

## CONSTANT MEAN CURVATURE HYPERSURFACES WITH CONSTANT $\delta$ -INVARIANT

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We completely classify constant mean curvature hypersurfaces (CMC) with constant  $\delta$ -invariant in the unit 4-sphere  $S^4$  and in the Euclidean 4-space  $\mathbb{E}^4$ .

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**1. Introduction.** A hypersurface in the unit round sphere  $S^{n+1}$  is called isoparametric if it has constant principal curvatures. It is known from [1] that an isoparametric hypersurface in  $S^4$  is either an open portion of a 3-sphere or an open portion of the product of a circle and a 2-sphere, or an open portion of a tube of constant radius over the Veronese embedding. Because every isoparametric hypersurface in  $S^4$  has constant mean curvature (CMC) and constant scalar curvature, it is interesting to determine all hypersurfaces with CMC and constant scalar curvature. In [2], it was proved that a closed hypersurface with CMC and constant scalar curvature in  $S^4$  is isoparametric. Furthermore, complete hypersurfaces with CMC and constant scalar curvature in  $S^4$  or in  $\mathbb{E}^4$  have been completely classified in [9].

For each Riemannian  $n$ -manifold  $M^n$  with  $n \geq 3$ , the first author defined in [3, 4] the Riemannian invariant  $\delta$  on  $M$  by

$$\delta(p) = \tau(p) - \inf K(p), \quad (1.1)$$

where  $\tau = \sum_{i < j} K(e_i \wedge e_j)$  is the scalar curvature and  $\inf K$  is the function assigning to each  $p \in M^n$  the infimum of  $K(\pi)$ ,  $\pi$  running over all planes in  $T_p M$ . Although the invariant  $\delta$  and the scalar curvature are both Riemannian scalar invariants, they are very much different in nature.

It is known that the invariant  $\delta$  plays some important roles in recent study of Riemannian manifolds and Riemannian submanifolds (see, e.g., [4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 16]). In particular, it was proved in [3] that for any submanifold of a real space form  $R^m(\epsilon)$  of constant curvature  $\epsilon$ , one has the following general sharp inequality:

$$\delta \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)\epsilon, \quad (1.2)$$

where  $H^2$  is the squared mean curvature function and  $n$  is the dimension of the submanifold.

Clearly, every isoparametric hypersurface in  $S^4$  or in  $\mathbb{E}^4$  has constant mean curvature and constant  $\delta$ -invariant. So, it is a natural problem to study hypersurfaces in  $S^4$  and  $\mathbb{E}^4$  with CMC and constant  $\delta$ -invariant. The purpose of this paper is thus to classify such hypersurfaces.

Our main results are the following theorems.

**THEOREM 1.1.** *A CMC hypersurface in the Euclidean 4-space  $\mathbb{E}^4$  has constant  $\delta$ -invariant if and only if it is one of the following:*

- (1) *an isoparametric hypersurface;*
- (2) *a minimal hypersurface with relative nullity greater than or equal to 1;*
- (3) *an open portion of a hypercylinder  $N \times \mathbb{R}$  over a surface  $N$  in  $\mathbb{E}^3$  with CMC and nonpositive Gauss curvature.*

**THEOREM 1.2.** *A CMC hypersurface  $M$  in the unit 4-sphere  $S^4$  has constant  $\delta$ -invariant if and only if one of the following two statements holds:*

- (1)  *$M$  is an isoparametric hypersurface;*
- (2) *there is an open dense subset  $U$  of  $M$  and a nontotally geodesic isometric minimal immersion  $\phi : B^2 \rightarrow S^4$  from a surface  $B^2$  into  $S^4$  such that  $U$  is an open subset of  $NB^2 \subset S^4$ , where  $NB^2$  is defined by*

$$N_p B^2 = \left\{ \xi \in T_{\phi(p)} S^4 : \langle \xi, \xi \rangle = 1, \langle \xi, \phi_*(T_p B^2) \rangle = 0 \right\}. \tag{1.3}$$

In contrast to [2, 9], we do not make any global assumption on the hypersurfaces in Theorems 1.1 and 1.2.

As an immediate application of Theorem 1.1, we have the following corollary.

**COROLLARY 1.3.** *Let  $M$  be a complete hypersurface of Euclidean 4-space  $\mathbb{E}^4$ . Then  $M$  has constant  $\delta$ -invariant and nonzero CMC if and only if  $M$  is one of the following hypersurfaces:*

- (1) *an ordinary hypersphere;*
- (2) *a spherical hypercylinder:  $\mathbb{R} \times S^2$ ;*
- (3) *a hypercylinder over a circle:  $\mathbb{E}^2 \times S^1$ .*

**2. Preliminaries.** Let  $R^m(\epsilon)$  denote the complete simply connected space form  $R^4(\epsilon)$  of constant curvature  $\epsilon$ . Let  $M$  be a hypersurface of an  $R^4(\epsilon)$ . Denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi-Civita connections of  $M^n$  and  $R^4(\epsilon)$ , respectively. Then the Gauss and Weingarten formulas of  $M^n$  in  $R^4(\epsilon)$  are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -AX \tag{2.1}$$

for tangent vector fields  $X, Y$ , and unit normal vector field  $\xi$ , where  $h$  denotes the second fundamental form and  $A$  the shape operator. The second fundamental form and the shape operator are related by

$$\langle AX, Y \rangle = \langle h(X, Y), \xi \rangle. \tag{2.2}$$

The mean curvature  $H$  of  $M$  in  $R^4(\epsilon)$  is defined by  $H = (1/3)\text{trace}A$ . A hypersurface is called a CMC hypersurface if it has CMC.

Denote by  $R$  the Riemann curvature tensor of  $M$ . Then the *equation of Gauss* is given by

$$R(X, Y; Z, W) = (\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle) \epsilon + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{2.3}$$

for vectors  $X, Y, Z$ , and  $W$  tangent to  $M$ . The Codazzi equation is given by

$$(\nabla_X A)Y = (\nabla_Y A)(X). \tag{2.4}$$

Since  $A$  is a symmetric endomorphism of  $T_pM$ ,  $p \in M$ , we have three eigenvalues  $a, b$ , and  $c$  with three independent unit eigenvectors  $e_1, e_2$ , and  $e_3$  so that

$$Ae_1 = ae_1, \quad Ae_2 = be_2, \quad Ae_3 = ce_3, \tag{2.5}$$

where  $A = A_{e_4}$ . The functions  $a, b$ , and  $c$  are called the principal curvatures and  $e_1, e_2$ , and  $e_3$  the principal directions.

With respect to the frame fields  $e_1, e_2$ , and  $e_3$  of  $M$  chosen above, let  $\omega^1, \omega^2$ , and  $\omega^3$  be the field of dual frames and let  $\omega_B^A, A, B = 1, 2, 3, 4$ , be the connection forms associated with  $e_1, e_2, e_3$ , and  $e_4$ . Then the structure equations of  $M$  in  $R^4(\epsilon)$  are given by

$$d\omega^i = - \sum_{j=1}^3 \omega_j^i \wedge \omega^j, \quad \omega_j^i + \omega_i^j = 0, \tag{2.6}$$

$$d\omega_j^i = \sum_{k=1}^3 \omega_k^i \wedge \omega_k^j + \omega_i^4 \wedge \omega_j^4 + \epsilon \omega^i \wedge \omega^j, \tag{2.7}$$

$$d\omega_i^4 = \sum_{j=1}^3 \omega_j^4 \wedge \omega_j^i, \quad i, j = 1, 2, 3. \tag{2.8}$$

Moreover, from (2.5), we have

$$\omega_1^4 = a\omega^1, \quad \omega_2^4 = b\omega^2, \quad \omega_3^4 = c\omega^3. \tag{2.9}$$

Without loss of generality, we may choose  $e_1, e_2$ , and  $e_3$  such that  $a \geq b \geq c$ . It is well known that  $a, b$ , and  $c$  are continuous on  $M$  and differentiable on the

open subset  $U = \{p \in M : a(p) > b(p) > c(p)\}$ . The principal directions  $e_1, e_2$ , and  $e_3$  can be chosen to be differentiable on  $U$ .

Let  $p$  be any given point in  $M$ . If  $0 > b \geq c$  at  $p$ , then, after replacing  $\xi$  by  $-\xi$  and interchanging  $a$  and  $c$ , we obtain  $a \geq b > 0$  and  $b \geq c$ .

**3. Lemmas.** We follow the notations given in Section 2. Throughout this paper, we will choose  $e_1, e_2, e_3$ , and  $e_4$  so that  $a \geq b \geq 0$  and  $b \geq c$ .

**LEMMA 3.1.** *For each point  $p \in M$ , either*

- (a)  $\inf K = bc + \epsilon$  with  $c \geq 0$  at  $p$ , or
- (b)  $\inf K = ac + \epsilon$  with  $c \leq 0$  at  $p$ .

**PROOF.** Recall that we have assumed that  $a \geq b \geq 0$  and  $b \geq c$  at  $p$ . Let  $P$  be any 2-plane in  $T_pM$ . Then  $P$  must intersect the 2-plane  $\text{Span}\{e_1, e_2\}$ . Thus, there exists an orthonormal basis  $\{X, Y\}$  of  $P$  such that  $X \in P \cap \text{Span}\{e_1, e_2\}$  and

$$\begin{aligned} X &= \cos \theta e_1 + \sin \theta e_2, \\ Y &= \pm \sin \theta \cos \phi e_1 \mp \cos \theta \cos \phi e_2 + \sin \phi e_3 \end{aligned} \tag{3.1}$$

for some  $\theta$  and  $\phi$  with  $\theta \in [0, \pi)$ ,  $\phi \in [0, \pi]$ . It is easy to see that the sectional curvature  $K(P)$  of  $P$  is given by

$$K(P) = ab \cos^2 \phi + c(a \cos^2 \theta + b \sin^2 \theta) \sin^2 \phi + \epsilon. \tag{3.2}$$

We regard the sectional curvature at  $p$  as a function  $K(\theta, \phi)$  of  $\theta$  and  $\phi$ .

If  $c \geq 0$ , (3.2) can be expressed as

$$K(\theta, \phi) = ac + a(b - c) \cos^2 \phi - c(a - b) \sin^2 \theta \sin^2 \phi + \epsilon, \tag{3.3}$$

which implies that  $K(\theta, \phi) \geq bc + \epsilon$  with the equality holding at  $(\theta, \phi) = (\pi/2, \pi/2)$ .

If  $c \leq 0$ , we can express (3.2) as

$$K(\theta, \phi) = bc + b(a - c) \cos^2 \phi + c(a - b) \cos^2 \theta \sin^2 \phi + \epsilon, \tag{3.4}$$

which implies that  $K(\theta, \phi) \geq ac + \epsilon$  with the equality holding at  $(\theta, \phi) = (0, \pi/2)$ . □

**LEMMA 3.2.** *On the open subset  $U$  on which  $M$  has three distinct principal curvatures, the following equations hold:*

$$e_2a = (a - b)\omega_1^2(e_1), \tag{3.5}$$

$$e_3a = (a - c)\omega_1^3(e_1), \tag{3.6}$$

$$e_3b = (b - c)\omega_2^3(e_2), \tag{3.7}$$

$$e_1b = (b - a)\omega_2^1(e_2), \tag{3.8}$$

$$e_1c = (c - a)\omega_3^1(e_3), \tag{3.9}$$

$$e_2c = (c - b)\omega_3^2(e_3), \tag{3.10}$$

$$(c - b)\omega_3^2(e_1) = (c - a)\omega_3^1(e_2), \tag{3.11}$$

$$(b - c)\omega_2^3(e_1) = (b - a)\omega_2^1(e_3), \tag{3.12}$$

$$(a - b)\omega_1^2(e_3) = (a - c)\omega_1^3(e_2). \tag{3.13}$$

**PROOF.** The proof follows from Codazzi's equation and is a straightforward computation. □

**4. Proofs of Theorems 1.1 and 1.2.** We use the same notations as before. Let  $M$  be a (connected) CMC hypersurface with constant  $\delta$ -invariant in  $R^4(\epsilon)$ . Then the scalar curvature  $\tau$  of  $M$  is given by

$$\tau = ab + bc + ac + 3\epsilon. \tag{4.1}$$

From the constancy of the mean curvature, we have

$$a + b + c = r_1 \tag{4.2}$$

for some constant  $r_1$ . By combining Lemma 3.1 with (1.1) and (4.1), we obtain

- (i)  $\delta = a(b + c) + 2\epsilon$  with  $c \geq 0$ , or
- (ii)  $\delta = b(a + c) + 2\epsilon$  with  $c \leq 0$ .

When  $U = \{p \in M : a(p) > b(p) > c(p)\}$  is empty,  $M$  is an isoparametric hypersurface since the mean curvature and the  $\delta$ -invariant are both constant. Thus, from now on, we may assume that  $U$  is nonempty and work on  $U$ .

We will treat Cases (i) and (ii) on  $U$  separately.

**CASE (i)** ( $\delta = a(b + c) + 2\epsilon, c \geq 0$ ). Since  $\delta$  is constant, we get  $a(b + c) = r_2 - 2\epsilon$  for some constant  $r_2$ . Combining this with (4.2) yields

$$a = c_1, \quad b + c = c_2 \tag{4.3}$$

for some constants  $c_1$  and  $c_2$ . For simplicity, let

$$\omega_3^2(e_1) = \mu, \quad \omega_1^2(e_2) = f, \quad \omega_2^3(e_2) = g, \quad \omega_2^3(e_3) = h. \tag{4.4}$$

If  $b$  and  $c$  are constant, then  $M$  is isoparametric. So, we assume that  $b$  and  $c$  are nonconstant on  $U$ . Using (4.3), we get  $e_j b = -e_j c, j = 1, 2, 3$ . Thus,

Lemma 3.2 gives

$$\omega_1^3(e_1) = \omega_1^2(e_1) = 0, \quad (4.5)$$

$$e_1b = (a-b)f = (c-a)\omega_1^3(e_3), \quad (4.6)$$

$$e_2b = (b-c)h, \quad (4.7)$$

$$e_3b = (b-c)g. \quad (4.8)$$

From (4.5), we know that the integral curves of  $e_1$  are geodesics in  $U$ . Applying (3.12), (3.13), (4.6), (4.7), and (4.8), we find

$$\begin{aligned} e_1b &= (a-b)f, & e_2b &= (c-b)h, \\ e_3b &= (b-c)g, & e_ja &= 0, & e_jc &= -e_jb, \end{aligned} \quad (4.9)$$

$$\omega_1^2 = f\omega^2 + \frac{b-c}{b-a}\mu\omega^3, \quad (4.10)$$

$$\omega_1^3 = \frac{b-c}{c-a}\mu\omega^2 + \frac{a-b}{c-a}f\omega^3, \quad (4.11)$$

$$\omega_2^3 = -\mu\omega^1 + g\omega^2 + h\omega^3, \quad (4.12)$$

for  $j = 1, 2, 3$ . By applying (2.6), (4.9), (4.10), (4.11), and (4.12), we find

$$\begin{aligned} d\omega^1 &= \left( \frac{b-c}{a-b} + \frac{b-c}{c-a} \right) \mu\omega^2 \wedge \omega^3, \\ d\omega^2 &= f\omega^1 \wedge \omega^2 + \frac{a-c}{b-a}\mu\omega^1 \wedge \omega^3 + g\omega^2 \wedge \omega^3, \\ d\omega^3 &= \frac{a-b}{a-c}\mu\omega^1 \wedge \omega^2 - \frac{a-b}{a-c}f\omega^1 \wedge \omega^3 + h\omega^2 \wedge \omega^3. \end{aligned} \quad (4.13)$$

Using  $(\nabla_{e_2}e_1 - \nabla_{e_1}e_2 - [e_2, e_1])b = 0$ , we get

$$(a-b)e_2f + (b-c)e_1h = 2(c-a)fh + \frac{(b-a)(b-c)}{c-a}\mu g. \quad (4.14)$$

Similarly, from  $(\nabla_{e_3}e_1 - \nabla_{e_1}e_3 - [e_3, e_1])b = (\nabla_{e_3}e_2 - \nabla_{e_2}e_3 - [e_3, e_2])b = 0$ , we get

$$\begin{aligned} (a-b)e_3f + (c-b)e_1g &= \frac{(a-c)(c-b)}{b-a}\mu h + \frac{2a(b+c) - b^2 - c^2 - 2a^2}{c-a}fg, \\ e_3h + e_2g &= \frac{c-b}{a-c}\mu f. \end{aligned} \quad (4.15)$$

By computing  $d\omega_1^2$  and applying (4.10), (4.11), (4.12), (4.13), and Cartan's structure equations, we obtain

$$e_1 f = \frac{2(b-c)}{a-c} \mu^2 - f^2 - ab - \epsilon, \tag{4.16}$$

$$e_1 \left( \frac{b-c}{b-a} \mu \right) = \left\{ \frac{b-c}{a-b} + \frac{2(a-b)}{a-c} \right\} \mu f, \tag{4.17}$$

$$e_3 f + e_2 \left( \frac{b-c}{a-b} \mu \right) = \frac{b+c-2a}{c-a} \left\{ fg - \frac{b-c}{a-b} \mu h \right\}. \tag{4.18}$$

Similarly, by computing  $d\omega_1^3$  and  $d\omega_2^3$ , and by applying (4.10), (4.11), (4.12), (4.13), and Cartan's structure equations, we obtain

$$e_1 \left( \frac{b-c}{c-a} \mu \right) = \left\{ \frac{2a^2 + 2c^2 + b^2 - ab - 3ac - bc}{(a-c)^2} \right\} \mu f, \tag{4.19}$$

$$e_1 \left( \frac{a-b}{c-a} f \right) = -ac - \epsilon - \left( \frac{a-b}{c-a} \right)^2 f^2 + \frac{2(b-c)}{b-a} \mu^2, \tag{4.20}$$

$$e_3 \left( \frac{c-b}{c-a} \mu \right) + e_2 \left( \frac{a-b}{c-a} f \right) = \frac{2a-b-c}{a-c} \left\{ fh + \frac{b-c}{a-b} \mu g \right\}, \tag{4.21}$$

$$e_2 \mu + e_1 g = \frac{a-b}{c-a} \mu h - fg, \tag{4.22}$$

$$e_1 h + e_3 \mu = \frac{a-b}{a-c} fh + \frac{a-c}{a-b} \mu g, \tag{4.23}$$

$$e_2 h - e_3 g = \frac{2(b-c)^2 \mu^2}{(a-b)(a-c)} + \frac{a-b}{a-c} f^2 - g^2 - h^2 - bc - \epsilon. \tag{4.24}$$

Combining (4.9), (4.16), and (4.20) yields

$$2(2a-b-c)(a-b)^2 + 2(2a-b-c)(b-c)^2 \mu^2 + (a-b)(a-c) \{ ab(a-b) + ac(a-c) + (2a-b-c)\epsilon \} = 0, \tag{4.25}$$

which is impossible unless  $\epsilon < 0$ , since we assume that  $a > b > c \geq 0$  in Case (i).

**CASE (ii)** ( $\delta = b(a+c) + 2\epsilon$ ,  $c \leq 0$ ). Since  $\delta$  is constant, we get  $b(a+c) = r_2 - 2\epsilon$  for some constant  $r_2$ . Combining this with (4.2) yields

$$b = c_3, \quad a + c = c_4, \tag{4.26}$$

for some constants  $c_3$  and  $c_4$ . For simplicity, let

$$\omega_3^1(e_2) = \tilde{\mu}, \quad \omega_2^1(e_1) = \tilde{f}, \quad \omega_1^3(e_1) = \tilde{g}, \quad \omega_1^3(e_3) = \tilde{h}. \tag{4.27}$$

If  $a$  and  $c$  are constant, then  $M$  is isoparametric. So, from now on, we may assume that  $a$  and  $c$  are nonconstant on  $U$ . Using (4.26), we get

$$e_j a = -e_j c, \quad j = 1, 2, 3. \tag{4.28}$$

Thus, [Lemma 3.2](#) yields

$$\omega_2^3(e_2) = \omega_2^1(e_2) = 0, \tag{4.29}$$

$$e_1a = (c - a)\tilde{h}, \quad e_2a = (b - a)\tilde{f}, \quad e_3a = (a - c)\tilde{g}. \tag{4.30}$$

Equation [\(4.28\)](#) shows that the integral curves of  $e_2$  are geodesics in  $U$ . Applying [\(3.12\)](#), [\(3.13\)](#), [\(4.29\)](#), and [\(4.30\)](#), we find

$$\omega_1^2 = -\tilde{f}\omega^1 - \frac{a-c}{a-b}\tilde{\mu}\omega^3, \tag{4.31}$$

$$\omega_1^3 = \tilde{g}\omega^1 - \tilde{\mu}\omega^2 + \tilde{h}\omega^3, \tag{4.32}$$

$$\omega_2^3 = \frac{a-c}{c-b}\tilde{\mu}\omega^1 + \frac{a-b}{b-c}\tilde{f}\omega^3. \tag{4.33}$$

By applying [\(2.6\)](#), [\(4.31\)](#), [\(4.32\)](#), and [\(4.33\)](#), we find

$$\begin{aligned} d\omega^1 &= -\tilde{f}\omega^1 \wedge \omega^2 + \tilde{g}\omega^1 \wedge \omega^3 + \frac{b-c}{a-b}\tilde{\mu}\omega^2 \wedge \omega^3, \\ d\omega^2 &= -\left(\frac{a-c}{a-b} + \frac{a-c}{b-c}\right)\tilde{\mu}\omega^1 \wedge \omega^3, \\ d\omega^3 &= \frac{a-b}{b-c}\tilde{\mu}\omega^1 \wedge \omega^2 + \tilde{h}\omega^1 \wedge \omega^3 + \frac{a-b}{b-c}\tilde{f}\omega^2 \wedge \omega^3. \end{aligned} \tag{4.34}$$

Using  $(\nabla_{e_2}e_1 - \nabla_{e_1}e_2 - [e_2, e_1])a = 0$ , we find

$$(a - b)e_1\tilde{f} - (a - c)e_2\tilde{h} = 2(b - a)\tilde{f}\tilde{h} + \frac{(a - b)(a - c)}{b - c}\tilde{\mu}\tilde{g}. \tag{4.35}$$

Similarly, from  $(\nabla_{e_3}e_1 - \nabla_{e_1}e_3 - [e_3, e_1])a = (\nabla_{e_3}e_2 - \nabla_{e_2}e_3 - [e_3, e_2])a = 0$ , we get

$$\begin{aligned} (a - b)e_3\tilde{f} + (a - c)e_2\tilde{g} &= \frac{(a - c)(b - c)}{a - b}\tilde{\mu}\tilde{h} + \frac{2ab + 2bc - 2b^2 - a^2 - c^2}{b - c}\tilde{f}\tilde{g}, \\ e_3\tilde{h} + e_1\tilde{g} &= \frac{c - a}{b - c}\tilde{\mu}\tilde{f}. \end{aligned} \tag{4.36}$$

By computing  $d\omega_1^2$  and applying [\(4.31\)](#), [\(4.32\)](#), and [\(4.33\)](#) and Cartan's structure equations, we obtain

$$e_2\tilde{f} = \frac{2(a-c)}{b-c}\tilde{\mu}^2 - \tilde{f}^2 - ab - c, \tag{4.37}$$

$$e_2\left(\frac{a-c}{a-b}\tilde{\mu}\right) = \left\{\frac{a-c}{b-a} - \frac{2(a-b)}{b-c}\right\}\tilde{\mu}\tilde{f}, \tag{4.38}$$

$$e_3\tilde{f} + e_1\left(\frac{a-c}{b-a}\tilde{\mu}\right) = \frac{a+c-2b}{c-b}\left\{\tilde{f}\tilde{g} + \frac{a-c}{a-b}\tilde{\mu}\tilde{h}\right\}. \tag{4.39}$$



Similarly, by computing  $d\omega_1^3$ ,  $d\omega_2^3$ , and by applying (4.31), (4.32), and (4.33) and Cartan's structure equations, we obtain

$$e_2\left(\frac{a-c}{c-b}\tilde{\mu}\right) = \left\{\frac{2b^2+2c^2+a^2-ab-3bc-ac}{(b-c)^2}\right\}\tilde{\mu}\tilde{f}, \tag{4.40}$$

$$e_2\left(\frac{a-b}{b-c}\tilde{f}\right) = -bc - \epsilon - \left(\frac{a-b}{c-b}\right)^2\tilde{f}^2 + \frac{2(a-c)}{a-b}\tilde{\mu}^2, \tag{4.41}$$

$$e_3\left(\frac{a-c}{b-c}\tilde{\mu}\right) + e_1\left(\frac{a-b}{b-c}\tilde{f}\right) = \frac{2b-a-c}{b-c}\left\{\tilde{f}\tilde{h} - \frac{a-c}{a-b}\tilde{\mu}\tilde{g}\right\}, \tag{4.42}$$

$$e_1\tilde{\mu} + e_2\tilde{g} = \frac{a-b}{b-c}\tilde{\mu}\tilde{h} - \tilde{f}\tilde{g}, \tag{4.43}$$

$$e_2\tilde{h} + e_3\tilde{\mu} = \frac{b-a}{b-c}\tilde{f}\tilde{h} - \frac{b-c}{a-b}\tilde{\mu}\tilde{g}, \tag{4.44}$$

$$e_1\tilde{h} - e_3\tilde{g} = \frac{2(a-c)^2\tilde{\mu}^2}{(b-a)(b-c)} - \frac{a-b}{b-c}\tilde{f}^2 - \tilde{g}^2 - \tilde{h}^2 - ac - \epsilon. \tag{4.45}$$

Applying (4.30), (4.37), and (4.41) yields

$$2(2b-a-c)(a-b)^2\tilde{f}^2 + 2(2b-a-c)(a-c)^2\tilde{\mu}^2 + (b-a)(b-c)\{ab(b-a) + bc(b-c) + (2b-a-c)\epsilon\} = 0. \tag{4.46}$$

Using (4.26), (4.30), and (4.38), we find

$$e_2\tilde{\mu} = \frac{2[(a-b)^2 + (b-c)^2]}{(a-c)(c-b)}\tilde{\mu}\tilde{f}. \tag{4.47}$$

On the other hand, by differentiating (4.46) with respect to  $e_2$  and using (4.26), (4.30), and (4.37), we obtain

$$\begin{aligned} &4(a+c-2b)(a-c)^2\tilde{\mu}(e_2\tilde{\mu}) \\ &= b(a-b)(3a^3 - 13a^2b + 10ab^2 + 7a^2c - 4abc - 2b^2c - 3ac^2 + bc^2 + c^3)\tilde{f} \\ &\quad - 8\frac{(a+c-2b)^2(a-b)(a-c)}{b-c}\tilde{\mu}^2\tilde{f} + 8(a+c-2b)(a-b)^2\tilde{f}^3 \\ &\quad - (a-b)(a+c-2b)(4b-3a-c)\epsilon\tilde{f}. \end{aligned} \tag{4.48}$$

Replacing  $\tilde{f}^2$  in (4.48) by using (4.46) yields

$$\begin{aligned} &4(a+c-2b)(a-c)^2\tilde{\mu}(e_2\tilde{\mu}) \\ &= 3(a-b)(a-c)(a+c-2b)\tilde{\epsilon}\tilde{f} \\ &\quad + 3b(a-b)(a-c)(a^2 - 3ab + 2b^2 + 2ac - 3bc + c^2)\tilde{f} \\ &\quad - 8\frac{(a+c-2b)(a-c)[(a-b)^2 + (b-c)^2]}{b-c}\tilde{\mu}^2\tilde{f}. \end{aligned} \tag{4.49}$$

Substituting (4.47) into (4.49) yields

$$\tilde{f}(a+c-2b)\{b(a+c-b)+\epsilon\}=0. \tag{4.50}$$

**CASE (ii-a) ( $\tilde{f} = 0$ ).** In this case, (4.37) and (4.41) imply that

$$2(a-c)\mu^2 = (ab+\epsilon)(b-c) = (bc+\epsilon)(a-b). \tag{4.51}$$

The equality in (4.51) yields

$$b(ab+bc-2ac) = (a+c-2b)\epsilon. \tag{4.52}$$

**CASE (ii-a.1) ( $\tilde{f} = 0, b \neq 0$ ).** In this case, (4.26) and (4.52) imply that  $ac$  is constant. Hence, by (4.26), we know that both  $a$  and  $c$  are constant. Thus,  $M$  is isoparametric.

**CASE (ii-a.2) ( $b = \tilde{f} = 0, \epsilon = 1$ ).** In this case, (4.52) reduces to  $a+c=2b$ . So,  $M$  satisfies the equality case of inequality (1.2). Therefore, by applying [7, Theorem 2], we know that  $M$  is given by Theorem 1.2 (2).

**CASE (ii-a.3) ( $b = \tilde{f} = \epsilon = 0$ ).** In this case, (4.37) implies that  $\tilde{\mu} = 0$ . Thus, by (4.31) and (4.33), we obtain  $\omega_1^2 = \omega_2^3 = 0$ . On the other hand, from (4.29), we have  $\nabla_{e_2}e_2 = 0$ . Therefore,  $\mathfrak{D}_1 = \text{Span}\{e_1, e_3\}$  and  $\mathfrak{D}_2 = \text{Span}\{e_2\}$  are integrable distributions in  $M$  with totally geodesic leaves. Hence,  $M$  is locally the Riemannian product of a line and a Riemannian 2-manifold  $N^2$ . Moreover, because the second fundamental form  $h$  of  $M$  in  $\mathbb{E}^4$  satisfies  $h(\mathfrak{D}_1, \mathfrak{D}_2) = \{0\}$ , Moore’s lemma [13] implies that  $M$  is an open portion of a hypercylindrical  $\mathbb{R} \times N^2 \subset \mathbb{E} \times \mathbb{E}^3 = \mathbb{E}^4$ . Furthermore, from the assumption on the shape operator of  $M$  in  $\mathbb{E}^4$ , we know that the mean curvature of  $N$  in  $\mathbb{E}^3$  is constant and the Gauss curvature of  $N$  is nonpositive. Thus, we obtain case (3) of Theorem 1.1.

**CASE (ii-b) ( $\tilde{f} \neq 0, b = 0$ ).** In this case, (4.50) yields  $(a+c)\epsilon = 0$ .

If  $\epsilon = 1$ , then  $a+c=0$ . Hence,  $M$  is a minimal hypersurface satisfying the equality case of inequality (1.2). Thus, by applying [7, Theorem 2], we obtain case (2) of Theorem 1.2.

If  $\epsilon = 0$ , then (4.46) implies that  $a+c-2b=0$  due to  $b=0$  and  $a \neq b$ . Hence,  $M$  satisfies the equality case of inequality (1.2). Since  $M$  has CMC, [7, Theorem 1] implies that  $M$  is either an isoparametric hypersurface or a minimal hypersurface which satisfies the equality  $\delta = 0$ . Hence, we obtain either case (1) or case (2) of Theorem 1.1.

**CASE (ii-c) ( $b \neq 0, \tilde{f} \neq 0$ ).** In this case, (4.50) yields

$$(a+c-2b)\{b(a+c-b)+\epsilon\}=0. \tag{4.53}$$

If  $a+c-2b=0$  holds, then (4.46) implies that  $a(a-b)-c(b-c)=0$  which is impossible, since  $a \geq 0, c \leq 0$ , and  $a > b > 0$  by assumption. Therefore, we

must have

$$\epsilon = b(b - a - c). \quad (4.54)$$

From (4.54),  $\epsilon \geq 0$ , and  $b > 0$ , we get

$$b \geq a + c. \quad (4.55)$$

On the other hand, by substituting (4.54) into (4.46), we find

$$(a + c - 2b)[(a - c)^2 \tilde{\mu}^2 + (a - b)^2 \tilde{f}^2] = b(b - a)^2 (b - c)^2. \quad (4.56)$$

In particular, we obtain  $a + c > 2b$ . Combining this with (4.55) gives  $b < 0$  which is a contradiction. Thus, this case is impossible.

The converse follows from [7, Theorem 2] and from direct computation.

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