0))$ and $\mathcal{N}_\ell(L(\lambda_0))$ are related as the left minimal bases in Theorem 6.5(b) with λ replaced by λ_0. The proofs of these facts follow a pattern similar to the proofs of Theorem 6.5 but are much simpler, since all the arguments concerning the degrees are not needed.

Once the relationships between the bases of $\mathcal{N}_r(G(\lambda_0))$ and $\mathcal{N}_\ell(G(\lambda_0))$ and the bases of the corresponding null-spaces of the strong block minimal bases linearizations of $G(\lambda)$ are known, the same argument as in the proof of Corollary 8.11 proves that if $L_G(\lambda)$

is a FP, or a proper GFP or a GFPR linearization of $G(\lambda)$, then bases of $\mathcal{N}_r(G(\lambda_0))$ and $\mathcal{N}_\ell(G(\lambda_0))$ can be obtained from those of $\mathcal{N}_r(L_G(\lambda_0))$ and $\mathcal{N}_\ell(L_G(\lambda_0))$ as in Corollary 8.11 with the only change of replacing again λ by λ_0. This provides alternative proofs to those in [2,12,14] for the recovery of eigenvectors of a regular rational matrix from those of its Fiedler-like linearizations.

10. Conclusions

In this paper a complete theory about the relationship between the minimal bases and indices of a rational matrix and those of its polynomial system matrices, as well as those of its strong linearizations, has been developed.

The original contributions of this paper are organized into two clearly different parts. On the one hand those in Sections 3, 4 and 5, which deal with general polynomial system matrices, general linearizations and general strong linearizations of rational matrices. On the other hand those in Sections 6, 7 and 8, which deal with specific (though large) families of strong linearizations and establish connections among them. More precisely that Fiedler-like linearizations are particular cases of strong block minimal bases linearizations modulo permutations. In the case of polynomial system matrices, we have shown that, under the standard assumption of minimality and a certain additional condition of properness, the minimal indices of the polynomial system matrices and their transfer functions are exactly the same and their minimal bases are easily related to each other. These results are connected to pioneer results by Verghese, Van Dooren and Kailath [39,38], who proved similar results under different and nonequivalent assumptions. In contrast, we have shown that the minimal bases and indices of a rational matrix and those of its linearizations and strong linearizations are not related to each other in general, and that only the sums of the left and the right minimal indices are determined by each other in the case of strong linearizations. This latter result is based on the fundamental index sum theorem obtained by Paul Van Dooren in [39].

In the case of the families of strong block minimal bases linearizations and $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices, we have proved that the minimal indices and bases of the linearizations and the rational matrices are easily related to each other and that any of them can be obtained from the others and vice versa. The results for strong block minimal bases linearizations are obtained by using those for polynomial system matrices in Section 3, and they imply easily the results for $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations. In the case of Fiedler-like pencils, we have shown how they allow to recover the minimal indices and bases of a rational matrix as a consequence of the results for strong block minimal bases linearizations. This approach gives alternative proofs to the results in [12,14]. In this context, it is worth to emphasize the important unifying role played by strong block minimal bases linearizations of rational matrices. In addition, to compute minimal bases and indices by applying algorithms for pencils to strong block minimal bases linearizations allow to deal with rectangular matrices.

Declaration of competing interest

There is no competing interest.

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