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ORIGINAL ARTICLE

Preferences, actions and voting rules

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Abstract In this paper we address several issues related to collective *dichotomous* decision-making by means of *quaternary* voting rules, i.e., when voters may choose between four actions: voting yes, voting no, abstaining and not turning up-which are aggregated by a voting rule into a dichotomous decision: acceptance or rejection of a proposal. In particular we study the links between the actions and preferences of the actors. We show that quaternary rules (unlike binary rules, where only two actions -yes or no- are possible) leave room for "manipulability" (i.e., strategic behaviour). Thus a preference profile does not in general determine an action profile. We also deal with the notions of success and decisiveness and their ex ante assessment for quaternary voting rules, and discuss the role of information and coordination in this context.

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1 Introduction

In classical social choice models the inputs of either social welfare functions (Arrow 1963) or social decision functions (Gibbard 1973; Satterthwaite 1975) are social preference profiles. As is well known, the seminal results established by these authors are negative "impossibility" theorems. In particular, the Gibbard–Sattherwaite theorem shows that *when more than two* alternatives are at stake it may be ingenuous to expect individuals to provide their actual preferences whatever the decision mechanism in use may be. Perhaps for this reason most committees or decision-making bodies use dichotomous voting rules, even in cases where they must often face decisions more complex than just dichotomous choices. Moreover, in actual collective decision-making individuals are most often not asked directly to report their preferences. When collective decisions are made by means of a voting rule, individuals are offered a certain set of actions (e.g. voting yes or no, or perhaps also abstaining, etc.) and each individual may choose her/his preferred action given her/his preferences and whatever information s/he may have about other individuals.

In this paper, based on the model of quaternary voting rules provided in Laruelle and Valenciano (2010), we address several issues related to collective *dichotomous* decision-making by means of *quaternary* voting rules. That is, when voters may choose between four actions—voting yes, voting no, abstaining and not turning up—which are aggregated by a voting rule into a dichotomous decision: acceptance or rejection of a proposal. First, we further disentangle the concepts of decisiveness and success, often conflated in voting power literature. The difference between these two notions becomes apparent in the framework of quaternary voting rules. Decisiveness depends only on the profile of actions, while success has to do with the voters' preferences. We relate these notions to that of manipulability in this context. It turns out that the possibility of manipulation arises in spite of the dichotomous character of decisions, as a result of the voters having more than two choices. We show the difficulty of extending the notions of success and decisiveness ex ante. Nevertheless, a natural extension is possible when preferences are strict and common knowledge and coordination between voters is feasible.

Among the most closely related literature the following may be mentioned: Felsenthal and Machover (1997), Braham and Steffen (2002) and Pongou et al. (2010) consider ternary rules where the three actions are voting yes, voting no and abstaining. They define voters' power in this context. Uleri (2002), Côrte-Real and Pereira (2004), Herrera and Mattozzi (2010), Maniquet and Morelli (2010) and Pauly (2010) deal with ternary rules where the third action is not participating. They study the strategic aspect induced by quorums. Côrte-Real and Pereira (2004) show that an agent may be better off abstaining than voting for her/his preferred choice under a simple majority with a participation quorum. The simple majority and the simple majority with approval quorum do not cause this paradox, which they refer to as the “no-show paradox”. Maniquet and Morelli (2010) confirm this finding. Dougherty and Edward (2010) compare the simple majority and the absolute majority in quaternary rules, in one case assuming that voters are sincere. Freixas and Zwicker (2003) study quaternary rules and more generally rules with many actions and possible outcomes. Their model is however more specific as they only consider ordered actions. In particular their rules never display any sort of no show paradox because of the linear order of the actions. Tchantcho et al. (2010) study influence in this context.

The rest of the paper is organized as follows. Section 2 briefly reviews some basic notions for binary rules as a term of reference. Section 3 summarizes the formalization of quaternary voting rules. In the context of quaternary voting rules *ex post* success and decisiveness are formulated in Sect. 4, while strategic issues are dealt with in Sect. 5. Section 6 deals with the connection between preferences and outcomes, and Sect. 7 addresses the ex ante extension of success and decisiveness.

2 Preferences and actions in binary rules

In this section we briefly review the simplest voting situations, where voters face a *dichotomous* decision (acceptance or rejection of a proposal) according to the specifications of a *binary* voting rule where *two* actions are allowed for each voter: voting yes and voting no (see, e.g., Laruelle and Valenciano 2008).

If the number of voters is n, let $\mathcal{N} = \{1, \ldots, n\}$ be the set of labels used indifferently to label the *seats* or the *voters* occupying them. A *vote configuration* lists the votes cast by the voters occupying each seat. So, we denote by the vote configuration $S = (S^Y, S^N)$ the result of a vote where S^Y is the set of *yes-voters* (i.e. the voters who vote yes) and S^N is the set of *no-voters* (i.e. the voters who vote no). For binary rules we have $S^N = \mathcal{N} \backslash S^Y$, thus there are 2^n possible vote configurations and each configuration can be represented by the set of yes-voters S^Y. A *binary voting rule* $\mathcal{V}$ specifies which vote configurations lead to the acceptance of a proposal. A vote configuration is *yes-winning* if $S^Y \in \mathcal{V}$, and *no-winning* if $S^Y \notin \mathcal{V}$. In order to discard inconsistent rules $\mathcal{V}$ is usually assumed to satisfy the following conditions: (i) $\mathcal{N} \in \mathcal{V}$ (a unanimous yes implies acceptance); (ii) $\emptyset \notin \mathcal{V}$ (a unanimous no entails rejection); (iii) if $S^Y \in \mathcal{V}$, then $T^Y \in \mathcal{V}$ whenever $S^Y \subseteq T^Y$ (monotonicity: if a vote configuration is yes-winning, then any other configuration with a wider set of yes-voters is also yes-winning); (iv) if $S^Y \in \mathcal{V}$ then $\mathcal{N} \backslash S^Y \notin \mathcal{V}$ (the possibility of a proposal and its negation both being accepted should be prevented).

Ex post success or *satisfaction* is usually defined as the correspondence between the final result and the voter’s vote. After a decision is made according to a binary voting rule $\mathcal{V}$, if the resulting vote profile is S^Y voter i is said to have been *successful* iff

$$(i \in S^Y \in \mathcal{V}) \text{ or } (i \notin S^Y \notin \mathcal{V}). \quad (1)$$

A voter is said to have been *decisive* if her/his vote was crucial for the final outcome; that is, had s/he changed her/his vote the outcome would have been different.

Definition 1 In a binary voting rule $\mathcal{V}$, voter i is decisive in a vote configuration S^Y if[1]

$$[(i \in S^Y \in \mathcal{V}) \text{ and } (S^Y \setminus i \notin \mathcal{V})] \text{ or } [(i \notin S^Y \notin \mathcal{V}) \text{ and } (S^Y \cup i \in \mathcal{V})]. \tag{2}$$

Note that a yes-voter can only be decisive for the acceptance of the proposal (which we will refer to as *yes-decisiveness*), while a no-voter can only be decisive for the rejection of the proposal (which we will refer to as *no-decisiveness*). This is a consequence of the monotonicity condition. Therefore being successful is a necessary condition for being decisive. A voter is decisive in a vote if her/his vote is crucial for her/his success.

These notions can be extended *ex ante* (Laruelle and Valenciano 2005) if it is assumed that for any action profile S^Y that may arise the probability $p(S^Y)$ of voters voting in such a way that S^Y emerges is known. That is, $p(S^Y)$ gives the probability that voters in S^Y will vote yes, and those in $\mathcal{N} \setminus S^Y$ will vote no. The *ex ante* version of success is defined as the the probability of i being successful:

$$\sum_{S^Y : i \in S^Y \in \mathcal{V}} p(S^Y) + \sum_{S^Y : i \notin S^Y \notin \mathcal{V}} p(S^Y), \tag{3}$$

while the *ex ante* version of decisiveness is the probability of i being decisive:

$$\sum_{\substack{S^Y : i \in S^Y \in \mathcal{V} \\ S^Y \setminus i \notin \mathcal{V}}} p(S^Y) + \sum_{\substack{S^Y : i \notin S^Y \notin \mathcal{V} \\ S^Y \cup i \in \mathcal{V}}} p(S^Y). \tag{4}$$

In the preceding formulations, the distinction between preferences and actions is not explicit. Two implicit assumptions underlie the statements and definitions there: *Preferences are strict*, i.e., no voter is indifferent between acceptance and rejection of a proposal, and *voters vote according to their preferences*, i.e. yes (no) iff they prefer the proposal to be accepted (rejected). We now examine this more carefully and revise the notions of success and decisiveness, clarifying the distinction between actions and preferences. A vote configuration $S = (S^Y, S^N)$ is an *action profile* as it lists the action (yes or no) chosen by each voter.

In general, a voter may be in any of three different positions for a given proposal: A voter is a *supporter* (*rejecter*) if s/he is in favor of the acceptance (rejection) of the proposal, otherwise we say that the voter is *indifferent*. Let $P^+ \subseteq \mathcal{N}$ denote the set of supporters, P^0 the set of indifferent voters, and P^- the set of rejecters. The voters' preferences can be represented by a 3-partition $P = (P^+, P^0, P^-)$ of $\mathcal{N}$ that we refer to as a *preference profile*. A preference profile is *strict* if no one is indifferent, that is, $P^0 = \varnothing$.

A supporter (rejecter) is said to vote sincerely if s/he votes yes (no). As to indifferent voters, the weakest assumption about them is the indeterminacy of their behavior. So a sincere *action profile* is defined as follows:

[1] We write $S^Y \setminus i$ and $S^Y \cup i$ instead of $S^Y \setminus \{i\}$ or $S^Y \cup \{i\}$ consistently to simplify notation.

Definition 2 An action profile S is sincere w.r.t. a preference profile $P = (P^+, P^0, P^-) \in 3^{\mathcal{N}}$, if $P^+ \subseteq S^Y$ and $P^- \subseteq S^N$.

While in general there is more than one sincere action profile, for strict preferences ($P^0 = \varnothing$) there is only one: $S^Y = P^+$ and $S^N = P^-$.

Making explicit the difference between preferences and actions permits us to redefine more properly the notion of success. If being *successful ex post* means obtaining the preferred outcome, it should be stated as follows.

Definition 3 In a binary rule $\mathcal{V}$, with preference profile P, voter i is *successful* in the vote configuration S^Y if the decision coincides with i's preference, that is, iff

$$(i \in P^+ \text{ and } S^Y \in \mathcal{V}) \text{ or } (i \in P^- \text{ and } S^Y \notin \mathcal{V}). \tag{5}$$

This definition is equivalent to (1) when $S^Y = P^+$ and $S^N = P^-$. That is, when the preferences are strict and voters vote sincerely. Although in general sincere and rational behavior may not coincide, the monotonicity condition guarantees that sincere voting is always rational in the context of dichotomous binary rules. This is a corollary of the following

Proposition 1 *Given a binary voting rule $\mathcal{V}$, for any preference profile P, a voter who is not successful in the vote configuration P^+ will not be successful by changing her/his vote.*

Proof Consider first a supporter i. If $i \in P^+$ is not successful in P^+ it means that $P^+ \notin \mathcal{V}$. Then by monotonicity we have $P^+ \setminus i \notin \mathcal{V}$ and supporter i is not successful in $P^+ \setminus i$ either. Similarly if $i \in P^-$ is not successful in P^+ it means that $P^+ \in \mathcal{V}$. Then by monotonicity we have $P^+ \cup i \in \mathcal{V}$ and rejecter i is not successful in $P^+ \cup i$ either. □

In other words, as a rational voter's objective is to be successful, sincere voting is a weakly dominant strategy if the voter is decisive for at least one vote configuration. In any case, a voter never has an incentive not to vote sincerely. As will be seen below, for some quaternary voting rules a voter may have an incentive to vote strategically rather than sincerely. When a voter is not successful in a sincere vote configuration, but may be successful if s/he unilaterally changes her/his vote, we say that the sincere vote configuration is *manipulable by that voter*. This concept can be extended to groups of voters. If a group of voters with the same preference could be successful by jointly changing their sincere vote, we would say that the corresponding sincere vote is *manipulable by the group of voters*. This is also impossible in the context of binary voting rules because of monotonicity. We then have the following definition and result:

Definition 4 A voting rule is not manipulable if no sincere vote configuration is manipulable by a group of voters whatever the preference profile P.

Proposition 2 *No binary voting rule is manipulable.*

Thus, assuming strict preferences, vote configurations follow immediately from preference profiles and, as (1) is then equivalent to (5), we can define the notion of success on the basis of action profiles. But in general the notion of success also depends on preference profiles as the definition of success shows. By contrast, decisiveness is the capability to reverse the final outcome. The definition depends on the action profile as can be seen from (2).

3 Quaternary voting rules

Now we consider the case where dichotomous decisions (acceptance or rejection of proposals) are made by means of a *quaternary* voting rule, where *four* possible actions are allowed for each voter: voting yes (Y), abstaining (A), staying home (H) and voting no (N). The actions chosen by all voters can be represented by a 4-partition $S = (S^Y, S^A, S^H, S^N)$ of $\mathcal{N}$, where S^Y is the set of yes-voters, S^A the set of abstainers, S^H the set of those who stay at home and S^N the set of no-voters. We refer to S as an *action profile* and denote by $4^{\mathcal{N}}$ the set of all such action profiles. A quaternary voting rule $\mathcal{W}$ specifies which action profiles lead to the acceptance of a proposal:

$$\mathcal{W} = \left\{(S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}} : S \text{ leads to the acceptance of the proposal}\right\}.$$

We say that an action profile S is *yes-winning* if $S \in \mathcal{W}$, and *no-winning* if $S \notin \mathcal{W}$. A set $Q \subseteq \mathcal{N}$ is *yes-enforcing* if for all $S \in 4^{\mathcal{N}}$ such that $S^Y \supseteq Q$ we have $S \in \mathcal{W}$. A set $Q \subseteq \mathcal{N}$ is *no-enforcing* if for all $S \in 4^{\mathcal{N}}$ such that $S^N \supseteq Q$ we have $S \notin \mathcal{W}$.

A quaternary voting rule is *anonymous* if only the number of voters who have chosen each of the different options matters, not their identities. Anonymous rules can be specified in terms of the number of voters who choose each option. In other words whether an action profile is yes-winning or no-winning only depends on vector (s^Y, s^A, s^H, s^N), where s^Y is the number of yes-voters, s^A is the number of abstaining voters, etc.

Given any two options $B, C \in \{Y, A, H, N\}$, a quaternary voting rule is *BC-monotonic* if the following condition is satisfied: if the set of C -voters is exclusively extended at the expense of the set of D-voters a yes-winning action profile does not become no-winning. Formally, if $S \in \mathcal{W}$, then $T \in \mathcal{W}$ for any T such that $S^C \subseteq T^C$ and $S^D = T^D$ for $D \in \{Y, A, H, N\} \setminus \{B, C\}$.

Following Laruelle and Valenciano (2010), we propose the following conditions for $\mathcal{W}$ to specify a sound voting rule: *Full-support*: if all voters vote yes the proposal is accepted: $S^Y = \mathcal{N} \Rightarrow S \in \mathcal{W}$; *Null-support*[2]: if no voter votes yes the proposal is rejected: $S^Y = \varnothing \Rightarrow S \notin \mathcal{W}$. In addition to these, we assume *AY-monotonicity*, *NY -monotonicity*, *HY-monotonicity*, and *NA-monotonicity*. But note

[2] In Laruelle and Valenciano (2010) this condition is replaced by a weaker one ("weak null-support") in order to cover rules that isolated would not be reasonable, but when intersected with others yield reasonable ones that *always satisfy null-support condition*. As here we are concerned with actual decision rules, we assume this stronger condition without loss of generality.

that NY-monotonicity is implied by AY-monotonicity together with NA-monotonicity. A voting rule is thus defined on the basis on the three minimal monotonicities:

Definition 5 A "quaternary dichotomous voting rule" is a set $\mathcal{W}$ of 4-partitions of $\mathcal{N}$ that satisfies full-support, null-support, NA-monotonicity, AY-monotonicity and HY -monotonicity.

The monotonicity conditions can be summarized by the following diagram where an arrow indicates the transition of action that does not change the yes-winning character of a yes-winning action profile:

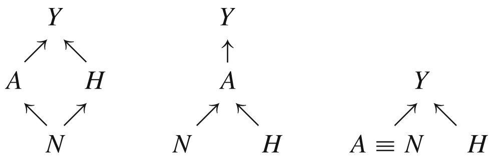
$$\begin{array}{ccc} & Y & \\ \nearrow & & \nwarrow \\ A & & H \\ \uparrow & & \\ N & & \end{array} \qquad (6)$$

Other reasonable monotonicities can also be found in real-world examples (Laruelle and Valenciano 2011). In some cases the options of staying at home and voting no can be compared in terms of being more or less preferable for the acceptance or rejection of the proposal. In this case, in all examples we have found, staying at home is at least as favorable to acceptance as voting no, in other words we have NH-monotonicity. The possible comparison of abstaining and staying at home may result in one of the options being better than the other. So we can have either AH-monotonicity or HA-monotonicity, but it may also be the case that these two options cannot be distinguished in terms of their effects on the final result. In this case we say that the two options are equivalent. Considering the different combinations of monotonicities and equivalences, we obtain six sub-classes of *monotonic* quaternary voting rules that are particular classes of Freixas and Zwicker (2003) (j, k)-voting rules[3]:

$$\begin{array}{cccccc} Y & Y & & & & \\ \uparrow & \uparrow & & & & \\ A & H & Y & Y & Y & \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ H & A & A \equiv H & A & H & Y \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ N & N & N & H \equiv N & A \equiv N & A \equiv H \equiv N \\ \text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} & \text{(e)} & \text{(f)} \end{array} \qquad (7)$$

[3] Namely, (a) and (b) are (4, 2)-voting rules, (c), (d) and (e) are essentially (3, 2)-voting rules, and those in (f) are essentially (2, 2)-voting rules. Subclasses (c) and (d) correspond to the ternary rules considered by Felsenthal and Machover (1997).

The three following subclasses contain rules where some options cannot be compared and consequently are not covered by the (j, k)-model[4].



(g) (h) (i) (8)

Sub-class (i) is the class of rules studied by Côrte-Real and Pereira (2004) and Maniquet and Morelli (2010).

4 Decisiveness and success

We now examine the notions of decisiveness and success for quaternary voting rules. We distinguish between *yes-decisiveness*, when a voter can turn acceptance into rejection, and *no-decisiveness,* when a voter can turn rejection into acceptance. In a binary rule a yes-voter (and only a yes-voter) can only be yes-decisive, while a no-voter (and only a no-voter) can only be no-decisive. This is no longer the case in the context of quaternary voting rules, as can be illustrated in the following rule:

Example 1 Let $\mathcal{W}$ be a majority of present voters with a quorum of 75%, with $n = 11$:

$$\mathcal{W} = \left\{(s^Y, s^A, s^H, s^N) : s^Y > s^A + s^N \text{ and } s^H < 4\right\}.$$

In the action profile $(5, 2, 3, 1) \in \mathcal{W}$ any yes-voter, any abstainer and the no-voter are yes-decisive (by staying home they would reverse the decision). In the action profile $(4, 2, 3, 2) \notin \mathcal{W}$ any abstainer, any no-voter and any voter who stays at home is no-decisive (by voting they would reverse the decision). Note also that now whatever the action chosen by an individual s/he may change her/his vote in three different ways. The way may matter: in the action profile $(5, 2, 3, 1)$ the no-voter turns acceptance into rejection only if the change is to staying at home.

In the sequel we use the following notation: If $Q \subseteq \mathcal{N}$ we denote

$$S_{-Q} := (S^Y \setminus Q, S^A \setminus Q, S^H \setminus Q, S^N \setminus Q),$$

and when Q consists of a single individual, i.e. $Q = \{i\}$, we write S_{-i}. Then a voter i is decisive in an action profile S if there exists another action profile T such that $T_{-i} = S_{-i}$ where S and T lead to different final outcomes.

Definition 6 For a quaternary voting rule $\mathcal{W}$ a voter i is yes-decisive in the action profile $S \in \mathcal{W}$ if there exists $T \notin \mathcal{W}$ such that $T_{-i} = S_{-i}$; and a voter i is no-decisive in the action profile $S \notin \mathcal{W}$ if there exists $T \in \mathcal{W}$ such that $T_{-i} = S_{-i}$.

[4] Classess (a)–(i) are some of these classes preordered by inclusion. Thus, it should be taken into account that, for instance, (c) is contained in (a) and (b), and (f) in all the others.

In the example above there is no action profile where a yes-voter is no-decisive, while it may be the case that a no-voter is yes-decisive. As the following proposition shows, the first fact is a general result, while the second is ruled out if the rule is NH-monotonic.

Proposition 3 *For any quaternary voting rule $\mathcal{W}$, (i) a yes-voter is never no-decisive; (ii) if $\mathcal{W}$ is NH-monotonic then a no-voter is never yes-decisive.*

Proof Let S be an action profile. (i) If $i \in S^Y$ and $S \notin \mathcal{W}$ then all T such that $T_{-i} = S_{-i}$ and $T^A = S^A \cup i$ will also be no-winning by the AY-monotonicity displayed by any $\mathcal{W}$. Similarly, by HY-monotonicity and NY-monotonicity all T such that $T_{-i} = S_{-i}$ are no-winning. (ii) A similar argument applies to all no-voters if $\mathcal{W}$ is NH-monotonic. □

Let us now focus on success. As for binary rules, being *successful* means obtaining one's preferred outcome. As with binary rules, success is defined only for supporters or rejecters.

Definition 7 In a voting rule $\mathcal{W}$, with preference profile P, voter i is *successful* in the action profile S if the decision coincides with i's preference, that is, iff

$$(i \in P^+ \text{ and } S \in \mathcal{W}) \text{ or } (i \in P^- \text{ and } S \notin \mathcal{W}).$$

5 Strategic voting or manipulation

When decisions are made by means of binary rules actions immediately follow strict preferences and no sincere action profile is manipulable by a group of supporters or of rejecters. Things become more complicated with quaternary voting rules. Recall (see Definition 2) that an action profile S is *sincere* w.r.t. a preference profile P if $P^+ \subseteq S^Y$ and $P^- \subseteq S^N$. The monotonicity conditions guarantee that sincere voting is a rational behavior for supporters. More precisely, we have the following result.

Proposition 4 *For any quaternary voting rule $\mathcal{W}$ and any preference profile P, a supporter i who is not successful in the sincere action profile S is not successful either in any T where $T_{-i} = S_{-i}$.*

Proof For a supporter i who votes sincerely, we have $i \in P^+$ and $i \in S^Y$. Voter i not being successful means that $S \notin \mathcal{W}$. By Proposition 3 a yes-voter is never no-decisive. In other words, for whatever T where $T_{-i} = S_{-i}$ we have $T \notin \mathcal{W}$. □

Thus, for a rational supporter whose objective is to be successful, sincere voting is a weakly dominant strategy as far as s/he is decisive for at least one vote configuration. This can be extended to groups of supporters. We have the following definition and result:

Definition 8 Given a voting rule $\mathcal{W}$ and a preference profile P, a sincere no-winning (yes-winning) action profile $S \notin \mathcal{W}$ ($S \in \mathcal{W}$) is manipulable by a group of supporters (rejecters) Q if there exists $T \in \mathcal{W}$ ($T \notin \mathcal{W}$) such that $T_{-Q} = S_{-Q}$.

Proposition 5 *Whatever the quaternary voting rule $\mathcal{W}$, whatever the preference profile P, no sincere action profile is manipulable by a group of supporters.*

Proof Let S be a vote configuration and Q a group of supporters who vote sincerely, i.e. $Q \subset P^+$ and $Q \subset S^Y$. Either supporters are successful and have no incentive to manipulate or they are not successful and $S \notin \mathcal{W}$. But then all T such that $T_{-Q} = S_{-Q}$ will also be no-winning by the monotonicities displayed by any $\mathcal{W}$. Thus, supporters are not successful in T either. □

By contrast, sincere voting is no longer a weakly dominant strategy for a rejecter whose objective is to be successful: a rejecter may be unsuccessful when s/he votes no, and successful if s/he stays at home. In Example 1, consider a preference profile with 5 supporters, 1 rejecter, and 5 indifferent voters. We have that $(5, 2, 3, 1) \in \mathcal{W}$ is a sincere profile. In this action profile the rejecter and no-voter is not successful but is yes-decisive, as $(5, 2, 4, 0) \notin \mathcal{W}$. This sincere action profile is manipulable by the rejecter.

A voting rule is not manipulable if no sincere action profile is manipulable by a group of supporters nor by a group of rejecters whatever the preference profile P. Given that no sincere action profile can be manipulated by a group of supporters or by a group of rejecters if the rule is NH-monotonic, we have the following:

Proposition 6 *If a quaternary voting rule $\mathcal{W}$ is NH-monotonic, then $\mathcal{W}$ is not manipulable.*

When the rule is NH-monotonic voting no is a weakly dominant strategy for rejecters as long as they are decisive for at least one vote configuration. This may however not be the only weakly dominant strategy. For rejecters in subclasses (a)–(c) and (g) voting no is always a weakly dominant strategy, in subclasses (d) voting no is equivalent to stay at home and weakly dominates abstaining and voting yes, in subclass (e) voting no is equivalent to abstaining and weakly dominates staying at home and voting yes, in subclass (f) voting no is equivalent to stay at home and abstaining and weakly dominates voting yes.

For rules which are not NH-monotonic[5] only supporters are sure to vote sincerely, while for some sincere action profiles rejecters may not be successful but are decisive. This happens when the action profile is yes-winning and a rejecter is yes-decisive because by choosing to stay at home instead of voting no s/he may change the result. This possibility is ruled out if the set of supporters can guarantee the acceptance of the proposal by itself. Thus we have an obvious conclusion:

Proposition 7 *If the preference profile P is such that P^+ is yes-enforcing in the quaternary voting rule $\mathcal{W}$, then no sincere action profile is manipulable by a group of rejecters.*

[5] That is to say, rules in the general class or in classes (h)–(i) which are not contained in any smaller class (see footnote 3).

6 Preferences and final outcome

By contrast with binary rules, when decisions are made by means of a quaternary rule it may be the case that the action profile and consequently the outcome are not fully determined, even when the preference profile is strict. As will be seen presently, this may critically depend on information and/or the possibility of coordination.

When decisions are made by means of a quaternary rule sincere voting is always a weakly dominant strategy for supporters. Thus if P^+ is yes-enforcing, the final outcome is sure to be the acceptance of the proposal. For rejecters, sincere voting is a weakly dominant strategy only if the rule is NH-monotonic. Thus if P^- is no-enforcing and the rule is NH-monotonic the final outcome is sure to be the rejection of the proposal. But if the rule is not NH-monotonic the final outcome is not necessarily the rejection of the proposal, even if P^- is no-enforcing. Indeed, if the preference profile is not common knowledge and/or coordination is not possible rejecters may not have a dominant strategy. Consider again Example 1, where $n = 11$ and

$$\mathcal{W} = \left\{ S \in 4^{\mathcal{N}} : s^Y > s^A + s^N \text{ and } s^H < 4 \right\}.$$

As the rule is anonymous, we will only need the numbers of supporters (p^+), indifferent voters (p^0) and rejecters (p^-), and we summarize the preference profile by the vector (p^+, p^0, p^-). With only this information about the rule, rejecters cannot choose a weakly dominant strategy, even if the preference profile is such that P^- is no-enforcing. Now let us examine the case where the preference profile is common knowledge. This knowledge does not influence the behavior of supporters: they will still vote yes, but for rejecters knowledge of the preference profile may or may not be enough for them to choose an optimal action taking into account the action of the supporters. In some cases an equilibrium can be obtained by elimination of dominated strategies. Assume that the preference profile $(p^+, p^0, p^-) = (7, 0, 4)$ is commonly known. Knowing that the 7 supporters vote 'yes' (and thus eliminating the other options for these supporters) the weakly dominant strategy for rejecters is to stay at home. Indeed the only no-winning vote profile where $s^Y = 7$ is when $s^H = 4$. If the preference profile is $(p^+, p^0, p^-) = (6, 0, 5)$, the same argument applies: knowing that $s^Y = 6$ (and thus eliminating the other options for these supporters) the weak dominant strategy for the rejecters is staying at home. Here there is more than one no-winning profile with $s^Y = 6$ but the option of staying at home is always as good as voting 'no' and in one case (when $s^H = 3$ and $s^N = 2$) it is strictly better to stay home than to vote no. Interestingly enough, if $(p^+, p^0, p^-) = (5, 0, 6)$ the option of staying at home no longer dominates any more voting no, and nor does voting no dominate staying at home. Here rejecters will have a problem of coordination, even if the subset of rejecters is no-enforcing.

Finally if coordination is possible between voters with identical preferences, the final outcome will be the acceptance of the proposal if P^+ is yes-enforcing and the rejection of the proposal if P^- is no-enforcing, *or* if the group of rejecters can coordinate in such a way that they obtain the rejection of the proposal. Otherwise it will depend on how the indifferent voters vote.

It can be concluded that information about the preferences and the possibility of coordination favor rejecters, while the less the information and the greater the difficulty of coordinating the better for supporters.

7 Success and decisiveness ex ante

The extension of the notions of decisiveness and success *ex ante* to the context of quaternary voting rules comes up against several difficulties. In the context of binary rules, assuming strict preferences and rational voters, the vote profile is determined and identical to the preference profile. Thus, each probability distribution on preference profiles determines a probability distribution on voting profiles. In this way the probability of each individual being successful or decisive can be calculated for any binary rule. Now, when four options are available to voters even a strict preference profile does not always determine a rational action profile: supporters will vote yes, but rational rejecters, unless the rule is NH-monotonic, may not have a weakly dominant strategy. However, this difficulty disappears when preferences are common knowledge and coordination is possible. Under these conditions, if the preference profile is P (with $P^0 = \varnothing$) then the action profile would be an S s.t. $S^+ = P^+$, and if S^+ is yes-enforcing the result will be acceptance, otherwise rejecters would coordinate and have the proposal rejected. Thus, in practical terms, under these conditions it is as if the actual rule were the binary rule given by

$$\mathcal{V}(\mathcal{W}) = \{S \subseteq \mathcal{N} : S \text{ is yes-enforcing}\}.$$

This is in fact the *core binary rule* associated with $\mathcal{W}$ (see Laruelle and Valenciano 2010). Thus, assuming strict preferences, a consistent way of calculating the *ex ante* success of a voter based on an estimate p (i.e. a probability distribution over strict *preference* profiles) would be by means of (3), where $p(S^Y)$ is to be replaced by $p(P^+ = S^Y)$. Similarly, *ex ante* decisiveness can be calculated by means of (4).

It is worth noting that a direct approach to decisiveness based on actual 4 -action profiles encounters conceptual difficulties. Consider Example 1 once more. Assume the preference profile $(p^+, p^0, p^-) = (5, 0, 6)$. If coordination is possible rejecters may coordinate their actions so that any of the following action profiles result (among others, assuming supporters play their dominant strategy): $(s^Y, s^A, s^H, s^N) = (5, 0, 0, 6)$ or $(5, 0, 1, 5)$ or $(5, 1, 0, 5)$ or $(5, 2, 0, 4)$ or $(5, 1, 1, 4)$ or $(5, 3, 0, 3)$ or $(5, 2, 1, 3)$ or $(5, 4, 0, 2)$ or $(5, 3, 1, 2)$ or $(5, 0, 4, 2)$ or $(5, 1, 4, 1)$ or $(5, 2, 4, 0)$ or even $(7, 0, 4, 0)$. Rational rejecters looking for success would be indifferent between these 13 possible ways of coordinating their actions as in all cases the proposal would be rejected. As the reader may check, these profiles are completely different from the point of view of which rejecters are decisive and which are not in each of them. But why should a rational rejecter interested in a successful rejection care about these differences? These irrelevant differences are disregarded by the proposed approach based on the underlying core rule that ignores them. Moreover, *there is no convincing way to derive a probability distribution over* 4-*action profiles based on a probability over preference profiles.*

The suggested approach also ignores the possibility of indifferent voters, but this is also the case in the traditional approach for binary rules (where often abstention is counted as a no). What can reasonably be assumed about the behavior of indifferent voters? Felsenthal and Machover (1997), in the context of ternary voting rules, assuming that indifferent voters abstain, assume that the probability of a voter choosing any of the three actions (yes, no, abstention) is 1/3. Is this reasonable? Then why not assume the same for binary rules which might actually result in voting yes with probability 1/3 and no with probability 2/3? Their rather ad hoc assumption pointed to the extension of Banzhaf–Penrose index, an a priori evaluation of decisiveness interpreted as a measure of "voting power". In our opinion (see Laruelle and Valenciano 2008), unless one form or another of bargaining precedes decisions, the very notion of voting power merely based on the likelihood of being decisive is conceptually inconsistent.

Finally, there remains the case where preferences are not common knowledge and/or coordination is not possible. In this case ex ante evaluation of success by means of the core rule would not be realistic in general. It would provide a low or pessimistic evaluation for supporters, because the calculation based on the core rule assumes that rejecters can always do their best for their interest. At least it can be seen as a reasonable lower bound calculated on well specified conditions.

8 Conclusion

Dichotomous decision-making by means of quaternary voting rules provides a context where the distinction between preferences and actions is obvious and necessary. This distinction permits the difference between success and decisiveness to be stressed again (if necessary). Success relates preferences and the final outcome, while decisiveness relates actions and the final outcome. This distinction also questions some extensions in the literature on power indices. If a distribution of probability is assumed, it should be on the preferences and not on actions. Indeed the choice of actions may be strategic and depends on the rule.

This paper suggests the following issues that need to be addressed. First, if we assume that the rule maximizes the sum of utilities, we can infer from the choice of the quorum/quota the implicit assumption which is made about the relation between preferences and actions. So the question is: what assumption on the agents behavior justifies a certain quorum? Second, can we make comparisons in terms of how a voting rule will fit best the average success of a voter? When there are two alternatives to vote on, the rule that maximizes the probability of a voter getting the outcome s/he favors is the simple majority. Can we extend results of this kind to more general rules?

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