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Research Article

Hyers-Ulam-Rassias Stability of Functional Differential Systems with Point and Distributed Delays

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This paper investigates stability and asymptotic properties of the error with respect to its nominal version of a nonlinear time-varying perturbed functional differential system subject to point, finite-distributed, and Volterra-type distributed delays associated with linear dynamics together with a class of nonlinear delayed dynamics. The boundedness of the error and its asymptotic convergence to zero are investigated with the results being obtained based on the Hyers-Ulam-Rassias analysis.

1. Introduction

The background literature on Hyers-Ulam-Rassias analysis is abundant and many different problems have been solved with it under the basis that there is a perturbation of a nominal equation and that a norm upper-bounding function of the error is obtained, [1–13]. A variety of results in this field have been obtained, in particular, for perturbations of additive and subadditive functions [5–7]. Special attention to the asymptotic properties of the Cauchy equation is paid in [5]. On the other hand, some inequalities related to the exponential function are proposed and investigated in [1]. Closed problems related to the slopes and mean values of exponential functions are discussed in [2]. Also, extensions to functions of several variables and to the study of approximate homomorphisms are discussed and solved in [3, 4], which are nowadays classical studies in the field of Hyers-Ulam-Rassias stability. A discussion with several results of asymptotic aspects is given in [5] close to the asymptotic derivability which is a very important issue in nonlinear analysis. The cubic function is studied in [8] from the Hyers-Ulam stability point of view while its relations to the related stability quadratic functional functions, symmetric biadditive functions are also commented. Different kinds of perturbed differential equations of first order are investigated in [11–13] in the light of the Hyers-Ulam-Rassias stability analysis.

On the other hand, it is well known that time-delay dynamic systems are a very relevant field of research in dynamic systems and functional differential equations because of their intrinsic theoretical interest since the required formalism lies in that of functional differential equations, then infinite dimensional, and since there are a wide range of applicability issues in modelling aspects of physical systems, like queuing systems, teleoperated systems, war and peace and biological models and transportation systems, also finite impulse response filtering, and so forth. Another important useful application is the inclusion of delays in the description of epidemic models so as to obtain richer information about the disease propagation and to take it into account in the design of vaccination laws. See, for instance, [14–23] and references therein. The stability and the derivation of approximate solutions of some kinds of functional equations have been also investigated in [24, 25] and references therein, under close analysis methods.

This paper is concerned with the study of the solutions of perturbed time-delay differential systems and their comparison and asymptotic properties of convergence to those of the corresponding unperturbed ones. The differential systems involve a combined fashion linear dynamics of point delays, finitely distributed time-delays, and infinitely distributed Volterra-type delays as well as perturbations involving non-linear dynamics depending on further delays, in general, and which can be unknown with just slight "a priori" knowledge on an upper-bounding function on the supremum of the trajectory solution norm. External nonnecessarily identical forcing terms can be also present in both the nominal and the current differential functional equations. The number of delays of the perturbed equation and that of its nominal versions might be distinct and the matrices describing the linear delayed and delay-free dynamics of both differential systems might be also distinct. There are two problems focused on in the paper; namely, firstly the paper focuses on the asymptotic convergence to zero of the error between both nominal and current solutions irrespective of the stability properties of the nominal differential system, if any, and, secondly such a problem is revisited together with the stability or asymptotic stability of both the nominal and the perturbed functional differential systems.

Notation. Consider

$$\begin{aligned} &\mathbf{R}_{0+} := \mathbf{R}_{+} \cup \{0\}, \qquad \mathbf{R}_{+} := \{z \in \mathbf{R} : z > 0\}, \\ &\mathbf{R}_{0-} := \mathbf{R}_{-} \cup \{0\}, \qquad \mathbf{R}_{-} := \{z \in \mathbf{R} : z < 0\}, \\ &\mathbf{C}_{0+} := \{z \in \mathbf{C} : \operatorname{Re} z \geq 0\}, \qquad \mathbf{C}_{+} := \{z \in \mathbf{C} : \operatorname{Re} z > 0\}, \\ &\mathbf{C}_{0-} := \{z \in \mathbf{C} : \operatorname{Re} z \leq 0\}, \qquad \mathbf{C}_{-} := \{z \in \mathbf{C} : \operatorname{Re} z < 0\}, \\ &\mathbf{Z}_{0+} := \mathbf{Z}_{+} \cup \{0\}, \qquad \mathbf{Z}_{+} := \{z \in \mathbf{Z} : z > 0\}, \end{aligned} \tag{1}$$

where $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{Z}$ are the sets of real, complex, and integer numbers, respectively. The complex imaginary unity is $\mathbf{i} = \sqrt{-1}$. A finite subset of j consecutive positive integers starting with 1 is denoted by $\overline{j} := \{1, 2, \ldots, j\}$. The set $\mathbf{R}_{-h}\theta := [-\theta, 0) \cup \mathbf{R}_{0+}$ will be used to define the solution of functional differential equations on $\mathbf{R}_{0+}$ including its initial condition on $[-\theta, 0]$.

$C_e(\mathbf{R}_{0+}) := C(\mathbf{R}_{0+}, \mathbf{C}^n)$ is the Banach space of continuous functions from $\mathbf{R}_{0+}$ into $\mathbf{C}^n$ endowed with the supremum norm $|x|_\alpha = \sup_{0 \leq \tau < \infty}(\|x(\tau)\|_\alpha)$.

$C(\mathbf{R}_{-\theta}) := C([-\theta, \infty), \mathbf{C}^n)$ is the Banach space of continuous functions from $[-\theta, \infty)$ into $\mathbf{C}^n$ endowed with the supremum norm $|\phi|_\alpha = \sup_{-\theta \leq \tau < \infty}(\|\phi(\tau)\|_\alpha)$; $\forall \phi \in C_e(-\theta) := C([-\theta, 0), \mathbf{C}^n)$ (defined below) is an initial condition, for some given vector norm $\|\cdot\|_\alpha$.

$C_e(-\theta) := \{\phi = \phi_1 + \phi_2 : \phi_1 \in C(-\theta),\ \phi_2 \in B^0(-\theta)\}$, $\phi(0) = x_0$, with $C(-\theta) := \{C^0([-\theta, 0], X)\}$, that is, the set of continuous mappings from $[-h, 0]$ into the Banach space X with norm $\overline{\phi}_\alpha := |\phi|_\alpha = \sup\{\|\phi(t)\|_\alpha : -\theta \leq t \leq 0\}$; $\|\cdot\|$ denotes the Euclidean norm of vectors in $\mathbf{C}^n$ and matrices in $\mathbf{C}^{n \times n}$, and $B^0(-\theta) := \{\phi : [-\theta, 0] \to X\}$ is the set of real bounded vector functions on X endowed with the supremum norm having support of zero measure.

x_t denotes the solution string within $[t - \theta, t]$ pointwise defined by the solution $x(t)$.

It is said that the delays associated with Volterra-type dynamics are infinitely distributed because the contribution of the delayed dynamics is made under an integral over $[0, \infty)$ as $t \to \infty$; that is, $x(t - \tau - h_i')$ acts on the dynamics of $x(t)$ from $\tau = 0$ to $\tau = t$ for finite t and as $t \to \infty$.

$\operatorname{Dom}(H)$ is the definition domain of the operator H.

2. Perturbed and Nominal Differential System

We now consider a functional nth order differential system with point and, in general, both infinite-type Volterra-type and finite-distributed delays in a more general context that is the approaches of [16, 21–23], since it includes the contributions of both structured and unstructured delayed dynamics with point and finite- and infinite-distributed delays, as well as the presence of nonlinear dynamics. Such a differential equation obeys the widely general structure:

$$\dot{x}(t) - Lx_t - g(t) = 0, \tag{2}$$

where

$$Lx_t \equiv L_0 x_t + f\left(t, x_{t-\theta_b}\right), \tag{3}$$

$$\begin{aligned} L_0 x_t &\equiv A_0 x(t) + \sum_{i=1}^{m_0} A_i(t)\, x\left(t - h_i\right) \\ &\quad + \sum_{i=0}^{m_0'} \int_0^t d\alpha_i(\tau)\, A_{\alpha_i}(\tau)\, x\left(t - \tau - h_i'\right) \\ &\quad + \sum_{i=m_0'+1}^{m_0+m_0''} \int_{t-h_i'}^{t} d\alpha_i(t-\tau)\, A_{\alpha_i}(\tau)\, x\left(\tau - \theta_0\right) + g_0(t) \end{aligned} \tag{4}$$

$$\begin{aligned} f\left(t, x_{t-\theta_b}\right) &= f_0\left(t, x_{t-\theta_a}\right) + f_1\left(t, x_{t-\theta_3}\right); \\ g(t) &= g_0(t) + \widetilde{g}(t), \end{aligned} \tag{5}$$

$$\begin{aligned} f_1\left(t, x_{t-\theta_3}\right) &= \sum_{i=1}^{m} \widetilde{A}_i(t)\, x\left(t - h_i\right) \\ &\quad + \sum_{i=0}^{m'} \int_0^t d\alpha_i(\tau)\, \widetilde{A}_{\alpha_i}(\tau)\, x\left(t - \tau - h_i'\right) \\ &\quad + \sum_{i=m'+1}^{m'+m''} \int_{t-h_i'}^{t} d\alpha_i(t-\tau)\, \widetilde{A}_{\alpha_i}(\tau)\, x\left(\tau - \theta_1\right), \end{aligned} \tag{6}$$

$x(t) \equiv \phi(t)$; $t \in [-\theta, 0]$ for some given initial condition vector function $\phi \in C_e(-\theta)$, where $\theta_0^* = \max(\max_{1 \leq i \leq m_0}(h_i), \max_{1 \leq i \leq m_0'}(h_i'), \max_{m_0'+1 \leq i \leq m_0'+m_0''}(h_i') + \theta_0)$ and $\theta^* = \max(\theta_0^*, \theta_a)$ are, respectively, the maximum delays of the unperturbed (or perfectly modelled nominal system, being associated with the operators $L_0 x_t$) and the nominal system subject to unmodeled and perhaps nonlinear non-structured dynamics, of maximum delay θ_a, which includes the contribution to the dynamics of the possibly nonlinear function $f_0(t, x_{t-\theta_a})$ while the maximum delay of the current system is

$$\theta = \max\left(\max_{1\le i\le m}(h_i), \max_{1\le i\le m'+m''}(h_i'), \max_{m'+1\le i\le m'+m''}(h_i')+\theta_1, \theta_b\right), \tag{7}$$

where $\theta_b = \max(\theta_a, \theta_3)$ is the maximum delay of its unmodeled dynamics with

$$\theta_3 = \max\left(\max_{1\le i\le m_0}(h_i), \max_{1\le i\le m_0'}(h_i'), \max_{m_0'+1\le i\le m_0'+m_0''}(h_i')+\theta_1\right). \tag{8}$$

The nonnegative real constants θ_0 and θ_1, if they are not zero in (4) and (6), modulate the finitely distributed delays with the functions that configure their contributions under the integral symbols. The objective is the comparison of the solution of the current dynamic functional equation (2), subject to (3)–(6), to that of its nominal version $\dot{x}(t) - L_0 x_t - f_0(t, x_{t-\theta_a}) - g_0(t) = 0$. The following, rather weak, hypotheses are made.

(1) $L : C_e(\mathbf{R}_{-\theta}) \to \mathbf{C}^n$ is a bounded linear functional defined by the right hand side of (2).

(2) h_k and h_ℓ' and $h_j' + \theta_{0,1}$ ($k = 1, 2, \ldots, m$; $\ell = 0, 1, \ldots, m'$, $j = m'+1, \ldots, m'+m''$) are nonnegative real point delays, infinite-time distributed Volterra-type delays (i.e., the first m' distributed delays), and finite time-interval distributed delays with $h_0 = h_0' = 0$ such that $m \ge m_0$, $m' \ge m_0'$, and $m'' \ge m_0''$, where the 0 subscripts stand for the nominal equation. The finitely distributed nominal and current delays have an increasing factor $\theta_0 \ge 0$ and $\theta_1 \ge 0$ for formulation generality purposes.

(3) $g_0, g : \mathbf{R}_{0+} \to \mathbf{C}^n$ is piecewise continuous, $f_0 : \mathbf{R}_{0+} \times C(\mathbf{R}_{-\theta}) \to \mathbf{C}^n$ describes a perturbed linear dynamics, and

$$x_t^{C(\mathbf{R}_{-\theta})} := \begin{cases} x : [-\theta, \tau) \longrightarrow X, & \tau \le t \\ 0, & \tau > t, \end{cases} \tag{9}$$

satisfying $x(t) = \phi(t)$, $\forall t \in [-\theta, 0]$, is a string of the solution of (2).

The following, rather nonrestrictive in practice, hypotheses are made.

(H.1) The initial condition of (2) is $\phi \in C_e(-\theta)$. Roughly speaking, $\phi \in B^0(-\theta)$ if and only if it is almost everywhere zero except at isolated discontinuity points within $[-\theta, 0]$ where it is bounded. Thus, $\phi \in C_e(-\theta)$ if and only if it is almost everywhere continuous in $[-\theta, 0]$ except possibly on a set of zero measure of bounded discontinuities. $C_e(-\theta)$ is also endowed with the supremum norm since $\phi = \phi_1 + \phi_2$, some $\phi_1 \in C(-\theta)$, $\phi_2 \in B^0(-\theta)$ for each $\phi \in C_e(-\theta)$. In the following, the supremum norms on $L(X)$ are also denoted by $|\cdot|$.

(H.2) All the linear operators $A_k, \widetilde{A}_k$ $(0 \le k \le m)$, $A_{\alpha_k}, \widetilde{A}_{\alpha_k}$ $(0 \le k \le m' + m'')$, with the abbreviated notation $A_{\alpha_0} = A_\alpha$, are in $L(X) := L(X, X)$, the set of linear operators on X, of dual X^*, which are closed and densely defined with respective domain and range $D(A_{\alpha_i})$ and $R(A_{\alpha_i}) \subset X$ $(i = 0, 1, \ldots, m')$. The functions $\alpha_i \in C^0([0,\infty), \mathbf{C}) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbf{C}_+)$ $(i = 0, 1, \ldots, m')$ and $\alpha_i \in C^0([-h, 0), \mathbf{C})$ $(i = 0, 1, \ldots, m' + m'')$ are everywhere differentiable with possibly bounded discontinuities on subsets of zero measure of their definition domains with $\int_0^\infty e^{\nu t}|d\alpha_i(t)| < \infty$ for some nonnegative real constant ν $(i = 0, 1, \ldots, m')$. If $\alpha_i(\cdot)$ is a matrix function $\alpha_i : [0,\infty)\times X^* \to L(X, X^*)$ then it is in $C^0([0,\infty), \mathbf{C}^{n\times n}) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbf{C}_+^{n\times n})$ with $\int_0^\infty e^{\nu t}|d\alpha_i(t)| < \infty$ and its entries are everywhere time differentiable with possibly bounded discontinuities within a subset of zero measure of their definition domains.

3. Main Results

There is an interesting set of references on the application of Hyers-Ulam method to stability of differential equations (c.f. [11–13]). The first and third ones are first-order equations, respectively, linear and nonlinear while the second one [12] is of linear time-varying type. Such differential equations in those references are delay-free, so that they are of a nonfunctional type. In the following, we develop a related formal stability analysis of functional differential equations with internal delays under the forms (2)–(6) and satisfying hypotheses (H.1)-(H.2). Note that the studied equations have several types of time-varying linear delayed dynamics (as, e.g., point delays, finitely distributed delays, and Volterra-type infinitely distributed delays). Note also that, furthermore, nonautonomous nonlinear dynamics can be considered in (3) under the generic structure of (4)-(5), that is, involving if suited any of the various types of delays plus nonlinear unmodeled terms under the functional f_0 which can be unstructured and unmeasurable. In that case, an upper-bounding function of generic structure, the supremum of the norm of the state, will be assumed to be known in the formal subsequent developments. This general structure of the functional equation of dynamics and its stability study under the Hyers-Ulam-Rassias formalism are the main contribution of the paper. It turns out while it is a well-known feature that Hyers-Ulam-Rassias method of analysis in differential equations relies basically on comparing the perturbed differential equation with the unperturbed (or nominal) one. In [11–13], the analysis is performed based on the error in between both differential equations while, in the current paper, it is based on the direct analysis of the error between both of them. Other important characteristics of the proposed formalism are that it is based on a dynamic system description rather than on simple differential equations and that the fundamental matrix can be defined based on different comparison systems which can be delay-free or it can contain a number of delays. The formal treatment proceeds in such a way that the delayed dynamics which is not accounted for, if any, to define the fundamental matrix is considered as a contribution to the forcing part of the differential system. The analysis is neither based in a formalism stated on the Hilbert space framework, as in [15], nor in Lyapunov stability theory, including the use of matrix equalities and inequalities as in [17, 18, 21], but on the analysis of the solution error in between the nominal and the current differential systems for all time. In particular, note that the approach used in [16] is not directly applicable in this context since the differential system at hand is time-varying and the Laplace transform methods cannot be used.

The main results of this section consist of a main theorem and three corollaries related to sufficiency-type conditions for guaranteeing the theorem and particular cases as well as several remarks related to their applications to further potential particular cases. The basic main result follows below.

Theorem 1. *Assume that* $\tilde{g} \equiv 0$ *and* $f_0 : \mathbf{R}_{0+} \times \mathbf{C}^n \to \mathbf{C}^n$ *is subadditive,* $\theta \geq \theta^*$, *and* $\|f_0(t, x_{t-\theta_b})\| \leq K_{00} \sup_{0\leq\delta\leq h}\|x_{t-\theta_b-\delta}\| + K_{01}$ *for some* $K_{00}, K_{01} \in \mathbf{R}_{0+}$; $\forall t \in \mathbf{R}_{0+}$. *Let* $\mu(t_n) = (\mu_1, \mu_2, \ldots, \mu_{m_0})(t_n)$ *be any* m_0*-tuple defined from a piecewise constant* m_0*-binary vector function* $\mu : [t_n, t_{n+1})\times\mathbf{Z}_{0+} \to \{\mu_1(t_n), \ldots, \mu_{m_0}(t_n)\}$ *for any combination of values of the set of binary variables* $\mu_i(t_n) \in \{0,1\}$; $\forall i \in \overline{m}_0 = \{1, 2, \ldots, m_0\}$ *defining a fundamental matrix of the nominal unforced differential system* $\dot{x}(t) = L_0 x_t$ *of the form*

$$\begin{aligned}\Psi_{\mu(t_n)}(t,t_n) &= e^{A_0(t-t_n)} \\ &\quad\times\left(I + \sum_{i=1}^{m_0}\mu_i(t_n)\right. \\ &\qquad\left.\times\int_{t_n}^{t} e^{A_0(t_n+\tau)}A_i(\tau)\,\Psi_{\mu(t_n)}(\tau-h_i)\,U(\tau-h_i)\,d\tau\right); \\ &\qquad \forall t \in [t_n, t_{n+1});\end{aligned} \tag{10}$$

$\forall n \in \mathbf{Z}_{0+}$ *with initial conditions with* $\Psi_{\mu(t_n)}(t,t) = I$; $\forall t \in \mathbf{R}_0$ *and* $\Psi_{\mu(t_n)}(t,t_n) = 0$ *for* $t < t_n$, *where* I *is the n-identity matrix and* $U(t)$ *is the Heaviside function, which satisfies the differential system:*

$$\begin{aligned}\dot{\Psi}_{\mu(t_n)}(t,t_n) &= A_0\Psi_{\mu(t_n)}(t,t_n) \\ &\quad+\sum_{i=1}^{m_0}\mu_i(t_n)\,A_i(t)\,U(t-h_i)\,\Psi_{\mu(t_n)}(t-h_i,t_n); \\ &\qquad \forall t \in [t_n, t_{n+1}),\quad \forall n \in \mathbf{Z}_{0+}.\end{aligned} \tag{11}$$

Then, the following properties hold.

(i) The error norm is in between the current solution and the nominal one on $[t_n, \infty)$; $\forall n \in \mathbf{Z}_{0+}$ *is upper-bounded by a prescribed positive norm bound* E *if* $\|e(t_0)\| \leq E$; *the fundamental matrix defined from a binary vector function* $\mu : [t_n, t_{n+1}) \times \mathbf{Z}_{0+} \to \{\mu_1(t_n), \ldots, \mu_{m_0}(t_n)\}$ *fulfills for a matrix induced vector norm* $\|\Psi_{\mu(t_n)}(t_n,t)\| \leq \psi_{\mu(t_n)}(t_n,t) < 1$; $\forall t \in (t_n, t_{n+1}]$, $\forall n \in \mathbf{Z}_{0+}$, *such that the upper-bounding function* $\psi_{\mu(\cdot)}(\cdot,\cdot) : \mathbf{Z}_{0+}\times\mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *satisfies* $\psi_{\mu(t_n)}(t_n,t_n) = 1$; $\forall n \in \mathbf{Z}_{0+}$, *and the subsequent constraint holds for some sufficiently small real constant* $T^* \in \mathbf{R}_+$:

$$\begin{aligned}t - t_n &\leq \left(\left(1-\psi_{\mu(t_n)}(t_n,t)\right)\right. \\ &\quad\times\left(\max_{t_n\leq\tau\leq t}\left(\psi_{\mu(t_n)}(t_n,\tau)\right.\right. \\ &\qquad\times\left\|\sum_{i=1}^{m_0}\left(1-\mu_i(t_n)\right)A_i(\tau)\right. \\ &\qquad+\sum_{i=0}^{m_0'}\int_0^{\tau}d\alpha_i(\sigma)\,A_{\alpha_i}(\sigma) \\ &\qquad+\sum_{i=m_0'+1}^{m_0+m_0''}\int_{\tau-h_i'}^{\tau}d\alpha_i(\tau-\sigma)\,A_{\alpha_i}(\sigma) \\ &\qquad\left.\left.\left.\left.+f\left(\tau,e_{\tau-\theta_b}\right)\right\|\right)\right)^{-1}\right); \\ &\qquad \forall t \in [t_n, t_{n+1}),\quad \forall n \in \mathbf{Z}_{0+}\end{aligned} \tag{12}$$

for some strictly increasing sequence $\{t_0, t_1, \ldots\}$ *satisfying*

$$\begin{aligned}T^* &\leq T_n := t_{n+1} - t_n \\ &\leq \min\left(h^*(t_n), \max\left(\arg r_n \in \mathbf{R}_+ : r_n = z_n\right)\right); \\ &\qquad \forall n \in \mathbf{Z}_{0+},\end{aligned} \tag{13}$$

where

$$\begin{aligned}z_n &\leq \left(\left(1-\psi_{\mu(t_n)}(t_n,t_n+z_n)\right)\right. \\ &\quad\times\left(\psi_{\mu(t_n)}(t_n,t_n+z_n)\right. \\ &\qquad\times\max_{t_n\leq\tau\leq t_{n+1}}\left(\left\|\sum_{i=1}^{m_0}\left(1-\mu_i(t_n)\right)A_i(\tau)\right.\right. \\ &\qquad+\sum_{i=0}^{m_0'}\int_0^{\tau}d\alpha_i(\sigma)\,A_{\alpha_i}(\sigma)\end{aligned}$$

$$+\sum_{i=m_0'+1}^{m_0+m_0''}\int_{\tau-h_i'}^{\tau}d\alpha_i(\tau-\sigma)A_{\alpha_i}(\sigma)$$
$$+f\left(\tau,e_{\tau-\theta_b}\right)\Big\|\Big)\Big)^{-1}\Big);$$
$$\forall n\in\mathbf{Z}_{0+} \tag{14}$$

$$h^*(t_n):=\min\left(\min_{1\le i\le m_0}(1-\mu_i(t_n))h_i,\ \min_{m_0<i\le m}h_i,h_1'\right);$$
$$\forall n\in\mathbf{Z}_{0+}; \tag{15}$$

$\forall\phi\in C_e(-\theta):=C([-\theta,t_0),\mathbf{C}^n)$ with $\theta=\max(\max_{1\le i\le m}(h_i),\max_{1\le i\le m'+m''}(h_i'),\ \max_{m'+1\le i\le m'+m''}(h_i')+\theta_1,\theta_b)\ge\theta^*$, with $x(t_0)=\varphi(t_0)$ for any given $t_0\in\mathbf{R}_{0+}$. If $\sup_{0\le t\le t_0}\|e(t)\|\le E$ for any given absolutely continuous vector function of initial conditions $\varphi:[-\theta,0]\to\mathbf{C}^n$ then $\sup_{0\le t<\infty}\|e(t)\|\le E$.

(ii) Assume that property (i) holds by

(a) *replacing the constraint $\|\Psi_{\mu(t_n)}(t_n,t)\|\le\psi_{\mu(t_n)}(t_n,t)<1$ with $\|\Psi_{\mu(t_n)}(t_n,t)\|\le\psi_{\mu(t_n)}(t_n,t)<K_n$; $\forall t\in(t_n+\gamma_n,t_{n+1}]$ for some real sequence $\{\gamma_n\}$ with $\gamma_n\ge\gamma>0$; $\forall n\in\mathbf{Z}_{0+}$;*

(b) *replacing E by a nonnegative real sequence $\{E_n\}$ satisfying $E_{n+1}=K_nE_n$ for each $n\in\mathbf{Z}_+$ and some decreasing sequence of nonnegative real numbers $\{K_n\}$ such that $\limsup_{n\to\infty}K_n=K$ for some real constant $K\in[0,1)$;*

(c) *replacing "1" in the numerator of (12) and (14) by K_n; $\forall n(\ge n_0)\in\mathbf{Z}_{0+}$ for some given $\varepsilon(<1-K)\in\mathbf{R}_+$ and $n_0=n_0(\varepsilon)\in\mathbf{Z}_{0+}$ such that $K_n\le K+\varepsilon<1$.*

Then, $e(t)\to 0$ exponentially as $t\to\infty$.

Proof. The nominal and current unique solutions of (2)–(4) [16, 17] are, respectively, given by

$$\begin{aligned}
x_0(t)&=\Psi_{\mu(t_n)}(t,t_0)x_0(t_0)+\int_{t_0}^{t}\Psi_{\mu(t_n)}(t,\tau)\\
&\times\Bigg[\sum_{i=1}^{m_0}(1-\mu_i)A_i(\tau)x_0(\tau-h_i)+\sum_{i=0}^{m_0'}\int_0^{\tau}d\alpha_i(\sigma)A_{\alpha_i}(\sigma)x_0\left(\tau-\sigma-h_i'\right)\\
&+\sum_{i=m_0'+1}^{m_0+m_0''}\int_{\tau-h_i'}^{\tau}d\alpha_i(\tau-\sigma)A_{\alpha_i}(\sigma)x_0(\sigma-\theta_0)\Bigg]d\tau\\
&+\int_{t_0}^{t}\Psi_{\mu}(t,\tau)g(\tau)d\tau;\quad\forall t(\ge t_0)\in\mathbf{R}_{0+}\\
x(t)&=\Psi_{\mu(t_n)}(t,t_0)x(t_0)+\int_{t_0}^{t}\Psi_{\mu(t_n)}(t,\tau)\\
&\times\Bigg[\sum_{i=1}^{m_0}(1-\mu_i)A_i(\tau)x(\tau-h_i)+\sum_{i=0}^{m_0'}\int_0^{\tau}d\alpha_i(\sigma)A_{\alpha_i}(\sigma)x\left(\tau-\sigma-h_i'\right)\\
&+\sum_{i=m_0'+1}^{m_0+m_0''}\int_{\tau-h_i'}^{\tau}d\alpha_i(\tau-\sigma)A_{\alpha_i}(\sigma)x(\sigma-\theta)+f\left(\tau,x_{\tau-\theta_b}\right)\Bigg]d\tau\\
&+\int_{t_0}^{t}\Psi_{\mu(t_n)}(t,\tau)g(\tau)d\tau;\quad\forall t(\ge t_0)\in\mathbf{R}_{0+}
\end{aligned}\tag{16}$$

and, since $f_1:\mathbf{R}_{0+}\times\mathbf{C}^n\to\mathbf{C}^n$ is structurally additive in its second argument, then also subadditive, and $f_0:\mathbf{R}_{0+}\times\mathbf{C}^n\to\mathbf{C}^n$ is subadditive in its second argument; then $f_1:\mathbf{R}_{0+}\times\mathbf{C}^n\to\mathbf{C}^n$ is subadditive in its second argument so that $h:\mathbf{R}_{0+}\times\mathbf{C}^n\to\mathbf{C}^n$ is also subadditive in its second argument. Thus,

$$\begin{aligned}
e(t)&:=x(t)-x_0(t)\\
&=\Psi_{\mu(t_n)}(t,t_n)e(t_n)+\int_{t_n}^{t}\Psi_{\mu(t_n)}(t_n,\tau)\\
&\times\Bigg[\sum_{i=1}^{m_0}(1-\mu_i)A_i(\tau)e(\tau-h_i)+\sum_{i=0}^{m_0'}\int_0^{\tau}d\alpha_i(\sigma)A_{\alpha_i}(\sigma)e\left(\tau-\sigma-h_i'\right)\\
&+\sum_{i=m_0'+1}^{m_0+m_0''}\int_{\tau-h_i'}^{\tau}d\alpha_i(\tau-\sigma)A_{\alpha_i}(\sigma)e(\sigma-\theta)+f\left(\tau,e_{\tau-\theta_b}\right)\Bigg]d\tau;
\end{aligned}\tag{17}$$

$\forall t \in [t_n, t_{n+1})$ for any strictly increasing sequence $\{t_0, t_1, \dots, t_n, \dots\}$ and any absolutely continuous initial condition $\varphi(t)$ on $[t_0 - h, t_0]$ with $x(t_0) = \varphi(t_0)$ for any given $t_0 \in \mathbf{R}_{0+}$ where $\mu(t_n) = (\mu_1(t_n), \mu_2(t_n), \dots, \mu_{m_0}(t_n))$ is any m_0-tuple defined for any combination of values of the set of binary variables $\mu_i \in \{0,1\}$; $\forall i \in \overline{m}_0 = \{1, 2, \dots, m_0\}$ with

$$
\begin{aligned}
f\left(t, x_{t-\theta_b}\right) = {} & f_0\left(t, x_{t-\theta_a}\right) \\
& + \sum_{i=1}^{m} \widetilde{A}_i(t)\, x\left(t - h_i\right) \\
& + \sum_{i=0}^{m'} \int_0^t d\alpha_i(\sigma)\, \widetilde{A}_{\alpha_i}(\sigma)\, x\left(t - \sigma - h_i'\right) \\
& + \sum_{i=m'+1}^{m'+m''} \int_{t-h_i'}^{t} d\alpha_i(t-\sigma)\, \widetilde{A}_{\alpha_i}(\sigma)\, x\left(\sigma - \theta_1\right) \\
& \qquad \forall t\left(\geq t_0\right) \in \mathbf{R}_{0+};
\end{aligned}
$$

$$
\begin{aligned}
e(t) := {} & x(t) - x_0(t) \\
= {} & \Psi_{\mu(t_n)}\left(t, t_n\right) e\left(t_n\right) \\
& + \int_{t_n}^{t} \Psi_{\mu(t_n)}\left(t_n, \tau\right) \\
& \times \left[\sum_{i=1}^{m_0} \left(1 - \mu_{in}\right) A_i(\tau)\, e\left(\tau - h_i\right) \right. \\
& + \sum_{i=0}^{m_0'} \int_0^{\tau} d\alpha_i(\sigma)\, A_{\alpha_i}(\sigma)\, e\left(\tau - \sigma - h_i'\right) \\
& + \sum_{i=m_0'+1}^{m_0+m_0''} \int_{\tau-h_i'}^{\tau} d\alpha_i(\tau-\sigma)\, A_{\alpha_i}(\sigma)\, e(\sigma - \theta) \\
& \left. + f\left(\tau, x_{\tau-\theta_b}\right) \right] d\tau.
\end{aligned} \tag{18}
$$

Assume for any $n \in \mathbf{Z}_{0+}$ that $\sup_{t_0 \leq t \leq t_n} \|e(t)\| \leq E$, $\|\Psi_{\mu(t_n)}(t_n, t)\| \leq \psi_{\mu(t_n)}(t_n, t) < 1$ (note that $\Psi_{\mu(t_n)}(t_n, t_n) = I$; $\forall n \in \mathbf{Z}_{0+}$; $\forall t \in (t_n, t_{n+1}]$, $\forall n \in \mathbf{Z}_{0+}$). Note also that $T_n = t_{n+1} - t_n$; $\forall n \in \mathbf{Z}_{0+}$, and define

$$
\begin{aligned}
h\left(t_n\right) = \max\Bigg(& \max_{1 \leq i \leq m_0} \left(1 - \mu_i\left(t_n\right)\right) h_i, \max_{1 \leq i \leq m_0'} \left(h_i'\right), \\
& \max_{m_0'+1 \leq i \leq m_0'+m_0''} \left(h_i'\right) + \theta_0, \theta \Bigg).
\end{aligned} \tag{19}
$$

Note that (12) holds with a bounded denominator for $0 \leq \tau \leq T_n \leq h^*(t_n)$; $\forall n \in \mathbf{Z}_{0+}$ since $\|f_0(t, x_{t-\theta_a})\| \leq K_{00} \sup_{0 \leq \delta \leq h} \|x_{t-\theta_a-\delta}\| + K_{01}$ then $\|f(t, x_{t-\theta_b})\| \leq K_0 \sup_{0 \leq \delta \leq h} \|x_{t-\theta_b-\delta}\| + K_1$; $\forall t \in \mathbf{R}_{0+}$ from (5)-(6) note the fact that linearity of f_1 grows non faster than linearity with respect to $\sup_{0 \leq \delta \leq h} \|x_{t-\theta_b-\delta}\|$; $\forall t \in \mathbf{R}_{0+}$. As a result, if $\sup_{t_0 \leq t \leq t_n} \|e(t)\| \leq E$, then

$$
\left\|\dot{e}(t) - L e_t\right\| = \left\|L_0 e_t + f\left(t, x_{t-\theta_b}\right)\right\| = \left\|\left(L - L_0\right) x_{0t}\right\|, \tag{20a}
$$

$$
\begin{aligned}
& \left\|e\left(t - t_n\right)\right\| \\
& \leq \left\|\Psi_{\mu(t_n)}\left(t_n, t\right) e\left(t_n\right)\right\| \\
& \quad + \left(t - t_n\right) \sup_{t_n - h(t_n) \leq \tau \leq t - h^*(t_n)} \|e(\tau)\| \\
& \quad \times \max_{t_n \leq \tau \leq t} \Bigg(\left\|\Psi_{\mu(t_n)}\left(t_n, t\right)\right\| \\
& \qquad \times \Bigg\| \sum_{i=1}^{m_0} \left(1 - \mu_i\left(t_n\right)\right) A_i(\tau) \\
& \qquad + \sum_{i=0}^{m_0'} \int_0^{\tau} d\alpha_i(\sigma)\, A_{\alpha_i}(\sigma) \\
& \qquad + \sum_{i=m_0'+1}^{m_0+m_0''} \int_{\tau-h_i'}^{\tau} d\alpha_i(\tau-\sigma)\, A_{\alpha_i}(\sigma) \\
& \qquad + f\left(\tau, x_{\tau-\theta_b}\right) \Bigg\| \Bigg) \\
& \leq E \psi_{\mu(t_n)}\left(t, t_n\right) \\
& \quad + \left(t - t_n\right) E \\
& \quad \times \max_{t_n \leq \tau \leq t} \Bigg(\psi_{\mu(t_n)}\left(\tau, t_n\right) \\
& \qquad \times \Bigg\| \sum_{i=1}^{m_0} \left(1 - \mu_i\left(t_n\right)\right) A_i(\tau) \\
& \qquad + \sum_{i=0}^{m_0'} \int_0^{\tau} d\alpha_i(\sigma)\, A_{\alpha_i}(\sigma) \\
& \qquad + \sum_{i=m_0'+1}^{m_0+m_0''} \int_{\tau-h_i'}^{\tau} d\alpha_i(\tau-\sigma)\, A_{\alpha_i}(\sigma) \\
& \qquad + f\left(\tau, x_{\tau-\theta_b}\right) \Bigg\| \Bigg) \\
& \leq E; \quad \forall t \in \left(t_n, t_{n+1}\right]; \ \forall n \in \mathbf{Z}_{0+}.
\end{aligned} \tag{20b}
$$

Thus, if $\sup_{t_0 \leq t \leq t_{n+1-h_n^*}} \|e(t)\| \leq E$ and $\|\Psi_{\mu(t_n)}(t_n, t)\| < 1$ for $t \in (t_n, t_{n+1}]$, $\forall n \in \mathbf{Z}_{0+}$ then, for any given $n \in \mathbf{Z}_{0+}$, $\sup_{t_0 \leq t \leq t_n} \|e(t)\| \leq E < +\infty \Rightarrow \sup_{t_0 \leq t \leq t_{n+1}} \|e(t)\| \leq E$; if constraint (12) holds, subject to (13)–(15) (note that the second min-max part of (13) comes from (12) for $\tau = t_{n+1}$, and then $t_{n+1-h_n^*} \leq t_n$, $\forall n \in \mathbf{Z}_{0+}$, and that the upper-bound of z_n is continuous and is zero if $z_n = 0$; $\forall n \in \mathbf{Z}_{0+}$), the matrix function $\Psi_{\mu(t_n)}(t_n, t)$ from $(\mathbf{Z}_{0+} \times \mathbf{R}) \times \mathbf{R}$ to $\mathbf{C}^{n\times n}$ is everywhere continuous in $(\mathbf{Z}_{0+} \times \mathbf{R}) \times \mathbf{R}$ even if $\mu(t_n^+) \neq \mu(t_n)$ (i.e., if the matrices of the dynamics used to define the fundamental matrix change at $t = t_n$ for some $n \in \mathbf{Z}_{0+}$). Then, property (i) has been proved by complete induction. The proof of property (ii) is direct from property (i) together with $\sup_{t_{n-1}\le t\le t_n}\|e(t)\| \le E_n$; $\forall n \in \mathbf{Z}_+$ and $\{E_n\} \to 0$ at exponential rate as $n \to \infty$ since $\|\Psi_{\mu(t_n)}(t_n, t)\| \le \psi_{\mu(t_n)}(t_n, t) < K_n$, $\limsup_{n\to\infty} K_n = K < 1$. □

Theorem 1 involves the assumption that $\theta \ge \theta^*$; that is, the total delay involved in the current system is not less than that of the nominal one subject to unmodeled dynamics. The above assumption is made for presentation clarity. Its removal is not difficult by replacing $\theta \to \overline{\theta} = \max(\theta, \theta^*)$ in the proof which would be the maximum delay appearing in the error in between the solutions $x(t)$ and $x_0(t)$ if any of the two situations $\theta \ge \theta^*$ or $\theta < \theta^*$ holds. Note that the definition of appropriate terms in the function f_1, θ_1, equal to or distinct from θ_0, or the integers defining maximum point and distributed numbers of delays in both differential equations, that is, $m_{(\cdot)}, m'_{(\cdot)}, m''_{(\cdot)}$ versus its nominal values, might allow the cancellation of some of the delayed dynamics contributions in the nominal system if suited. As a result of the case, $\theta < \theta^*$ can be also easily considered in the formulation.

Remark 2. It turns out that all fundamental matrices of form (10), satisfying (11), for any values of μ in the set $\{0,1\} \times \{0,1\} \overbrace{\cdots \quad \cdots}^{m_0} \{0,1\}$ are useful to construct the solution since the dynamics of the point delays which do not contribute to the homogeneous part of the solution are transferred to the forced solutions through contributing coefficients of the form $(1-\mu_i)$; $\forall i \in \overline{m}_0 = \{1, 2, \ldots, m_0\}$. Note also that the definition of the fundamental matrix dynamics (10) as a solution of (11) allows dealing with the stability conditions under more general condition than the approaches used in [14–20]. See also Remark 3 and Corollaries 4 and 5.

Remark 3. Note that if $\psi_{\mu(\cdot)}(\cdot,\cdot) : \mathbf{Z}_{0+} \times \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ is nonincreasing in $[t_n + \eta_n, t_{n+1}]$ and strictly decreasing in $(t_n, t_n + \eta_n)$, $\forall n \in \mathbf{Z}_{0+}$ for some positive real sequence $\{\eta_n\}$, with $\eta_n \le T^* \le T_n$, then the right-hand side of (12) is positive for all $n \in \mathbf{Z}_{0+}$ since $\psi_{\mu(t_n)}(t_n, t_n) = 1$. Thus, Theorem 1(i) is applicable if its fundamental matrix has a norm $\|\Psi_{\mu(t_n)}(t_n, t)\| < 1$ being nonincreasing in $[t_n + \eta_n, t_{n+1}]$ and strictly decreasing in $(t_n, t_n + \eta_n)$; $\forall n \in \mathbf{Z}_{0+}$. Theorem 1(ii) is applicable if $\|\Psi_{\mu(t_n)}(t_n, t)\| < K_n$; $\forall t \in [t_n, t_{n+1})$; $\forall n \in \mathbf{Z}_{0+}$ and it is strictly decreasing in time subintervals $(t_n, t_n + \eta_n)$; $\forall n \in \mathbf{Z}_{0+}$.

Direct sufficient conditions for the fulfilment of Theorem 1 are given in the subsequent result by using the upper-bounding function of the constraint for the potentially unknown nonlinear nonautonomous unmodeled dynamics contribution $\|f_0(t, x_{t-\theta_a})\| \le K_0 \sup_{0\le\delta\le h}\|x_{t-\theta_a-\delta}\| + K_1$.

Corollary 4. *The following properties hold.*

(i) Theorem 1(i) holds if the upper-bound of (12) is replaced with the following one:

$$\begin{aligned}&\left(1 - K_0 - \psi_{\mu(t_n)}(t_n, t)\right)\\&\times\left(\max_{t_n\le\tau\le t}\left(\psi_{\mu(t_n)}(t_n, \tau)\right.\right.\\&\qquad\times\left(K_1 + \sum_{i=1}^{m_0}\left(1-\mu_i(t_n)\right)\|A_i(\tau)\|\right.\\&\qquad+\sum_{i=0}^{m'_0}\left\|\int_0^\tau d\alpha_i(\sigma)\,A_{\alpha_i}(\sigma)\right\|\\&\qquad\left.\left.\left.+\sum_{i=m'_0+1}^{m_0+m''_0}\left\|\int_{\tau-h'_i}^{\tau} d\alpha_i(\tau-\sigma)\,A_{\alpha_i}(\sigma)\right\|\right)\right)\right)^{-1};\\&\qquad\forall t \in [t_n, t_{n+1}),\quad \forall n \in \mathbf{Z}_{0+}\end{aligned} \tag{21}$$

provided that $\psi_{\mu(t_n)}(t_n, t) < 1 - K_0$ with $K_0, K_1 \in \mathbf{R}_+$ being such that $\|f(t, x_{t-\theta_b})\| \le K_0 \sup_{0\le\delta\le h}\|x_{t-\theta_b-\delta}\| + K_1$; $\forall t \in \mathbf{R}_{0+}$.

(ii) Theorem 1(ii) holds if the upper-bound of (12) is replaced with

$$\begin{aligned}&\left(K - K_0 - \varepsilon - \psi_{\mu(t_n)}(t_n, t)\right)\\&\times\left(\max_{t_n\le\tau\le t}\left(\psi_{\mu(t_n)}(t_n, \tau)\right.\right.\\&\qquad\times\left(K_1 + \sum_{i=1}^{m_0}\left(1-\mu_i(t_n)\right)\|A_i(\tau)\|\right.\\&\qquad+\sum_{i=0}^{m'_0}\left\|\int_0^\tau d\alpha_i(\sigma)\,A_{\alpha_i}(\sigma)\right\|\\&\qquad\left.\left.\left.+\sum_{i=m'_0+1}^{m_0+m''_0}\left\|\int_{\tau-h'_i}^{\tau} d\alpha_i(\tau-\sigma)\,A_{\alpha_i}(\sigma)\right\|\right)\right)\right)^{-1};\\&\qquad\forall t \in [t_n, t_{n+1}),\quad \forall n \in \mathbf{Z}_{0+}\end{aligned} \tag{22}$$

provided that $\psi_{\mu(t_n)}(t_n, t) < K + \varepsilon - K_0$ with $K = \limsup_{n\to\infty} K_n < 1$; $\forall t \in [t_n + \gamma_n, t_{n+1})$ with $\gamma_n \ge \gamma > 0$; $\forall n (\ge n_0 = n_0(\varepsilon)) \in \mathbf{Z}_{0+}$ for any given $\varepsilon (< 1 - K) \in \mathbf{R}_+$.

A particular stability result of the perturbed system under that of the nominal one follows.

Corollary 5. *Consider the perturbed differential system $Lx_t \equiv L_0 x_t + f(t, x_{t-\theta})$ with linear nominal version subject to a single point delay $h_1 > 0$ of dynamics $L_0 x_t \equiv \sum_{i=0}^{1} A_i x(t - h_i)$, with $h_0 = 0$ and A_0 being a stability matrix of stability* *abscissa* $(-\rho_0) < 0$, *under any initial vector function satisfying hypothesis (H.1) with* $f(t, x_{t-\theta b}) = f_0(t, x_{t-\theta_a}) + f_1(t, x_{t-\theta_3})$ *with* $\|f(t, x_{t-\theta_b})\| \le K_0 \sup_{0\le\delta\le h}\|x_{t-\theta_b-\delta}\| + K_1$ *for some* $K_0, K_1 \in \mathbf{R}_{0+}$; $\forall t \in \mathbf{R}_{0+}$, *and* $f_0(t, x_{t-\theta_0})$ *being subadditive. Assume that*

(1) $|\mu_2(A_1)| < \rho_0$ *(or if* $\|\mu_2(A_1)\|_2 < \rho_0$*), where* $\mu_2(A_1)$ *is the 2-matrix measure of* A_1;

(2) $\psi_1(t_n, t) \le Ke^{-(\rho_0-\delta_0)(t-t_n)}$; $\forall t \in [t_n, t_{n+1})$, $\forall n \in \mathbf{Z}_{0+}$ *for any arbitrary* $\delta_0 \in \mathbf{R}_+$, $K = K_{\Psi_0}(1 + \|A_1\|_2/\rho_0)$ *and* $K_{\Psi_0} \in \mathbf{R}_+$ *being such that* $\|e^{A_0 t}\|_2 \le K_{\Psi_0} e^{-\rho_0 t}$; $\forall t \in \mathbf{R}_{0+}$;

(3) $0 \le K_0 < K(1 - e^{-\rho_0\gamma})$ *for some* $\gamma \in \mathbf{R}_+$;

(4) $\{T_n\}$ *is subject to* (13) *so that* (14) *takes the form*

$$z_n \le \frac{K - K_0 - \psi_1(t_n, t)}{K_1 \max_{t_n\le\tau\le t}\psi_1(t_n, \tau)}; \qquad \forall t \in [t_n + \gamma_n, t_{n+1}) \quad \text{with } \gamma_n \ge \gamma > 0 \tag{23}$$

for sufficiently large $n_0 \in \mathbf{Z}_{0+}$ *and* (15) *becomes* $h^*(t_n) = h_1$; $\forall n \in \mathbf{Z}_{0+}$.

Then, the nominal and perturbed solutions are bounded for all time and $x_0(t) \to 0$, $x(t) \to 0$, *and* $e(t) \to 0$ *exponentially fast as* $t \to \infty$ *so that the nominal and the perturbed differential systems are globally exponentially Lyapunov stable. The condition that* $\|e(t)\|$ *is bounded for all time and* $e(t) \to 0$ *exponentially fast as* $t \to \infty$ *still holds if condition 1 (global asymptotic stability of the nominal differential system) is removed.*

Proof. It turns out that if A_0 is a stability matrix of stability abscissa $(-\rho_0) < 0$ and $|\mu_2(A_1)| < \rho$ (or if $\|\mu_2(A_1)\|_2 < \rho_0$ since $|\mu_2(A_1)| \le \|A_1\|_2$) then the solution of $L_0 x_t \equiv \sum_{i=0}^{1} A_i x(t - h_i)$ is bounded for any initial solution subject to hypothesis (H.1). Furthermore, $x_0(t) \to 0$ as $t \to \infty$ exponentially for the fundamental matrix function for $\mu_1(t) \equiv 1$:

$$\begin{aligned}\Psi_1(t, t_n) &= e^{A_0(t-t_n)} \times \left(I + \int_{t_n}^{t} e^{A_0(t_n+\tau)} A_1 \Psi_1(\tau - h_1) U(\tau - h_1)\, d\tau\right); \\ &\forall t \in [t_n, t_{n+1}); \quad \forall n \in \mathbf{Z}_{0+}\end{aligned} \tag{24}$$

being the unique solution of

$$\begin{aligned}\dot{\Psi}_1(t, t_n) &= A_0 \Psi_1(t, t_n) + A_1(t) U(t - h_1) \Psi_1(t - h_1, t_n); \\ &\forall t \in [t_n, t_{n+1}), \forall n \in \mathbf{Z}_{0+}\end{aligned} \tag{25}$$

with $\Psi_1(t, t) = I$; $\forall t \in \mathbf{R}_0$ and $\Psi_1(t, t_n) = 0$ for $t < t_n$; $\forall n \in \mathbf{Z}_{0+}$ [16, 21–23] with $\|\Psi_1(t, t_n)\|_2 \le \psi_1(t_n, t)$; $\forall t \in [t_n, t_{n+1})$, $\forall n \in \mathbf{Z}_{0+}$. Furthermore,

(a) $\|e^{A_0 t}\|_2 \le K_{\Psi_0} e^{-\rho_0 t}$ for some $K_{\Psi_0} \in \mathbf{R}_+$; $\forall t \in \mathbf{R}_{0+}$ since A_0 is a stability matrix, and

(b)

$$\begin{aligned}&\left\|\Psi_1(t, t')\right\|_2 \\ &\le \left\|e^{A_0(t-t')}\right\|_2 \times \left(1 + \left\|\int_{t'}^{t} e^{A_0(t'+\tau)} A_1 \Psi_1(\tau - h_1) U(\tau - h_1)\, d\tau\right\|_2\right) \\ &\le K_{\Psi_0} e^{-\rho_0(t-t')}\left(1 + \frac{K_{\Psi_0}\|A_1\|_2}{\rho_0}\right) \\ &\le \left(K_{\Psi_0} + \widetilde{K}\right) e^{-(\rho_0-\delta_0)(t-t')} \\ &\le K_{\Psi_0} e^{-\rho_0(t-t')} + \widetilde{K} e^{-(\rho_0-\delta_0)(t-t')}\end{aligned} \tag{26}$$

for all $t(\ge t') \in \mathbf{R}_{0+}$ and any arbitrary $\delta_0 \in \mathbf{R}_+$ if $\widetilde{K} \ge K_{\Psi_0}\|A_1\|_2/\rho_0$. Thus,

$$\begin{aligned}&\left\|\Psi_1(t, t_n)\right\|_2 \\ &\le \psi_1(t_n, t) \le Ke^{-(\rho_0-\delta_0)(t-t_n)} \\ &\le K - K_0 = K_{\Psi_0}\left(1 + \frac{\|A_1\|_2}{\rho_0}\right) - K_0;\end{aligned} \tag{27}$$

$\forall t \in [t_n + \gamma_n, t_{n+1})$, $\forall n \in \mathbf{Z}_{0+}$ for $\gamma_n \ge \gamma > 0$ and sufficiently large $\gamma \in \mathbf{R}_+$. On the other hand, from Corollary 4 for $\mu_1(t) \equiv 1$, then leading to (23), one gets that $\|e(t)\|$ is bounded for all $t \in \mathbf{R}_{0+}$ for any initial solution of the perturbed differential system subject to hypothesis (H.1) and $e(t) \to 0$ as $t \to \infty$ exponentially fast. As a result, $\|x(t)\|_2$ is bounded for all $t \in \mathbf{R}_{0+}$ and $x(t) \to 0$ as $t \to \infty$ exponentially fast. □

Remark 6. Note that a close result to Corollary 5 may be formulated for the special case when $f_1(t, x_{t-\theta}) = \widetilde{A}_1(t)x(t - h_1)$ by replacing $K_0 \to K_{00}$, $K_1 \to K_{01}$ in (23) with $K_{00} \le K_0$ and $K_{01} \le K_1$ being such that $\|f_0(t, x_{t-\theta_a})\| \le K_{00} \sup_{0\le\delta\le h}\|x_{t-\theta_a-\delta}\| + K_{01}$.

Remark 7. A close result to Corollary 5 may be formulated if $(A_0 + A_1)$ is a stability matrix of stability abscissa $(-\rho) < 0$ by replacing $\rho_0 \to \rho$ in conditions 1–3 and $K_1 \to 2K_1$ in condition 4. In the case that $f_1(t, x_{t-\theta}) = \widetilde{A}_1(t)x(t - h_1)$ (see Remark 6) a further close result is obtained by replacing $K_1 \max_{t_n\le\tau\le t}\psi_1(t_n, \tau) \to 2 \max_{t_n\le\tau\le t}(\psi_1(t_n, \tau) + \|\widetilde{A}_1(\tau)\|_2)$ in condition 4 of Corollary 5 since $\|A_1(x(t - h_1) - x(t))\|_2 \le 2\|A_1\|_2\|x_t\|_2$ with x_t being the string of solution in $[t - h_1, t]$. The associated proofs are direct from Corollary 5 by rewriting the nominal differential system as $L_0 x_t \equiv (A_0 + A_1)x(t) + A_1(x(t - h_1) - x(t))$.

Remark 8. A further close result to Corollary 5 can be obtained by considering the nominal differential system to be $L_0 x_t \equiv A_0 x(t)$ so that $A_1 x(t-h_1)$ defines a perturbed delayed dynamics. In this case, we consider the fundamental matrix to be $\Psi_0(t,t_n)=e^{A_0(t-t_n)}$ with $\|\Psi_0(t_n,t)\|_2 \le \psi_0(t,t_n) \le K_{\psi_0}e^{-\rho_0 t}$; $\forall t \in [t_n,t_{n+1})$, $\forall n \in \mathbf{Z}_{0+}$. Conditions 1–3 are restated with the replacement $\psi_1(t_n,\tau) \to \psi_0(t_n,\tau)$ for $t \in [t_n,t_{n+1})$, $\forall n \in \mathbf{Z}_{0+}$ while constraint (23) of condition 4 is changed to

$$z_n \le \frac{K-K_0-\psi_0(t_n,t)}{\max_{t_n\le\tau\le t}\left[\psi_0(t_n,\tau)\left(K_1+\left\|A_1+\widetilde{A}_1(\tau)\right\|_2\right)\right]}; \quad \forall t \in [t_n,t_{n+1}), \quad \forall n \in \mathbf{Z}_{0+}. \tag{28}$$

Then, the nominal and perturbed solutions are bounded for all time and $x_0(t) \to 0$, $x(t) \to 0$, and $e(t) \to 0$ exponentially fast as $t \to \infty$ so that the nominal and the perturbed differential systems are globally exponentially Lyapunov stable under conditions 1–3 and condition 4 modified with (28). The condition that $\|e(t)\|_2$ is bounded for all time and $e(t) \to 0$ exponentially fast as $t \to \infty$ still holds if condition 1 (global asymptotic stability of the nominal differential system) is removed. Variants of this result under the considerations of Remarks 6 and 7 are direct.

Note that the above results imply that both the nominal system and the current perturbed one have trajectory solutions which converge asymptotically to zero under any initial conditions. It is easy to see that $x = 0$ is a globally asymptotically stable equilibrium point and also a fixed point of the state-trajectory solution, under the various conditions of Corollary 5 as well as their variants in Remarks 3–8; that is, it is a globally stable attractor for all the trajectory solutions. The relevance of fixed point theory in stability of perturbed differential systems has been also emphasized in some background literature. See, for instance, [10, 20, 24, 26–28] and references therein. On the other hand, Hyers-Ulam stability has been also invoked in difference-type linear and nonlinear equations as, for instance, in [25] and several background references therein. Note that difference equations are sometimes got from the discretization of continuous-time system either via the use of numerical tools or by the use of physical sampling and hold devices and that the stability of such discretized systems can be, in general, either studied independently of that of their continuous-time counterparts, via "ad hoc" discrete analysis methods, or based with the stability properties of the continuous-time version with extra conditions on the sequence of sampling instants (see, e.g., [22]).

If $\widetilde{g} = g - g_0$ is nonzero, then the extension of all the above results is direct for the cases that it is either bounded on $\mathbf{R}_{0+}$, if the fundamental matrix is absolutely integrable, or square-integrable, if the fundamental matrix is absolutely integrable on $\mathbf{R}_{0+}$. Note that if the nominal differential system is exponentially stable, then a fundamental matrix is of exponential negative order, and then both absolutely integrable and square-integrable exist. Then one has the following result.

Corollary 9. *Assume that the nominal differential system is globally exponentially stable and $\widetilde{g} : \mathbf{R}_{0+} \to \mathbf{R}^n$ is either bounded or integrable or square-integrable. Then, a fundamental matrix $\Psi_{\mu(t_n)}(t_n,t)$ exists such that Theorem 1, Corollaries 4 and 5, and their extensions of Remarks 3–8 still hold for some $E > \|\int_{t_n}^{t}\Psi_\mu(t,\tau)\widetilde{g}(\tau)d\tau\|$ if the needed denominator of (21) to (23) is, in each case, corrected with the additive term $\|\int_{t_n}^{t}\Psi_\mu(t,\tau)\widetilde{g}(\tau)d\tau\|$ or with any of its upper-bounds:*

$$\max_{t_n\le\tau\le t}\psi_\mu(t,\tau)\left\|\int_{t_n}^{t}\widetilde{g}(\tau)\,d\tau\right\|;$$
$$\max_{t_n\le\tau\le t}\|g(\tau)\|\left\|\int_{t_n}^{t}\Psi_\mu(t,\tau)\,d\tau\right\|; \tag{29}$$
$$\left(\left\|\int_{t_n}^{t}g(\tau)\,d\tau\right\|^2\right)^{1/2}\left(\left\|\int_{t_n}^{t}\Psi_\mu(t,\tau)\,d\tau\right\|^2\right)^{1/2}.$$

Outline of Proof. Note that (20a) is modified as follows:

$$\begin{aligned}\|\dot{e}(t)-Le_t\| &= \left\|\int_{t_n}^{t}\Psi_\mu(t,\tau)\,\widetilde{g}(\tau)\,d\tau\right\| \\ &\le \left\|(L-L_0)\,x_{0t}\right\| + \Lambda(t_n,t)\end{aligned} \tag{30}$$

for some nonnegative real function $\Lambda : (\mathbf{Z}_{0+}\times\mathbf{R}_{0+})\times\mathbf{R}_{0+} \to \mathbf{R}_{0+}$ such that $\Lambda(t_n,t)$ is of some of the forms of (29) leading to $\Lambda(t_n,t) \to 0$ and $\|(L-L_0)x_{0t}\| + \Lambda(t_n,t) \to 0$ for $t \in [t_n,t_{n+1})$ as $n \to \infty$; the nominal differential system is globally exponentially stable and $\widetilde{g} : \mathbf{R}_{0+} \to \mathbf{R}^n$ is either bounded or integrable or square-integrable.

Conflict of Interests

The author declares that he has no conflict of interests.

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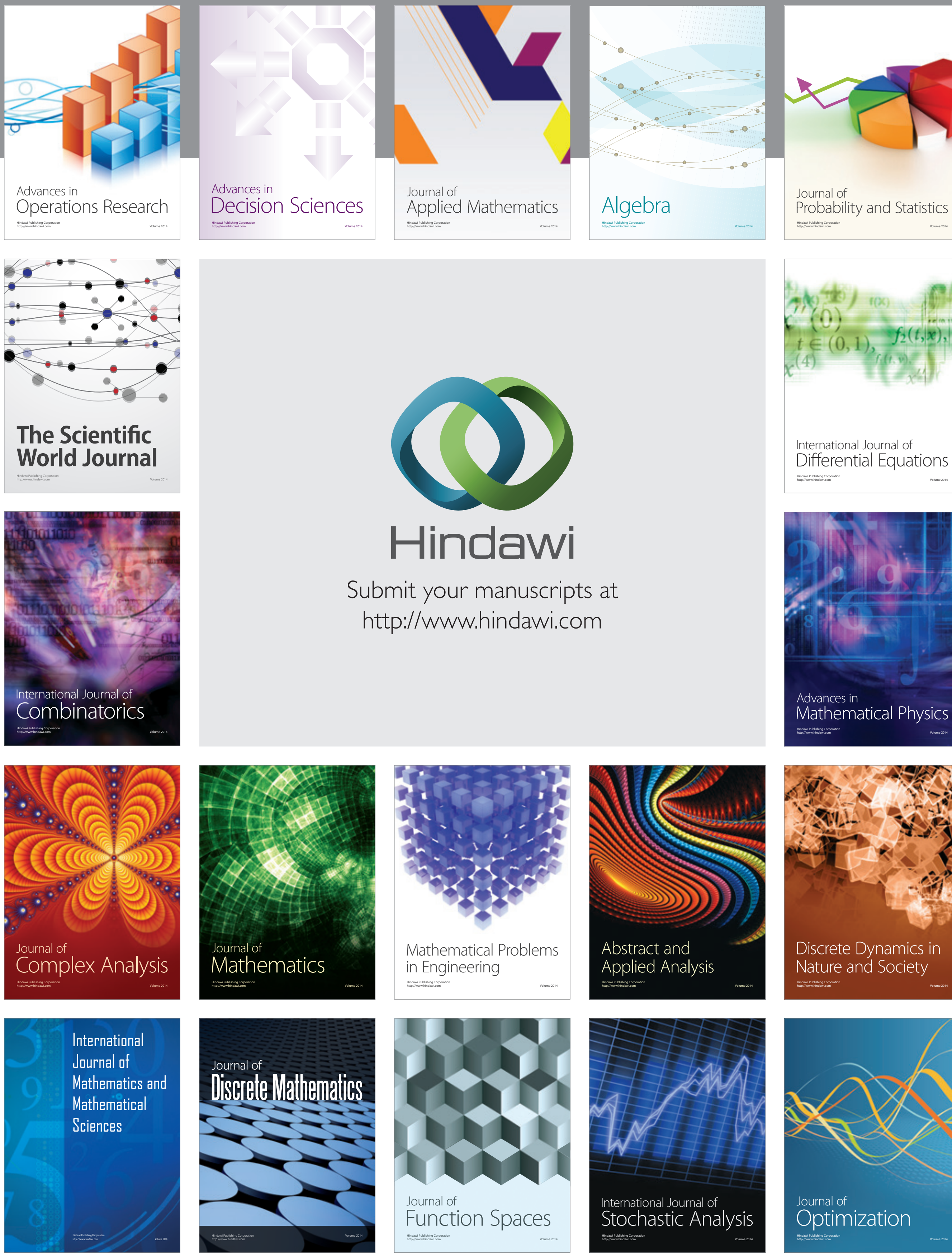

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