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THE SD-PREKERNEL FOR TU GAMES

by

J. Arín and I. Katsev

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University of the Basque Country

The SD-prekernel for TU games

J. Arin *and I. Katsev†

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Abstract

We introduce and analyze a new solution concept for TU games: The Surplus Distributor Prekernel. Like the prekermel, the new solution is based on an alternative motion of complaint of one player against other with respect to an allocation. The SD-prekernel contains the SD-prenucleolus and they coincide in the class of convex games. This result allows us to prove that in bankruptcy problems the SD-prekernel and the Minimal Overlapping rule select the same allocation.

Keywords: TU games, prenucleolus, prekernel.

*Dpto. Ftos. A. Económico I, University of the Basque Country, L. Agirre 83, 48015 Bilbao, Spain. Email: franciscojavier.arin@ehu.eus.

†National Research University Higher School of Economics, Soyza Pechatnikov str., 16, St. Petersburg, Russian Federation. Email: katsev@yandex.ru

1 Introduction

In the literature of TU games, the prenucleolus (Schmeidler, 1969) is one of the most important single-valued solution concepts. The prenucleolus of a game is contained in its prekernel, which also makes the study of this solution concept attractive.

The prenucleolus for TU games is a lexicographic value that selects the vector of satisfactions of coalitions, which lexicographically dominates any other vector of satisfactions of coalitions. When this vector is selected its associated allocation is automatically selected and this proves to be the prenucleolus of the game.

The prekernel selects allocations where each pair of players is fairly treated since a complaint by a player i against player j equals a complaint by j against i. A complaint by i against j is the minimal satisfaction obtained by a coalition containing i and excluding j. In some cases the prekernel is single-valued and therefore coincides with the prenucleolus, which provides an alternative definition of the prenucleolus. This is the case for convex games (Maschler*et al* (1972) and veto balanced games (Arin and Feltkamp, 1997).

When the satisfactions of coalitions are weighted by using a system of weights for the size of the coalitions different weighted prenucleoli and weighted prekernels may be defined. In the per capita prenucleolus satisfactions are divided by the size (cardinality) of the coalition. In this way, the satisfaction of the coalition is divided equally among its members. Each member of the coalition receives the same part of the total surplus of the coalition (the difference between the total payoff received by the coalition and its worth).

Arin and Katsev (2014) propose a different way of computing the satisfactions of coalitions given an allocation. Similarly to the per capita prenucleolus, a division of the surplus of the coalition among its members is proposed. By contrast with the per capita prenucleolus, we do not consider an equal division of the surplus of coalitions.

Once a new way of computing the vector of satisfactions associated with an allocation is drawn up, the SD-prenucleolus is defined in identical terms by considering the allocations whose vectors of satisfactions dominate any other vector of satisfactions.

A new solution concept, the SD-prekernel, can immediately be defined by considering the notion of a complaint associated with the new vector of satisfactions. The aim of this paper is to introduce and analyze this new solution for TU games: The SD-prekernel.

The new solution, in general, is neither single-valued nor a subset of the core. We prove that in the class of totally relevant games the SD-prekernel and the SD-prenucleolus coincide. Convex games are totally relevant games. These results are used to prove that the SD-prenucleolus of a bankruptcy game coincides with the outcome provided by the rule known as the Minimal Overlapping rule. Aumann and Maschler (1985) show that the prenucleolus of bankruptcy games and the Talmud rule provide the same outcomes.

The rest of the paper is organized as follows: Section 2 introduces TU games and the SD-prenucleolus. In Section 3 we define the SD-prekernel and show that, in general, it is neither single-valued nor a subset of the core. Section 4 proves that in the class of totally relevant games the SD-prekernel is single-valued. Section 5 ends the paper by showing that in the class of bankruptcy games the SD-prekernel and the Minimal Overlapping rule coincide.

2 Preliminaries

2.1 TU Games

A *cooperative n-player game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \rightarrow \mathbb{R}$ is a real-valued function in the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v is called the characteristic function of the game. Any subset S of N is called a *coalition*. Singletons are coalitions that contain only one player. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players, an allocation, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(N, v)$. We denote $\sum_{i \in S} x_i$ by $x(S)$. The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(N,v) = \{x \in X(N,v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

It has been shown that a game with a nonempty core is balanced (Shapley (1967) and Bondareva (1963)), and games with nonempty core are therefore called balanced games. *Player i is a veto player if* $v(S) = 0$ for all S where player i is not present. A balanced game with at least one veto player is called a veto balanced game.

A game is *balanced* when it has a nonempty core (Bondareva (1963) and Shapley (1967)). We say that a game (N, v) is *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subseteq N.$$

A solution φ on a class of games Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in $\mathbb{R}^N$with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a *single* element for each game in the class.

Given $x \in \mathbb{R}^N$ the *satisfaction of a coalition S with respect to x* in a game (N, v) is defined as $f(S, x) := x(S) - v(S)$.

Given $x \in \mathbb{R}^N$ the *satisfaction of a coalition S with respect to x* in a game (N, v) is defined as $f(S, x) := x(S) - v(S)$. Let $\theta(x)$ be the vector of all satisfactions at x arranged in nondecreasing order. The weak lexicographic order $\succeq_L$ between two vectors x and y is defined by $x \succeq_L y$ if there is an index k such that $x_l = y_l$ for all $l < k$ and $x_k > y_k$ or $x = y$.

Schmeidler (1969) introduced the *prenucleolus* of a game (N, v), denoted by $PN(N, v)$, as the unique allocation that lexicographically maximizes the vector of nondecreasingly ordered satisfactions[1] over the set of allocations. In formula:

$$PN(N,v) = \{x \in X(N,v)\, | \theta(x) \succeq_L \theta(y) \text{ for all } y \in X(N,v)\}.$$

The prenucleolus is single-valued and selects a core allocation whenever the game is balanced.

[1]The same solution concept can be defined using the notion of excess instead of satisfaction. Given a game (N, v) and an allocation x, the *excess of a coalition S with respect to x* in game (N, v) is defined as follows: $e(S, x) := v(S) - x(S)$.

The per capita prenucleolus is defined in a similar way by using the notion of per capita satisfaction instead of satisfaction. Given S, $S \neq \emptyset$, and x the per capita satisfaction of S at x is

$$f^{pc}(S,x) := \frac{x(S) - v(S)}{|S|}.$$

Other weighted prenucleoli can be defined in a similar way whenever a weighted satisfaction function is defined.

Given a TU game (N,v) and an allocation $x \in X(N,v)$ the complaint of player i against player j is defined as follows:

$$s_{ij}(x) = \min_{S: i \in S, j \notin S} f(S,x).$$

The prekernel of a TU game (N,v) is:

$$PK(N,v) = \{x \in X(N,v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i \neq j\}$$

Similarly, the per capita prekernel can be defined using per capita satisfactions when defining the complaints.

Some convenient and well-known properties of a solution concept φ on Γ_0 are the following:

- φ satisfies **anonymity** if for each (N,v) in Γ_0 and each bijective mapping $\tau : N \longrightarrow N$ such that $(N,\tau v)$ in Γ_0 it holds that $\varphi(N,\tau v) = \tau(\varphi(N,v))$ (where $\tau v(\tau T) = v(T)$, $\tau x_{\tau(j)} = x_j$ $(x \in \mathbb{R}^N, j \in N, T \subseteq N)$). In this case (N,v) and $(N,\tau v)$ are equivalent games.

- φ satisfies the **equal treatment property (ETP)** if for each (N,v) in Γ_0 and for every $x \in \varphi(N,v)$ interchangeable players i,j are treated equally, i.e. $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$.

Given a game (N,v), a real number $\alpha > 0$ and a vector $\beta \in \mathbb{R}^N$ we define game $(N, v^{\alpha,\beta})$ as follows:

$$v^{\alpha,\beta}(S) = \alpha v(S) + \sum_{i \in S} \beta_i \text{ for all } S \subseteq N.$$

- φ satisfies **covariance** if the following condition is satisfied. If $\alpha > 0$, $\beta \in \mathbb{R}^N$ and $(N,v), (N,v^{\alpha,\beta}) \in \Gamma_0$ then $\varphi(N,v^{\alpha,\beta}) = \alpha\varphi(N,v) + \beta$.

2.2 The SD-prenucleolus

Arin and Katsev (2014) introduce the SD-prenucleolus, a single-valued solution concept for TU games. In this section we recall some definitions and results that are needed in the present paper. The definition of the SD-prenucleolus is based on the concept of satisfaction of a coalition with respect to an allocation. Given a game (N, v) and an allocation $x \in X(N, v)$ we calculate a new satisfaction vector $(F(S, x))_{S\subset N}$. We define the components of this vector recursively by defining an algorithm. The algorithm has several steps (at most $2^{|N|} - 2$) and at each step we identify the collection of coalitions $\mathcal{H}$ that has obtained the new satisfaction. In the first step this collection $\mathcal{H}$ is empty. The algorithm ends when $\mathcal{H} = 2^N \setminus \{N\}$.

For a collection $\mathcal{H} \subset 2^N \setminus \{N\}$ and a function $F : \mathcal{H} \to \mathbb{R}$ we will define the function $F_{\mathcal{H}} : 2^N \setminus \{\mathcal{H} \cup \{N\}\} \to \mathbb{R}$. To that end, we introduce some notation. For $\mathcal{H} \subset 2^N \setminus \{N\}$ and $S \subset N$, we denote

$$\sigma_{\mathcal{H}}(S) = \bigcup_{T\in\mathcal{H}, T\subset S} T.$$

For $S \subset N$ we denote by $f_{\mathcal{H},F}(i, S)$ the satisfaction of player i with respect to a coalition S and a collection $\mathcal{H}$ ($i \in \sigma_{\mathcal{H}}(S)$):

$$f_{\mathcal{H},F}(i, S) = \min_{T:T\in\mathcal{H}, i\in T\subset S} F(T).$$

Now we define a value $F_{\mathcal{H}}(S)$ for all $S \in 2^N \setminus \{\mathcal{H} \cup \{N\}\}$. We consider two cases (since it is evident that $\sigma_{\mathcal{H}}(S) \subseteq S$):

1. Relevant coalitions. $\sigma_{\mathcal{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum\limits_{i\in\sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S)}{|S| - |\sigma_{\mathcal{H}}(S)|}. \tag{1}$$

Note that if collection $\mathcal{H}$ is empty then the current satisfaction of coalition S coincides with its per capita satisfaction:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}.$$

2. Completed coalitions. $\sigma_{\mathcal{H}}(S) = S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i\in S} f_{\mathcal{H},F}(i, S) + \max_{i\in S} f_{\mathcal{H},F}(i, S). \tag{2}$$

The algorithm that computes the new satisfaction vector, whose components are denoted by $F(S) = F(S, x)$, is the following:

Algorithm 1 *Let (N, v) be a TU game and $x \in X(N, v)$.*

Step 1: *Set $k = 0$, $\mathcal{H}_0 = \emptyset$ and $F_\emptyset(S) = \frac{x(S)-v(S)}{|S|}$.*

Step 2: *Set*

$$\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{S \notin \mathcal{H}_k : F_{\mathcal{H}_k}(S) = \min_{T \notin \mathcal{H}_k} F_{\mathcal{H}_k}(T)\}.$$

Define for each $S \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$:

$$F(S, x) = F_{\mathcal{H}_k}(S).$$

Step 3: *If $\mathcal{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k + 1$ and go to Step 2. Otherwise go to Step 4.*

Step 4: *Stop. Return the vector $(F(S, x))_{S \subset N}$.*

Given a TU game (N, v) and $x \in X(N, v)$, we say that a coalition $S \subset N$ is *relevant* (*completed*) with respect to x if S was a relevant (completed) coalition at the step where its satisfaction $F(S, x)$ was determined. Given a TU game (N, v), a player $i \in S \subset N$ and $x \in X(N, v)$ we denote by $f_i(S, x) = \min_{T:i \in T \subseteq S} F(T, x)$ and $z_i(S, x) = x_i - f_i(S, x)$ the *satisfaction* and the *coalitional payoff of player i in coalition S at x*. If there is no confusion we write $f_i(S)$ and $z_i(S)$ instead of $f_i(S, x)$ and $z_i(S, x)$. Arin and Katsev (2014) prove the following lemma which is used in the proof of Lemma **??**.

Definition 1 *Let (N, v) be a TU game. Then $x \in SD(N, v)$ if and only if for any $y \in X(N, v)$ it holds that $F^x \succeq_L F^y$.*

A TU game is relevant with respect to an allocation x (we say that the game is x-relevant) if all coalitions are relevant with respect to x. Formally,

Definition 2 *Let (N, v) be a TU game and x be an allocation. We say that (N, v) is relevant with respect to x (x-relevant) if all coalitions are relevant with respect to x.*

Definition 3 *A TU game (N, v) is totally relevant if the game is x-relevant for each $x \in X(N, v)$..*

The two notions play a central role in the proof of the main theorem of this paper.

3 The SD-prekernel

The SD-prekernel arises naturally whenever this new satisfaction vector is considered. We define the complaint by a player i against a player j as the minimal satisfaction obtained with coalitions that contain player i but not player j.

Given a TU game (N,v) and an allocation $x \in X(N,v)$ the complaint by player i against player j, denoted by $s_{ij}(x)$, is defined as follows:,

$$s_{ij}(x),(x) = \min_{S:i\in S, j\notin S} F(S,x).$$

Unlike the prekernel, the following remarks identify coalitions that may be used as complaints by the players.

Remark 1 *Let (N,v) be a game and $x \in X(N,v)$ Then* $\min_{S:i\in S, j\notin S} F(S,x) = f_i(N\backslash\{j\},x)$.

Remark 2 *If $T \in \arg\min_{S:i\in S, j\notin S} F(S,x)$ then T is relevant with respect to x.*

We now introduce the SD-prekernel, denoted by $SD\text{-}PK$.

Definition 4 *Let (N,v) be a TU game. Then*

$$SD\text{-}PK(N,v) = \{x \in X(N,v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i \neq j\}$$

The SD-prekernel satisfies equal treatment of equals, covariance and anonymity. The SD-prenucleolus of a game is contained in its SD-prekernel. In some cases, this inclusion is strict.

Example 1 *Consider a 4-player game (N,v) defined as follows:*

$$v(S) = \begin{cases} 4 & \text{if } S = N \\ 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1,2\},\{3,4\}\} \\ 2 & \text{otherwise.} \end{cases}$$

It can be checked that $SD\text{-}PK(N,v) = \{(x,x,2-x,2-x) : 0 \leq x \leq 2\}$ and that the $SD(N,v) = (1,1,1,1)$.

Next example shows that in some cases the SD-prekernel is multivalued while the prekernel is single-valued.

Example 2 *Consider a 4-person glove game (N, v) defined as follows:*

$$v(S) = \begin{cases} 24 & \text{if } S = N \\ 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1,2\}, \{3,4\}\} \\ 2 & \text{otherwise.} \end{cases}$$

Note that $SD\text{-}PK(N, v) = \{(5 + x, 5 + x, 7 - x, 7 - x) : 0 \leq x \leq 2\}$ [2] and that $SD(N, v) = (6, 6, 6, 6)$ that is the only allocation contained in the prekernel..

In the class of two-player games the SD-prekernel coincides with the SD-prenucleolus and the prenucleolus. Let $(\{i, j\}, v)$ be a two-player game Then

$$SD\text{-}PK(\{i, j\}, v) = \{(v(\{i\} + \alpha, v(\{j\} + \alpha)\}$$

where $\alpha = \frac{v(\{i,j\} - v(\{i\}) - v(\{j\})}{2}$.

This result follows from the fact that the SD-prekernel satisfies efficiency, equal treatment of equals and covariance.

Finally, we show that, in general, the SD-prekernel does not need to be a subset of the core.

Example 3 *Let $N = M_1 \cup M_2 = \{1, 2, 3\} \cup \{4, 5\}$ and consider a 5-person glove game (N, v) defined as follows:*

$$v(S) = \begin{cases} 0 & \text{if } S = N \\ 0 & \text{if } |S| = 3 \text{ and } |S \cap M_2| = 1 \\ -10 & \text{otherwise.} \end{cases}$$

The game is balanced since $(0, 0, 0, 0, 0) \in C(N, v)$. Note that $x = (-2, -2, -2, 3, 3) \in SD\text{-}PK(N, v)$ since $F(S, x) = -\frac{1}{6}$ if $v(S) = 0$, $S \neq N$, and the rest of coalitions have a positive satisfaction.

4 Totally relevant games

In this section we prove the coincidence of the SD-prekernel and the SD-prenucleolus in the class of totally relevant games . This class contains the class of convex games that includes the class of bankruptcy games.

[2] Let $(N, v), (N, w)$ be two games such that $w(N) > v(N)$ and $w(S) = v(S)$ if $S \neq N$. If $x \in SD\text{-}PK(N, v)$ then $x + (\frac{w(N)-v(N)}{|N|}, .., \frac{w(N)-v(N)}{|N|}) \in SD\text{-}PK(N, w)$.

The proof of the result requires some previous results that are introduced below in two subsections. The first subsection investigates the SD-reduced game property, a property that is satisfied by the SD-prekernel. The second subsection establishes that when totally relevant games are considered, the set of coalitions with minimal satisfaction with respect to a prekernel allocation contains an antipartition.

4.1 The SD-reduced game property

Arin and Katsev (2014) introduce the notions of SD-reduced game and SD-reduced game property.

Definition 5 *Let (N,v) be a TU game, $S \subset N$ and $x \in X(N)$. A game (S, v_S^x) is the SD-reduced game with respect to S and x if*

1. $v_S^x(S) = v(N) - x(N \setminus S)$

2. for every $T \subsetneq S$

$$F^{(S,v_S^x)}(T, x_S) = \min_{U \in N \setminus S} F^{(N,v)}(U \cup T, x).$$

For any game (N,v), $S \subset N$ and $x \in X(N)$ the SD-reduced game (S, v_S^x) exists and is unique.

Definition 6 *A solution ϕ satisfies the SD-reduced game property on Γ, SD-RGP, if for every game $(N,v) \in \Gamma$ then $(x_i)_{i \in S} \in \phi(S, v^x)$ for any $S \subset N$ and any $x \in \phi(N,v)$.*

This type of property[3] plays a determinant role in the characterization of lexicographic values such as the prenucleolus (Sobolev, 1975) and the per capita prenucleolus (Kleppe, 2010). The reduced games associated with the prenucleolus and the per capita prenucleolus can be reformulated explicitly.

The fact that the SD-prekernel satisfies the SD-RGP is immediately aperient and widely use din the proofs of the main results of this paper.

[3]Note that the definition of this reduced game depends on the definition of the vector of satisfactions. If the vector of satisfactions considered is $(x(S) - v(S))_{S \subset N}$ then the associated reduced game property is satisfied by the prenucleolus (see Theorem 5.2.7 in Peleg and Sudholter (2007)). See also Peleg (1986) for a characterization of the prekernel using this reduced game property.

Lemma 1 *The SD-prekernel satisfies the SD-reduced game property.*

Below, we show that SD-reduced games of totally relevant games are totally relevant.

Lemma 2 *Let (N, v) be a game, x be an allocation, S be a coalition and $x_S = (x_i)_{i \in S}$. If (N, v) is x-relevant then (S, v^x) is x_S-relevant.*

Proof. Let $x_S = (x_i)_{i \in S}$ and let P and M be two subsets of S such that $M \cup P \neq S$. Assume, without loss of generality, that $F(M, x_S) \geq F(P, x_S)$. We seek to prove that

$$F(M \cup P, x_S) \leq \max \{F(M, x_S), F(P, x_S)\} = F(M, x_S).$$

Let $F(M, x_S) = F(M \cup Q, x)$ where $Q \subseteq N \backslash S$ and let $F(P, x_S) = F(P \cup T, x)$ where $T \subseteq N \backslash S$. Note that $(M \cup Q) \cup (P \cup T) \neq N$.

Since all coalitions in the game (N, v) are relevant with respect to x,

$$F((M \cup Q) \cup (P \cup T), x) \leq$$

$$\max \{F(M \cup Q, x), F(P \cup T, x)\} = F(M \cup Q, x).$$

Note that $(M \cup Q) \cup (P \cup T) = (M \cup P) \cup (Q \cup T)$. Therefore,

$$F(M \cup P, x_S) \leq F((M \cup Q) \cup (P \cup T), x) \leq$$

$$\leq F(M \cup Q, x) = F(M, x_S).$$

Consequently, (S, v^x) is y-relevant. ■

An immediate consequence of this lemma is that if a game is totally relevant then its SD-reduced games (with respect to any allocation) are totally relevant.

Lemma 2 allows for a different interpretation of the SD-reduced game of an x-relevant game. SD-reduced games with respect to x can be easily computed according to the result established by the following lemma.

Lemma 3 *Let (N, v) be a relevant game with respect to x and $S \subset N$. Consider the SD-reduced game (S, v^x) and $T \subset S$. Then*

$$v^x(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S)) = \sum_{i \in T} z_i(T \cup (N \setminus S)).$$

Proof. By Lemma 2, (S, v^x) is x_S-relevant. For game (S, v^x) we denote by $f^{x,S}$ the analog of function f in Algorithm 1. By definition of $f_i^{x,S}(T)$ it holds that

$$f_i^{x,S}(T) = \min_{i\in U\subseteq T} F^{(S,v^x)}(U) = \min_{i\in U\subseteq T} \min_{R\subseteq N\setminus S} F(U\cup R) =$$
$$= \min_{i\in M\subseteq T\cup(N\setminus S)} F(M) = f_i(T\cup(N\setminus S)).$$

Hence,

$$v^x(T) = x(T) - \sum_{i\in T} f_i^{x,S}(T) = x(T) - \sum_{i\in T} f_i(T\cup(N\setminus S)) =$$
$$= \sum_{i\in T} z_i(T\cup(N\setminus S)).$$

Since coalition $T\cup(N\setminus S)$ is relevant in the game (N, v), by Lemma **??** $v(T\cup(N\setminus S)) = \sum_{i\in T\cup(N\setminus S)} z_i(T\cup(N\setminus S))$. Therefore,

$$\sum_{i\in T} z_i(T\cup(N\setminus S)) = v(T\cup(N\setminus S)) - \sum_{i\in N\setminus S} z_i(T\cup(N\setminus S)).$$

■

The next corollary presents a simple formula for computing some SD-reduced games. This result is used in the proof of the main theorem.

Let (N, v) be a TU game and $x \in X(N, v)$. We denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction with respect to x.

Corollary 1 *Let (N, v) be an x-relevant game. Let $S \in \mathcal{B}(x)$ and consider the SD-reduced game $(N\setminus S, v^x)$ and coalition $T \subset N\setminus S$. Then $v^x(T) = v(T\cup S) - v(S)$.*

Proof. Since $S \in \mathcal{B}(x)$, it holds that $f_i(S) = \frac{x(S)-v(S)}{|S|}$. Since (N, v) is x-relevant, for any $i \in T$ such that $S \subset T$ it holds that $f_i(T) = f_i(S)$. By applying Lemma 3,

$$v(T\cup S) - v^x(T) = \sum_{i\in S} z_i(T\cup S) = \sum_{i\in S} x_i - \sum_{i\in S} f_i(T\cup S) =$$
$$\sum_{i\in S}(x_i - f_i(S)) = x(S) - |S|\,\frac{x(S)-v(S)}{|S|} = v(S).$$

■

4.2 Antipartition

The notion of antipartition (Arin and Inarra, 1998) also plays a central role in the proof of the main result of this paper.

A collection of sets $\mathcal{C} = \{S : S \subset N\}$ is called *antipartition* if the collection of sets $\{N \backslash S : S \in \mathcal{C}\}$ is a partition of N. An antipartition is a balanced collection of sets.

In order to balance an antipartition $\mathcal{Q}$ each coalition receives the same weight, i.e. $\frac{1}{|\mathcal{C}|-1}$.

For any game (N, v) the *satisfaction of an antipartition* $\mathcal{C}$ is defined by

$$\mathcal{F}(\mathcal{C}, v) = \frac{v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S)}{|N|}.$$

If there is no confusion we write $\mathcal{F}(\mathcal{C})$ instead of $\mathcal{F}(\mathcal{C}, v)$.

Let (N, v) be a TU game and x be an allocation. Denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction at x.

The next lemma, proved in Arin and Katsev (2015), shows that the satisfaction of an antipartition can be easily computed.

Lemma 4 *Let (N, v) be a TU game and $x \in X(N, v)$. If $\mathcal{B}(x)$ contains an antipartition $\mathcal{C}$ then $F(S) = \mathcal{F}(\mathcal{C})$ for all $S \in \mathcal{B}(x)$.*

Note that if the set of coalitions with minimal satisfaction with respect to an allocation of the SD-prekernel of the game contains an antipartition then the satisfaction of those coalitions only depends on the characteristic function of the game.

Lemma 5 *Let (N, v) be a TU game and let $x \in SD\text{-}PK(N, v)$. If (N, v) is x-relevant then $\mathcal{B}(x)$ contains an antipartition.*

Proof. Let S be a maximal coalition in $\mathcal{B}(x)$, that is, there is no coalition T in $\mathcal{B}(x)$ such that $S \subset T$. Since $x \in SD\text{-}PK(N, v)$, for each $i \in S$ there exists a coalition, T^i, such that $i \notin T^i$ and $T^i \in \mathcal{B}(x)$. Since (N, v) is x-relevant, the maximality of S implies that $N \backslash S \subset T^i$. Let $\{T^i : i \in S\}$ be the set of maximal coalitions for each i in S ((it may occur that $T^i = T^j$ for players i, j). Then $\{T^i : i \in S\} \cup \{S\}$ is an antipartition. It is immediately apparent that $(N \backslash T^i) \cap (N \backslash S)$ is empty. If for any $i, j \in S$ it holds that $(N \backslash T^i) \cap (N \backslash T^j)$ is nonempty then $T^i \cup T^j \neq N$ which contradicts the maximality of T^i and T^j since the fact that (N, v) is x-relevant implies that $T^i \cup T^j$ is in $\mathcal{B}(x)$. ■

4.3 The SD-prekernel and the SD-prenucleolus

With the results above, we are in a position to establish one of the main theorem of this paper.

Theorem 2 *Let (N, v) be a totally relevant game. Then SD-$PK(N, v) = \{SD(N, v)\}$.*

Proof. *Let $x \in SD$-$PK(N, v)$ and assume that there exists y, $y \neq x$, such that $y \in SD$-$PK(N, v)$. By Lemma 5 $\mathcal{B}(y)$ and $\mathcal{B}(x)$ contain an antipartition. Since the satisfaction of an antipartition only depends on the characteristic function, $\mathcal{B}(x)$ and $\mathcal{B}(y)$ contain the same antipartition. And for any coalition S in the antipartition it must hold that $y(S) = x(S)$. Finally, the SD-reduced games (S, v^y) and (S, v^x) coincide and since the SD-prekernel satisfies SD-reduced game property, $(x_i)_{i \in S}$ and $(y_i)_{i \in S}$ belong to SD-$PK(S, v^x)$. By Lemma 3 (S, v^x) is relevant with respect to x and y. Consider the game (S, v^x). By Lemma 5 $\mathcal{B}((y_i)_{i \in S})$ and $\mathcal{B}((x_i)_{i \in S})$ contain an antipartition. Since the satisfaction of an antipartition only depends on the characteristic function, they contain the same antipartition. And for any coalition T in the antipartition it must hold that $y(S) = x(S)$. Finally, the SD-reduced games (T, v^y) and (T, v^x) coincide and since the SD-prekernel satisfies the SD-reduced game property, $(x_i)_{i \in S}$ and $(y_i)_{i \in S}$ belong to SD-$PK(T, v^x)$. By Lemma 3 (T, v^x) is relevant with respect to x and y. The argument is repeated until the SD-reduced games are two-player games.* ■

Arin and Katsev (2015) prove that convex games are totally relevant.

As a corollary of the proof of Theorem 2, note that if given a TU game (N, v), if $x, y \in SD$-$PK(N, v)$ then (N, v) is either not x-relevant or not y-relevant.

Finally, we show that there are not totally relevant TU games with a single-valued SD-prekernel.

Example 4 *Consider a 3-player game (N, v) defined as follows:*

$$v(S) = \begin{cases} 3 & \text{if } S = N \\ 0 & \text{if } |S| = 1 \\ -2 & \text{otherwise.} \end{cases}$$

It can be checked that SD-$PK(N, v) = \{(1, 1, 1)\}$ and that the the game is not relevant with respect to $(1, 1, 1)$.

5 Bankruptcy games

In this section we prove that in the class of bankruptcy games (O'Neill, 1982) the SD-prenucleolus coincides with the Minimal Overlapping rule. Next, we introduce bankruptcy problems, bankruptcy games and the Minimal Overlapping rule.

Bankruptcy problems model situations where a finite set of claimants need to share an endowment that is not enough to satisfy fully all their claims. Formally, consider an infinite set of potential claimants, indexed by the natural numbers $\mathbb{N}$. Each given bankruptcy problem involves a finite number of claimants. Let $\mathcal{N}$ denote the class of non-empty finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $i \in N$, let c_i be claimant i's claim and $c \equiv (c_i)_{i\in N}$ the claims vector and let E be the endowment to be divided among the claimants in N. A *bankruptcy problem* (or just *problem*) is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $N \in \mathcal{N}$, such that $\sum_{i\in N} c_i \geq E$. Let $\mathcal{B}^N$ denote the class of all problems with claimants set N. An *allocation* for $(c, E) \in \mathcal{B}^N$ is a vector $x \in \mathbb{R}^N$ such that satisfies the non-negativity and claim boundedness conditions, *i.e.* $0 \leqq x \leqq c$ and the efficiency condition $\sum_{i\in N} x_i = E$.[4] Let $X(c, E)$ be the set of allocations of (c, E). A *bankruptcy rule* (or just *rule*) is a function defined on $\cup_{N\in\mathcal{N}}\mathcal{B}^N$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$ an allocation in $X(c, E)$. For each $N' \subset N$ we denote by $c_{N'}$ the claim vector of claimants in N'. Similarly, $\phi_{N'}(c, E) = (\phi_i(c, E)_{i\in N'}$.

Given a problem $(c, E) \in \mathcal{B}^N$, we denote by $C = \sum_{i\in N} c_i$ and $L = C - E$ the total claim and total loss respectively.

- The constrained equal awards rule divides the endowment equally among the claimants under the constraint that no claimant receives more than his/her claim.

  Constrained equal awards rule, CEA: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

  $$CEA_i(c, E) \equiv \min\{\beta, c_i\} \text{ where } \beta \in \mathbb{R}_+ \text{ solves } \textstyle\sum_{i\in N} \min\{\beta, c_i\} = E.$$

- The constrained equal losses rule divides the total loss equally among the claimants under the constraint that no claimant receives a negative amount.

[4]The notation $x \leqq y$ means that for each $i \in N$, $x_i \leq y_i$.

Constrained equal losses rule, CEL: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$CEL_i(c, E) \equiv \max\{0, c_i - \beta\} \text{ where } \beta \in \mathbb{R}_+ \text{ solves } \sum_{i \in N} \max\{0, c_i - \beta\} = E.$$

The Minimal Overlapping rule for Bankruptcy problems.

- The **Minimal overlapping rule, MO:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$MO_i(c, E) = \begin{cases} CEA_i(SR(c), E) & \text{if } E \leq c_n \\ SR_i(c) + CEL_i(c - SR(c), E - c_n) & \text{otherwise.} \end{cases}$$

where the vector of claims has been ordered such that $c_1 \leq ... \leq c_n$ and $SR_i(c) = \sum\limits_{j=0}^{l-1} \frac{c_{j+1} - c_j}{n - j}$ for all $l \in N$.

According to this definition, $SR(c)$ is a kind of boundedness. No claimant receives less (more) that $SR_i(c)$ if the Endowment is greater (smaller) than the highest claim.

Finally we introduce the TU game associated with a bankruptcy problem (N, c, E) as a pair (N, v) where $v(S) = \max\left\{E - \sum_{l \notin S} c_l, 0\right\}$ for all $S \subset N$.

The following example considers several bankruptcy problems with the same vector of claims.

Example 5 *Let* $N = \{1, 2, 3, 4\}$ *and* $c = (4, 7, 9, 10)$. *Then* $SR(c) = (1, 2, 3, 4)$. *Then,*

$$\begin{array}{rc} MO(c, 4) = & (1, 1, 1, 1) = CEA(c, 4) \\ MO(c, 7) = & (1, 2, 2, 2) \\ MO(c, 10) = & (1, 2, 3, 4) = SR(c, 10) \\ MO(c, 12) = & (1, 2, 4, 5) \\ MO(c, 22) = & (2, 5, 7, 8) = CEL(c, 22) \end{array}$$

The main result of this section follows.

Theorem 3 *Let* (N, c, E) *be a problem. Then* $MO(N, c, E)$ *coincides with the SD-prenucleolus of the associated TU game.*

The proof is a consequence of the fact that in the class of convex games the SD-prekernel coincides with the SD-prenucleolus and the following lemma.

Lemma 6 *Let* $(c, E) \in \mathcal{B}^N$ *and* (N, v) *be its associated TU game. Let* $MO(N, c, E) = x$*. Then* $s_{ij}(x) = f_i(N\backslash\{j\}, x) = \min(x_i, F(N\backslash\{j\})$ *for each pair* $i, j \in N$*.*

Proof. Assume on the contrary that $s_{ij}(x) = F(S, x) < \min(x_i, F(N\backslash\{j\})$. Clearly, in this case $v(S) > 0$.Consider claimant $l \in N\backslash S$ and $F(S \cup \{l\}, x)$. Since $v(S) > 0$ and $x_l < c_l$, it is not difficult to check that $F(S \cup \{l\}, x) \leq F(S, x)$. Following the same argument, if $S \cup \{l\} \neq N\backslash\{j\}$, another claimant p can be added, such that $F(S \cup \{l\} \cup \{p\}, x) \leq F(S \cup \{l\}, x)$. At the end a contradiction is obtained. ■

Lemma 7 *Let* (c, E) *be a problem. Then* $MO(c, E)$ *belongs to the SD-prekernel of the associated TU game.*

Proof. Let (N, v) be the associated TU game and let $i, j \in N$. Note that:

1. 1) If $x_i = SR_i(c)$ then $F(N\backslash\{i\}) = x_i$. 2) If $x_i < SR_i(c)$ then $v(N\backslash\{i\}) = 0$. 3) If $x_i > SR_i(c)$ then $F(N\backslash\{i\}) < x_i$.

Assume w. l. o. g. that $x_{i.} \leq x_j$. We distinguish three cases.

1) $x_i < SR_i(c)$. Then $v(N\backslash\{i\}) = v(N\backslash\{j\}) = 0$ and $x_i = x_j = F(N\backslash\{i\}) = F(N\backslash\{j\})$.

2) $x_i = SR_i(c)$. Then $F(N\backslash\{i\}) = x_i \leq F(N\backslash\{j\})$.

3) $x_i > SR_i(c)$. Then $x_j \geq SR_j(c)$ and $F(N\backslash\{i\}) = F(N\backslash\{j\})$. ■

Aumann and Maschler (1985) prove that the nucleolus of bankruptcy games and the Talmud rule coincide. Recently, Huijink *et al.* (2015) introduce the claim-and-right rules family. They provide a formula to compute the per capita nucleolus and show that it is a member of the aforementioned family that also includes the Talmud rule and the Minimal Overlapping rule.

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