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by

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The von Neumann-Morgenstern Stable Sets for 2×2 Games*

E. Inarra[†], C. Larrea[‡], and A. Saracho[§]

Abstract

We analyze the von Neumann and Morgenstern stable sets for the mixed extension of 2×2 games when only single profitable deviations are allowed. We show that the games without a strict Nash equilibrium have a unique *vN&M* stable set and otherwise they have infinite sets.

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1 Introduction

We start discussing the matching pennies game whose payoff matrix is

	heads	*tails*
heads	$+1, -1$	$-1, +1$
tails	$-1, +1$	$+1, -1$

This game has a unique Nash equilibrium (*NE*) strategy profile in which each player chooses heads and tails with equal probability and the expected payoff of each player is zero. Trembling out of this equilibrium leads players to cycle indefinitely. That is, if players in a non-equilibrium profile alternate taking a better response to their opponent's strategy so that it increases their utility, the *NE* will never be reached. In contrast, let us select the set of strategy profiles A formed by the strategy profiles in which the two players get zero payoffs. In any other strategy profile one player gets a negative payoff, while her opponent gets a positive payoff. If the latter maintains her strategy the former increases her payoff by deviating to a profile in A. Specifically *(i)* once the two players are playing a strategy profile in A, no one has a single profitable deviation to move to another profile in this set and *(ii)* if players are playing a strategy profile not in A, then one of them increases her payoff by deviating to a profile in A. Hence, this set A satisfies the internal and the external stability conditions and, therefore, it is a von Neumann-Morgenstern (*vN&M*) stable set. Further, as we shall see, it is unique. Thus the *vN&M* stable sets solution stands out as a good candidate to solve this game.

Stable sets were initially defined by von Neumann and Morgenstern in [6] as solution for cooperative games. Greenberg [3] associates a normal form game with an abstract system (a set and a binary relation defined over it). He argues that normal form games should capture the notion of negotiation among players when modeling social environments. One of the proposed negotiations is based on the idea that a single player can object to the prevailing strategy profile by threatening the opponents with the possibility of using a different strategy that gives her a higher payoff. For this type of negotiation and for finite normal form games with pure strategies Greenberg shows that *vN&M* stable sets exist if there are either at most two players or n players with at most two strategies each. (See *Theorems 7.4.5* and *7.4.6* in Greenberg [3]).[1]

Along this line of research our objective in this paper is to analyze the *vN&M* stable sets for the mixed extension of 2×2 games when only single profitable deviations are allowed. We show that all 2×2 games without a strict *NE* have a unique *vN&M* stable set and otherwise they have infinite sets. We also characterize the strategy profiles belonging to the *vN&M* stable sets. It turns out that only strategy profiles which are not dominated by *NE* strategy profiles or by strategy profiles in which one player's payoff does not depend on his own strategy belong to the *vN&M* stable sets[2]. Finally, we conclude by relating our results with the usual taxonomy of 2×2 games into generic (dominant solvable, strictly competitive and coordination) and non-generic.

The rest of the paper is organized as follows. Section 2 compiles the preliminaries of the paper. Section 3 contains the results and two illustrative examples. Section 4 concludes.

2 Notation and definitions

Let $G = \langle N, \{S_i\}_{i=1,2}, \{u_i\}_{i=1,2} \rangle$ be a normal form 2×2 game where $N = \{1, 2\}$ is the set of players, S_i is the set of two actions for player i, and $u_i : S{=}S_1 \times S_2 \longrightarrow \mathbb{R}$, is player i's payoff function. The *mixed extension of game* G is $< N, \{\Delta(S_i)\}_{i=1,2}, \{U_i\}_{i=1,2} >$ where $\Delta(S_i)$ is the simplex of the mixed strategies for player i, and $U_i : \Delta(S) = \Delta(S_1) \times \Delta(S_2) \longrightarrow \mathbb{R}$, assigns to $\sigma \in \Delta(S)$ the expected value under u_i of the lottery over S that is induced by σ, so that $U_i(\sigma) = \sum\limits_{s \in S} (\prod\limits_{j \in N} \sigma_j(s_j) u_i(s))$ where $\sigma_j(s_j)$ is player j's probability of playing action s_j.

A player $i's$ strategy σ_i is a *best response* to player $j's$ strategy σ_j if $U_i(\sigma_i, \sigma_j) \geq U_i(\sigma'_i, \sigma_j)$ for all $\sigma'_i \in \Delta(S_i)$. We denote by $BR_i(\sigma_j)$ player i's set of best responses to σ_j.

A strategy profile $\sigma^* = (\sigma^*_1, \sigma^*_2)$ is a *NE*, see Nash [5], if σ^*_i is a best response to σ^*_j, *i.e.,* if. $\sigma^*_i \in BR_i(\sigma^*_j)$ for $i = 1, 2$ and $j \neq i$. A *NE* strategy profile σ^* is

[1] For a n-player prisoners' dilemma game Nakanishi [4] has shown the existence and the efficiency of the *vN&M* stable sets.

[2] A strategy profile dominates another profile if there is a player who can profitable deviate given her opponet's strategy.

strict if σ_i^* is the unique best response to σ_j^*, *i.e.*, if $BR_i(\sigma_j^*) = \{\sigma_i^*\}$ for $i = 1, 2$ and $j \neq i$. The set of *NE* strategy profiles is denoted by Σ^*.

Let $\sigma_i, \sigma_i' \in \Delta(S_i)$. Then σ_i strictly dominates σ_i' if $U_i(\sigma_i, \sigma_j) > U_i(\sigma_i', \sigma_j)$ for all $\sigma_j \in \Delta(S_j)$ $(j \neq i)$. A strategy σ_i is strictly dominant if σ_i strictly dominates σ_i' for all $\sigma_i' \in \Delta(S_i)$ $(\sigma_i' \neq \sigma_i)$.

Abusing notation we denote strategies σ_1 and σ_2 as p and q where p and q are, respectively, player 1 and 2's probabilities of playing their "first" action of game G. (By default, $(1 - p)$ and $(1 - q)$ are player 1 and 2's probabilities of playing their "second" action.)

As indicated in the introduction we follow the negotiation procedure proposed by Greenberg. Suppose that strategy profile σ is proposed to players and that only one player can deviate. Then player i can object σ inducing σ' if she is better off in σ' than in σ. Thus an *abstract system* associated to the mixed extension of a game G is the pair $(\Delta(S), \succ)$ where $\succ$ is the binary relation defined on $\Delta(S)$ so that $\sigma' \succ \sigma$ if there exists player i such that $\sigma_j' = \sigma_j$ for $j \neq i$ and $U_i(\sigma') > U_i(\sigma)$.

Let $A \subseteq \Delta(S)$. Then A is a *vN&M* stable set of $(\Delta(S), \succ)$ if it satisfies the following two conditions:

(i) Internal stability: For all $\sigma \in A$ there not exists $\sigma' \in A$ such that $\sigma' \succ \sigma$.

(ii) External stability: For all $\sigma \notin A$ there exists $\sigma' \in A$ such that $\sigma' \succ \sigma$.

Denoting by $D(A) = \cup_{\sigma \in A} D(\sigma)$ where $D(\sigma) = \{\sigma' \in \Delta(S) : \sigma \succ \sigma'\}$ we have that A is a *vN&M* stable set if and only if $D(A) = \Delta(S) \backslash A$.

3 The vN&M stable set for 2×2 games

In this section we present our results and two illustrative examples.

Theorem 1 *Let $(\Delta(S), \succ)$ be the system associated to game G. $(\Delta(S), \succ)$ has infinite vN&M stable sets if and only if G has a strict NE strategy profile and otherwise it has a unique vN&M stable set.*

Proof. Let us consider two cases:

Case 1. *Game G has a strict NE strategy profile σ^*.*

Let us assume without loss of generality. that $\sigma^* = (1, 1)$. Three sub-cases can be distinguished:

a) *Both players have a strictly dominant strategy.* Then $BR_1(q) = \{1\}$ and $BR_2(p) = \{1\}$ for all $p, q \in [0,1]$. Let $\widetilde{\sigma} = (0,0)$ and define $A = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$. We first show that A is a *vN&M* stable set and then that there exist infinite *vN&M* stable sets.

Since U_1 is increasing in p for all $q \in [0,1]$ and U_2 is increasing in q for all $p \in [0,1]$, then for each $\sigma \in A$, $D(\sigma) = \{\sigma' \in \Delta(S) : q' = q,\ p' < p\} \cup \{\sigma' \in \Delta(S) : p' = p,\ q' < q\}$. Thus, for all $\sigma, \sigma' \in A$ we have that $\sigma' \notin D(\sigma)$. Moreover for all $\sigma' \notin A$, there exists $\sigma \in A$ such that $\sigma' \in D(\sigma)$. Hence, $D(A) = \Delta(S) \backslash A$ and consequently A is a *vN&M* stable set.

To show that there are infinite *vN&M* stable sets consider a strategy profile $\widehat{\sigma} \notin A$; $1 > \widehat{p} > 0$ and $1 > \widehat{q} > 0$.[3] Reasoning as above we conclude that $\widehat{A} = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widehat{\sigma},\ \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\widehat{\sigma} + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$ is also a *vN&M* stable set. Therefore, $(\Delta(S), \succ)$ associated to a game in which both players have a strictly dominant strategy has infinite *vN&M* stable sets.

b) *Only one player has a strictly dominant strategy.* Suppose without loss of generality that $BR_1(q) = \{1\}$ for all $q \in [0,1]$. In this case there exists $\overline{p} \in [0,1)$ such that[4]

$$BR_2(p) = \begin{cases} \{0\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{1\} & if \quad p > \overline{p} \end{cases} .$$

Let $\overline{\sigma} = (\overline{p}, 0)$ and let $A = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\overline{\sigma},\ \lambda \in [0,1]\}$. Given that U_1 is increasing in p for all $q \in [0,1]$ and U_2 is increasing in q if $p > \overline{p}$, then arguing as in case **a)** we conclude that A is a *vN&M* stable set.

To show that there are infinite *vN&M* stable sets let us consider a strategy profile $\widehat{\sigma} \notin A$; $1 > \widehat{p} \geq \overline{p}$ and $1 > \widehat{q} > 0$. Reasoning as above we have that $\widehat{A} = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widehat{\sigma},\ \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\widehat{\sigma} + (1-\lambda)\overline{\sigma},\ \lambda \in [0,1]\}$ is also a *vN&M* stable set. Therefore $(\Delta(S), \succ)$ associated to these games have infinite *vN&M* stable sets.

c) *None of the two players has a strictly dominant strategy.* Then there exist $\overline{p}, \overline{q} \in [0,1)$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{0\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{1\} & if \quad p > \overline{p} \end{cases} .$$

[3] Note that there are infinite strategy profiles satisfying these conditions.

[4] Given that σ^* is a strict NE strategy profile then $\overline{p} \neq 1$.

Let $\overline{\sigma} = (\overline{p}, \overline{q})$, $\widetilde{\sigma} = (0,0)$ and $A = A_1 \cup A_2$ where $A_1 = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\overline{\sigma},\ \lambda \in [0,1]\}$ and $A_2 = \{\sigma \in \Delta(S) : \sigma = \lambda\overline{\sigma} + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$. In this case U_1 is increasing in p if $q > \overline{q}$ and U_2 is increasing in q if $p > \overline{p}$, while U_1 is decreasing in p if $q < \overline{q}$ and U_2 is decreasing in q if $p < \overline{p}$. It is easy to verify that $D(A) = \Delta(S)\backslash A$. Hence, A is a *vN&M* stable set.

To show that there are infinite *vN&M* stable sets consider a strategy profile $\widehat{\sigma} \notin A$; $1 > \widehat{p} \geq \overline{p}$ and $1 > \widehat{q} \geq \overline{q}$. Reasoning as above we have that

$\widehat{A} = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widehat{\sigma}, \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\widehat{\sigma} + (1-\lambda)\overline{\sigma}, \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\overline{\sigma} + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$ is also a *vN&M* stable set. Therefore $(\Delta(S), \succ)$ associated to these games have infinite *vN&M* stable sets.

Case 2. *Game G does not have a strict NE strategy profile.*

In trivial games, where all payoffs of at least one player are identical, it is immediate that Σ^* is the unique *vN&M* stable set. Suppose that game G is non-trivial. Let $I_i = \{\sigma \in \Delta(S) : \sigma_j = \sigma_j^*$ and $U_i(\sigma) = U_i(\sigma^*)$ for some $\sigma^* \in \Sigma^*\}$ for $i = 1, 2$ and $j \neq i$. We show that $A = \cup_{i \in N} I_i$ is the unique *vN&M* stable set. Two sub-cases are distinguished:

a) *Only one player has a strictly dominant strategy.* Suppose, as above, without loss of generality that $BR_1(q) = \{1\}$ for all $q \in [0,1]$. Since the game does not have a strict *NE* strategy profile then $BR_2(1) = [0,1]$. Thus $\Sigma^* = \{\sigma \in \Delta(S) : p = 1\}$. As $A = \Sigma^*$ and U_1 is increasing in p for all $q \in [0,1]$ then $D(A) = \Delta(S)\backslash A$ and the result follows.

b) *None of the two players has a strictly dominant strategy.* Without loss of generality two sub-cases are distinguished:

(i) There exist $\overline{p}, \overline{q} \in [0,1]$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{0\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{1\} & if \quad p > \overline{p} \end{cases}.$$

Since game G does not have a strict *NE* strategy profile then either $\overline{p} = 1$ and $\overline{q} = 0$ or $\overline{p} = 0$ and $\overline{q} = 1$. Thus $A = \Sigma^*$ and as $D(\Sigma^*) = \Delta(S)\backslash\Sigma^*$ the result follows.

(ii) There exist $\overline{p}, \overline{q} \in [0,1]$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{1\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{0\} & if \quad p > \overline{p} \end{cases}.$$

Let $\overline{\sigma} = (\overline{p}, \overline{q})$. Then $\overline{\sigma} \in \Sigma^*$ and we have $I_1 = \{\sigma \in \Delta(S) : q = \overline{q}\}$ and $I_2 = \{\sigma \in \Delta(S) : p = \overline{p}\}$. In this case, U_1 is increasing in p if $q > \overline{q}$ and U_2 is decreasing in q if $p > \overline{p}$. In contrast U_1 is decreasing in p if $q < \overline{q}$ and U_2 is increasing in q if $p < \overline{p}$. It is easy to see that $D(A) = \Delta(S)\backslash A$, *i.e.*, that A is a *vN&M* stable set. To prove that A is unique consider a *vN&M* stable set B. We show by contradiction that $B \subseteq A$. Suppose that there is $\sigma \in B$ such that $\sigma \notin A$. Since A satisfies external stability there exists $\sigma' \in A$ such that $\sigma' \succ \sigma$. Given that $\sigma' \in A$ we have $\sigma' \in I_i$ for some $i \in \{1, 2\}$. Then $\sigma'_j = \overline{\sigma}_j$ for $j \neq i$ and since $\sigma' \succ \sigma$ we have $\sigma'_i = \sigma_i$ and $U_j(\sigma') > U_j(\sigma)$. As $\sigma' \notin B$ and B satisfies the external stability condition, there exists $\sigma'' \in B$ such that $\sigma'' \succ \sigma'$. Given that if $\sigma''_j = \sigma'_j$ then $U_i(\sigma'') = U_i(\sigma')$ we have $\sigma''_i = \sigma'_i$ and $U_j(\sigma'') > U_j(\sigma')$. Thus $\sigma'' \succ \sigma$, contradicting the internal stability of B. Consequently $B \subseteq A$ and given that A satisfies external stability it follows that $B = A$. ■

The following remark establishes the strategy profiles belonging to *vN&M* stable sets:

Remark 2 *Only strategy profiles which are not dominated by some NE strategy profile or by strategy profiles in which the payoff of one player does not depend on his own strategy belong to the vN&M stable sets.*

Notice that in games without a strict *NE* strategy profile the dominion of the set A formed by the strategy profiles in which one player's payoff does not depend on his own strategy is $\Delta(S)\backslash A$, and therefore A is the unique *vN&M* stable set.

The following two examples illustrate our result.

Example 1 Consider the mixed extension of the following game:

	s_2^1	s_2^2
s_1^1	1,1	0,0
s_1^2	0,0	1,0

In this example the players' best responses are:

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < 1/2 \\ [0,1] & if \quad q = 1/2 \\ \{1\} & if \quad q > 1/2 \end{cases} \text{ and } BR_2(p) = \begin{cases} [0,1] & if \quad p = 0 \\ \{1\} & if \quad p > 0 \end{cases}$$

It is easy to check that the set of *NE* strategy profiles is $\Sigma^* = \{(1,1)\} \cup \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$. In *Figure 1* the bold point indicates the unique strict *NE* strategy profile and the thick line segment indicates the non-strict ones. Thus, by the previous theorem, $(\Delta(S), \succ)$ has infinite *vN&M* stable sets. In this game none of the two players has a strictly dominant strategy (see **Case 1 c** in the proof of *Theorem 1*). To give a *vN&M* stable set A, it suffices to consider the set of strategy profiles A_1 such that $(p,q) = \lambda(1,1) + (1-\lambda)(0,\frac{1}{2})$ $(\lambda \in [0,1])$ *i.e.,* $A_1 = \{(p,q) : q = \frac{1}{2} + \frac{1}{2}\ p,\ 0 \leq p \leq 1\}$, and the set of strategy profiles A_2 such that $(p,q) = \lambda(0,\frac{1}{2}) + (1-\lambda)(0,0)$, $\lambda \in [0,1]$ *i.e.,* $A_2 = \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$. Then $A = A_1 \cup A_2$ (see *Figure 1*). To obtain a second *vN&M* stable set $\widehat{A}$, we consider $\widehat{\sigma} \notin A$; $1 > \widehat{p} \geq 0$ and $1 > \widehat{q} \geq \frac{1}{2}$. For example, taking $\widehat{\sigma} = (\frac{1}{2}, \frac{1}{2})$ we have $\widehat{A} = \{(p,q) : q = p,\ \frac{1}{2} \leq p \leq 1\} \cup \{(p,\frac{1}{2}) : 0 \leq p \leq \frac{1}{2}\} \cup \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$.

Example 2 Consider the mixed extension of the following game:

	s_2^1	s_2^2
s_1^1	1,0	0,1
s_1^2	0,0	1,0

In this example player 1's best response coincides with the one in *Example 1* while player 2´s best response is:

$$BR_2(p) = \begin{cases} [0,1] & if \quad p = 0 \\ \{0\} & if \quad p > 0 \end{cases}$$

It is easy to check that the set of *NE* strategy profiles is $\Sigma^* = \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$ which is represented by the thick line segment in *Figure 2*. Since the game does not have a strict *NE* strategy profile, $(\Delta(S), \succ)$ has a unique *vN&M* stable set. None of the two players has a strictly dominant strategy (**Case 2 b** *(ii)* in the proof of *Theorem 1*) and in this case $\overline{p} = 0$ and $\overline{q} = \frac{1}{2}$. To give the only *vN&M* stable set A, we consider the sets $I_1 = \{(p,\frac{1}{2}) : 0 \leq p \leq 1\}$ and $I_2 = \{(0,q) : 0 \leq q \leq 1\}$. Thus, $A = \{(p,\frac{1}{2}) : 0 \leq p \leq 1\} \cup \{(0,q) : 0 \leq q \leq 1\}$ and it is the unique *vN&M* stable set.

4 Relationship between 2×2 games classifications

Consider a 2×2 game. It is well known (see, for instance, Calvo-Armengol [1] and Eichberger *et al.* [2]) that transforming the players' payoff functions as follows:

$$u_1'\left(s_1,s_2^1\right) = u_1\left(s_1,s_2^1\right)-u_1\left(s_1^2,s_2^1\right),\ u_1'\left(s_1,s_2^2\right)=u_1\left(s_1,s_2^2\right)-u_1(s_1^1,s_2^2) \text{ for } s_1\in S_1 \text{ and}$$
$$u_2'\left(s_1^1,s_2\right) = u_2\left(s_1^1,s_2\right)-u_2\left(s_1^1,s_2^2\right),\ u_2'\left(s_1^2,s_2\right)=u_2\left(s_1^2,s_2\right)-u_2\left(s_1^2,s_2^1\right) \text{ for } s_2\in S_2$$

preserves the best response structure of the game. The transformed 2×2 game becomes:

	s_2^1	s_2^2
s_1^1	a_1,b_1	$0,0$
s_1^2	$0,0$	a_2,b_2

where $a_1=u_1'\left(s_1^1,s_1^1\right), a_2=u_1'\left(s_1^2,s_2^2\right), b_1=u_2'\left(s_1^1,s_2^1\right)$ and $b_2=u_2'\left(s_1^2,s_2^2\right)$. This transformation permits the following classification of 2×2 games in terms of their number and nature of *NE* strategy profiles by just examining payoff parameters a_1,a_2,b_1 and b_2 which for generic games are all different from 0 and for non-generic games at least one is zero[5]:

Generic games	Conditions	
Dominant solvable	$a_1a_2<0$ or $b_1b_2<0$	One pure NE
Coordination	$a_1,a_2,b_1,b_2>0$ or $a_1,a_2,b_1,b_2<0$	Two pure and one mixed NE
Strictly competitive	$a_1,a_2>0$ and $b_1,b_2<0$ or $a_1,a_2<0$ and $b_1,b_2>0$	One mixed NE

Non-generic games	Conditions	
	$a_i=0$ or $b_i=0$ for some i	Two or infinite NE

Let us relate these classes of games with our result:

1. For generic games only strictly competitive games have a unique *vN&M* stable set and it does not coincide with the set of *NE* strategy profiles. Dominant solvable and coordination games however, have infinite *vN&M* stable sets.

2. If a non-generic game has a strict *NE* strategy profile then $(\Delta(S),\succ)$ has infinite *vN&M* stable sets. Otherwise $(\Delta(S),\succ)$ has a unique *vN&M* stable set. Leaving aside trivial games, there are games in which this set coincides with the set of *NE* strategy profiles, see **Sub-case 2 a** and **Sub-case 2 b** (i) in the proof of *Theorem 1.*

[5] See for instance von Stengel [?]. Roughly speaking, a game is *generic* if it has some neighborhood whose elements have the same number of *NE* as the original game.

Summing up, we have provided a simple classification of the 2×2 games. The nonexistence of a strict *NE* strategy profile in game G guarantees that $(\Delta(S), \succ)$ has a unique *vN&M* stable set. Otherwise $(\Delta(S), \succ)$ have infinite *vN&M* stable sets.

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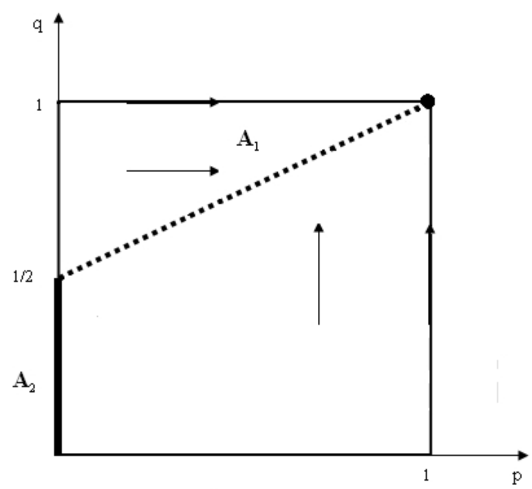

Figure 1

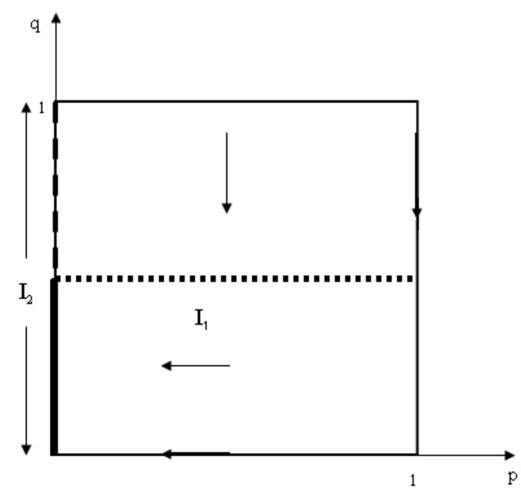

Figure 2