Sarriko-On

Imperfect Competition

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Notes on

Imperfect Competition

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Chapter 1. **Monopoly**

Introduction

We say that a firm is a monopoly if it is the only seller of a good (or goods) in a market.

Problem: it is not easy to define *good* and *market*.

A firm may become a monopoly by various reasons:

- Control over raw materials.
- Acquisition of the exclusive selling rights (by a patent, by a public auction etc.).
- Better access to the capital market.
- Increasing returns of scale etc.

In contrast with a perfectly competitive firm which faces a perfectly elastic demand (taking price as given), a monopolist faces the market demand. Therefore, a firm with monopolistic power in a market it is aware of the amount of output that it is be able to sell it is a continuous function of the price charged. Put differently, the monopolistic firm takes into account that a reduction in output will increase the price that can be charged. In consequence, a monopolist has the power to set the market price. While we can consider a competitive firm as a "price taker", a monopolist is price decision-maker or price setter.

1.1. **Profit maximization**

(i) The problem of profit maximization in prices and in quantities. First order conditions. Second order conditions. A graphical interpretation of the profit maximization problem.

(ii) Interpretation of marginal revenue.

(iii) Marginal revenue equals marginal cost condition.

(iv) Output and demand elasticity.

(v) Lerner Index of monopolistic power.

(vi) Graphical analysis.

(vii) Second order conditions.

(i) *The problem of profit maximization in prices and in quantities*

There are two types of constraint that restrict the behaviour of a monopolist:

a) Technological constraints summarized in the cost function $C(x)$.

b) Demand constraints: $x(p)$.

We can write the profit function of the monopolist in two alternative ways:

- $\Pi(p) = px(p) - C(x(p))$ by using the demand function.

- $\Pi(x) = p(x)x - C(x)$ by using the inverse demand function.

The demand, x (p), and the inverse demand, $p(x)$, represent the same relationship between price and demanded quantity from different points of view. The demand function is a complete description of demanded quantity at each price whereas the inverse demand gives us the maximum price at which a given output x may be sold in the market.

$$\begin{array}{ccc} \max\limits_{p} \Pi(p) & & \max\limits_{x \geq 0} \Pi(x) \\ \Downarrow p^m & \equiv & \Downarrow x^m \\ x^m = x(p^m) & & p^m = p(x^m) \end{array}$$

The problem of profit maximization as a function of price

$$\max_{p} \Pi(p) \equiv \max_{p} px(p) - C(x(p))$$

$$\Pi'(p) = x(p) + px'(p) - C'(x(p))x'(p) = 0$$

$$\Pi''(p) = 2x'(p) + px''(p) - C''(x(p))\left[x'(p)\right]^2 - C'(x(p))x''(p) < 0$$

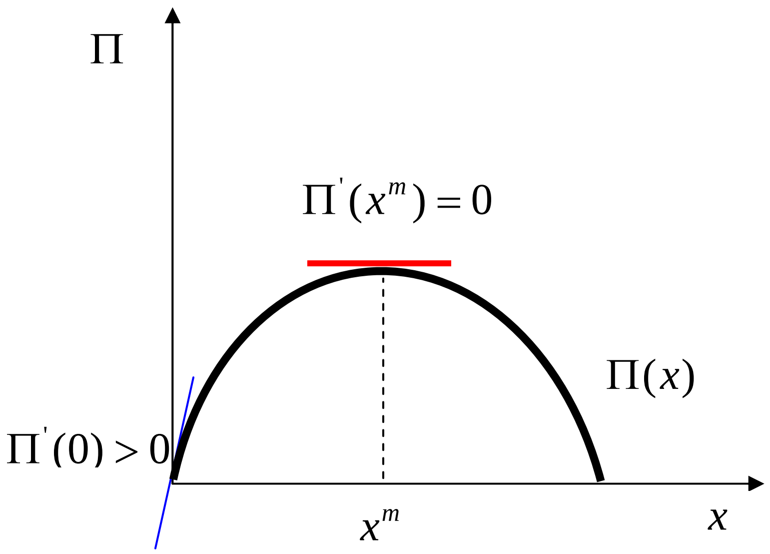
The problem of profit maximization as a function of the output

$$\max_{x\geq 0} \Pi(x) \equiv \max_{x\geq 0} p(x)x - C(x)$$

$$\Pi'(0) = p(0) - C'(0) > 0 \Rightarrow p(0) > C'(0)$$

$\Pi'(x) = p(x) + xp'(x) - C'(x) = 0 \Leftrightarrow \Pi'(x^m) = 0$ First order condition.

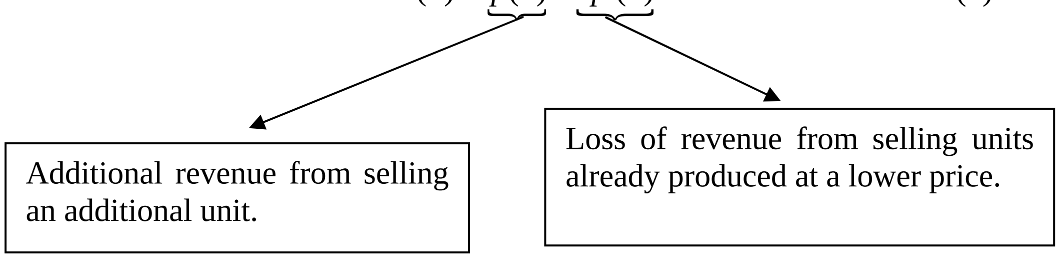
$\Pi''(x) = 2p'(x) + xp''(x) - C''(x) < 0$ Strictly concave profit function (regular case).



(ii) *Interpretation of marginal revenue*

Marginal revenue, $r'(x)$, is:

$$r'(x) = \underbrace{p(x)} + \underbrace{xp'(x)} \qquad (1)$$



(iii) *Marginal revenue equals marginal cost condition*

The profit-maximizing output level (interior solution) satisfies:

$$\Pi'(x^m) = r'(x^m) - C'(x^m) = p(x^m) + xp'(x^m) - C'(x^m) = 0 \qquad (2)$$

At the monopolistic optimal output the marginal profit is zero, $\Pi'(x^m) = 0;$ that is, an infinitesimal change in the level of output maintains profit unchanged. An output level such that $\Pi'(.) > 0$ does not maximize profits: an (infinitesimal) increase in output would increase profits. In a similar way, a level of output such that $\Pi'(.) < 0$ does not maximize profits: a (infinitesimal) decrease in output would increase profits.

At the profit-maximizing level of output marginal revenue equals marginal cost, $r'(x^m) = C'(x^m);$ that is, an infinitesimal change in the level of output changes revenue and cost equally. (In other words, an infinitesimal increase in the level of output increases revenue and cost by the same amount and an infinitesimal decrease in the level of output reduces revenue and cost by the same amount). An output level such that $r'(.) > C'(.)$ does not maximize profits: an (infinitesimal) increase in output would increase revenue more than cost (therefore increasing profits). Likewise, a level of output such that $r'(.) < C'(.)$ does not maximize profits: a (infinitesimal) decrease in output would reduce cost more than revenue (therefore increasing profits).

(iv) *Output and elasticity:* $|\varepsilon(x)| \geq 1$

We seek to show that at the monopoly output the price-elasticity of demand is 1 or more. First, we define the price-elasticity of demand in absolute value:

- as a function of price: $\left|\varepsilon(p)\right| = -x^{'}(p)\dfrac{p}{x(p)}$, (3)

- as a function of output : $\left|\varepsilon(x)\right| = -\dfrac{1}{p^{'}(x)}\dfrac{p(x)}{x}$. (4)

We next represent marginal revenue as a function of the price-elasticity of demand:

$$r^{'}(x) = p(x) + xp^{'}(x) \qquad (5)$$

$$r^{'}(x) = p(x)\left[1 + x\frac{p^{'}(x)}{p(x)}\right] \qquad (6)$$

$$r^{'}(x) = p(x)\left[1 - \frac{1}{\left|\varepsilon(x)\right|}\right] \qquad (7)$$

In the monopoly output marginal revenue and marginal cost are equal:

$$r^{'}(x) = p(x)\left[1 - \frac{1}{\left|\varepsilon(x)\right|}\right] = C^{'}(x). \quad (8)$$

Given that the marginal cost is by definition non-negative (zero or more) then the marginal revenue must be non-negative. This occurs when the price-elasticity of demand in absolute value is 1 or more. That is:

$$C^{'}(x) \geq 0 \Rightarrow p(x)\left[1 - \frac{1}{\left|\varepsilon(x)\right|}\right] \geq 0 \underset{p(x)\geq 0}{\Rightarrow} \left|\varepsilon(x)\right| \geq 1.$$

(v) *Lerner index of market power*

Now we obtain the Lerner index of monopoly power (or market power) also called the relative price-marginal cost margin. From condition (8) we obtain:

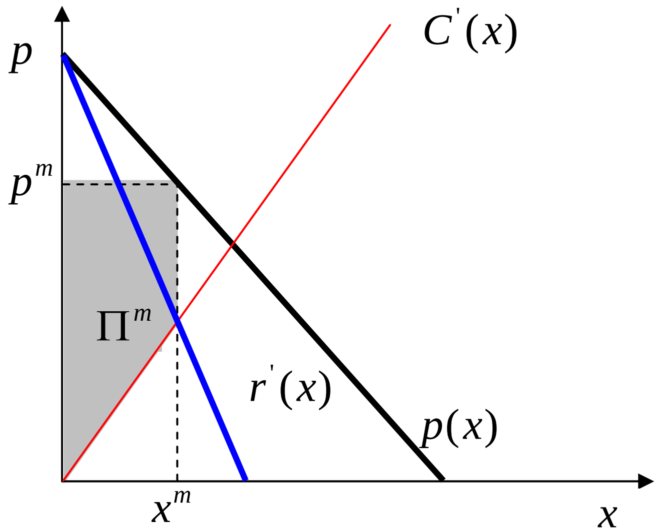
$$p(x) - \frac{p(x)}{\left|\varepsilon(x)\right|} = C^{'}(x).$$

By rearranging we get:

$$\frac{p(x)-C'(x)}{p(x)}=\frac{1}{|\varepsilon(x)|}. \quad (9)$$

Therefore, the Lerner index is a decreasing function of the price-elasticity of demand in absolute value. In particular, when $|\varepsilon(x)|=0$ monopoly power would be $\frac{p(x)-C'(x)}{p(x)}=\infty$ and when $|\varepsilon(x)|=\infty$ (as it would occur if the firm behaved as a perfectly competitive firm) market power would be zero, $\frac{p(x)-C'(x)}{p(x)}=0.$.

(vi) *Graphical representation*



Marginal revenue, $r'(x)=p(x)+xp'(x)$, is located below inverse demand given that the inverse demand function is downward sloping, $p'(x)<0$. That is, $r'(x)<p(x)$ for $x>0$, but both functions have the same intercept, $r'(0)=p(0)$. The profit of the monopolist (when there is no fixed cost) is given by:

$$\Pi^m = \Pi(x^m) = p^m x^m - C(x^m) = p^m x^m - \int_0^{x^m} C'(z)dz = \left[p^m - \frac{C(x^m)}{x^m} \right] x^m$$

(vii) *Second order conditions*

Interpretation

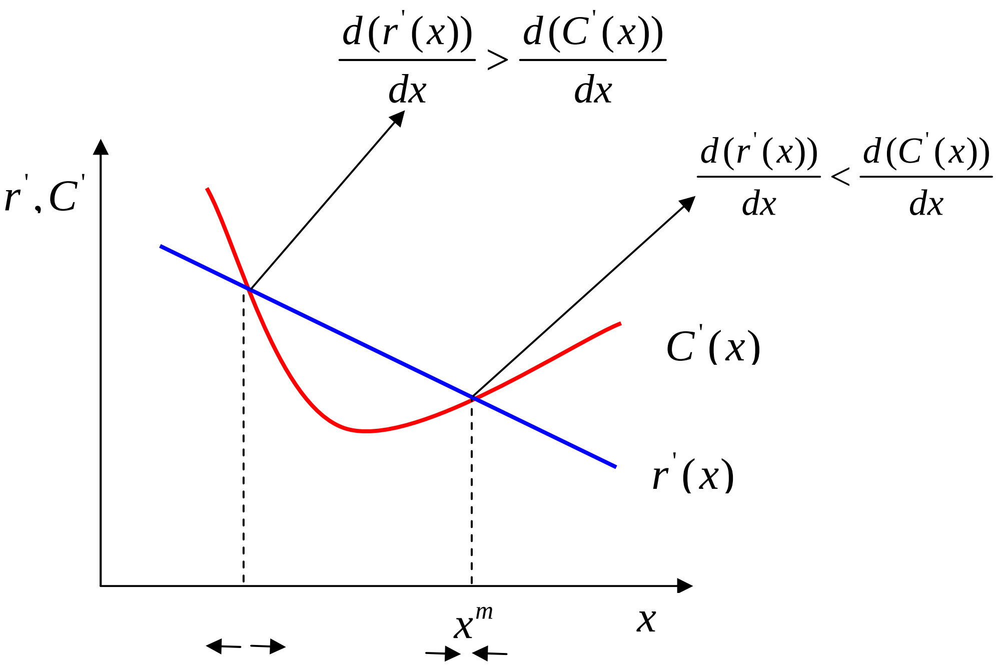
We assume for the sake of simplicity that the profit function is strictly concave.

$$\Pi''(x) = r''(x) - C''(x) = 2p'(x) + xp''(x) - C''(x) < 0 \qquad (10)$$

Condition (10) is equivalent to saying that the slope of the marginal revenue has to be lower than the slope of the marginal cost:

$$\frac{d(r'(x))}{dx} < \frac{d(C'(x))}{dx}$$

In other words, the marginal revenue curve must cross marginal cost from above.



Cases

1. Strictly convex cost or linear cost: $C^{''}(x) \geq 0$ (increasing or constant marginal cost)

a) Strictly concave demand or liner demand: $p^{''}(x) \leq 0$

$$\Pi^{''}(x) = 2\underbrace{p^{'}(x)}_{<0} + x\underbrace{p^{''}(x)}_{\leq 0}\underbrace{-C^{''}(x)}_{\leq 0} < 0$$

b) Strictly convex demand: $p^{''}(x) > 0$

$$r^{''}(x) = 2\underbrace{p^{'}(x)}_{<0} + x\underbrace{p^{''}(x)}_{>0}.$$ We need to check $r^{''}(x) < C^{''}(x)$.

2. Strictly concave cost: $C^{''}(x) < 0$ (decreasing marginal cost)

We always have to check whether $r^{''}(x) < C^{''}(x)$.

1.2. **Linear demand, constant elasticity demand and constant marginal cost**

(i) *Linear demand and constant marginal cost*

Inverse demand: $p(x) = a - bx$ ($a > 0, b > 0$).

Production cost: $C(x) = cx$ ($c \geq 0$). ($a > c$)

Marginal revenue: $r^{'}(x) = a - 2bx$.

Slope of inverse demand: $p^{'}(x) = -b$

Slope of marginal revenue: $\dfrac{d(r^{'}(x))}{dx} = -2b$

Strictly concave profit function: $\Pi^{''}(x) = r^{''}(x) = -2b < 0$.

Marginal profit at zero: $\Pi^{'}(0) = p(0) - C^{'}(0) = a - c > 0$.

Profit maximization: $r^{'}(x^m) = C^{'}(x^m) \Rightarrow a - 2bx^m = c \Rightarrow \quad x^m = \dfrac{a-c}{2b}$

Monopoly price: $p^m = p(x^m) \Rightarrow p^m = a - bx^m \Rightarrow \quad p^m = \dfrac{a+c}{2}$

Monopoly profits: $\Pi^m = \Pi(x^m) = [p(x^m) - c]x^m = [p^m - c]x^m = \dfrac{a-c}{2}\dfrac{a-c}{2b} = \dfrac{(a-c)^2}{4b}$

(ii) *Constant elasticity demand and constant marginal cost*

Demand: $x(p) = Ap^{-b}$ $\;(A > 0, b > 1)$.

Production cost: $C(x) = cx$ $\;(c > 0)$.

Price-elasticity of demand: $|\varepsilon(p)| = -x'(p)\dfrac{p}{x(p)} = bAp^{-(b+1)}\dfrac{p}{Ap^{-b}} = b.$

Inverse demand: $p(x) = A^{\frac{1}{b}}x^{-\frac{1}{b}}$.

Marginal revenue: $r'(x) = A^{\frac{1}{b}}\dfrac{(b-1)}{b}x^{-\frac{1}{b}}$.

Slope of marginal revenue: $r''(x) = -A^{\frac{1}{b}}\dfrac{(b-1)}{b^2}x^{-\frac{(1+b)}{b}}$.

Strictly concave profit function:

$$\Pi''(x) = r''(x) = -A^{\frac{1}{b}}\frac{(b-1)}{b^2}x^{-\frac{(1+b)}{b}} < 0 \Leftrightarrow b > 1.$$

Marginal profit at zero: $\Pi'(0) = \infty > 0$.

Profit maximization:

$$r'(x^m) = C'(x^m) \Rightarrow r'(x) = A^{\frac{1}{b}}\frac{(b-1)}{b}(x^m)^{-\frac{1}{b}} = c \Rightarrow \quad (x^m)^{-\frac{1}{b}} = A^{-\frac{1}{b}}\frac{b}{(b-1)}c$$

$$\left((x^m)^{-\frac{1}{b}}\right)^{-b} = \left(A^{-\frac{1}{b}}\frac{b}{(b-1)}c\right)^{-b} \Rightarrow x^m = A\left(\frac{b}{(b-1)}\right)^{-b}c^{-b}$$

Monopoly price:

$$p^m = p(x^m) \Rightarrow p^m = A^{\frac{1}{b}}(x^m)^{-\frac{1}{b}} = A^{\frac{1}{b}}\left(A\left(\frac{b}{(b-1)}\right)^{-b} c^{-b}\right)^{-\frac{1}{b}} \Rightarrow p^m = \frac{b}{(b-1)}c$$

Monopoly profits:

$$\Pi^m = \Pi(x^m) = [p(x^m) - c]x^m = [p^m - c]x^m = \frac{c}{b-1}A\left(\frac{b}{(b-1)}\right)^{-b} c^{-b} = A\frac{b^{-b}}{(b-1)^{-(b-1)}}c^{-(b-1)}$$

By solving the problem of profit maximization as a function of price, we obtain the Lerner index:

$$\frac{p^m - c}{p^m} = \frac{1}{|\varepsilon(p)|}$$

Under constant elasticity demand the Lerner index becomes $\frac{p^m - c}{p^m} = \frac{1}{b}$ and it is straightforward to obtain the monopoly price. Then it is easy to obtain the monopoly output and the monopoly profits.

1.3. **Comparative statics**

We now study how the monopoly price and output respond to a change in production costs. Economic intuition tells us that an increase in marginal cost should entail a reduction in output and an increase in price. We assume that the marginal cost is constant (and there is no fixed cost). The cost function is given by $C(x) = cx$.

$$\max_{x\geq 0} \Pi(x) \equiv \max_{x\geq 0}\ p(x)x - C(x)$$

$$\Pi^{'}(0) = p(0) - C^{'}(0) > 0 \Rightarrow\ p(0) > C^{'}(0)$$

$\Pi^{'}(x) = p(x) + xp^{'}(x) - c = 0 \quad (11) \Rightarrow\ x^m(c) \rightarrow$ the monopoly output is an implicit function of the marginal cost.

$\Pi''(x) = 2p'(x) + xp''(x) - C''(x) < 0$ Strictly concave profit function (regular case).

We can analyze the change in monopoly output due to a change in marginal cost in two equivalent ways:

(i) By completely differentiating condition (11) with respect to *x* and *c*.

$\left[2p'(x) + xp''(x)\right]dx - dc = 0$

We get:

$$\frac{dx}{dc} = \frac{1}{\underbrace{2p'(x) + xp''(x)}_{\substack{<0 \\ \text{C2ºO}}}} < 0 \qquad (12)$$

Therefore, an infinitesimal increase in marginal cost reduces output and an infinitesimal reduction in marginal cost increases output.

(ii) By using the fact that the optimal output for the monopolist $x^m(c)$ is an implicit function of marginal cost. Therefore, by definition, $x^m(c)$ satisfies the first order condition; that is,

$$p(x^m(c)) + x^m(c)p'(x^m(c)) - c = 0$$

By differentiating with respect to marginal cost:

$2p'(x^m(c))x^{m'}(c) + x^m(c)p''(x^m(c))x^{m'}(c) = 1$

$\left[2p'(x^m(c)) + x^m(c)p''(x^m(c))\right]x^{m'}(c) = 1$

Rearranging: $x^{m'}(c) = \dfrac{1}{\left[2p'(x^m(c)) + x^m(c)p''(x^m(c))\right]} < 0$

Finally the change in price due to the change in marginal cost is:

$$\frac{dp}{dc} = \frac{dp}{dx}\frac{dx}{dc} = \frac{\overbrace{p^{'}(x)}^{<0}}{\underbrace{2p^{'}(x) + xp^{''}(x)}_{\substack{<0 \\ \text{C2ºO}}}} > 0 \qquad (13)$$

Examples

(i) Linear demand

$$p^{m} = \frac{a+c}{2} \rightarrow \frac{dp^{m}}{dc} = \frac{1}{2}$$

$$\frac{dp}{dc} = \frac{p^{'}(x)}{2p^{'}(x) + x\underbrace{p^{''}(x)}_{=0}} = \frac{1}{2}$$

Under linear demand the change in price is half the change in marginal cost: $dp = \frac{1}{2}dc$

(ii) Constant elasticity demand

$$p^{m} = \frac{b}{b-1}c \rightarrow \frac{dp^{m}}{dc} = \frac{b}{b-1} > 1$$

$$p(x) = A^{\frac{1}{b}}x^{-\frac{1}{b}} \rightarrow p^{'}(x) = -A^{\frac{1}{b}}\frac{1}{b}x^{-\frac{(1+b)}{b}} \rightarrow p^{''}(x) = A^{\frac{1}{b}}\frac{(1+b)}{b^{2}}x^{-\frac{(1+2b)}{b}}$$

$$\frac{dp}{dc} = \frac{1}{2 + x\frac{p^{''}(x)}{p^{'}(x)}} = \frac{1}{2 + x\frac{A^{\frac{1}{b}}\frac{(1+b)}{b^{2}}x^{-\frac{(1+2b)}{b}}}{-A^{\frac{1}{b}}\frac{1}{b}x^{-\frac{(1+b)}{b}}}} = \frac{1}{2 - \frac{(1+b)}{b}} = \frac{b}{b-1} > 1$$

Under constant elasticity demand the increase in the monopoly price is greater than the increase in marginal cost: $dp > dc$.

1.4. **Welfare and output**

(i) The representative consumer approach. Quasi-linear utility.

(ii) Maximum willingness to pay. Marginal willingness to pay.

(iii) The demand function is independent of income.

(iv) Social welfare function and social welfare maximizing output.

(v) Total surplus, consumer surplus and producer surplus.

(vi) Efficiency conditions in the presence of several consumers or markets.

(vii) A comparison between monopoly output and efficient output by using the profit maximization problem.

(viii) A comparison between monopoly output and efficient output by using the social welfare maximization problem.

(ix) Irrecoverable efficiency loss.

(i) *The representative consumer approach. Quasi-linear utility*

We will follow *the representative consumer approach* to analyze welfare and evaluate monopoly from a social welfare point of view. Under this approach, it is assumed that market demand *x*(*p*) is generated by maximizing the (quasi-linear) utility of a representative consumer.

Consider an economy with two goods, *x* and *y*. Good *x* is produced in the monopolistic market while we can interpret the good as the amount of money to be spent on the other good by the consumer once he/she has spent the optimal amount of money on good *x*. We assume that the representative consumer has a Quasi-linear Utility Function:

$$U(x,y) = u(x) + y \qquad (u(0) = 0; u^{'}(.) > 0; u^{''}(.) < 0)$$

(ii) *Maximum willingness to pay and marginal willingness to pay*

Maximum willingness to pay, $R(x)$: the maximum amount of money that the consumer is willing to pay for x units of the good. He/she pays the maximum if he/she is indifferent between consuming x units by paying $R(x)$ and not consuming the good, thus using all his/her income endowment to consume the other goods.That is:

$$U(x, m - R(x)) = U(0, m)$$

Note that the consumer must be indifferent and, therefore, the above condition must be satisfied with equality. If for example $U(x, m - \widetilde{R}(x)) > U(0, m)$ then the consumer would wish to pay a greater amount to $\widetilde{R}(x)$ and if $U(x, m - \widetilde{R}(x)) < U(0, m)$ then $\widetilde{R}(x)$ would be higher than his/her maximum willingness to pay.

Given that the utility function is quasi-linear then:

$$U(x, m - R(x)) = U(0, m)$$

$$u(x) + m - R(x) = u(0) + m$$

$$R(x) = u(x)$$

Therefore, under quasi-linear utility:

$u(x) \rightarrow$ **Maximum willingness to pay**

Marginal willingness to pay: this is the change in maximum willingness to pay due to an infinitesimal change in the quantity consumed.

$u^{'}(x) \rightarrow$ ***Marginal willingness to pay***

(iii) *The demand function is independent of income*

$$\begin{array}{l}\max\limits_{x,y} u(x)+y \\ s.a \quad y+px=m\end{array} \equiv \max_{x,y,\lambda} \overbrace{u(x)+y+\lambda\left[m-y-px\right]}^{L(x,y,\lambda)}$$

$$\left.\begin{array}{l}\dfrac{\partial L}{\partial x}=u^{'}(x)-\lambda p=0 \\ \dfrac{\partial L}{\partial y}=1-\lambda=0\end{array}\right\} \rightarrow p=u^{'}(x) \rightarrow \text{ Inverse demand funtion}$$

$$\frac{\partial L}{\partial \lambda}=m-y-px=0$$

The demand function $x(p)$ is the inverse of this function and therefore satisfies the first order condition:

$$p=u^{'}(x(p)) \rightarrow \text{Demand function}$$

Property of the quasi-linear utility function: the demand function is independent of income.

By differentiating with respect to p we get:

$$1=u^{''}(x(p))x^{'}(p)$$

$$x^{'}(p)=\frac{1}{\underbrace{u^{''}(x(p))}_{<0}}<0 \rightarrow \quad \text{negative sloping demand}$$

(iv) *Social welfare function and social welfare maximizing output*

In this subsection we justify the use of $W(x)=u(x)-C(x)$ as the social welfare function. We consider the problem of obtaining the allocation that maximizes the utility of the representative consumer with a resources constraint: we interpret the production cost of good x as the amount of good y to which must be given up in order to have the good x.

$$\max_{x,y} u(x) + y$$
$$s.a \quad y = m - C(x)$$

By replacing *y* in the objective function we get:

$$\max_x u(x) + \underbrace{m}_{constante} - C(x) \equiv \max_x u(x) - C(x)$$

Therefore the social welfare maximizing problem becomes:

$$\max_{x \geq 0} W(x) \equiv \max_{x \geq 0} u(x) - C(x)$$

$$W^{'}(0) = u^{'}(0) - C^{'}(0) > 0$$

$W^{'}(x) = u^{'}(x) - C^{'}(x) = 0 \Leftrightarrow \; W^{'}(x^e) = 0$ (13) First order condition.

$W^{''}(x) = u^{''}(x) - C^{''}(x) < 0$ Strictly concave welfare function (regular case).

Therefore, the welfare maximizing output or efficient output satisfies $W^{'}(x^e) = 0 \Leftrightarrow u^{'}(x^e) = C^{'}(x^e)$. Under constant marginal cost the efficiency condition becomes:

$$u^{'}(x^e) = c,$$

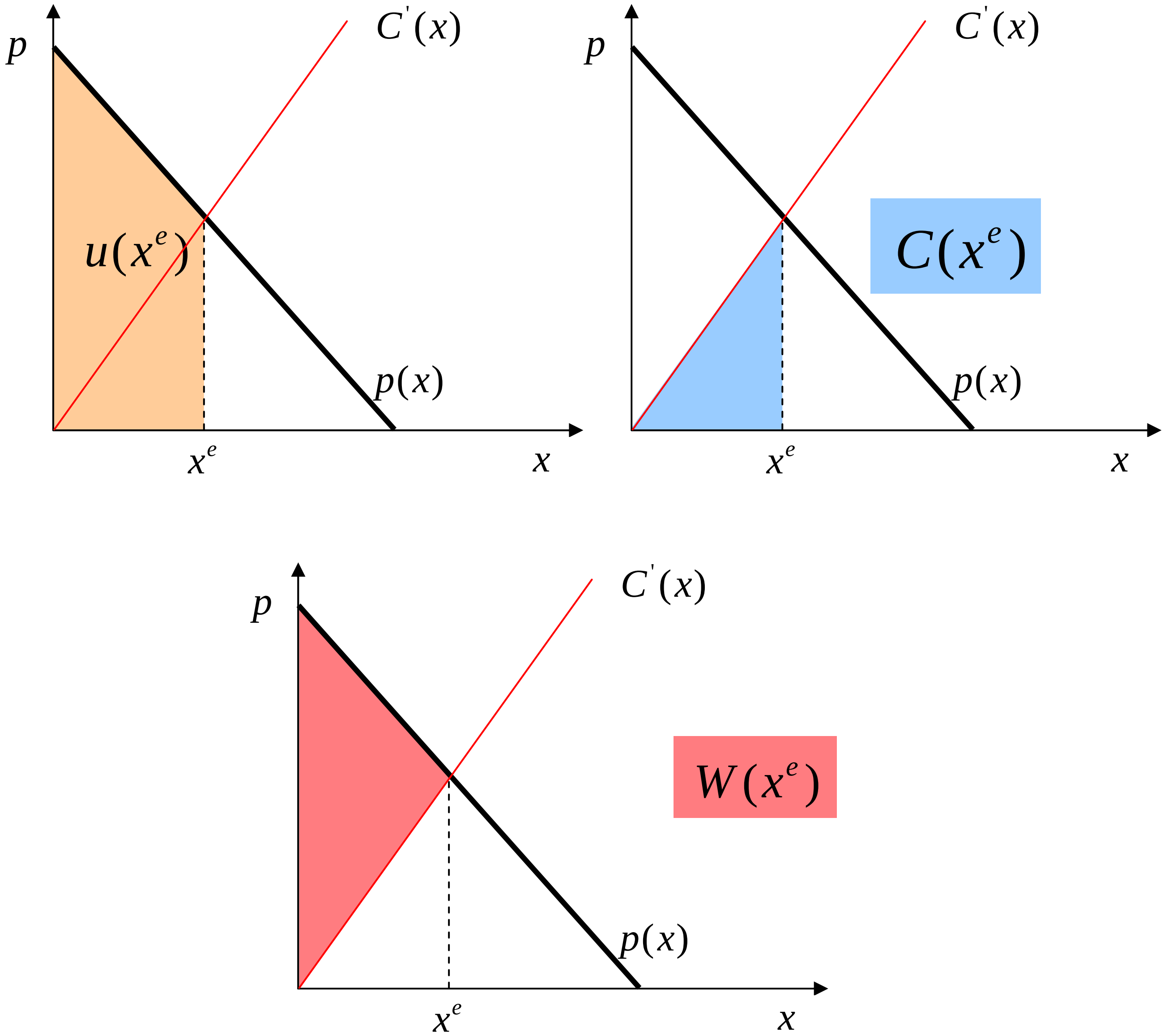
That is, at the efficient output marginal willingness to pay equals marginal cost.

(v) *Total surplus, consumer surplus and producer surplus*

The function $W(x) = u(x) - C(x)$ can be interpreted as the total surplus; that is, the difference between maximum willingness to pay and production cost. By definition the following is satisfied:

$$u(x) - \underbrace{u(0)}_{=0} = \int_0^x u^{'}(z)dz \qquad\qquad C(x) - \underbrace{C(0)}_{=F=0} = \int_0^x C^{'}(z)dz$$

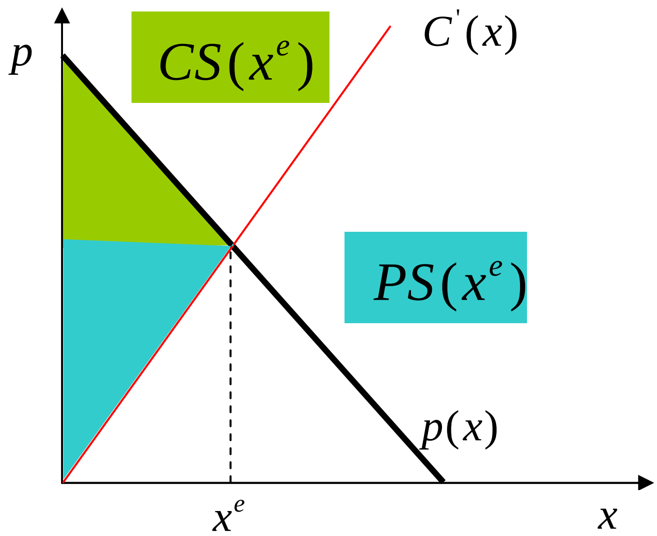
Therefore maximizing $u(x) - C(x)$ is equivalent to chosing the level of output that maximizes the area below the inverse demand and above the marginal cost.



By adding and subtracting *px* we can rewrite the total surplus as:

$$W(x) = u(x) - C(x) = \underbrace{\left[u(x) - px\right]}_{EC(x)} + \underbrace{\left[px - cx\right]}_{EP(x)}$$

The consumer surplus, *CS*(*x*), measures the difference between maximum willingness to pay and the amount of money actually paid. The producer surplus, *PS*(*x*), measures the profits of the firm (when there are no fixed costs). Therefore, efficient production also maximizes the addition of the consumer surplus and the producer surplus.



(vi) *Efficiency conditions in the presence of several consumers or markets*

We next analyze the problem of obtaining a Pareto efficient allocation when we consider an economy with two consumers under quasi-linear utility, $u_i(x_i) + y_i$, and an endowment of m_i, $i = 1, 2.$. We maximize the utility of one agent (for example consumer 1) while maintaining constant the utility of the other (consumer 2), given a resource constraint (marginal cost, *c*, is assumed to be constant).

$$\max_{x_1, y_1, x_2, y_2} u_1(x_1) + y_1$$
$$s.a \;\; u_2(x_2) + y_2 = \bar{u}_2$$
$$y_1 + y_2 = m_1 + m_2 - c.(x_1 + x_2)$$

By substituting y_1 and y_2 in the objective function the problem becomes:

$$\max_{x_1, x_2} u_1(x_1) + u_2(x_2) - c.(x_1 + x_2) + m_1 + m_2 - \bar{u}_2$$

From the first order conditions we get:

$$\left.\begin{array}{l} u_1^{'}(x_1^e) - c = 0 \\ \\ u_2^{'}(x_2^e) - c = 0 \end{array}\right\} \rightarrow u_1^{'}(x_1^e) = u_2^{'}(x_2^e) = c \rightarrow \text{Efficiency condition} \quad (14)$$

(vii) *A comparison between monopoly output and efficient output by using the profit maximization problem.*

$$\max_{x \geq 0} \Pi(x) \equiv \max_{x \geq 0} p(x)x - C(x)$$

$$\Pi^{'}(0) = p(0) - C^{'}(0) > 0 \Rightarrow \; p(0) > C^{'}(0)$$

$$\Pi^{'}(x) = p(x) + xp^{'}(x) - C^{'}(x) = 0 \Leftrightarrow \; \Pi^{'}(x^m) = 0 \quad \text{First order condition.}$$

$$\Pi^{''}(x) = 2p^{'}(x) + xp^{''}(x) - C^{''}(x) < 0 \;\; \text{Strictly concave profit function (regular case).}$$

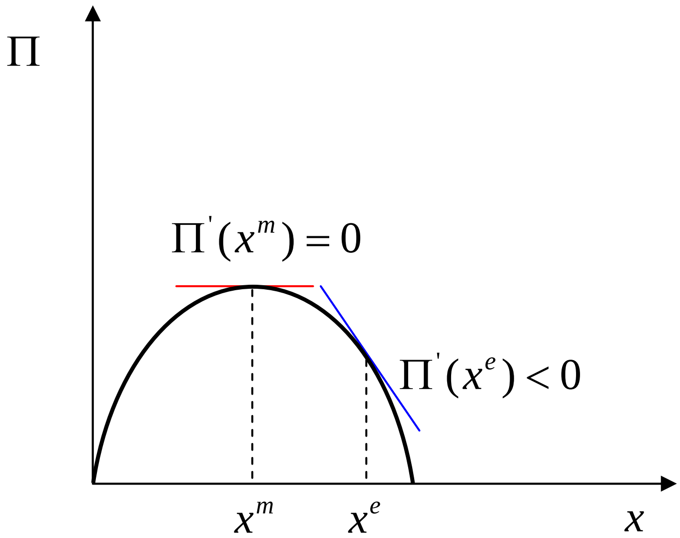
$$\left\{\begin{array}{c} \Pi^{'}(x^m) = 0 \\ \Pi^{'}(x^e)? \\ \Pi^{''}(x) < 0 \end{array}\right\}$$

$$\Pi^{'}(x^e) = \underbrace{p(x^e)}_{=u^{'}(x^e)} + x^e p^{'}(x^e) - C^{'}(x^e) = \underbrace{\left[u^{'}(x^e) - C^{'}(x^e)\right]}_{=0} + x^e \underbrace{p^{'}(x^e)}_{<0} < 0$$

By definition of efficient output.

$$\left.\begin{cases} \Pi'(x^m) = 0 \\ \Pi'(x^e) < 0 \\ \Pi''(x) < 0 \end{cases}\right\} \rightarrow \Pi'(x^e) < \Pi'(x^m) \rightarrow x^e > x^m$$

$$\boxed{\Pi''(x) < 0 \Leftrightarrow \frac{d\Pi'(x)}{dx} < 0 \rightarrow \uparrow x \; \downarrow \Pi'(x)}$$



(vii) *A comparison between monopoly output and efficient output by using the social welfare maximization problem.*

$$\max_{x \geq 0} W(x) \equiv \max_{x \geq 0} u(x) - C(x)$$

$$W'(0) = u'(0) - C'(0) > 0 \Rightarrow \; p(0) > C'(0)$$

$W'(x) = u'(x) - C'(x) = 0 \Leftrightarrow \; W'(x^e) = 0$ First order condition.

$W''(x) = u''(x) - C''(x) < 0$ Strictly concave welfare function.

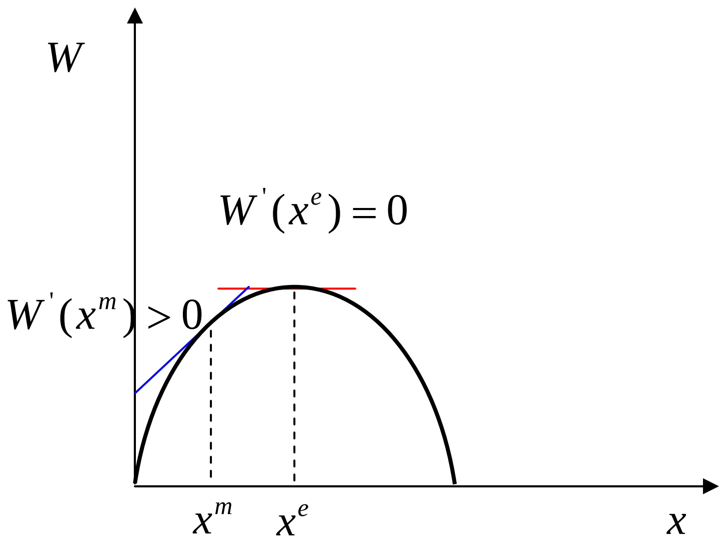
$$\begin{cases} W'(x^e) = 0 \\ W'(x^m) ? \\ W''(x) < 0 \end{cases}$$

$$W^{'}(x^m) = \underbrace{u^{'}(x^m)}_{p(x^m)} - C^{'}(x^m) = -x^m \overbrace{\underbrace{p^{'}(x^m)}_{<0}}^{u^{''}(x^m)} > 0$$

By definition of monopoly output.

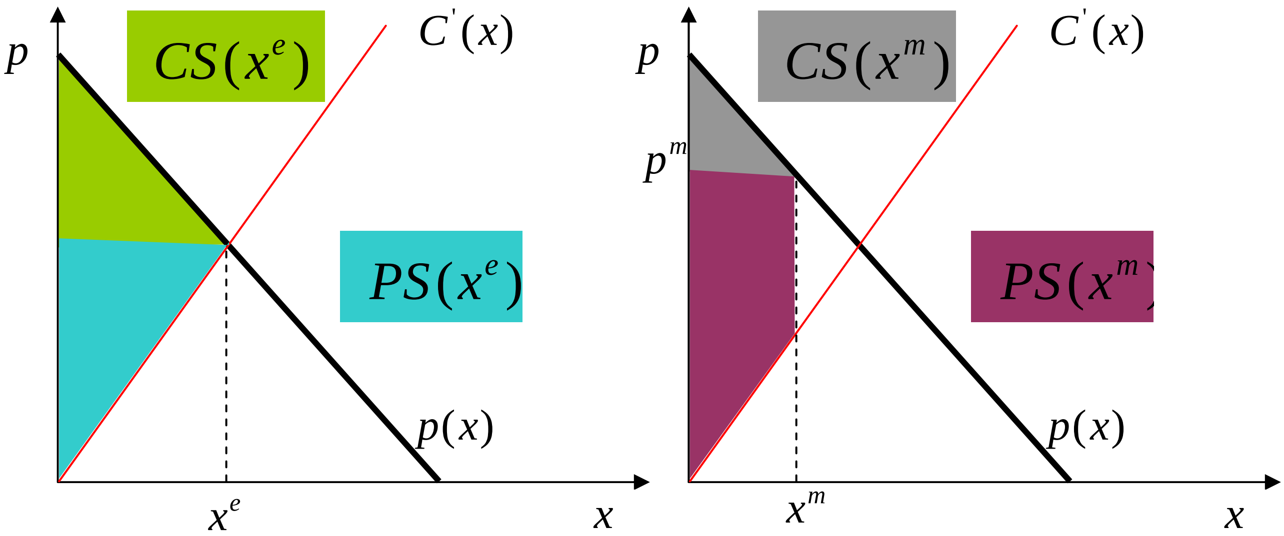
$$\left.\begin{matrix} W^{'}(x^e) = 0 \\ W^{'}(x^m) > 0 \\ W^{''}(x) < 0 \end{matrix}\right\} \rightarrow W^{'}(x^e) < W^{'}(x^m) \rightarrow \ x^e > x^m$$

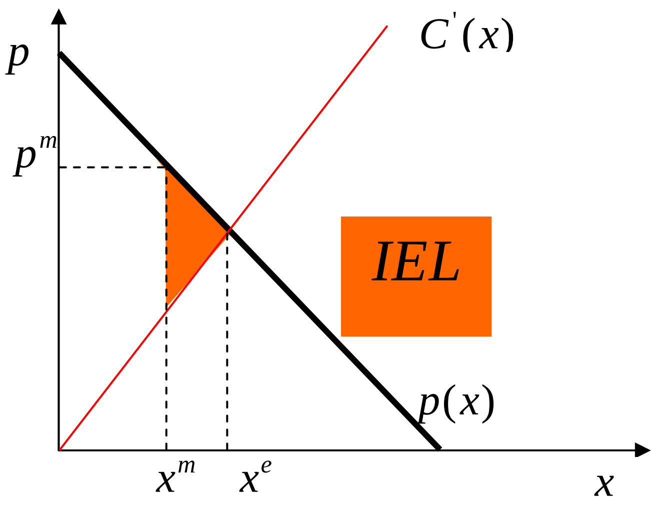
$$W^{''}(x) < 0 \Leftrightarrow \frac{dW^{'}(x)}{dx} < 0 \rightarrow \uparrow x \ \downarrow W^{'}(x)$$



(vii) *Irrecoverable efficiency loss (IEL).*

$$IEL = W(x^e) - W(x^m) = \int_0^{x^e} [u'(z) - C'(z)]dz - \int_0^{x^m} [u'(z) - C'(z)]dz = \int_{x^m}^{x^e} [u'(z) - C'(z)]dz$$





1.5. ***Price discrimination***

(i) Definition.

(ii) The incentive to discriminate prices.

(iii) Conditions.

(iv) Types of price discrimination (Pigou, 1920).

(v) Examples.

(vi) The model.

(i) *Definition*

"There exists price discrimination when different units of the same good are sold at different prices either to the same consumer or to different consumers".

Discussion

- Differences in quality: passenger transport, cultural or sporting events etc.
- A single price may be discriminatory and different prices not. We say that there is no price discrimination when the difference between the prices paid by two consumers for a unit of the good exactly responds to the difference in the cost of providing them with the good.

(ii) *The incentive to discriminate prices*

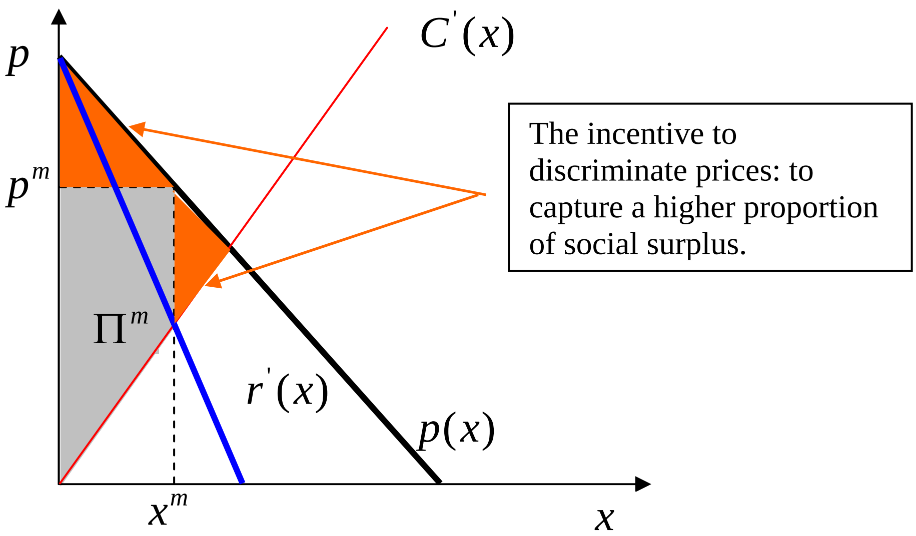
At the profit-maximizing level of output marginal revenue equals marginal cost, $r'(x^m) = C'(x^m);$ i.e., an infinitesimal change in the level of output changes revenue and cost equally. That is:

$$\underbrace{p(x^m)}+\underbrace{x^m p'(x^m)}=C'(x^m) \qquad (1)$$

Additional revenue from selling an additional unit.

Loss of revenue from selling units already produced at a lower price.

The monopolist would be reluctant to sell more units if it does not have to reduce the price. Therefore, there are incentives to try to capture a higher proportion of the consumer surplus $\rightarrow$ incentives to discriminate prices.



(iii) *Conditions*

Two conditions are needed for a firm to be able to discriminate prices:

a) The firm must be able to classify consumers (which depends on information).

b) The firm must be capable of preventing the resell of the good (which depends on the possibilities of arbitrage and on transaction costs).

The simplest case occurs when a firm receives an exogenous sign (age, location, occupation, etc.) which allows it to classify consumers into different groups.

It is more difficult to classify according to an endogenous category (e.g., quantity purchased or the time of purchase). In that case the monopolist must establish prices in such a way that consumers classify themselves in the correct categories.

(iv) *Types of price discrimination* (Pigou, 1920)

1) **First-degree price discrimination or perfect discrimination**.

The seller charges a different price for each unit equal to the maximum willingness to pay for that unit. This requires *full information* concerning consumer preferences and no arbitrage. The monopolist succeeds in extracting the complete consumer surplus.

2) **Second-degree price discrimination** (or nonlinear pricing).

Prices differ depending on the number of units of the good but not across consumers. Each consumer faces the same price catalogue but prices depend on the quantity purchased (or on another variable, e.g., product quality). Examples: volume discounts. Self selection.

3) **Third-degree price discrimination**.

Different prices are charged to different consumers but each consumer pays a constant amount (the same price) for each unit. The firm receives an exogenous sign which allows it to classify consumers into different groups. This is the most frequent type of price discrimination. Examples: discounts for students, senior citizens, etc. Identification.

Another way of classifying price discrimination is to distinguish between direct price discrimination and indirect price discrimination. Second-degree price discrimination is a case of indirect discrimination (consumers face a unique price schedule and they classify themselves by their choices) while first-degree price discrimination and third-degree price discrimination would be direct discrimination. In the case of third-degree price discrimination the firm gives different price menus for consumers belonging to different groups or markets.

(v) *Examples*

It is more difficult to find real markets where there is no price discrimination than markets where such discrimination exists. Although it is often not possible to distinguish clearly what type of price discrimination exists it is an interesting exercise to think about what type of price discrimination is been practiced in the following cases.

- Two-part tariffs: telephone, Internet, electricity, cable television, etc.
- Different electricity rates for industrial use and domestic use.
- Discounts in museums, magazine subscriptions, cultural and sporting events, for children, young people or senior citizens.
- Volume discount in public transport.
- Quality differences: different prices depending on the quality of the product in cultural or sporting events, passenger transport (trains, etc.).
- Discounts for repeated buying.
- 2x1, 3x2, etc. in supermarkets, etc.
- Home-service food, tele-shopping etc.

(vi) *The model*

We study the three types of price discrimination by using a very simple model. Assume that there are two potential consumers with quasi-linear utility functions: $u_i(x_i) + y_i, \; i = 1, 2.$

$u_i(0) = 0, \; i = 1, 2.$

$u_i(x_i)$: maximum willingness to pay of consumer $i = 1, 2.$

$u_i^{'}(x_i)$: marginal willingness to pay of consumer $i = 1, 2.$

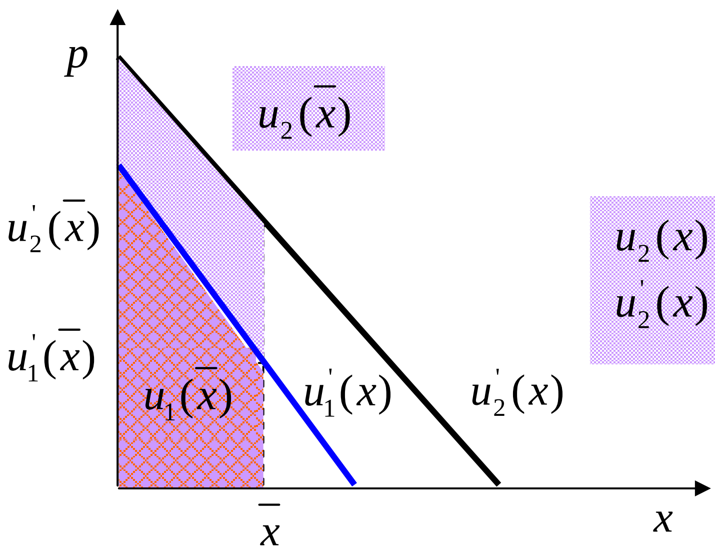
We say that the *consumer 2* is a *high-demand consumer* and that the *consumer 1* is a *low-demand consumer* if the following is satisfied:

$$u_2(x) > u_1(x) \quad \forall x$$

$$u_2^{'}(x) > u_1^{'}(x) \quad \forall x$$

Thus, consumer 2 is a *high-demand consumer* and *consumer 1* is a *low-demand consumer* if both the maximum willingness to pay and the marginal willingness to pay are higher for consumer 2 than for consumer 1 for any quantity of the good.

The comparison between consumers of maximum willingness to pay and marginal willingness to pay only makes sense for the same level of output. Moreover, the comparison has to be made for any level of output.



The marginal cost of the monopolist is assumed to be constant (and there are no fixed costs) $c > 0$. In an equivalent way, we can see the cost function as: $C(x) = c.x = c.(x_1 + x_2)$.

1.6. ***First-degree price discrimination or perfect price discrimination***

(i) Definition and context.

(ii) The case of a single consumer.

(iii) Observations. Is the quantity supplied by the monopolist efficient?

(iv) The case of two consumers.

(v) Does the monopolist supply efficient outputs to consumers? The monopolist supplies a higher quantity to the high-demand consumer (proof).

(vi) What would happen if the monopolist were not able to identify consumers?

(i) *Definition and context*

The seller charges a different price for each unit of product and equals the maximum willingness to pay for that unit.

This requires *full information* on consumer preferences and *no arbitrage* of any kind. In particular, the monopolist needs to be able to identify consumers when they buy the good. (Classic example: a village doctor).

(ii) *The case of a single consumer*

The monopolist supplies a price-quantity bundle (r^*, x^*) which maximizes profits. The monopolist proposes a "take it or leave it" choice: $\left\langle \begin{matrix} (r^*, x^*) \\ (0,0) \end{matrix} \right.$. The consumer either pays r^* for x^* units or does not receive the good. The maximization problem of the monopolist is:

$$\max_{r,x} r - cx$$
$$s.a\ u(x) \geq r \quad (1)$$

Constraint (1) can be equivalently written as $u(x) - r \geq 0$: the consumer has to obtain a non-negative surplus from good *x*. This type of constraint is known as *participation restriction* or *individual rationality restriction*.

Given that the monopolist wishes to maximize profits it will choose the highest possible tariff *r* and, therefore, condition (1) will be satisfied as equality: $r = u(x)$. The problem thus consists of:

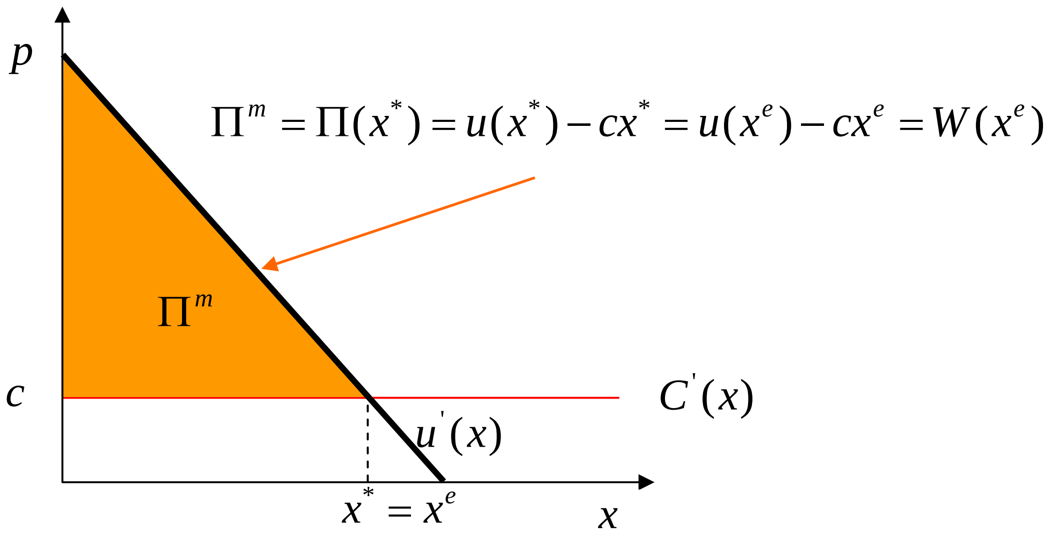
$$\max_{x} \overbrace{u(x) - cx}^{\Pi(x)}$$
$$\frac{d\Pi}{dx} = u^{'}(x) - c = 0 \rightarrow \quad u^{'}(x^*) = c$$
$$\frac{d^2\Pi}{dx^2} = u^{''}(x) < 0$$

Given this level of output the tariff will be: $r^* = u(x^*)$.

(iii) *Observations*

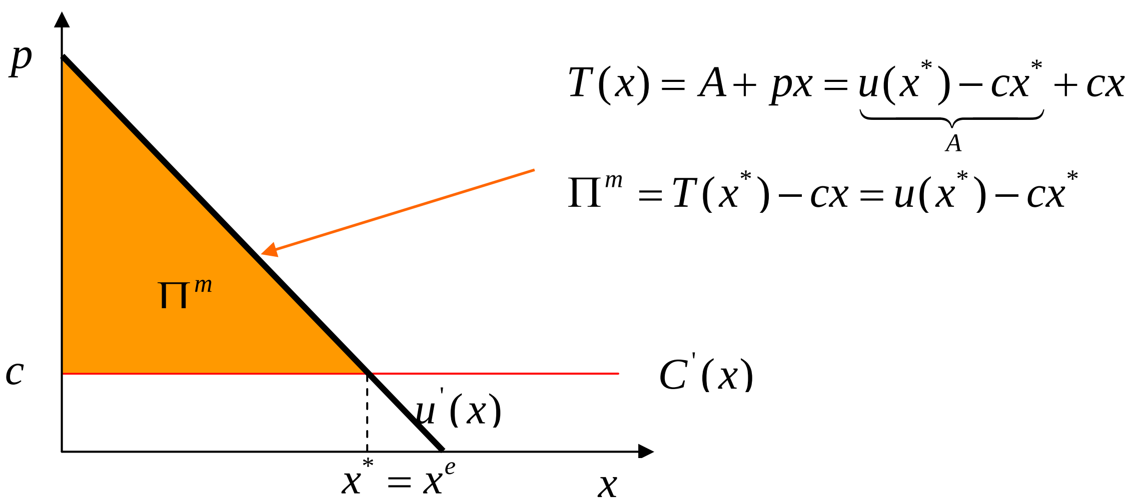
a) **Is the quantity supplied by the monopolist efficient?**

The monopolist produces a Pareto-efficient output, $x^* = x^e$, given that it supplies a quantity such that the marginal willingness to pay equals the marginal cost. (Review the problem of maximizing social welfare and compare with the problem we have just solved). However, the monopolist obtains the entire social surplus.



b) The monopolist produces the same quantity that it would produce if it behaved as a perfectly competitive firm. If it took price as a parameter then its output decision would be $p(x) = c$ but given that utility is quasi-linear then $p(x) = u'(x)$ and consequently $u'(x) = c$. However, the distribution of trade gains would be just the opposite.

c) We might obtain the same results by using a **two-part tariff**.



d) We would obtain the same result if the monopolist sold each unit to the consumer at a different price equal to his/her maximum willingness to pay for that unit. Assume that we break production down into *n* equal portions of size Δx so as $x = n\Delta x$.

The maximum willingness to pay for the first unit (of size Δx) is given by:

$$u(0)+m = u(\Delta x)+m-p_1 \rightarrow u(0) = u(\Delta x)-p_1$$

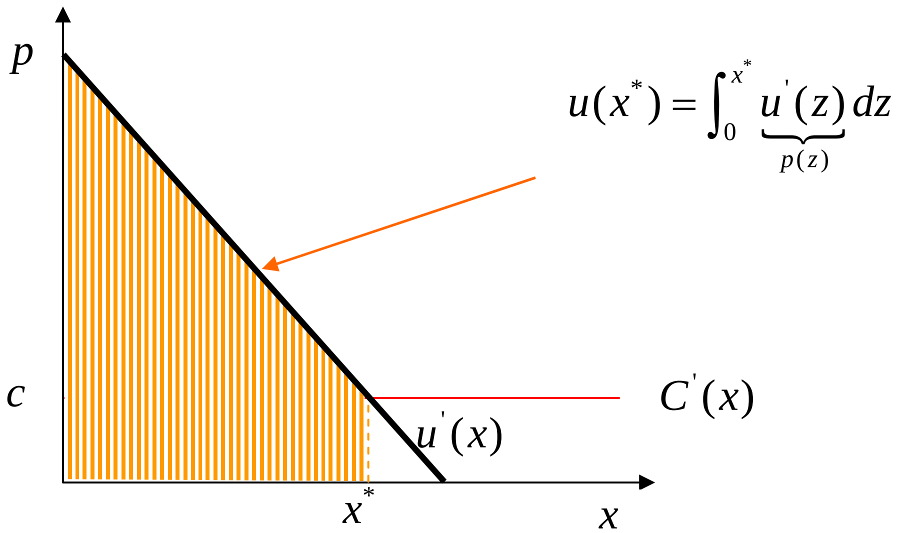
The maximum willingness to pay for the second unit is:

$$u(\Delta x)+m-p_1 = u(2\Delta x)+m-p_1-p_2 \rightarrow u(\Delta x) = u(2\Delta x)-p_2$$

And so on. We would obtain the following sequence of equations:

$$\begin{aligned} u(0) &= u(\Delta x)-p_1 \\ u(\Delta x) &= u(2\Delta x)-p_2 \\ u(2\Delta x) &= u(3\Delta x)-p_3 \\ &\ldots\ldots\ldots\ldots \\ u((n-1)\Delta x) &= u(n\Delta x)-p_n \end{aligned}$$

Adding and taking into account that $u(0)=0$ we get $u(\underbrace{n\Delta x}_{x}) = \sum_{i=1}^{n} p_i$. When the size of the units becomes infinitesimal, we obtain that proposing a "take it or leave it" choice to the consumer is equivalent to selling him/her each (infinitesimal) unit at a price equal to the marginal willingness to pay for it.



(iv) *The case of two consumers*

The monopolist supplies consumer i, $i = 1, 2$, with a price-output bundle (r_i^*, x_i^*) in order to maximize profits. The monopolist gives consumer i, $i = 1, 2$, a "take it or leave it" choice: $\left\langle \begin{matrix} (r_i^*, x_i^*) \\ (0,0) \end{matrix} \right.$. Consumer i, $i = 1, 2$, either pays r_i^* for x_i^* units or does not receive the good. The maximization problem of the monopolist is:

$$\begin{aligned} &\max_{r_1, x_1, r_2, x_2} r_1 + r_2 - c.(x_1 + x_2) \\ &s.a \quad u_1(x_1) - r_1 \geq 0 \\ &\qquad u_2(x_2) - r_2 \geq 0 \end{aligned} \quad \underset{\text{profit maximization}}{\Longrightarrow} \quad \begin{aligned} r_1 &= u_1(x_1) \\ r_2 &= u_2(x_2) \end{aligned}$$

Therefore, the problem becomes:

$$\max_{x_1, x_2} u_1(x_1) + u_2(x_2) - c.(x_1 + x_2)$$

$$\left. \begin{aligned} \frac{\partial \Pi}{\partial x_1} &= u_1'(x_1) - c = 0 \\ \frac{\partial \Pi}{\partial x_2} &= u_2'(x_2) - c = 0 \end{aligned} \right\} \rightarrow u_1'(x_1^*) = u_2'(x_2^*) = c$$

Given these levels of output the tariffs are: $r_1^* = u_1(x_1^*)$ and $r_2^* = u_2(x_2^*)$.

(v) *Does the monopolist supply efficient outputs to consumers? The monopolist supplies a higher quantity to the high-demand consumer (proof)*

The monopolist offers efficient outputs: $x_1^* = x_1^e$ and $x_2^* = x_2^e$. (Review the problem of obtaining a Pareto-efficient allocation and compare with the problem we have just solved)

We next demonstrate that the monopolist offers a higher quantity to the high-demand consumer: $x_2^* > x_1^*$.

$$\left.\begin{array}{l} u_1'(x_1^*) = c \\ \\ u_2'(x_2^*) = c \end{array}\right\} u_2'(x_2^*) = u_1'(x_1^*) < u_2'(x_1^*)$$

Consumer 2 is the high-demand consumer: $u_2'(x) > u_1'(x) \quad \forall x$

Therefore, $u_2'(x_2^*) < u_2'(x_1^*)$ but given that function u_2 is strictly concave then $\frac{d(u_2'(x))}{dx} < 0$ and in consequence $x_2^* > x_1^*$.

(vi) *What would happen if the monopolist were not able to identify consumers?*

(This subsection serves to introduce the analysis of second-degree price discrimination). Assume now that the monopolist is not able to identify consumers when they go to buy the good. That is, the monopolist cannot propose personalized supplies and is therefore restricted to stating a single price menu. Assume that it states a price menu by using the tariffs and quantities which are optimal under perfect price discrimination:

$$\left\langle \begin{array}{c} (r_1^*, x_1^*) \\ (r_2^*, x_2^*) \\ (0,0) \end{array}\right.$$

where $r_1^* = u_1(x_1^*)$ and $r_2^* = u_2(x_2^*)$. We can see that the high-demand consumer has incentives to buy the bundle designed for the low-demand consumer.

$$0 = \underbrace{u_2(x_2^*) - r_2^*} < \underbrace{u_2(x_1^*) - r_1^*} = \underbrace{u_2(x_1^*) - u_1(x_1^*)}_{>0}$$

Incentive to engage in personal arbitrage.

The surplus obtained by consumer 2 if he/she buys the bundle designed for him.

The surplus that consumer 2 would obtain if he/she buys the bundle designed for consumer 1.

1.7. ***Second-degree price discrimination (or non-linear pricing)***

(Keywords: no identification, unique price menu and self selection).

(i) Definition and context.

(ii) Participation restrictions and self selection restrictions. Interpretation.

(iii) Demonstration of what constraints are satisfied with equality. Interpretation.

(iv) The profit maximization problem.

(v) Observations. Does the monopolist supply efficient quantities? The monopolist offers a lower-than- efficient quantity to the low-demand consumer (Proof).

(vi) Under what conditions does the monopolist offer the good to both consumers?

(vii) Graphic representation.

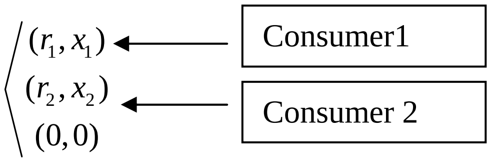
(i) *Definition and context*

The prices differ depending on the number of units bought but not from one consumer to another.

We consider a context where the monopolist knows the preferences of the consumers (it knows the preference distribution function) but is unable to identify consumers when they go to buy the good. So the firm is obliged to establish a unique price menu and to allow consumers to self classify or self select. In this sense we can say that there is indirect price discrimination. The consumers face the same price schedule but prices depend on quantity (or some other variable, e.g. the quality of the good) bought.

(ii) *Participation restrictions and self selection restrictions. Interpretation*

The objective of the monopolist is to optimally design the price menu in such a way that each consumer chooses the price-quantity bundle designed for him/her.



Restrictions for the monopolist

- **Participation restrictions** (or individual rationality constraints)

$$u_1(x_1) - r_1 \geq 0 \quad (1)$$

$$u_2(x_2) - r_2 \geq 0 \quad (2)$$

These restrictions guarantee that each consumer wishes to buy the good. Each consumer obtains at least as much utility by consuming the good as by not consuming. Put differently, each consumer obtains a non-negative surplus by purchasing the good.

- **Self selection restrictions** (or incentive compatibility constraints)

$$u_1(x_1) - r_1 \geq u_1(x_2) - r_2 \quad (3)$$

$$u_2(x_2) - r_2 \geq u_2(x_1) - r_1 \quad (4)$$

These restrictions guarantee that each consumer prefers the price-quantity bundle designed for him/her to the price-quantity bundle designed for the other consumer. Put differently, these constraints avoid personal arbitrage: each consumer gets as least as great a surplus by choosing the bundle designed for him/her as he/she does by choosing the bundle designed for the other consumer.

(iii) *Demonstration of what constraints are satisfied with equality. Interpretation*

We now arrange constraints according to each consumer.

$$\text{(1) y (3)} \rightarrow \begin{cases} r_1 \le u_1(x_1) & (1)' \\ r_1 \le u_1(x_1) - u_1(x_2) + r_2 & (2)' \end{cases}$$

$$\text{(2) y (4)} \rightarrow \begin{cases} r_2 \le u_2(x_2) & (3)' \\ r_2 \le u_2(x_2) - u_2(x_1) + r_1 & (4)' \end{cases}$$

The monopolist wishes to maximize profits and will therefore choose the highest possible r_1 and r_2. As a consequence, only one of the first two inequalities and only one of the second two inequalities will be binding (that is, they will be satisfied with equality). The assumption that consumer 2 is the high-demand consumer and consumer 1 the low-demand consumer ($u_2(x) > u_1(x) \ \ \forall x$ and $u_2^{'}(x) > u_1^{'}(x) \ \ \forall x$) is sufficient to determine what constraints are binding.

1) **Demonstration that (4)’ is satisfied with equality and (3)’ with strict inequality**.

Assume that (3)’ is satisfied with equality and, therefore, that $r_2 = u_2(x_2)$. Then $(4)' \rightarrow r_2 \le r_2 - u_2(x_1) + r_1 \rightarrow r_1 \ge u_2(x_1)$. Given that consumer 2 is the high-demand consumer $u_2(x) > u_1(x) \ \ \forall x$ then $r_1 \ge u_2(x_1) > u_1(x_1)$. That is, $r_1 > u_1(x_1)$ which means that restriction

(1)' would not be satisfied which is a contradiction. (The fact that the participation constraint of the high-demand consumer is satisfied with equality is not compatible with the fact that the low-demand consumer buys the good). As a conclusion, (3)' is not binding and (4)' is satisfied with equality:

$$r_2 = u_2(x_2) - u_2(x_1) + r_1 \quad (5)$$

2) **Demonstration that (1)' is satisfied with equality and (2)' with strict inequality**

Assume that condition (2)' is satisfied with equality and, therefore, that $r_1 = u_1(x_1) - u_1(x_2) + r_2$. By substituting r_2 from condition (5) we get:

$$\cancel{r_1} = u_1(x_1) - u_1(x_2) + \underbrace{u_2(x_2) - u_2(x_1) + \cancel{r_1}}_{=r_2}$$

which implies

$$u_2(x_2) - u_2(x_1) = u_1(x_2) - u_1(x_1)$$

$$\int_{x_1}^{x_2} u_2^{'}(t)dt = \int_{x_1}^{x_2} u_1^{'}(t)dt$$

$$\int_{x_1}^{x_2} [u_2^{'}(t) - u_1^{'}(t)]dt = 0$$

But this contradicts the assumption that consumer 2 is the high-demand consumer, $u_2^{'}(x) > u_1^{'}(x) \ \ \forall x.$ Therefore, (2)' is not binding and (1)' is satisfied with equality:

$$r_1 = u_1(x_1) \quad (6)$$

Interpretation

The monopolist charges consumer 1 a tariff equal to his maximum willingness to pay given that the low-demand consumer has no incentive to engage in personal arbitrage. Given that the high demand consumer has incentive to engage in personal arbitrage (and to mimic the low-demand consumer) the monopolist charges him/her the maximum price that induces him/her to choose the bundle designed for him/her (the amount of money that just leaves him/her indifferent between his/her bundle and that designed for the low-demand consumer).

We now show (in a different more intuitive way) why the monopolist must provide a positive surplus to the high-demand consumer. Consider the self selection constraint for the high-demand consumer:

$$u_2(x_2) - r_2 \geq u_2(x_1) - r_1 \quad (4)$$

Note that the right side of this constraint is positive conditional on the low-demand consumer's wishing to buy the good. That is, if we choose the maximum value for r_1 condition (4) would be:

$$u_2(x_2) - r_2 \geq u_2(x_1) - u_1(x_1) > 0$$

given that consumer 2 is the high-demand consumer (which implies that the participation restriction of consumer 2 cannot be satisfied with equality). But given that the monopolist must allow the high-demand consumer to obtain a positive surplus, it decides to leave the consumer with the minimum possible surplus, just that amount such that the high-demand consumer is indifferent between his/her bundle and the bundle designed for consumer 1. That is, rearranging restriction (5):

$$u_2(x_2) - r_2 = u_2(x_1) - u_1(x_1) > 0$$

Given that the low-demand consumer has no incentive to engage in personal arbitrage the monopolist charges him/her the maximum that he/she is willing to pay $r_1 = u_1(x_1)$.

(iv) *The profit maximization problem*

$$\max_{r_1,x_1,r_2,x_2} r_1 + r_2 - c.(x_1 + x_2) \qquad\qquad \max_{r_1,x_1,r_2,x_2} r_1 + r_2 - c.(x_1 + x_2)$$

$$s.a \quad \begin{array}{l} u_1(x_1) - r_1 \geq 0 \quad (1) \\ u_2(x_2) - r_2 \geq 0 \quad (2) \\ u_1(x_1) - r_1 \geq u_1(x_2) - r_2 \quad (3) \\ u_2(x_2) - r_2 \geq u_2(x_1) - r_1 \quad (4) \end{array} \quad \Rightarrow \quad s.a \quad \begin{array}{l} r_1 = u_1(x_1) \quad (6) \\ r_2 = u_2(x_2) - [u_2(x_1) - r_1] \quad (5) \end{array}$$

By substituting we get:

$$\max_{x_1,x_2} \overbrace{u_1(x_1) + u_2(x_2) - [u_2(x_1) - u_1(x_1)] \; - c.(x_1 + x_2)}^{\Pi(x_1,x_2)}$$

$$\frac{\partial \Pi}{\partial x_1} = u_1^{'}(\tilde{x}_1) - c - [u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)] = 0 \quad (7)$$

$$\frac{\partial \Pi}{\partial x_2} = u_2^{'}(\tilde{x}_2) - c = 0 \quad (8)$$

The tariffs are given by:

$$\tilde{r}_1 = u_1(\tilde{x}_1)$$
$$\tilde{r}_2 = u_2(\tilde{x}_2) - [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)]$$

(v) *Observations*

1) **The monopolist provides the high-demand consumer with the efficient quantity and leaves him/her with a positive surplus**.

Condition (8) implies $u_2^{'}(\tilde{x}_2) = c$ and, therefore, the monopolist offers the efficient quantity to the high-demand consumer $\tilde{x}_2 = x_2^e$ (review Pareto-efficiency conditions). Moreover, the monopolist charges him/her a price (a tariff) lower than his/her maximum willingness to pay leaving him/her with a positive surplus equals to that which he/she would obtain if he/she chose the bundle designed for consumer 1. $\tilde{r}_2 = u_2(\tilde{x}_2) - [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)]$ and his/her surplus would thus be: $u_2(\tilde{x}_2) - \tilde{r}_2 = [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)].$

2) **The monopolist offers the low-demand consumer a quantity lower than the efficient quantity and leaves him/her with no surplus**.

$$\frac{\partial \Pi}{\partial x_1} = u_1^{'}(\tilde{x}_1) - c - \underbrace{[u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)]}_{>0} = 0 \quad (7)$$

Given that consumer 2 is the high-demand consumer $[u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)] > 0$ and then from condition (7) we get $u_1^{'}(\tilde{x}_1) > c.$ By definition, the efficient output satisfies $u_1^{'}(x_1^e) = c,$ and as a consequence $u_1^{'}(\tilde{x}_1) > u_1^{'}(x_1^e).$ The maximum willingness to pay is a strictly concave function:

$$\left.\begin{array}{l} u_1^{'}(\tilde{x}_1) > u_1^{'}(x_1^e) \\ \\ \dfrac{d(u_1^{'}(x_1))}{dx_1} < 0 \end{array}\right\} \rightarrow \tilde{x}_1 < x_1^e$$

We next look at the intuition of this result. We interpret the marginal profit of x_1 and evaluate it at different production levels.

$$\frac{\partial \Pi}{\partial x_1} = \underbrace{u_1^{'}(x_1) - c}_{>0\,(x_1 < x_1^*)} \underbrace{- \underbrace{[u_2^{'}(x_1) - u_1^{'}(x_1)]}_{>0}}$$

Marginal profit from consumer 1: a change in the quantity supplied to this consumer implies a change in the profit obtained by the monopolist from him/her.

Marginal profit from consumer 2: a change in the quantity supplied to consumer 1 implies a change in the surplus the monopolist must leave consumer 2 to avoid personal arbitrage.

$$\frac{\partial \Pi(x_1^*)}{\partial x_1} = \underbrace{u_1^{'}(x_1^*) - c}_{=0} - \underbrace{[u_2^{'}(x_1^*) - u_1^{'}(x_1^*)]}_{>0} < 0$$

Starting from x_1^* a reduction in the quantity supplied to consumer 1 increases the profit because the surplus that the monopolist must leave consumer 2 to avoid arbitrage is reduced. An output such that $\tilde{x}_1 < x_1 < x_1^*$ satisfies the following:

$$\frac{\partial \Pi(x_1)}{\partial x_1} = \underbrace{u_1^{'}(x_1) - c}_{>0} - \underbrace{[u_2^{'}(x_1) - u_1^{'}(x_1)]}_{>0} < 0$$

It is worthwhile for the monopolist to continue reducing x_1 because the increase in profits from the high-demand consumer (obtained by leaving him/her with a lower surplus) offsets the loss of profits from the low-demand consumer obtained by supplying him/her a lower quantity.

$$\frac{\partial \Pi(\tilde{x}_1)}{\partial x_1} = u_1^{'}(\tilde{x}_1) - c - \underbrace{[u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)]}_{>0} = 0$$

In output $\tilde{x}_1$ the marginal gain, from an infinitesimal reduction in x_1, from the high-demand consumer by leaving him/her with lower surplus is just equal to the marginal loss from the low-demand consumer as a result of offering a lower quantity.

Moreover, the monopolist charges the low-demand consumer a price (tariff) equal to the maximum willingness to pay, thus leaving him/her with no surplus: $\tilde{r}_1 = u_1(\tilde{x}_1)$.

(vi) *Under what conditions does the monopolist offer the good to both consumers?*

The monopolist will decide to offer the good to both consumers if it obtains more profits than by selling the good only to the high-demand consumer. That is, the monopolist supplies the good to both consumers if the following is satisfied:

$$\Pi(0,x_2^*) \leq \Pi(\tilde{x}_1,\tilde{x}_2)$$

$$u_2(x_2^*) - cx_2^* \leq \underbrace{u_1(\tilde{x}_1)}_{\tilde{r}_1} - c\tilde{x}_1 + \underbrace{u_2(x_2^*) - [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)]}_{\tilde{r}_2} - cx_2^*$$

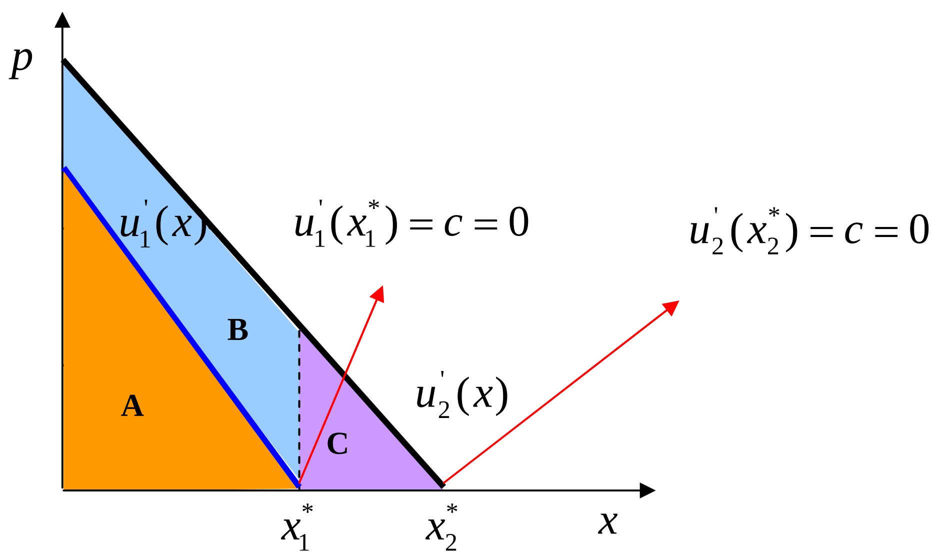
$$[u_2(\tilde{x}_1) - u_1(\tilde{x}_1)] \leq u_1(\tilde{x}_1) - c\tilde{x}_1$$

If this condition is not satisfied, the monopolist offers the good only to the high-demand consumer. Another equivalent way of looking at the problem consists of considering the marginal profit of x_1. If it were negative for any level of x_1

$$\frac{\partial \Pi(x_1)}{\partial x_1} = \underbrace{u_1^{'}(x_1) - c}_{>0} - \underbrace{[u_2^{'}(x_1) - u_1^{'}(x_1)]}_{>0} < 0 \quad \forall x_1$$

then the monopolist would decide not to sell the good to the low-demand consumer given that for any level of x_1 it would increase profits by reducing the quantity supplied to the low-demand consumer.

(vi) *Graphic representation (zero marginal cost)*



Perfect price discrimination

$$\left\langle \begin{matrix} (r_i^*, x_i^*) \\ (0,0) \end{matrix} \right. \quad i = 1,2$$

$$u_1^{'}(x_1^*) = u_2^{'}(x_2^*) = \underbrace{0}_{c}$$

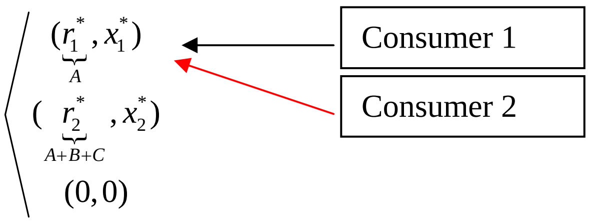
$$r_1^* = u_1(x_1^*) \equiv A$$

$$r_2^* = u_2(x_2^*) \equiv A + B + C$$

$$\Pi^* = u_1(x_1^*) + u_2(x_2^*) \equiv \underbrace{A}_{r_1^*} + \underbrace{A + B + C}_{r_2^*}$$

No identification

Assume that the monopolist does not know the identity of the consumer and that it states a unique price menu where it maintains the price-quantity bundles which were optimal under perfect price discrimination. Consumer 2 would have incentives to engage in personal arbitrage.



$$0 = \underbrace{A + B + C}_{u_2(x_2^*)} - \underbrace{(A + B + C)}_{r_2^*} < \underbrace{A + B}_{u_2(x_1^*)} - \underbrace{A}_{r_1^*} = B$$

Second-degree price discrimination

The following conditions are satisfied with equality:

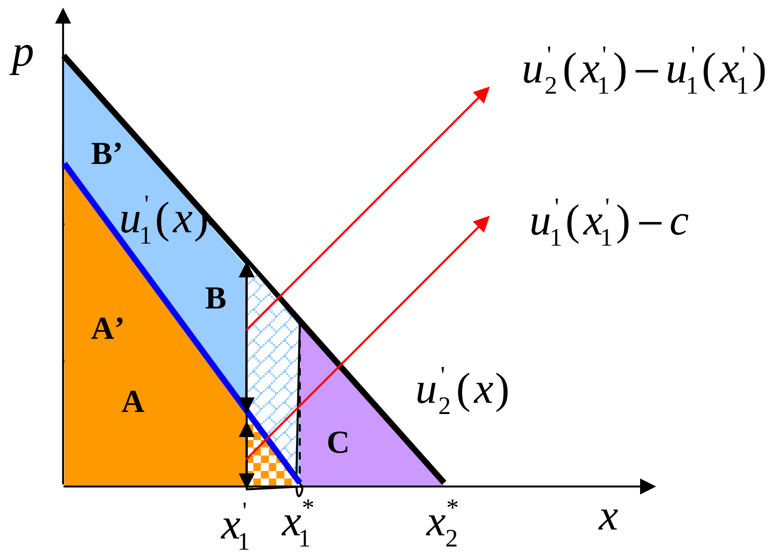
$r_1 = u_1(x_1) \equiv A(x_1) \rightarrow$ the monopolist charges consumer 1 the area below his/her inverse demand function.

$u_2(x_2) - r_2 = u_2(x_1) - r_1 \equiv B(x_1) \rightarrow$ the monopolist leaves consumer 2 with a surplus $B(x_1)$ (the minimum) in order to avoid arbitrage.

Firstly, we maintain quantities and only adjust the tariffs.

$$\left\langle \begin{array}{l} (\underbrace{\overline{r}_1}_{A}, x_1^*) \\ (\underbrace{\overline{r}_2}_{A+C}, x_2^*) \\ (0,0) \end{array} \right. \qquad \Pi(x_1^*, x_2^*) = 2A + C$$

$$\Pi(x_1^{'}, x_2^*) = A' + A + B + C - B'$$
$$\Pi(x_1^{'}, x_2^*) - \Pi(x_1^*, x_2^*) \equiv -(A - A') + (B - B') > 0$$

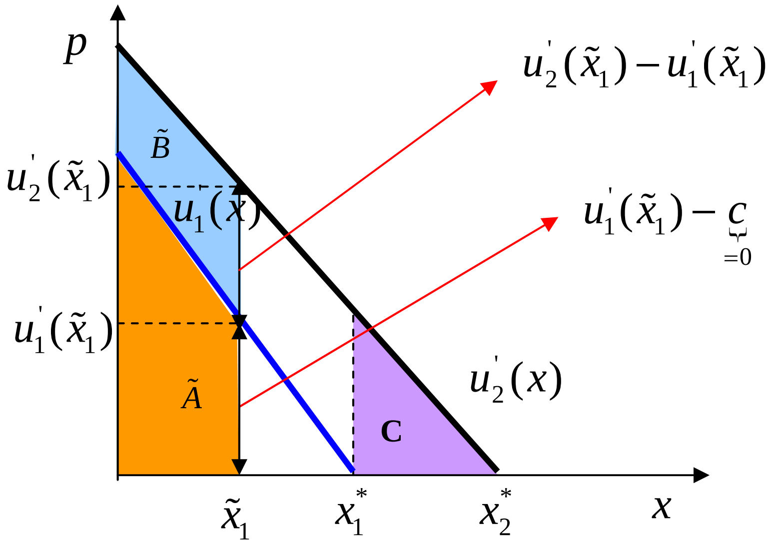


$$\left\langle \begin{array}{l} (\tilde{r}_1, \tilde{x}_1) \\ (\tilde{r}_2, \tilde{x}_2) \\ (0,0) \end{array} \right.$$

$$\frac{\partial \Pi(\tilde{x}_1)}{\partial x_1} = u_1^{'}(\tilde{x}_1) - c - \underbrace{[u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)]}_{>0} = 0$$

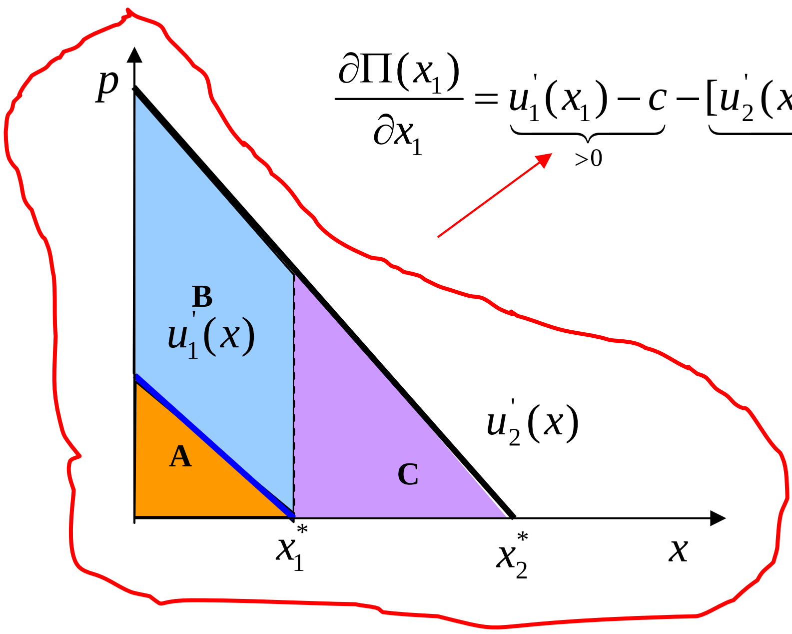
As we are assuming that the marginal cost is zero:

$$\frac{\partial \Pi(\tilde{x}_1)}{\partial x_1} = u_1^{'}(\tilde{x}_1) - \underbrace{[u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)]}_{>0} = 0 \rightarrow u_1^{'}(\tilde{x}_1) = u_2^{'}(\tilde{x}_1) - u_1^{'}(\tilde{x}_1)$$



$$\Pi(\tilde{x}_1,\underbrace{\tilde{x}_2}_{x_2^*})=u_1(\tilde{x}_1)-\underbrace{c}_{0}\tilde{x}_1+u_2(x_2^*)-[u_2(\tilde{x}_1)-u_1(\tilde{x}_1)]-\underbrace{c}_{0}x_2^*\equiv\tilde{A}+A+B+C-\tilde{B}$$

The decision to supply the good only to the high-demand consumer.



$$\frac{\partial\Pi(x_1)}{\partial x_1}=\underbrace{u_1'(x_1)-c}_{>0}-\underbrace{[u_2'(x_1)-u_1'(x_1)]}_{>0}<0\quad\forall x_1$$

$$\left\langle\begin{array}{l}(\underbrace{r_2^*}_{A+B+C},x_2^*)\\(0,0)\end{array}\right.$$

1.8. ***Third-degree price discrimination***

(i) Definition and context.

(ii) Profit maximization. The rule of the inverse of elasticity.

(iii) A comparison of profits with the case of uniform pricing (single monopoly pricing).

(iv) Effects on social welfare.

(i)*Definition and context*

There is third-degree price discrimination when consumers belonging to different groups or submarkets are charged different prices, although each consumer pays the same price for each unit bought. This is probably the most common type of price discrimination. Examples: discounts to students, senior citizens etc.

The monopolist receives an exogenous sign which allows it to distinguish m perfectly separated markets or submarkets: $\frac{\partial x_i}{\partial p_j} = 0.$ This is a type of direct discrimination: the monopolist states different price menus for consumers belonging to different groups or markets. *Identification*: the monopolist classifies each consumer in a group.

(ii) *Profit maximization. The rule of the inverse of elasticity*

We consider the simple case of $m = 2$: the monopolist classifies consumers in two groups with inverse demand functions $p_1(x_1)$ and $p_2(x_2)$, with $p_i^{'}(x_i) < 0,\ i = 1, 2.$ The monopolist can establish different prices in the two markets but within a market it is not possible to discriminate prices. The maximization problem is:

$$\max_{x_1,x_2} \overbrace{p_1(x_1)x_1 + p_2(x_2)x_2 - c.(x_1 + x_2)}^{\Pi(x_1,x_2)}$$

$$\left.\begin{array}{l} \dfrac{\partial \Pi}{\partial x_1} = p_1(x_1) + x_1 p_1^{'}(x_1) - c = 0 \quad (1) \\ \\ \dfrac{\partial \Pi}{\partial x_2} = p_2(x_2) + x_2 p_2^{'}(x_2) - c = 0 \quad (2) \end{array}\right\} (i) \rightarrow MR_1 = MR_2 = c$$

$$(i) \rightarrow p_i(x_i) + x_i p_i^{'}(x_i) = c$$

$$p_i(x_i)[1 + \frac{x_i p_i^{'}(x_i)}{p_i(x_i)}] = c$$

$$p_i(x_i)[1 + \frac{1}{\varepsilon_i(x_i)}] = c$$

$$p_i(x_i)[1 - \frac{1}{|\varepsilon_i(x_i)|}] = c$$

$$p_i(x_i) = \frac{c}{1 - \dfrac{1}{|\varepsilon_i(x_i)|}} \qquad i = 1, 2.$$

Therefore, $p_1(x_1) > p_2(x_2)$ iff $|\varepsilon_1(x_1)| < |\varepsilon_2(x_2)|$. As a consequence, the monopolist charges the highest price in the market with the lower price elasticity (in absolute value).

(iii) *A comparison of profits with the case of uniform pricing (single monopoly pricing)*

The monopolist's profit under third-degree price discrimination is at least as high as the profit under uniform pricing. The reason is simple: under third-degree price discrimination the firm can always choose equal prices if that is the most profitable option.

(iv) *Effects on social welfare*

1) What is the problem?

2) Bounds of the change in social welfare.

3) Applications:

a) Linear demand.

b) Opening of markets.

1) **What is the problem?**

This section compares third-degree price discrimination and uniform pricing from a social welfare point of view. In general, a movement from uniform pricing to third-degree price discrimination benefits some agents and harms others.

Benefited by T-DPD: the monopolist and the consumers in the higher-elasticity market (given that the price is reduced by discrimination).

Harmed by T-DPD: the consumers in the lower-elasticity market (given that the price is increased by discrimination).

Therefore, the effect on social welfare is indeterminate.

2) **Bounds of the change in social welfare**

Assume for the sake of simplicity that there are only two markets and we start from an aggregate utility function $u_1(x_1)+u_2(x_2)+y_1+y_2$, where x_1 and x_2 are the consumptions of good x by the two groups and y is the money to be spent on other goods ($y = y_1 + y_2$).

u_1 and u_2 are strictly concave. The inverse demand functions are given by $p_1(x_1) = u_1^{'}(x_1)$ and $p_2(x_2) = u_2^{'}(x_2).$

If $C(x_1, x_2)$ is the cost of supplying x_1 and x_2 we can measure the social welfare as:

$$W(x_1, x_2) = u_1(x_1) + u_2(x_2) - C(x_1, x_2)$$

Consider two configurations of output (x_1^0, x_2^0) and (x_1^1, x_2^1) whose prices are (p_1^0, p_2^0) and $(p_1^1, p_2^1),$ respectively. Assume that the initial set of prices corresponds to uniform pricing (the monopoly single price) $p_1^0 = p_2^0 = p^0$ and that p_1^1 and p_2^1 are the prices under third-degree price discrimination. Consider the movement from $\boldsymbol{x}^0$ to $\boldsymbol{x}^1.$ Due to the strictly concavity of u we have (see Appendix):

$$\left.\begin{array}{l} u_1(x_1^1) < u_1(x_1^0) + \overbrace{u_1^{'}(x_1^0)}^{p_1(x_1^0)=p_1^0}\overbrace{(x_1^1 - x_1^0)}^{\Delta x_1} \quad (1) \to \Delta u_1 < p_1^0 \Delta x_1 \\ u_1(x_1^0) < u_1(x_1^1) + \underbrace{u_1^{'}(x_1^1)}_{p_1(x_1^1)=p_1^1}\underbrace{(x_1^0 - x_1^1)}_{-\Delta x_1} \quad (1)' \to \Delta u_1 > p_1^1 \Delta x_1 \end{array}\right\} \to p_1^0 \Delta x_1 > \Delta u_1 > p_1^1 \Delta x_1 \quad (3)$$

$$\left.\begin{array}{l} u_2(x_2^1) < u_2(x_2^0) + \overbrace{u_2^{'}(x_2^0)}^{p_2(x_1^0)=p_2^0}\overbrace{(x_2^1 - x_2^0)}^{\Delta x_2} \quad (2) \to \Delta u_2 < p_2^0 \Delta x_2 \\ u_2(x_2^0) < u_2(x_2^1) + \underbrace{u_2^{'}(x_2^1)}_{p_2(x_1^1)=p_2^1}\underbrace{(x_2^0 - x_2^1)}_{-\Delta x_2} \quad (2)' \to \Delta u_2 > p_2^1 \Delta x_2 \end{array}\right\} \to p_2^0 \Delta x_2 > \Delta u_2 > p_2^1 \Delta x_2 \quad (4)$$

By adding (3) and (4) we get

$$p_1^0 \Delta x_1 + p_2^0 \Delta x_2 > \Delta u_1 + \Delta u_2 > p_1^1 \Delta x_1 + p_2^1 \Delta x_2$$

where

$$\Delta u = \Delta u_1 + \Delta u_2;\ \Delta x_1 = x_1^1 - x_1^0;\ \Delta x_2 = x_2^1 - x_2^0$$
$$p_1^0 = p_1(x_1^0) = u_1^{'}(x_1^0);\ p_2^0 = p_2(x_2^0) = u_2^{'}(x_2^0);$$
$$p_1^1 = p_1(x_1^1) = u_1^{'}(x_1^1);\ p_2^1 = p_2(x_2^1) = u_2^{'}(x_2^1).$$

The change in social welfare is given by:

$$\Delta W = W(x_1^1, x_2^1) - W(x_1^0, x_2^0) = \underbrace{u_1(x_1^1) - u_1(x_1^0)}_{\Delta u_1} + \underbrace{u_2(x_2^1) - u_2(x_2^0)}_{\Delta u_2} - [\underbrace{C(x_1^1, x_2^1) - C(x_1^0, x_2^0)}_{\Delta C}]$$
$$= \Delta u_1 + \Delta u_2 - \Delta C$$

Therefore

$$p_1^0 \Delta x_1 + p_2^0 \Delta x_2 - \Delta C > \Delta W > p_1^1 \Delta x_1 + p_2^1 \Delta x_2 - \Delta C$$

If marginal cost is constant:

$$\Delta C = c(x_1^1 + x_2^1) - c(x_1^0 + x_2^0) = c\Delta x_1 + c\Delta x_2$$

Therefore the bounds of the change in social welfare become:

$$\underbrace{(p_1^0 - c)\Delta x_1 + (p_2^0 - c)\Delta x_2}_{\text{Upper bound}} > \Delta W > \underbrace{(p_1^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2}_{\text{Lower bound}} \quad (5)$$

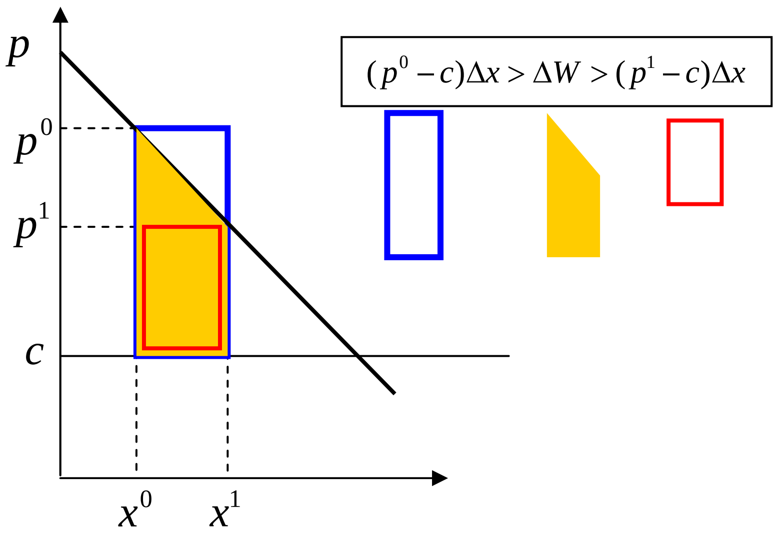
Given that $p_1^0 = p_2^0 = p^0$ the bounds of the change in social welfare are:

$$\underbrace{(p^0 - c)\overbrace{(\Delta x_1 + \Delta x_2)}^{\Delta x}}_{\text{Upper bound}} > \Delta W > \underbrace{(p_1^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2}_{\text{Lower bound}} \quad (6)$$

- Upper bound: this implies that a necessary condition for third-degree price discrimination to increase social welfare, $\Delta W > 0,$ is that it should increase total output. Assume on the contrary that $\Delta x = \Delta x_1 + \Delta x_2 \leq 0.$ Given that $(p^0 - c) > 0$ then $(4) \rightarrow \Delta W < 0.$

- Lower bound: this indicates that a sufficient condition for third-degree price discrimination to increase social welfare is that the sum of the changes in output weighted by the difference between the price under discrimination and the marginal cost must be positive.

Graphically, for the case of a single market, the bounds would be:



3) **Applications**

a) ***Linear demands***

Assume that the demands are given by $x_i(p_i) = \frac{a_i}{b_i} - \frac{1}{b_i} p_i,\ i = 1,2,$ and that the marginal cost is zero, $c = 0$. The profit maximization problem under third-degree price discrimination is:

$$\max_{p_1, p_2} p_1 x_1(p_1) + p_2 x_2(p_2)$$

$$\frac{\partial \Pi}{\partial p_1} = x_1(p_1) + p_1 x_1'(p_1) = 0 \rightarrow \frac{a_1}{b_1} - \frac{1}{b_1} p_1 - \frac{1}{b_1} p_1 = 0 \rightarrow p_1^1 = \frac{a_1}{2};\ x_1^1 = \frac{a_1}{2b_1}$$

$$\frac{\partial \Pi}{\partial p_2} = x_2(p_2) + p_2 x_2'(p_2) = 0 \rightarrow \frac{a_2}{b_2} - \frac{1}{b_2} p_2 - \frac{1}{b_2} p_2 = 0 \rightarrow p_2^1 = \frac{a_2}{2};\ x_2^1 = \frac{a_2}{2b_2}$$

The total output is:

$$x^1 = x_1^1 + x_2^1 = \frac{a_1}{2b_1} + \frac{a_2}{2b_2} = \frac{a_1 b_2 + a_2 b_1}{2 b_1 b_2}$$

Under uniform pricing:

$$\max_{p} p x_1(p) + p x_2(p)$$

$$\frac{\partial \Pi}{\partial p} = x_1(p) + x_2(p) + p x_1'(p) + p x_2'(p) \rightarrow \frac{a_1}{b_1} - \frac{1}{b_1} p + \frac{a_2}{b_2} - \frac{1}{b_2} p - \frac{1}{b_1} p - \frac{1}{b_2} p = 0$$

$$\rightarrow p^0 = \frac{a_1 b_2 + a_2 b_1}{2(b_1 + b_2)};$$

$$x_1^0 = \frac{a_1}{b_1} - \frac{1}{b_1}\frac{a_1b_2 + a_2b_1}{2(b_1+b_2)} = \frac{2a_1b_1 + 2a_1b_2 - a_1b_2 - a_2b_1}{2b_1(b_1+b_2)} = \frac{2a_1b_1 + a_1b_2 - a_2b_1}{2b_1(b_1+b_2)}$$
$$x_1^0 = \frac{a_2}{b_2} - \frac{1}{b_2}\frac{a_2b_1 + a_1b_2}{2(b_1+b_2)} = \frac{2a_2b_2 + 2a_2b_1 - a_2b_1 - a_1b_2}{2b_2(b_1+b_2)} = \frac{2a_2b_2 + a_2b_1 - a_1b_2}{2b_2(b_1+b_2)}$$

The total output is:

$$x^0 = x_1^0 + x_2^0 = \frac{2a_1b_1 + a_1b_2 - a_2b_1}{2b_1(b_1+b_2)} + \frac{2a_2b_2 + a_2b_1 - a_1b_2}{2b_2(b_1+b_2)}$$
$$= \frac{2a_1b_1b_2 + a_1(b_2)^2 - a_2b_1b_2 + 2a_2b_1b_2 + a_2(b_1)^2 - a_1b_1b_2}{2b_1b_2(b_1+b_2)}$$
$$= \frac{a_1b_1b_2 + a_1(b_2)^2 + a_2b_1b_2 + a_2(b_1)^2}{2b_1b_2(b_1+b_2)} = \frac{(a_1b_2 + a_2b_1)(b_1+b_2)}{2b_1b_2(b_1+b_2)} = \frac{a_1b_2 + a_2b_1}{2b_1b_2}$$

Therefore, total output is the same under both pricing policies. That is, $\Delta x = \Delta x_1 + \Delta x_2 = 0,$ or, equivalently, $\Delta x_1 = -\Delta x_2.$ The bounds would be

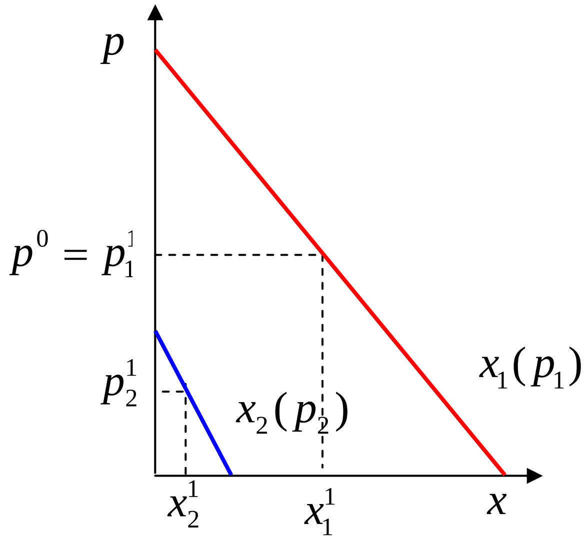
$$(p^0 - c)\underbrace{\overbrace{(\Delta x_1 + \Delta x_2)}^{\Delta x}}_{=0} > \Delta W > \underbrace{(p_1^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2}_{<0} \quad (6)$$

Social welfare therefore decreases: $\Delta W < 0$.

As we show below, the above result depends crucially on the assumption that all markets are served under uniform pricing.

b) ***Opening of markets***

Imagine that the two markets demands are like those in the graphic.



If the monopolist had to sell at a uniform price, it would have to reduce the price in market 1 by such an amount that the decrease in profits in that market would not be offset. Therefore,

$$\underbrace{(p_1^1}_{p^0} - c)\underbrace{(\underbrace{\Delta x_1}_{=0} + \underbrace{\Delta x_2}_{>0})}_{>0} > \Delta W > \underbrace{\underbrace{(p_1^1 - c)\underbrace{\Delta x_1}_{=0}}_{=0} + \underbrace{(p_2^1 - c)\underbrace{\Delta x_2}_{>0}}_{>0}}_{>0} \quad (6)$$

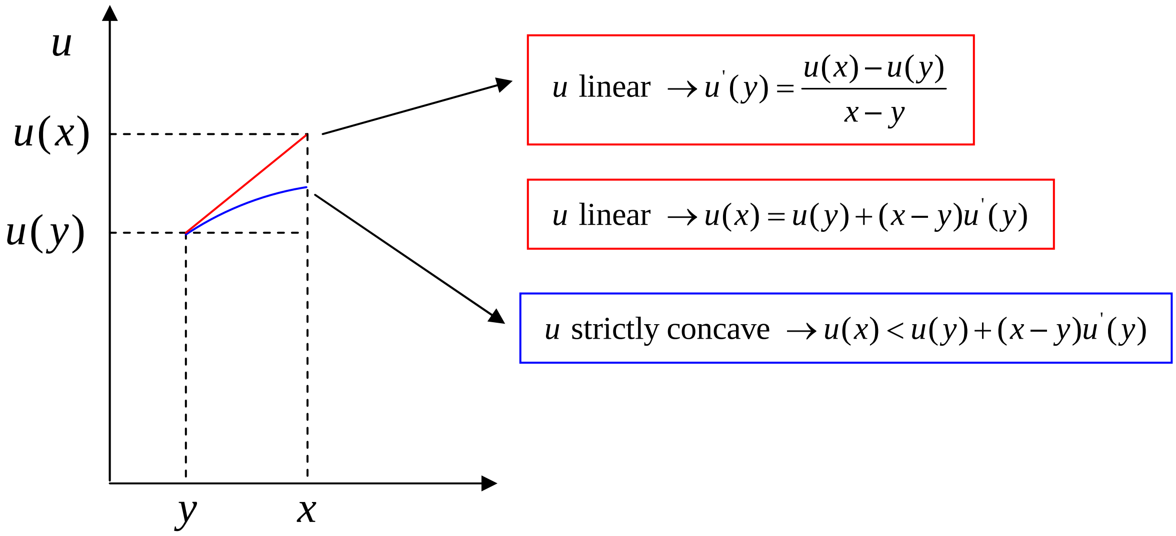
Given that the lower bound in (4) is positive then $\Delta W > 0$. But not only does social welfare increase, in fact third-degree price discrimination Pareto dominates uniform pricing. A move from uniform pricing to third-degree price discrimination implies an increase in the monopolist's profits, an improvement for consumers in market 2 and no change for consumers in market 1.

Appendix

If u is a strictly concave function for any $\boldsymbol{x}$ and $\boldsymbol{y}$ the following is satisfied:

$$u(x) < u(y) + u^{'}(y)(x-y).$$

The tangents always remain above the function when it is strictly concave.



Chapter 2. Non-Cooperative Game Theory

Introduction

The Theory of Non-Cooperative Games studies and models *conflict situations* among economic agents; that is, it studies situations where the profits (gains, utility or payoffs) of each economic agent depend not only on his/her own acts but also on the acts of the other agents.

We assume *rational players* so each player will try to maximize his/her profit function (utility or payoff) given his/her conjectures or beliefs on how the other players are going to play. The outcome of the game will depend on the acts of all the players.

A fundamental characteristic of non-cooperative games is that it is *not* possible to sign *contracts* between players. That is, there is no external institution (for example, courts of justice) capable of enforcing the agreements. In this context, co-operation among players only arises as an equilibrium or solution proposal if the players find it in their best interest.

For each game we try to propose a “solution”, which should be a reasonable prediction of *rational behaviour* by players (OBJECTIVE).

We are interested in Non-Cooperative Game Theory because it is very useful in modelling and understanding *multi-personal* economic problems characterized by *strategic interdependency*. Consider, for instance, competition between firms in a market. Perfect competition and pure monopoly (not threatened by entry) are special non-realistic cases. It is more frequent in real life to find industries with not many firms (or with a lot of firms but with just a few of them producing a large part of the total production). With few firms, competence between them is characterized by strategic considerations: each firm takes its decisions (price, output, advertising, etc.) taking into account or conjecturing the behaviour of the others. Therefore, competition in an oligopoly can be seen as a non-cooperative game where the firms are the players. Many predictions or solution proposals arising from Game

Theory prove very useful in understanding competition between economic agents under strategic interaction.

Section 2 defines the main notions of Game Theory. We shall see that there are two ways of representing a game: the extensive form and the strategic form. In Section 3 we analyze the main solution concepts and their problems; in particular, we study the Nash equilibrium and its refinements. Section 4 analyzes repeated games and, finally, Section 5 offers concluding remarks.

2.1. **Basic notions**

There are two ways of representing a game: the extensive form and the strategic form. We start by analyzing the main elements of an extensive form game.

2.1.1. **Games in extensive form** (dynamic or sequential games)

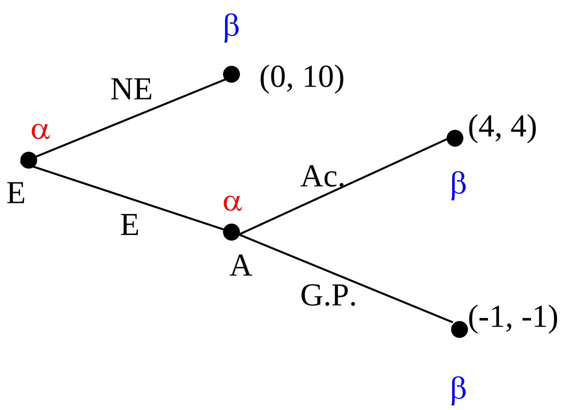
An extensive from game specifies:

1) The players.
2) The order of the game.
3) The choices available to each player at each turn of play (at each decision node).
4) The information held by each player at each turn of play (at each decision node).
5) The payoffs of each player as a function of the movements selected.
6) Probability distributions for movements made by nature.

An extensive form game is represented by a decision tree. A decision tree comprises nodes and branches. There are two types of node: decision nodes and terminal nodes. We have to assign each decision node to one player. When the decision node of a player is reached the player chooses a move. When a terminal node is reached the players obtain payoffs: an assignment of payoffs for each player.

EXAMPLE 1: Entry game

Consider a market where there are two firms: an incumbent firm, A, and a potential entrant, E. At the first stage, the potential entrant decides whether or not to enter the market. If it decides "not to enter" the game concludes and the players obtain payoffs (firm A obtains the monopoly profits) and if it decides "to enter" then the incumbent firm, A, has to decide whether to accommodate entry (that is, to share the market with the entrant) or to start a mutually injurious war price. The extensive form game can be represented as follows:



Players: E and A.

Actions: *E* (to enter), *NE* (not to enter), *Ac.* (to accommodate), *G.P.* (price war).

Decision nodes: α.

Terminal nodes: β.

(x, y): vector of payoffs. x: payoof of player E; y: payoff of player A.

At each terminal node we have to specify the payoffs of each player (even though some of them have not actually managed to play).

Assumptions:

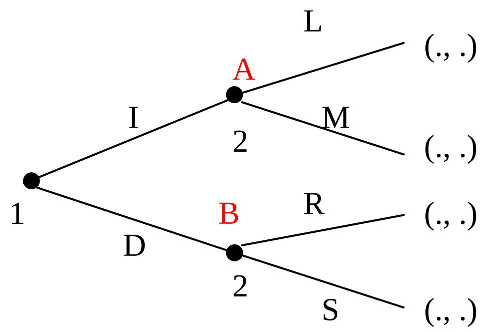
(i) All players have the same perception of how the game is.

(ii) Complete information: each player knows the characteristics of the other players: preferences and strategy spaces.

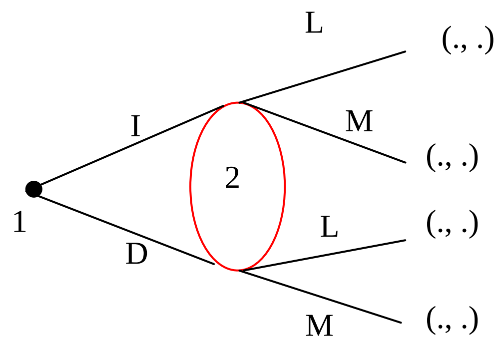
(iii) Perfect recall: each player remembers his/her previous behaviour in the game.

Definition 1: **Information set**

"The information available to each player at each one of his/her decision nodes".



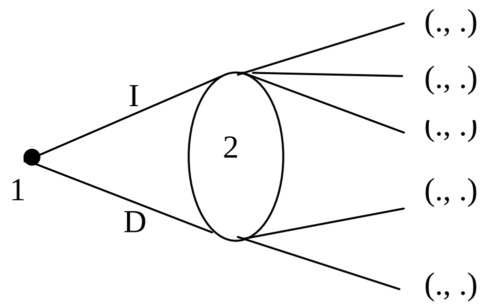
Game 1



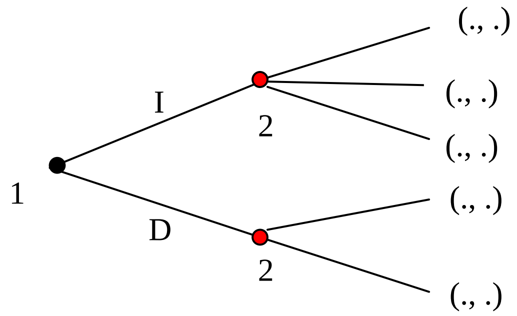
Game 2

In game 1, player 2 has different information at each one of his/her decision nodes. At node *A*, if he/she is called upon to play he/she knows that the player 1 has played *I* and at *B* he/she knows that player 1 has played *D*. We say that these information sets are singleton sets consisting of only one decision node. *Perfect information game*: a game where all the information sets are singleton sets or, in other words, a game where all the players know everything that has happened previously in the game. In game 2, the player 2 has the same information at both his/her decision nodes. That is, the information set is composed of two decision nodes. Put differently, player 2 does not know which of those nodes he or she is at. A game in which there are information sets with two or more decision nodes is called an *imperfect information game*: at least one player does not observe the behaviour of the other(s) at one or more of his/her decision nodes.

The fact that players know the game that they are playing and the perfect recall assumption restrict the situations where we can find information sets with two or more nodes.

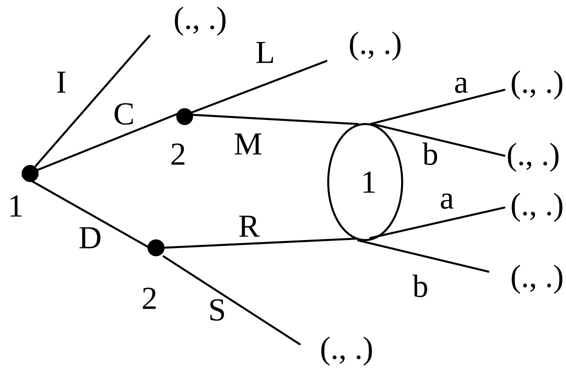


Game 3

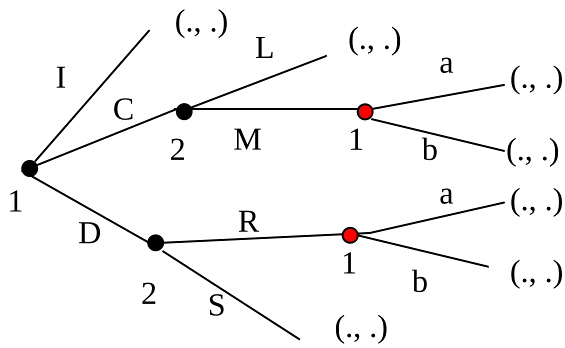


Game 4

Game 3 is poorly represented because it would not be an imperfect information game. Assuming that player 2 knows the game, if he/she is called on to move and faces three alternatives he/she would inmediatley deduce that the player 1 has played I. That is, the game should be represented like game 4. Therefore, *if an information set consists of two or more nodes the number of alternatives, actions or moves at each one should be the same*.



Game 5



Game 6

The assumption of perfect recall avoids situations like that in game 5. When player 1 is called on to play at his/her second decision node perfectly recall his/her behaviour at his/her first decision node. The extensive form should be like that of game 6.

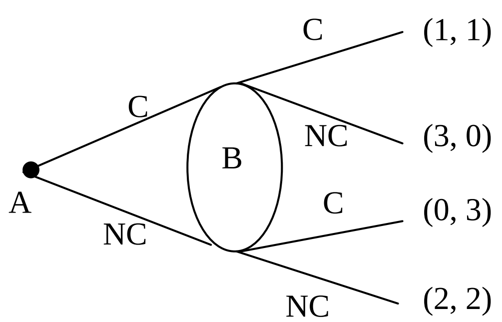
Definition 2: **Subgame**

"It is what remains to be played from a decision node with the condition that what remains to be played does not form part of an information set with two or more decision nodes. To build subgames we look at parts of the game tree that can be constructed without breaking any information sets. An information set starts at a singleton information set and all the decision nodes of the same information set must belong to the same subgame."

EXAMPLE 2: The Prisoner's Dilemma

Two prisoners, A and B, are being held by the police in separate cells. The police know that the two (together) committed a crime but lack sufficient evidence to convict them. So the police offer each of them separately the following deal: each is asked to implicate his partner. Each prisoner can "confess" (*C*) or "not confess" (*NC*). If neither confesses then each player goes to jail for one month. If both players confess each prisoner goes to jail for three months. If one prisoner confesses and the other does not confess, the first player goes free while the second goes to jail for six months.

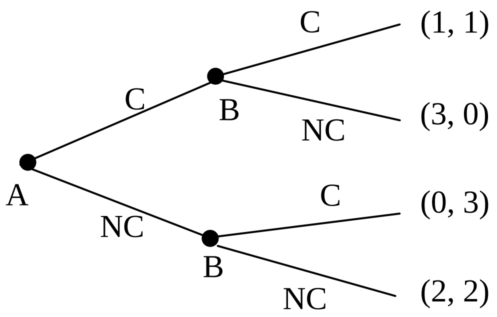
- **Simultaneous case**: each player takes his decision with no knowledge of the decision of the other.



PD1

There is an information set with two decision nodes. This is an imperfect information game. There is a subgame which coincides with the proper game.

- **Sequential game**: the second player observes the choice made by the first.



PD2

Game PD2 is a perfect information game and there are three subgames. “In perfect information games there are as many subgames as there are decision nodes”.

Definition 3: **Strategy**

“A player’s strategy is a complete description of what he/she would do if he/she were called on to play at each one of his/her decision nodes. It needs to be specified even in those nodes not attainable by him/her given the current behavior of the other(s) player(s)”. It is a *behaviour plan* or *conduct plan*. (Examples: consumer demand, supply from a competitive firm.). It is a player’s function which assigns an action to each of his/her decision nodes (or to each of his/her information sets). A player’s strategy has as many components as the player has information sets.

Definition 4: **Action**

“A choice (decision or move) at a decision node”.

Actions are physical while strategies are conjectural.

Definition 5: **Combination of strategies or strategy profile**

"A specification of one strategy for each player". The result (the payoff vector) must be unequivocally determined.

EXAMPLE 1: The entry game

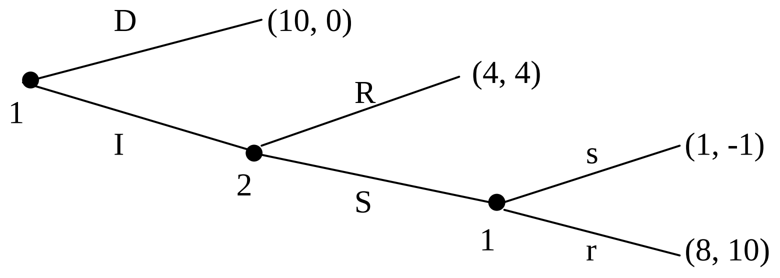
This is a perfect information game with two subgames. Each player has two strategies: $S_E = \{NE, E\}$ and $S_A = \{Ac., G.P.\}$. Combinations of strategies: (*NE*, *Ac.*), (*NE*, *G.P.*), (*E*, *Ac.*) and (*E*, *G.P.*).

EXAMPLE 2: The Prisoner's Dilemma

PD1: This is an imperfect information game with one subgame. Each player has two strategies: $S_A = \{C, NC\}$ and $S_B = \{C, NC\}$. Combinations of strategies: (*C*, *C*), (*C*, *NC*), (*NC*, *C*) and (*NC*, *NC*).

PD2: This is a perfect information game with three subgames. Player A has two strategies $S_A = \{C, NC\}$ but player B has four strategies $S_B = \{CC, CNC, NCC, NCNC\}$. Combinations of strategies: (*C*, *CC*), (*C*, *CNC*), (*C*, *NCC*), (*C*, *NCNC*), (*NC*, *CC*), (*NC*,*CNC*), (*NC*, *NCC*) and (*NC*, *NCNC*).

EXAMPLE 3



Player 1 at his/her first node has two possible actions, *D* and *I*, and two actions also at his/her second: *s* and *r*. $S_1 = \{Ds, Dr, Is, Ir\}$ and $S_2 = \{R, S\}$.

2.1.2. **Games in normal or strategic form** (simultaneous or static games)

A game in normal or strategic form is described by:

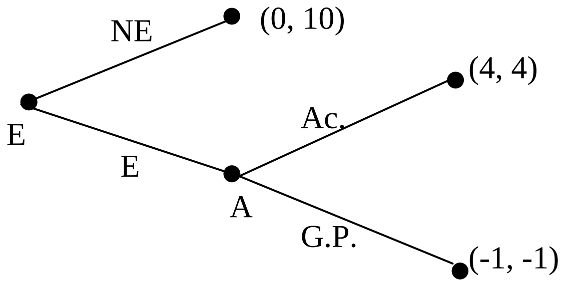
1) The players.

2) The set (or space) of strategies for each player.

3) A payoff function which assigns a payoff vector to each combination of strategies.

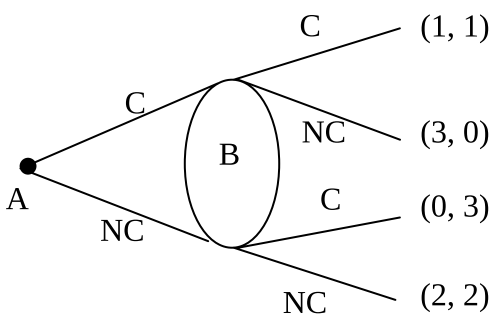
The key element of this way of representing a game is the description of the payoffs of the game as a function of the strategies of the players, without explaining the actions taken during the game. In the case of two players the usual representation is a bimatrix form game where each row corresponds to one of the strategies of one player and each column corresponds to one strategy of the other player.

EXAMPLE 1: The entry game

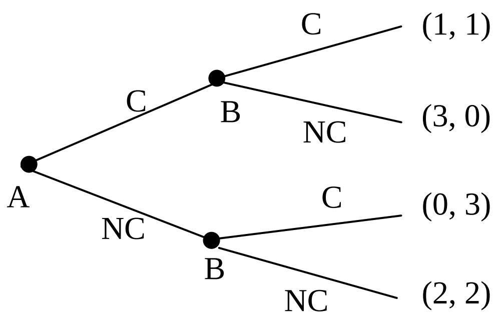


		A: Ac.	A: G.P.
E	NE	(0, 10)	(0, 10)
E	E	(4, 4)	(-1, -1)

EXAMPLE 2: The Prisoner's Dilemma

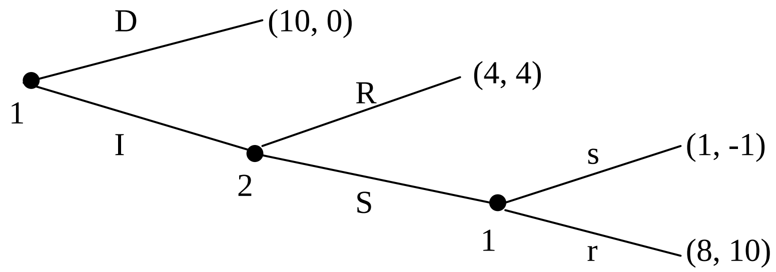


		B: C	B: NC
A	C	(1, 1)	(3, 0)
A	NC	(0, 3)	(2, 2)



		B			
		CC	CNC	NCC	NCNC
A	C	(1, 1)	(1, 1)	(3, 0)	(3, 0)
	NC	(0, 3)	(2, 2)	(0, 3)	(2, 2)

EXAMPLE 3



		2	
		R	S
1	Ds	(10, 0)	(10, 0)
	Dr	(10, 0)	(10, 0)
	Is	(4, 4)	(1, -1)
	Ir	(4, 4)	(8, 10)

Link between games in normal form and games in extensive form

a) For any game in extensive form there exists a unique corresponding game in normal form. This is due to the game in normal form being described as a function of the strategies of the players.

b) (Problem) Different games in extensive form can have the same normal (or strategic) form. (Example: in the prisoner's dilemma, PD1, if we change the order of the game then the game in extensive form also changes but the game in normal form does not change).

2.2. **Solution concepts (criteria) for noncooperative games**

The general objective is to predict how players are going to behave when they face a particular game. NOTE: "A solution proposal is (not a payoff vector) a combination of strategies, one for each player, which lead to a payoff vector". We are interested in predecting behaviour, not gains.

Notation

i: Representative player, $i = 1,\ldots, n$

S_i: set or space of player *i*'s strategies.

$s_i \in S_i$: a strategy of player *i*.

$s_{-i} \in S_{-i}$: a strategy or combination of strategies of the other player(s).

$\Pi_i(s_i, s_{-i})$: the profit or payoff of player *i* corresponding to the combination of strategies $s \equiv (s_1, s_2, \ldots.., s_n) \equiv (s_i, s_{-i})$.

2.2.1. **Dominance criterion**

Definition 6: **Dominant strategy**

"A strategy is strictly dominant for a player if it leads to strictly better results (more payoff) than any other of his/her strategies no matter what combination of strategies is used by the other players".

"s_i^D is a strictly dominant strategy for player *i* if $\Pi_i(s_i^D, s_{-i}) > \Pi_i(s_i, s_{-i}), \forall s_i \in S_i, s_i \neq s_i^D; \forall s_{-i}$"

EXAMPLE 2: The Prisoner's Dilemma

In game PD1 "confess", *C*, is a (strictly) dominant strategy for each player. Independently of the behavior of the other player the best each player can do is "confess".

The presence of dominant strategies leads to a solution of the game. We should expect each player to use his/her dominant strategy. The solution proposal for game DP1 is the combination of strategies (*C*, *C*).

Definition 7: **Strict dominance**

"One strategy strictly dominates another when it leads to strictly better results (more payoff) than the other no matter what combination of strategies is used by the other players".

"If $\Pi_i(s_i^d, s_{-i}) > \Pi_i(s_i^{dd}, s_{-i}), \forall s_{-i}$, then s_i^d strictly dominates s_i^{dd}".

Obviously, one strategy is strictly dominated for a player when there is another strategy which dominates it. The dominance criterion consists of the iterated deletion of strictly dominated strategies.

EXAMPLE 4

1 \ 2	t_1	t_2	t_3
s_1	(4, 3)	(2, 7)	(0, 4)
s_2	(5, 5)	(5, -1)	(-4, -2)

In this game there are no dominant strategies. However, the existence of dominated strategies allows us to propose a solution. We next apply the dominance criterion. Strategy t_3 is strictly dominated by strategy t_2 so player 1 can conjecture (predict) that player 2 will never use that strategy. Given that conjecture, which assumes rationality on the part of player 2, strategy s_2 is better than strategy s_1 for player 1. Strategy s_1 would be only used in the event that player 2 used strategy t_3. If player 1 thinks player 2 is racional then he/she assigns zero probability to the event of player 2 playing t_3. In that case, player 1 should play s_2 and if player 2 is rational the best he/she can do is t_1. The criterion of iterated deletion of strictly dominated strategies (by eliminating dominated strategies and by computing the reduced games) allows us to solve the game.

EXAMPLE 5

		2: t_1	t_2
1	s_1	(10, 0)	(5, 2)
	s_2	(10, 1)	(2, 0)

In this game there are neither dominant strategies nor (strictly) dominated strategies.

Definition 8: **Weak dominance**

"One strategy weakly dominates another for a player if the first leads to results at least as good as those of the second no matter what combination of strategies is used by the other players and to strictly better results for any combination of strategies of the other players".

"If $\Pi_i(s_i^{db}, s_{-i}) \geq \Pi_i(s_i^{ddb}, s_{-i}), \forall s_{-i}$, and $\exists s_{-i}$ such that $\Pi_i(s_i^{db}, s_{-i}) > \Pi_i(s_i^{ddb}, s_{-i})$, then s_i^{db} weakly dominates s_i^{ddb}".

Thus, a strategy is weakly dominated if another strategy does at least as well for all s_{-i} and strictly better for some s_{-i}.

In example 5, strategy s_1 weakly dominates s_2. Player 2 can conjecture that player 1 will play s_1 and given this conjecture the best he/she can do would be to play t_2. By following the criterion of weak dominance (iterated deletion of weakly dominated strategies) the solution proposal would be (s_1, t_2).

However, the criterion of weak dominance may lead to problematic results, as occurs in example 6, or to no solution proposal as occurs in example 7 (because there are no dominant strategies, no dominated strategies and no weakly dominated strategies).

EXAMPLE 6

		2		
		t_1	t_2	t_3
1	s_1	(10, 0)	(5, 1)	(4, -200)
	s_2	(10, 100)	(5, 0)	(0, -100)

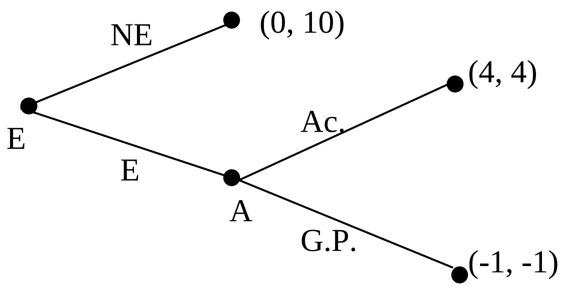
EXAMPLE 7

		2		
		t_1	t_2	t_3
1	s_1	(4, 10)	(3, 0)	(1, 3)
	s_2	(0, 0)	(2, 10)	(10, 3)

2.2.2. **Backward induction criterion**

We next use the dominance criterion to analyze the extensive form. Consider example 1.

EXAMPLE 1: The entry game



		A: Ac.	A: G.P.
E	NE	(0, 10)	(0, 10)
E	E	(4, 4)	(-1, -1)

In the game in normal form, player A has a weakly dominated strategy: *G.P.*. Player E might conjecture that and play *E*. However, player E might also have chosen *NE* in order to obtain a certain payoff against the possibility of player A playing *G.P.*.

In the game in extensive form, the solution is obtained more naturally by applying backward induction. As he/she moves first, Player E may conjecture, correctly, that if he/she plays *E* then player A (if rational) is sure to choose *Ac.*. By playing before A, player E may anticipate the rational behavior of player A. In the extensive form of the game we have more information because when player A has to move he already knows the movement of player E. The criterion of backward induction lies in applying the criterion of iterated dominance backwards starting from the last subgame(s). In example 1 in extensive form the criterion of backward induction proposes the combination of strategies (*E*, *Ac.*) as a solution.

Result: *In perfect information games with no ties, the criterion of backward induction leads to a unique solution proposal.*

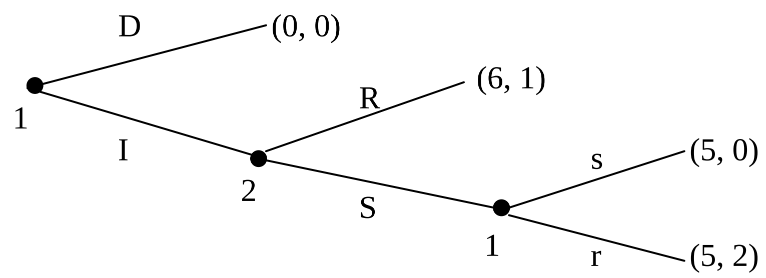
Problems

(i) *Ties*.

(ii) *Imperfect information*. Existence of information sets with two or more nodes.

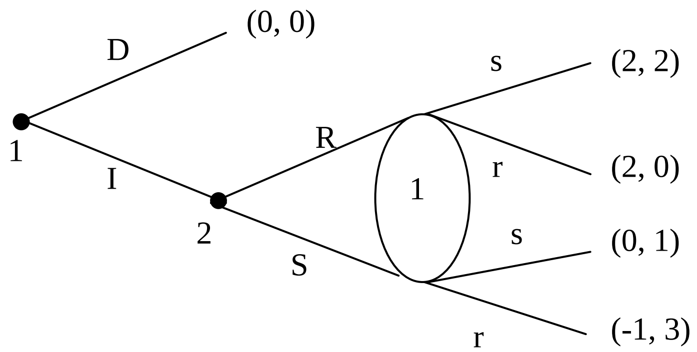
(iii) The success of backward induction is based on all conjectures about the rationality of agents checking out exactly with independence of how long the backward path is. (It may require *unbounded rationality*).

EXAMPLE 8



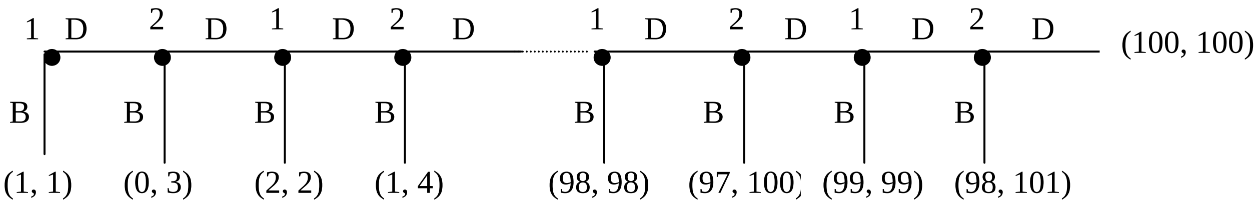
Backward induction does not propose a solution because in the last subgame player 1 is indifferent between *s* and *r*. In the previous subgame, player 2 would not have a dominated action (because he/she is unable to predict the behavior of player 1 in the last subgame).

EXAMPLE 9



We cannot apply the criterion of backward induction.

EXAMPLE 10: Rosenthal's (1981) centipede game



In the backward induction solution the payoffs are (1, 1). ¿Is another rationality possible?

2.3. **Nash equilibrium**

Player *i*, *i* = 1,…, *n*, is characterized by:

(i) A set of strategies: S_i.

(ii) A profit function, $\Pi_i(s_i, s_{-i})$ where $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

Each player will try to maximize his/her profit (utility or payoff) function by choosing an appropriate strategy with knowledge of the strategy space and profit functions of the other players but with no information concerning the current strategy used by rivals. Therefore, each player must conjecture the strategy(ies) used by his/her rival(s).

Definition 9: **Nash equilibrium**

"A combination of strategies or strategy profile $s^* \equiv (s_1^*, \ldots, s_n^*)$ constitutes a Nash equilibrium if the result for each player is better than or equal to the result which would be obtained by playing another strategy, with the behaviour of the other players remaining constant.

$s^* \equiv (s_1^*, \ldots, s_n^*)$ is a Nash equilibrium if: $\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*)\ \forall s_i \in S_i, \forall i, i = 1, \ldots, n$."

At equilibrium two conditions must be satisfied:

(i) The *conjectures* of players concerning how their rivals are going to play must be *correct*.

(ii) No player has incentives to change his/her strategy given the strategies of the other players. This is an element of *individual rationality*: do it as well as possible given what the rivals do. Put differently, no player increases his/her profits by *unilateral deviation*.

Being a Nash equilibrium is a necessary condition or minimum requisite for a solution proposal to be a reasonable prediction of rational behaviour by players. However, as we shall see it is not a sufficient condition. That is, being a Nash equilibrium is not in itself sufficient for a combination of strategies to be a prediction of the outcome for a game.

Definition10: **Nash equilibrium**

"A combination of strategies or strategy profile $s^* \equiv (s_1^*, ..., s_n^*)$ constitutes a Nash equilibrium if each player's strategy choice is a best response to the strategies actually played by his/her rivals". That is, $s^* \equiv (s_1^*, ..., s_n^*)$ is a Nash equilibrium if: $s_i^* \in MR_i(s_{-i}^*) \forall i, i = 1, ..., n$ where $MR_i(s_{-i}^*) = \{s_i' \in S_i : \Pi_i(s_i', s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*),\ \forall s_i \in S_i, s_i \neq s_i'\}$

A simple way of obtaining the Nash equilibria for a game is to build the best response sets of each player to the strategies (or combinations of strategies) of the other(s) player(s) and then look for those combinations of strategies being mutually best responses.

EXAMPLE 11

		2: h	i	j
	a	(5, 3)	(5, 11)	(20, 5)
1	b	(9, 11)	(2, 8)	(15, 6)
	c	(3, 10)	(10, 2)	(0, 5)

s_1	MR_2	s_2	MR_1
a	i	h	b
b	h	i	c
c	h	j	a

The strategy profile (*b*, *h*) constitutes the unique Nash equilibrium.

EXAMPLE 7

		2: t_1	t_2	t_3
1	s_1	(4, 10)	(3, 0)	(1, 3)
	s_2	(0, 0)	(2, 10)	(10, 3)

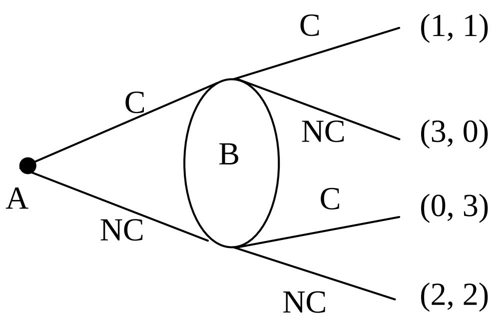
Note that the dominance criterion did not propose any solution for this game. However, the combination of strategies (s_1, t_1) constitutes the unique Nash equilibrium.

2.3. **Problems and refinements of the Nash equilibrium**

2.3.1. **The possibility of inefficiency**

It is usual to find games where Nash equilibria are not Pareto optimal (efficient).

EXAMPLE 2: The Prisoner's Dilemma



		B	
		C	NC
A	C	(1, 1)	(3, 0)
	NC	(0, 3)	(2, 2)

(*C*, *C*) is a Nash equilibrium based on dominant strategies. However, that strategy profile is the only profile which is not Pareto optimal. In particular, there is another combination of strategies, (*NC*, *NC*), where both players obtain greater payoffs.

2.3.2. **Inexistence of Nash equilibrium (in pure strategies)**

EXAMPLE 12

		2	
		t_1	t_2
1	s_1	$(\underline{1}, 0)$	$(0, \underline{1})$
	s_2	$(0, \underline{1})$	$(\underline{1}, 0)$

This game does not have Nash equilibria in pure strategies. However, if we allow players to use mixed strategies (probability distributions on the space of pure strategies) the result obtained is that "for any finite game there is always at least one mixed strategy Nash equilibrium".

2.3.3. **Multiplicity of Nash equilibria**

We distinguish two types of games.

2.3.3.1. ***With no possibility of refinement or selection***

EXAMPLE 13: The Battle of the Sexes

		Na: C	Na: T
No	C	($\underline{3}$, $\underline{2}$)	(1, 1)
	T	(1, 1)	($\underline{2}$, $\underline{3}$)

This game has two Nash equilibria: (*C*, *C*) and (*T*, *T*). There is a *pure coordination problem*.

2.3.3.2. ***With possibility of refinement or selection***

a) *Efficiency criterion*

This criterion consists of choosing the Nash equilibrium which maximizes the payoff of players. In general this is not a good criterion for selection.

b) *Weak dominance criterion*

This criterion consists of eliminating Nash equilibria based on weakly dominated strategies. Although as a solution concept it is not good, the weak dominance criterion allows us to select among the Nash equilibria.

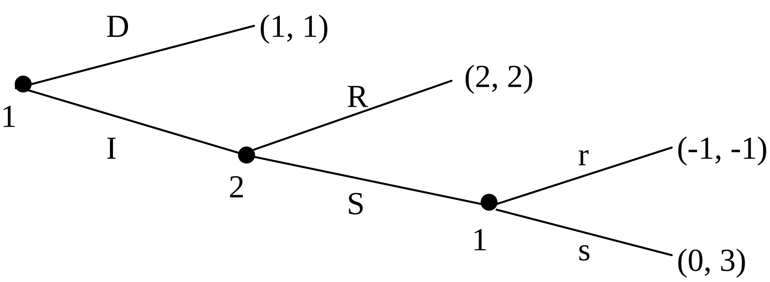
EXAMPLE 14

		2: D	2: I
1	D	(1, 1)	(0, 0)
	I	(0, 0)	(0, 0)

Nash equilibria: (*D*, *D*) and (*I*, *I*). Strategy *I* is a weakly dominated strategy for each player. By playing strategy *D* each player guarantees a payoff at least as high (and sometimes a higher) than that obtained by playing *I*. So we eliminate equilibrium (*I*, *I*) because it is based on weakly dominated strategies. So we propose the strategy profile (*D*, *D*) as the outcome of the game.

c) *Backward induction criterion and subgame perfect equilibrium*

EXAMPLE 15



s_1	MR_2	s_2	MR_1
Dr	R, S	R	Ir, Is
Ds	R, S	S	Dr, Ds
Ir	R		
Is	S		

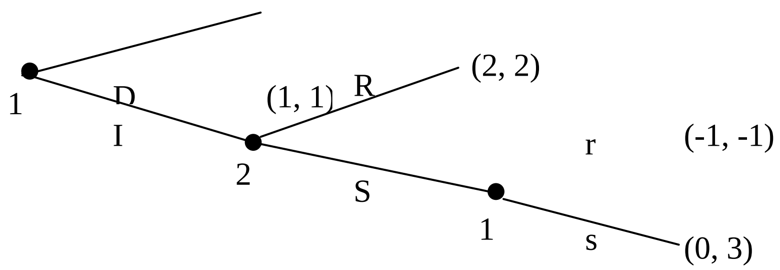
There are three Nash equilibria: (*Dr*, *S*), (*Ds*, *S*) and (*Ir*, *R*). We start by looking at the efficient profile: (*Ir*, *R*). This Nash equilibrium has a problem: at his/her second decision node, although it is an unattanaible given the behavior of the other player, player 1 announces that he/she would play *r*.

By threatening him/her with *r* player 1 tries to make player 2 play *R* and so obtain more profits. However, that equilibrium is based on a non credible threat: if player 1 were called on to play at his/her second node he/she would not choose *r* because it is an action (a non credible threat) dominated by *s*. The refinement we are going to use consists of eliminationg those equilibria based on non credible threats (that is, based on actions dominated in one subgame). From the joint use of the notion of Nash equilibrium and the backward induction criterion the following notion arises:

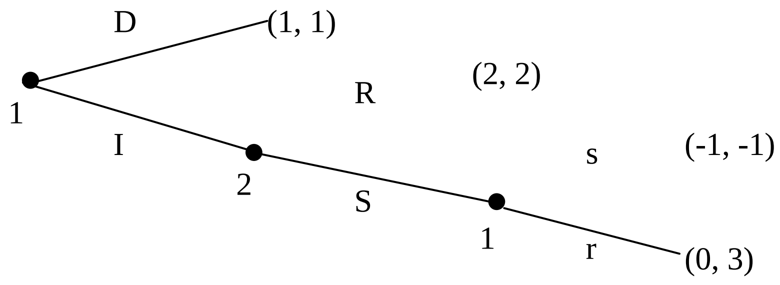
Definition 11: **Subgame perfect equilibrium**

"A combination of strategies or strategy profile $s^* \equiv (s_1^*, ..., s_n^*)$, which is a Nash equilibrium, constitutes a *subgame perfect equilibrium* if the relevant parts of the equilibrium strategies of each player are also an equilibrium in each of the subgames".

In example 15 (*Dr*, *S*) and (*Ir*, *R*) are not subgame perfect equilibria. Subgame perfect equilibria may be obtained by backward induction. We start at the last subgame. In this subgame *r* is a dominated action (a non credible threat); therefore, it cannot form part of player 1's strategy in the subgame perfect equilibrium, so we eliminate it and compute the reduced game

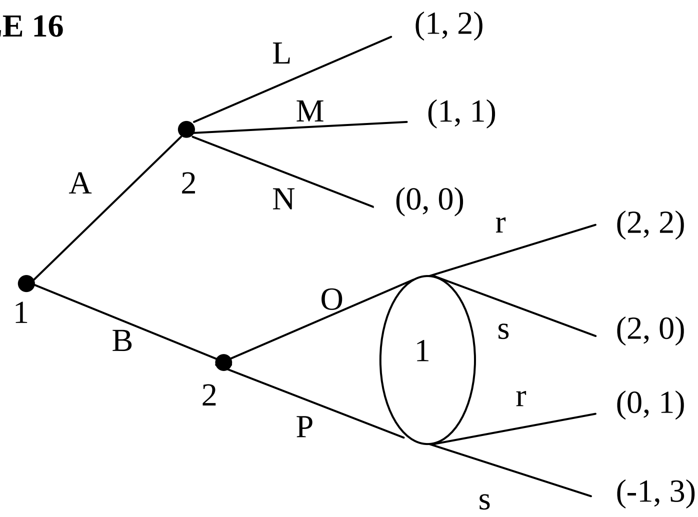


In the second stage of the backward induction we go to the previous subgame which starts at the decision node of player 2. In this subgame *R* is a dominated action for player 2. Given that player 2 anticipates that player 1 is not going to play *r* then *R* is a dominated action or non credible threat. We therefore eliminate it and compute the reduced game



At his/her first node player 1 has *I* as a dominated action (in the reduced game) and, therefore, he/she will play *D*. Then the subgame perfect equilibrium is (*Ds*, *S*). We can interpret the logic of backward induction in the following way. When player 2 has to choose he/she should conjecture that if he/she plays *S* player 1 is sure to play *s*. Player 2 is able to predict the rational behavior of player 1 given that player 1 observes the action chosen by him/her. If player 1 is equally rational he should anticipate the behavior (and the reasoning) of player 2 and play *D*.

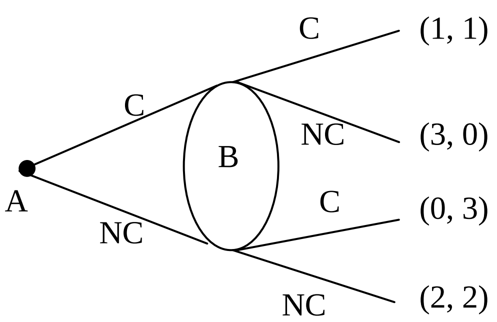
EXAMPLE 16



In this game there is a multiplicity of Nash equilibria and we cannot apply backward induction because there is a subgame with imperfect information. We shall use the definition of subgame perfect equilibrium and we shall require that the relevant part of the equilibrium strategies to be an equilibrium at the subgames. What we can do is solve the lower subgame (which starts at the lower decision node of player 2) and it is straightforward to check that the Nash equilibrium is *O*, *r*. At the upper subgame the only credible threat by player 2 is *L*. At his/her first decision node player 1 therefore has to choose between *A* and *B* anticipating that if he/she chooses *A* then player 2 will play *L* and if he/she chooses *B*, then they will both play the Nash equilibrium (of the subgame) *O*, *r*. Therefore, the subgame perfect equilibrium is (*Br*, *LO*): the relevant part of the equilibrium strategies are also an equilibrium at each of the subgames.

2.4. **Repeated games**

EXAMPLE 2: The Prisoner's Dilemma



		B: C	B: NC
A	C	(1, 1)	(3, 0)
	NC	(0, 3)	(2, 2)

When the game is played once the strategy profile (*C*, *C*) is the Nash equilibrium in dominant strategies and cooperation or collusion between players cannot hold as an equilibrium. Even though both players obtain more profits in the combination of strategies (*NC*, *NC*), both players would have incentives to deviate by using the dominant strategy. In this section, we are going to study the possibilities of cooperation or collusion when the players play the game repeatedly.

2.4.1. **Finite temporal horizon**

Assume that the game (the Prisoner's Dilemma) is repeated a finite number of times **T** (known by both players). We know that if T = 1 the unique Nash equilibrium is (*C*, *C*).

The first point to note is that when the game is repeated T times, a player's strategy for the repeated game should indicate what the player would do at each stage of the game, contingent upon past history.

We shall use a *backward induction* argument to show that in the unique subgame perfect equilibrium of this repeated game each player (independently of past history) will choose "confess" at each atage of the game. Consider T, t = 1, 2,….., T, iterations of the Prisoner's Dilemma.

We start by looking at the last period **T**: in this last stage of the game what has happened earlier (the past history) is irrelevante (because there is now no future) and all that remains is to play the Prisoner's Dilemma once. Therefore, as each player has "confess" as his/her dominant strategy (when the game is played once) in the last period each player will choose "confess". The only reason for playing "not confess" in any stage of the game would be to try to improve in the future given that such behaviour might be interpreted as a sign of goodwill by the other player so as to gain his/her cooperation. However, at the last stage of the game there is no future so (*C*, *C*) is unavoidable.

Now consider period T-1. Given that players anticipate that in the last period they are not going to cooperate, the best they can do in period T-1 is follow the short term dominant strategy, that is, “confess”. The only reason for playing “not confess” in this stage of the game would be to try to improve in the future, but in period T the players will choose (*C*, *C*). The same reasoning applies from periods T-2, T-3,….to period 1. Therefore, the unique subgame perfect equilibrium of the finitely repeated Prisoner’s Dilemma simply involves T repetitions of the short term (static) Nash equilibrium.

Therefore, if the game is repeated a finite (and known) number of times, in the subgame perfect equilibrium each player would choose his/her short term dominant strategy at each stage of the game. As a consequence, cooperation between players cannot hold as an equilibrium when the temporal horizon is finite.

2.4.2. **Infinite temporal horizon**

There are two ways of interpreting an infinite temporal horizon:

(i) *Literal interpretation*: the game is repeated an infinite number of times. In this context, to compare two alternative strategies a player must compare the discounted present value of the respective gains. Let δ be the discount factor, $0 < \delta < 1$, and let r be the discount rate ($0 < r < \infty$) where $\delta = \dfrac{1}{1+r}$.

(ii) *Informational interpretation*: the game is repeated a finite but unknown number of times. At each stage there is a probability $0 < \delta < 1$ of the game continuing. In this setting, each player must compare the expected value (which might be also discounted) of the different strategies.

In this repeated game, a player's strategy specifies his/her behaviour in each period ***t*** as a function of the game's past history. Let $H_{t-1} = \{s_{1\tau}, s_{2\tau}\}_{\tau=1}^{t-1}$, where $s_{i\tau} \in \{C, NC\}$, represents the past history (of the game).

Note first that there is a subgame perfect equilibrium of the infinitely repeated game where each player plays C (his/her short term dominant strategy) in each period. The strategy of each player would be "confess in each period independently of past history".

We now determine under what conditions there is also a subgame perfect equilibrium where the two players cooperate. Consider the following combination of long term strategies: $s_i^c \equiv \{s_{it}(H_{t-1})\}_{t=1}^{\infty}$, i =1,2.

where

$$s_{it}(H_{t-1}) = \begin{cases} NC & \text{if all elements of } H_{t-1} \text{ equal } (NC, NC) \text{ or } t = 1 \\ C & \text{otherwise} \end{cases}$$

Note that these long term strategies incorporate "implicit punishment threats" in the case of breach of the (implicit) cooperation agreement. The threat is credible because "confess" in each period (independently of the past history) is a Nash equilibrium of the repeated game.

To check whether it is possible to maintain cooperation as an equilibrium in this context, we have to check that players have no incentives to deviate; that is, we have to check that the combination of strategies (s_1^c, s_2^c) constitutes a Nash equilibrium of the repeated game. The discounted present value for player *i* in the strategy profile (s_1^c, s_2^c) is given by:

$$\pi_i(s_i^c, s_j^c) = 2 + 2\delta + 2\delta^2 + ... = 2(1 + \delta + \delta^2 + ...) = \frac{2}{1-\delta}$$

Assume that player *i* deviates, and does so from the first period. Given that the other player punishes him/her (if the other player follows his/her strategy) for the rest of the game, the best that player *i* can do is also "confess" for the rest of the game. The discounted present value of deviating is:

$$\pi_i(s_i, s_j^c) = 3 + 1\delta + 1\delta^2 + ... = 3 + \delta(1 + \delta + \delta^2 + ...) = 3 + \delta\frac{1}{1-\delta}$$

Cooperation will be supported as a Nash equilibrium if no player has incentives to deviate; that is, when $\pi_i(s_i^c, s_j^c) \geq \pi_i(s_i, s_j^c)$. It is straightforward to check that if $\delta \geq \frac{1}{2}$ no player has any incentive to break the (implicit) collusion agreement.

We next see how that Nash equilibrium is also subgame perfect: that is, threats are credible. Consider a subgame arising after a deviation has occurred. The strategy of each player requires "confess" for the rest of the game independently of the rival's behaviour. This pair of strategies is a Nash equilibrium of an infinitely repeated Prisoner's Dilemma because each player would obtain

$$\delta^{T-1}(1+\delta+\delta^2+...)=\frac{\delta^{T-1}}{1-\delta}$$

if he/she does not deviate, while he/she would obtain 0 in each period in which he/she deviates from the cooperative strategy.

The above analysis serves as an example of a general principle arising in repeated games with an infinite temporal horizon. In these games it is possible to support as equilibria behaviours that are not equilibria in the short term. This occurs because of the "implicit punishment threat" that in the case of breach of the agreement one will be punished for the rest of the game. So the increase in profits (from a breach of the agreement) does not offset the loss of profits for the rest of the game.

2.5. **Conclusions**

We have analyzed different ways of solving games, although none of them is exempt from problems. The dominance criterion (elimination of dominated strategies) is useful in solving some games but does not serve in others because it provides no solution proposal. The weak version of this criterion (elimination of weakly dominated strategies) is highly useful in selecting among Nash equilibria, especially in games in normal or strategic form. The backward induction criterion allows solution proposals to be drawn up for games in extensive form. This criterion has the important property that in perfect information games without ties it leads to a unique outcome. But it also presents problems: the possibility of ties, imperfect information and unbounded rationality. This criterion is highly useful in selecting among Nash equilibria in games in extensive form. The joint use of the notion of Nash equilibrium and backward induction give rise to the concept of subgame perfect equilibrium, which is a very useful criterion for proposing solutions in many games.

Although it also presents problems (inefficiency, nonexistence and multiplicity) the notion of the Nash equilibrium is the most general and most widely used solution criterion for solving games. Being a Nash equilibrium is considered a necessary (but not sufficient) condition for a solution proposal to be a reasonable prediction of rational behaviour by players. If, for instante, we propose as the solution for a game a combination of strategies which is not a Nash equilibrium, that prediction would be contradicted by the development of the game itself. At least one player would have incentives to change his/her predicted strategy. In conclusion, although it presents problems, there is quasi-unanimity that all solution proposals must at least be Nash equilibria.

Chapter 3. **Oligopoly**

Introduction

Non-Cooperative Game Theory is very useful for modelling and understanding the *multi-agent* economic problems characterized by *strategic interdependency*, in particular for analyzing competition between firms in a market. Perfect competition and pure monopoly (not threatened by entry) are special non-realistic cases. It is more frequent in the real life to find industries with not many firms or with a lot of firms but with a smaller number of them producing a large proportion of the total production. With few firms, competition is characterized by strategic considerations: each firm takes its decisions (price, output, advertising, etc.) taking into account or conjecturing the behaviour of the others. Competition in an oligopoly can be seen as a game where firms are the players. So we shall adopt a Game Theory perspective to analyze the different models of oligopoly. For each case, we shall wonder what game firms are playing (information, order of playing, strategies, etc.), and what the equilibrium notion is. An important difference between the games of the previous chapter and the games we shall solve in this chapter is that the former were finite games while the latter are infinite games.

3.1. **The Cournot model**

3.1.1. ***Duopoly***

(i) Context.

(ii) Representation of the game in normal form.

(iii) Notion of equilibrium.

(iv) Best response function. Characterization of equilibrium.

(v) Example. Graphic representation.

(i) *Context*

The Cournot model has four basic characteristics:

a) We consider a market served by *2 firms*.

b) *Homogeneous product*. That is, from the consumers' point of view, the goods produced by the two firms are perfect substitutes.

c) *Quantity competition.* The variable of choice of each firm is output level. Denote by x_1 and x_2 the production levels of firm 1 and firm 2, respectively.

d) *Simultaneous choice*. The two firms have to choose their outputs simultaneously. That is, each firm has to choose its output without knowing the rival's choice. Simultaneous Choice does not mean that choices are made at the same instant in time. An equivalent context would be one where one firm chooses its output first and tthe other firm chooses its output afterwards but without observing the decision adopted by the first firm. In other words, sequential choice together with imperfect information (the player moving second does not observe what the first mover does) is equivalent to simultaneous choice.

The inverse demand function is $p(x),$ where $p^{'}(x) < 0.$ As the product is homogeneous, the price at which a firm can sell will depend on the total output: $p(x) = p(x_1 + x_2).$

The production cost of firm i is $C_i(x_i),$ i=1,2.

(ii) *Representation of the game in normal form*

1) $i = 1, 2.$ (Players).

2) $x_i \geq 0.$ Any non-negative amount would serve as a strategy for player *i*. Equivalently, we can represent the set of strategies of player *I* as $x_i \in [0, \infty),\ i = 1, 2.$

3) The payoff obtained by each firm at the combination of strategies (x_1, x_2) is:

$$\left.\begin{array}{l} \Pi_1(x_1, x_2) = p(x_1 + x_2)x_1 - C_1(x_1) \\ \Pi_2(x_1, x_2) = p(x_1 + x_2)x_2 - C_2(x_2) \end{array}\right\} \equiv \Pi_i(x_i, x_j) = p(x_i + x_j)x_i - C_i(x_i),\ i, j = 1, 2,\ j \neq i.$$

(iii) *Notion of equilibrium. Cournot-Nash equilibrium*

It is very easy to adapt the definition of the Nash equilibrium in the previous chapter to this new context.

“ $s^* \equiv (s_1^*, .., s_n^*)$ is a Nash equilibrium if : $\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*)\ \forall s_i \in S_i, \forall i, i = 1, ..., n$.”

For the Cournot duopoly game we say:

“ (x_1^*, x_2^*) is a Cournot-Nash equilibrium if $\Pi_i(x_i^*, x_j^*) \geq \Pi_i(x_i, x_j^*)\ \forall x_i \geq 0, i, j = 1, 2,\ j \neq i$ ”.

The second definition based on best responses proves more useful.

“ $s^* \equiv (s_1^*, .., s_n^*)$ is a Nash equilibrium if: $s_i^* \in MR_i(s_{-i}^*)\ \forall i, i = 1, .., n$ where $MR_i(s_{-i}^*) = \left\{ s_i' \in S_i : \Pi_i(s_i', s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*),\ \forall s_i \in S_i, s_i \neq s_i' \right\}$”.

For the Cournot duopoly game:

“ (x_1^*, x_2^*) is a Cournot-Nash equilibrium if $x_i^* = f_i(x_j^*),\ i, j = 1, 2,\ j \neq i$ ”,

where $f_i(x_j)$ is the best-response function of firm *i* to firm *j*’s output.

(iv) *Best response function. Characterization of equilibrium*

The proceedure that we follow to obtain the Nash equilibrium is similar to that used in the previous chapter. First we calculate the best response of each player to the possible strategies of its rival and then we look for combinations of strategies which are mutually best responses to each other.

Given a strategy of firm *j* we look for the strategy that provides most profit for firm *i*. That is,, given the strategy $x_j \geq 0$ firm *i*'s best response is to choose a strategy x_i such that:

$$\max_{x_i \geq 0} \Pi_i(x_i, x_j) \equiv p(x_i + x_j)x_i - C_i(x_i)$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i + x_j) + x_i p'(x_i + x_j) - C_i'(x_i) = 0 \quad (1) \rightarrow \overline{f}_i(x_j)$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = 2p'(x_i + x_j) + x_i p''(x_i + x_j) - C_i''(x_i) < 0$$

Taking into account the non-negativity constraint, $x_i \geq 0,$ or in terms of game theory taking into account that the best response must belong to the strategy space of the player, the best response function is: $f_i(x_j) = \max\left\{ \overline{f}_i(x_j), 0 \right\}$.

The Cournot-Nash equilibrium is a strategy profile (x_1^*, x_2^*) such that the strategy of each player is its best response to the rival's strategy. That is,

$$\left.\begin{aligned} x_1^* &= f_1(x_2^*) = \max\left\{ \overline{f}_1(x_2^*), 0 \right\} \\ x_2^* &= f_2(x_1^*) = \max\left\{ \overline{f}_2(x_1^*), 0 \right\} \end{aligned}\right\} \leftrightarrow x_i^* = f_i(x_j^*) = \max\left\{ \overline{f}_i(x_j^*), 0 \right\}, \; i, j = 1, 2, \; j \neq i.$$

Let us now forget the non-negativity constraint and we are going to assume that the best response function is broadly characterized by condition (1) (interior solution). By definition, the best response must satisfy the first order condition: $\frac{\partial \Pi_i(f_i(x_j),x_j)}{\partial x_i}=0 \rightarrow$ firm i's best response to $x_j \geq 0$ is $f_i(x_j)$. The Cournot-Nash equilibrium satisfies $\frac{\partial \Pi_i(x_i^*,x_j^*)}{\partial x_i}=0$ given that $x_i^*=f_i(x_j^*),\ i=1,2.$ We have a simple way of checking whether a combination of strategies is a Nash equilibrium: by calculating each firm's marginal profit corresponding to that strategy profile. If any one of them is other than zero the equilibrium condition is not met.

$\frac{\partial \Pi_i(\hat{x}_i,\hat{x}_j)}{\partial x_i}>0 \rightarrow f_i(\hat{x}_j)>\hat{x}_i \rightarrow (\hat{x}_i,\hat{x}_j)$ is not a Cournot-Nash equilibrium.

$\frac{\partial \Pi_i(\hat{x}_i,\hat{x}_j)}{\partial x_i}<0 \rightarrow f_i(\hat{x}_j)<\hat{x}_i \rightarrow (\hat{x}_i,\hat{x}_j)$ is not a Cournot-Nash equilibrium.

(v) *Example. Graphic representation*

Consider the case of linear demand and constant marginal cost: $p(x)=a-bx$ and $C_i(x_i)=c_i x_i,\ i=1,2.$ Assume for the sake of simplicity that the marginal cost is the same for the two firms: $c_i=c>0,\ i=1,2.$ ($a>c$ for the example to make sense).

We first obtain the best response function for firm $i,\ i=1,2.$

$$\max_{x_i\geq 0} \Pi_i(x_i,x_j) \equiv p(x_i+x_j)x_i - C_i(x_i) \equiv [a-b(x_i+x_j)]x_i - cx_i \equiv [a-c-b(x_i+x_j)]x_i$$

$$\frac{\partial \Pi_i}{\partial x_i}=p(x_i+x_j)+x_i p'(x_i+x_j)-C_i'(x_i)=a-2bx_i-bx_j-c=0 \rightarrow \overline{f}_i(x_j)\frac{a-c-bx_j}{2b}$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2}=-2b<0$$

So the best response function is:

$$f_i(x_j)=\max\left\{\overline{f}_i(x_j),0\right\}=\max\left\{\frac{a-c-bx_j}{2b},0\right\}.$$

The Cournot-Nash equilibrium satisfies:

$$x_1^* = f_1(x_2^*) = \max\left\{\frac{a-c-bx_2^*}{2b}, 0\right\} \underbrace{\geq 0}_{\text{given that } a>c}$$

$$x_2^* = f_2(x_1^*) = \max\left\{\frac{a-c-bx_1^*}{2b}, 0\right\} \underbrace{\geq 0}_{\text{given that } a>c}$$

By solving the system:

$$x_1^* = f_1(x_2^*) = f_1(\underbrace{f_2(x_1^*)}_{x_2^*})$$

$$x_1^* = \frac{a-c-bx_2^*}{2b} = \frac{a-c-b\left(\frac{a-c-bx_1^*}{2b}\right)}{2b} = \frac{\frac{a-c+bx_1^*}{2}}{2b} = \frac{a-c+bx_1^*}{4b} \rightarrow x_1^* = \frac{a-c}{3b}.$$

$$\rightarrow x_2^* = \frac{a-c-bx_1^*}{2b} = \frac{a-c-b\left(\frac{a-c}{3b}\right)}{2b} = \frac{2(a-c)}{6b} = \frac{a-c}{3b}.$$

The Cournot-Nash total output is: $x^* = x_1^* + x_2^* = \frac{2(a-c)}{3b}$ and the equilibrium price

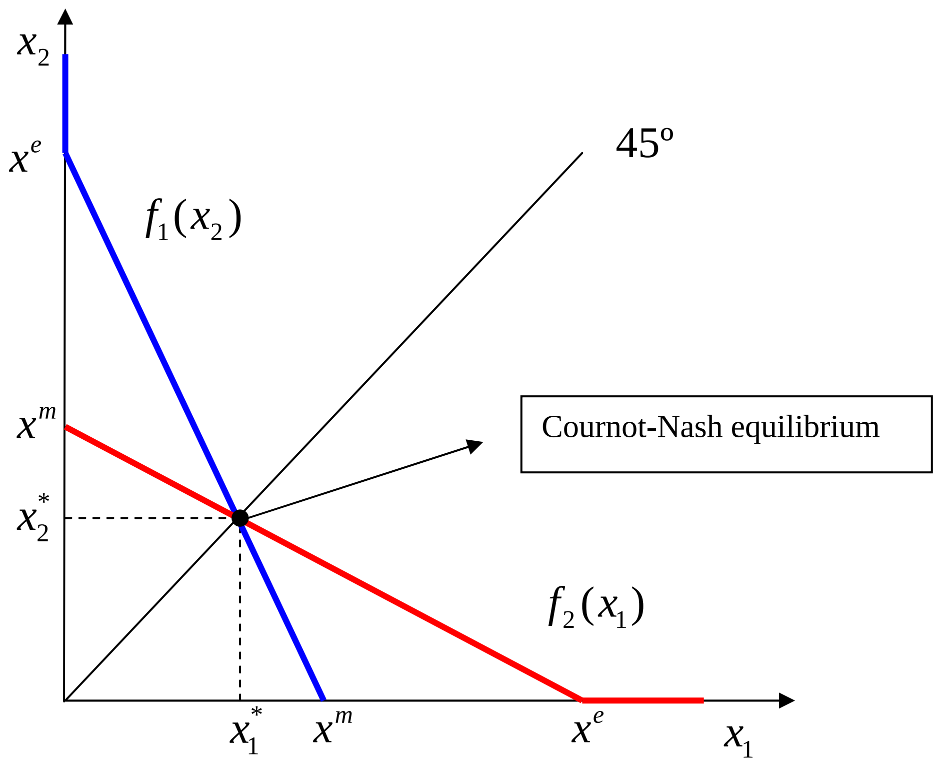
$$p^* = p(x_1^* + x_2^*) = a - b\frac{2(a-c)}{3b} = \frac{a+2c}{3}.$$

Profits are:

$$\Pi_1^* = \Pi_1(x_1^*, x_2^*) = [p(x_1^* + x_2^*) - c]x_1^* = \frac{a-c}{3}\frac{a-c}{3b} = \frac{(a-c)^2}{9b}$$

$$\Pi_2^* = \Pi_2(x_1^*, x_2^*) = [p(x_1^* + x_2^*) - c]x_2^* = \frac{a-c}{3}\frac{a-c}{3b} = \frac{(a-c)^2}{9b}.$$

Graphic representation



3.1.2. ***Oligopoly***

(i) Representation of the game in normal form.

(ii) Notion of equilibrium. Best response function. Cournot-Nash equilibrium.

(iii) Lerner index.

(iv) Special cases. Constant marginal cost.

(i) *Representation of the game in normal form*

1) $i = 1, 2, ..., n.$ (Players)

2) $x_i \geq 0.$ Similarly, $x_i \in [0, \infty),\ i = 1, 2, .., n.$

3) The profit of each firm corresponding to strategy profile (x_i, x_{-i}) is:

$$\Pi_i(x_i, x_{-i}) = p(\underbrace{x_i + x_{-i}}_{x})x_i - C_i(x_i),\ i = 1, 2, ..., n.$$

The way of representing the game in normal form has changed slightly. Given the strategy profile $(x_1, x_2, ..., x_n)$ the relevant point for firm i, $i = 1, 2, ..., n,$ is the total output produced by the other firms, $x_{-i} = \sum_{j \neq i} x_j$. Therefore, (x_i, x_{-i}) is not really a strategy profile and $\Pi_i(x_i, x_{-i})$ is firm *i*'s profit associated with every combination of strategies where firm *i* produces x_i and the other firms produce in aggregate x_{-i} (it being irrelevant to firm *i* how that production x_{-i} is distributed among the *n* -1 firms).

(iii) *Notion of equilibrium. Best response functions. Cournot-Nash equilibrium*

In the Cournot oligopoly game we say that

"$(x_1^*, x_2^*, .., x_n^*) \equiv (x_i^*, x_{-i}^*)$ is a Cournot-Nash equilibrium if:

$$\Pi_i(x_i^*, x_{-i}^*) \geq \Pi_i(x_i, x_{-i}^*)\ \forall x_i \geq 0,\ \forall i, i = 1, 2, ..., n."$$.

In terms of best response functions the definition is:

"$(x_1^*, x_2^*, .., x_n^*) \equiv (x_i^*, x_{-i}^*)$ is a Cournot-Nash equilibrium if $x_i^* = f_i(x_{-i}^*),\ \forall i, i = 1, 2, .., n.$",

where $f_i(x_{-i})$ is firm *i*'s best response function to all those combinations of strategies whose total output is x_{-i}.

We next obtain the best response of firm *i* to all those combinations of strategies (of the other firms) whose total output is x_{-i}. Firm *I*'s best response is to choose a strategy x_i such that:

$$\max_{x_i \geq 0} \Pi_i(x_i, x_{-i}) \equiv p(x_i + x_{-i})x_i - C_i(x_i)$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i + x_{-i}) + x_i p^{'}(x_i + x_{-i}) - C_i^{'}(x_i) = 0 \quad (1) \rightarrow \overline{f}_i(x_{-i})$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = 2p^{'}(x_i + x_{-i}) + x_i p^{''}(x_i + x_{-i}) - C_i^{''}(x_i) < 0$$

Taking into account the non-negativity constraint, $x_i \geq 0,$ or in terms of game theory that the best response must belong to the player's strategy space, the best response function is: $f_i(x_{-i}) = \max\left\{\overline{f}_i(x_{-i}), 0\right\}.$

The Cournot-Nash equilibrium is a strategy profile $(x_1^*, x_2^*, .., x_n^*) \equiv (x_i^*, x_{-i}^*)$ such that $x_i^* = f_i(x_{-i}^*), \; \forall i, i = 1, 2, .., n.$".

Let us now forget the non-negativity constraint and assume that the best response function is broadly characterized by condition (1) (interior solution). By definition, the best response must satisfy the first order condition: $\frac{\partial \Pi_i(f_i(x_{-i}), x_{-i})}{\partial x_i} = 0 \rightarrow$ firm *i*'s best response to $x_{-i} \geq 0$ is $f_i(x_{-i})$. The Cournot-Nash equilibrium satisfies $\frac{\partial \Pi_i(x_i^*, x_{-i}^*)}{\partial x_i} = 0$ given that $x_i^* = f_i(x_{-i}^*), \; i = 1, 2, ..., n.$ Here again there is a simple way of checking whether a combination of strategies is a Nash equilibrium: calculating each firm's marginal profit corresponding to that strategy profile. If any one is other than zero the equilibrium condition is not met.

$\frac{\partial \Pi_i(\hat{x}_i, \hat{x}_{-i})}{\partial x_i} > 0 \rightarrow f_i(\hat{x}_{-i}) > \hat{x}_i \rightarrow (\hat{x}_i, \hat{x}_{-i})$ is not a Cournot-Nash equilibrium.

$\frac{\partial \Pi_i(\hat{x}_i, \hat{x}_{-i})}{\partial x_i} < 0 \rightarrow f_i(\hat{x}_{-i}) < \hat{x}_i \rightarrow (\hat{x}_i, \hat{x}_{-i})$ is not a Cournot-Nash equilibrium.

(iii) *Lerner index*

By assuming an interior solution we are going to transform condition (1) to obtain the Lerner index of market power.

$$p(\underbrace{x_i + x_{-i}}_{x}) + x_i p'(x_i + x_{-i}) - C_i'(x_i) = 0$$

$$p(x)[1 + x_i \frac{p'(x)}{p(x)}] - C_i'(x_i) = 0$$

$$p(x)[1 + \frac{x_i}{x} \underbrace{\frac{xp'(x)}{p(x)}}_{-\frac{1}{|\varepsilon(x)|}}] - C_i'(x_i) = 0$$

Defining firm *i*'s share as $s_i = \frac{x_i}{x}$ we get:

$$p(x)[1 - \frac{s_i}{|\varepsilon(x)|}] - C_i'(x_i) = 0$$

Then the Lerner index of market power for firm *i* is

$$\frac{p(x) - C_i'(x_i)}{p(x)} = \frac{s_i}{|\varepsilon(x)|}$$

Then the Cournot model is located between the case of pure monopoly ($s_i = 1$) and perfect competition ($\lim_{s_i \to 0} \frac{p - C'}{p} = 0$).

(iv) *Special cases*. *Constant marginal costs*

a) **Constant marginal cost:** $c_i > 0,\ i = 1,..,n.$

At equilibrium the first order condition for each firm (interior solution) must be satisfied:

$$p(\underbrace{x_i^* + x_{-i}^*}_{x^*}) + x_i^* p'(x_i^* + x_{-i}^*) - c_i = 0 \quad i = 1, 2.., n.$$

By adding up the *n* first order conditions:

$$np(x^*) + \underbrace{\sum_{i=1}^{n} x_i^*}_{x^*} p'(x^*) - \sum_{i=1}^{n} c_i = 0$$

That is

$$np(x^*) + x^* p'(x^*) = \sum_{i=1}^{n} c_i$$

Then the total output in Cournot-Nash equilibrium depends exclusively on the sum of the marginal costs, not on their distribution across firms (in an interior solution with all *n* firms producing positive quantities).

b) **Common constant marginal cost:** $c_i = c > 0, \; i = 1, .., n.$

The Lerner index is:

$$\frac{p(x) - c}{p(x)} = \frac{s_i}{|\varepsilon(x)|}$$

Taking into account that if the product is homogeneous and the marginal cost is the same for all firms then the Cournot-Nash equilibrium should be symmetric:

$$s_i = \frac{x_i^*}{x^*} = \frac{\overline{x}^*}{n\overline{x}^*} = \frac{1}{n}, \quad i = 1, .., n.$$

If the price-elasticity is constant then:

$$\frac{p(x) - c}{p(x)} = \frac{1}{n|\varepsilon|}$$

Therefore, when the number of firms increases the relative price-marginal cost margin (the Lerner index) decreases and at the limit when $n \rightarrow \infty$, $p \rightarrow c$.

3.1.3. ***Welfare analysis***

We consider the simplest case where marginal cost is constant and common to all firms.

$$p(\underbrace{x_i^* + x_{-i}^*}_{x^*}) + x_i^* p'(x_i^* + x_{-i}^*) - c = 0 \quad i = 1, 2.., n.$$

By adding the *n* first order conditions:

$$np(x^*) + x^* p'(x^*) - nc = 0$$

We follow a similar approach to that in the chapter on monopolies to compare the Cournot output with the efficient output.

(*Review the obtaining of the welfare function and the problem of maximizing social welfare*)

$$\max_{x \geq 0} W(x) \equiv \max_{x \geq 0} u(x) - C(x)$$

$$W'(0) = u'(0) - C'(0) > 0 \Rightarrow \; p(0) > C'(0)$$

$$W'(x) = u'(x) - C'(x) = 0 \Leftrightarrow \; W'(x^e) = 0 \quad \text{First order condition.}$$

$$W''(x) = u''(x) - C''(x) < 0 \quad \text{Strictly concave welfare function.}$$

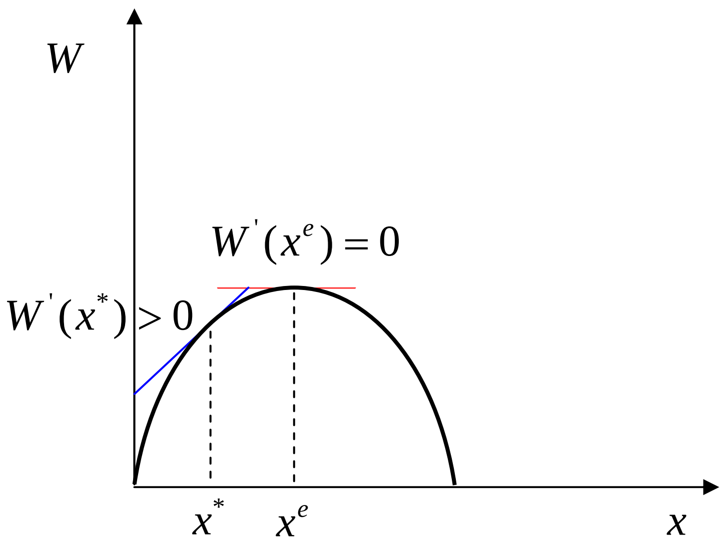
$$\left\{ \begin{matrix} W'(x^e) = 0 \\ W'(x^*)? \\ W''(x) < 0 \end{matrix} \right\}$$

$$W'(x^*) = \underbrace{u'(x^*)}_{p(x^*)} - C'(x^*) = -\frac{x^*}{n} \overbrace{\underbrace{p'(x^*)}_{<0}}^{u''(x^*)} > 0$$

By definition of Cournot output.

$$\left.\begin{cases} W'(x^e)=0 \\ W'(x^*)>0 \\ W''(x)<0 \end{cases}\right\} \rightarrow W'(x^e)<W'(x^*) \rightarrow \; x^e > x^*$$

$$\boxed{W''(x)<0 \Leftrightarrow \frac{dW'(x)}{dx}<0 \rightarrow \uparrow x \; \downarrow W'(x)}$$



3.2. **The Bertrand model**

3.2.1. ***Homogeneous product***

(i) Context.

(ii) Residual demand.

(iii) Representation of the game in normal form. Notion of equilibrium.

(iv) The Bertrand Paradox. Characterization of equilibrium and uniqueness.

(i) *Context*

The Bertrand model is characterized by the following elements

1) We consider a market with *2 firms*.

2) Firms sell a *homogeneous product*.

3) *Price competition*.

4) *Simultaneous choice*. Each firm has to choose its price with no knowledge of the rival's decision. Again, simultaneous choice does not mean that choices are made at the same instant in time; the relevant point is that although one firm may play first the other does not observe its decision.

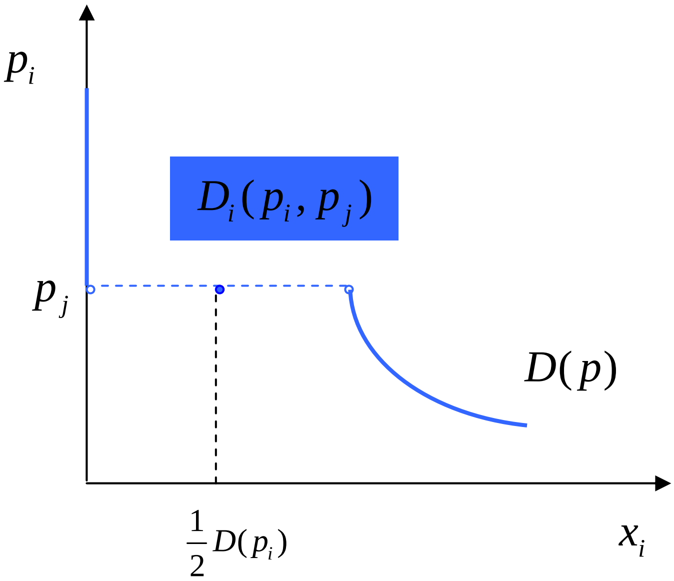
5) *Constant marginal cost and common*: $c_1 = c_2 = c > 0$.

(ii) *Residual demand*

Firms produce a homogeneous product and compete on price. Then from the consumers' point of view the only relevant point is the relationship between the prices of the two firms; consumers buy the product from the firm that sets the lower price. That is, if one firm charges a lower price than the other, it captures the entire market and the second firm sells nothing. If both firms give the same price then consumers are indifferent between buying from one or the other. For the sake of simplicity we assume that in the case of equal prices each firm sells to the half of the market.

Firm *i*'s residual demand, $i, j = 1, 2,\ j \neq i,$ is:

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & p_i < p_j \\ \frac{1}{2} D(p_i) & p_i = p_j \\ 0 & p_i > p_j \end{cases}$$



(iii) *Representation of the game in normal form. Notion of equilibrium*

The game in **normal form** is:

1) $i = 1, 2.$ (Players)

2) $p_i \geq 0.$ Any non negative price serves as a strategy for player *i*. Equivalently, we can represent the player *i*'s strategy space as $p_i \in [0, \infty),\ i = 1, 2.$

3) The firms profits corresponding to the strategy profile (p_1, p_2) are:

$$\left.\begin{array}{l} \Pi_1(p_1, p_2) = (p_1 - c)D_1(p_1, p_2) \\ \\ \Pi_2(p_1, p_2) = (p_2 - c)D_2(p_1, p_2) \end{array}\right\} \equiv \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j),\ i, j = 1, 2,\ j \neq i,$$

where the residual demand for firm $i,\ i, j = 1, 2,\ j \neq i,$ is:

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & p_i < p_j \\ \frac{1}{2}D(p_i) & p_i = p_j \\ 0 & p_i > p_j \end{cases}.$$

In the Bertrand game we say that

"(p_1^*, p_2^*) is a Bertrand-Nash equilibrium if:

$$\Pi_i(p_i^*, p_j^*) \geq \Pi_i(p_i, p_j^*)\ \forall p_i \geq 0,\ i, j = 1, 2,\ j \neq i \text{ ''}.$$

To simplify the analysis we use this definition exclusively because the fact that the residual demand of each firm is a discontinuos function of its own price means that we cannot use standard optimization techniques (in fact instead of having best response functions we would have best response correspondences and the analysis would be more complex).

(iv) *The Bertrand Paradox. Characterization of the equilibrium and uniqueness*

We demonstrate here that the unique Bertrand-Nash equilibrium is:

$$p_1^* = p_2^* = c$$

This result is known as the *Bertrand Paradox*:

"*Two firms competing on prices suffice to obtain a competitive outcome*"

Demonstration

We demonstrate that the strategy profile $p_1^* = p_2^* = c$:

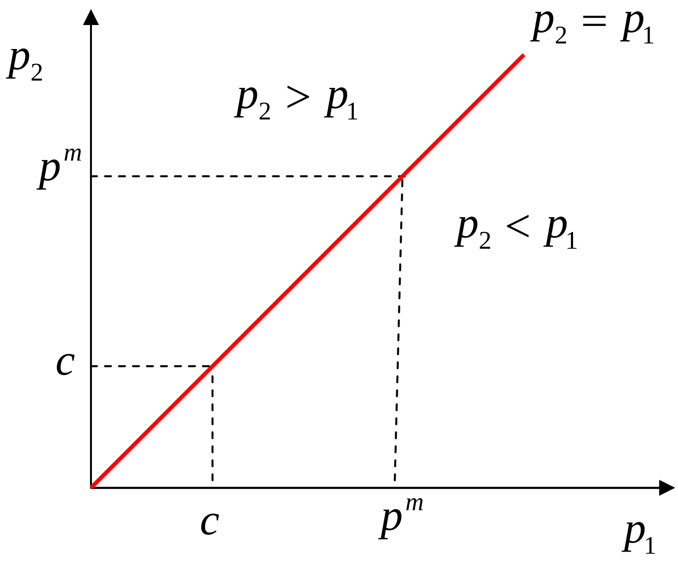
a) is a Nash equilibrium.

b) is the unique Nash equilibrium.

a) The profit of each firm under strategy profile (c,c) is: $\Pi_i(c,c) = (c-c)\frac{1}{2}D(c),\ i = 1,2.$ If firm *i* unilaterally deviates by charging a price $p_i > c$ then its profit will be zero because it will make no sales. By reducing its price below marginal cost $p_i < c$ it would capture the entire market but incur losses. Therefore,

$$\Pi_i(c,c) \geq \Pi_i(p_i,c)\ \forall p_i \geq 0,\ i,j = 1,2,\ j \neq i$$

b) We demonstrate that no other combination of strategies can be a Nash equilibrium.The graph below shows the different types of strategy profile. We check whether a strategy profile is an equilibrium or not by calculating the profit of each player corresponding to this combination of strategies and we wonder if any of the players has an incentive to unilaterally deviate. To eliminate a strategy profile as an equilibrium it suffices to show that at least one player can improve by deviating unilaterally.



1) Equal prices: $p_i = p_j$

a) ¿ $p_i = p_j > c$ NE? NO. In a strategy profile like this the profit of each firm would be: $\Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i)$. Firm i =1, 2 would have an incentive to deviate unilaterally. For example, we can choose a price $p_i^{'} = p_i - \varepsilon$ (where ε is an arbitrary positive amount as small as required):

$$(p_i^{'} - c)D(p_i^{'}) = (p_i^{'} - c)D_i(p_i^{'}, p_j) = \Pi_i(p_i^{'}, p_j) > \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i).$$

In fact there would be inifinite deviations such that firm i improves with a unilateral deviation.

b) ¿ $p_i = p_j < c$ NE? NO. Firm *i*'s profit in such a strategy profile would be: $\Pi_i(p_i, p_j) = \underbrace{(p_i - c)}_{<0} D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i) < 0.$ Firm *i* would have an incentive to deviate unilaterally. For example, any price $p_i^{'} > p_i$:

$$0 = (p_i^{'} - c)\underbrace{D_i(p_i^{'}, p_j)}_{=0} = \Pi_i(p_i^{'}, p_j) > \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i).$$

2) Different prices: $p_i \neq p_j$

c) ¿ $p_i > p_j > c$ NE? NO. Firm *i*'s profit in such strategy profile would be zero: $\Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = 0$ and the profit of the other firm, firm *j*, would be $\Pi_j(p_i, p_j) = (p_j - c)D_j(p_i, p_j) = (p_j - c)D(p_j) > 0.$ For firm *i* any unilateral deviation $p_i^{'}$ such that $c < p_i^{'} \leq p_j$ increases profits:

$$(p_i^{'} - c)\underbrace{D(p_i^{'})}_{\text{si } p_i^{'} < p_j} = (p_i^{'} - c)D_i(p_i^{'}, p_j) = \Pi_i(p_i^{'}, p_j) > \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)0 = 0.$$

Although we have already proved that the strategy profile (p_i, p_j), with $p_i > p_j > c$ cannot be an equilibrium we can also show that in many cases firm *j* would also have an incentive to deviate unilaterally. (For example, if $p^m \geq p_i > p_j > c$ any unilateral deviation $p_i > p_j^{'} > p_j$ increases the profits of firm *j*. For the cases $p_i > p_j > p^m > c$ and $p_i > p^m > p_j > c$ it is also straightforward to find deviations such the profits of firm *j* increase. The only situation where firm *j* would have no incentive to deviate would be one such that $p_i > p^m = p_j > c$).

d) Other cases:

- ¿ $p_i > c \geq p_j$ NE? NO. Firm *i* would have no incentive to deviate unilaterally while for firm *j* when $p_i > c > p_j$ any $p_j^{'} > p_j$ increases profits and when $p_i > c = p_j$ firm *j* increases profits by conveniently increasing the price. For example, if $p^m \geq p_i > c = p_j$ any price $p_i > p_j^{'} > c$ increases the profits of firm *j*. When $p_i > p^m > c = p_j$ any price $p^m > p_j^{'} > c$ (and others) increases the profits of firm *j*.

- ¿ $c \geq p_i > p_j$ NE? NO. Firm *i* would have no incentive to deviate while for firm *j* any price $p_j^{'} > p_j$ increases profits.

3.2.2. ***Heterogeneous products*** (differentiated products)

(i) Heterogeneous product. Residual demand.

(ii) Representation of the game in normal form.

(iii) Notion of equilibrium. Best response function. Bertrand-Nash equilibrium.

(i) *Heterogeneous product.Residual demand*

We maintain the rest of the assumptions of the Bertrand model (two firms, simultaneous choice, constant and common marginal cost, price competition) but now we consider that the two firms sell heterogeneous products. That is, firms sell products that are close but imperfect substitutes. The demand for the product of firm *i*, the residual demand, is given by $D_i(p_i, p_j)$. Assume that $\frac{\partial D_i}{\partial p_i} < 0$, $\frac{\partial D_i}{\partial p_j} > 0$ and $\left|\frac{\partial D_i}{\partial p_i}\right| > \frac{\partial D_i}{\partial p_j}$; that is, the demand for product

i is a decreasing function of its own price, products are substitutes and the own effect is larger than the cross effect.

(iii)*Representation of the game in normal form. Notion of equilibrium*

The game in normal form is:

1) $i = 1, 2.$ (Players)

2) $p_i \geq 0$ or equivalently $p_i \in [0, \infty),\ i = 1, 2.$

3) The profit of each firm corresponding to (p_1, p_2) is:

$$\left.\begin{array}{l} \Pi_1(p_1, p_2) = (p_1 - c)D_1(p_1, p_2) \\ \Pi_2(p_1, p_2) = (p_2 - c)D_2(p_1, p_2) \end{array}\right\} \equiv \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j),\ i, j = 1, 2,\ j \neq i$$

Now the residual demand of each firm is a continous function of its price.

(iv) *Notion of equilibrium. Best response function. Bertrand-Nash equilibrium*

In terms of best responses the definition of the Bertrand-Nash equilibrium is:

"(p_1^*, p_2^*) is a Bertrand-Nash equilibrium if $p_i^* = g_i(p_j^*),\ \forall i, j = 1, 2,\ j \neq i.$",

where $g_i(p_j)$ is firm i's best response to the price p_j of its rival. The best response of firm i consists of choosing p_i such that:

$$\max_{p_i \geq 0} \Pi_i(p_i, p_j) \equiv (p_i - c)D_i(p_i, p_j)$$

$$\frac{\partial \Pi_i}{\partial p_i} = D_i(p_i, p_j) + (p_i - c)\frac{\partial D_i}{\partial p_i} = 0 \quad (1) \rightarrow g_i(p_j)$$

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = 2\frac{\partial D_i}{\partial p_i} + (p_i - c)\frac{\partial^2 D_i}{\partial p_i^2} < 0.$$

3.3. **Leadership in the choice of output. The Stackelberg model**

(i) Context.

(ii) Two-stage game. Perfect information. Notion of strategy.

(iii) Backward induction. Subgame perfect equilibrium.

(iv) Example: linear demand and constant marginal cost.

(v) Other Nash equilibria which are not subgame perfect.

(i) *Context*

The Stackelberg duopoly model has four basic characteristics:

a) We consider a market with *2 firms*.

b) *Homogeneous product*. That is, from the consumers' point of view the two firms produce products which are perfect substitutes.

c) *Quantity competition*. Let x_1 and x_2 be the outputs of firm 1 and 2, respectively.

d) *Sequential choice*. One of the firms (the leader), firm 1, chooses its output level first. Next the other firm (the follower), firm 2, chooses its output after observing the output chosen by firm 1. From a game theory perspective this is a perfect information game.

(ii) *A two-stage game. Perfect information. Notion of strategy*

Firms play a two-stage game:

Stage 1: firm 1 chooses its output $x_1 \geq 0$.

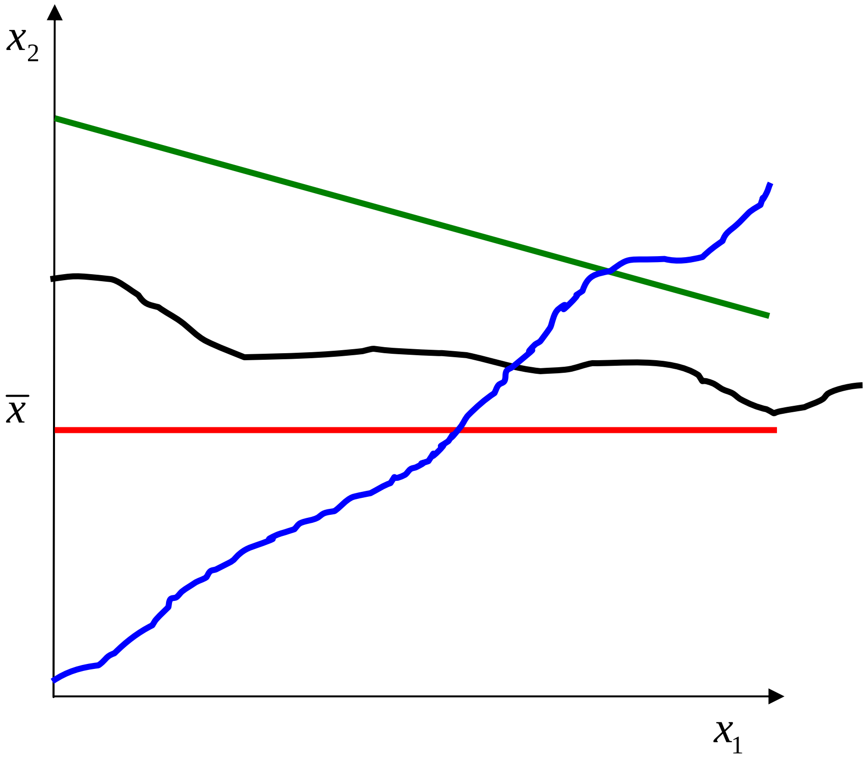
Stage 2: firm 2 chooses its output $x_2 \geq 0$ after observing the output chosen by firm 1.

Given that both players must have the same perception of the game not only does player 2 observes the choice made by player 1 but also player 1 knows that player 2 observes its choice. That is, there is perfect information and both players have the same perception of the game.

(Note: a game with two stages, i.e. a sequential game, in which the second mover does not observe the output chosen by the first mover –i.e. an imperfect information game– would be equivalent to a simultaneous game, such as the Cournot game).

The strategy spaces of the players are as follows:

- $x_1 \geq 0$: any non-negative quantity serves as a strategy for player 1; equivalently $x_1 \in [0, \infty)$.



- The description of the strategies for player 2 is more complex. Recall that a strategy is a complete description of what a player would do if he/she were called on to play at each one of his/her decision nodes, independently of whether they are attaniable or not given the current behavior of the other(s) player(s). In the Stackelberg game, each possible output of firm 1 generates a different decision node for firm 2. Therefore, firm 2's strategy is a function $x_2(x_1)$ which tells us how much firm 2 is going to produce for each possible production of firm 1.

(iii) *Backward induction. Subgame perfect equilibrium*

Although the game seems too complex to be solved, we know that in perfect information games without ties the backward induction criterion proposes a unique solution which coincides with the unique subgame perfect equilibrium. The proceedure is similar to that used with finite games in the previous chapter.

We start from the last subgames, that is, at stage 2.

Stage 2

We eliminate the non credible threats or dominated actions in each subgame. Given an output of firm 1 (a subgame) x_1 the only credible threat is for firm 2 to choose a profit maximizing output level:

$$\max_{x_2 \geq 0} \Pi_2(x_1, x_2) \equiv p(x_1 + x_2)x_2 - C_2(x_2)$$

$$\frac{\partial \Pi_2}{\partial x_2} = p(x_1 + x_2) + x_2 p'(x_1 + x_2) - C_2'(x_2) = 0 \quad (1) \rightarrow \overline{f}_2(x_1)$$

$$\frac{\partial^2 \Pi_2}{\partial x_2^2} = 2p'(x_1 + x_2) + x_2 p''(x_1 + x_2) - C_2''(x_2) < 0$$

Taking into account the non-negativity constraint, $x_2 \geq 0$, we have:

$f_2(x_1) = \max\left\{ \overline{f}_2(x_1), 0 \right\} \rightarrow$ ***Firm 2's strategy in the subgame perfect equilibrium***.

In finite games, the proceedure continues by eliminating all the non credible threats and computing the reduced game. In the Stackelberg game eliminating all the incredible threats is equivalent to eliminating player 1's strategies other than $f_2(x_1) = \max\left\{ \overline{f}_2(x_1), 0 \right\}$.

Stage 1

Player 1 anticipates that firm 2 will behave at each subgame according to the strategy $f_2(x_1) = \max\left\{ \overline{f}_2(x_1), 0 \right\}$. Firm 1's profit function in reduced form is:

$\Pi_1(x_1, f_2(x_1)) \equiv p(x_1 + f_2(x_1))x_1 - C_1(x_1).$

Therefore, the problem for firm 1 becomes:

$$\max_{x_1 \geq 0} \Pi_1(x_1, f_2(x_1)) \equiv p(\underbrace{x_1 + f_2(x_1)}_{x})x_1 - C_1(x_1).$$

$$\frac{d\Pi_1}{dx_1} = p(x_1 + x_2) + x_1[1 + f_2'(x_1)]p'(x_1 + x_2) - C_1'(x_1) = 0 \quad (2) \rightarrow x_1^L$$

$$\frac{d^2\Pi_1}{dx_1^2} < 0$$

Therefore, the **subgame perfect equilibrium** is the strategy profile

$$(x_1^L, f_2(x_1)).$$

(iv) *Example: linear demand and constant marginal cost*.

Stage 2

We eliminate the non credible threats or dominated actions in each subgame. Given an output of firm 1 (a subgame) x_1 the only credible threat is for firm 2 to choose a profit maximizing output level:

$$\max_{x_2 \geq 0} \Pi_2(x_1, x_2) \equiv p(x_1 + x_2)x_2 - C_2(x_2) \equiv [a - b(x_1 + x_2)]x_2 - cx_2$$

$$\frac{\partial \Pi_2}{\partial x_2} = p(x_1 + x_2) + x_2 p^{'}(x_1 + x_2) - C_2^{'}(x_2) = 0 \;\; (1) \rightarrow \overline{f}_2(x_1) = \frac{a - c - bx_1}{2b}$$

$$\frac{\partial^2 \Pi_2}{\partial x_2^2} = 2p^{'}(x_1 + x_2) + x_2 p^{"}(x_1 + x_2) - C_2^{"}(x_2) = -2b < 0$$

Taking into account the non-negativity constraint, $x_2 \geq 0,$ we get:

$$f_2(x_1) = \max\left\{\overline{f}_2(x_1), 0\right\} = \max\left\{\frac{a - c - bx_1}{2b}, 0\right\}$$ ***Firm 2's strategy in the SPE***.

Stage 1

Player 1 anticipates that firm 2 will behave at each subgame according to the strategy $f_2(x_1) = \max\left\{\overline{f}_2(x_1), 0\right\} = \max\left\{\frac{a - c - bx_1}{2b}, 0\right\}$. Firm 1's profit function in reduced form is:

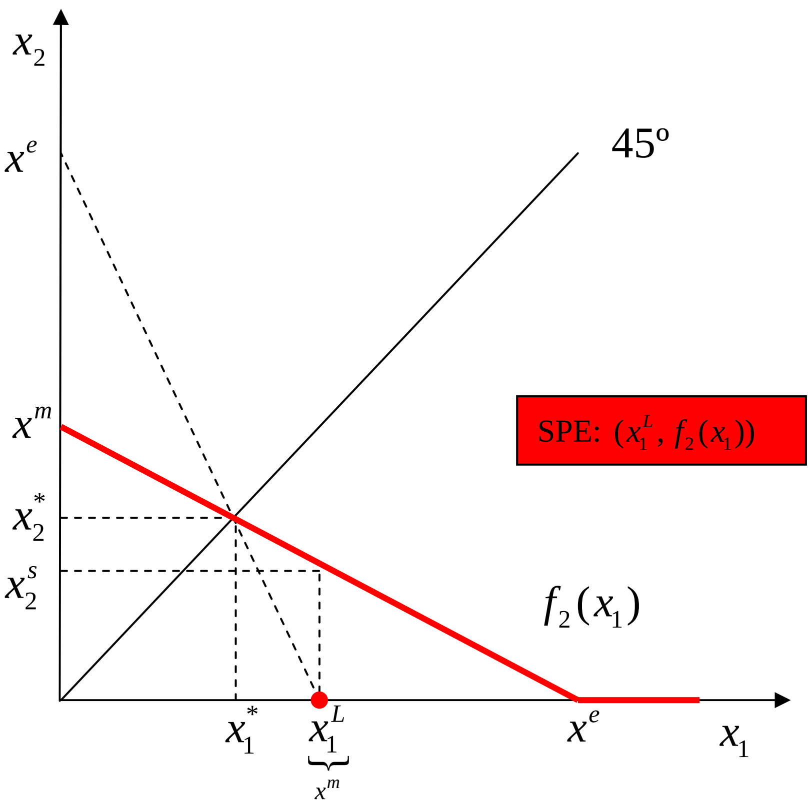
$\Pi_1(x_1, f_2(x_1)) \equiv p(x_1 + f_2(x_1))x_1 - C_1(x_1).$ Then firm 1's problem is:

$$\max_{x_1 \geq 0} \Pi_1(x_1, f_2(x_1)) \equiv [a - c - b(x_1 + f_2(x_1))]x_1 \equiv [a - c - b(x_1 + \frac{a - c - bx_1}{2b})]x_1 \equiv [\frac{a - c - bx_1}{2}]x_1$$

$$\frac{d\Pi_1}{dx_1} = p(x_1 + x_2) + x_1[1 + f_2^{'}(x_1)]p^{'}(x_1 + x_2) - C_1^{'}(x_1) = a - c - 2bx_1 = 0 \;\; (2) \rightarrow x_1^L = \frac{a - c}{2b}$$

$$\frac{d^2\Pi_1}{dx_1^2} < 0$$

Therefore, the **subgame perfect equilibrium is** $(x_1^L, f_2(x_1)).$



In order to obtain the profits of the firms we have to play the game.

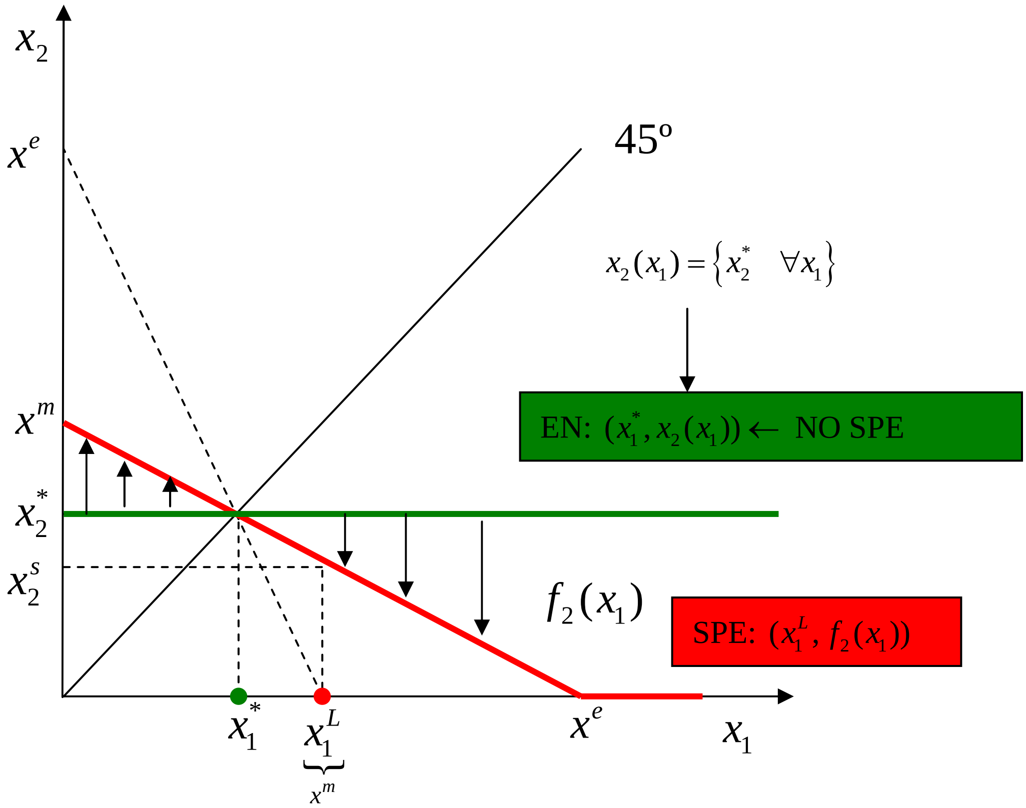
$$x_2^s = f_2(x_1^L) = \frac{a-c-bx_1^L}{2b} = \frac{a-c-b\left(\frac{a-c}{2b}\right)}{2b} = \frac{a-c}{4b}$$

$$x^s = x_1^L + x_2^s = \frac{a-c}{2b} + \frac{a-c}{4b} = \frac{3(a-c)}{4b}$$

$$p^s = p(x^s) = a - bx^s = a - b\frac{3(a-c)}{4b} = \frac{a+3c}{4};\; p^s - c = \frac{a-c}{4}$$

$$\Pi_1^L = (p^s - c)x_1^L = \frac{(a-c)}{4}\frac{(a-c)}{2b} = \frac{(a-c)^2}{8b} =;\; \Pi_2^s = (p^s - c)x_2^s = \frac{(a-c)}{4}\frac{(a-c)}{4b} = \frac{(a-c)^2}{16b}.$$

(v) *Other Nash equilibria which are not subgame perfect*



3.4. **Collusion and stability of agreements**

3.4.1. ***Short-term collusion***

(i) Cournot model. The collusion agreement is not a short-term equilibrium.

(ii) Bertrand model. The collusion agreement is not a short-term equilibrium.

(i) *Cournot model. The collusion agreement is not a short-term equilibrium*

If firms colluded they would be interested in maximizing aggregate profits.

$$\max_{x_1,x_2} \Pi_1(x_1,x_2)+\Pi_2(x_1,x_2) \equiv p(x_1+x_2)x_1 - C_1(x_1) + p(x_1+x_2)x_2 - C_2(x_2)$$

$$\left.\begin{array}{l} \dfrac{\partial \Pi}{\partial x_1} = p(x_1^m + x_2^m) + (x_1^m + x_2^m)p'(x_1^m + x_2^m) - C_1'(x_1^m) = 0 \quad (1) \\ \dfrac{\partial \Pi}{\partial x_2} = p(x_1^m + x_2^m) + (x_1^m + x_2^m)p'(x_1^m + x_2^m) - C_2'(x_2^m) = 0 \quad (2) \end{array}\right\} MR_I = C_1' = C_2'$$

When marginal costs are constant and equal across firms conditions (1) and (2) are identical. The two-equation system would have infinite solutions: any pair of outputs such that $x_1 + x_2 = x^m$ would maximize the industry profit. For these cases we also refer to the symmetric collusion agreement where each firm produces a half of the monopoly output: $x_i^m = \frac{x^m}{2},\ i = 1, 2.$

We now show that the collusion agreement (implicit of course) cannot be supported as an equilibrium when the game is played once. In other words, we demonstrate that the strategy profile (x_1^m, x_2^m) is not a Nash equilibrium in the Cournot game.

Given the strategy $x_j \geq 0$ the best response of firm *i* is to choose a strategy x_i such that:

$$\max_{x_i \geq 0} \Pi_i(x_i, x_j) \equiv p(x_i + x_j)x_i - C_i(x_i)$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i + x_j) + x_i p^{'}(x_i + x_j) - C_i^{'}(x_i) = 0 \quad (1) \rightarrow \overline{f}_i(x_j)$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = 2p^{'}(x_i + x_j) + x_i p^{"}(x_i + x_j) - C_i^{"}(x_i) < 0$$

So the best response function is: $f_i(x_j) = \max\left\{\overline{f}_i(x_j), 0\right\}.$

To check that the combination of strategies (x_1^m, x_2^m) is not a Nash equilibrium we calculate the marginal profit for each firm:

$$\frac{\partial \Pi_i(x_i^m, x_j^m)}{\partial x_i} = p(x_i^m + x_j^m) + x_i^m p^{'}(x_i^m + x_j^m) - C_i^{'}(x_i^m) = -x_j^m \underbrace{p^{'}(x_i^m + x_j^m)}_{<0} > 0$$

By definition of collusion agreement.

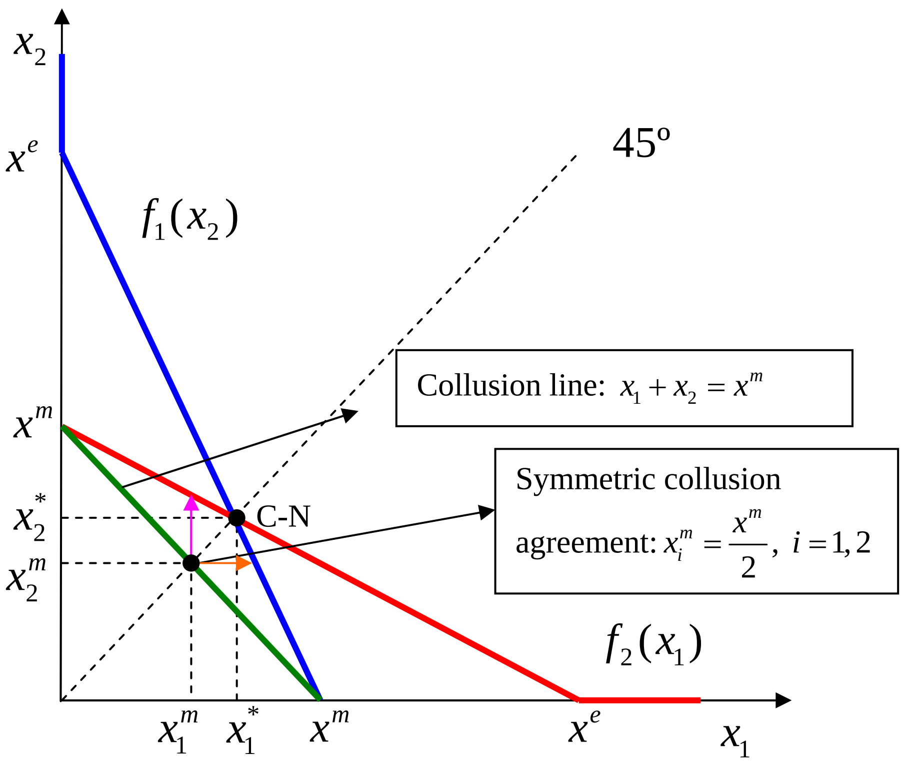
Then starting from the collusion agreement an increase in output leads to an increase in the firm i's profit and, therefore, firm i would have an incentive to break the collusion agreement. Put differently, given the definition of best response function $\frac{\partial \Pi_i(f_i(x_j^m), x_j^m)}{\partial x_i} = 0$ and as

$\frac{\partial \Pi_i(x_i^m, x_j^m)}{\partial x_i} > 0$ then $f_i(x_j^m) > x_i^m$.

As we know what the optimal deviation for firm i is if it decides to break the collusion agreement, $f_i(x_j^m)$, we denote by $\overline{\Pi}_i$ the profit that i would obtain if it deviates optimally and the other firms do not deviate. That is,

$$\overline{\Pi}_i = \Pi_i(f_i(x_j^m), x_j^m).$$

Graphic analysis: linear demand and constant marginal cost



Oligopoly

It is easy to generalize the above result to the case of *n* firms. The condition which defines the collusion agreement (the strategy profile maximizing the aggregate profit) is:

$$p(x_i^m + x_{-i}^m) + (x_i^m + x_{-i}^m)\, p^{'}(x_i^m + x_{-i}^m) - C_i^{'}(x_i^m) = 0 \quad i = 1,..,n.$$

To show that the strategy profile $(x_1^m,..,x_n^m)$ is not a Nash equilibrium we obtain the marginal profit of each firm:

$$\frac{\partial \Pi_i(x_i^m, x_{-i}^m)}{\partial x_i} = p(x_i^m + x_{-i}^m) + x_i^m\, p^{'}(x_i^m + x_{-i}^m) - C_i^{'}(x_i^m) = -x_{-i}^m \underbrace{p^{'}(x_i^m + x_{-i}^m)}_{<0} > 0$$

By definition of the collusion agreement.

Then starting from the collusion agreement an increase in the output of firm *i* also increases its profit and, therefore, firm *I* would have an incentive to break the collusion agreement. In other words, given the definition of best response function $\frac{\partial \Pi_i(f_i(x_{-i}^m), x_{-i}^m)}{\partial x_i} = 0$ and as

$\frac{\partial \Pi_i(x_i^m, x_{-i}^m)}{\partial x_i} > 0$ then $f_i(x_{-i}^m) > x_i^m$.

As we know what the optimal deviation for firm *i* is if it decides to break the collusion agreement, $f_i(x_{-i}^m)$, we denote by $\overline{\Pi}_i$ the profit that *i* would obtain if it deviates optimally and the other firms do not deviate. That is,

$$\overline{\Pi}_i = \Pi_i(f_i(x_{-i}^m), x_{-i}^m).$$

(ii) *Bertrand mode*l. *The collusion agreement is not a short-term equilibrium*

Consider the Bertrand model with homogeneous product and constant (and common) marginal cost. The strategy profile which represents the symmetric collusion agreement is (p^m, p^m). The profit of each firm is:

$$\Pi_i^m = \Pi_i(p^m, p^m) = (p^m - c)\frac{1}{2}D(p^m) = \frac{1}{2}\Pi^m$$

We know (it has been demonstrated) that a combination of strategies of the type $p_i = p_j > c$ is not a Nash equilibrium. Any firm would have an incentive to deviate unilaterally. For example, we can choose $p_i^{'} = p^m - \varepsilon$ (where ε is an arbitrary positive amount as small as necessary). Of course, there are infinite deviations such that firm *i* is better off.

Finding the optimal deviation for firm *i* is more problematical. The best that it can do is to undercut its rival's price by the lowest amount possible, $\varepsilon > 0, \varepsilon \to 0$. Although we do not have that optimal deviation well-defined we can be as near as we wish to monopoly price. Let $\overline{\Pi}_i$ be firm *i*'s profit when it optimally breaks the collusion agreement and the rival keeps it. That is,

$$\overline{\Pi}_i = \Pi_i(p^m - \varepsilon, p^m) = (p^m - \varepsilon - c)D(p^m - \varepsilon) \underbrace{\simeq}_{\varepsilon \to 0} (p^m - c)D(p^m) = \Pi^m$$

3.4.2. *Stability of agreements. Finite temporal horizon and infinite temporal horizon*

We have shown that in the short term the collusion (cooperation) agreement cannot hold as an equilibrium in either the Cournot game or the Bertrand game. In this section, we study the possibilities of collusion or cooperation when the game is repeated.

(i) ***Finite temporal horizon***

Backward induction argument $\rightarrow$ collusion (cooperation) cannot be supported as an equilibrium (at each stage firms behave as in the one-shot game). The reasoning is similar to that in the Prisoner's Dilemma.

(ii) ***Infinite temporal horizon***

There are two ways of interpreting an infinite temporal horizon:

(a) *Literal interpretation*: the game is repeated an infinite number of times. In this context, to compare two alternative strategies a player must compare the discounted present value of the respective gains. Let δ be the discount factor, $0 < \delta < 1$, and let r be the discount rate ($0 < r < \infty$) where $\delta = \dfrac{1}{1+r}$.

(b) *Informational interpretation*: the game is repeated a finite but unkonwn number of times. At each stage, there is a probability $0 < \delta < 1$ of the game continuing. In this setting, each player must compare the expected value (which might be also discounted) of the different strategies.

We shall see that the existence of *implicit punishment threats* may serve to maintain collusion as an equilibrium of the repeated game.

Note first that there is a subgame perfect equilibrium of the infinitely repeated game where each player plays his/her short-term Nash equilibrium strategy in each period. In the Cournot model such a strategy would consist of "producing in each period the Cournot quantity independently of past history". In the Bertrand model that strategy would consist of "charging a price equals to marginal cost independently of past history".

We next study the possibility that there may be another subgame perfect equilibrium where players cooperate with each other. Consider the following combination of long term strategies: $s_i^c \equiv \{s_{it}(H_{t-1})\}_{t=1}^{\infty}$, i =1,2.

where,

$$s_{it}^c(H_{t-1}) = \begin{cases} \overbrace{\text{"cooperate"}}^{\text{to collude}} \text{ if all elements of } H_{t-1} \text{ are equal to ("cooperate","cooperate") or } t=1 \\ \text{"not cooperate"(the short-term NE strategy)} \qquad \text{otherwise} \end{cases}$$

(in Cournot:

$$s_{it}^c(H_{t-1}) = \begin{cases} x_i^m & \text{if all elements of } H_{t-1} \text{ are equal to } (x_i^m, x_{-i}^m) \text{ or } t=1 \\ x_i^* & \text{otherwise} \end{cases})$$

(in Bertrand:

$$s_{it}^c(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ are equal to } (p^m, p^m) \text{ or } t=1 \\ c & \text{otherwise} \end{cases})$$

Note that these long term strategies incorporate "implicit punishment threats" in case of breach of the (implicit) cooperation agreement. The threat is credible because "confess" in each period (independently of the past history) is a Nash equilibrium of the repeated game.

To check whether it is possible in this context to maintain cooperation as an equilibrium, we must check that players have no incentive to deviate; that is, we must check that the combination of strategies (s_1^c, s_2^c) constitutes a Nash equilibrium of the repeated game.

Notation

$\Pi_i^m \rightarrow$ Firm *i*'s profit under collusion at each stage of the game.

$\Pi_i^* \rightarrow$ Firm *i*'s profit in the short-term Nash equilibrium at each stage of the game.

$\overline{\Pi}_i \rightarrow$ Firm *i*'s profit if the other firms cooperate and it optimally deviates.

$$\overline{\Pi}_i > \Pi_i^m > \Pi_i^*$$

The discounted present value for firm *i* in the strategy profile (s_1^c, s_2^c) is given by:

$$\pi_i(s_i^c, s_j^c) = \Pi_i^m + \delta\Pi_i^m + \delta^2\Pi_i^m + ... = \Pi_i^m(1+\delta+\delta^2+...) = \frac{\Pi_i^m}{1-\delta}$$

If firm *i* deviates in the first period its gains are:

$$\pi_i(\overline{s}_i, s_j^c) = \overline{\Pi}_i + \delta\Pi_i^* + \delta^2\Pi_i^* + ... = \overline{\Pi}_i + \delta(1+\delta+\delta^2+...)\Pi_i^* = \overline{\Pi}_i + \delta\frac{\Pi_i^*}{1-\delta}$$

Cooperation is supported as a Nash equilibrium if no player has any incentive to deviate; that is, when $\pi_i(s_i^c, s_j^c) \geq \pi_i(s_i, s_j^c)$. It is straightforward to check that if $\delta \geq \overline{\delta}$ no firm has an incentive to break the collusion agreement, where

$$\overline{\delta} = \frac{\overline{\Pi}_i - \Pi_i^m}{\overline{\Pi}_i - \Pi_i^*}.$$

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