Approximate Best Proximity Points of Cyclic Self-maps in Metric Spaces and Cyclic Asymptotic Regularity

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Abstract. This manuscript investigates some relations and properties between 2-cyclic self-mappings. In particular, a formal development is given to link some concepts and their basic related properties like those of approximate best proximity points, approximate best proximity point property and cyclic asymptotic regularity.

1. Introduction

Fixed point theory is receiving important attention in the last decades because of its applicability to many physical and engineering problems. See [1-13] and references therein. In [4-5] and other cited papers by the same authors and also some references therein, the problem of existence of approximate fixed points of functions in metric spaces is described and formalized as well as its links with asymptotic regularity of such functions including cases where contractive conditions are fulfilled . It is not required either that the metric space be complete by the same reasons. This paper extends such a formalism to 2-cyclic self-mappings. In this way, a simple formal development is given which brings together the concepts of approximate best proximity points of 2-cyclic self-mappings, approximate best proximity (respectively, partial best proximity) point property and cyclic asymptotic regularity of 2-cyclic self-mappings. Let us remember that 2-cyclic self-mappings are defined on the union of two nonempty subsets of the set X where (X, d) is a metric space. Some related properties are proved while certain particular results are derived for contractive cyclic self-mappings f on $A \cup B$ with A and B being nonempty, in general disjoint, subsets of X. There are other interesting directly induced results available for composite self-mapping f^2 in $A \cup B$. It can be pointed out that the obtained results are applicable very easily to the stability and approximate stability of dynamic systems since equilibrium points of such systems are also fixed points of the mapping defining the trajectory solution from given initial conditions.

2. Problem statement

Let (X, d) be a metric space and let $f: A \cup B \to A \cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Since there are only two subsets involved, the self-mapping will be referred to simply as a cyclic self-mapping. The following definition will be then used:

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Definitions 2.1. Let (X, d) be a metric space and let $f: A \cup B \to A \cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Then:

(1) $x \in A \cup B$ is a ε-best proximity point of f (in A or in B) for a given $\varepsilon \in \mathbf{R}_{0+}$ if $d(x, fx) \le D + \varepsilon$, where $D = d(A, B) = \inf_{x \in A,\, y \in B} d(x, y)$.

(2) $x \in A$ is a ε-best proximity point of f in A for a given $\varepsilon \in \mathbf{R}_{0+}$ if $d(x, fx) \le D + \varepsilon$.

It turns out that $x \in A \cup B$ is an ε- best proximity point of f if and only if

$$x \in BP_{\varepsilon}(f) = \{x \in A \cup B : d(x, fx) \le D + \varepsilon\}$$

Also $x \in A$ is an ε- best proximity point of f in A if and only if

$$x \in BP_{A\varepsilon}(f) = \{x \in A : d(x, fx) \le D + \varepsilon\}$$

The following results are direct to prove:

Proposition 2.2. Let (X, d) and $f: A \cup B \to A \cup B$ be a metric space and a 2-cyclic self-mapping (referred to in the sequel as cyclic self-mappings) where A and B are nonempty subsets of X. Then, if $x \in A \cup B$ is an ε-best proximity point of f then it is an ε_1-best proximity point of f for any real $\varepsilon_1 \ge \varepsilon$.

Proposition 2.3. Let (X, d) be a metric space and let $f: A \cup B \to A \cup B$ be a 2-cyclic self-mapping on the union of two bounded nonempty subsets A and B of X. Then $x \in A \cup B$ is an ε-best proximity point of f then fx is an ε'- best proximity point for some $\varepsilon' \in \mathbf{R}_{+}$, then, $fx \in BP_{\varepsilon'}(f)$.

Definitions 2.4. Let (X, d) be a metric space and let $f: A \cup B \to A \cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Then,

(1) $f: A \cup B \to A \cup B$ has the approximate best proximity point property if $BP_{\varepsilon}(f) \ne \varnothing$ for all $\varepsilon \in \mathbf{R}_{0+}$.
(2) $f: A \cup B \to A \cup B$ has the approximate best proximity point property in A if $BP_{A\varepsilon}(f) \ne \varnothing$ for all $\varepsilon \in \mathbf{R}_{0+}$.
(3) $f: A \cup B \to A \cup B$ has the ε_0-partial approximate best proximity point property if $BP_{\varepsilon}(f) \ne \varnothing$ for all real $\varepsilon \ge \varepsilon_0$ and a given $\varepsilon_0 \in \mathbf{R}_{0+}$.
(4) $f: A \cup B \to A \cup B$ has the ε_0-partial approximate best proximity point property in A if $BP_{A\varepsilon}(f) \ne \varnothing$ for all real $\varepsilon \ge \varepsilon_0$ and a given $\varepsilon_0 \in \mathbf{R}_{0+}$.
(5) $f^2: A \cup B \to A \cup B$ has the ε_0-partial approximate fixed point property if $FP_{\varepsilon}(f) = \{x \in A \cup B : d(x, f^2 x) \le \varepsilon\} \ne \varnothing$; $\forall \varepsilon (\in \mathbf{R}_{+})$ for some given $\varepsilon_0 \in \mathbf{R}_{0+}$.
(6) $f^2: A \cup B \to A \cup B$ has the ε_0-partial approximate fixed point property in A if $FP_{A\varepsilon}(f) = \{x \in A : d(x, f^2 x) \le \varepsilon\} \ne \varnothing$; $\forall \varepsilon (\in \mathbf{R}_{+})$ for some given $\varepsilon_0 \in \mathbf{R}_{0+}$.
(7) $f^2: A \cup B \to A \cup B$ has the approximate fixed point property if it has 0-partial approximate fixed point property.

Definitions 2.5. Let (X, d) be a metric space and let A and B be nonempty subsets of X with $d(A, B) = D$. Then,

(1) $f:A\cup B\to A\cup B$ is cyclic asymptotically regular if it is cyclic and

$$d\left(f^{n}x\,,\,f^{n+1}x\right)\to D \text{ as } n\to\infty\,;\ \forall x\in A\cup B$$

(2) $f:A\cup B\to A\cup B$ is cyclic asymptotically ε_0-regular,respectively, ε_0-regular in A, if it is cyclic and

$$d\left(f^{n}x\,,\,f^{n+1}x\right)\to D+\varepsilon_0 \text{ as } n\to\infty\,;\ \forall x\in A\cup B \text{ respectively } \forall x\in A$$

Proposition 2.6. Let $(X\,,d)$ be a metric space and let A and B be nonempty subsets of X with $d(A,B)=D$. Then any strictly contractive cyclic self-mapping $f:A\cup B\to A\cup B$ is cyclic asymptotically regular, and equivalently, it has the approximate best proximity point property.

Proof: It is direct since, if $f:A\cup B\to A\cup B$ then

$$d\left(f^{2}x\,,\,fx\right)\le Kd(fx,x)+(1-K)D \tag{2.1}$$

for some $K\in[0\,,1)$; $\forall x\in A\cup B$. Thus, it is cyclic asymptotically regular since

$$D\le d\left(f^{n+1}x\,,\,f^{n}x\right)\le K^{n}d(fx,x)+\left(1-K^{n}\right)D \tag{2.2}$$

and $d\left(f^{n+1}x,f^{n}x\right)\to D$ as $n\to\infty$; $\forall x\in A\cup B$. Also, since $f:A\cup B\to A\cup B$ is cyclic asymptotically regular then there is $n_0=n_0(\varepsilon)>0$ for any given $\varepsilon\in\boldsymbol{R}_{+}$ such that $D\le d\left(f^{n+1}x,f^{n}x\right)\le D+\varepsilon$ so that $BP_{\varepsilon}(f)\neq\varnothing$; $\forall\varepsilon\in\boldsymbol{R}_{+}$, $\forall x\in A\cup B$. As a result, $f:A\cup B\to A\cup B$ has the approximate best proximity point property. Equivalently, if $BP_{\varepsilon}(f)\neq\varnothing\ \Leftrightarrow\left(D\le d\left(f^{n+1}x,f^{n}x\right)\le D+\varepsilon;\forall\varepsilon\in\boldsymbol{R}_{+},\forall x\in A\cup B\right)$ then $\lim\limits_{n\to\infty}d\left(f^{n+1}x,f^{n}x\right)=D$; $\forall x\in A\cup B$ so that $f:A\cup B\to A\cup B$ is cyclic asymptotically regular.

Lemma 2.7. Let $(X\,,d)$ be a metric space and let A and B be nonempty subsets of X with $d(A,B)=D$. If $f:A\cup B\to A\cup B$ is cyclic asymptotically regular then it has the approximate best proximity point property.

Lemma 2.8. Let $(X\,,d)$ be a metric space and let A and B be nonempty bounded subsets of X with $d(A,B)=D$. If $f^{2}:A\cup B\to A\cup B$ is asymptotically regular then $f:A\cup B\to A\cup B$ has the ε_0-partial best proximity point property for some threshold $\varepsilon_0\in\boldsymbol{R}_{0+}$ with $\varepsilon_0\le min\left(diam\,A\,,diam\,B\right)$.

Proof: One has $\left|d\left(f^{2n}x\,,\,f^{2n+1}x\right)-d\left(f^{2n+2}x\,,\,f^{2n+1}x\right)\right|\to 0$ as $n\to\infty$ so that $d\left(f^{2n}x\,,\,f^{2n+1}x\right)\to D_0(\ge D)$ as $n\to\infty$ (2.3)
; $\forall x\in A\cup B$.Thus, $D_0\le D+min\left(diam\,A\,,diam\,B\right)<\infty$ since A and B are bounded and each $f^{2n}x$, $f^{2n+1}x$ is one of them in A and the other one in B. Then, fix $x_0=f^{2n}x$ there is $n_0\in\boldsymbol{Z}_{0+}$ such that $d(x_0\,,fx_0)\le D+\varepsilon$ so that $BP_{\varepsilon}(f)\neq\varnothing$ for any real $min\left(diam\,A\,,diam\,B\right)\ge\varepsilon\ge\varepsilon_0$ and some $\varepsilon_0\in\boldsymbol{R}_{0+}$.

Theorem 2.9. Let $(X\,,d)$ be a complete metric space and let A and B be nonempty closed subsets of X with $d(A,B)=D$. Assume that A is approximatively compact with respect to B. Then $f:A\cup B\to A\cup B$ is cyclic asymptotically regular iff $f^{2}:A\cup B\to A\cup B$ is asymptotically regular.

Proof: Note that, since $BP_\varepsilon(f)\neq\varnothing;\forall\varepsilon\in\mathbf{R}_{0+}$, then $f:A\cup B\to A\cup B$ has 0-best proximity points in A and in B for any $\varepsilon\in\mathbf{R}_{0+}$ so that, in particular, $BP_0(f)\neq\varnothing$ and $f:A\cup B\to A\cup B$ has 0-best proximity points. Since A is approximatively compact with respect to B, the set $\{y\in B:d(x,A)=D\}$ is nonempty and, also, if $d(y,x_n)\to d(y,A)=D$ for some $y\in B$ and some sequence $\{x_n\}\subset A$, then there is a convergent subsequence $\{x_{n_k}\}\subset A$ of $\{x_n\}$.

a) First, it is proved that if $f:A\cup B\to A\cup B$ is cyclic asymptotically regular then $f^2:A\cup B\to A\cup B$ is asymptotically regular. Assume that $f:A\cup B\to A\cup B$ is cyclic asymptotically regular so that it has the approximate best proximity point property so that $BP_\varepsilon(f)\neq\varnothing;\ \forall\varepsilon\in\mathbf{R}_{0+}$ and $d\left(f^{n+1}x,f^nx\right)\to D$ as $n\to\infty;\forall x\in A\cup B$. Now, take $x\in A$. Since A is approximatively compact with respect to B, then $\{y\in B:d(x,A)=D\}\neq\varnothing$, and there is a convergent subsequence in A, $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with the properties $d(x_n,x_{n+1})\to D$, $d(x_{2n},y)\to D$, $d(x_{2n_k},y)\to D$ as $n\to\infty$ for $x\in A$, and $\{x_{2n_k}=f^{2n_k}x\}\to z(\in A)$ with $x_{n+1}=fx_n=f^{n+1}x(\in A\cup B)$, $x_{2n}=f^{2n}x(\in A)$ since $x\in A$. Proceed by contradiction by assuming that $f^2:A\cup B\to A\cup B$ is not asymptotically regular. Then, there is $\varepsilon\in\mathbf{R}_{0+}$ and a sequence of positive integers $\{n_k\}$ such that $d\left(f^{2n_k+2}x,f^{2n_k}x\right)>\varepsilon;\ \forall x\in A$ with $z\in A$ and $fz\in B$ being best proximity points of A and B which are then 0-best proximity points (note that if $z\in A$ is a 0-best proximity point then $d(z,fz)=D$ so that $fz\in B$ is also a 0-best proximity point). Then, the following contradiction follows:

$$0<\varepsilon\le\liminf_{k\to\infty}d\left(f^{2n_k+2}x,f^{2n_k}x\right)\le=\liminf_{k\to\infty}d\left(\lim_{k\to\infty}f^{2n_k+2}x,\lim_{k\to\infty}f^{2n_k}x\right)=0 \tag{2.4}$$

For $x'\in B$, we can repeat all the above reasoning for $x=fx'\in A$.In conclusion, $f^2:A\cup B\to A\cup B$ is asymptotically regular and $BP_\varepsilon(f)\neq\varnothing\Rightarrow\left[F_\varepsilon\left(f^2\right)=\left\{x\in A\cup B:d\left(x,f^2x\right)\le\varepsilon\right\}\neq\varnothing\right]$ for any given $\varepsilon\in\mathbf{R}_{0+}$ if $f:A\cup B\to A\cup B$ is cyclic asymptotically regular.

b) Now, the converse implication is proved, that is, if $f^2:A\cup B\to A\cup B$ is asymptotically regular then $f:A\cup B\to A\cup B$ is cyclic asymptotically regular, equivalently,

$$\left[F_\varepsilon\left(f^2\right)=\left\{x\in A\cup B:d\left(x,f^2x\right)\le\varepsilon\right\}\neq\varnothing\right]\Rightarrow BP_\varepsilon(f)\neq\varnothing \text{ for any given } \varepsilon\in\mathbf{R}_{0+}$$

or, equivalently, we prove its equivalent contrapositive logic proposition, that is, $BP_\varepsilon(f)=\varnothing\Rightarrow F_\varepsilon\left(f^2\right)=\varnothing$ for any given $\varepsilon\in\mathbf{R}_{0+}$. Assume on the contrary that $BP_\varepsilon(f)=\varnothing\Rightarrow F_\varepsilon\left(f^2\right)\neq\varnothing$.Then, $d(x,fx)\ge D+\varepsilon;\ \forall x\in A\cup B$ and $d\left(x,f^2x\right)<\varepsilon_1$ for some $\varepsilon\in\mathbf{R}_+$, some $x\in A\cup B$ and any $\varepsilon_1\in\mathbf{R}_+;\ \forall x\in A\cup B$. Note that, although $BP_\varepsilon(f)=\varnothing$ for $\varepsilon\in\mathbf{R}_+$ is being assumed, $BP_0(f)\neq\varnothing$ so that there are $z\in A$ and $fz\in B$ such that $d(z,fz)=D$ since $f:A\cup B\to A\cup B$ has 0-best proximity points in A and in B. As a result, one has for some $x\in A\cup B$:

$$d\left(f^2x,fx\right)+\varepsilon_1>d\left(x,f^2x\right)+d\left(f^2x,fx\right)\ge d(x,fx)\ge D+\varepsilon \tag{2.5}$$

By applying the above chain of inequalities to the 0-best proximity points z and fz satisfying $d(z,fz)=D$. It is proved that $d\left(f^2z,fz\right)=D$. Assume not. Then, the sequence of points $\left(z,fz,f^2z\right)$is

generated through $f:A\cup B\to A\cup B$. If $f^2z=z$ then $d\left(f^2z,\ fz\right)=d\left(z,\ fz\right)=D$ holds. Assume that $f^2z\neq z$ with $d\left(z,\ f^2z\right)<\varepsilon_1$; $\forall\varepsilon_1\in\boldsymbol{R}_{0+}$. Then, it follows from (2.5) that

$$D<d\left(fz,\ f^2z\right)\leq d\left(z,\ fz\right)+d\left(z,\ f^2z\right)<\varepsilon_1+D \tag{2.6}$$

fails for $\varepsilon_1=0$ so that $d\left(f^2z,\ fz\right)=d\left(z,\ fz\right)=D$.Thus, $d\left(f^2z,\ fz\right)=d\left(z,\ fz\right)=D$ then $D+\varepsilon_1>D+\varepsilon$ from (2.2) and $\varepsilon_1>\varepsilon$. This constraint fails for $\varepsilon\in\boldsymbol{R}_+$ and $0\leq\varepsilon_1\leq\varepsilon$ which contradicts that ε_1 is arbitrary.

Theorem 2.10. Let (X,d) be a metric space and let A and B be nonempty bounded closed subsets of X with $d(A,B)=D$. Assume that the cyclic self-mapping $f:A\cup B\to A\cup B$ satisfies:

$$d\left(f^2x,\ fx\right)\leq Kd\left(fx,x\right)+\left(1-K\right)\left(D+\delta\left(x\right)\right) \tag{2.7}$$

for $x\in A\cup B$ and some $K\in[0,1)$, where $\delta(x)=\varepsilon_{0A}$ if $x\in A$ and $\delta(x)=\varepsilon_{0B}$ if $x\in B$. Then, $f:A\cup B\to A\cup B$ is cyclic asymptotically ε_{0A}-regular in A and cyclic asymptotically ε_{0B}-regular in B and it has both the ε_{0A}-partial best proximity point property in A and the $\varepsilon_{0B}(\neq\varepsilon_{0A})$-partial best proximity point property in B. Also, $f^2:A\cup B\to A\cup B$ has not the approximate fixed point property and, equivalently, it is not cyclic asymptotically regular.

3. Concluding remarks

In this paper, a formal study is given which compares the close concepts of approximate best proximity points of 2-cyclic self-mappings and approximate best proximity point property and cyclic asymptotic regularity. Some related properties are proved while particular results are given for contractive cyclic self-mappings. There are other derived results available in the paper concerned with composite self-mappings of the original mapping.

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