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ORIGINAL RESEARCH

A Qualitative Approach to Conceptual Spaces: Prototypes as Qualitative Atoms

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Abstract
The aim of this paper is to propose a qualitative approach to the theory of conceptual spaces, in contrast to the usual metric framework. This requires qualitative notions of similarity, simple concepts, prototypes and conceptual categorisation. For this purpose, I will introduce three mathematical models for conceptual spaces. The first one is topological and has been proposed by Mormann. The other two are new and are based on atomistic orders and similarity relations. I will discuss how each of them deals with the Design Principles proposed by Douven and Gärdenfors and with further Adequacy Conditions. Despite being apparently different, I will show that these three models are mathematically equivalent. Finally, I will address three objections to the present approach. The first one says that the qualitative notion of a prototype is a bad analogue of the metric one. The second one suggests that, in contrast to the Voronoi construction, the function qualitatively representing the conceptual categorisation process is arbitrary. The last one appeals to Goodman's companionship and imperfect community problems to show that there is a flaw in defining simple concepts from similarity relations.

1 Introduction

The aim of this paper is to propose a qualitative approach to conceptual spaces that stands to Gärdenfors's metric framework Gärdenfors (2000) and Gärdenfors (2014) as qualitative models of belief stand to Bayesian ones. This requires introducing qualitative notions of similarity, simple concepts, prototypes and conceptual categorisation. For this purpose, I will introduce three mathematical models for conceptual spaces based on topology (Mormann's polar and WSA spaces in Mormann (2020) and Mormann (2021)), atomistic orders and similarity relations, respectively (the latter two are new). In order to show that these are reasonable counterparts of the

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metric approach, I will introduce some Adequacy Conditions that concern the features that similarity, prototypes and concepts should have. Then I will discuss how each of these models deals with these and with the Design Principles proposed by Douven and Gärdenfors in Douven and Gärdenfors (2019). Despite being apparently different, these three models will be shown to be mathematically equivalent to each other. Finally, I will address three objections that purport to show that there are important differences between the two approaches.

More specifically, the outline of the paper is as follows. In Sect. 2, I review the basic features of the standard metric framework of Gärdenfors' conceptual spaces, including Douven's and Gärdenfors' Design Principles. I add some Adequacy Conditions that a qualitative approach to conceptual spaces should satisfy. In Sect. 3, I introduce the first qualitative model for conceptual spaces, namely Mormann's polar and weakly-scattered spaces (WSA spaces). At first, his approach will be seen to be at odds with some of the Design Principles. In Sects. 4 and 5, I introduce two new qualitative models for conceptual spaces, namely the atomistic model and the similarity model, and I argue that they score better with respect to the Design Principles and Adequacy Conditions. In Sect. 6, I prove that the polar, similarity and atomistic models are mathematically equivalent (in a sense to be explained later on). I also provide several examples of conceptual spaces and of how to translate notions from model to another in Sect. 6.2. The equivalence shows how each of these models can make use of notions introduced by the other two to solve the apparent mismatches with the Design Principles and Adequacy Conditions. In Sect. 7, I address three objections to the qualitative approach that purport to show that the explications of the notions of prototype, conceptual categorisation and simple concepts are either defective or too different from the metric ones. The first objection is that the qualitative notion of a paradigm is a bad analogue of the metric one. The second one suggests that, in contrast to the Voronoi construction, the choice of the function qualitatively representing the conceptual categorisation process is arbitrary. The last objection appeals to Goodman's companionship and imperfect community problems to show that there is a flaw in the representation of simple concepts. After replying to these objections, the paper concludes with some general remarks and an Appendix 1 with the proofs for the main results.

2 The Theory of Conceptual Spaces

2.1 The Metric Approach to Conceptual Spaces

Let us recall first the core features of the conceptual spaces framework. Gärdenfors introduced in Gärdenfors (2000) his conceptual spaces to deal empirically with the problem of natural properties. These spaces are called 'domains'. The most studied example is the 3-dimensional colour solid whose dimensions represent the hue, saturation and brightness of colours (see Douven, 2019). In recent years, the theory has been successfully applied to other philosophical issues, such as vagueness Douven et al. (2013), cognitive semantics Gärdenfors (2014), confirmation Sznajder (2016), explication Benedetto (2020) and the structure of scientific theories Zenker and Gärdenfors (2014).

Although the original framework by Gärdenfors was quite liberal regarding which mathematical structures to use as representing conceptual spaces, most applications have used at least a *metric* space:

Definition 2.1 Let S be a set and $d : S^2 \to \mathbb{R}$ a real valued function. Then (S, d) is a *metric space* iff $\forall x, y, z \in S$:

i $d(x, y) \geq 0$. [Positiveness]
ii $d(x, y) = 0 \Leftrightarrow x = y$. [Indiscernibility]
iii $d(x, y) = d(y, x)$. [Symmetry]
iv $d(x, z) \leq d(x, y) + d(y, z)$. [Triangle Inequality]

Usually the space is the familiar space $\mathbb{R}^n$ with the Euclidean or taxicab metric. The distance between two points represents the *degree of dissimilarity* between objects. Then (ii) and (iii) are requirements analogous to the reflexivity and symmetry of a categorical similarity relation. Actually, (ii) is stronger than reflexivity, it says that two objects are maximally similar iff they are identical, which is a version of the Identity of Indiscernibles. Given that the metric distance is a function, by (i)-(ii) any two different objects will be dissimilar to each other *up to some degree of dissimilarity*. (iv) says that degrees of similarity can be added to each other and puts a bound to the similarities between any three objects. By taking products of spaces one can get spaces with several factors, which represent the respects of comparison. A point in each factor space represents a specific determinate attribute and an object is represented as an n-tuple of points. In other words, the framework represents objects as bundles of attributes and assumes a principle of Identity of Indiscernibles, for any two objects having the same determinate attributes will be represented by the same point. For instance, an object a corresponds to a pair (x, y) where x is its size and y is its colour.

Originally, natural properties (e.g. *Red*) were conjectured to be adequately represented by *convex regions* in a conceptual space. Using the convexity criterion, the geometric structure has a way to distinguish between natural and non-natural concepts. In contrast, concepts (e.g. *Apple*) would be represented as collections of convex regions from possibly different domains (e.g. *Green*, *Sweet*, *Round*, and so on). In order to introduce the notion of convexity, one needs to assume a betweenness relation among points. In the Euclidean space the usual choice is the *affine betweenness* $B(x, y, z)$ iff $y = (1 - t)x + tz$ for some $t \in [0, 1]$. Thus y is between x and z iff y is in the shortest segment that joins x and z. From a betweenness relation one can define the notion of convexity standardly as follows. A subset of the space A is *convex* iff it includes every point z that is between any two points x, y that are already in A. In the Euclidean space, convex regions have those nice round-like shapes we usually associate with the notion of convexity. Although arbitrary intersections of convex sets are convex, neither the unions nor the complements of convex sets need to be convex. Moreover, the spatial representation allows for the interpretation of several operations among concepts as spatial relations, providing a basic treatment of inferential relations between concepts. For instance, two concepts are co-instantiated by a common object iff the corresponding regions overlap, one concept implies another concept iff the region corresponding to the former one is included in the region representing the latter, and so on.

Gärdenfors proposed convexity as an empirical hypothesis and combined the model with the theory of prototypes pioneered by E. Rosch in Rosch (1975). To explain conceptual categorization, first several points of the space are chosen as representing prototypical objects. Then the space is divided into regions induced by the distances from the points of the space to each of these prototypes, by applying the Voronoi construction. Each region is a class of points fixed by a prototypical item and contains all those items that are more (or equally) similar to it than to the other prototypes. The points that are at equal distance from several prototypes form the *boundary* of the tessellation[1]. The resulting division represents the categorisation process in terms of prototypes: each object x is compared by similarity to each prototype p, and if it is sufficiently similar to p it is included under the corresponding concept. The cells of the tessellation represent the simple concepts, since with a suitably chosen metric (say the Euclidean one) the resulting Voronoi cells are convex. Although the empirical evidence originally suggested that convexity is necessary, this claim has been vigorously disputed (see Hernández-Conde, 2017).

Apart from convexity, Douven and Gärdenfors have argued for several 'design principles' that a system of natural concepts should satisfy. Such principles are chosen by analogy with an optimal conceptual scheme that would be developed to allow for a system to make correct, sufficiently fine-grained and successful classifications under limited constraints. Let us quote directly Douven and Gärdenfors (2019):

Design Principles

(1) **Parsimony**: The conceptual structure should not overload the system's memory.
(2) **Informativeness**: The concepts should be informative, meaning that they should jointly offer good and roughly equal coverage of the domain of classification cases.
(3) **Representation**: The conceptual structure should be such that it allows the system to choose for each concept a prototype that is a good representative of all items falling under the concept.
(4) **Contrast**: The conceptual structure should be such that prototypes of different concepts can be so chosen that they are easy to tell apart.
(5) **Learnability**: The conceptual structure should be learnable, ideally from a small number of instances.
(6) **Well-Formedness**: The concepts should be "well-formed" in that the items falling under any one of them are maximally similar to each other and maximally dissimilar to the items falling under the other concepts represented in the same space.

[1] I will ignore the problems related to the thickness of boundaries, see Douven et al. (2013).

Parsimony and *Informativeness* are holistic features, they concern the system of concepts as a whole. As the authors note, these conditions pull in opposite directions. Parsimony requires the overall system of concepts not to have too many concepts, to avoid overloading the memory. However, the system needs to have enough concepts to allow for informative inferences involving them. The more concepts we have, the more informative and less parsimonious the system of concepts will be, and vice versa. At the level of the prototypes, we have *Representation* and *Contrast*. Whereas Representation requires maximizing the similarities among the instances of a concept, Contrast requires maximizing the dissimilarities between prototypes of different concepts. *Well-Formedness* follows from the other two and is the most important criterion for this paper. Objects falling under the same concept should be maximally similar to each other and maximally dissimilar to objects falling under other concepts. By linking concepts to prototypes, the concept *Learning* process is easier, since one can develop the concepts from those prototypes and then recall the concepts again by recalling the prototypes.

The use of the terms 'properties' and 'concepts' in this approach suggests that there is an important ontological difference between the two. But in the context of conceptual spaces both properties and concepts seem to be the same sort of entities, the latter being just combinations of the former (which is needed to account for the phenomenon of compositionality). For example, *APPLE* is a concept obtained by combining properties such as *Sweet*, *Round*, and so on, which are convex regions from different domains (e.g. TASTE, SHAPE, and so on). But these latter 'properties' are Voronoi cells obtained by comparing how similar each object is to the prototypes. Thus these properties are the entities that result from the sort of conceptual categorisation process described by the prototype theory of concepts and arguably are concepts too. It is just that these concepts are simpler constituents of *APPLE*. For these reasons, I think it is just safer to call the former *simple natural concepts* and the latter *complex natural concepts*[2].

To sum up, according to the *metric* approach to conceptual spaces, natural simple concepts are represented as regions in a metric space. Inferences among simple concepts are given in terms of spatial relations among them. Concepts are structured around prototypical instances and conceptual categorisation works by constructing Voronoi tessellations from prototypes and comparing how similar new items are to the prototypical instances stored in memory. Moreover, an *optimal* system of concepts involves a few informative concepts with corresponding prototypical instances, in such a way that similarities among the instances of a common concept and dissimilarities among prototypes of different concepts are maximized.

2.2 The Qualitative Approach to Conceptual Spaces

The differences between the former approach and the models to be introduced below can be explained by appealing to Carnap's distinction between categorical,

[2] We will only consider natural concepts, so I will drop the adjective 'natural' from now on.

comparative and quantitative concepts in Hempel (1972). To take the usual example, there are at least three main conceptions of belief. First, we have the logical approach that takes belief to be a *categorical* notion including three propositional attitudes, namely believing a proposition, disbelieving a proposition and suspending judgement over a proposition. Second, we have the order-theoretic approach that takes belief to be a *comparative* notion, and thus allows for comparisons of the sort "S is more confident that *p* than she is that *q*". These two could be called *qualitative* notions. In contrast, Bayesian approaches take belief to be a *quantitative* notion admitting of many degrees that can be added. Now consider by analogy the case of conceptual spaces. We can take the metric formulation of conceptual spaces to be analogous to the Bayesian approaches, in the sense of making use of quantitative notions of similarity, prototypicality and so on. But we could also approach the topic by making use of categorical and comparative notions. The main aim of this paper is precisely to outline and discuss the strengths and possible limitations of such an approach.

Although Douven and Gärdenfors introduced the Design Principles with the metric approach in mind, I think that the criteria are general enough to apply to the qualitative models too. In particular, in the following sections I will make use of them to test the adequacy of the models proposed. Furthermore, it would be desirable that the qualitative picture was as similar to the metric one as possible. We should require categorical notions of similarity, simple concepts, prototypes and so on to have features analogous to those of the metric approach. To check whether this requirement is met, let us introduce the following Adequacy Conditions:

Adequacy Conditions

Categorical Similarity: Categorical similarity relations should be reflexive and symmetric. In some cases they also satisfy the Identity of Indiscernibles[3].

Categorical Concepts: Categorical simple concepts should be distinguished regions in a space and should satisfy the Design Principles.

Categorical Prototypes: Categorical prototypes should determine the simple concepts in virtue of their similarities to the objects falling under them.

Bridge Principles: To explain conceptual categorisation by prototypes, some 'bridge principles' that link similarity, paradigms and concepts are required.

In the following sections, I will introduce three qualitative models based on topology, order theory and the theory of similarity, respectively. The last two are new. I will go on highlighting how each of these addresses the Design Principles and

[3] According to this Identity of Indiscernibles, any two objects that fall under the same simple natural concepts are identical. Note that a similar principle is assumed by the metric approach too, since objects are represented as points, which are n-tuples of properties.

the Adequacy Conditions. Later on I will show that the three of them are mathematically equivalent to each other. Note that to explain compositionality any interesting account of conceptual spaces has to deal with complex concepts alongside simple ones. Nevertheless, for reasons of space, I will only focus on *simple* concepts. To account for simplicity, I will take simple concepts to be *maximal*, in the sense that no simple concept will be properly included in another. Since Parsimony and Informativeness are features of the whole system of concepts, I will be concerned only with Representation, Contrast, Learnability and Well-Formedness.

3 The Qualitative Approach to Conceptual Spaces I: Polarity

3.1 The Topological Model

Thomas Mormann has proposed in Mormann (2020) and Mormann (2021) a *topological* model for conceptual spaces. It was first introduced by Rumfitt to deal with the Sorites paradox in Rumfitt (2015). These *polar distributions* involve a classification of objects in terms of paradigmatic or prototypical objects, called *poles*. A simple example is the colour circle used to represent hues. The most prototypical exemplars of red, yellow, orange and so on are the poles. Any other colour in between, such as an orangish red, gets mapped to the poles to which it is similar (orange, red, ...). Thus the approach matches the *prototype* and *exemplar* models of psychological categorisation, but makes use of a categorical notion of a prototype, whereas the former usually assume that objects can be more or less prototypical. To distinguish between the two, we will use 'pole' or 'paradigm' for the categorical notion, and 'more or less prototypical' for the comparative and quantitative notions[4]. For the main notions I follow Mormann (2020):

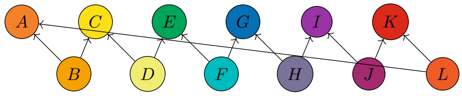
Definition 3.1 Let S be a non-empty set and $P \subseteq S$. A *polar distribution* over S is a function $m : S \to \wp(P)$ that satisfies (1)-(2). A distribution is (*PII*) iff it also satisfies (3):

(1) $\forall x \in S\ m(x) \neq \emptyset$.
(2) $\forall x \in S\ \forall p \in P\ m(x) = \{p\} \Leftrightarrow x = p$.
(3) $\forall x, y \in S\ m(x) = m(y) \Rightarrow x = y$. (PII)

Elements in P are *poles* or *paradigms*. The first axiom says that every element gets classified by some paradigm. The second one says that the paradigms are exactly the elements that are classified by just one element. The third one is an optional indiscernibility constraint, it requires distinct elements to be mapped to distinct paradigms. For an example, consider the following digital version of the

4 The notion of polar distribution I will make use of is the one from Mormann's paper Mormann (2020), not that from Mormann (2021). The latter is slightly more general, see Sect. 7.1. The condition 'PII' (Identity of Indiscernibles) does not occur in Mormann's writings.

Fig. 1 Colour Circle: Polar Distribution



colour circle. Right now it does not have the shape of a circle, but do not worry about this fact. Our space has some coloured spots $COLOUR = \{A, B, C, D, E, F, G, H, I, J, K, L\}$ as points and the paradigms are $P = \{A, C, E, G, I, K\}$. The assignment is pictured by arrows in Figure 1, for example, the spot B gets mapped to A and C, which are the paradigms of orange and yellow, respectively, i.e. $m(B) = \{A, C\}$. It can be checked that the circle is in fact polar (PII).

The crucial insight by Rumfitt was that polar distributions have *spatial structure*. To explain this we need to introduce some basic notions of topology[5]:

Definition 3.2 Let S be a set and $O(S) \subseteq \wp(S)$ a family of sets, to be called the family of *open sets*. Then $(S, O(S))$ is a *topological space* iff it satisfies:

(1) S and $\varnothing$ are open.
(2) If A, B are open, then their intersection $A \cap B$ is open.
(3) If $A_1, A_2, \ldots$ are open, then their union $\bigcup_i A_i$ is open.

A set $B \subseteq S$ is *closed* iff B^c is open. The family of closed sets is $C(S)$. A set is *clopen* iff is both open and closed. Given an element x, an *open neighbourhood of* x is an open set $N(x) \in O(S)$ which is such that $x \in N(x)$. In particular, we will say that a point x is open (closed) iff $\{x\}$ is an open (closed) set. Loosely put, one can think about a topological space as a set of points and families (open, closed, ...) of regions. Since conceptual spaces will be topological spaces, our points will be objects and some of the regions will be concepts.

The simplest examples of spaces are the *indiscrete space* and the *discrete space*. The indiscrete space has as open sets the whole space and the empty set, it is so coarse that one cannot use regions to distinguish between the points (i.e. it only has trivial concepts). At the other extreme, the discrete space has as open sets every subset of the space, it is so fine-grained that each collection of points counts as a region (i.e. every subset counts as a concept). The fundamental notions are:

Definition 3.3 Let $(S, O(S))$ be a topological space and $A \subseteq S$. Then:

(1) $Cl(A) := \bigcap\{B \in C(S) \mid A \subseteq B\}$ is the *closure* of A.
(2) $Int(A) := \bigcup\{B \in O(S) \mid B \subseteq A\}$ is the *interior* of A.
(3) A is *open regular* iff $A = IntCl(A)$.
(4) $x \leq y \Leftrightarrow \forall A \in O(S)(x \in A \Rightarrow y \in A)$ is the *specialization preorder* of S.

[5] The concepts are standard and can be found in any textbook on topology, say Willard (2004). The reader can find many examples of Alexandroff spaces in the Appendix of Mormann (2021).

The closure (interior) of a set is the smallest (biggest) closed (open) set including (included in) it. The last property says that in every topological space the points are preordered by the open sets they belong to. The topological spaces we are interested in are those that can be completely described by this preorder:

Definition 3.4 Let $(S, \leq)$ be a preorder and $A \subseteq S$. Then A is *up* iff if $x \in A$ and $x \leq y$ then $y \in A$. The set $O(S) = \{A \subseteq S \mid A \text{ is up}\}$ is the *Alexandroff topology* over S. The closed sets are exactly the *down*-s, i.e if $x \in B$ and $y \leq x$ then $y \in B$.

The most important and well-known fact about Alexandroff spaces is that they can be equivalently described as preorders. In other words, if we start from a preorder and take the topology of ups, then its specialization preorder will be the preorder we started from, and vice versa. In addition, well-behaved spaces need to satisfy an indiscernibility principle (a so-called 'separation axiom') in order to have enough resources to distinguish the points by making use of the open regions:

Proposition 3.1 *Let* $(S, O(S))$ *be a topological space. The following conditions are equivalent*:

(1) $O(S)$ *is a* T_0 *space.*
(2) $\leq$ *is a partial order.*
(3) *For all* x, y *in* S, *there is an open neighbourhood of* x *which is not a neighbourhood of* y, *or vice versa.*

It turns out that every polar distribution induces an Alexandroff topological space as follows Rumfitt (2015), Mormann (2020):

Proposition 3.2 *Let* (S, P, m) *be a polar distribution. Let* $O(S) := \{A \subseteq S \mid \forall x \in A\ (p \in m(x) \Rightarrow p \in A)\}$. *Then* $O(S)$ *is a* T_0 *Alexandroff topology over* S *called the polar topology.*

So a polar distribution is a space whose fundamental regions, the open sets, are 'centered around' the paradigms. The topological closure of a paradigm p is $Cl(p) := \{x \in S \mid p \in m(x)\}$. The specialization order is $x \leq^* y$ iff $x = y$ or $y \in m(x)$. The smallest open set for each x is $N_x = \{x\} \cup m(x)$. For example, if we consider the 'colour circle' in Figure 1 again, we have that whereas the closure of the orange spot A is $Cl(A) = \{L, A, B\}$, which contains all the orangish spots, the closure of the blue spot G is the set $Cl(G) = \{F, G, H\}$, which contains all the bluish spots. The specialization order is reflected by the arrows, for instance, we have that $F \leq G$, since the spot F has as a paradigm the blue spot G.

There is a fundamental property that characterizes polar spaces. First note that the set of open points in the space is exactly P, the set of poles or paradigms.

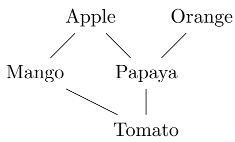


Fig. 2 Toy Example: Non-Polar WSA Space

Definition 3.5 Let $(S, O(S))$ be a topological space and $A \subseteq S$. Then A is *dense* iff $Cl(A) = S$. Moreover, $O(S)$ is a *Weakly-Scattered Alexandroff space* (WSA space) iff the set of open points is dense.

A dense region in a space is a set of points that are 'everywhere', so to speak. Whichever point in the space we choose, we will always be able to find a point in the dense region that is as close to it as we wish. The most important fact about polar spaces is that they are weakly-scattered. This is proven in Mormann (2020). According to Mormann, this is the fundamental topological property that makes the idea of there being enough prototypical elements precise: whichever object we choose, we will always be able to find a paradigm that is arbitrarily close to it.

For several reasons Mormann suggests taking as a model for conceptual spaces the more general class of weakly-scattered Alexandroff spaces. For instance, the conceptual spaces approach usually makes use of *product* spaces, but whereas the product of polar spaces is not necessarily polar, the product of WSA spaces is WSA. Moreover, whereas the specialization order induced by polar spaces only allows for a categorical distinction between prototypical and non-prototypical elements, that of WSA spaces allows for a comparative notion of prototypicality (we will consider this in depth later on). In particular, this means that we can have more complex orders of several 'layers' of prototypicality. For instance, the space in Figure 2 is a toy example of a WSA space that is not a polar order. The points are particular fruits. Orange is more prototypical than papaya, which is more prototypical than tomato. The maximal elements are the most prototypical fruits, namely the apple and the orange. Note that some fruits, like the mango and the papaya, are incomparable.

Due to scatteredness, one can still take the maximal elements in the order to be the maximally prototypical elements in WSA-spaces. So these play the role that poles play in polar spaces. Therefore, the polar spaces can be seen to be a very special case of WSA-spaces, those where the comparative notion of prototypicality collapses into a categorical one. Finally, several examples of digital spaces used in computer science, which are the 'discretized' analogues of our familiar continuous spaces like the Euclidean space (e.g. the digital circle we just saw), are non-polar WSA spaces.

3.2 Polarity and the Design Principles

In this section, I want to point at some apparent mismatches between the topological and metric approaches to conceptual spaces. I will focus on the Adequacy

Conditions and the design principles of Representativeness, Contrast, Learnability and Well-formedness. Briefly put, the model is this one:

Topological Model: According to the topological model, a conceptual space is represented as a weakly-scattered Alexandroff space $(S, O(S))$. Objects are points, most prototypical objects are open points and concepts are open regular sets.

First, the metric approach represents degrees of similarity as distances, but it is not clear how similarity is represented by the topological model. Similarity relations are at least reflexive and symmetric[6]. These properties are well captured by the axioms of a metric space. However, they seem to be absent in both polar and WSA spaces. In other words, the model breaks the Categorical Similarity Condition.

Second, Mormann takes concepts to be represented by open regular sets, which are a special sort of open sets. His motivation for choosing open sets concerns vagueness. Take polar open sets to be the semantic values of predicates. If A is open and $a \in A$, then a has an open neighbourhood $N(a) \subseteq A$. The explanation seems to be the following one. If an object x falls under a concept A, then one can always find a more specific concept $N(x)$ under which x falls too. The more specific concept $N(x)$ contains 'minor variations' of x. Thus the concept A is open iff it is stable under small changes to its instances, which could be used to explain the phenomenon of vagueness. But it is not clear to me that *polar* open sets properly preserve basic intuitions about similarity and therefore I am unconvinced that they are the appropriate choice to model natural concepts. An open set can contain two objects x and y that lack common poles, so long as it includes all the poles of x and all the poles of y. Arguably, if x and y are not similar to a common pole, they are not similar to each other. For instance, take the colour circle. L is orange-red and I is purple, so they are not similar enough, because they lack common poles, but the set $\{L, I, A, K\}$ including L, I and their poles will be polar open. There is no common pole we can point at that will serve as a basis for a similarity judgment. One could reply that these are examples of complex concepts, which are unions of simpler ones. However, the point also concerns simple concepts. For instance, the minimal open set $N(B) = \{A, B, C\}$ of B contains B alongside all its poles. What natural concept does this open set correspond to? In what sense are all of B-s poles similar to each other? The crucial point is that a good classification should maximize the similarities among the members of a common class (by Well-Formedness) while minimizing similarities among members of different classes, specially among poles (by Contrast). Choosing the polar open sets to represent simple concepts seems to be at odds with these two principles. In contrast, consider the *closures* of paradigms. If we take these to be the simple concepts, every pair of objects falling under a concept will be mapped to a common paradigm. This is the choice that I will make in the following

[6] Tversky famously argued in Tversky (1977) against the symmetry assumption. I cannot address this objection here, see Decock and Douven (2009) for a brief defence of the spatial model.

sections. For the case of the colour circle, it gives the expected results. For example, the closure of E is $Green = Cl(E) = \{D, E, F\}$, which gives us the green spots.

Finally, one of the main reasons for considering general WSA spaces instead of polar spaces is that, whereas the former provide a comparative notion of prototypicality by making use of the specialization preorder, the latter collapse prototypicality to a categorical notion. However, in this move from polar to WSA spaces an interesting feature of polarity is lost. In polar spaces that satisfy (PII), an object can be uniquely described by pointing at those poles the object is mapped to, or equivalently, by listing which concepts the object falls under. To put it differently, giving the poles exhausts all the relevant information about an object. But some conceptual spaces seem to satisfy this condition. For instance, in the colour circle, one can uniquely describe each coloured spot by listing the paradigms it is similar to. Apart from the colour circle, many other examples of this phenomenon, such as classifications of geographic location by latitude and longitude, or classifications of dates by day, month and year, can be found in section 6.2. Although WSA spaces provide a comparative notion of prototypicality, unless we add other constraints this feature of polar spaces is lost. To avoid this conclusion, we can add one last condition to be satisfied by some spaces:

Atomism: For some spaces, there should be a way to explain how an object can be completely described by pointing at the prototypes it is similar to.

In the following sections I will introduce two new models. Whereas the atomistic model can account for this last problem, the similarity model solves the other two. However, I will also show that the three models are mathematically equivalent. More specifically, polar distributions are equivalent to certain similarities, and by adding the indiscernibility axiom we will get an equivalence between the three models. A fortiori, the mismatches I just pointed out will be simply apparent since they can be dealt with by introducing by definition in this model the successful features of the others. Moreover, in a sense, the atomistic structures form a special subclass of WSA spaces that still allow us to represent non-trivially prototypicality comparatively. Thus my point will be that we can have the best of both worlds: we can keep the polar model and still have a special class of WSA spaces that allows for a comparative notion of prototypicality that satisfies all the desiderata.

4 The Qualitative Approach to Conceptual Spaces II: Atomism

4.1 The Atomistic Model

The second model I will introduce is order-theoretic. Objects will be in relations of more or less prototypicality. So the notion of prototypicalness will be comparative. Nevertheless, there will also be some maximally prototypical objects, which will correspond to the poles we discussed before. The simple concepts will be the sets of all objects that are less prototypical than these maximally prototypical objects. A fortiori, the less prototypical an object is, the more concepts it will fall under, and vice versa. Each object will also be describable as a 'combination' of its maximally prototypical objects.

In the literature on prototype theories of concepts, it is often highlighted that the notion of a prototype is comparative. As was mentioned, Mormann proposes to replace the categorical notion of a prototype by a comparative notion, by making use of the specialization order that describes the Alexandroff spaces. Since WSA spaces require the existence of maximal elements in this order, these represent the maximally prototypical objects. The closer an element is to a maximal element, the more prototypical it will be. If we restrict our attention to these maximal elements, the categorical and comparative notions of prototypicality coincide.

In the following, I will modify this basic idea in two ways. First, I will establish a correspondence between how prototypical an object is and how many concepts the object falls under. Let us estipulate that y is *qualitatively richer* than x iff y falls under all the concepts that x falls and possibly more. I propose to make the order of prototypicality to correspond to the *dual* of this order. In other words, x is more prototypical than y iff y is qualitatively richer than x. The idea is that in order to satisfy Representativeness we will require prototypes to fall under as few concepts as possible, so that the most representative instance of a concept turns out to be an object that only falls under that concept. Formally, we have to put the WSA order suggested by Mormann *upside-down*, so that the maximally prototypical objects are now the *minimal* elements of the order. Second, following the discussion in the previous section 3.2, I will require that the information about each object be completely exhausted by the maximally prototypical objects it is similar to. In other words, we will require some sort of *qualitative atomism*, where each object is the 'qualitative sum' of its paradigms. Both examples and counterexamples to this condition will be given in section 6.2.

Recall that z is an *upper bound* (resp. *lower bound*) of x and y iff $x, y \leq z$ (resp. $z \leq x, y$). An element x is *minimal* (resp. *maximal*) iff if $z \leq x$ then $x = z$ (resp. if $x \leq z$ then $x = z$) (see Davey and Priestly (2012)). Our class of orders will be:

Definition 4.1 Let L be a poset. Then L is *atomic* (resp. co-atomic) iff for each element x there is an element z such that z is minimal (resp. maximal) and $z \leq x$ (resp. $x \leq z$). If L is atomic (resp. co-atomic), then L is an *atomistic poset* (resp. co-atomistic poset) iff each element in L is the smallest upper bound (resp. greatest lower bound) of all its minimal (resp. maximal) elements.

It is immediate that an Alexandroff T_0 space is WSA iff its specialization order is co-atomic. Thus the only difference between the model we will introduce now and the WSA spaces is the stronger requirement of co-atomism. Recall that an element in a poset is the *bottom* iff it is the smallest element. Note that a poset with a bottom element is atomistic iff it has just one element. So atomistic posets do not have bottoms. Some examples of atomistic posets are *atomic Boolean algebras* and *complete atomistic lattices* (with the corresponding bottom elements removed).

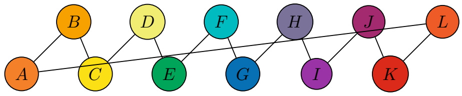


Fig. 3 Colour Circle: Atomistic Order

From now on, we will interpret $x \leq y$ as "x is more prototypical than y" or equivalently as "y falls under all concepts that x falls and possibly more". Let us denote the set $\uparrow x = \{y \in L \mid x \leq y\}$ as usual. The following is an easy topological reformulation:

Proposition 4.1 *Let L be a poset. Then the following conditions are equivalent*:

(1) *L is an atomistic poset.*
(2) $\uparrow x = \bigcap\{\uparrow p \mid p \in \downarrow x \text{ and } p \text{ is minimal}\}$ *for every x in L.*

Let $Atom(L)$ be the set of minimal elements (atoms) in L and $atom(x)$ the set of minimal elements below x. We have that if $x \leq y$ then $atom(x) \subseteq atom(y)$ by transitivity and conversely, if $atom(x) \subseteq atom(y)$ then $x = \vee atom(x) \leq \vee atom(y) = y$. From this fact the previous proposition follows.

To represent concepts we will use the sets of the form $\uparrow p$ where p is an atom, which are known as *principal ultrafilters*. A concept is the set of all the objects that are qualitatively richer (less prototypical) than a given paradigmatic object. For instance, take the colour circle again. We will model it as an atomistic poset (see Figure 3). As we can see, the polar order is now upside down. The atoms are $\{A, C, E, G, I, K\}$, the concepts are now the ultrafilters such as $Blue =\uparrow G = \{F, G, H\}$ or $Red =\uparrow K = \{J, K, L\}$. We see, for example, that the spot A is more prototypical than B. Each coloured spot is the least upper bound of its paradigms. This particular example is misleading in one sense. We will later on show that the order and topological models are equivalent. Nevertheless, this equivalence is not that the polar order and the atomistic order are converses to each other. Usually, the atomistic order induced by a polar space is much richer than the polar order. It is accidentally the case that in this particular example both are converses.

Before we move on let us consider the topology once more. Recall that Alexandroff spaces and preorders are mathematically equivalent. The weakly-scattered Alexandroff spaces of Mormann's model are exactly the co-atomic posets. So we can consider a more specific class of weakly-scattered Alexandroff spaces, the ones that correspond to the co-atomistic posets. We just need the topological separation condition that is dual to the one we mentioned before, namely:

Definition 4.2 Let $(S, O(S))$ be a weakly-scattered Alexandroff topological space. Then S is *co-atomistic* iff $Cl(x) = \bigcap\{Cl(p) \mid p \in N_x \text{ and p is open}\}$ for every $x \in S$.

What the previous condition says is that the set of objects that have the same properties (and possibly more) of a given object x is exactly the set of objects that belong to all the properties induced by the paradigms of x. Topologically, this says that every object z that is sufficiently close to x is sufficiently close to every paradigm of x. The spaces I propose to consider are the *duals* of co-atomistic spaces, namely those satisfying the topological condition corresponding to atomism.

4.2 Atomism and the Design Principles

It is time to check how well this model satisfies the Adequacy Conditions and the principles of Representativeness, Contrast, Learnability and Well-formedness. Let me give first a brief description:

Order-Theoretic Model: According to the order-theoretic model, a conceptual space is represented as an atomistic order $(S, \leq)$. Objects are elements, maximally prototypical objects are atoms (minimal elements) and simple concepts are principal ultrafilters.

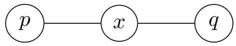
Since simple concepts are represented as ultrafilters, it is worth analysing their features closely. First, note that every simple concept F is maximal, it is not properly included in another concept. Thus the concepts considered are 'simple', in the sense of not being the result of combinations of other concepts. In particular, the collection of all objects is not taken to be a simple concept. Second, since we know that the Alexandroff T_0 topologies are exactly the partial orders and simple concepts are ups, they are open regions in the corresponding space. Third, the following hold for every simple concept F:

i If x is F and y is qualitatively richer than x (x is more prototypical than y), then y is also F.
ii There is an object p which is the most prototypical instance of F.

These conditions say that a concept is a collection of objects that can be 'refined' until one reaches a most prototypical instance. This instance is precisely the paradigm of the concept. Therefore, all the requirements we put on simple concepts and prototypes seem to hold. According to Representativeness, each concept should be well represented by a prototype. Since we put the order upside down, x is more prototypical than y iff y is qualitatively richer than x. In other words, the fewer concepts an object falls under, the more prototypical it will be. Given the bijective correspondence between simple concepts and paradigms (the atoms), by the previous features it follows that each paradigm falls exactly under one simple concept. Thus according to this model, paradigms are *qualitatively atomic* entities. A paradigm of a concept is its most representative instance, because it is an object that *only falls under that concept*.

Furthermore, we have the following bridge principle that says that having a paradigm as a common atom is equivalent to being an instance of a common concept (let F be a principal ultrafilter):

Fig. 4 Simple Similarity



$$atom(x) \cap atom(y) \neq \emptyset \Leftrightarrow \exists F \; x, y \in F$$

The previous requirement of Atomism that failed in the WSA setting now is satisfied, for the information about an object is completely exhausted by giving all its atoms. To put it pictorially, each object is the *qualitative sum* of its paradigms.

Just as in the topological case though, there are no similarity relations. So it would seem that Contrast, Well-formedness and the remaining Adequacy Conditions fail. But this complaint would be unfair. We already have the simple concepts, so if we defined a similarity relation of “falling under a common concept" these principles would be satisfied. However, this move still leaves us with a difference between the metric and the qualitative approaches. According to the former, similarity relations are taken as given and simple concepts are represented as spatial regions induced by them. But in this model simple concepts are induced by the prototypicality relations between objects and similarity is defined from the concepts.

5 The Qualitative Approach to Conceptual Spaces III: Resemblance

5.1 The Similarity Model

Regrettably, the previous models make no mention at all of similarity relations between objects. In contrast, I will provide now a third model based on *similarity structures*. Objects will be in relations of categorical similarity. A paradigm will be an object which is such that any two objects that are similar to it are similar to each other. The simple concepts will be the sets of objects that are similar to a common paradigm.

Some of the notions I will be making use of are from the literature on Carnap’s quasianalysis (see Carnap (1923), Carnap (1967), Mormann (1994), Leitgeb (2007), Mormann (2009))[7]. The quasianalysis provided a method to reconstruct properties common to objects in similarity relations and then represent those objects as bundles of properties. Since we want to represent concepts as classes of similar objects, the quasianalysis is precisely the tool we need.

A *similarity structure* is a set S with a binary reflexive and symmetric relation $\sim \subseteq S \times S$. The transitive similarities are the *equivalence relations*[8]. Examples of similarities are “the distance between x and y is less than ε", “x and y fall under a common concept" and “x and y share a common prototype". We can depict each finite similarity structure geometrically as an undirected graph. Objects will be

[7] The similarity model is based on Carnap’s pre-Aufbau work. It is different from his later theory of *attribute spaces*. For the latter, see Sznajder (2016).

[8] Similarity structures are also called ‘tolerance structures’ and ‘reflexive undirected graphs’.

pictured as dots and similarities as undirected edges connecting these dots. For example, the simple similarity $p \sim x \sim q$ will be pictured as in Figure 4:

We just need some basic notions from Mormann (2009):

Definition 5.1 Let $(S, \sim)$ be a similarity structure and $A \subseteq S$. Then we define:

(i) $co(A) := \{x \in S \mid \exists y \in A\ x \sim y\}$ is the *similarity neighbourhood* of A.
(ii) A is a *clique* $\Leftrightarrow \forall x, y \in A\ x \sim y$.
(iii) A is *maximal* $\Leftrightarrow \forall z \in S\ ((\forall x \in A\ z \sim x) \Rightarrow z \in A)$.
(iv) A is a *similarity circle* $\Leftrightarrow A$ is a maximal clique.

The *set of all the similarity circles* is $SC(S)$. In the previous example, $co(x) = \{p, x, q\}$ and $SC(S) = \{\{p, x\}, \{x, q\}\}$. We will take similarity circles to represent natural simple concepts. The model I will propose is an extension of some ideas by Carnap in Carnap (1923):

Definition 5.2 Let $(S, \sim)$ be a similarity structure and $T \subseteq S$ and $p \in S$. Then p is a *generator of order 1* iff if $x \sim p \sim y$ then $x \sim y$ for any two $x, y \in S$. T is a *similarity circle of order 1* iff $T = co(p)$ for some generator of order 1 p.

It is immediate that p is a generator of order 1 iff $co(p)$ is a similarity circle of order 1. The set $Gen(S)$ is the *set of generators of order 1* and $SC_1(S)$ is the *set of similarity circles of order 1*. Carnap called these generators 'representing elements'. Whereas generators will represent paradigms, similarity circles of order 1 will represent simple concepts. The following class of similarities provides the model for conceptual spaces:

Definition 5.3 Let $(S, \sim)$ be a set S with a binary relation $\sim \subseteq S \times S$. Then S is a *pure similarity structure of order 1* iff for all $x, y \in S$ and $p, q \in Gen(S)$:

(i) $x \sim x$. [Reflexivity]
(ii) If $x \sim y$ then $y \sim x$. [Symmetry]
(iii) If $p \sim q$ then $p = q$. [Pure]
(iv) If $x \sim y$ then $x \sim p \sim y$ for some generator $p \in Gen(S)$. [Order 1]

The third axiom (Pure) says that any two similar paradigms are identified. In other words, paradigms must be 'maximally dissimilar to one another'. Equivalently, a similarity is pure iff each similarity circle T of order 1 has a unique generator p. The fourth axiom is crucial, it says that two similar objects must be similar to a common paradigm. This captures the idea that paradigms classify objects. Equivalently, a similarity is of order 1 iff if any two similar objects x and y belong to a common similarity circle of order 1, $x, y \in T \in SC_1(S)$, i.e. fall under a common simple concept. In order to represent conceptual categorisation, we introduce the following functions:

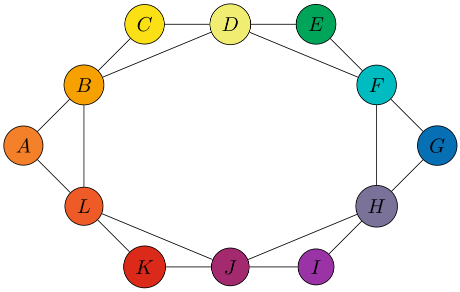


Fig. 5 Colour Circle: Similarity

$$gen : S \to \wp(Gen(S))\ gen(x) := \{p \in Gen(S) \mid x \sim p\}$$
$$q : S \to \wp(SC_1(S))\ q(x) := \{T \in SC_1(S) \mid x \in T\}$$

Whereas the function $gen(x)$ represents each object as the set of its generators (its paradigms), the function $q(x)$ represents each object as the set of concepts under which it falls. I will come back to these in the following sections.

Finally, in some spaces a stronger indiscernibility axiom holds. We define a new relation of duplication that holds between those entities that are similar to the same entities. It will follow that two duplicates belong to the same similarity circles. We will consider structures for which this relation coincides with identity, those that satisfy Mormann (2009):

Definition 5.4 Let $(S, \sim)$ be a similarity structure. Then S satisfies the *Similarity Neighbourhood Indiscernibility Axiom* (SNI) iff if $co(x) = co(y)$ then $x = y$.

This axiom is the similarity analogue of the *Identity of Indiscernibles*. It says that if two entities are similar to the same entities, then they are identical. Every (SNI) similarity of order 1 is pure, but the converse is false. For instance, $y \sim p \sim x$ & $y \sim q \sim x$ is pure, but is not (SNI) since x and y are indiscernible.

The previous notions can be checked again using the colour circle, now in Figure 5. It will now be clearer in what sense this is allegedly a *circle*.

In this example, the generators of order 1 (paradigms) are A, C, E, G, I and K and the similarity circles of order 1 (the concepts) are the triangles, such as $Yellow = co(C) = \{B, C, D\}$ or $Green = co(E) = \{D, E, F\}$. The previous functions represent each spot by the paradigms it is similar to, like $gen(B) = \{A, C\}$ or by the concepts it falls under, like $q(B) = \{Orange, Yellow\} = \{\{L, A, B\}, \{B, C, D\}\}$. It can be checked that the circle is in fact (SNI) of order 1.

5.2 Resemblance and the Design Principles

Let us now check how the similarity model fits the design principles of Representativeness, Contrast, Learnability and Well-formedness. The model can be briefly described as follows:

Similarity Model: According to the similarity model, a conceptual space is represented as an (SNI) similarity structure of order 1 $(S, \sim)$. Objects are elements, maximally prototypical objects are generators and simple concepts are similarity circles of order 1.

Consider the basic axioms and definitions. On the one hand, Representativeness is guaranteed by combining the axiom of order 1, which says that two similar objects are similar to a common paradigm, with the definition of a paradigm and the definition of concepts as similarity circles of order 1, which makes all the objects similar to a common paradigm similar to each other and the resulting collection to be maximal. On the other hand, Contrast is assumed by the axiom of purity, which states that no two paradigms are sufficiently similar to each other. In other words, one can interpret the assumptions made by the similarity model directly as the requirements that the conditions of Representation and Contrast be satisfied. The condition of Well-Formedness is immediately satisfied by the definition of a similarity circle, since any pair of objects falling under a concept will be similar to each other. Finally, since each simple concept corresponds uniquely to a paradigm, a fewer amount of paradigms will make concept learning easier for the agent[9].

What about the Adequacy Conditions? The similarity model provides plausible categorical analogues of objects, prototypes, similarity relations, simple concepts and the process of conceptual categorisation. It gives formal properties which are categorical analogues of those of metric spaces, namely reflexivity, symmetry and the identity of indiscernibles. Finally, note that the following bridge principle holds:

Similarity-Paradigm-Concept: Two objects are similar iff they fall under a common concept iff they are similar to a common paradigm.

This bridge principle is supported both by the Design Principles and the Prototype Theory of Concepts. According to this theory, every concept is generated by some prototype. The similarity to this prototype is what determines the conditions under which something falls under the concept. So two objects fall under a common concept iff they are similar enough to the prototype of the concept. But by the design principle of Well-Formedness, any two objects falling under a common concept must be similar to each other. Moreover, since the concepts are developed to explain the similarities observed between the objects, if two objects are similar then this similarity must be explained by some concept that they both fall under.

Note that, in exchange, we lost the comparative notion of prototypicality, atomicity and the condition that concepts should be regions in a space. These problems will be fixed in the next section.

[9] I grant that requiring just one paradigm for each concept is a quite strong idealization. However, the Voronoi cells in the metric approach are also induced by one object.

6 Equivalence Between the Models

6.1 Similarity-Topology-Order Equivalence

In this section I will show that the three models are mathematically equivalent. More specifically, I will show that by making use of the primitive notions of one model (e.g. poles, polar distribution and so on) we will be able to define all the primitive notions of the other models (e.g. similarity, similarity circles, and so on). This is done by defining a function that sends each model of one kind (e.g. each polar model) to a model of the other kind (e.g. a similarity model) in such a way that by composing these transformations we will get back to the model we started from. That being so, the problems about the previous approaches vanish, since we will be able to introduce the required conditions by making use of similarity, order and topology as wished.

First I will show that the similarity model and the polar model are equivalent. Note that given any similarity structure, the function:

$$gen : S \rightarrow \wp(Gen(S)) \; gen(x) := \{p \in Gen(S) \mid x \sim p\}$$

that represents each object as a bundle of paradigms is a polar distribution. Conversely, a polar distribution induces a similarity as follows:

$$x \sim^{*} y \Leftrightarrow m(x) \cap m(y) \neq \emptyset$$

This shows that there was indeed a similarity hidden in the original polar model. It is the similarity of having a common pole. This basic observation provides us the resources to show that the two structures are equivalent.

Theorem 6.1 *Let* (S, P, m) *be a polar distribution. Then* $(S, \sim)$ *defined as* $x \sim y := \exists p \in P \; p \in m(x) \cap m(y)$ *is a pure order* 1 *similarity. It induces a polar distribution* $(S, Gen(S), gen)$ *which is such that* $P = Gen(S)$ *and* $m = gen$. *Conversely, let* $(S, \sim)$ *be a pure order* 1 *similarity. Then* $(S, Gen(S), gen)$ *is a polar distribution where* $(S, \sim')$ *is such that* $x \sim y$ *iff* $x \sim' y$.

Corollary 6.1 *For a polar distribution and its similarity, the following hold*:

1. *The polar distribution is* (*PII*) $\Leftrightarrow$ *The similarity is* (*SNI*).
2. *The poles are exactly the generators of order* 1.
3. *The closures of poles are exactly the similarity circles of order* 1.

The second result establishes that, assuming the indiscernibility axiom, the similarity model and the order-theoretic model are equivalent. In any partial order one can define the following similarity of having a common lower bound:

$$x \sim^{*} y \Leftrightarrow \exists z \; z \leq x, y$$

In any similarity structure one can define the following preorder Mormann (2009):

$$x \leq_{co} y \Leftrightarrow co(x) \subseteq co(y)$$

The similarity is (SNI) iff this preorder is anti-symmetric. Notice that this order corresponds to the idea of qualitative richness we mentioned before. Object y is qualitatively richer than x iff every object similar to x is similar to y too. Given the correspondence between being similar and sharing a common concept, we have that object y is qualitatively richer than x iff y falls under every concept under which x falls, and possible more.

These two structures are usually not in unique correspondence. However, we will show now that for the case of similarities of order 1 and atomistic orders this holds:

Theorem 6.2 *Let* $(S, \sim)$ *be a* (*SNI*) *similarity structure of order* 1. *Then* $(S, \leq_{co})$ *is an atomistic poset where the minimal elements are* $Gen(S)$. *Moreover*, $(S, \sim^*)$, *defined as* $x \sim^* y := \exists z \in Min(S)\ z \leq x, y$, *is identical to* $(S, \sim)$. *Conversely*, *if* $(S, \leq)$ *is an atomistic poset*, *then* $(S, \sim^*)$ *is a* (*SNI*) *similarity structure of order* 1 *such that* $x \leq y$ *iff* $x \leq_{co^*} y$.

Corollary 6.2 *For an atomistic order and its similarity*, *the following hold*:

1. *The generators of order* 1 *are exactly the atoms* (*minimal elements*).
2. *The similarity circles of order* 1 *are exactly the principal ultrafilters*.

In other words, the class of (SNI) similarities of order 1 is the class of atomistic posets. The generators of order 1 are exactly the minimal elements and the similarity circles of order 1 are the principal filters $\uparrow p$ of the minimal elements.

This result has an interesting corollary. Whereas WSA spaces give a comparative notion of prototypicality, polar spaces are restricted to a categorical one. Nevertheless, under the assumption of co-atomism both models are equivalent:

Corollary 6.3 *Let* $(S, O(S))$ *be a co-atomistic weakly-scattered Alexandroff space*. *Then* $(S, Max(S), m')$ *where* $m'(x) := \{p \in S \mid x \in Cl(p) \text{ and p is open}\}$, *is a* (*PII*) *polar distribution such that* $(S, \leq)$ *defined as* $x \leq y \Leftrightarrow m'(y) \subseteq m'(x)$ *is the specialization order of the original space*. *Conversely*, *if* (S, P, m) *is a* (*PII*) *polar distribution*, *then* $(S, \leq)$ *is the specialization order of a co-atomistic weakly-scattered Alexandroff space which is such that* $Max(S) = P$ *and* $m' = m$.

So if we put their order upside-down, co-atomistic WSA spaces are equivalent to the polar spaces. In other words, an object x is more prototypical than an object y iff y is similar to all the poles that x is similar to. This is, I think, a surprising result: even though generally WSA spaces are richer in their order, coatomistic WSA spaces are those spaces where the comparative notion of prototypicality is completely fixed by the categorical one.

Lastly, it is illuminating to see how the fundamental notions are 'translated' between the models:

Similarity

x is similar to *y* iff
x and *y* are similar to a common paradigm iff
x and *y* fall under a common simple concept iff
x and *y*, as bundles of simple concepts/paradigms, overlap.

Indiscernibility

x and *y* are indiscernible iff
x and *y* are similar to the same objects iff
x and *y* are similar to the same paradigms iff
x and *y* fall under the same simple concepts iff
x and *y*, as bundles of simple concepts/paradigms, are identical.

Paradigms

p is a paradigm iff
Any two objects similar to *p* are similar to each other iff
p falls under a unique concept iff
p is a qualitatively minimal or maximally prototypical object.

Simple Concepts

T is a simple concept iff
T is the collection of all the objects similar to a paradigm iff
T is a maximal collection of similar objects generated by an object.

Prototypicality

y is qualitatively richer than *x* iff
y is similar to every object to which *x* is similar iff
y is similar to every paradigm of *x* iff
y falls under all the simple concepts that *x* does iff
x is more prototypical than *y* is.

6.2 Examples of Conceptual Spaces

We have seen three equivalent descriptions of the colour circle. A similar structure with fewer paradigms for sweetness, sourness, bitterness and saltiness would give us a *taste circle* (a digital version of the surface of the taste tetrahedron)[10]. Let us take a look at other new spaces.

[10] For empirical purposes a more detailed model of the colour solid may be required, e.g. the one in Douven (2019). If so, we need a weaker notion of polar distribution, see Section 7.1.

$$
\begin{tikzcd}
0 & & 2 & & 4 & & 6 & & \dots \\
1 \arrow[u] \arrow[urr] & & 3 \arrow[u] \arrow[urr] & & 5 \arrow[u] \arrow[urr] & & \dots
\end{tikzcd}
$$

Fig. 6 Natural Numbers

6.2.1 Dates and Historical Events

Historical events are usually recalled by appealing to the dates when they occurred. Each date can be completely described (e.g. using the Gregorian calendar) by combining the corresponding day, month and year. We can take as points in our space specific dates such as:

$$m(\text{end of WWII}) = \{2_d, \textit{September}, 1945_y\}$$
$$m(\text{Rudolf Carnap's birthday}) = \{18_d, \textit{May}, 1891_y\}$$

If we choose a paradigm like the year 1945 (e.g. any specific day of 1945), its simple concept $Cl(1945)$ will include all the dates of the events that happened in 1945. Two dates are similar iff they concern the same day, month or year. If our space only classifies dates then it must be (PII). If the aim is to represent historical events and there are at least two events that occurred at the same day, then the space will not be (PII) but it will still be polar.

6.2.2 Natural and Integer Numbers

Another interesting example is given by the *digital* or *Khalimsky line* (see Mormann (2020)) used in computer science. This is a polar topology that can be defined over the *integers*. Explicitly, the space has the form $(\mathbb{Z}, Even^0(\mathbb{Z}), m)$, where the set of paradigms $Even^0(\mathbb{Z})$ contains zero and the even numbers and where $m(n) = \{n-1, n+1\}$ iff n is odd, and $m(n) = \{n\}$ iff n is even or zero. For example, $m(0) = \{0\}$ and $m(3) = \{2, 4\}$. Moreover, this particular polar distribution is even PII and the *natural numbers* $\mathbb{N}$ (with zero) inherit the topological structure (see Figure 6).

6.2.3 Words and Alphabets

There are many interesting symbols that we can use for purposes of representation. Most of these can be considered to be letters of our *alphabets*, such as the Latin alphabet $Latin = \{a, b, c, \dots, z\}$, the Roman numerals Roman Numerals $= \{I, II, III, IV, \dots\}$ or the Morse code $Morse = \{., , -\}$. The crucial feature of alphabets is that we can obtain *strings* by *concatenating* the symbols. Starting from the latin alphabet we get strings such as *b*, *tree*, *love*, *helloworld*, *blabla*, and so on. We can classify words by taking as paradigms the symbols in the alphabet. The concept induced by a symbol contains all the words that are made up by it. Two words are similar iff they share a common symbol.

Since two different words such as 'risen' and 'siren' can be constituted by the same symbols, the space is not (PII). Nevertheless, the space is polar. Mathematically, the previous examples are all cases of the free monoid construction of strings from an alphabet (all the strings obtained by concatenating symbols, including the empty string ε). By taking as points all the strings, as paradigms the empty string ε and all the symbols in the alphabet and by defining $m(x) = \{x\}$ if x is in P and $m(x) = \{p \in P \mid p \text{ is substring of } x\}$ otherwise, we get that every free monoid is a polar distribution. For example, $m(aaa) = m(aa) = \{a, \varepsilon\}$, $m(tree) = \{\varepsilon, e, r, t\}$, $m(helloworld) = \{\varepsilon, e, d, h, l, o, r, w\}$, $m(a) = \{a\}$ and $m(\varepsilon) = \{\varepsilon\}$.

6.2.4 Classifications by Coordinates

Each location of the surface of the Earth can be described uniquely by giving its latitude and its longitude. We take latitudes and longitudes as paradigms for geographic classifications, here are some examples:

$$m(\mathit{Machu\ Picchu}) = \{-13^\circ_{lat}, -72^\circ_{long}\}$$
$$m(\mathit{Athens}) = \{37^\circ_{lat}, 23^\circ_{long}\}$$

If we want to use the Euclidean plane $\mathbb{R}^2$ to represent the previous locations we can select as paradigms the pairs in the 'X and Y axes' $P := \{(x, 0), (0, y) \mid x, y \in \mathbb{R}\}$. Reformulating the examples we would have $m(\mathit{Machu\ Picchu}) = \{(-13, 0), (0, -72)\}$. The paradigms would then be the locations of the Earth that have a latitude or longitude of 0°, namely those at the Equator or at the Prime Meridian. The concepts would be the lines of latitude and the meridians. Two locations are similar iff they have the same latitude or the same longitude.

For another example, we are used to representing concrete objects as points in the 3-dimensional Euclidean space $\mathbb{R}^3$, by providing their length, width and depth according to a certain measurement scale (e.g. cm, m, km, ...). We can consider again each of the points in the X, Y and Z axes as paradigms. E.g. (0, 3, 0) is a paradigm and $m(2, 3, 2) = \{(2, 0, 0), (0, 3, 0), (0, 0, 2)\}$. Two objects will be similar iff they have the same length, width or depth. For a more abstract example, each complex number is usually represented by a pair of real numbers, its real part and its imaginary part, and we can use the previous representation to show this e.g. $m(2 + 3i) = \{(2, 0), (0, 3)\}$.

These are all special cases of a more general phenomenon that I will call *classification by coordinates*, which is a clear example of a polar classification. It is crucial to these examples that the same entity (location, object) is mapped to *several paradigms* and not just to one. Note also that there is nothing special about $\mathbb{R}^2$ in these examples. The Euclidean plane can be replaced by any finite n-dimensional vector space F^n over some field F, by taking first the canonical basis $B = \{(1, 0, \ldots, 0, 0), \ldots, (0, 0, \ldots, 0, 1)\} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and then choosing as paradigms the vectors obtained by scalar multiplication over basis vectors $P := \{c\mathbf{e}_k \mid \mathbf{e}_k \in B, c \in F\}$ (includes the zero vector $\mathbf{0}$). The distribution is $m(\mathbf{v}) = \{\mathbf{v}\}$ if $\mathbf{v}$ is in P and $m(\mathbf{v}) = \{c\mathbf{e}_k \mid c = v_k\}$ otherwise. E.g. $m(2, -3, 0) = \{2(1, 0, 0), -3(0, 1, 0), 0(0, 0, 1)\}$. The well-known fact that every vector is uniquely described as the linear combination of its basis vectors guarantees that it is a (PII) polar distribution, which reflects the atomistic assumption. In these examples, every object is the linear combination of its paradigms.

7 Objections to the Qualitative Approach

In the following subsections I will address some objections to the qualitative approach. The first one says that the metric prototypes do not have the core features that paradigms have. According to the second one, there are many mappings that could be said to represent conceptual categorisation processes and thus our choice was an arbitrary one. Finally, simple concepts have been represented by maximal collections of pairwise similar objects. But there are some famous objections by Nelson Goodman that would seem to show that such a strategy must fail[11].

7.1 Mismatches between Paradigms and Prototypes

The first objection questions the claim that the paradigms from the similarity model are good categorical analogues of the prototypes of the metric approach. This is due to a clash between the metric and the categorical notions of similarity.

Let us go back to the metric approach for a moment. In order to represent the conceptual categorisation process, several points in the space are selected as representing prototypical elements. Then the distances of each point to these prototypes are measured, resulting in a Voronoi tessellation:

Definition 7.1 Let (S, d) be a metric space and $A \subseteq S$. Then the *Voronoi tessellation* induced by A is defined as the family of the sets of the form $V_p := \{x \in S \mid \forall q \in A - \{p\}\ d(x,p) \leq d(x,q)\}$, for each $p \in A$, called the *Voronoi cells*.

Each class is fixed by a prototypical object and contains all those objects that are more (or equally) similar to it than to the other prototypes. Given that an object can be at equal distance to several prototypes, it can belong to many such classes. The set of all the points that are at equal distance from various prototypes is the *boundary* of the tessellation. Although the boundary is usually removed from the tessellation in order to get a partition, we will not do this here, for an object can fall under several simple concepts. Given a metric space and a number $\varepsilon > 0$, a similarity can be defined as (points are similar if their distance is less than ε):

$$x \sim y := d(x,y) \leq \varepsilon$$

[11] These are only some possible objections but for reasons of space, I cannot address the rest in this paper. For an example, each concept is mapped to a *unique* paradigm. But an exemplar model of categorisation plausibly appeals to several paradigms for each concept.

Thus two objects are ε-*similar* iff they are sufficiently close (ε-close) to each other. Recall that an object is a paradigm iff any two objects that are similar to it are similar to each other. However, under this similarity it may happen that $x \sim p \sim y$ for some prototype $p \in A$, while we do not have $x \sim y$. This means that the core feature of paradigms in the similarity model is not shared by the prototypes of the metric model. Moreover, the notion of paradigm seems to clash with other properties of the metric similarity too, like the Triangular Inequality. Suppose that x and y are at distance $\frac{\varepsilon}{2}$ from each other and are ε-*similar* to a paradigm p_{xy}, whereas y and z are at distance $\frac{\varepsilon}{2}$ from each other and are ε-*similar* to a paradigm p_{yz}. Suppose that there is no paradigm to which both x and z are ε-*similar*. Nevertheless, by the Triangular Inequality x and z are at distance less than ε and therefore are ε-*similar*. Based on this definition of categorical similarity one may conclude that paradigms are bad categorical analogues of prototypes.

But this is not the only way in which we can define a categorical similarity based on the metric one. If the simple concepts are to be represented by the cells of the tessellation and if we want to keep the bridge principle that says that two objects are similar iff they fall under a common concept (see Section 5.2), we should choose a different similarity. In the previous counterexample, x and z do not fall under a common concept, for if they did they would be similar to the corresponding paradigm. But then it is the notion of ε-*similarity* that is at fault, for it entails the possibility that two objects are categorically similar even though they do not fall under a common concept. Suppose that every element belongs to at least one cell. Consider instead the similarity:

$$x \sim^* y := x, y \in V_p, \text{ for some Voronoi cell } V_p$$

Under this choice it is easy to prove that $\sim^*$ is a similarity of order 1 where every prototype is a paradigm and every Voronoi cell is a similarity circle of order 1 (compare Mormann (2021)). That prototypes (generators of Voronoi cells) are paradigms follows from the fact that any two $*$-similar prototypes p and q must be identical, because if they were distinct we would have $0 \leq d(p,q) \leq d(p,p) = 0$. My point is that one is not forced to define a categorical similarity from the metric one as ε-*similarity*. By constructing the Voronoi tessellation through the prototypes the metric approach is *already* introducing a categorical similarity.

Still, this similarity need not be pure (and it definitely does not satisfy PII), because it might happen that some non-paradigmatic points belong only to one Voronoi cell and so any two such points would be identified. So the qualitative approach is not strictly speaking a *generalization* of the metric one.

On the one hand, one can object for this reason that the models are too strong. That tessellations are not a special case of the polar model cast doubts on the empirical fruitfulness of the qualitative models. The metric approach has already had many empirical applications. Given that some of those empirical results are based on tessellations where there are points that belong only to one cell, the models presented here cannot account for these empirical facts. But if the metric approach were a special case of the qualitative approach then no empirical conflict would arise. There is a way to subsume Voronoi tessellations as a special case of the qualitative one by a slight generalization of the notion of polar distribution, which is the one used by Mormann in Mormann (2021). The difference is subtle and concerns the second axiom of polar distributions:

$$\forall x \in S\ m(x) = \{x\} \Leftrightarrow x \in P\ [Polar_2]$$

This requires that the paradigms be exactly those elements that classify themselves, which is compatible with there being non-paradigmatic elements that are classified by just one paradigm. If we use this notion, every partition with a choice of representative elements, and so every tessellation, is a polar distribution. Voronoi cells are then the simple concepts and their generators are the paradigms (and the boundary of the tessellation is the union of the topological boundaries of the concepts). This makes the metric approach a special case of the qualitative one (the metric space has now two topologies, the metric topology and the polar one).

On the other hand, one may object that the models presented here are too weak to be used to explain empirical data. But so long as the previous axioms are satisfied this is not really a problem, for particular applications one can always add more structure if needed (recall the examples in Section 6.2). In any case, the most interesting claims that can be made using metric models can be made using qualitative models too. For example, by using a metric one can predict that two elements that are equally far from a prototype will be equally prototypical. But analogously, since the orders used here are *partial* orders, they allow the representation of cases where two items are incomparable in terms of their prototypicality (e.g. is a mango a more prototypical fruit than a papaya?).

In contrast, I would say that the models that are usually applied tend to be too strong. Although I introduced Gärdenfors approach as using metric spaces, many applications of his theory make use of much richer spaces, such as the Euclidean space. Since qualitative models make fewer assumptions than these do about the structure of the data involved, they bring additional benefits.

For instance, they can be used even in cases where there is no clear metric or convexity to be applied. If one is given a collection of entities as empirical data, there may not be any obvious choice of a metric or convexity (not to say, a Euclidean metric or convexity). Consider classifications of ordinary objects like clothes, faces or pieces of furniture. Which are the metrics involved? Empirical data is often obtained in the form of ordinal rankings or indiscriminability tests. Then it is transformed into distances between points and statistical techniques are used to embed these into a lower-dimensional Euclidean space. Finally, tessellations are constructed. This process of constructing layers of representations in order to spatially represent data is standard practice. Nevertheless, in some cases these steps are unnecessary. The qualitative models of similarity and order can be used to represent the data (be they sounds, colours, words, and so on) in a space directly without any previous numerical representation. The structures involved can still be represented by numbers if needed, so long as mostly ordinal behavior is preserved.

Furthermore, the qualitative models are specially well equipped to deal with *finite data*. For example, both similarities and partial orders are equivalently described as *graphs*, and computer scientists have developed many algorithms to deal with them. For another example, the topological structure of a finite metric space is trivial, since it is the discrete topology. In a discrete space, the boundary of every set is empty and the space is highly disconnected. In order to use non-trivially topological notions in a metric space, one has to assume that the domain of the space is infinite. Thus one is forced to use an infinite space to model finite data. In contrast, the notions of boundary and connectedness work fine both in finite and infinite polar spaces and they give us the expected results, e.g. an object belongs to the boundary of a simple concept iff it falls under other simple concepts as well iff it is similar to other prototypes as well. In fact, since every finite topological space must be a WSA space, this suggests that the topology doing the work in the conceptual categorisation process of finitely many objects is not the metric topology, but the polar one.

Finally, there is no chance of confusion regarding other structural features that rich spaces like the Euclidean space have, such as angles, norms, vector structure, completeness, and so on, which in many cases will be meaningless for the data. Qualitative models are structurally sparse, they include only those features needed to represent the main commitments of the conceptual spaces approach, namely paradigms, similarity, concepts, conceptual categorisation processes and the principles that link them together.

To sum up, one can use qualitative models to discuss the very same topics that the metric approach has been applied to, such as vagueness Douven et al. (2013), Mormann (2021) cognitive semantics of natural language Gärdenfors (2014) or confirmation Sznajder (2016). However, in contrast with metric models, the use of qualitative models allows for a more cautious representation of the data.

7.2 Non-Uniqueness of Conceptual Categorisations

The Voronoi tessellation represents the family of simple concepts the objects get mapped to. But the categorisation process itself is better represented by the *function* that maps each object to its classes[12]. In the Voronoi case this map is completely determined by an algorithm that says how to construct the cells from the prototypes. Granted that simple concepts are plausibly represented by similarity circles, the second objection says that selecting the similarity circles of order 1 was an arbitrary choice because there are many functions mapping objects to similarity circles that could be reasonably called 'conceptual categorisation processes'. In contrast to the Voronoi construction, our mapping is not unique.

Let us go back to the similarity model. Since each concept corresponds to a maximally prototypical object, this process is represented mathematically in two different ways. On the one hand, we have the arrow:

$$gen : S \to \wp(Gen(S))$$

[12] Note that the same object may fall under several simple concepts.

This is the polar distribution that represents each object by the set of the maximally prototypical objects that it is similar to. In other words, it represents each object as a *bundle of paradigms*. On the other hand, we have the arrow:

$$q : S \to \wp(SC_1(S))$$

This is the function that represents each object by the set of concepts it falls under. In other words, it represents each object as a *bundle of simple concepts*. The difference between them is that the representation in terms of prototypes is much more economical. Cognitively speaking, one can obtain a very good description of the object just by listing the prototypes it is similar to (again, qualitative atomism).

However, the objection is that there could be other functions $q : S \to \wp(SC(S))$ that deserve to be called 'conceptual categorisations'. My reply is that, under some plausible bridge principles, this is not the case: there is a unique function on a similarity model that maps each object to a family of similarity circles it belongs to, and therefore this function must be q. So what are these bridge principles? We have already seen the following:

Similarity-Concept: Two objects are similar iff they fall under a common concept.

Similarity-Order: An object is qualitatively richer than another object iff the former falls under all the concepts that the latter falls and possibly more.

The first condition says that the conceptual categorisation explains similarity exactly as falling under a common concept. The second condition says that the order of qualitative richness corresponds exactly to falling under more or less concepts. We also know that, at least in the similarity model, we get the correspondence between being similar to the same objects and falling under the same concepts. Finally, there is a sense in which a Parsimony condition can be imposed even for the family of simple concepts. It can be introduced as a bridge principle too:

Parsimony: Every concept is indispensable to the categorization process. If we deleted one of them some similarity would be left unexplained.

These conditions can be formalized by appealing to Carnap's quasianalysis as defined in Mormann (2009):

Definition 7.2 Let $(S, \sim)$ be a similarity structure, Q a non-empty set, $\sim^*$ the similarity relation $A \sim^* B := A \cap B \neq \emptyset$ on $\wp(Q)$, and $q : S \to \wp(Q)$ a function. Then q is a *strong quasianalysis* iff $Q \subseteq SC(S)$, $q(x) := \{T \in Q \mid x \in T\}$ and satisfies (i)-(iii) for any $q' : S \to \wp(Q')$ defined as follows:

i $x \sim y \Leftrightarrow q(x) \sim^* q(y)$. [Similarity-Concept]
ii $co(x) \subseteq co(y) \Leftrightarrow q(x) \subseteq q(y)$. [Similarity-Order]

iii If $Q' \subseteq Q$ is such that $q' : S \to \wp(Q')$, defined as $q'(x) := \{T \in Q' \mid x \in T\}$ satisfies (i)-(ii), then $Q' = Q$. [Parsimony]

In principle, there are many functions on a similarity model that could be taken to represent processes of conceptual categorisation. However, the following theorem shows that if we stick to the similarity model, there is a unique function satisfying the previous constraints. The result follows from a more general result first shown by Brockhaus in Brockhaus (1963) and refined by Mormann and J. A. Schreider:

Corollary 7.1 (Brockhaus-Mormann-Schreider) *Let S be a* (*SNI*) *similarity structure of order* 1. *Then it has a unique strong quasianalysis q.*

In other words, in a similarity model there is a *unique function representing the categorisation process* that satisfies the three principles just mentioned.

7.3 From Similarity to Concepts: Goodman's Objections

The reader familiar with the literature on quasianalysis, or with that related to resemblance nominalism, will be puzzled. Concepts are here reconstructed as maximal classes of pairwise similar objects. But this strategy should not work. Nelson Goodman in Goodman (1951) directed two devastating objections to Carnap's method of quasianalysis that showed that concepts cannot be reconstructed as similarity circles. Our last objection says that, according to these, the previous approach is flawed.

Hannes Leitgeb gives in Leitgeb (2007) a detailed analysis of these problems, which we now consider. Take a structure (S, Q) where S is a non-empty set and Q is a non-empty family of non-empty subsets of S that covers S. Let us think about the elements in Q as simple concepts. Any such structure induces a similarity over S as follows:

$$x \sim y \Leftrightarrow \exists R \in Q \; x, y \in R$$

In other words, two objects are similar iff they fall under a common concept. From this it follows immediately that every concept is a clique. But there are many cliques that do not correspond to concepts. A reasonable conjecture by Carnap was to take only the maximal ones, the similarity circles. For the special case of equivalences this will give us the equivalence classes, as expected. However, if we try to reconstruct the structure of concepts from the similarity by taking the concepts to be the similarity circles our strategy will fail for two reasons.

First, whereas some concepts can be properly included into others (e.g. *Magenta* is included into *Red*), similarity circles are maximal and cannot be properly included into one another. This means that some concepts are not similarity circles. This is the *companionship problem*. For instance, take $S = \{x, z\}$ and $Q = \{\{x\}, \{x, z\}\}$, then the pair $\{x\}, \{x, z\}$ forms a companionship. $\{x\}$ is not a similarity circle because it is not maximal. The method fails because one of the concept in Q is not constructed, in Leitgeb's terms, the similarity structure is not 'full'. Second, it sometimes happens that given three objects $x, y, z \in S$, while each pair instantiates a common concept, there is no concept which is instantiated by all of them. This means that some similarity circles are not concepts. This is the *imperfect community problem*. Imperfect communities are cliques and some of them can be similarity circles. For instance, suppose the structure has the form of a Goodman triangle $S = \{x, y, z\}$ and $Q = \{\{x, y\}, \{y, z\}, \{x, z\}\}$. Then $\{x, y, z\}$ is an imperfect community that does not correspond to a concept in Q. Now the method fails because the concept constructed was not there, the similarity structure is not 'faithful'. The puzzle can be summarized as follows:

(1) We start from an arbitrary structure of concepts (S, Q), where Q is a non-empty family of non-empty subsets of S that covers S.
(2) We define a binary categorical similarity relation in the domain of objects S as falling under a common concept.
(3) We select a certain class of cliques in the similarity structure to recover the concepts.
(4) The class of cliques selected must be identical to the original set Q of concepts.
(5) We select as the class of cliques the class of all similarity circles.

We know that if we follow these steps the strategy fails. Thus, in order to answer to Goodman's problems, at least one of them has to be rejected. For example, Leitgeb (2007) rejects the first one, while Mormann (2009) suggests rejecting steps one, four and five. I reject steps one and five. I select only a special class of structures of concepts to be reconstructed and a special class of similarity circles, namely those of order 1. The reason for this choice is that, as we have seen before, generators and circles of order 1 give qualitative representations of prototypes and simple concepts, respectively. The special class of the structures of simple concepts for which the previous reconstruction works is this one:

Definition 7.3 Let S be a non-empty set, $P \subseteq S$ and $\varnothing \notin Q \subseteq \wp(S)$ a covering of S by non-empty sets. Then (S, P, Q) is a *(PII) polar structure* iff:

(1) $\forall R \in Q\ \exists p \in P\ (p \in R\ \&\ \forall r \in P\ (r \in R \Rightarrow p = r))$.
(2) $\forall p \in P\ \forall R, T \in Q\ (p \in R \cap T \Rightarrow R = T)$.
(3) $\forall x, y \in S\ \forall R \in Q(x \in R \Leftrightarrow y \in R) \Rightarrow x = y$.

Members of P are once again *paradigms* and members of Q are called *simple concepts*. What (1)-(2) say is that in polar structures there is a bijection between concepts and paradigms. (3) Corresponds again to the Identity of Indiscernibles. Each concept is in some sense 'generated' by a unique element, which is the paradigm of the concept. These structures are also equivalent to the previous models, let i(x) be the set of simple concepts under which x falls:

Theorem 7.1 *Let* (S, P, m) *be a polar distribution. Then* $(S, P, Q^*, \in)$, *where* $Q^* := \{Cl(p) \subseteq S \mid p \in P\}$, *is a polar structure whose polar distribution* (S, P, n) *is such that* $n = m$. *Conversely, let* $(S, P, Q, \in)$ *be a polar structure. Then* (S, P, n), *where* $n(x) := \{p \in P \mid i(x) \cap i(p) \neq \varnothing\}$, *is a polar distribution whose polar structure* $(S, P, Q^*, \in)$ *is such that* $Q^* = Q$.

Why does not this class of similarities fall prey to Goodman's objections? If there is, for each concept, a unique object which is such that it only falls under that concept, then it is no mystery that no companionship problems arise. If all R-s are T-s, then the paradigm of R is also a T and since such an object falls under a unique concept, $R = T$. Analogously, there cannot be imperfect community problems. If there is a (maximal) imperfect community, it is not a concept and therefore there is no paradigm corresponding to it. The members of the imperfect community are pairwise similar to each other because each pair is similar to a paradigmatic object. But there is no paradigm to which all the objects are similar. Equivalently put, similarity circles of order 1 cannot be imperfect communities.

Finally, now that we have all the models in place we can combine the previous results to get the full picture:

Theorem 7.1 *The following qualitative models for conceptual spaces are mathematically equivalent*:

(1) (*SNI*) *Similarities of order* 1.
(2) *Atomistic posets*.
(3) (*PII*) *Polar distributions*.
(4) (*PII*) *Polar structures*.
(5) *Co-atomistic WSA spaces*.

This theorem summarizes several equivalent ways to present the qualitative approach to conceptual spaces. They can be introduced in terms of a similarity between objects, an order of qualitative richness, an assignment of prototypes to objects, a family of concepts and a space with a distinguished dense region of points.

8 Conclusion

The aim of this paper was to introduce a qualitative approach to conceptual spaces by providing several models and comparing them by how well they satisfied the Design Principles and Adequacy Conditions. Whereas the former were used to check that the qualitative models provide a plausible explication of simple natural concepts and conceptual categorisation, the latter required that the qualitative approach provided categorical analogues of the metric notions of similarity, prototypes, simple concepts and conceptual categorisation. I showed that these models are mathematically equivalent to each other and therefore notions introduced by one could be use by the others to satisfy the constraints. I also addressed three objections that purported to show that there are important differences between the two approaches. Most importantly, one can use qualitative models to discuss the same topics that the metric approach has been applied to, such as vagueness, confirmation or cognitive semantics. In contrast with metric models, qualitative models are structurally sparse, they include only those features needed to represent the main commitments of the conceptual spaces approach, namely paradigms, similarity, concepts and categorisation processes, which allows for a more faithful representation of the data.

Appendix

This Appendix contains the main results stated in the paper. The equivalence between similarities and polar topologies Theorem 6.1:

Theorem 6.1 *Let* (S, P, m) *be a polar distribution. Then* $(S, \sim)$ *defined as* $x \sim y := \exists p \in P\ p \in m(x) \cap m(y)$ *is a pure order* 1 *similarity. It induces a polar distribution* $(S, Gen(S), gen)$ *which is such that* $P = Gen(S)$ *and* $m = gen$. *Conversely, let* $(S, \sim)$ *be a pure order* 1 *similarity. Then* $(S, Gen(S), gen)$ *is a polar distribution where* $(S, \sim')$ *is such that* $x \sim y$ *iff* $x \sim' y$.

Proof Let (S, P, m) be polar and define $x \sim y := \exists p \in P\ p \in m(x) \cap m(y)$, which is symmetric. By polarity, $m(x) \neq \emptyset$ and reflexivity follows. Let $p \in P$, if $w \sim p \sim z$, then there are $r \in m(w) \cap m(p)$ and $s \in m(z) \cap m(p)$, so by polarity $r = p = s$. Therefore $p \in m(w) \cap m(z)$ and $w \sim z$. So p is a generator. If $x \sim y$ then there is a $p \in P \cap m(x) \cap m(y)$ and so the similarity is of order 1. Suppose that $p, r \in P$ then if $p \sim r$ we have $m(p) \cap m(r) = \{p\} \cap \{r\} \neq \emptyset$, so $p = r$ and the similarity is pure.

Now let $(S, Gen(S), gen)$. We have $Gen(S) \subseteq S$ and $gen : S \to \wp(Gen(S))$ which satisfies $gen(x) \neq \emptyset$ by order 1. If $p \in Gen(S)$, then $gen(p) = \{p\}$ because if $q \in Gen(S) \cap gen(p)$ we have $q \sim p$ and by purity $p = q$. And recall that if gen(x)={p} then if y~x~z there are generators shared by x one with y and one with z which must both be p and so y~z, which makes x a generator and thus x=p. So it is a polar distribution. We already showed that $P \subseteq Gen(S)$. Let $p \in Gen(S)$ we prove $m(p) = \{p\}$. If $q \in m(p)$ then $q \in m(p) \cap m(q)$ and so $p \sim q$ and again by purity $p = q$. Therefore $m(p) = \{p\}$ and so $p \in P$. It follows that $P = Gen(S)$. Therefore, $p \in m(x) \Leftrightarrow p \in P \& p \sim x \Leftrightarrow p \in Gen(S) \& p \sim x \Leftrightarrow p \in gen(x)$, because if $p \in P$ and $p \sim x$ then there is a $q \in m(p) \cap m(x)$ such that $q = p$.

Conversely, let $(S, \sim)$ be a pure similarity of order 1. We already proved that $(S, Gen(S), gen)$ is a polar distribution and that $(S, \sim')$ is pure of order 1. We show $\sim = \sim'$. By order 1, $x \sim y \Leftrightarrow \exists p \in Gen(S)\ p \in gen(x) \cap gen(y) \Leftrightarrow x \sim' y$. □

Equivalence between similarities and orders Theorem 6.2.

Proposition 8.1 *Let* $(S, \sim, \leq_{co})$ *be an* (*SNI*) *structure of order* 1. *Then*:

i p *is minimal* $\Leftrightarrow p$ *is a generator of order* 1.
ii $x = \bigvee gen(x)$.

Proof (i) Let p be minimal. It belongs to some similarity circle T of order 1 with unique generator z. Then $T = co(z) \subseteq co(p)$ and by minimality $p = z$. Conversely, let p be the generator of an order 1 circle T and $y \leq p$. Then $y \in co(y) \subseteq co(p) = T$ and therefore $co(y) = T$. It follows that $p = y$ and so p is minimal. (ii) x is greater than each of its generators. Let $p \leq y \leq x$ for each generator p of x. Then $gen(x) \subseteq gen(y) \subseteq gen(x)$ and therefore $x = \bigvee gen(x)$. □

Theorem 6.2 *Let* $(S, \sim)$ *be a* (*SNI*) *similarity structure of order* 1. *Then* $(S, \leq_{co})$ *is an atomistic poset where the minimal elements are* $Gen(S)$. *Moreover*, $(S, \sim^*)$, *defined as* $x \sim^* y := \exists z \in Min(S)\ z \leq x, y$, *is identical to* $(S, \sim)$. *Conversely*, *if* $(S, \leq)$ *is an atomistic poset*, *then* $(S, \sim^*)$ *is a* (*SNI*) *similarity structure of order* 1 *such that* $x \leq y$ *iff* $x \leq_{co^*} y$.

Proof Let $(S, \sim)$ be the similarity. We already proved that the generators are exactly the minimal elements and that every element is the join of its generators. Now $x \sim y$ iff $gen(x) \cap gen(y) \neq \emptyset$ iff $x \sim^* y$. Conversely, let $(S, \leq)$ be an atomistic poset. If p is minimal and $x \sim^* p \sim^* y$ then $z \leq x, p$ and $w \leq y, p$ and by minimality $z = p = w$ and so $x \sim^* y$. Conversely, if p is a generator and $x \leq p$ then $min(x) \subseteq min(p)$. If $q \in min(p)$ and $y \sim^* q$ then $y \sim^* p$ therefore $co^*(q) \subseteq co^*(p)$. Since p is a generator of order 1, $x \in co^*(p) = co^*(q)$ and therefore $q \sim^* x$, from which it follows that $q \leq x$ by minimality and therefore $min(x) = min(p)$. Thus $x = p$ and so p is minimal. So the minimals are exactly the generators and by definition S is of order 1. Let $co^*(x) = co^*(y)$, then by order 1 $min(x) = gen^*(x) = gen^*(y) = min(y)$ and therefore $x = y$. Finally, $x \leq y$ iff $min(x) \subseteq min(y)$ iff $gen^*(x) \subseteq gen^*(y)$ iff $x \leq_{co^*} y$. □

Equivalence between polar distributions and polar structures Theorem 7.1:

Theorem 7.1 *Let* $(S,\ P,\ m)$ *be a polar distribution. Then* $(S, P, Q^*, \in)$, *where* $Q^* := \{Cl(p) \subseteq S \mid p \in P\}$, *is a polar structure whose polar distribution* $(S,\ P,\ n)$ *is such that* $n = m$. *Conversely*, *let* $(S, P, Q, \in)$ *be a polar structure. Then* $(S,\ P,\ n)$, *where* $n(x) := \{p \in P \mid i(x) \cap i(p) \neq \emptyset\}$, *is a polar distribution whose polar structure* $(S, P, Q^*, \in)$ *is such that* $Q^* = Q$.

Proof $(S, P, Q^*, \in)$ is a covering, since $p \in m(x) \neq \emptyset$ implies $x \in Cl(p) \neq \emptyset$. If $R \in Q^*$, then $p \in m(p) = \{p\}$ iff $p \in Cl(p) = R$. Suppose $p, r \in P \cap R = P \cap Cl(q)$. Then $q \in m(p) \cap m(r) = \{p\} \cap \{r\}$, so $p = q = r$. Let $p \in P \cap R \cap T = P \cap Cl(r) \cap Cl(t)$, then $r, t \in m(p) = \{p\}$, so $R = Cl(r) = Cl(p) = Cl(t) = T$, which proves that the structure is polar. We prove n is polar. Since $p \in m(x) \neq \emptyset$ for all x, $x, p \in Cl(p)$. Let $p \in P$ and $q \in n(p)$, then $p, q \in Cl(r)$, therefore $r \in m(p) \cap m(q) = \{p\} \cap \{q\}$ and so $n(p) = \{p\}$. The converse follows from $n = m$, which we prove. Let $p \in n(x)$, then $x, p \in Cl(q)$ for some $Cl(q) \in Q^*$. Therefore $q \in m(p) = \{p\}$, which implies $p \in m(x)$.

Conversely, if $p \in m(x)$ then $x, p \in Cl(p)$ so $p \in n(x)$. Let $n : S \to \wp(P)$ be defined as $n(x) := \{p \in P \mid \exists R \in Q\ x, p \in R\}$. By assumption, for every x we have $x, p \in R$ for some $p \in P, R \in Q$ and thus $p \in n(x)$. Let $p \in P$ and $q \in n(p)$. Then $p, q \in R$ so by polarity $p = q$, which makes n polar. As before, it follows that $(S, P, Q^*, \in)$ is polar. We prove now that $Q = Q^*$. Let $Cl(p) \in Q^*$. Since p is a paradigm, it corresponds to a unique $R \in Q$. If $x \in Cl(p)$, then $p \in n(x)$, so there is a $T \in Q$ such that $x, p \in T = R$. If $x, p \in R$ then $p \in n(x)$ so $x \in Cl(p)$. So $Cl(p) = R$. It follows that $Q^* \subseteq Q$. Let $R \in Q$, then it corresponds to a unique paradigm $p \in R$, we analogously prove that $Cl(p) = R$. □

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