Sarriko-On

Choice under risk: Theory and applications

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Choice under Risk: Theory and Applications.*

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1 Introduction

Risk is a central dimension of the decision-making environment and many important economic decisions involve risk. In this work I analyze the economic theory of the characterization of risk and the modeling of economic agents' responses to it. The work is completed with several applications of decision making under risk to different economic problems. An analytically convenient unified framework to incorporate risk in economic modeling is used in this document: Expected Utility Theory.

The presentation is organized as follows: Section 2 is devoted to the Expected Utility Theory. The properties of the preference relation defined on the set of risky alternatives that are required for the Expected Utility Theorem are analyzed in that section. Section 3 centers on risk aversion and its measurement. The concepts of certainty equivalent, risk premium, probability premium, absolute risk aversion and relative risk aversion, and the "more risk averse than" relation are discussed in that section. Section 3 concludes with the analysis of wealth effects in some utility functions that are often used in economic analysis. Section 4 studies the comparisons of risky alternatives in terms of return and risk. First and second-order stochastic dominance and the index of riskiness of Aumann and Serrano are explained in that section.

The last section includes thirteen Exercises that apply the analyses developed in previous sections to a great variety of situations: insurance, investment in risky assets and portfolio selection, risk sharing, taxes and income underreporting, deposit insurance and bank loans. Full solutions of the exercises are provided.

2 Expected utility theory

Consider that a decision maker faces a choice among a number of risky alternatives (or lotteries). The possible outcomes of these alternatives are monetary payoffs. There are N possible outcomes. The outcome that will occur with each alternative is uncertain. A risky alternative is characterized by the vector of probabilities of the outcomes in that alternative. The decision maker knows the probability of each outcome in each alternative.[1]

A lottery L is simple if it is given by $L =(p_1, p_2, ..., p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring. A lottery is compound if some (or all) outcomes of that lottery are themselves lotteries. The lottery $(L_1, L_2,..., L_h; q_1, q_2, ..., q_h)$ is a compound lottery that yields the lottery $L_i =(p_1^i, p_2^i, ..., p_N^i)$ with probability q_i. The reduced lottery of a compound lottery $(L_1, L_2,..., L_h; q_1, q_2, ..., q_h)$ is a simple lottery $L' =(p'_1, p'_2, ..., p'_N)$ where

$$p'_n = q_1 p_n^1 + q_2 p_n^2 + ... + q_h p_n^h$$

for $n = 1, ..., N$.[2] Let us assume that for any lottery or risky alternative, only the reduced lottery over final outcomes is of relevance to the decision maker (note that simple lotteries are already defined in reduced form).

The decision maker has a preference relation defined on the set $\pounds$ of simple lotteries (or lotteries in reduced form). When lottery $L =(p_1, p_2, ..., p_N)$ is at least as good as lottery $L' =(p'_1, p'_2, ..., p'_N)$ we write $L \succeq L'$. When lottery $L =(p_1, p_2, ..., p_N)$ is preferred (indifferent) to lottery $L' =(p'_1, p'_2, ..., p'_N)$ we write $L \succ L'$ $(L \sim L')$.

Let us assume that the preference relation $\succeq$ possesses the following properties:

[1]For the moment, let us include in the set of risky alternatives those (risk-free) alternatives where the probability of one of the outcomes is equal to 1.

[2]This reduction of a compound lottery to a reduced lottery requires that all the lotteries in the compound lottery be independent of each other.

i) Completeness: For any L, $L' \in \pounds$, we have that $L \succeq L'$ or $L' \succeq L$ (or both),

ii) Transitivity: For any L, L', $L'' \in \pounds$, if $L \succeq L'$ and $L' \succeq L''$, then $L \succeq L''$,

iii) Continuity: For any L, L', $L'' \in \pounds$ such that $L \succeq L' \succeq L''$, there exists $\alpha \in [0, 1]$ such that $L' \sim \alpha L + (1-\alpha)L''$, and

iv) Independence axiom: For any L, L', $L'' \in \pounds$ and $\alpha \in (0, 1)$ we have

$$L \succeq L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succeq \alpha L + (1-\alpha)L''.$$

When the preference relation $\succeq$ is complete and transitive we say that it is a rational preference relation. Continuity means that small changes in probabilities do not change the nature of the ordering between two risky alternatives. When the preference relation $\succeq$ is complete, transitive and continuous there exists a utility function representing $\succeq$ (a function U such that $L \succeq L'$ if and only if $U(L) \geq U(L')$). The independence axiom states that if we mix in the same way each of two risky alternatives with a third one, then the preference ordering of the two resulting mixtures will be independent of the particular third risky alternative used.

A utility function over risky alternatives has an expected utility form when there is an assignment of numbers (u_1, u_2, ..., u_N) to the outcomes such that for any risky alternative L :(p_1, p_2, ..., p_N) we have:

$$U(L) = u_1 p_1 + u_2 p_2 + ... + u_N p_N$$

Hence, the utility of a risky alternative is the expected value of the utilities of the outcomes under that alternative. Note that while the outcomes themselves are objective, their utility is subjective and may differ among decision makers.

In this work a utility function over risky alternatives with the expected utility form is going to be called an expected utility function.[3] If U is an

[3]The first analysis of the expected utility theory is developed in von-Neumann-Morgenstern (1944). However, Bernoulli (1954, translation from 1738) was the first to suggest that a risky alternative should be valued according to the expected utility that it provides.

expected utility function that represents the preference relation $\succeq$ on the set of risky alternatives, then V is another expected utility function representing $\succeq$ if and only if $V = \alpha + \beta U$, with $\beta > 0$.

A preference relation complete, transitive and continuous, that satisfies the independence axiom is representable by a utility function with the expected utility form (this is the Expected Utility Theorem).[4] For a utility function U with the expected utility form that represents those preferences it is

$$L \succeq L' \Leftrightarrow U(L) = u_1 p_1 + u_2 p_2 + ... + u_N p_N \geq u_1 p_1' + u_2 p_2' + ... + u_N p_N' = U(L').$$

Sometimes the utility function of the decision maker can be transformed to obtain another utility function that represents the same preferences and has the expected utility form. Consider, for instance, that the utility of lottery L :(p_1, p_2, ..., p_N) is

$$U(L) = \Pi_{i=1}^{N}(1 + x_i)^{p_i}.$$

Then we may define the utility function

$$V(L) = \ln U(L) = \sum_{i=1}^{N} p_i \ln(1 + x_i)$$

that represents the same preferences and has the expected utility form. The utility of outcome x_i is $\ln(1 + x_i)$ in utility function V.

The Theory of Expected Utility is very convenient analytically and provides useful information as to how decision makers choose under risk. However, its plausibility has been challenged in some situations. Among the criticisms of the expected utility theory we can include the Allais Paradox, the frame dependence, the Ellsberg Paradox and ambiguity aversion, the Prospect Theory and loss aversion. These topics are not analyzed in this work.[5]

[4] See A. Mas-Colell et al. (1995, section 6.B) for a proof of the Expected Utility Theorem.

[5] Some references for these topics are Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979) and Machina (1987).

3 Risk aversion and its measurement

Consider in sections 3 and 4 that the monetary outcomes may be represented by a continuous variable $\tilde{x}$. A risky alternative will be characterized by a density function $f(.)$, or by the corresponding distribution function $F(.)$, defined on $\tilde{x}$ ($F(x) = \Pr(\tilde{x} \leq x)$. The utility of a risky alternative L, or the utility of the distribution function $F_L(.)$ that characterizes that alternative, is

$$U(L) = U(F_L) = \int u(x)dF_L(x) = \int u(x)f_L(x)dx,$$

where $u(x)$ is the utility of outcome x. Let us call the function u the von-Neumann-Morgenstern (v.N-M) utility function. The power of the expected utility approach rests on the ability to use that theory with many functional forms for u. Let us consider that u is increasing and continuous.

3.1 Risk aversion

A decision maker is risk averse if and only if, for every distribution function $F(.)$,

$$\int u(x)dF(x) \leq u(\int xdF(x)) \tag{1}$$

The decision maker is strictly risk averse if this inequality is strict. It is risk neutral if $\int u(x)dF(x) = u(\int xdF(x))$ for every $F(.)$ and it is (strictly) risk lover if $\int u(x)dF(x) > u(\int xdF(x))$ for every $F(.)$.

Inequality (1) is equivalent to the concavity of u ($u''(x) \leq 0$ for every x, with $u''(x) < 0$ for some x). For a risk neutral (strictly risk lover) decision maker, u is linear (convex) and $\int u(x)dF(x) = u(\int xdF(x))$ ($>$).

A decision maker may be neither risk averse nor strictly risk averse nor risk neutral nor risk lover, as the corresponding conditions in the previous paragraphs have to be satisfied for every distribution function $F(.)$.

Consider the v.N-M utility functions $u_1(x) = \ln x$, $u_2(x) = x^2$, $u_3(x) = 20 + 7x$ and $u_4(x) = 2 - e^{-x}$. All these functions are increasing functions as their first derivatives are positive for every x. The signs of the second derivatives are, for every x, $u_1''(x) = -\frac{1}{x^2} < 0$, $u_2''(x) = 2 > 0$, $u_3''(x) = 0$ and $u_4''(x) = -e^{-x} < 0$. Therefore, a decision maker with v.N-M utility function $u_1(x)$ or $u_4(x)$ is strictly risk averse, a decision maker with v.N-M utility function $u_2(x)$ is (strictly) risk lover, and a decision maker with v.N-M utility function $u_3(x)$ is risk neutral.

As $\int x dF(x)$ $(=\int xf(x)dx)$ is the mean (or expected payoff) of a risky alternative with distribution function $F(.)$, a risk averse agent always prefers receiving the expected payoff of a lottery with certainty (and obtaining utility $u(E(\tilde{x})) = u(\int x dF(x)))$, rather than bearing the risk $\tilde{x}$ (and obtaining expected utility $E(u(\tilde{x})) = \int u(x)dF(x))$. Hence, a risk averse decision maker dislikes every risky alternative with a expected payoff of zero.

In the rest of this work it is going to be considered that the decision maker is risk averse.

3.2 Certainty equivalent, risk premium and probability premium

The certainty equivalent of a risky alternative L with distribution function $F_L(.)$ is $c(F_L, u)$ such that

$$u(c(F_L, u)) = \int u(x)dF_L(x)$$

The certainty equivalent of L is the amount of money that leaves the decision maker indifferent between receiving for sure that amount of money and playing the risky alternative L.

The risk premium of a risky alternative L with distribution function $F_L(.)$ is $m(F_L, u)$ such that

$$m(F_L, u) = \int x dF(x) - c(F_L, u)$$

The risk premium of L is the maximum amount of money that the decision maker is willing to pay to avoid the risky alternative L.

The previous definitions of certainty equivalent and risk premium are adequate when the initial wealth of the individual is 0. If the decision maker has initial wealth w, the certainty equivalent of a risky alternative L will be $c(F_L, u, w)$ such that

$$u(w + c(F_L, u, w)) = \int u(w + x) dF_L(x)$$

and the risk premium of a risky alternative L will be $m(F_L, u, w)$ such that

$$m(F_L, u, w) = \int x dF_L(x) - c(F_L, u, w).$$

The probability premium $\pi(x, \varepsilon, u)$ for any amount of money x and positive number ε is

$$u(x) = (\frac{1}{2} + \pi(x, \varepsilon, u))u(x + \varepsilon) + (\frac{1}{2} - \pi(x, \varepsilon, u))u(x - \varepsilon)$$

The probability premium is the excess of winning probability over fair odds that makes the individual indifferent between a certain outcome and a gamble where he may win or lose some amount of money (ε) with respect to that outcome.

When the decision maker is risk averse it is, for any w, $c(F, u, w) \leq \int x dF(x)$ for every $F(.)$ ($\Rightarrow m(F, u, w) \geq 0$) and $\pi(x, \varepsilon, u) \geq 0$ for all x and ε.

If the risk of a lottery (or risky asset) increases while its mean remains unchanged the certainty equivalent decreases and the risk premium and the probability premium increase. Consider a consumer with initial wealth equal to 0 and v.N-M utility function $u(x) = \sqrt{x}$ (note that it is risk averse). The lottery (36,16;$\frac{1}{2}, \frac{1}{2}$) has mean=26, certainty equivalent (C)=25 ($\sqrt{C} = \frac{1}{2}\sqrt{36} + \frac{1}{2}\sqrt{16} = 5$), risk premium=26-25=1 and probability premium: $\sqrt{26} = (\frac{1}{2} + \pi)\sqrt{36} + (\frac{1}{2} - \pi)\sqrt{16} = (\frac{1}{2} + \pi)\sqrt{26 + 10} + (\frac{1}{2} - \pi)\sqrt{26 - 10} \Rightarrow \pi = 0.0495$. However, the lottery (48,4;$\frac{1}{2}, \frac{1}{2}$), with the same mean but a higher risk, has certainty equivalent=19.928, risk premium=26-19.928=6.072 and probability premium: $\sqrt{26} = (\frac{1}{2}+\pi)\sqrt{48} + (\frac{1}{2}-\pi)\sqrt{4} = (\frac{1}{2}+\pi)\sqrt{26+22} + (\frac{1}{2}-\pi)\sqrt{26-22} \Rightarrow \pi = 0.12883$.

3.3 Measurement of risk aversion

3.3.1 Absolute risk aversion

Given a (twice-differentiable) v.N-M utility function, the Arrow-Pratt coefficient of absolute risk aversion at x is defined as $r_A(x,u) = -\frac{u''(x)}{u'(x)}$.[6]

The Arrow-Pratt coefficient of absolute risk aversion tries to capture the idea that the faster the marginal utility of wealth declines, the more risk averse the individual is. Hence, the degree of risk aversion of the decision maker is related to the curvature of $u(x)$. One possible measure of the curvature of the v.N-M utility function is $u''(x)$. But this is not an adequate measure because it is not invariant to positive linear transformations of the utility function. To obtain an invariant measure, it is used $-\frac{u''(x)}{u'(x)}$ (with a $-$ sign to have a positive number).

The v.N-M utility function exhibits decreasing (constant; increasing) absolute risk aversion if $r_A(x,u)$ is a decreasing (constant; increasing) function of x. We say that a decision maker with a v.N-M utility function that exhibits decreasing (constant) absolute risk aversion is a DARA (CARA) decision maker.

A decision maker with a v.N-M utility function that exhibits decreasing absolute risk aversion is willing to accept more risky alternatives as her wealth increases. This decision maker is also willing to invest a greater amount in a risky asset when her wealth increases. The following conditions are equivalent:

i) The v.N-M utility function exhibits decreasing absolute risk aversion

[6] The measures of risk aversion studied in this section were proposed in Arrow (1963) and Pratt (1964).

ii) $m(F, u, w)$ is decreasing in w

iii) $\pi(x, \varepsilon, u)$ is decreasing in x

iv) If $w_1 > w_2$ then $\int u(w_2 + x)dF(x) \geq u(w_2) \Rightarrow \int u(w_1 + x)dF(x) \geq u(w_1)$.

If the decision maker had a v.N-M utility function that exhibits constant absolute risk aversion she would be willing to accept the same set of risky alternatives and to invest the same amount in a risky asset independently of her level of wealth. For a decision maker with a v.N-M utility function that exhibits constant absolute risk aversion, $m(F, u, w)$ does not depend on w and $\pi(x, \varepsilon, u)$ does not depend on x.

3.3.2 Comparisons across decision makers

Consider two decision makers 1 and 2 with, respective, v.N-M utility functions $u_1(x)$ and $u_2(x)$. The following conditions are equivalent:

i) Decision maker 2 is more risk averse than decision maker 1

ii) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x

iii) $u_2(.)$ is a concave transformation of $u_1(.)$

iv) $c(F, u_2) \leq c(F, u_1)$ for any $F(.)$

v) $m(F, u_2) \geq m(F, u_1)$ for any $F(.)$

vi) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε

vii) $\int u_2(x)dF(x) \geq u_2(\bar{x}) \Rightarrow \int u_1(x)dF(x) \geq u_1(\bar{x})$ for any $F(.)$ and $\bar{x}$.

There is also a strict version of this equivalence of conditions. The equivalence of conditions remains valid when the inequalities in ii), iv), v) vi) and vii) are strict, the decision maker 2 is strictly more risk averse than the decision maker 1, and $u_2(.)$ is a strictly concave transformation of $u_2(.)$.

The more risk aversion than relation is a partial ordering of v.N-M utility functions as it is not complete.

3.3.3 Relative risk aversion

Given a (twice-differentiable) v.N-M utility function, the coefficient of relative risk aversion at x is defined as $r_R(x,u) = -x\frac{u''(x)}{u'(x)}$. The v.N-M utility function exhibits decreasing (constant; increasing; non-increasing) relative risk aversion if $r_R(x,u)$ is a decreasing (constant; increasing; non-increasing) function of x.

The property of non-increasing relative risk aversion is stronger than the property of decreasing absolute risk aversion. A risk-averse decision maker with non-increasing relative risk aversion will exhibit decreasing absolute risk aversion, but the converse might not be true: $r_R = xr_A \Rightarrow r'_R = r_A + xr'_A$; then $r'_R \leq 0 \Rightarrow r'_A < 0$ but $r'_A < 0 \nRightarrow r'_R \leq 0$.

A decision maker with a v.N-M utility function that exhibits decreasing relative risk aversion is willing to accept more risky alternatives as her wealth increases. This decision maker is also willing to invest a greater proportion of her wealth in a risky asset when her wealth increases. If the decision maker had a v.N-M utility function that exhibits constant relative risk aversion she would be willing to invest the same proportion of her wealth in a risky asset independently of her level of wealth (hence, she would be willing to invest a greater amount of her wealth in a risky asset as her wealth increases). If we consider risky projects whose outcomes are percentage gains or losses of current wealth, a decision maker with a v.N-M utility function that exhibits constant relative risk aversion will be willing to accept the same set of risky projects as her wealth changes.

3.4 Types of von-Neumann-Morgenstern utility functions and wealth effects

The general form of a v.N-M utility function with a coefficient of absolute risk aversion equal to the constant $a > 0$, for every x, is $u(x) = \alpha - \beta e^{-ax}$, where $\beta > 0$. For this utility function it is: $u'(x) = a\beta e^{-ax} > 0$, $u''(x) = -a^2$

$\beta e^{-ax} < 0$ and $r_A(x,u) = -\frac{-a^2\beta e^{-ax}}{a\beta e^{-ax}} = a$. Note also that

$$r_A(x,u) = -\frac{u''(x)}{u'(x)} = a \Rightarrow -u''(x) = au'(x) \Rightarrow -u'(x) = au(x)$$

$$\Rightarrow \frac{u'(x)}{u(x)} = -a \Rightarrow \ln u(x) = -ax + k \Rightarrow u(x) = -\beta e^{-ax},$$

and a linear transformation of this function is $u(x) = \alpha - \beta e^{-ax}$, a v.N-M utility function increasing and concave. A decision maker with this utility function will invest the same amount in a risky asset, independently of her level of wealth.

The general form of a v.N-M utility function with a coefficient of relative risk aversion for every x equal to the constant ρ, where $0 < \rho \neq 1$, is $u(x) = \alpha + \beta x^{1-\rho}$. For this utility function it is: $u'(x) = (1-\rho)\beta x^{-\rho}$, $u''(x) = -\rho(1-\rho)\,\beta x^{-\rho-1}$ and $r_R(x,u) = -x\frac{-\rho(1-\rho)\beta x^{-\rho-1}}{(1-\rho)\beta x^{-\rho}} = \rho$; for a concave function we require $\beta(1-\rho) > 0 \Rightarrow$ either $\beta > 0$ and $0 < \rho < 1$, or $\beta < 0$ and $\rho > 1$. Moreover

$$r_R(x,u) = \rho \Rightarrow -x\frac{u''(x)}{u'(x)} = \rho \Rightarrow \frac{u''(x)}{u'(x)} = -\frac{\rho}{x}$$

$$\Rightarrow \ln u'(x) = -\rho \ln x + \ln k = \ln \kappa x^{-\rho}$$

$$\Rightarrow u'(x) = \kappa x^{-\rho} \Rightarrow u(x) = \alpha + (1-\rho)\kappa x^{1-\rho};$$

in general, $u(x) = \alpha + \beta x^{1-\rho}$. A decision maker with this utility function will invest the same proportion of her wealth in a risky asset, independently of her level of wealth.

The general form of a v.N-M utility function with a coefficient of relative risk aversion equal to 1 for every x is $u(x) = \alpha + \beta \ln x$, where $\beta > 0$. For this utility function it is: $u'(x) = \frac{\beta}{x} > 0$, $u''(x) = -\frac{\beta}{x^2} < 0$ and $r_R(x,u) = -x\frac{-\frac{\beta}{x^2}}{\frac{\beta}{x}} = 1$. Note also that

$$r_R(x,u) = -x\frac{u''(x)}{u'(x)} = 1 \Rightarrow -xu''(x) = u'(x) \Rightarrow xu''(x) + u'(x) = 0$$

$$\Rightarrow xu'(x) = \beta \Rightarrow u'(x) = \frac{\beta}{x} \Rightarrow u(x) = \alpha + \beta \ln x.$$

The v.N-M utility function $u(x) = \alpha + \beta\sqrt{x}$, with $\beta > 0$, is used to illustrate several results and applications in this work. For this utility function it is $u'(x) = \frac{\beta}{2\sqrt{x}} > 0$, $u''(x) = -\frac{\beta x^{-\frac{3}{2}}}{4} < 0$, $r_A(x,u) = \frac{1}{2x}$ and $r_R(x,u) = \frac{1}{2}$. Hence, a decision maker with v.N-M utility function $u(x) = \alpha + \beta\sqrt{x}$ has decreasing absolute risk aversion and constant relative risk aversion. She will be willing to accept more risky alternatives and to invest a greater amount of her wealth in a risky asset as her level of wealth increases. However, she would invest the same proportion of her wealth in a risky asset, independently of her level of wealth.

The v.N-M utility function $u(x) = \alpha + \beta \ln x$, with $\beta > 0$, is also used below to illustrate several results and applications. For that utility function it is $r_A(x,u) = \frac{1}{x}$ and $r_R(x,u) = 1$. Hence, a decision maker with v.N-M utility function $u(x) = \alpha + \beta \ln x$ has decreasing absolute risk aversion and constant relative risk aversion. She will be willing to accept more risky alternatives and to invest the same proportion of her wealth in a risky asset as her level of wealth increases. Moreover, note that a decision maker with v.N-M utility function $u(x) = \alpha + \beta \ln x$ is more risk averse than a decision maker with v.N-M utility function $u(x) = \alpha + \beta\sqrt{x}$ as $r_A(x, \alpha + \beta \ln x) = \frac{1}{x} > r_A(x, \alpha + \beta\sqrt{x}) = \frac{1}{2x}$ for every x.

Some of those wealth effects are discussed in the following two exercises:

Exercise A: Wealth, certainty equivalent, risk premium and probability premium

Consider a consumer with initial wealth equal to w and v.N-M utility function $u(x) = \sqrt{x}$. For the lottery $(36{,}16;\frac{1}{2},\frac{1}{2})$ how do the certainty equivalent, the risk premium and the probability premium depend on w?

Solution

The lottery $(36{,}16;\frac{1}{2},\frac{1}{2})$ has mean=26 and certainty equivalent (C) that increases with w ($u(x) = \sqrt{x}$ implies decreasing absolute risk aversion):

$$\sqrt{w+C} = \frac{1}{2}\sqrt{w+36} + \frac{1}{2}\sqrt{w+16}$$

$$\Rightarrow C = (\frac{1}{2}\sqrt{w+36} + \frac{1}{2}\sqrt{w+16})^2 - w$$

and $\frac{dC}{dw} > 0$.[7] Hence, the risk premium is $\sqrt{w+26} - (\frac{1}{2}\sqrt{w+36} + \frac{1}{2}\sqrt{w+16})^2$ and it decreases with w. Finally the probability premium solves

$$\sqrt{w+26} = (\frac{1}{2} + \pi)\sqrt{w+26+10} + (\frac{1}{2} - \pi)\sqrt{w+26-10}$$

$\Rightarrow \pi = \frac{\sqrt{w+26} - 0.5(\sqrt{w+26+10} + \sqrt{w+26-10})}{\sqrt{w+26+10} - \sqrt{w+26-10}}$, and it may be shown that $\frac{d\pi}{dw} < 0$.

Exercise B: Wealth, participation in a lottery, certainty equivalent and risk premium

A decision maker with wealth w and v.N-M utility function $u(x) = k - e^{-ax}$ may participate in a lottery where he may win z_1 (final wealth $w + z_1$) with probability p or lose z_2 (final wealth $w - z_2$) with probability $1 - p$. Discuss the effect of w on his decision to participate in that lottery, considering that $z_1 > z_2$.

Do the certainty equivalent and the risk premium depend on the wealth of the decision maker? Calculate the certainty equivalent and the risk premium when $a = 2$, $p = 0.4$, $z_1 = 5$ and $z_2 = 3$ and when $a = 0.5$, $p = 0.9$, $z_1 = 5$ and $z_2 = 3$.

Solution

The decision maker would participate in the lottery if

$$p(k - e^{-a(w+z_1)}) + (1-p)(k - e^{-a(w-z_2)}) > k - e^{-aw} \Leftrightarrow -pe^{-az_1} - (1-p)e^{az_2} > -1,$$

which is independent of w (note that the v.N-M utility function implies constant absolute risk aversion and then any risk is either accepted at all levels of wealth or rejected at all levels of wealth).

[7] Note that $\frac{d((\frac{1}{2}\sqrt{w+36} + \frac{1}{2}\sqrt{w+16})^2 - w)}{dw} = \frac{1}{2}\frac{-\sqrt{(w+36)}\sqrt{(w+16)} + w + 26}{\sqrt{(w+36)}\sqrt{(w+16)}} > 0$
$\Leftrightarrow w + 26 > \sqrt{(w+36)}\sqrt{(w+16)} \Leftrightarrow w^2 + 52w + 676 > w^2 + 52w + 576$.

The certainty equivalent C of that lottery is

$$k - e^{-a(w+C)} = p(k - e^{-a(w+z_1)}) + (1-p)(k - e^{-a(w-z_2)})$$

$$\Leftrightarrow -e^{-a(w+C)} = -e^{-aw}(pe^{-az_1} + (1-p)e^{az_2})$$

$$\Leftrightarrow -e^{-aC} = -(pe^{-az_1} + (1-p)e^{az_2}) \Leftrightarrow aC \ln e = \ln(pe^{-az_1} + (1-p)e^{az_2})^{-1}$$

$$\Rightarrow C = \frac{1}{a} \ln \frac{1}{pe^{-az_1} + (1-p)\, e^{az_2}}$$

Hence, the risk premium is $pz_1 + (1-p)(-z_2) - \frac{1}{a} \ln \frac{1}{pe^{-az_1}+(1-p)e^{az_2}}$. The certainty equivalent and the risk premium do not depend on w (the v.N-M utility function implies constant absolute risk aversion).

If $a = 2$, $p = 0.4$, $z_1 = 5$ and $z_2 = 3$, we have $C = -2.7446$. As $C < 0$ the decision maker does not want to participate in that lottery. In this case the risk premium is $0.4(5) + 0.6(-3) - (-2.7446) = 2.9446$.

If $a = 0.5$, $p = 0.9$, $z_1 = 5$ and $z_2 = 3$, it is $C = 1.3$ and the risk premium is $0.9(5) + 0.1(-3) - 1.3 = 2.9$.

4 Comparison of risky alternatives in terms of return and risk

Any decision maker may compare any pair of risky assets using her v.N-M utility function and the expected utility theory. However, in practice it is often difficult to find an agent's utility function. In this section I focus on comparisons among monetary payoff distributions, without considering particular utility functions.

4.1 First-order stochastic dominance

Let us assume in this section and in section 4.2 that all payoff distributions $F(.)$ are such that $F(0) = 0$ and $F(\hat{x}) = 1$ for some $\hat{x}$.

First-order stochastic dominance captures the idea of dominance through higher returns. We say that the distribution $F(.)$ first-order stochastically dominates the distribution $G(.)$ ($F(.)$ FSD $G(.)$) if every expected utility maximizer who values more over less prefers the distribution $F(.)$ to the distribution $G(.)$. Hence, $F(.)$ FSD $G(.)$ if, for every nondecreasing v.N-M utility function $u(x)$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x). \tag{2}$$

If $F(.)$ FSD $G(.)$ it is not true that every possible outcome under $F(.)$ is greater than every possible outcome under $G(.)$. However, $F(.)$ FSD $G(.)$ if and only if $F(x) \leq G(x)$ for every x.

If $F(.)$ FSD $G(.)$ then the mean of x under $G(.)$ ($\int xdG(x)$) cannot exceed that under $F(.)$ ($\int xdF(x)$). This can be proved taking $u(x) = x$ in (2). The contrary, however, is not true: we may have $\int xdF(x) > \int xdG(x)$ but $F(.)$ does not first-order stochastically dominate $G(.)$. To prove this latter result consider the following distributions

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4} & 1 \leq x < 3 \\ 1 & 3 \leq x \leq \hat{x} \end{cases}$$

and

$$G(x) = \begin{cases} 0 & x < 2 \\ 1 & 2 \leq x \leq \hat{x} \end{cases}$$

We have $\int xdF(x) = \frac{1}{4}1 + \frac{3}{4}3 = 2.5 > \int xdG(x) = 2$, but $F(.)$ does not stochastically dominate $G(.)$.

If some probability mass is transferred from a low monetary outcome to higher monetary outcomes, the initial distribution is first-order stochastically dominated by the latter one. Such a shift of probability mass is known as a first-order stochastically dominating shift.

First-order stochastic dominance is a partial ordering: there are pairs of payoff distributions such that no one first-order stochastically dominates the other. If distributions $F(.)$ and $G(.)$ cannot be ordered according to the first-order stochastic dominance ordering, there exist monotone increasing v.N-M utility functions such that, under those utility functions, the expected utility of $F(.)$ is greater than the expected utility of $G(.)$, and there also exist other monotone increasing v.N-M utility functions such that, under these other utility functions, the expected utility of $G(.)$ is greater than the expected utility of $F(.)$.[8]

4.2 Second-order stochastic dominance

Second-order stochastic dominance captures the idea of dominance through lower risk. If every risk averter prefers the distribution $F(.)$ to the distribution $G(.)$ we say that $F(.)$ second-order stochastically dominates $G(.)$ ($F(.)$ SSD $G(.)$). Hence, for any two distributions $F(.)$ and $G(.)$ with the same mean, $F(.)$ SSD $G(.)$ if, for every nondecreasing concave utility function, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

It may be proved that $F(.)$ SSD $G(.)$ if and only if[9]

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt \tag{3}$$

for every $x \leq \tilde{x}$. Moreover, if $F(.)$ SSD $G(.)$ then it is $\int xdF(x) \geq \int xdG(x)$.

If $F(.)$ FSD $G(.)$ then $F(.)$ SSD $G(.)$. However, the contrary is not true. First-order stochastic dominance is a stronger requirement than second-order stochastic dominance.

[8] Other stochastic orderings, as the likelihood ratio and hazard-rate orderings, are stronger requirements than first-order stochastic dominance. It may be proved that if $F(.)$ is larger than $G(.)$ in the sense of the monotone hazard-rate condition, then $F(.)$ is larger than $G(.)$ in the sense of the monotone likelihood ratio condition. Moreover, if $F(.)$ is larger than $G(.)$ in the sense of the monotone likelihood ratio condition, then $F(.)$ FSD $G(.)$. See Wolfstetter (1999), section 4.3.3.

[9] See Hadar and Rusell (1969) and Rothschild and Stiglitz (1970).

Second-order stochastic dominance is a partial ordering: there are pairs of payoff distributions such that no one second-order stochastically dominates the other.[10]

If $H(.)$ is a distribution independent of distributions $F(.)$ and $G(.)$ then for any $a > 0$ and $b \geq 0$ we have:[11]

i) $F(.)$ FSD $G(.) \Rightarrow aF(.) + bH(.)$ FSD $aG(.) + bH(.)$

ii) $F(.)$ SSD $G(.) \Rightarrow aF(.) + bH(.)$ SSD $aG(.) + bH(.)$.

Consider now payoff distributions with the same mean. If $F(.)$ and $G(.)$ have the same mean (i.e., if $\int xdF(x) = \int xdG(x)$) it is $\int_0^{\hat{x}} F(x)dx = \int_0^{\hat{x}} G(x)dx$ as, integrating by parts, we have

$$\int_0^{\hat{x}} (F(x) - G(x))dx = -\int_0^{\hat{x}} xd(F(x) - G(x)) + (F(\hat{x}) - G(\hat{x}))\hat{x} = 0.$$

The distribution $G(.)$ is a mean-preserving spread of the distribution $F(.)$ if the only difference between the two distributions is that the distribution $G(.)$ has been obtained from the distribution $F(.)$ randomizing each outcome x in $F(.)$ (or randomizing some outcomes in $F(.)$), so that instead of that outcome in $G(.)$ we have the final payoff $x + z$, where z has a distribution function $H_x(z)$ with a mean of zero. If the distribution $G(.)$ is a mean-preserving spread of the distribution $F(.)$, then $F(.)$ SSD $G(.)$.

The distribution $G(.)$ is an elementary increase in risk from the distribution $F(.)$ if $G(.)$ is generated from $F(.)$ by taking all the probability assigned in $F(.)$ to an interval $[x_i,x_j]$ and transferring it to the endpoints x_i and x_j in a way that the mean is preserved (hence, an elementary increase in risk is mean preserving). If $G(.)$ is an elementary increase in risk from $F(.)$, then $F(.)$ SSD $G(.)$.

Stochastic dominance theory allows us to compare some risky alternatives regardless of whose utility function we consider. This may be relevant for

[10] There are other dominance criteria weaker than SSD that allow to order more pairs of payoffs distributions. For instance, Caballé and Pomansky (1996) define a partial ordering that uses the mixed utility functions, an important subset of the nondecreasing concave utility functions.

[11] See Hadar and Rusell (1971).

decision making designed to benefit a group, such as a corporate manager making decisions on behalf of the company's shareholders.

4.3 The index of riskiness of Aumann and Serrano

Aumann and Serrano (2008) define an index of riskiness (R_{AS}) on risky alternatives that completes the partial orderings on risky alternatives given by FSD and by SSD. For any risky alternative L such that $\int x dF_L(x) > 0$ and $\Pr_L(x < 0) > 0$, $R_{AS}(L)$ is the solution to

$$\int e^{-\frac{x}{R_{AS}(L)}} dF_L(x) = 1.$$

Hence, $R_{AS}(L) = \frac{1}{a(L)}$, where $a(L)$ is the coefficient of absolute risk aversion of the individual of the CARA decision maker which is indifferent between accepting and rejecting the risky alternative: $\int e^{-a(L)(w+x)} dF_L(x) = e^{-a(L)w}$ ($\int e^{-a(L)x} dF_L(x)$ is the expected utility of that decision maker when he accepts the risky alternative L and $e^{-a(L)w}$ is his utility when he does not accept the risky alternative L: see section 3.4).

Note that this index of riskiness depends only on the distribution of the risky alternative and not on the utility of the decision maker or his wealth. Some properties of R_{AS} are the following:

i) R_{AS} is continuous (when two risky alternatives are likely to be close, their riskiness levels are close).

ii) If the distribution of L first-order stochastically dominates the distribution of L', it will be $R_{AS}(L) < R_{AS}(L')$.

iii) If the distribution of L second-order stochastically dominates the distribution of L', it will be $R_{AS}(L) < R_{AS}(L')$.

iv) If L and L' are (statistically) independent and $R_{AS}(L) < R_{AS}(L')$ then $R_{AS}(L) \leq R_{AS}(L + L') \leq R_{AS}(L')$, with equalities when $R_{AS}(L) = R_{AS}(L')$. If L and L' are not (statistically) independent then $R_{AS}(L + L') < R_{AS}(L) + R_{AS}(L')$, with equality only when L' is a positive multiple of L.

v) R_{AS} is more sensitive to the loss side of a risky alternative than to its gain side.

If we measure riskiness using the A&S index we also have that:[12]

i) given two risky alternatives L and L' such that $R_{AS}(L) < R_{AS}(L')$, a risk averse decision maker may prefer L' to L,

ii) given two risky alternatives L and L' such that $R_{AS}(L) < R_{AS}(L')$, a non-CARA risk averse decision maker may accept L' and reject L,

iii) a risk averse decision maker may not be indifferent between two risky alternatives with the same level of riskiness, and

iv) given two risky alternatives with the same level of riskiness, a risk averse decision maker may accept one of those risky alternatives and reject the other.

As a consequence of i) to iv), R_{AS} is a quantification of riskiness that, for the moment, does not help most investors and other decision makers make their decisions. Further research on this index is required.

5 Applications

This section analyzes some applications of the previous definitions and results. The applications considered are insurance, investment in risky assets and portfolio selection, risk sharing, taxes and income underreporting, deposit insurance, and bank loans. The applications are presented by means of Exercises and their solutions. In all situations considered it is assumed that the decision maker's preferences satisfy the axioms of expected utility.

To shorten the presentation I do not include in the solutions of the Exercises the second order conditions for the interior solutions obtained. However, it is easy to check that those second order conditions are satisfied in every interior solution. Often, the problem to solve is a maximization problem with only one decision variable. In that case the second order condition for the interior solution is satisfied when the second derivative of the objective function with respect to the decision variable is negative for every value of that decision variable.

[12] See Usategui (2008).

5.1 Two types of decision problems under risk

Most applications presented in this work may be included within two general problems of decision making under risk: the problem where the decision maker invests in a risky alternative and the problem where the decision maker buys insurance to reduce his exposure to risk.

5.1.1 Investment in a risky alternative

A risk averse decision maker with v.N-M utility function $u(x)$ wants to invest an amount of wealth equal to w at time 0. She must decide the amount s to invest in a risky alternative that pays $1 + z_1$ at time 1 with probability p per unit invested and $1 + z_2$ at time 1 with probability $1 - p$ per unit invested, where $z_1 > z_2$. The wealth $w - s$ not invested in the risky alternative is invested in a riskless alternative that at time 1 pays always $1 + r$ per unit invested. Consider that there is no discounting of the future.

As the decision maker is risk averse, a necessary condition for an strictly positive investment in the risky alternative is $pz_1 + (1 - p)z_2 > r$. Moreover, a necessary condition for an strictly positive demand of the safe, or riskless, alternative is $\min\{z_1, z_2\} < r$. Hence, $z_1 > r > z_2$ is required.

To obtain s the decision maker solves

$$\max_s p.u(w + sz_1 + r(w - s)) + (1 - p).u(w + sz_2 + r(w - s))$$

subject to $s \leq w$. The first order condition for an interior solution is

$$p(z_1 - r).u'(w + sz_1 + r(w - s)) + (1 - p)(z_2 - r).u'(w + sz_2 + r(w - s)) = 0.$$

5.1.2 Investment to reduce risk

A risk averse decision maker with v.N-M utility function $u(x)$ and wealth w faces a risky situation where he may lose L with probability p. The decision maker may buy insurance or invest in a risk shifting contract. He has to choose the level α of partial coverage to buy, with $0 \leq \alpha \leq 1$. When $\alpha = 1$ there will be full coverage of the possible loss and when $0 < \alpha < 1$ there will be partial coverage of that loss. The price of a level α of partial coverage is αm.

The expected wealth of the decision maker without loss coverage is $w - pL$. To decide on the amount of coverage to buy, the decision maker will solve

$$\max_{\alpha}(1-p)u(w-\alpha m) + pu(w - \alpha m - L + \alpha L)$$

The first order condition is

$$-m(1-p)u'(w-\alpha m) + p(L-m)u'(w - L + \alpha(L-m)) \leq 0$$

with equality if $\alpha > 0$.[13]

5.2 Insurance

Exercise C: Full and partial insurance

An individual owns a house with value equal to 300000. There is a probability equal to 0.05 that the house will burn down completely in a fire. The individual can insure his house against a loss from this fire. The premium for full insurance is 17000. The individual has other wealth equal to 100000 in non-risky assets.

[13] Note that the second order condition is satisfied as $u''(x) < 0$.

i) What is the expected profit to the insurance company from this full insurance policy? Is this insurance policy actuarially fair?

ii) If the individual were risk neutral, would he buy this full insurance policy?

iii) If the individual had v.N-M utility function $u(x) = \sqrt{x}$, would he buy this full insurance policy? Which is the maximum premium that he is willing to pay for full insurance? If, instead, the individual had v.N-M utility function $u(x) = \ln x$, which would be the maximum premium that he would be willing to pay for full insurance?

iv) Consider that the individual could buy partial insurance β, with $0 \leq \beta \leq 1$, such that if he paid a premium equal to 17000β he would receive a compensation from the insurance company, in the event of fire, equal to 300000β. What level of partial insurance would select an individual with v.N-M utility function $u(x) = \sqrt{x}$? If, instead, the individual had v.N-M utility function $u(x) = \ln x$, would he select a greater level of partial insurance?

v) If the individual had other wealth equal to 140000 in non-risky assets and his v.N-M utility function were $u(x) = \sqrt{x}$, what level of partial insurance would he select? How does α depend on the total wealth of the individual?

vi) If the individual had reduced the risk of loss to $(1 - \gamma)300000$, with $0 \leq \gamma \leq 1$, by investing in fire protection, what level of partial insurance would he select when his v.N-M utility function is $u(x) = \sqrt{x}$ and the insurance premium is $(1 - \gamma)17000$? (Consider that the individual has other wealth equal to 100000 in non-risky assets).

vii) If the insurance policy included a deductible equal to D and the individual had v.N-M utility function $u(x) = \sqrt{x}$, which would be the maximum premium that he would be willing to pay for full insurance?

Solution

i) Insurance is actuarially fair when the insurance premium is equal to the expected compensation from the insurance company. As $17000 > 0.05(300000) = 15000$, this insurance policy is not actuarially fair. The expected profit of the insurance company is $17000 - 0.05(300000) = 2000$.[14]

ii) As $0.95(400000) + 0.05(100000) = 385000 > 400000 - 17000 = 383000$, the individual would not buy that insurance policy. The expected monetary value is smaller under the insurance policy.

iii) As $0.95\sqrt{400000} + 0.05\sqrt{100000} = 616.64$ and $\sqrt{383000} = 618.87$, the individual would buy that insurance policy. For this risk averse individual the reduction in exposure to risk makes up for the loss in expected monetary value under the insurance policy. The maximum premium m he is willing to pay for full insurance is

$$\sqrt{400000 - m} = 0.95\sqrt{400000} + 0.05\sqrt{100000} \Leftrightarrow m = 19750.$$

As $0.95\ln(400000) + 0.05\ln(100000) = 12.83$ and $\ln(383000) = 12.856$, the individual would also buy that insurance policy when $u(x) = \ln x$. The maximum premium m that the individual would pay for full insurance in this case would solve

$$\ln(400000 - m) = 0.95\ln(400000) + 0.05\ln(100000) \Leftrightarrow m = 26787.$$

The individual is willing to pay more for full insurance when $u(x) = \ln x$. An individual with $u(x) = \ln x$ is more risk averse than an individual with $u(x) = \sqrt{x}$, as $r_A(x, \ln x) = \frac{1}{x} > \frac{1}{2x} = r_A(x, \sqrt{x})$, and, therefore, is willing to pay more for full insurance (the risk premium of the risky alternative is greater for an individual with $u(x) = \ln x$ than for an individual with $u(x) = \sqrt{x}$).

[14] We can consider that insurance companies are risk neutral as they insure many different and independent risks, and there is a very high probability that the profits they obtain are very close to the expected profits from insuring all those risks.

iv) This is a particular case of section 5.1.2 where $w = 400000$, $L = 300000$, $p = 0.05$ and $m = 17000$. Hence, when $u(x) = \sqrt{x}$ the individual will solve

$$\max_{\alpha}(0.95\sqrt{400000 - 17000\alpha} + 0.05\sqrt{100000 + 283000\alpha})$$

The first order condition for an interior solution (first derivative of the objective function with respect to α equal to 0) is in this case

$$-2.5\frac{323\sqrt{(1000 + 2830\alpha)} - 283\sqrt{(4000 - 170\alpha)}}{\sqrt{(4000 - 170\alpha)}\sqrt{(1000 + 2830\alpha)}} = 0.$$

Therefore, the individual will decide $\alpha^* = 0.699$.

When $u(x) = \ln x$ the individual will solve

$$\max_{\alpha}(0.95\ln(400000 - 17000\alpha) + 0.05\ln(100000 + 283000\alpha))$$

The first order condition for an interior solution is

$$\frac{-4045 + 4811\alpha}{(-400 + 17\alpha)(100 + 283\alpha)} = 0 \Rightarrow \alpha^* = 0.841.$$

The individual buys a greater level of insurance when $u(x) = \ln x$. An individual with $u(x) = \ln x$ is more risk averse than an individual with $u(x) = \sqrt{x}$, and, therefore, buys a greater amount of insurance (prefers to buy a less risky asset).

v) With other wealth equal to 140000 in non-risky assets the individual will solve

$$\max_{\alpha}(0.95\sqrt{440000 - 17000\alpha} + 0.05\sqrt{140000 + 283000\alpha})$$

From the first order condition for an interior solution it is $\alpha^* = 0,668$.

As there is decreasing absolute risk aversion when $u(x) = \sqrt{x}$, the individual buys a lower level of partial insurance when his wealth increases. This analysis of the effect of the change in wealth on the level of insurance bought considers that the size of the risk is fixed as wealth changes (the individual does not move to a more expensive house as his wealth increases). However, we know that a DARA decision maker invests more in risky assets when he becomes wealthier and, as he assumes more risk, he may decide to buy more insurance (in total, even though he buys less insurance for each risk he faces). As the v.N-M utility functions considered imply constant relative risk aversion, the individual would select the same value of α if the value of the house, the amount of other wealth held by the individual and the insurance premium changed in the same proportion.

vi) The individual will solve

$$\max_{\alpha}(0.95\sqrt{400000-\gamma 17000\alpha}+0.05\sqrt{100000+300000(1-\gamma)+\gamma 283000\alpha})$$

From the first order condition for an interior solution it is $\alpha^* = 9.713\frac{-0.323+1.043(1-\gamma)}{10(1-\gamma)}$.

As α^* decreases with γ and $9.713\frac{-0.323+1.043(1-\gamma)}{10(1-\gamma)} = 0 \Rightarrow \gamma = .690$, the decision maker will not buy insurance when $\gamma \geq 0.690$ and he will buy partial insurance $\alpha = 9.713\frac{-0.323+1.043(1-\gamma)}{10(1-\gamma)}$ when $\gamma < 0.690$.

vii) The maximum premium m that the individual would pay for full insurance would solve

$$0.95\sqrt{400000-m}+0.05\sqrt{400000-m-D} = 0.95\sqrt{400000}+0.05\sqrt{100000}$$

$$\Leftrightarrow m = -24832 + 2.7778 \times 10^{-3}D + 21.697\sqrt{(4225\,000-10D)}$$

As m decreases with D, the maximum premium that the individual is willing to pay for full insurance diminishes with the level of the deductible.

Exercise D: Insurance actuarially fair and insurance coverage

A risk averse decision maker with v.N-M utility function $u(x)$ and wealth w runs a risk of a loss equal to L. The probability of the loss is p. The decision maker may buy insurance and the insurance premium is e per unit of loss covered. The decision maker has to decide the number β of units of loss to insure. If $\beta = L$ there will be full insurance and if $\beta < L$ there will be partial insurance. Solve the maximization problem of the decision maker in the following contexts:

i) Insurance is actuarially fair

ii) Insurance is not actuarially fair

Solution

This situation is analogous to the one considered in section 5.1.2, with $\alpha = \frac{\beta}{L}$ and $m = eL$.

i) When insurance is actuarially fair it is $e = p$. Then from the first order condition in section 5.1.2 we have

$$-u'(w - \beta p) + u'(w - L + \beta(1 - p)) \leq 0$$

with equality if $\beta > 0$.

The solution cannot be $\beta = 0$ as $u'(w - L) > u'(w)$. Then in the solution it is $\beta > 0$ and

$$-u'(w - \beta p) + u'(w - L + \beta(1 - p)) = 0$$

As $u'(.)$ is strictly decreasing this implies $w - \beta p = w - L + \beta(1 - p) \Rightarrow \beta = L$.

When insurance is actuarially fair, the decision maker buys full insurance. The final wealth of the decision maker is $w - pL$, regardless of the occurrence of the loss.

Note that when $e = p$ the expected wealth of the decision maker is

$$(1 - p)(w - \beta p) + p(w - \beta p - L + \beta) = w - \beta p - pL + \beta p = w - pL$$

As this expected wealth is independent of β and it is reached with certainty when $\beta = L$, full insurance is selected by the risk averse decision maker.

ii) When insurance is not actuarially fair it is $e > p$. Note that $e > p \Rightarrow e - pe > p - ep \Rightarrow e(1-p) > p(1-e)$. In this case a possible solution is $\beta = 0$. However, if $\beta > 0$ it is

$$-e(1-p)u'(w - \alpha e) + p(1-e)u'(w - L + \beta(1-e)) = 0$$

$$\Rightarrow u'(w - L + \beta(1-e)) = \frac{e(1-p)}{p(1-e)}u'(w - \beta e) > u'(w - \beta e)$$

$$\Rightarrow w - L + \beta(1-e) < w - \beta e \Rightarrow \beta < L.$$

When insurance is not actuarially fair, the decision maker does not buy full insurance. The final wealth of the decision maker would be $w - \beta e$ if there were no loss and $w - \beta e - L + \beta$ if the loss occurred. His expected wealth is

$$(1-p)(w - \beta e) + p(w - \beta e - L + \beta) = w - pL - \beta(e - p)$$

This expected wealth is smaller than the expected wealth without insurance. However, the decision maker reduces the dispersion of his final wealth when he buys partial insurance.

When insurance is not actuarially fair, the decision maker will not buy full insurance even if his degree of risk aversion were very high. The reason is that for a very small level of risk, individual behavior towards risk approaches risk neutrality. Any risk averse decision maker prefers to retain some risk, buying partial insurance, and increase his wealth by saving in policy premium payment.

Exercise E: Insurance policies and second order stochastic dominance

The wealth of a decision maker is invested in an asset with actual value equal to 90000. This asset is subject to a random loss that has a uniform density in $[0, 90000]$. When the insurance policies are actuarially fair:

i) Which is the premium corresponding to a deductible of $D = 30000$?

ii) Which is the level of partial insurance that would induce the same premium?

iii) If there were a maximum indemnity I that the insurance company would pay, which value of I would yield the same premium as in i)?

iv) Which of the three policies will be preferred by a risk averter?

Solution

i) The maximum that the insurance company will pay in this case is 60000. Hence, the actuarially fair premium (the premium that is equal to the expected indemnity to be paid by the insurance company) will be $\int_{30000}^{90000} \frac{x-30000}{90000} dx = 20000.$

ii) Let α be the level of partial insurance, with $0 \leq \alpha \leq 1$, that the decision maker buys. It is

$$\int_0^{90000} \frac{\alpha x}{90000} dx = \frac{\alpha (90000)^2}{2(90000)} = 20000 \Rightarrow \alpha^* = \frac{4}{9}.$$

iii) It is

$$\int_0^I \frac{x}{90000} dx + \int_I^{90000} \frac{I}{90000} dx = -\frac{1}{180\,000} I^2 + I = 20000 \Rightarrow I^* = 22918.$$

iv) A risk averter prefers the deductible to partial insurance and he also prefers partial insurance to the policy with a maximum indemnity. To prove this let us compare the wealth distributions of the decision maker in each case. It occurs that the wealth distribution in the case of deductible second order stochastically dominates the wealth distribution with partial insurance, and the wealth distribution with partial insurance second order stochastically dominates the wealth distribution with the indemnity.

The distribution function of wealth of the decision maker with the deductible is

$$F_D = \begin{cases} 0 & x < 60000 \\ \frac{x}{90000} & 60000 \leq x \leq 90000 \end{cases}$$

The distribution function of wealth of the decision maker with partial insurance is

$$F_\alpha = \begin{cases} 0 & x < 90000(\frac{4}{9}) = 40000 \\ \frac{x-40000}{90000-40000} & 40000 \leq x \leq 90000 \end{cases}$$

The distribution function of wealth of the decision maker with the indemnity is

$$F_I = \begin{cases} 0 & x < 22918 \\ \frac{x-22918}{90000} & 22918 \leq x < 90000 \\ 1 & x = 90000 \end{cases}$$

From these distribution functions we have:

$$\int_0^x F_\alpha(t)dt - \int_0^x F_D(t)dt$$

$$= \begin{cases} 0 & x < 40000 \\ \int_{40000}^x \frac{t-40000}{90000-40000}dt & 40000 \leq x < 60000 \\ \int_{40000}^x \frac{t-40000}{90000-40000}dt - \int_{60000}^x \frac{t}{90000}dt & 60000 \leq x \leq 90000 \end{cases}$$

and

$$\int_0^x F_I(t)dt - \int_0^x F_\alpha(t)dt =$$

$$= \begin{cases} 0 & x < 22918 \\ \int_{22918}^{x} \frac{t-22918}{90000} dt & 22918 \leq x < 40000 \\ \int_{22918}^{x} \frac{t-22918}{90000} dt - \int_{40000}^{x} \frac{t-40000}{90000-40000} dt & 40000 \leq x < 90000 \\ \int_{22918}^{90000} \frac{t-22918}{90000} dt - \int_{40000}^{90000} \frac{t-40000}{90000-40000} dt = 0 & x = 90000 \end{cases}$$

Note that, for $60000 \leq x \leq 90000$, it is

$$\int_0^x F_\alpha(t)dt - \int_0^x F_D(t)dt = \int_{40000}^x \frac{t-40000}{90000-40000} dt - \int_{60000}^x \frac{t}{90000} dt$$

$$= \frac{1}{225\,000} x^2 - \frac{4}{5} x + 36\,000$$

and it is not difficult to prove that this expression is positive for $60000 \leq x < 90000$. Hence, from (3) in section 4.2, the deductible second order stochastically dominates partial insurance and a risk averter prefers the deductible to partial insurance.

We also have, for $40000 \leq x < 90000$:

$$\int_0^x F_I(t)dt - \int_0^x F_\alpha(t)dt = \int_{22918}^x \frac{t-22918}{90000} dt - \int_{40000}^x \frac{t-40000}{90000-40000} dt$$

$$= -\frac{1}{225\,000} x^2 + \frac{24\,541}{45\,000} x - \frac{588\,691\,319}{45\,000}$$

and it is not difficult to prove that this expression is positive for $40000 \leq x < 90000$. Hence, from (3) in section 4.2, partial insurance second order stochastically dominates a policy with a maximum indemnity and a risk averter prefers partial insurance to that fixed maximum indemnity.

Exercise F: Insurance and investment to reduce the probability of loss

A decision maker with v.N-M utility function $u(x) = \sqrt{x}$ has wealth equal to 10000 and runs a risk of a loss of 3600. The probability of this loss is 0.2.

i) The decision maker may buy insurance to cover totally this loss at a price m. Which is the maximum amount that she is willing to pay for that insurance?

ii) Consider now that the decision maker cannot buy insurance but she has the possibility of reducing the probability of loss to 0.1 by investing H in internal security against that risk. When $H = 600$ will the decision maker be willing to make that investment? If the probability of loss were $\pi(H)$, with $\pi'(H) < 0$, state the problem that the decision maker would solve to decide on the amount to invest in internal security.

Solution

i) She will buy the insurance if

$$0.8\sqrt{10000 - m} + 0.2\sqrt{10000 - m} > 0.8\sqrt{10000} + 0.2\sqrt{10000 - 3600}$$

i.e., if $\sqrt{10000 - m} > 96 \Leftrightarrow 10000 - m > 9216 \Leftrightarrow m < 10000 - 9216 = 784$. The expected indemnity of the insurer is $0.2(3600) = 720$. If $m > 720$ the insurer would obtain positive profits. The decision maker is willing to pay for the insurance more than the expected loss (720) because he is risk averse (he is willing to pay a positive risk premium to insure against the loss he faces).

ii) The investment H will be made when

$$0.9\sqrt{10000 - H} + 0.1\sqrt{10000 - H - 3600}$$

$$> 0.8\sqrt{10000} + 0.2\sqrt{10000 - 3600} = 96$$

When the required H were 600, the investment would not be made as $0.9\sqrt{10000 - 600} + 0.1\sqrt{10000 - 600 - 3600} = 94.874 < 96$. The maximum investment that the decision maker would be willing to make to reduce the probability of loss to 0.1 would be

$$0.9\sqrt{10000 - H} + 0.1\sqrt{10000 - H - 3600} = 96 \Rightarrow H^{*} = 386.17.$$

In general, we can consider that the probability of the loss depends on the amount invested (amount of costly effort incurred to reduce that probability): $p(H)$, with $p'(H) < 0$ and $p''(H) > 0$. With a probability of loss given by the function $p(H)$ the decision maker would solve

$$\max_{H}(1 - p(H))\sqrt{10000 - H} + p(H)\sqrt{10000 - H - 3600}.$$

Exercise G: Insurance policies with several independent risks

A decision maker owns two assets. Each asset has a value of 100 and is subject to risk of full loss with probability 0.2. However, the two risks are independent. The decision maker may buy actuarially fair insurance premia to cover the risks.

i) If the decision maker had a budget of 18 to spend on insurance premia in order to cover (partially) the risks, analyze how the insurance budget would be distributed between the two risks.

ii) Consider that the decision maker is offered full insurance with a deductible of 55 for each risky asset. Calculate the premium and the final wealth with these deductibles.

iii) Consider that the decision maker is offered full insurance with a joint deductible on the aggregate loss for the same premium as in ii). Which will be the amount of this joint deductible on the aggregate loss? Will the decision maker prefer this joint deductible to the two independent deductibles of ii)? Why?

Solution

i) Let us denote by α_i the proportion of the loss in asset i that is insured by the decision maker, with $i = 1, 2$. The insurance premium that the decision maker will have to pay to insure asset i is $0.2\alpha_i(100) = 20\alpha_i$, with $i = 1, 2$. As insurance is actuarially fair the decision maker will spend all his insurance budget (insurance reduces dispersion without changing the expected wealth of the decision maker): $20\alpha_1 + 20\alpha_2 = 18 \Rightarrow \alpha_2 = 0.9 - \alpha_1$. Let us consider without loss of generality that $\alpha_1 \geq \alpha_2$. The expected wealth of the decision maker is

$$0.04(100\alpha_1 + 100\alpha_2) + 0.16(100 + 100\alpha_2) + 0.16(100\alpha_1 + 100) + 0.64(200) - 18$$

$$= 0.04(90) + 0.32(145) + 0.64(200) - 18 = 160$$

and it is independent of α_1 and α_2. However, the dispersion of the wealth of the consumer depends on α_1 and α_2.

The distribution function of the wealth of the decision maker is

$$F(x, \alpha_1, \alpha_2) = \begin{cases} 0 & x < 100\alpha_1 + 100\alpha_2 - 18 = 72 \\ 0.04 & 72 \leq x < 82 + 100\alpha_2 \\ 0.20 & 82 + 100\alpha_2 \leq x < 82 + 100\alpha_1 \\ 0.36 & 82 + 100\alpha_1 \leq x < 200 - 18 = 182 \\ 1 & x \leq 182 \end{cases}$$

and depends on the pair (α_1, α_2). A set of distributions is obtained from all pairs (α_1, α_2) such that $\alpha_1 + \alpha_2 = 0.9$. Among these distributions, we have from (3) in section 4.2 that partial insurance with proportions $\alpha_1 = \alpha_2 = \frac{0.9}{2}$ second order stochastically dominates all other partial insurance policies with pairs (α_1,α_2) such that $\alpha_1 + \alpha_2 = 0.9$ and $\alpha_1 > \alpha_2$. The distribution corresponding to any situation where $\alpha_1 > \alpha_2$ is second order stochastically dominated by the distribution corresponding to a situation with $\hat{\alpha}_1$ and $\hat{\alpha}_2$ such that $\hat{\alpha}_1 = \alpha_1 - \varepsilon$, $\hat{\alpha}_2 = \alpha_2 + \varepsilon$ and $\varepsilon \leq \frac{\alpha_1 - \alpha_2}{2}$.

Hence, a risk averse decision maker will select $\alpha_1 = \alpha_2 = \frac{0.9}{2} = 0.45$.

ii) The total premium for full insurance with those deductibles is $2(0.2(100 - 55)) = 18$. The distribution of final wealth of the decision maker is

$$F(x,55,55)=\begin{cases} 0 & x<2(45)-18=72 \\ 0.04 & 72\leq x<100+45-18=127 \\ 0.36 & 127\leq x<200-18=182 \\ 1 & x\leq 182 \end{cases}$$

iii) The joint deductible D will satisfy $0.04(200-D)+0.32(100-D)=18 \Rightarrow D=61.111$.

With this joint deductible the expected wealth of the decision maker is

$$0.36(200-61.111)+0.64(200)-18=160$$

The distribution of final wealth of the decision maker is

$$F(x,61.111)=\begin{cases} 0 & x<200-61.111-18=120.89 \\ 0.36 & 120.89\leq x<200-18=182 \\ 1 & x\leq 182 \end{cases}$$

Hence, from (3) in section 4.2 we have that this joint deductible second order stochastically dominates the independent deduction for each risky asset of ii). Hence, a risk averse decision maker will prefer this joint deductible to the independent deduction for each risky asset of ii).

5.3 Investment in risky assets and portfolio selection

Exercise H: Value of risky assets. Investment in risky assets.

A decision maker has v.N-M utility function $u(x)=\sqrt{x}$ and wealth $w=500$.

i) If the decision maker accepts the lottery (100,-100; $p,1-p$), which is the minimum value of p?

ii) If $p=\frac{2}{3}$ and the decision maker owns the lottery (100,-100; $p,1-p$), what is the minimum price he will sell it for?

iii) If $p = \frac{2}{3}$ and the decision maker does not own the lottery (100,-100; $p, 1-p$), what is the maximum price he will be willing to pay for it?

iv) Which is the minimum value of H required to make the lottery $(H, -100;\ p, 1-p)$ acceptable?

v) What is the minimum amount M that has to be paid to the decision maker to induce him to accept the lottery (100,-100;$\frac{1}{2}, 1-\frac{1}{2}$)?

vi) Determine the amount s that this decision maker would invest in a risky asset that pays $2s$ with probability $\frac{1}{2}+\gamma$ and 0 with probability $\frac{1}{2}-\gamma$, where $0 < \gamma < \frac{1}{2}$. Explain the variation of s with γ.

vii) How would a change in the level of wealth of the decision maker affect the values obtained in your answers to questions i) to vi)?

Solution

i) It must be $p\sqrt{600} + (1-p)\sqrt{400} > \sqrt{500} \Rightarrow p > \frac{\sqrt{500}-\sqrt{400}}{\sqrt{600}-\sqrt{400}} = 0.525.$

ii) The decision maker will be willing to sell the lottery at any price S such that $\sqrt{500+S} \geq \frac{2}{3}\sqrt{600} + \frac{1}{3}\sqrt{400}$. The minimum selling price will be such that $\sqrt{500+S} = \frac{2}{3}\sqrt{600} + \frac{1}{3}\sqrt{400} \Rightarrow S = 28.84$ (at this price the decision maker is indifferent between selling and not selling the lottery).

iii) The decision maker will be willing to buy the lottery at any price B such that $\sqrt{500} \leq \frac{2}{3}\sqrt{600-B} + \frac{1}{3}\sqrt{400-B}$. The maximum price he would be willing to pay for the lottery will be such that $\sqrt{500} = \frac{2}{3}\sqrt{600-B} + \frac{1}{3}\sqrt{400-B} \Rightarrow B = 28.565$.

Note that as $28.565 < 28.84$, an owner of the lottery with $w = 500$ would not be able to sell the lottery to a buyer that also has $w = 500$.

iv) The lottery is acceptable if $p\sqrt{w+H} + (1-p)\sqrt{w-100} > \sqrt{w}$. The minimum value of H that makes the lottery acceptable is such that $p\sqrt{w+H} + (1-p)\sqrt{w-100} = \sqrt{w} \Rightarrow H = -w + \frac{1}{p^2}\left(\sqrt{w} - (1-p)\sqrt{w-100}\right)^2$. If $w = 500$ it is $H = \frac{1}{p^2}(20p + 2.361)^2 - 500$.

v) As $w = 500$ it is
$\frac{1}{2}\sqrt{500+100+M} + (1-\frac{1}{2})\sqrt{500-100+M} = \sqrt{500} \Rightarrow M = 5$.

vi) This is particular case of the problem discussed in section 5.1.1 where $z_1 = 1$, $z_2 = -1$ and $r = 0$, and, hence, $(\frac{1}{2}+\gamma)z_1 + (\frac{1}{2}-\gamma)z_2 > r$ and $z_1 > r > z_2$, as required. From that section we know that the first order condition for an interior solution is

$$\frac{\frac{1}{2}+\gamma}{2\sqrt{500+s}} - \frac{\frac{1}{2}-\gamma}{2\sqrt{500-s}} = 0$$

and the interior solution is $s^* = \frac{2000}{1+4\gamma^2}\gamma$. Note that $\gamma > 0 \Rightarrow s^* > 0$ (this is a general result: if a risk is actuarially favorable and the decision maker may decide the amount of the risky asset to buy, then a risk averter will always accept at least a small amount of it)[15] Moreover, $\frac{ds^*}{d\gamma} > 0$.

vii) The v.N-M utility function $u(x) = \sqrt{x}$ implies decreasing absolute risk aversion (when w increases the decision maker is willing to accept more risks).[16]

The minimum value of p in part i) decreases with wealth (when w increases the decision maker is willing to accept more risks, i.e., to accept a lottery with smaller winning probability in this case). To accept the lottery it must be

$$p\sqrt{w+100} + (1-p)\sqrt{w-100} > \sqrt{w} \Rightarrow p > \frac{\sqrt{w} - \sqrt{w-100}}{\sqrt{w+100} - \sqrt{w-100}}$$

It is easy to check that $\frac{dp}{dw} < 0$.

[15] See, for instance, Mas-Colell et al. (1995), Example 6.C.2.

[16] I refer to risks that imply absolute gains and losses from current wealth

The minimum selling price in part ii) increases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case his willingness to get rid of any risk is smaller and he only accepts to get rid of the risk for a higher selling price). For instance, if $w = 600$ the minimum selling price is such that $\sqrt{600+S} = \frac{2}{3}\sqrt{700} + \frac{1}{3}\sqrt{500} \Rightarrow S = 29.604$ and if $w = 800$ it is $\sqrt{800+S} = \frac{2}{3}\sqrt{900} + \frac{1}{3}\sqrt{700} \Rightarrow S = 30.545$.

The maximum buying price of part iii) increases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he is willing to pay more to participate in the lottery proposed). For instance, if $w = 600$ the maximum buying price is such that $\sqrt{600} = \frac{2}{3}\sqrt{700-B} + \frac{1}{3}\sqrt{500-B} \Rightarrow B = 29.408$ and if $w = 800$ it is $\sqrt{800} = \frac{2}{3}\sqrt{900-B} + \frac{1}{3}\sqrt{700-B} \Rightarrow B = 30.433$.

The value of H in iv) decreases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he is willing to accept a lottery that pays less in the case of good outcome). From iv) it may be shown that $\frac{dH}{dw} < 0$.

The value of M in v) decreases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he has to be paid less to accept the lottery). From v) it is

$$\frac{1}{2}\sqrt{w+M+100} + (1-\frac{1}{2})\sqrt{w+M-100} = \sqrt{w}$$

and it may be shown that $\frac{dM}{dw} < 0$.

From vi) the first order condition for an interior solution is (see section 5.1.1) $\frac{\frac{1}{2}+\gamma}{2\sqrt{w+s}} - \frac{\frac{1}{2}-\gamma}{2\sqrt{w-s}} = 0$ and the interior solution is $s^* = \frac{4w}{1+4\gamma^2}\gamma$. We have that s increases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he is willing to invest more in the risky asset).

Exercise I: Degree of risk aversion and investment in a risky asset

A decision maker with wealth w and v.N-M utility function $u(x) = \ln(x)$ must decide the amount s she will invest in a lottery, or risky asset, with a probability p of receiving y_1 per unit invested (final wealth $w + s(y_1 - 1)$) and a probability $1-p$ of receiving y_2 per unit invested (final wealth $w+s(y_2-1)$), with $y_1 > y_2$. The wealth not invested in the risky asset is invested in a safe or riskless asset that pays always 1 per unit invested. Solve for z in the general case and analyze how z changes with the parameters of the problem. Obtain z when $p = 0.5$, $z_1 = 3$ and $z_2 = 0$. Analyze how z depends on w.

Solve again the exercise considering that the v.N-M utility function is, instead, $u(x) = \sqrt{x}$. Use the coefficients of absolute risk aversion to explain why the decision maker invests more in the risky asset with one utility function than with the other.

Solution

This is a particular case of the problem discussed in section 5.1.1 where $z_1 = y_1 - 1$, $z_2 = y_2 - 1$ and $r = 0$. From that section we have that $p(y_1 - 1) + (1 - p)(y_2 - 1) > 0$ is a necessary condition for an strictly positive investment in the risky asset and $y_1 - 1 > 0 > y_2 - 1$ is a necessary condition for an strictly positive demand of the safe, or riskless, asset.

From section 5.1.1 the first order condition for an interior solution is

$$\frac{p(y_1 - 1)}{w + s(y_1 - 1)} + \frac{(1 - p)(y_2 - 1)}{w + s(y_2 - 1)} = 0$$

and the interior solution is $s^* = w\frac{(py_1+(1-p)y_2)-1}{(y_1-1)(1-y_2)} > 0$. The solution will be $s^* = w\frac{(py_1+(1-p)y_2)-1}{(y_1-1)(1-y_2)}$ if $\frac{(py_1+(1-p)y_2)-1}{(y_1-1)(1-y_2)} < 1$ and $s^{**} = w$ (corner solution) if $\frac{(py_1+(1-p)y_2)-1}{(y_1-1)(1-y_2)} \geq 1$.

The interior solution s^* increases with p and it also increases with w (the v.N-M utility function implies decreasing absolute risk aversion; hence, when

w increases the decision maker is willing to accept more risks, i.e., in this case she is willing to invest more in the risky asset, or to keep less money in the riskless asset). However, we know that this utility function implies constant relative risk aversion (equal to 1). Therefore, the decision maker invests the same proportion of her wealth in the risky asset, independently of her level of wealth: that proportion is $\frac{(py_1+(1-p)y_2)-1}{(y_1-1)(1-y_2)}$ of his wealth in this case. Moreover, $\frac{ds^*}{dy_1} > 0$ and $\frac{ds^*}{dy_2} > 0$.

When $p = 0.5$, $y_1 = 3$ and $y_2 = 0$, it is $s = \frac{w}{4}$. The decision maker invests $\frac{1}{4}$ of her wealth in the risky asset and maintains $\frac{3}{4}$ of her wealth in the safe, or riskless, asset.

When $u(x) = \sqrt{x}$, from section 5.1.1 we know that the first order condition for an interior solution is

$$\frac{p(y_1-1)}{2\sqrt{w+s(y_1-1)}} + \frac{(1-p)(y_2-1)}{2\sqrt{w+s(y_2-1)}} = 0$$

and the interior solution is

$$s^* = \frac{w\left(py_1+1-y_2-2p+py_2\right)\left((py_1+(1-p)y_2)-1\right)}{\left(y_1-1\right)\left(1-y_2\right)\left(-y_2+1+2py_2-2p-p^2y_2+p^2y_1\right)}.$$

In this case we may have a corner solution $s = 0$. When $y_1 = 3$ and $y_2 = 0$ it is $s = \frac{1}{2}w\frac{-1+2p+3p^2}{1-2p+3p^2}$. When $p < \frac{1}{3}$ the solution will be $s = 0$, as $p < \frac{1}{3} \Leftrightarrow \frac{-1+2p+3p^2}{1-2p+3p^2} < 0$.

If $p = 0.5$, $y_1 = 3$ and $y_2 = 0$, it is $s = \frac{w}{2}$. We have that s increases with w (the v.N-M utility function implies decreasing absolute risk aversion; hence, when w increases the decision maker is willing to accept more risks, i.e., in this case she is willing to invest more in the risky asset, or to keep less money in the riskless asset). However, we know that this utility function implies constant relative risk aversion (equal to $\frac{1}{2}$); hence, the decision maker invests the same proportion of her wealth in the risky asset, independently of her level of wealth: that proportion is $\frac{1}{2}$ in this case).

The decision maker invests less in the risky asset when $u(x) = \ln x$ because a decision maker with $u(x) = \ln x$ is more risk averse than a decision maker with $u(x) = \sqrt{x}$, as $r_A(x, \ln x) = \frac{1}{x} > \frac{1}{2x} = r_A(x, \sqrt{x})$.

Exercise J: Portfolio selection (1)

A decision maker with v.N-M utility function $u(x) = \ln(x)$ wants to invest an amount of wealth equal to w at time 0. She must decide the amount s to invest in a risky asset that pays $1 + z_1$ at time 1 with probability p per unit invested and $1 + z_2$ at time 1 with probability $1 - p$ per unit invested, where $z_1 > z_2$. The wealth $w - s$ not invested in the risky asset is invested in a government bond that at time 1 pays always $1+r$ per unit invested. Consider that there is no discounting of the future. Solve for s in the general case and analyze how s changes with p and with w. Obtain s when $p = 0.5$, $r = 0.1$, $z_1 = 0.3$ and $z_2 = -0.06$. If the v.N-M utility function were $u(x) = \sqrt{x}$, would s be greater than when the v.N-M utility function is $u(x) = \ln x$? Why?

Solution

From 5.1.1 the first order condition for an interior solution is

$$\frac{p(z_1 - r)}{w + sz_1 + r(w - s)} + \frac{(1 - p)(z_2 - r)}{w + sz_2 + r(w - s)} = 0$$

and the interior solution is $s^* = w\frac{(1+r)(pz_1+(1-p)z_2-r)}{(z_1-r)(r-z_2)} > 0$. The solution will be $s^* = w\frac{(1+r)(pz_1+(1-p)z_2-r)}{(z_1-r)(r-z_2)}$ if $\frac{(1+r)(pz_1+(1-p)z_2-r)}{(z_1-r)(r-z_2)} < 1$ and $s^{**} = w$ (corner solution) if $\frac{(1+r)(pz_1+(1-p)z_2-r)}{(z_1-r)(r-z_2)} \geq 1$.

The interior solution s^* increases with p and it also increases with w (the v.N-M utility function implies decreasing absolute risk aversion; hence, when w increases the decision maker is willing to accept more risks, i.e., in this case she is willing to invest more in the risky asset, or to keep less money in the riskless asset). However, we know that this utility function implies constant relative risk aversion (equal to 1). Therefore, the decision maker invests the same proportion of her wealth in the risky asset, independently of her level of wealth: that proportion is $\frac{(1+r)(pz_1+(1-p)z_2-r)}{(z_2-r)(r-z_1)}$ of his wealth in this case. Moreover,$\frac{ds^*}{dz_1} > 0$ and $\frac{ds^*}{dz_2} > 0$.

When $p = 0.5$, $r = 0.1$, $z_1 = 0.3$ and $z_2 = -0.06$ it is $s^* = .687\,5w$ (note that it is $pz_1 + (1-p)z_2 > r$ and $z_1 > r > z_2$ as required: see section 5.1.1) The decision maker invests 68.75% of her wealth in the risky asset and 31.25% of her wealth in bonds.

When the v.N-M utility function is $u(x) = \sqrt{x}$, the decision maker will invest more in the risky asset because a decision maker with $u(x) = \ln x$ is more risk averse than a decision maker with $u(x) = \sqrt{x}$, as $r_A(x, \ln x) = \frac{1}{x} > \frac{1}{2x} = r_A(x, \sqrt{x})$.

Exercise K: Portfolio selection (2)

A decision maker with wealth w and v.N-M utility function $u(x) = \ln(x)$ wants to invest an amount of wealth equal to w at time 0. She must decide the amount z she will invest in a lottery, or risky asset, with a probability p of winning t_1% at time 1 and a probability $1-p$ of winning t_2% at time 1, where $t_1 > t_2$. The wealth $w - z$ not invested in the risky asset is invested in a government bond that at time 1 pays always $1 + r$ per unit invested. Solve for z in the general case and analyze how z changes with the parameters of the problem. Obtain z when $p = 0.6$, $t_1 = 0.3$ and $t_2 = -0.2$. Analyze how z depends on w.

Solve again the exercise considering that the v.N-M utility function is, instead, $u(x) = \sqrt{x}$. Use the coefficients of absolute risk aversion to explain why the decision maker invests more in the risky asset with one utility function than with the other.

Solution

Proceed as in the previous exercise (it is the same situation).

5.4 Other applications

5.4.1 Risk sharing

Exercise L: Risk sharing

Investor A has wealth equal to 30000 and v.N-M utility function $u(x) = \sqrt{x}$. Will this investor invest all her wealth (30000) in a project that returns 0 with probability 1/2 (the initial outlay is lost) and returns 110000 with probability 1/2? If there is another investor B with the same wealth and v.N-M utility function as investor A, will investors A and B want to share the project (sharing means that each investor puts up 15000 and they split the proceeds of the investment)?

Solution

Investor A will not invest in the project as
$\sqrt{30000} - (\frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{110000}) = 7.3738 > 0.$

Investors A and B will want to share the project as
$\sqrt{30000} - (\frac{1}{2}\sqrt{15000} + \frac{1}{2}\sqrt{15000 + 55000}) = -20.320 < 0.$
Risk sharing makes the project feasible.

5.4.2 Taxes and income underreporting

Exercise M: Taxes and income underreporting

The income of a risk averse individual is taxed at a rate t. He has earned some extra income in an amount equal to y and he is considering not to report that extra income to avoid the corresponding tax payment. If he is caught underreporting his income he will have to pay αt for every unit of income he failed to report, with $\alpha > 1$ (the taxes owed plus a fine). If he underreports his income, the probability of being caught is p. Answer the following questions, in the context of the expected utility theory:

i) To avoid underreporting of earned income the government is considering two policies: an increase of a 10% in p and an increase of 10% in α. Which policy has more possibilities of reducing underreporting by the individual we have considered?

ii) If the v.N-M utility function of the individual is $u(x) = \ln(x)$, $y = 10000$, $t = 0.40$ and the initial wealth of the individual is 0, obtain the amount of extra income that the individual will fail to report as a function of α and p.

iii) Consider that the individual has initial wealth equal to w. Otherwise, the situation is the same as in ii). How does the extra income that the individual will fail to report depends on w? Obtain that extra income.

Solution

i) Consider that the individual, in case of income underreporting, does not report his extra income at all. The cases where the individual may underreport a fraction of his extra income will be considered in ii) and iii) below. With the old levels of the policies the expected net extra income of the individual, if he decided not to report his extra income, was: $(1-p)y + p(y - \alpha t y) = y - p\alpha t y$. With the two new levels of the policies the expected net extra income of the individual, if he decided not to report his extra income, would be: $y - 1.1p\alpha t y$ (the same for the two new levels of the policies). However, when there is a 10% increase in p the extra income of the individual if he is caught underreporting will be $y - \alpha t y$, and when there is a 10% increase in α the extra income of the individual if he is caught underreporting will be $y - 1.1\alpha t y$. We know that a risk averse individual who faces two alternatives with the same expected gains selects the alternative with less dispersion of outcomes. Hence, the individual in this case prefers a 10% increase in p. As a consequence, the government must select a 10% increase in α as this policy will have more possibilities of reducing underreporting.[17] If the individual were risk neutral, the effect on income disclosure of the two policy changes would be the same.

ii) Let s be the amount of extra income that the individual will fail to report. If he is not caught underreporting, his final wealth will be $(y-s)(1-0.4)+s = y(1-0.4)+0.4s$. If he is caught underreporting, his final wealth will be $(y-s)(1-0.4)+s-0.4\alpha s = y(1-0.4)+0.4(1-\alpha)s$. Therefore, this situation is a particular case of the problem discussed in section 5.1.1 where $w = y(1-0.4)$, $z_1 = 0.4$, $z_2 = 0.4(1-\alpha)$, $r = 0$, and the risky alternative (underreporting the income y) pays $1+z_1$ with probability $1-p$ per unit invested and $1+z_2$ with probability p per unit invested (the individual faces a risky gamble that provides a net gain of 0.4 per unit invested with probability $1-p$ and a net loss of $0.4(\alpha-1)$ per unit invested with probability p). Note that $z_1 > r > z_2$, as required, and that $(1-p)z_1 + (p)z_2 > r \Leftrightarrow \alpha p < 1$.

In this case the individual will solve

$$\max_z (1-p)\ln(6000+0.4s) + p\ln(6000+0.4(1-\alpha)s)$$

subject to $z \leq y$. From the first order condition the interior solution is $z^* = \frac{15000(1-\alpha p)}{\alpha-1}$. Note that $z^* > 0 \Leftrightarrow \alpha p < 1$. This interior solution requires $0 < \frac{15000(1-\alpha p)}{\alpha-1} < y$. If $\alpha p \geq 1$ the individual will report all his income. If $\frac{15000(1-\alpha p)}{\alpha-1} \geq y$, the individual will not report any income to the tax administration. At the interior solution it is $\frac{\partial z^*}{\partial p} < 0$ and $\frac{\partial z^*}{\partial \alpha} = 15000\frac{p-1}{(-1+\alpha)^2} < 0$.

Let us compare, as in i), the case where p is increased in 10% and the case where α is increased in 10%. The interior solutions in those cases will be $z^*(1.1p,\, \alpha) = \frac{15000(1-1.1\alpha p)}{\alpha-1}$ and $z^*(p, 1.1\alpha) = \frac{15000(1-1.1\alpha p)}{1.1\alpha-1}$. As $z^*(1.1p,\, \alpha) > z^*(p, 1.1\alpha)$ the government prefers a 10% increase in α, as in i).

iii) When the individual has initial wealth equal to w, he will solve

$$\max_z (1-p)\ln(w+6000+0.4z) + p\ln(w+6000+0.4(1-\alpha)z)$$

[17] If the costs of implementation of the two policies were different, the government might take into account the difference in implementation costs in the selection of the policy to reduce income underreporting.

From the first order condition the solution is $z^* = \frac{(2.5w+15000)(1-\alpha p)}{\alpha-1}$. Note that $z^* > 0 \Leftrightarrow \alpha p < 1$. Moreover, $\frac{\partial z^*}{\partial p} < 0$ and $\frac{\partial z^*}{\partial \alpha} = (2.5w + 15000)\frac{p-1}{(-1+\alpha)^2} < 0$. Finally, $\frac{\partial z^*}{\partial w} > 0$ if $\alpha p < 1$: the individual will fail to report more extra income when his wealth increases. As this individual has a v.N-M utility function with decreasing absolute risk aversion, he will invest more in the risky gamble (he will underreport more income and take the risk to pay a greater fine) as his wealth increases.

The result of the comparison of an increase of a 10% in p and an increase of 10% in α is as in part ii).

5.4.3 Deposit insurance

Exercise N: Bank solvency and deposit insurance

A decision maker with v.N-M utility function $u(x) = \sqrt{x}$ has a deposit of 20000 in a bank. The depositor thinks that a year from now the bank will be solvent with a probability of 0.99. The deposit could be withdrawn from the bank at a cost of 100 at any time. The depositor is insured for γ per cent of the deposit. Study the decision to withdraw the deposit as a function of γ.

Solution

The decision maker will withdraw the deposit if

$$\sqrt{20000 - 100} > 0,99\sqrt{20000} + 0.01\sqrt{\gamma 20000}$$

$$\Leftrightarrow 141,07 > 140,01 + 1,414\sqrt{\gamma} \Leftrightarrow \gamma < 0.562,$$

and he will maintain the deposit if $\gamma \geq 0.562$.

5.4.4 Bank loan

Exercise O: Bank loan and collateral

An investor with v.N-M utility function $u(x) = \sqrt{x}$ may undertake a project that requires an initial investment of 80000, and that returns 0 with probability 1/2 (the initial investment is lost) and 200000 with probability 1/2. The investor only owns A in assets, with $A < 80000$. She may borrow 80000 from a bank to finance the project. The bank requires the investor to bring A as collateral (the situation would be the same if the investor borrows $80000 - A$ from the bank and the bank does not recover the loan if the project fails). Which is the maximum interest rate that this investor will be willing to pay to the bank?

Solution

The maximum interest rate that the investor will be willing to pay to the bank will solve

$$0.5\sqrt{0} + 0.5\sqrt{A + 200000 - 80000(1+r)} = \sqrt{A} \Leftrightarrow r = 1.5 - 3.75 \times 10^{-5} A$$

As r decreases with A, the maximum interest rate that the investor is willing to pay to the bank diminishes with the level of the collateral. However, we have

$$r = 1.5 - 3.75 \times 10^{-5} A > 0 \Leftrightarrow A < 40000$$

Hence, the investor would not undertake the project (as she would not be willing to pay a positive interest rate for a bank loan) when $A > 40000$. Risk aversion and limited responsibility on the side of the investor explain that she is willing to undertake the project only when $A < 40000$.

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