Intermediate serial cost-sharing rules

M. Josune Albizuri, Díez Henar, de Sarachu Amaia[*]

June 4, 2014

Abstract.   In this paper we give a generalization of the serial cost-sharing rule defined by Moulin and Shenker (1992) for cost sharing problems. According to the serial cost sharing rule, agents with low demands of a good pay cost increments associated with low quantities in the production process of that good. This fact might not always be desirable for those agents, since those cost increments might be higher than others, for example with concave cost functions. In this paper we give a family of cost sharing rules which allocates cost increments in all the possible places in the production process. And we characterize axiomatically each of them by means of an axiomatic characterization related to the one given for the serial cost-sharing rule by Moulin and Shenker (1994).

JEL classification: C71

Key words: cost sharing problems, serial cost sharing-rule.

[*]Department of Applied Economics IV, Faculty of Business and Economics, University of the Basque Country, Avda Lehendakari Agirre, 83, 48015 Bilbao, Spain, e-mail: mj.albizuri@ehu.es, henar.diez@ehu.es, amaia.sarachu@ehu.es

1. Introduction

We consider cost-sharing problems in which there is a production process of a private good shared by n agents. Each agent demands a quantity q_i of the good. The cost function is denoted C and a cost-sharing rule allocates the total production cost, that is $C\left(\sum_i q_i\right)$, among all the agents.

In this paper we propose a family of cost-sharing rules which is related to the serial cost-sharing rule defined by Moulin and Shenker (1992). This rule has caught much attention and several related rules have been defined until now. Among others we can highlight the decreasing serial mechanism defined by de Frutos (1998) and the concave serial rule and the convex serial rule introduced by Koster (2002). Generalizations of the serial cost-sharing rule to heterogeneous cost-sharing models (Koster (2006)) and when agents require bundle of goods (Koster et al. (1998)) have also been carried on.

The family defined in this paper contains the serial cost-sharing rule defined by Moulin and Shenker (1992), and the dual serial cost-sharing rule (Albizuri and Zarzuelo, 2007), which is also connected to the former rule. All the mentioned cost-sharing rules, including the ones defined in this paper, will give more or less suitable cost shares, depending on the particular cost-sharing problem they are applied to. To better understand the new rules let us describe the serial cost-sharing rule and the dual serial cost-sharing rule.

The serial cost-sharing rule is as follows. There are two agents i and j, and $q_i \leq q_j$. When the production starts, each unit of the good is equally divided among the two agents, who share equally the incurred cost. This continues until $2q_i$ is produced, that is, until agent i is given q_i. At this point agent i leaves the system and the process continues as before, that is, agent j receives the remaining quantity and pays the associated cost. Consequently, agent i pays $C\left(2q_i\right)/2$ and j pays the rest, that is, $C\left(q_i + q_j\right) - C\left(2q_i\right)/2$. In Fig. 1 we draw the associated production path. The serial cost-sharing rule is obtained when we generalize this process to n agents.
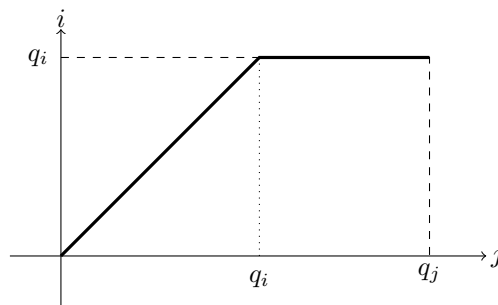

<small>Figure 1</small>

Albizuri and Zarzuelo (2007) define the dual serial cost-sharing which equalizes the quantities left to be allocated to agents. So when the good production starts

each unit goes to agent j, that is, the agent with the highest demand, who pays the incurred cost. When agent j is served $q_j - q_i$ units, that is, when both i and j are short of the same quantity q_i, agent j pays $C(q_j - q_i)$ and agent i enters the picture. The production process continues and both agents are served simultaneously and pay equally the cost, that is, each of the agents pays $\frac{C(q_j+q_i)-C(q_j-q_i)}{2}$. Therefore, agent i pays $\frac{C(q_j+q_i)-C(q_j-q_i)}{2}$ and agent j pays $\frac{C(q_j+q_i)+C(q_j-q_i)}{2}$. In Fig. 2 we see the associated path. Generalizing this procedure to n agents the dual serial cost-sharing rule is obtained.
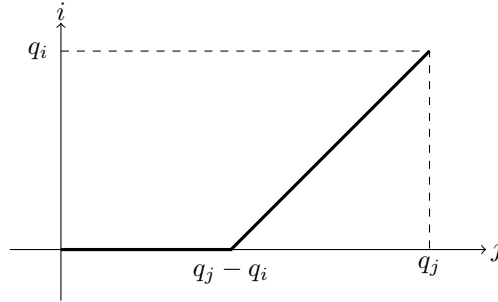

FIGURE 2

Notice that in the first case agent i pays only the cost increments of lowest quantities and in the second only the cost increments of highest quantities. In this paper we consider the cases when agent i pays the cost increments in any position in between 0 and the total production $q_i + q_j$. As with the serial cost-sharing rule (and the dual serial cost-sharing rule) agents i and j share equally the quantity and cost increment associated with a production process interval and in that production process agent i meets his demand q_i. Agent j is the only one who pays the rest. If we denote by $\left[a_{q_i+q_j}(2q_i), a_{q_i+q_j}(2q_i)+2q_i\right]$ the production interval paid by both agents, then agent i pays $\frac{C\left(a_{q_i+q_j}(2q_i)+2q_i\right)-C\left(a_{q_i+q_j}(2q_i)\right)}{2}$. And agent j pays the rest, that is, $\frac{C\left(a_{q_i+q_j}(2q_i)\right)-C\left(a_{q_i+q_j}(2q_i)+2q_i\right)}{2}+C(q_i+q_j)$. We see in Fig. 3 the associated production path. Generalizing this procedure to n agents we obtain the family of rules defined in this paper. We call them *intermediate serial cost-sharing rules*. If $a_{q_i+q_j}(2q_i) = 0$ we get the serial cost-sharing rule, while if $a_{q_i+q_j}(2q_i) = q_j - q_i$ the dual serial cost-sharing rule arises.
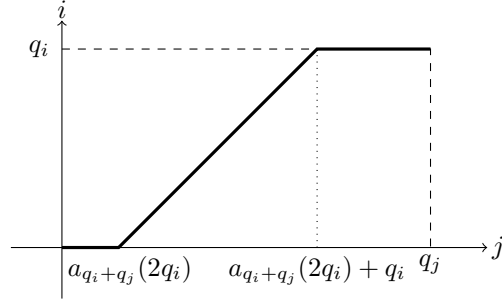
FIGURE 3

Observe that if C is convex then agent i will pay less with the serial cost-sharing rule than with the dual serial cost-sharing rule. And on the contrary, if C is concave agent i will pay more with the serial cost-sharing rule. The rules of the new family are determined by $a_{q_i+q_j}(2q_i) \in [0, q_j - q_i]$. The nearer this number is from 0 the better agent i is if the cost function is convex and the worse agent i is if the cost function is concave.

With the serial cost-sharing rule agents with low demands pay cost increments associated with low quantities. This fact might not always be desirable for them, since those cost increments might be higher than others, for example when the cost function is concave, and agents with low demands might have imposed a positive externality on the others which is not reflected in their cost shares according to the serial cost-sharing rule. On the contrary, with convex cost functions agents with low demands might have imposed a negative externality on the others which is not reflected in their allocations determined by the serial cost-sharing rule. The intermediate serial cost-sharing rules might reflect these externalities.

Take for example the concave cost function $C(t) = \min\{t, 8\}$, and $q_i = 4$ and $q_j = 6$. According to the serial cost-sharing rule, the allocation for agent i is 4 and for agent j is also 4. Since C is concave, the allocations determined by the concave serial rule by Koster (2002) coincide with those given by the serial cost-sharing rule. If we consider the decreasing serial mechanism by de Frutos (1998), the allocation for j is $\frac{C(2 \cdot 6)}{2} = 4$ and for i is the rest, that is, 4. The cost share of 4 for each agent does not seem a good allocation since agent i pays his entire stand alone cost 4 and agent j pays 2 units less than his stand alone cost 6. Agent i has imposed a positive externality to agent j: total cost of 10 units is 8 because agent i has demanded 4 units. Therefore, it seems sensible that agent i also gains and pays less than his stand alone cost. According to the dual serial cost-sharing rule, agent i pays 3 and agent j pays 5. These are the same allocations given by the convex serial rule by Koster (2002) since C is concave. Notice that in this case both agents pay 1 unit less than their stand alone costs. We can think that agent i has gained too much. The rules of our family give precisely to agent i all the numbers in between 3 and 4 as possible

cost shares, depending on the elected $a_{10}(8) \in [0,2]$. If $a_{10}(8) = 0$ the cost share is 4 and if $a_{10}(8) = 2$ the cost share is 3.

As another example, consider the convex cost function $C(t) = \max\{t - 8, 0\}$ with the previous demands: $q_i = 4$ and $q_j = 6$. The serial cost-sharing rule allocates 0 and 2 for i and j respectively. Since the cost function is convex, those are the allocations provided by the convex serial rule by Koster (2002). The decreasing serial mechanism by de Frutos (1998) gives the same cost shares since the allocation for j is $\frac{C(2 \cdot 6)}{2} = 2$, and i pays the rest, that is, 0. Notice that the stand alone cost for both agents is 0. And total cost is 2 because both players are present. So it seems sensible that agent j pays also less than the entire cost 2. The dual serial cost-sharing rule gives cost share 1 for both agents. And the concave serial rule by Koster (2002) coincides with the dual serial cost-sharing rule for convex cost functions. If the cost share for each agent is 1, agent i pays as much as j, although q_i is smaller than q_j. With the rules of our family, cost share for i ranges from 0 to 1, and hence i can pay not so few and not so much.

Needless to say, we are not saying that these new rules give better cost shares than the other ones in any case. We just say that there are cases in which the cost shares determined by them seem suitable.

In this paper we define and give some formulas for the intermediate serial cost-sharing rules. We also provide an axiomatic characterization for each of them related to the one for the serial cost-sharing rule given by Moulin and Shenker (1994). In their characterization some standard axioms and free lunch, a kind of consistency axiom, are employed. Free lunch deals with cost functions that vanish identically at the beginning of production process of the good. It turns out that it serves to determine the serial cost-sharing rule. In this paper we consider cost functions which are flat in any production process interval. And we prove that with an axiom related to free lunch we get an intermediate serial cost-sharing rule. When we consider different positions for the flat part in the axiom related to free lunch, the intermediate cost-sharing rules are determined, since such a position (joint with the other axioms) determines the place of the cost increments given to agents. In the particular case in which the production has not cost at the end of the production process, the dual serial cost-sharing rule arises.

The paper is structured as follows. Section 2 is a preliminary one. In Section 3 we define and give a formula for the intermediate serial cost-sharing rules. And we present the intermediate serial cost-sharing rule of a problem as a serial cost-sharing rule of an associated problem. In Section 4 we characterize each intermediate serial cost-sharing rule by means of a generalization of the characterization for the serial cost-sharing rule provided by Moulin and Shenker (1994). In Section 5 we identify a subfamily by requiring the well known scale invariance axiom. Finally, the paper

finishes with some examples in Section 6.

2. Preliminaries

Let U denote a set of potential *agents*. Given a non-empty finite subset N of U, by $\mathbb{R}^N$ we write the $|N|$-dimensional euclidean space whose axes are labelled with the members of N, $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_i \geq 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. If $x \in \mathbb{R}$ then $x_+ = \max\{x, 0\}$. Given $q \in \mathbb{R}_+^N$, we denote $Q = \sum_{i \in N} q_i$, and if $N = \{1, 2, \ldots, n\}$ and $q_1 \leq q_2 \leq \cdots \leq q_n$, we write $q^0 = 0$ and $q^j = (n - j + 1) q_j + q_{j-1} + \ldots + q_1$ for every $j \in N$. If $S \subseteq N$ then $q_S \in \mathbb{R}_+^S$ satisfies $(q_S)_i = q_i$ for all $i \in S$.

A triple (N, q, C) is called a *cost-sharing problem*, if N is a non-empty finite subset of U (the set of agents involved in the problem), $q \in \mathbb{R}_+^N$ (the demand profile of the cost-sharing problem) and C is a nondecreasing function defined on $[0, Q]$ such that $C(0) = 0$ (the cost function of the cost-sharing problem).

Let Γ_U denote the set of all cost-sharing problems with the foregoing properties.

A *cost-sharing rule* σ on a subset Γ of Γ_U associates each $(N, q, C) \in \Gamma$ with a vector $\sigma(N, q, C) \in \mathbb{R}_+^N$ satisfying

$$\sum_{i \in N} \sigma_i(N, q, C) = C(Q) \qquad \text{(efficiency)}.$$

Thus a cost-sharing rule must allocate total cost among the n agents.

Moulin and Shenker (1992) define *the serial cost-sharing rule*. Its functional form (as well as the following ones) is presented assuming $N = \{1, 2, \ldots, n\}$ and $q_1 \leq q_2 \leq \cdots \leq q_n$. The serial cost-sharing rule of (N, q, C), denoted φ, is defined by

$$\varphi_i(N, q, C) = \sum_{j=1}^{i} \frac{C_j^q - C_{j-1}^q}{n - j + 1} \tag{1}$$

for all $i \in \{1, \ldots, n\}$, where

$$C_j^q = C\left(q^j\right) \tag{2}$$

for all $j \in \{0, \ldots, i\}$.

3. Intermediate serial cost-sharing rules

We describe first the serial cost-sharing rule of Moulin and Shenker (1992). Let (N, q, C) be a cost-sharing problem with $q_1 \leq q_2 \leq \cdots \leq q_n$. When the production starts, each unit of the good is equally divided among the agents, who share equally the incurred cost. When quantity q^1 is produced, since agent 1 has met all his demand, he stops receiving the good and leaves the picture. And the process goes on in the same way. The production continues and each additional unit is divided equally

among the remaining $n-1$ agents, who share equally the incurred cost. When agent 2 has met his demand, that is, when quantity q^2 is produced, agent 2 stops receiving the good, he leaves the picture and the production continues for the remaining agents. These agents pay equally until agent 3 has met his demand and so on.

The dual serial cost-sharing rule defined by Albizuri and Zarzuelo (2007) is as follows. This rule equalizes quantities left to be allocated to agents, who share equally the incurred cost. When the production starts, each unit is given to agent n, the agent with the highest demand. When agent n is given $q_n - q_{n-1}$ units, that is, $Q - q^{n-1}$ units, then both n and $n-1$ are short of the same quantity. Agent n pays the incurred cost, that is, $C(Q - q^{n-1})$, and the production process continues by sharing the good equally among agents n and $n-1$, who pay equally the incurred cost. When each of them is given $q_{n-1} - q_{n-2}$ then agents n, $n-1$ and $n-2$ are left to be allocated the same quantity of good. So n and $n-1$ share equally the corresponding cost, that is, each one pays $\frac{C(Q-q^{n-2})-C(Q-q^{n-1})}{2}$ and the process continues in the same way. Units are given simultaneously to agents n, $n-1$ and $n-2$ until they are short of the same quantity of demand and so on.

If we look at the serial cost-sharing rule, all the agents pay the cost associated with the cost production interval $[0, q^1]$ and each one gets q_1 units of good. The remaining cost associated with $[0, q^2]$ is paid by $2, ..., n$ and each of them gets $q_2 - q_1$ units of good. The remaining cost associated with $[0, q^3]$ is paid by $3, ..., n$ to receive $q_3 - q_2$ units each, and so on until the cost associated with $[0, Q]$ is paid. Regarding the dual serial cost-sharing rule all the agents pay the cost associated with $[Q - q^1, Q]$ and each one gets q_1 units. The remaining cost associated with $[Q - q^2, Q]$ is paid by $2, ..., n$ and each of them receives $q_2 - q_1$ units, and this process continues until the cost associated with $[0, Q]$ is paid. We can see in Fig. 4 and Fig. 5 the intervals whose associated cost is paid in both cases.

FIGURE 4   FIGURE 5

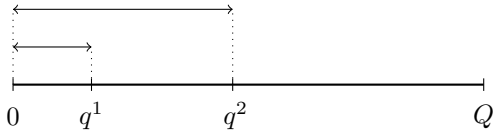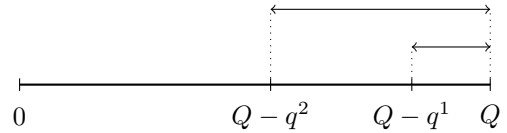We define an intermediate serial cost-sharing rule by taking any sequence of embedded intervals of length $q^1, q^2, ...$ whose associated cost is divided respectively among all the agents, who get q_1 units each of them, among agents $2, ..., n$, who receive $q_2 - q_1$ units each of them, and so on. Fig. 6 shows the sequence of intervals, denoted $[a_Q(q^j), a_Q(q^j) + q^j]$, where $j = 0, ..., n$.
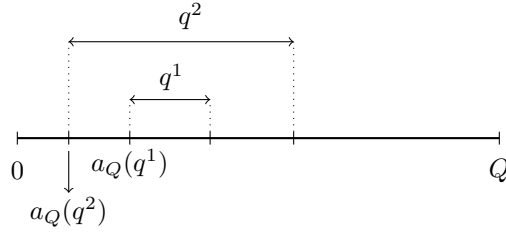
FIGURE 6

Notice that when we define a cost-sharing rule we have to determine cost shares for any cost-sharing problem, and hence we have to consider problems with different demand profiles $q, q', q \neq q'$. So it may hold $Q \neq Q'$. For (N, q', C') there is also a sequence of intervals $\left[a_{Q'}\left((q')^j \right), a_{Q'}\left((q')^j \right) + (q')^j \right]$, where $j = 0, ..., n$, whose associated costs determine the intermediate serial cost-sharing rule. In order to be consistent in our definition we suppose that $a_Q(x) = a_{Q'}(x')$ when $Q - x = Q' - x'$. Indeed, let $Q - x = Q' - x'$ and suppose that $Q' > Q$. Then (see Fig. 7) when the remaining cost associated with $[a_{Q'}(x'), a_{Q'}(x') + x']$ has to be paid, $x' - x$ units might have been already paid associated with a previous interval $([a_{Q'}(x' - x), a_{Q'}(x' - x) + x' - x])$. And therefore, in fact just only x units might have to be paid. Thus at this step Q units remain to be paid and x units have to be paid. Since we are defining a unique cost-sharing rule, cost makes sense to be paid in the same way as if there were Q units at the beginning and x units had to be paid, that is, $a_Q(x) = a_{Q'}(x')$. Hence, $a_Q(x) = a_{Q'}(x') = a(Q - x)$.
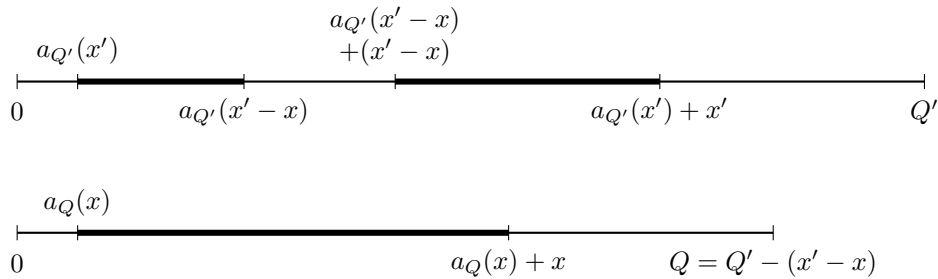
FIGURE 7

So an intermediate serial cost-sharing problem has associated a mapping $a : \mathbb{R}_+ \to \mathbb{R}_+$ (we write on page 9 the properties it has to satisfy). We formalize the definition by establishing payoffs for agents as units of good are produced.

When the production starts each unit goes to the agent with the highest demand, agent n, who pays for it. Then $Q - q^{n-1}$ units are considered, that is, the quantity beyond which agents n and $n-1$ are short of the same quantity of demand, and

a portion of those units, namely, $a\left(Q-q^{n-1}\right)$, is served to agent n. Agent n pays the incurred cost $C\left(a\left(Q-q^{n-1}\right)\right)$. At this point agent $n-1$ enters the picture and production is shared and paid equally among agents n and $n-1$. The quantity shared by them is a portion of $Q-q^{n-2}$, determined by a, that is, $a\left(Q-q^{n-2}\right)$. So it is considered the quantity beyond which agents n, $n-1$ and $n-2$ are short of the same quantity of demand, $Q-q^{n-2}$, and agents n and $n-1$ are served a portion of it. Since $a\left(Q-q^{n-1}\right)$ has already been given, each of n and $n-1$ receives $\frac{a\left(Q-q^{n-2}\right)-a\left(Q-q^{n-1}\right)}{2}$ and pays

$$\frac{C\left(a\left(Q-q^{n-2}\right)\right)-C\left(a\left(Q-q^{n-1}\right)\right)}{2}.$$

This procedure continues until $Q-q^0=Q$ units are considered and agents $1,...,n$ are given the same quantity of good and pay equally for it. Since at this point $a\left(Q-q^1\right)$ were already shared, $\frac{a(Q)-a\left(Q-q^1\right)}{n}$ is paid by each of them. Notice that $a\left(Q\right)$ units have been shared and paid in this way.

From now on all the agents are served equally until they meet their demands. So when production process continues quantities are equally divided among all agents who share equally the incurred cost until agent 1 receives the remaining part of his demand, that is, each agent is given $q_1-\frac{a(Q)-a\left(Q-q^1\right)}{n}$ units, that is, $\frac{a\left(Q-q^1\right)+q^1-a(Q)}{n}$ units. Since $a\left(Q\right)$ units were already paid, each agent pays

$$\frac{C\left(a\left(Q-q^1\right)+q^1\right)-C\left(a\left(Q\right)\right)}{n}.$$

At this point agent 1 has met his demand and leaves the production process. Then agents $2,...,n$ are served simultaneously and share equally the incurred cost until agent 2 gets the remaining part of his demand, that is, $\frac{a\left(Q-q^2\right)+q^2-\left(a\left(Q-q^1\right)+q^1\right)}{n-1}$ units. Since $a\left(Q-q^1\right)+q^1$ units were already served and paid, each of $2,...,n$ pays

$$\frac{C\left(a\left(Q-q^2\right)+q^2\right)-C\left(a\left(Q-q^1\right)+q^1\right)}{n-1}$$

and agent 2 leaves the system. The process continues in the same way until all the agents' demands are met.

By means of this procedure we define the *intermediate serial cost-sharing rule associated with $a:\mathbb{R}_+\rightarrow\mathbb{R}_+$*, denoted φ^a:

$$\varphi_i^a(N, q, C) = \sum_{j=1}^{i} \frac{C\left(a\left(Q - q^j\right) + q^j\right) - C\left(a\left(Q - q^{j-1}\right) + q^{j-1}\right)}{n - j + 1} \tag{3}$$

$$+ \sum_{j=1}^{i} \frac{C\left(a\left(Q - q^{j-1}\right)\right) - C\left(a\left(Q - q^j\right)\right)}{n - j + 1},$$

for all $i \in \{1, ..., n\}$.[1] Mapping a is continuous and satisfies the following three conditions.

$$a\left(x\right) \in [0, x], \tag{4}$$

$$a\left(x\right) \le a\left(y\right) \text{ for every } 0 \le x \le y, \tag{5}$$

$$a\left(y\right) - a\left(x\right) \le y - x \text{ for every } 0 \le x \le y. \tag{6}$$

The first condition means the sequence of intervals associated with the cost-sharing rule is inside $[0, Q]$, and the second and the third that the intervals fit together inside each other (by the second condition the beginnings fit and by the third one the ends do). We prove in the following proposition that those conditions are necessary and sufficient in order to get a cost-sharing rule. Though continuity for mapping a is not necessary for φ^a to be a cost-sharing rule, we think that it is a reasonable requirement in order agents to receive quantities which are near if agents' demands are also near.

Proposition 1. φ^a is a cost-sharing rule if and only if mapping $a : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies conditions (4), (5) and (6).

Proof. Take $a : \mathbb{R}_+ \to \mathbb{R}_+$ and let φ^a be the mapping defined in (3). Sufficiency of the three conditions easily follows. So let us prove necessity.

i) If function a does not satisfy condition (4), then there exists $x \in \mathbb{R}_+$ such that $a\left(x\right) > x$. Consider a cost-sharing problem (N, q, C) where the demand profile $q = (q_1, ...q_1, q_n) \in \mathbb{R}_+^N$ is such that $q_n = q_1 + x$. Notice that $Q - q^1 = x$, and therefore $a\left(Q - q^1\right) > Q - q^1$, that is, $a\left(Q - q^1\right) + q^1 > Q$. From (3) with $i = 1$ we have

$$\varphi_1^a(N, q, C) = \frac{C\left(a\left(Q - q^1\right) + q^1\right) - C\left(a\left(Q - q^1\right)\right)}{n},$$

and therefore this expression has no sense.

ii) If condition (5) does not hold, then there exist $x, y \in \mathbb{R}_+$ such that $x < y$ and $a\left(x\right) > a\left(y\right)$. Then we consider a cost-sharing problem (N, q, C) where $q =$

---

[1] Recall that $q^0 = 0$.

$(q_1, q_2, ...q_2, q_n)$ is such that $q_2 = q_1 + \frac{y-x}{n-1}$ and $q_n = q_2 + x$, and the cost function C is defined by

$$C(t) = \begin{cases} t & \text{if } t \leq a(Q - q^2), \\ a(Q - q^2) & \text{otherwise.} \end{cases}$$

Observe that $Q - q^1 = y$ and $Q - q^2 = x$. Then, from (3) with $i = 2$, we have

$$\varphi_2^a(N, q, C) = \frac{C(a(Q - q^1) + q^1) - C(a(Q - q^1))}{n}$$
$$+ \frac{C(a(Q - q^2) + q^2) - C(a(Q - q^1) + q^1)}{n - 1} + \frac{C(a(Q - q^1)) - C(a(Q - q^2))}{n - 1}.$$

And by definition of C,

$$\varphi_2^a(N, q, C) = \frac{C(a(Q - q^1)) - C(a(Q - q^1) + q^1)}{n(n - 1)}.$$

Since $a(Q - q^1) < a(Q - q^2)$ then $\varphi_2^a(N, q, C) < 0$. Hence φ^a is not a cost-sharing rule.

iii) If a does not satisfy (6), then there exist $x, y \in \mathbb{R}_+$ such that $0 \leq x < y$ and $a(y) - a(x) > y - x$. Then we consider a cost-sharing problem (N, q, C) where $q = (q_1, q_2, ...q_2, q_n)$ is such that $q_2 = q_1 + \frac{y-x}{n-1}$ and $q_n = q_2 + x$, and the cost function defined by

$$C(t) = \begin{cases} 0 & \text{if } t \leq a(Q - q^2) + q^2, \\ t - (a(Q - q^2) + q^2) & \text{otherwise.} \end{cases}$$

We have that $Q - q^1 = y$ and $Q - q^2 = x$. From (3) with $i = 2$ and the definition of C it follows that

$$\varphi_2^a(N, q, C) =$$
$$\frac{C(a(Q - q^1) + q^1) - C(a(Q - q^1))}{n} + \frac{C(a(Q - q^1)) - C(a(Q - q^1) + q^1)}{n - 1}.$$

And since $a(Q - q^1) - a(Q - q^2) > Q - q^1 - (Q - q^2) = q^2 - q^1$, then $\varphi_2^a(N, q, C) < 0$. Hence φ^a is not a cost-sharing rule. ■

The family of intermediate serial cost-sharing rules contains the serial cost-sharing rule defined by Moulin and Shenker (1992) $(a(x) = 0)$ and the dual serial cost-sharing rule defined by Albizuri and Zarzuelo (2007) $(a(x) = x)$.

Finally, we formalize in a proposition (which can be easily proved) that the formula (3) for the intermediate serial cost-sharing rule gives the allocations prescribed by the serial cost-sharing rule in the associated cost-sharing problem described at

the beginning of this section (Fig. 6). Given a cost-sharing problem (N, q, C), the associated problem is $\left(N, q, \widetilde{C}_a^Q\right)$ where $\widetilde{C}_a^Q$ is defined by[2]

$$\widetilde{C}_a^Q(t) = C\left(a\left(Q - t\right) + t\right) - C\left(a\left(Q - t\right)\right).\tag{7}$$

So $\widetilde{C}_a^Q(t)$ measures the cost of t units produced just after $a(Q - t)$. We can see that in Fig. 8 below.

FIGURE 8

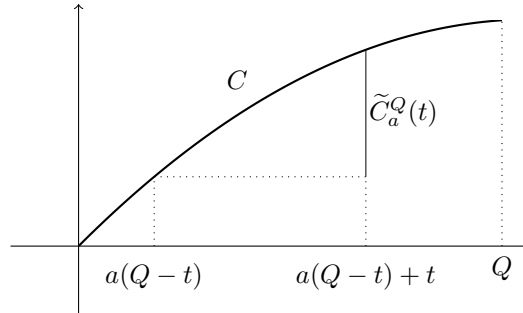**Proposition 2.**

$$\varphi_i^a(N, q, C) = \varphi_i\left(N, q, \widetilde{C}_a^Q\right)$$

where $\widetilde{C}_a^Q$ is defined in (7).

4. A characterization for the intermediate serial cost-sharing rules

In this section we present some properties fulfilled by intermediate serial cost-sharing rules which provide together a characterization for them. More precisely, we give a characterization for each intermediate serial cost-sharing rule.

First, we present the axioms employed by Moulin and Shenker (1994) to characterize the serial cost-sharing rule. They characterize the serial cost-sharing rule by means of five properties: continuity, additivity, ranking, separable costs and free lunch. The first axiom requires continuity for the topology of pointwise convergence and the others are as follows.

Let σ be a cost-sharing rule and (N, q, C) be a cost-sharing problem.

Additivity:

$$\sigma_i\left(N, q, C_1 + C_2\right) = \sigma_i\left(N, q, C_1\right) + \sigma_i\left(N, q, C_2\right) \text{ for all } (N, q, C_1), (N, q, C_2) \in \Gamma_U.$$

---

[2]We have defined cost function C on $[0, Q]$ not to define arbitrarily this cost function beyond Q. We could have chosen to define C and $\widetilde{C}_a^Q$ on $\mathbb{R}_+$ and all the results in this paper would be valid.

By means of ranking the order of cost shares coincides with the order of demands.

Ranking: If $q_i \leq q_j$, then

$$\sigma_i \left(N, q, C \right) \leq \sigma_j \left(N, q, C \right)$$

for all q and all $i, j \in N$.

Separable costs states that if costs are separable then they are so allocated.

Separable costs: If there exists $\lambda \geq 0$ such that $C \left(t \right) = \lambda t$ for all $t \geq 0$ then

$$\sigma_i \left(N, q, C \right) = \lambda q_i$$

for all $i \in N$.

For the last axiom, a kind of consistency axiom, we need some notation. Given a cost function C and $\alpha, \delta \in \mathbb{R}_+$ we define the cost function $C^{\alpha, \delta}$ by

$$C^{\alpha, \delta} \left(t \right) = \begin{cases} C \left(t \right) & \text{if } t \leq \alpha, \\ C \left(t + \delta \right) - C \left(\alpha + \delta \right) + C \left(\alpha \right) & \text{otherwise.} \end{cases}$$

We see in Fig. 9 that $C^{\alpha, \delta}$ is the cost function which results from C when δ units from α to $\alpha + \delta$ have been withdrawn, joint with their associated cost, that is, $C \left(\alpha + \delta \right) - C \left(\alpha \right)$.

FIGURE 9

Free lunch: If $C \left(n q_i \right) = 0$ for some $i \in N$, then $\sigma_i \left(N, q, C \right) = 0$ and

$$\sigma_j \left(N, q, C \right) = \sigma_j \left(N \backslash \{i\}, q_{N \backslash \{i\}}, C^{0, q_i} \right)$$

for all $j \in N \backslash \{i\}$.

According to free lunch, if nq_i units of the good have no cost, then any agent whose demand is q_i pays nothing. And moreover, if that agent leaves the system the cost shares for the remaining agents do not change. Observe that when q_i goes out the new cost function is C^{0,q_i} since agent i receives q_i and pays $C(q_i)$, that is, zero.

We characterize the intermediate serial cost-sharing rule associated with a mapping a by means of the same axioms which characterize the serial cost-sharing rule except free lunch, instead of which a modified axiom is given.

Observe that in free lunch Moulin and Shenker require the cost of the first nq_i units to be zero and agent i to be given q_i units from them. But this might not be the case. Agent i might be given q_i units which are not at the beginning. We propose an axiom in which we also suppose that the cost of nq_i units is zero, but not necessarily the cost of the first nq_i units. These units can be located anywhere in the production process. We suppose that for each Q there is a mapping a_Q defined on $[0, Q]$ which gives the location of units: if $x \in [0, Q]$ then these units are located in $[a_Q(x), a_Q(x) + x] \subseteq [0, Q]$. For example, if $a_Q(x) = 0$, units are at the beginning of $[0, Q]$. If $a_Q(x) = Q - x$, units are at the end of $[0, Q]$. And if $a_Q(x) = \frac{Q-x}{2}$ they are in the middle.

The new axiom states that given a_Q, if nq_i units located according to a_Q are costless, then agent i does not pay anything and if i is given q_i units located also according to a_Q then the other agents' payoffs do not change in the remaining cost-sharing problem.

Moreover, suppose that in a first step agent i is given q_i units (located according to a_Q) and leaves the picture, and that in a second step agent j is given q_j units (located according to a_{Q-q_i}). We assume that the $q_i + q_j$ units which have been given to those agents are the same units that would have been given if there were just one agent with demand $q_i + q_j$ and that agent would have been given his demand, that is, $a_Q(q_i + q_j) = a_{Q-q_i}(q_j)$ (see Fig. 7 with total demands Q and $Q - q_i$, and $x' = q_i + q_j$, $x = q_j$, $x' = x + q_i$). Therefore there exists a mapping a such that $a_Q(x) = a(Q - x)$ and we state the axiom by employing mapping a.

We formalize the property as follows. Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuos mapping such that $a(x) \in [0, x]$.
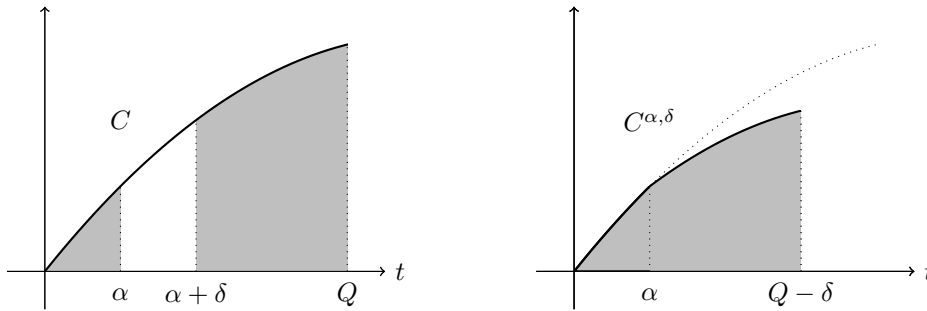
Free a-middle: If

$$C(a(Q - nq_i) + nq_i) - C(a(Q - nq_i)) = 0$$

for some $i \in N$, then $\sigma_i(N, q, C) = 0$ and

$$\sigma_j(N, q, C) = \sigma_j\left(N \setminus \{i\}, q_{N \setminus \{i\}}, C^{a(Q-q_i), q_i}\right)$$

for all $j \in N \setminus \{i\}$.

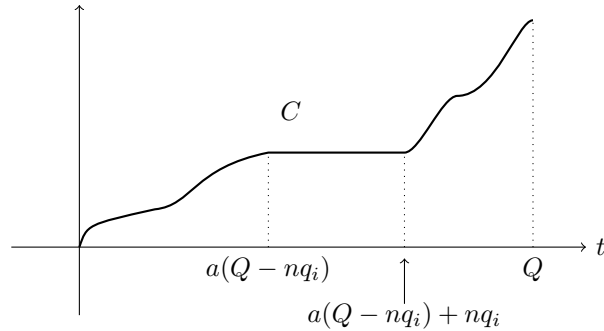We see in Fig. 10 the shape of the cost function.

FIGURE 10

We prove in the following proposition that if a cost-sharing rule satisfies free a-middle then conditions (5) and (6) hold, and therefore in $C^{a(Q-q_i),q_i}$ we withdraw q_i units with no cost. Observe that $C^{a(Q-q_i),q_i}(t) = \begin{cases} C(t) & \text{if } t \le a(Q - q_i), \\ C(t + q_i) & \text{otherwise.} \end{cases}$

In order free a-middle to make sense $Q - nq_i$ has to be non-negative. That occurs for agent with lowest demand but not in general. If we consider $C\left(a\left(Q - q^i\right) + q^i\right) - C\left(a\left(Q - q^i\right)\right) = 0$ in the free a-middle axiom, we would get the same characterization and the axiom could make sense for all the agents. We have chosen the first presentation to be closer to the one given by Moulin and Shenker (1994). Finally, when $a(x) = 0$ the new axiom coincides with free lunch.

Lemma 3. *Let σ be a cost-sharing rule that satisfies free a-middle. Then mapping $a : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies conditions (5) and (6).*

Proof. Let σ be a cost-sharing rule satisfying free a-middle.

i) If a does not satisfy condition (5), then there exist $x, y \in \mathbb{R}_+$ such that $x < y$ and $a(x) > a(y)$.

Consider the cost-sharing problem (N, q, C), where $q = (q_1, q_2, ..., q_2)$ is such that $q_1 = \frac{y-x}{n-1}$ and $q_2 = \frac{y}{n-1}$, and the cost function C is defined by

$$C(t) = \begin{cases} t & \text{if } t \le a(x), \\ a(x) & \text{if } a(x) < t \le a(x) + Q - x, \\ t - (Q - x) & \text{if } t \ge a(x) + Q - x. \end{cases}$$

Notice that $x = Q - nq_1$ and $y = Q - q_1$. Since $C\left(a\left(Q - nq_1\right) + nq_1\right) - C\left(a\left(Q - nq_1\right)\right) = 0$, by free a-middle we know that $\sigma_1(N, q, C) = 0$ and

$$\sigma_i(N, q, C) = \sigma_i\left(N \backslash \{1\}, q_{N \backslash \{1\}}, C^{a(Q-q_1),q_1}\right)$$

for all $i \in N \backslash \{1\}$. These equalities and efficiency imply

$$C(Q) = \sum_{i \in N \backslash \{1\}} \sigma_i(N, q, C) = \sum_{i \in N \backslash \{1\}} \sigma_i(N, q, C) = C^{a(Q-q_1), q_1}(Q - q_1). \quad (8)$$

On the other hand since $a(Q - q_1) < a(Q - nq_1)$, by definition of $C^{a(Q-q_1), q_1}$ we have that $C^{a(Q-q_1), q_1}(Q - q_1) = C(Q) - C(a(Q - q_1) + q_1) + C(a(Q - q_1)) < C(Q)$, which contradicts (8). Hence a must be non-decreasing

ii) If condition (6) does not hold there exist $x, y \in \mathbb{R}_+$ such that $x > y$ and $a(y) - a(x) > y - x$. We consider the demand profile q and the cost function C defined in point i). We argue as before and reach (8). Since now $a(Q - q_1) + q_1 > a(Q - nq_1) + nq_1$, then

$$C^{a(Q-q_1), q_1}(Q - q_1) < C(Q),$$

which contradicts (8). Hence a satisfies condition (6). $\blacksquare$

Let us prove the characterization theorem in three steps.

Lemma 4. *The intermediate serial cost-sharing rule associated with function a satisfies free a-middle.*

Proof. If

$$C(a(Q - nq_i) + nq_i) - C(a(Q - nq_i)) = 0$$

for some $i \in N$, then $\widetilde{C}_a^Q(nq_i) = 0$. Since the serial cost-sharing rule φ satisfies free lunch we know that $\varphi_i\left(N, q, \widetilde{C}_a^Q\right) = 0$ and

$$\varphi_j\left(N, q, \widetilde{C}_a^Q\right) = \varphi_j\left(N \backslash \{i\}, q_{N \backslash \{i\}}, \left(\widetilde{C}_a^Q\right)^{0, q_i}\right)$$

for all $j \in N \backslash \{i\}$.

So by Proposition 2 we get $\varphi_i^a(N, q, C) = 0$ and

$$\varphi_j^a(N, q, C) = \varphi_j\left(N \backslash \{i\}, q_{N \backslash \{i\}}, \left(\widetilde{C}_a^Q\right)^{0, q_i}\right)$$

for all $j \in N \backslash \{i\}$. Denote $C_a^{a(Q-q_i), q_i} = C^*$. Taking into account that $\left(\widetilde{C}_a^Q\right)^{0, q_i} = \left(\widetilde{C^*}\right)_a^{Q-q_i}$ it follows that

$$\varphi_j^a(N, q, C) = \varphi_j\left(N \backslash \{i\}, q_{N \backslash \{i\}}, \left(\widetilde{C^*}\right)_a^{Q-q_i}\right).$$

Applying again Proposition 2 it holds

$$\varphi_j^a\left(N,q,C\right) = \varphi_j^a\left(N\backslash\{i\}, q_{N\backslash\{i\}}, C^*\right),$$

as was to be proved. ■

In the following lemma and in the main theorem we employ the cost-sharing problems (N, q, Λ_α), (N, q, Λ'_α), such that $\alpha \in [0, Q]$, defined by

$$\Lambda_\alpha\left(t\right) = \left(t - \alpha\right)_+ \text{ and } \Lambda'_a\left(t\right) = \min\left\{t, \alpha\right\}.$$

The first ones are employed by Moulin and Shenker (1994) to characterize the serial cost-sharing rule. This type of problems was sufficient in order to characterize that rule since free lunch could be applied for all (N, q, Λ_α). In our case we also employ (N, q, Λ'_α) to approach all cost-sharing problems.

Lemma 5. *If a cost-sharing rule σ satisfies additivity, ranking, separable costs and free a-middle, then*

$$\sigma_i\left(N, q, \Lambda_{a(Q)}\right) = q_i - \sum_{j=1}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1},$$

$$\sigma_i\left(N, q, \Lambda'_{a(Q)}\right) = \sum_{j=1}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1}.$$

Proof. Let $N = \{1, ..., n\}$ and $q \in \mathbb{R}_+^N$, and assume without loss of generality that $q_1 \leq q_2 \leq \cdots \leq q_n$. We prove this lemma by induction on $i = 1, ..., n$. Consider

$$\Delta_{a,Q,q^1} = \Lambda_{a(Q)} - \Lambda_{a(Q-q^1)+q^1}. \tag{9}$$

By efficiency we have

$$\sum_{i=1}^{n} \sigma_i\left(N, q, \Delta_{a,Q,q^1}\right) = a\left(Q - q^1\right) + q^1 - a\left(Q\right),$$

and by ranking we get

$$\sigma_1\left(N, q, \Delta_{a,Q,q^1}\right) \leq q_1 - \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n}. \tag{10}$$

Since free a-middle implies

$$\sigma_1\left(N, q, \Delta_{a,Q,q^1}\right) = \sigma_1\left(N, q, \Lambda_{a(Q)}\right),$$

then together with (10) we have

$$\sigma_1\left(N, q, \Lambda_{a(Q)}\right) \leq q_1 - \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n}. \tag{11}$$

Now we define

$$\Delta'_{a,Q,q^1} = \Lambda'_{a(Q)} - \Lambda'_{a(Q-q^1)}. \tag{12}$$

Reasoning as above it follows that

$$\sigma_1\left(N, q, \Lambda'_{a(Q)}\right) \leq \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n}. \tag{13}$$

And additivity and separable costs imply

$$\sigma_1\left(N, q, \Lambda_{a(Q)}\right) + \sigma_1\left(N, q, \Lambda'_{a(Q)}\right) = \sigma_1\left(N, q, \Lambda_0\right) = q_1.$$

Using (11) and (13) in that equality we can deduce

$$\sigma_1\left(N, q, \Lambda_{a(Q)}\right) = q_1 - \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n},$$

$$\sigma_1\left(N, q, \Lambda'_{a(Q)}\right) = \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n}.$$

Now we suppose that the following equalities are true for all $i < k$,

$$\sigma_i\left(N, q, \Lambda_{a(Q)}\right) = q_i - \sum_{j=1}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1},$$

$$\sigma_i\left(N, q, \Lambda'_{a(Q)}\right) = \sum_{j=1}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1},$$

and we will prove them for agent k.

For that take

$$\Delta_{a,Q,q^k} = \Lambda_{a(Q)} - \Lambda_{a\left(Q-q^k\right)+q^k}.$$

Since σ satisfies efficiency we get

$$\sum_{i=1}^{n} \sigma_i\left(N, q, \Delta_{a,Q,q^k}\right) = a\left(Q - q^k\right) + q^k - a\left(Q\right). \tag{14}$$

Now we prove the following equality, which will be useful in the sequel,

$$\sigma_i\left(N, q, \Lambda_{a\left(Q-q^k\right)+q^k}\right) = 0 \tag{15}$$

for all $i \in \{1, ..., k\}$.

By free a-middle we know that $\sigma_1 \left(N, q, \Lambda_{a(Q-q^k)+q^k} \right) = 0$ and

$$\sigma_i \left(N, q, \Lambda_{a(Q-q^k)+q^k} \right) = \sigma_i \left(N \backslash \{1\}, q_{N \backslash \{1\}}, \Lambda_{a(Q-q^k)+q^k-q_1} \right)$$

if $i \in \{2, ..., k\}$. Applying again free a-middle,

$$\sigma_2 \left(N \backslash \{1\}, q_{N \backslash \{1\}}, \Lambda_{a(Q-q^k)+q^k-q_1} \right) = 0,$$

and thus $\sigma_2 \left(N, q, \Lambda_{a(Q-q^k)+q^k} \right) = 0$, and moreover

$$\sigma_i \left(N \backslash \{1\}, q_{N \backslash \{1\}}, \Lambda_{a(Q-q^k)+q^k-q_1} \right) = \sigma_i \left(N \backslash \{1, 2\}, q_{N \backslash \{1,2\}}, \Lambda_{a(Q-q^k)+q^k-q_1-q_2} \right)$$

if $i \in \{3, ..., k\}$. Taking into account free a-middle repeatedly, if $i \leq k$ we get

$$\sigma_i \left(N, q, \Lambda_{a(Q-q^k)+q^k} \right) =$$

$$= \cdots = \sigma_i \left(N \backslash \{1, 2, ..., i-1\}, q_{N \backslash \{1,2,...,i-1\}}, \Lambda_{a(Q-q^k)+q^k-q_1-q_2-...-q_{i-1}} \right) = 0,$$

and therefore equality (15) is proved.

Turning to the main proof, by additivity and equalities (15) for $i \in \{1, ..., k-1\}$ we have

$$\sum_{i=1}^{n} \sigma_i \left(N, q, \Delta_{a,Q,q^k} \right) = \sum_{i=1}^{k-1} \sigma_i \left(N, q, \Lambda_{a(Q)} \right) + \sum_{i=k}^{n} \sigma_i \left(N, q, \Delta_{a,Q,q^k} \right).$$

And taking into account expression (14),

$$\sum_{i=k}^{n} \sigma_i \left(N, q, \Delta_{a,Q,q^k} \right) = a \left(Q - q^k \right) + q^k - a \left(Q \right) - \sum_{i=1}^{k-1} \sigma_i \left(N, q, \Lambda_{a(Q)} \right).$$

Applying induction hypothesis we get

$$\sum_{i=k}^{n} \sigma_i \left(N, q, \Delta_{a,Q,q^k} \right) \leq (n - k + 1) \left(q_k - \sum_{j=1}^{k} \frac{a \left(Q - q^{j-1} \right) - a \left(Q - q^j \right)}{n - j + 1} \right)$$

and by ranking it follows that

$$\sigma_k \left(N, q, \Delta_{a,Q,q^k} \right) \leq q_k - \sum_{j=1}^{k} \frac{a \left(Q - q^{j-1} \right) - a \left(Q - q^j \right)}{n - j + 1}.$$

Applying additivity and equality (15) for $i = k$ we get

$$\sigma_k \left(N, q, \Lambda_{a(Q)} \right) \leq q_k - \sum_{j=1}^{k} \frac{a \left(Q - q^{j-1} \right) - a \left(Q - q^j \right)}{n - j + 1}. \tag{16}$$

If we take $\Delta'_{a,Q,q^k} = \Lambda'_{a(Q)} - \Lambda'_{a(Q-q^k)}$, reasoning in a similar way we prove that

$$\sigma_k \left(N, q, \Lambda'_{a(Q)} \right) \leq \sum_{j=1}^{k} \frac{a \left(Q - q^{j-1} \right) - a \left(Q - q^j \right)}{n - j + 1}.$$

And additivity, separable costs and (16) imply

$$\sigma_k \left(N, q, \Lambda_{a(Q)} \right) = q_k - \sum_{j=1}^{k} \frac{a \left(Q - q^{j-1} \right) - a \left(Q - q^j \right)}{n - j + 1},$$

$$\sigma_k \left(N, q, \Lambda'_{a(Q)} \right) = \sum_{j=1}^{k} \frac{a \left(Q - q^{k-1} \right) - a \left(Q - q^k \right)}{n - k + 1}.$$

as was to be proved. ∎

Theorem 6. *The intermediate serial cost-sharing rule associated with function a is the unique cost-sharing rule that satisfies continuity, additivity, ranking, separable cost and free a-middle .*

Proof.   By Lemma 4 the intermediate serial cost-sharing rule associated with function a satisfies free a-middle. And it is straightforward to prove the other axioms.

For uniqueness we consider an allocation rule σ satisfying the five properties above. We show that σ coincides with the intermediate serial cost-sharing rule associated with function a on (N, q, Λ_t) when $t \geq a(Q)$ and on (N, q, Λ'_t) when $t < a(Q)$.

Case 1. We are going to determine $\sigma(N, q, \Lambda_t)$ when $t \geq a(Q)$.

First we determine $\sigma \left(N, q, \Lambda_{a(Q-nq_1)+nq_1} \right)$. By free a-middle we get

$$\sigma_i \left(N, q, \Lambda_t \right) = 0 \tag{17}$$

for all $t \geq a(Q - nq_i) + nq_i$ and hence

$$\sigma_1 \left(N, q, \Lambda_{a(Q-nq_1)+nq_1} \right) = 0. \tag{18}$$

Considering the cost function defined in (9), additivity and (18), we have

$$\sigma_1 \left(N, q, \Delta_{a,Q,q^1} \right) = \sigma_1 \left(N, q, \Lambda_{a(Q)} \right)$$

and by the previous lemma

$$\sigma_1 \left(N, q, \Delta_{a,Q,q^1}\right) = q_1 - \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n}.$$

And therefore by ranking

$$\sigma_i \left(N, q, \Delta_{a,Q,q^1}\right) \geq q_1 - \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n} \tag{19}$$

for all $i \in N$. Since from efficiency we get

$$\sum_{i=1}^{n} \sigma_i \left(N, q, \Delta_{a,Q,q^1}\right) = a\left(Q - q^1\right) + q^1 - a\left(Q\right),$$

applying (19) it holds

$$\sigma_i \left(N, q, \Delta_{a,Q,q^1}\right) = q_1 - \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n} \tag{20}$$

for all $i \in N$. Hence, by Lemma 5, additivity and (20) we find

$$\sigma_i \left(N, q, \Lambda_{a(Q-nq_1)+nq_1}\right) = q_i - q_1 - \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1} \tag{21}$$

for all $i \in N$.

To determine $\sigma\left(N, q, \Lambda_t\right)$ when $a\left(Q\right) \leq t < a\left(Q - q^1\right) + q^1$, we increase the set of agents with an additional individual denoted 0. So let $N' = \{0, 1, ..., n\} = N \cup \{0\}$ and $q_0 \leq q_1$. Taking into account (21) we get

$$\sigma_0 \left(N', \left(q_0, q\right), \Lambda_{a((Q+q_0)-(n+1)q_0)+(n+1)q_0}\right) = 0$$

and

$$\sigma_i \left(N', \left(q_0, q\right), \Lambda_{a((Q+q_0)-(n+1)q_0)+(n+1)q_0}\right)$$

$$= q_i - q_0 - \frac{a\left(Q - nq_0\right) - a\left(Q - q^1\right)}{n} - \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1}$$

for all $i \geq 1$. Applying free a-middle by removing agent 0 from N' it holds

$$\sigma_i \left(N, q, \Lambda_{a(Q-nq_0)+nq_0}\right) = q_i - q_0 - \frac{a\left(Q - nq_0\right) - a\left(Q - q^1\right)}{n} - \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1}$$

for all $i \geq 1$. Then

$$\sigma_i(N, q, \Lambda_t) = q_i - \frac{t}{n} + \frac{a(Q - q^1)}{n} - \sum_{j=2}^{i} \frac{a(Q - q^{j-1}) - a(Q - q^j)}{n - j + 1} \qquad (22)$$

for all $i \geq 1$ and $t \leq q^1 + a(Q - q^1)$ (note that continuity of a implies that t takes any value from $a(Q)$ to $q^1 + a(Q - q^1)$). Moreover by (17) it holds $\sigma_1(N, q, \Lambda_t) = 0$ for all $t \geq q^1 + a(Q - q^1)$.

Now let us determine $\sigma(N, q, \Lambda_t)$ when $q^1 + a(Q - q^1) \leq t < q^2 + a(Q - q^2)$. If we drop agent 1, expression (22) and free a-middle yield

$$\sigma_i(N, q, \Lambda_t) = q_i - \frac{t - q_1}{n - 1} + \frac{a(Q - q^2)}{n - 1} - \sum_{j=3}^{i} \frac{a(Q - q^{j-1}) - a(Q - q^j)}{n - j + 1}$$

for all $i \geq 2$ and $q^1 + a(Q - q^1) \leq t < q^2 + a(Q - q^2)$. Moreover $\sigma_2(N, q, \Lambda_t) = 0$ for all $t \geq q^2 + a(Q - q^2)$ by (17).

Using the induction process we determine $\sigma_i(N, q, \Lambda_t)$ for all $t \geq a(Q)$ and for all $i \in N$.

Case 2. To determine $\sigma(N, q, \Lambda'_t)$ when $t < a(Q)$ we proceed in a similar way as in Case 1.

By free a-middle we get

$$\sigma_i(N, q, \Lambda'_t) = 0 \qquad (23)$$

for all $t \leq a(Q - nq_i)$ and therefore

$$\sigma_1\left(N, q, \Lambda'_{a(Q - nq_1)}\right) = 0. \qquad (24)$$

Considering the cost function defined in (12), additivity, expression (24) and the previous lemma we have

$$\sigma_1\left(N, q, \Delta'_{a, Q, q^1}\right) = \sigma_1\left(N, q, \Lambda'_{a(Q)}\right) - \sigma_1\left(N, q, \Lambda'_{a(Q - nq_1)}\right) = \frac{a(Q) - a(Q - q^1)}{n},$$

and hence ranking implies

$$\sigma_i\left(N, q, \Delta'_{a, Q, q^1}\right) \geq \frac{a(Q) - a(Q - q^1)}{n} \qquad (25)$$

for all $i \in N$. From efficiency it follows that

$$\sum_{i=1}^{n} \sigma_i\left(N, q, \Delta'_{a, Q, q^1}\right) = a(Q) - a(Q - q^1),$$

which together with (25) implies

$$\sigma_i\left(N, q, \Delta'_{a,Q,q^1}\right) = \frac{a\left(Q\right) - a\left(Q - q^1\right)}{n} \tag{26}$$

for all $i \in N$. Taking into account Lemma 5, additivity and (26),

$$\sigma_i\left(N, q, \Lambda'_{a(Q-nq_1)}\right) = \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1} \tag{27}$$

for all $i \in N$.

To determine $\sigma\left(N, q, \Lambda'_t\right)$ when $a\left(Q - q^1\right) < t < a\left(Q\right)$, we take as in Case 1 the set of agents $N' = \{0, 1, ..., n\} = N \cup \{0\}$ and $q_0 \leq q_1$ Applying (27) we get

$$\sigma_0\left(N', (q_0, q), \Lambda'_{a((Q+q_0)-(n+1)q_0)}\right) = 0$$

and

$$\sigma_i\left(N', (q_0, q), \Lambda'_{a((Q+q_0)-(n+1)q_0)}\right) = \frac{a\left(Q - nq_0\right) - a\left(Q - q^1\right)}{n} + \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1}$$

for all $i \geq 1$. And by free a-middle

$$\sigma_i\left(N, q, \Lambda'_{a(Q-nq_0)}\right) = \frac{a\left(Q - nq_0\right) - a\left(Q - q^1\right)}{n} + \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1}$$

for all $i \geq 1$. Then

$$\sigma_i\left(N, q, \Lambda'_t\right) = \frac{t - a\left(Q - q^1\right)}{n} + \sum_{j=2}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1} \tag{28}$$

for all $i \geq 1$ and $t > a\left(Q - q^1\right)$. Moreover expression (23) implies $\sigma_1\left(N, q, \Lambda'_t\right) = 0$ for all $t \leq a\left(Q - q^1\right)$,

Next let us determine $\sigma\left(N, q, \Lambda'_t\right)$ when $a\left(Q - q^2\right) < t \leq a\left(Q - q^1\right)$. If we drop agent 1, expression (28) and free a-middle yield

$$\sigma_i\left(N, q, \Lambda'_t\right) = \frac{t - a\left(Q - q^2\right)}{n - 1} + \sum_{j=3}^{i} \frac{a\left(Q - q^{j-1}\right) - a\left(Q - q^j\right)}{n - j + 1}$$

for all $i \geq 2$, and for all t such that $a\left(Q - q^2\right) < t \leq a\left(Q - q^1\right)$. And by (23) then $\sigma_2\left(N, q, \Lambda'_t\right) = 0$ for all $t \leq a\left(Q - q^2\right)$. By induction process we determine $\sigma_i\left(N, q, \Lambda'_t\right)$ for all $0 \leq t < a\left(Q\right)$ and for all $i \in N$.

Finally, using linear combinations of Λ_t and Λ'_t, additivity and continuity the proof is extended to any nondecreasing function. ■

Remark 1. *Note that when $a(x) = x$, free a-middle states that if the last nq_i units in $[0, Q]$ have no cost then agent i pays nothing and*

$$\sigma_j(N, q, C) = \sigma_j\left(N \setminus \{i\}, q_{N \setminus \{i\}}, C\right)$$

for all $j \in N \setminus \{i\}$. Thus, the dual serial cost-sharing rule is characterized by employing that axiom.

Remark 2. *Note that in the previous proof we obtain expression (22) from (21) since we have a mapping $a : \mathbb{R}_+ \to \mathbb{R}_+$ in free a-middle and not mappings a_Q.*

5. A subfamily of intermediate serial cost-sharing rules

In this section we consider the intermediate serial cost-sharing rules which satisfy scale invariance. This is a well known axiom according to which the payoffs do not depend on the scale in which goods are measured. If σ is a cost-sharing rule it is formalized as follows.

Scale invariance: For all (N, q, C), $i \in N$ and $\beta > 0$

$$\sigma_i(N, \beta q, C_\beta) = \sigma_i(N, q, C),$$

where C_β denotes the cost function defined by $C_\beta(t) = C(t/\beta)$.

In the next corollary we show that when scale invariance is required function $a : \mathbb{R}_+ \to \mathbb{R}_+$ in (3) must be linear, i.e., $a(x) = \alpha x$ with $\alpha \in [0, 1]$. Given a cost-sharing problem (N, q, C), those intermediate serial cost-sharing rules share αQ units by equalizing quantities to be allocated. Each agent i is given αq_i units in this way. From αQ units upwards cost is shared by giving equally quantities of the good to all the agents, who pay equally the incurred cost.

Obviously in this subfamily we have the serial and the dual cost-sharing rules when $\alpha = 0$ and $\alpha = 1$. If $\alpha = 1/2$ then the reverse self-dual cost-sharing rule defined by Albizuri et al. (2012) is obtained. That rule is a self-dual cost-sharing rule, it gives the same payoffs in a cost-sharing problem and in its dual one. In fact, it is the unique self-dual intermediate serial cost-sharing rule.

Corollary 7. *A cost-sharing rule satisfies continuity, additivity, scale invariance, ranking, separable cost and free a-middle if and only if it is an intermediate serial cost-sharing rule and there exists $\alpha \in [0, 1]$ such that $a(x) = \alpha x$.*

Proof.   By Theorem 6 a cost-sharing rule σ satisfies additivity, ranking, separable cost and free a-middle if and only if $\sigma(N, q, C) = \varphi^a(N, q, C)$. Then

$$\sigma_1(N, q, C) = \frac{C\left(a\left(Q - q^1\right) + q^1\right) - C\left(a\left(Q - q^1\right)\right)}{n}. \tag{29}$$

We have to prove that if σ satisfies scale invariance then there exits $\alpha \in \mathbb{R}_+$ such that

$$a(t) = \alpha t \qquad (30)$$

for all $t \in \mathbb{R}_+$.

Assume that there exists $t_0, t_1 \in \mathbb{R}_+$, $t_0 \neq t_1$, such that $a(t_0) = \alpha_1 t_0$ and $a(t_1) = \alpha_2 t_1$ with $\alpha_1 \neq \alpha_2$. Consider a cost function C' which satisfies $C'((1 + \alpha_1) t_0) - C'(\alpha_1 t_0) \neq C'((1 + \alpha_2) t_0) - C'(\alpha_2 t_0)$. Suppose that $t_0 \neq 0$ and let $\beta = t_1/t_0$, (N, q', C') and $(N, \beta q', C'_\beta)$ such that $Q' = 2t_0$ and $n q'_1 = t_0$. Equality (29) implies

$$
\begin{aligned}
\sigma_1(N, q', C') &= \frac{C'(a(Q' - n q'_1) + n q'_1) - C'(a(Q' - n q'_1))}{n} \\
&= \frac{C'(a(2t_0 - t_0) + t_0) - C'(a(2t_0 - t_0))}{n} \\
&= \frac{C'(\alpha_1 t_0 + t_0) - C'(\alpha_1 t_0)}{n}.
\end{aligned}
$$

Similarly,

$$\sigma_1(N, \beta q', C'_\beta) = \frac{C'_\beta(\alpha_2 \beta t_0 + \beta t_0) - C'_\beta(\alpha_2 \beta t_0)}{n}.$$

And taking into account the definition of C'_β,

$$\sigma_1(N, \beta q', C'_\beta) = \frac{C'(\alpha_2 t_0 + t_0) - C'(\alpha_2 t_0)}{n}.$$

By scale invariance $\sigma_1(N, q', C') = \sigma_1(N, \beta q', C'_\beta)$, and therefore

$$C'(\alpha_1 t_0 + t_0) - C'(\alpha_1 t_0) = C'(\alpha_2 t_0 + t_0) - C'(\alpha_2 t_0)$$

which is false. Hence $\alpha_1 = \alpha_2$ and (30) holds. Notice also that $a(x) \leq x$ implies $\alpha \in [0, 1]$. ∎

6. References

ALBIZURI, M. J. and ZARZUELO, J. M. (2007). "The Dual Serial Cost-Sharing Rule", Mathematical Social Sciences, 53, 150-163.

DE FRUTOS, M. A. (1998). "Decreasing serial cost sharing under economies of scale", Journal of Economic Theory, 79, 245-275.

KOSTER, M. (2002). "Concave and convex serial cost sharing", in Chapters in Game Theory, in honor of Stef Tijs, eds. P. Borm and H. Peters, Kluwer Academic Publishers, Chapter 7: 143-155.

KOSTER, M. (2006). "Heterogeneous cost sharing, the directional serial rule", Mathematical Methods of Operations Research, 64, 429-444.

KOSTER, M., TIJS, S. and BORM, P. (1998). "Serial cost sharing methods for multi-commodity situations", Mathematical Social Sciences, 36, 229-242.

MOULIN, H. and SHENKER, S. (1992). "Serial Cost Sharing", Econometrica, 60, 1009–1037.

MOULIN, H. and SHENKER, S. (1994). "Average cost Pricing versus Serial Cost Sharing: an Axiomatic Comparison", Journal of Economic Theory, 64, 178–201.