Derivatives of the diameter and the area of a connected component of the pseudospectrum.*

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Abstract

The paper concerns the relation between the following two quantities.

- Hölder condition number of an eigenvalue λ of a square complex matrix.
- The rate of growth of the diameter and the area of the connected component of the ε–pseudospectrum containing λ.

Keywords Matrices, Pseudospectra, Condition number, Eigenvalues, Derivatives.

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1 Introduction

By $\Lambda(A)$ we denote the spectrum of any $A \in \mathbb{C}^{n\times n}$. We denote by $\|\cdot\|$ the 2-norm. Let λ be an eigenvalue of A of algebraic multiplicity m. For $X \in \mathbb{C}^{n\times n}$, $\mathrm{sv}_{(A,\lambda)}(X)$ denotes the radius of the smallest circle centered at λ containing m of the eigenvalues of X counting multiplicities. The (Hölder) condition number of the eigenvalue λ of order $\omega > 0$ is defined as

$$\mathrm{cond}_\omega(A,\lambda) := \lim_{\varepsilon\to 0^+} \max_{0<\|X-A\|\leq\varepsilon} \frac{\mathrm{sv}_{(A,\lambda)}(X)}{\|X-A\|^\omega}.$$

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The index of an eigenvalue λ of A, $\nu = \nu(\lambda)$, is the size of the largest Jordan block associated with λ. The limit that defines $\mathrm{cond}_\omega(A, \lambda)$ is of interest just for $\omega = 1/\nu$.

On the other hand for $\varepsilon \geq 0$, the ε-pseudospectrum of A consists of the eigenvalues of all matrices X within an ε-neighborhood of A, *i.e.*,

$$\Lambda_\varepsilon(A) := \bigcup_{\substack{X \in \mathbb{C}^{n\times n} \\ \|X - A\| \leq \varepsilon}} \Lambda(X).$$

For any complex matrix M we denote by $\sigma_1(M) \geq \sigma_2(M) \geq \cdots$ its singular values arranged in decreasing order. It is well known that

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} :\ \sigma_n(zI - A) \leq \varepsilon\}.$$

The subset $\Lambda_\varepsilon(A)$ of the complex plane is a compact set consisting of at most r (disjoint) connected components, one around each eigenvalue, where $\Lambda(A) = \{\lambda_1, \dots, \lambda_r\}$. Denote the connected component of $\Lambda_\varepsilon(A)$ around the eigenvalue λ by $\mathcal{K}_\lambda(\varepsilon)$, and consider the diameter $\delta(\varepsilon)$ and the area $a(\varepsilon)$ of this component as a function of ε.

We denote by $\partial\mathcal{K}_\lambda(\varepsilon)$ the boundary of $\mathcal{K}_\lambda(\varepsilon)$. From [6, Proposition 2.6.5] we deduce that

$$c = \lim_{\varepsilon\to 0^+} \frac{\displaystyle\max_{z\in\partial\mathcal{K}_\lambda(\varepsilon)} |z - \lambda|}{\varepsilon^{1/\nu}},$$

where c denotes the Hölder condition number $\mathrm{cond}_{1/\nu}(A, \lambda)$. Thus, calling $\rho(\varepsilon) := \max_{z\in\partial\mathcal{K}_\lambda(\varepsilon)} |z - \lambda|$, we see that $\lim_{\varepsilon\to 0^+} \frac{\rho(\varepsilon)}{\varepsilon^{1/\nu}} = c$. We extend this result to $\delta(\varepsilon)$ and $a(\varepsilon)$ instead of $\rho(\varepsilon)$ in Thoerems 4 and 7.

Remark 1. *When* $\lim_{\varepsilon\to 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = \infty$ *[resp.* $\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon} = \infty$*], strictly speaking the function* $\delta(\varepsilon)$ *[resp.* $a(\varepsilon)$*] is not differentiable from the right-hand side at* 0*. However, in this case we put* $\delta'_+(0) = \infty$ *[resp.* $a'_+(0) = \infty$*] in order to grasp the geometric meaning of the results.*

The main results of the paper are as follows.

1. $\delta'_+(0) = 2c$ if $\nu = 1$. Otherwise $\delta'_+(0) = \infty$. Here δ'_+ denotes the right-derivative of δ. See Theorem 6.

2. $a'_+(0) = 0$ if $\nu = 1$, $a'_+(0) = \pi c^2$ if $\nu = 2$. Otherwise $a'_+(0) = \infty$. See Theorem 9.

In Section 2, we work four examples of matrices A for which both the condition number and the geometry of the ε-pseudospetrum are known in detail; this let us corroborate our results. We demonstrate that an important result by Karow [6] let us bound the ε-pseudospectrum by lower and upper closed disks; see Section 3. By the monotony of the diameter function, and its changes under a homotecy, we show how to transfer these inequalities with respect to the inclusion relation between sets to the diameters, in Section 4. In Section 5, we relate the first rigth-derivative of the diameter at $\varepsilon = 0$ with the condition number. In Section 6, the monotony of the area function, and its changes under a homotecy, let us translate the bounds in Section 3 to numeric inequalities. In Section 7, we relate the first rigth-derivative of the area at $\varepsilon = 0$ with the condition number c; moreover we establish a relation between c and the second right-derivative $a''_+(0)$ whenever this derivative exists. Finally, in Section 8 we formulate a conjecture about the semialgebraicity of the functions $\delta(\varepsilon)$ and $a(\varepsilon)$; if it were true, a de l'Hôpital inverse rule would let us prove the existence of $a''_+(0)$.

2 Examples

Next, we consider four examples where we compute the condition number of order $1/\nu$ of an eigenvalue λ and the right-derivatives at $\varepsilon = 0$ of the diameter $\delta(\varepsilon)$ and the area $a(\varepsilon)$ of the connected component $\mathcal{K}_\lambda(\varepsilon)$.

First Example. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Then the ε-pseudospectrum of A is the union of the closed disks of radius ε centered at the eigenvalues of A. So, for sufficiently small $\varepsilon \geq 0$, we have

$$\mathcal{K}_\lambda(\varepsilon) = \mathcal{D}(\lambda, \varepsilon).$$

Therefore, $\rho(\varepsilon) = \varepsilon$, and since the eigenvalues of a normal matrix are semisimple (i.e. of index 1),

$$\mathrm{cond}_1(A, \lambda) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\varepsilon} = 1.$$

The diameter of $\mathcal{D}(\lambda, \varepsilon)$ is 2ε. So, $\delta'(\varepsilon) = 2$ and $\delta'_+(0) = 2$. If we denote by $a(\varepsilon)$ the area of this circle, $a(\varepsilon) = \pi\varepsilon^2$; hence $a'(\varepsilon) = 2\pi\varepsilon, a''(\varepsilon) = 2\pi$. Therefore, $a'_+(0) = 0, a''_+(0) = 2\pi$.

Second Example. Let

$$J_2(\lambda, d) = \begin{bmatrix} \lambda & d \\ 0 & \lambda \end{bmatrix}$$

be like a Jordan block, with complex numbers λ, d and $d \neq 0$. Karow proved in [6, Theorem 5.4.1, p. 74] that for each $\varepsilon \geq 0$,

$$\Lambda_\varepsilon(J_2(\lambda, d))$$

is a closed disk centered at λ and with radius $r_2(\varepsilon) = \max\{r > 0 \,|\, \sigma_2\big(J_2(r, |d|)\big) \le \varepsilon\}$. As

$$\sigma_2\big(J_2(r, |d|)\big) = \sqrt{r^2 + \frac{|d|^2}{2} - \sqrt{r^2|d|^2 + \frac{|d|^4}{4}}},$$

solving the equation in the unknown r

$$r^2 + \frac{|d|^2}{2} - \sqrt{r^2|d|^2 + \frac{|d|^4}{4}} = \varepsilon^2,$$

we find that $r_2(\varepsilon) = \sqrt{\varepsilon^2 + |d|\varepsilon}$. This result has also been proved by Cui et al. [4, Proposition 2.1]. So, $\delta(\varepsilon) = 2\sqrt{\varepsilon^2 + |d|\varepsilon}$. Hence,

$$\delta'(\varepsilon) = \frac{2\varepsilon + |d|}{\sqrt{\varepsilon^2 + |d|\varepsilon}}, \text{ which implies } \delta'_+(0) = \infty.$$

It is obvious that $\nu(\lambda) = 2$. Let us remark that

$$\lim_{\varepsilon \to 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/2}} = 2\sqrt{|d|}. \tag{1}$$

Now, let us see that $c := \operatorname{cond}_{1/2}\big(J_2(\lambda, d), \lambda\big) = \sqrt{|d|}$. We need some previous considerations. For a general matrix $A \in \mathbb{C}^{n \times n}$, let $\Lambda(A) = \{\lambda_1, \dots, \lambda_r\}$. Let

$$A = \sum_{j=1}^{r} (\lambda_j P_j + N_j)$$

be the Jordan decomposition of A, where for each $j \in \{1, \dots, r\}$, P_j is the Riesz projector onto the root subspace (or generalized eigenspace) $\mathcal{R}_{\lambda_j}(A)$ of λ_j and along the sum of root subspaces associated with all eigenvalues of A different from λ_j; and $N_j := (A - \lambda_j I_n)P_j$ is the nilpotent matrix corresponding to λ_j. By [6, Theorem 5.4.4 (viii), p. 78], if $\nu_j := \nu(\lambda_j) > 1$, then

$$\operatorname{cond}_{1/\nu_j}(A, \lambda_j) = \|N_j^{\nu_j - 1}\|^{1/\nu_j}. \tag{2}$$

In our present example $A = J_2(\lambda, d) = \lambda I_2 + N$, where

$$N = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix},$$

thus, $c = \|N\|^{1/2} = \sqrt{|d|}$. From (1), we have

$$\lim_{\varepsilon \to 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/2}} = 2c.$$

The area $a(\varepsilon)$ is given by $\pi(\varepsilon^2 + |d|\varepsilon)$. So, $a'(\varepsilon) = \pi(2\varepsilon + |d|)$; hence, $a'_+(0) = \pi|d| = \pi c^2$. This concludes the Second Example.

Before discussing the third example, we need to introduce a result on the pseudospectra of nilpotent matrices of nilpotency index two. Here on, we denote by O_k the $k \times k$ zero matrix.

Proposition 1 ([5], Theorem 3)**.** *Let us assume that q, r are nonnegative integers such that $n = 2q + r$. Let $N \in \mathbb{C}^{n\times n}$ be a matrix such that $N^2 = O_n$, whose nonzero singular values are $\sigma_1(N) \geq \cdots \geq \sigma_q(N)$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$U^*NU = \begin{bmatrix} 0 & \sigma_1(N) \\ 0 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \sigma_q(N) \\ 0 & 0 \end{bmatrix} \oplus O_r.$$

By the Second Example, or [4, Proposition 2.1] by Cui et al., the ε–pseudospectrum of

$$\begin{bmatrix} 0 & \sigma_i(N) \\ 0 & 0 \end{bmatrix}$$

is $\mathcal{D}\big(0, \sqrt{\varepsilon^2 + \varepsilon\sigma_i(N)}\big)$ for $i = 1, \ldots, q$. So, by [6, Propositions 5.2.3 and 5.2.4], we have the following result.

Proposition 2. *Under the hypotheses of Proposition 1 for $\varepsilon \geq 0$,*

$$\begin{aligned} \Lambda_\varepsilon(N) &= \textstyle\bigcup_{i=1}^{q} \mathcal{D}\big(0, \sqrt{\varepsilon^2 + \varepsilon\sigma_i(N)}\big) \cup \mathcal{D}(0, \varepsilon) \\ &= \mathcal{D}\big(0, \sqrt{\varepsilon^2 + \varepsilon\sigma_1(N)}\big). \end{aligned} \tag{3}$$

We will also need that for any $\alpha \in \mathbb{C}$, $A \in \mathbb{C}^{n\times n}$ and $\varepsilon \geq 0$,

$$\Lambda_\varepsilon(\alpha I_n + A) = \alpha + \Lambda_\varepsilon(A). \tag{4}$$

Third Example. This example is a small generalization of the second one. Let A be any n-by-n complex matrix with a unique eigenvalue λ. Moreover, let us assume that $\nu(\lambda) = 2$. Let $N := A - \lambda I_n$; thus, $A = \lambda I_n + N$ is the Jordan decomposition of A. Hence, by (2), $c = \mathrm{cond}_{1/2}(A, \lambda) = \|N\|^{1/2} = \sqrt{\sigma_1(A - \lambda I_n)}$. By Proposition 2 and (4) we see that for $\varepsilon \geq 0$,

$$\Lambda_\varepsilon(A) = \mathcal{D}\big(\lambda, \sqrt{\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)}\big).$$

So, $\delta(\varepsilon) = 2\sqrt{\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)}$, $a(\varepsilon) = \pi\big(\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)\big)$. Therefore,

$$\delta'(\varepsilon) = \frac{2\varepsilon + \sigma_1(A - \lambda I_n)}{\sqrt{\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)}}, \text{ which implies } \delta'_+(0) = \infty,$$

and

$$a'(\varepsilon) = \pi\big(2\varepsilon + \sigma_1(A - \lambda I_n)\big), \text{ which implies } a'_+(0) = \pi\,\sigma_1(A - \lambda I_n) = \pi c^2.$$

Fourth Example. Let λ_1, λ_2 be two different complex numbers. Let $A \in \mathbb{C}^{2\times 2}$ whose eigenvalues are λ_1 and λ_2. Let us define

$$d(A) := \sqrt{\operatorname{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2}.$$

The number $d(A)$ is the departure from normality of A. In [6, Proposition 5.5.3, p. 80] was proved that for each $\varepsilon > 0$,

$$\Lambda_\varepsilon(A) = \mathcal{D}(\lambda_1, \varepsilon) \cup \mathcal{D}(\lambda_2, \varepsilon) \cup \mathcal{M}_{\lambda_1,\lambda_2}(d(A), \varepsilon) \tag{5}$$

where

$$\mathcal{M}_{\lambda_1,\lambda_2}(d, \varepsilon) := \left\{ z \in \mathbb{C} \colon \left(|z - \lambda_1|^2 - \varepsilon^2\right)\left(|z - \lambda_2|^2 - \varepsilon^2\right) \le \varepsilon^2 d^2 \right\}, \quad d \ge 0. \tag{6}$$

Let us consider Figure 1 that shows the ε-pseudospectrum for the values of $\varepsilon = 1.00, 1.50, 1.75$ of the matrix

$$A_1 := \begin{bmatrix} -1 - 3\mathrm{i} & 3 \\ 0 & 1 + 2\mathrm{i} \end{bmatrix}.$$

So, $\lambda_1 = -1-3\mathrm{i}, \lambda_2 = 1+2\mathrm{i}$, $\|A_1\|_{\mathrm{F}} = \sqrt{|-1-3\mathrm{i}|^2 + 3^2 + |1+2\mathrm{i}|^2} = \sqrt{10+9+5} = \sqrt{24}$. Thus, $d(A_1) = \sqrt{24 - 10 - 5} = \sqrt{9} = 3$.

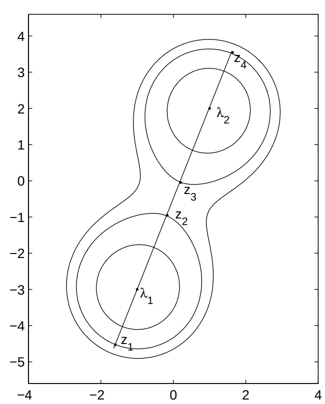

Figure 1: Pseudospectra of A_1 for $\varepsilon = 1.00, 1.50, 1.75$

We see that for sufficiently small values of ε the ε–pseudospectrum of A_1 has two connected components. Let us fix our attention on $\varepsilon = 1.50$. The figure contains a straight line that passes by λ_1 and λ_2. This line intersects the boundary $\partial\Lambda_\varepsilon(A_1)$ at the points z_1, z_2, z_3, z_4. If we consider the eigenvalue λ_2, the diameter $\delta(\varepsilon)$ of the connected component $\mathcal{K}_{\lambda_2}(\varepsilon)$ is equal to the distance between z_3 and z_4. From (5) and (6), we can deduce that

$$\delta(\varepsilon) = \frac{1}{\sqrt{116}}\left(\sqrt{116\varepsilon^2 + 116\sqrt{38}\varepsilon + 841} - \sqrt{116\varepsilon^2 - 116\sqrt{38}\varepsilon + 841}\right)$$

Hence, differentiating $\delta(\varepsilon)$ with respect to ε and evaluating at $\varepsilon = 0$, we have

$$\delta'(0) = 2\sqrt{\frac{38}{29}}.$$

On the other hand, let us compute the condition number of the eigenvalue $\lambda_2 = 1 + 2\mathrm{i}$,

$$\mathrm{cond}_1(A_1, \lambda_2) = \sqrt{1 + \left(\frac{d(A_1)}{|\lambda_2 - \lambda_1|}\right)^2} = \sqrt{1 + \left(\frac{3}{\sqrt{29}}\right)^2} = \sqrt{1 + \frac{9}{29}} = \sqrt{\frac{38}{29}}.$$

Let us remark that

$$\delta'(0) = 2\,\mathrm{cond}_1(A_1, \lambda_2).$$

Instead of making a special reasoning for the matrix A_1, we are going to find a general expression for the diameter $\delta(\varepsilon)$ of $\mathcal{K}_{\lambda_2}(\varepsilon)$ for any matrix $A \in \mathbb{C}^{2\times 2}$ with eigenvalues λ_1 and λ_2. By (6), the boundary $\partial\Lambda_\varepsilon(A)$ is formed by quasi-Cassini ovals with foci the points λ_1 and λ_2. In fact, the set $\Lambda_\varepsilon(A)$ is symmetric about the straight line that joints λ_1 and λ_2. This is a consequence of (5) and (6). Moreover, $\delta(\varepsilon) = |z_4 - z_3| = |z_2 - z_1|$. Later we will need the condition numbers of λ_1 and λ_2 of order 1. These numbers are equal. In fact, by [6, Proposition 5.5.8, p. 83], we have

$$\mathrm{cond}_1(A, \lambda_k) = \sqrt{1 + \left(\frac{d(A)}{|\lambda_2 - \lambda_1|}\right)^2}, \qquad k = 1, 2. \tag{7}$$

Thus, let $\varepsilon \geq 0$ be such that $\Lambda_\varepsilon(A)$ has two connected components. Now, we compute the intersection points z_1, z_2, z_3, z_4 of the line $z(t) := (1-t)\lambda_1 + t\lambda_2, t \in \mathbb{R}$ with the curve $\partial\Lambda_\varepsilon(A)$. By (6), this curve is given by the equation

$$\left(|z - \lambda_1|^2 - \varepsilon^2\right)\left(|z - \lambda_2|^2 - \varepsilon^2\right) - \varepsilon^2 d(A)^2 = 0. \tag{8}$$

For determining the values of the parameter t that correspond to the points z_1, z_2, z_3 and z_4 we substitute $z(t)$ into (8),

$$\begin{aligned} &|\lambda_1 - \lambda_2|^4 t^4 - 2\,|\lambda_1 - \lambda_2|^4 t^3 + \left(|\lambda_1 - \lambda_2|^4 - 2\,\varepsilon^2|\lambda_1 - \lambda_2|^2\right) t^2 \\ &\quad + 2\,\varepsilon^2|\lambda_1 - \lambda_2|^2 t - \varepsilon^2|\lambda_1 - \lambda_2|^2 + \varepsilon^4 - \varepsilon^2 d(A)^2 = 0. \end{aligned}$$

For simplicity, we write $\theta := |\lambda_1 - \lambda_2|$,

$$\theta^4 t^4 - 2\,\theta^4 t^3 + \left(\theta^4 - 2\,\varepsilon^2\theta^2\right) t^2 + 2\,\varepsilon^2\theta^2 t - \varepsilon^2\theta^2 + \varepsilon^4 - \varepsilon^2 d(A)^2 = 0.$$

With the command `solve` of MAPLE 13 we find the roots of this equation in t obtaining

$$\begin{aligned} t_1 &= \frac{1}{2} - \frac{1}{2\theta}\sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \\ t_2 &= \frac{1}{2} - \frac{1}{2\theta}\sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \\ t_3 &= \frac{1}{2} + \frac{1}{2\theta}\sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \\ t_4 &= \frac{1}{2} + \frac{1}{2\theta}\sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \end{aligned}$$

where $t_1 < 0 < t_2 < \frac{1}{2} < t_3 < 1 < t_4$. Hence, $z_3 = z(t_3)$ and $z_4 = z(t_4)$. So,

$$\delta(\varepsilon) = |z_4 - z_3| = \frac{1}{2}\sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} - \frac{1}{2}\sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}.$$

Therefore,

$$\delta'(\varepsilon) = \frac{2\varepsilon + \sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}}} - \frac{2\varepsilon - \sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}},$$

$$\delta'_+(0) = \frac{2\sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2}} = 2\sqrt{1 + \left(\frac{d(A)}{\theta}\right)^2}.$$

By (7), we have $\delta'_+(0) = 2\,\mathrm{cond}_1(A, \lambda_2)$.

Remark 2. *As discussed in the introduction, we will see that the results in these examples are not casual. In fact, we will prove that the condition number of λ of order $1/\nu$ is related with the functions δ and a.*

3 Bounds by closed disks

From Theorems 2.6.6 and 5.4.4 of Karow [6] we infer the following theorem.

Theorem 3. *Let λ be an eigenvalue of $A \in \mathbb{C}^{n\times n}$ of index ν. For each $\eta \in (0, 1]$ there exists an $\varepsilon_\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_\eta]$,*

$$\mathcal{D}\left(\lambda, \left((1-\eta)\varepsilon\right)^{1/\nu} c\right) \subset \mathcal{K}_\lambda(\varepsilon) \subset \mathcal{D}\left(\lambda, \left((1+\eta)\varepsilon\right)^{1/\nu} c\right) \tag{9}$$

c being the condition number of λ of order $1/\nu$.

From (9) we deduce that

$$\mathcal{D}\big(0, \big((1-\eta)\varepsilon\big)^{1/\nu} c\big) \subset \mathcal{K}_\lambda(\varepsilon) - \lambda \subset \mathcal{D}\big(0, \big((1+\eta)\varepsilon\big)^{1/\nu} c\big)$$

Considering the homotecy

$$z \mapsto \frac{z}{\varepsilon^{1/\nu}}, \qquad z \in \mathbb{C},$$

we see

$$\mathcal{D}\big(0, (1-\eta)^{1/\nu} c\big) \subset \frac{\mathcal{K}_\lambda(\varepsilon) - \lambda}{\varepsilon^{1/\nu}} \subset \mathcal{D}\big(0, (1+\eta)^{1/\nu} c\big). \tag{10}$$

4 Bounds by diameters

Since the diameter function is monotone increasing with respect to $\subset$, by (10),

$$2c(1-\eta)^{1/\nu} \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c(1+\eta)^{1/\nu}. \tag{11}$$

Thus, we arrive at the following theorem.

Theorem 4. *Let ν be the index of λ, and let c be the condition number of λ of order $1/\nu$. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} = 2c. \tag{12}$$

Proof. Let us define the functions

$$f_1(\eta) \quad := 1 - (1-\eta)^{1/\nu}, \tag{13}$$

$$f_2(\eta) \quad := (1+\eta)^{1/\nu} - 1, \tag{14}$$

where $\eta \in [0,1]$. If $\nu = 1$, then

$$f_1(\eta) \quad := 1 - 1 + \eta = \eta,$$
$$f_2(\eta) \quad := 1 + \eta - 1 = \eta;$$

so, $f_1(\eta) = f_2(\eta)$ on $[0,1]$.

If $\nu \geq 2$, we will deduce that $f_2(\eta) \leq f_1(\eta)$. This last inequality is equivalent to

$$(1+\eta)^{1/\nu} - 1 \leq 1 - (1-\eta)^{1/\nu}, \ \forall \eta \in [0,1]$$

$$\iff \quad (1+\eta)^{1/\nu} + (1-\eta)^{1/\nu} \leq 2, \ \forall \eta \in [0,1].$$

Let us define

$$g(\eta) := (1+\eta)^{1/\nu} + (1-\eta)^{1/\nu}, \quad \eta \in [0,1].$$

Then

$$g'(\eta) = \frac{1}{\nu}\big[(1+\eta)^{1/\nu-1} - (1-\eta)^{1/\nu-1}\big].$$

Since $x^\alpha := \mathrm{e}^{\alpha \ln x}$, when $\alpha < 0$ the function $x \mapsto x^\alpha$ is decreasing in $(0,\infty)$; therefore, $x \mapsto x^{1/\nu-1}$ is decreasing in $(0,\infty)$. Hence, if $0 < \eta < 1$, we see that $1-\eta < 1+\eta$; what implies

$$(1-\eta)^{1/\nu-1} > (1+\eta)^{1/\nu-1}.$$

Thus, $g'(\eta) < 0$. So, g is decreasing on $[0,1]$. Therefore, $g(\eta) < g(0) = 2$. That is, $f_2(\eta) \le f_1(\eta)$.

Given that $2c - 2cf_1(\eta) = 2c(1-\eta)^{1/\nu}$ and $2c + 2cf_2(\eta) = 2c(1+\eta)^{1/\nu}$, by (11), we have

$$2c - 2cf_1(\eta) \le \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \le 2c + 2cf_2(\eta).$$

For every $\nu \ge 1$, $f_2(\eta) \le f_1(\eta)$ for $\eta \in [0,1]$. Then,

$$2c - 2cf_1(\eta) \le \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \le 2c + 2cf_1(\eta)$$

$$\iff \quad -2cf_1(\eta) \le \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \le 2cf_1(\eta),$$

or

$$\left|\frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c\right| \le 2cf_1(\eta).$$

But $\lim_{\eta\to 0^+} f_1(\eta) = 0$ and $f_1(\eta) > 0$ for $\eta > 0$. Thus for a fixed $\eta_0 > 0$ there exists an $\eta_1 > 0$ such that $2cf_1(\eta_1) < \eta_0$. For this η_1, there is an $\varepsilon_{\eta_1} > 0$ such that for all $\varepsilon \in (0, \varepsilon_{\eta_1})$,

$$\left|\frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c\right| \le 2cf_1(\eta) < \eta_0.$$

So, there exists the limit

$$\lim_{\varepsilon\to 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}}$$

and it is equal to $2c$. □

5 Derivatives of the diameter

In this section we relate the right-derivative of the diameter δ at 0 with the condition number of the eigenvalue λ, when $\nu = 1$. First, we have the following lemma.

Lemma 5. *Let $A \in \mathbb{C}^{n\times n}$ and λ be an eigenvalue of A of index ν. Let $\delta(\varepsilon)$ be the diameter of the connected component of $\Lambda_\varepsilon(A)$ that contains λ. Let c be the condition number of λ of order $1/\nu$. Then, there exists the limit*

$$\lim_{\varepsilon\to 0^+} \frac{\delta(\varepsilon)}{\varepsilon}$$

and it is equal to

$$\begin{cases} 2c & \text{if} \quad \nu = 1, \\ \infty & \text{if} \quad \nu \geq 2. \end{cases}$$

Proof. If $\nu = 1$, Theorem 4 implies

$$\lim_{\varepsilon\to 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = 2c.$$

If $\nu \geq 2$, from the same Theorem,

$$\frac{\delta(\varepsilon)}{\varepsilon} = \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \frac{1}{\varepsilon^{1-1/\nu}} \to 2c \cdot \infty,$$

when $\varepsilon \to 0^+$. □

From this lemma the next theorem follows immediately.

Theorem 6. *Let $A \in \mathbb{C}^{n\times n}$ and λ be an eigenvalue of A of index ν. Let $\delta(\varepsilon)$ be the diameter of the connected component of $\Lambda_\varepsilon(A)$ that contains λ. Let c be the condition number of λ of order $1/\nu$. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = \begin{cases} 2c & \text{if} \quad \nu = 1, \\ \infty & \text{if} \quad \nu \geq 2. \end{cases}$$

Therefore, $\delta'_+(0) = 2c$ if $\nu = 1$.

6 Bounds by areas

Since the area function is monotone increasing with respect to $\subset$, by (10),

$$\pi c^2 (1-\eta)^{2/\nu} \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 (1+\eta)^{2/\nu}, \tag{15}$$

where $a(\varepsilon) :=$ area or Lebesgue measure of $\mathcal{K}_\lambda(\varepsilon)$. Thus, we arrive at the following theorem.

Theorem 7. *Let ν be the index of λ, and let c be the condition number of λ of order $1/\nu$. Then*

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} = \pi c^2. \tag{16}$$

Proof. Let us define the functions

$$\varphi_1(\eta) := 1 - (1-\eta)^{2/\nu}, \tag{17}$$
$$\varphi_2(\eta) := (1+\eta)^{2/\nu} - 1. \tag{18}$$

If $\nu = 1$, then

$$\begin{aligned}\varphi_1(\eta) &= 1 - (1-\eta)^2 = 2\eta - \eta^2,\\ \varphi_2(\eta) &= (1+\eta)^2 - 1 = 2\eta + \eta^2;\end{aligned}$$

it is obvious that $\varphi_1(\eta) \le \varphi_2(\eta)$ when $0 \le \eta$.

If $\nu = 2$, then

$$\varphi_1(\eta) = 1 - (1-\eta) = \eta = (1+\eta) - 1 = \varphi_2(\eta).$$

If $\nu \ge 3$, we will see that

$$\varphi_2(\eta) \le \varphi_1(\eta), \quad \eta \in [0,1],$$

or, equivalently,

$$\begin{aligned}&(1+\eta)^{2/\nu} - 1 \le 1 - (1-\eta)^{2/\nu}, \quad \eta \in [0,1]\\ \iff\ &(1+\eta)^{2/\nu} + (1-\eta)^{2/\nu} \le 2, \quad \eta \in [0,1].\end{aligned}$$

Let us define

$$\psi(\eta) := (1+\eta)^{2/\nu} + (1-\eta)^{2/\nu}, \quad \eta \in [0,1].$$

Then

$$\psi'(\eta) = \frac{2}{\nu}\left[(1+\eta)^{2/\nu-1} - (1-\eta)^{2/\nu-1}\right].$$

As the function $x \mapsto x^{2/\nu-1}$ is decreasing in $(0,\infty)$, if $0 < \eta < 1$, then $1-\eta < 1+\eta$ implies $(1-\eta)^{2/\nu-1} > (1+\eta)^{2/\nu-1}$ and $\psi'(\eta) < 0$. In consequence, ψ is decreasing in $[0,1]$. Thus, for $\eta \in (0,1], \psi(\eta) < \psi(0) = 2$. Accordingly,

$$\varphi_2(\eta) \le \varphi_1(\eta), \quad \eta \in [0,1].$$

By (17) and (18),

$$\begin{aligned}(1-\eta)^{2/\nu} &= 1 - \varphi_1(\eta),\\ (1+\eta)^{2/\nu} &= 1 + \varphi_2(\eta).\end{aligned}$$

Inequalities (15) imply

$$\pi c^2 - \pi c^2 \varphi_1(\eta) \le \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \le \pi c^2 + \pi c^2 \varphi_2(\eta). \tag{19}$$

The $\nu \leq 2$ case. Since $\varphi_1(\eta) \leq \varphi_2(\eta)$ in $[0, 1]$,

$$-\pi c^2 \varphi_2(\eta) \leq -\pi c^2 \varphi_1(\eta).$$

Hence, by (19),

$$\pi c^2 - \pi c^2 \varphi_2(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_2(\eta); \tag{20}$$

which is equivalent to

$$\left| \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} - \pi c^2 \right| \leq \pi c^2 \varphi_2(\eta).$$

Bearing

$$\lim_{\eta \to 0^+} \pi c^2 \varphi_2(\eta) = 0$$

in mind, we deduce that for a fixed $\eta_0 > 0$ there exists an $\eta_1 > 0$ such that $\pi c^2 \varphi_2(\eta_1) < \eta_0$. For this η_1 there is an $\varepsilon_{\eta_1} > 0$ such that for $\varepsilon \in (0, \varepsilon_{\eta_1}]$,

$$\left| \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} - \pi c^2 \right| \leq \pi c^2 \varphi_2(\eta_1) < \eta_0.$$

This proves that there exists the limit

$$\lim_{\varepsilon \to 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}}$$

and it is equal to πc^2.

The $\nu \geq 3$ case. As $\varphi_2(\eta) \leq \varphi_1(\eta)$ in $[0, 1]$, from (19) we deduce that

$$\pi c^2 - \pi c^2 \varphi_1(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_1(\eta)$$

and, as $\varphi_1(\eta) > 0$ if $\eta > 0$ and

$$\lim_{\eta \to 0^+} \varphi_1(\eta) = 0,$$

by a reasoning analogous to the former one we infer that there exists the limit

$$\lim_{\varepsilon \to 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} = \pi c^2. \ \square$$

7 Derivatives of the area

In this Section we establish the relation between the first and second right-derivatives of a at 0 and the condition number of the eigenvalue λ of order $1/\nu$, when $\nu = 1$ or 2. First, we prove the following lemma.

Lemma 8. *Let $A \in \mathbb{C}^{n\times n}$ and λ be an eigenvalue of A of index ν. Let $a(\varepsilon)$ be the area of the connected component of $\Lambda_\varepsilon(A)$ that contains λ. Let c be the condition number of λ of order $1/\nu$. Then, there exists the limit*

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon}$$

and it is equal to

$$\begin{cases} 0 & \textit{if} \quad \nu = 1, \\ \pi c^2 & \textit{if} \quad \nu = 2, \\ \infty & \textit{if} \quad \nu \geq 3. \end{cases}$$

Proof. First, let us suppose that $\nu = 1$. Then Theorem 7 implies

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon^2} = \pi c^2.$$

Hence,

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \varepsilon \frac{a(\varepsilon)}{\varepsilon^2} = \left(\lim_{\varepsilon\to 0^+} \varepsilon\right)\left(\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon^2}\right) = 0 \cdot \pi c^2 = 0.$$

Second, let us assume now that $\nu = 2$. From Theorem 7,

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon} = \pi c^2.$$

Finally, when $\nu \geq 3$, it is obvious, by Theorem 7 and

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \cdot \frac{1}{\varepsilon^{(\nu-2)/\nu}},$$

that

$$\lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon} = \infty. \qquad \square$$

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$ and λ be an eigenvalue of A of index ν. Let $a(\varepsilon)$ be the area of the connected component $\Lambda_\varepsilon(A)$ that contains λ. Then, there exists the right-derivative of a at 0, $a'_+(0)$, and*

$$a'_+(0) = \begin{cases} 0 & \textit{if} \quad \nu = 1, \\ \pi c^2 & \textit{if} \quad \nu = 2, \\ \infty & \textit{if} \quad \nu \geq 3. \end{cases}$$

Proof. By Lemma 8 we deduce that

if $\nu = 1$,

$$a'_+(0) = \lim_{\varepsilon\to 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon} = 0;$$

if $\nu = 2$,

$$a'_+(0) = \lim_{\varepsilon \to 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \pi c^2;$$

if $\nu \geq 3$,

$$a'_+(0) = \lim_{\varepsilon \to 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \infty. \square$$

Theorem 10. *Let $A \in \mathbb{C}^{n \times n}$ and λ be a semisimple eigenvalue. Let $a(\varepsilon)$ be the area of the connected component $\Lambda_\varepsilon(A)$ that contains λ. Let us assume that there exists $a''_+(0)$. Then*

$$a''_+(0) = 2\pi c^2.$$

Proof. Let us define the function

$$A(\varepsilon) := \begin{cases} a(\varepsilon) & \text{if } \varepsilon \geq 0, \\ a(-\varepsilon) & \text{if } \varepsilon < 0. \end{cases}$$

By 9, as $\nu = 1$, there exists $A'(0)$ and $A'(0) = 0$. If we suppose that there exists the derivative $A''(0)$, then

$$A''(0) = \lim_{\varepsilon \to 0} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2}.$$

But, the existence of this limit does not imply the existence of $A''(0)$. See [1, Exercise 5–20.].

When $\varepsilon > 0$, $A(\varepsilon) = a(\varepsilon), A(-\varepsilon) = a(-(-\varepsilon)) = a(\varepsilon)$; therefore, $A(\varepsilon) + A(-\varepsilon) = 2a(\varepsilon)$.

When $\varepsilon < 0$, $A(\varepsilon) = a(-\varepsilon)$, $A(-\varepsilon) = a(-\varepsilon)$; hence, $A(\varepsilon) + A(-\varepsilon) = 2a(-\varepsilon)$.

Then, by Theorem 7,

$$\lim_{\varepsilon \to 0^+} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \to 0^+} \frac{2a(\varepsilon)}{\varepsilon^2} = 2\pi c^2,$$

$$\lim_{\varepsilon \to 0^-} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \to 0^-} \frac{2a(-\varepsilon)}{(-\varepsilon)^2} = \lim_{\beta \to 0^+} \frac{2a(\beta)}{\beta^2} = 2\pi c^2.$$

Consequently, there exists the limit

$$\lim_{\varepsilon \to 0} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2}$$

and is equal to $2\pi c^2$. So, $A''(0) = 2\pi c^2$, and $a''_+(0) = 2\pi c^2$. $\square$

8 Conjecture

Let λ be a semisimple eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$. We need the definition of a semialgebraic set $S \subset \mathbb{R}^n$ and of a semialgebraic function $f : S \to \mathbb{R}$. These concepts can be seen in [6, Chapter 3, p. 39]. A classical reference is [2, Chapter 2, p. 23].

Conjecture 11. *The functions* $\delta, a\colon [0,\infty) \to \mathbb{R}$ *are semialgebraic.*

The following theorem is proved in [3, Lemma 3.1 (ii)]

Theorem 12 (de l'Hôpital inverse rule)**.** *If* $f, g\colon [0,\infty) \to \mathbb{R}$ *are semialgebraic functions,* $f(0) = g(0) = 0$*, and there is an* $\varepsilon_0 > 0$ *such that* $g'(\varepsilon) > 0$ *for* $\varepsilon \in (0, \varepsilon_0)$*, then*

$$\lim_{\varepsilon\to 0^+} \frac{f(\varepsilon)}{g(\varepsilon)} = \ell \in \mathbb{R} \quad \Longrightarrow \quad \lim_{\varepsilon\to 0^+} \frac{f'(\varepsilon)}{g'(\varepsilon)} = \ell.$$

If Conjecture 11 were true, then

$$\pi c^2 = \lim_{\varepsilon\to 0^+} \frac{a(\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon\to 0^+} \frac{a'(\varepsilon)}{2\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{a''(\varepsilon)}{2}.$$

Thus, the derivative $a''_+(0)$ would exist and it would be equal to $2\pi c^2$ because $a''_+(0) = \lim_{\varepsilon\to 0^+} a''(\varepsilon)$.

As the area of a region can be expressed by means of a line integral, and taking into account that a parametric integral is differentiable with respect to the parameter when the integrand is, the derivatives $a'(\varepsilon)$ and $a''(\varepsilon)$ exist for sufficiently small $\varepsilon > 0$.

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