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UNDER UNCERTAINTY

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Risk Management for Mathematical Optimization under Uncertainty

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Abstract

We present a general multistage stochastic mixed 0-1 problem where the uncertainty appears everywhere in the objective function, constraints matrix and right-hand-side. The uncertainty is represented by a scenario tree that can be a symmetric or a nonsymmetric one. The stochastic model is converted in a mixed 0-1 Deterministic Equivalent Model in compact representation. Due to the difficulty of the problem, the solution offered by the stochastic model has been traditionally obtained by optimizing the objective function expected value (i.e., mean) over the scenarios, usually, along a time horizon. This approach (so named risk neutral) has the inconvenience of providing a solution that ignores the variance of the objective value of the scenarios and, so, the occurrence of scenarios with an objective value below the expected one. Alternatively, we present several approaches for risk averse management, namely, a scenario immunization strategy, the optimization of the well known Value-at-Risk (VaR) and several variants of the Conditional Value-at-Risk strategies, the optimization of the expected mean minus the weighted probability of having a "bad" scenario to occur for the given solution provided by the model, the optimization of the objective function expected value subject to stochastic dominance constraints (SDC) for a set of profiles given by the pairs of threshold objective values and either bounds on the probability of not reaching the thresholds or the expected shortfall over them, and the optimization of a mixture of the VaR and SDC strategies.
Keywords: Multistage stochastic mixed 0-1 optimization, scenario analysis, mixed 0-1 Deterministic Equivalent Model, risk aversion measures, scenario immunization, VaR, CVAR, mean-risk, stochastic dominance constraints.

1 Introduction

Stochastic optimization is currently one of the most robust tools for decision making. It is broadly used in real-world applications in a wide range of problems from different areas such as finance, scheduling, production planning, industrial engineering, capacity allocation, energy, air traffic, logistics, etc. The integer problems under uncertainty have been studied in

[4, 13, 43, 48, 54, 56], just for citing a few references. An extended bibliography of Stochastic Integer Optimization has been collected in [60].

It is well known that a mixed 0-1 problem under uncertainty with a finite number of possible scenarios has a mixed 0-1 Deterministic Equivalent Model (DEM), where the risk of providing a wrong solution is included in the model, partially at least, via a set of representative scenarios. Let us assume that we are dealing with a maximization problem. Traditionally, special attention has been given to optimizing the DEM by maximizing the objective function expected value over the scenarios, subject to the satisfaction of all the problem constraints in the defined scenarios. Currently, we are able to solve huge DEMs by using different types of decomposition approaches, see [23] and, particularly, our Branch-and-Fix Coordination algorithm for multistage problems, BFC-MS, presented in [23, 26, 29]. However, the optimization of the so named risk neutral approach has the inconvenience of providing a solution that ignores the variance of the objective value of the scenarios and, so, the occurrence of scenarios with an objective value below the expected one, see e.g., [51].

Alternatively, we present in this work several approaches for risk management, i.e., risk averse strategies for multistage stochastic problems, namely, (1) a scenario immunization strategy, (2) the maximization of the well known Value-at-Risk, (3) the maximization of several variants of the Conditional Value-at-Risk (CVaR), (4) the maximization of the mean-risk, i.e., the expected objective minus the weighted probability of having a "bad" scenario occurring for the given solution provided by the model, (5) the maximization of the objective function expected value subject to first-order stochastic dominance constraints (SDC) for a set of profiles given by the pairs of threshold objective values and the probability of not reaching them, (6) the maximization of the objective function expected value subject to second-order SDC whose set of profiles is given by the pairs of threshold objective values and bounds on the expected shortfalls on reaching the thresholds, and (7) the maximization of the mixture of the VaR & SDC strategies.

The remainder of the paper is organized as follows. In Section 2 the uncertainty in the problem's coefficients and the scenario analysis methodology to use for dealing with the uncertainty are presented as well as the model for the risk neutral environment. Section 3 present the risk aversion strategies of our choice. Section 4 concludes.

2 Multistage mixed 0-1 stochastic problems

Without loss of generality, let us consider the following multistage deterministic mixed 0-1 model

$$\begin{aligned} \max & \sum_{t\in\mathcal{T}} a_t x_t + b_t y_t \\ \text{s.t. } & A'_t x_{t-1} + A_t x_t + B'_t y_{t-1} + B_t y_t = h_t \quad \forall t \in \mathcal{T} \\ & x_t \in \{0,1\}^{nx_t}, \quad y_t \in I\!R^{+ny_t} \qquad \forall t \in \mathcal{T}, \end{aligned} \tag{1}$$

where $\mathcal{T}$ is the set of stages, x_t and y_t are the nx_t and ny_t dimensional vectors of the 0-1 and continuous variables, respectively, a_t and b_t are the vectors of the objective function coefficients, A'_t, A_t, B'_t and B_t are the constraint matrices and h_t is the right-hand-side vector (rhs) for stage t.

However, some of the problem coefficients in the objective function, constraint matrix and rhs are frequently uncertain, mainly in dynamic domains (i.e, problems whose decisions to be made are based on data along a time horizon). There are several ways in which to express future uncertainty in the coefficients. One the most used consists of representing it by considering scenarios with known or estimated probabilities. For this purpose we need some definitions.

Definition 1 *A* ***stage*** *of a time horizon is a set of one or various time periods in which the random parameters are realized.*

Definition 2 *A* ***scenario*** *consists of a realization of all the random parameters in all stages, that is, a path through the scenario tree from the root to a leaf node.*

For representing the uncertainty we use a scenario tree approach in which uncertainty is modeled in terms of a set of scenarios.

Definition 3 *A* ***partial scenario*** *for a given stage consists of a realization of all the random parameters up to that stage. That is, the part of the path thorough that scenario from the root up to the intermediate node at that stage.*

Definition 4 *A* ***scenario group*** *for a given stage is the group of scenarios with the same partial scenario up to that stage.*

Notice that the partial scenarios for the last stage are the corresponding scenarios.

To illustrate the multistage scenario tree concept, let Fig. 1 depict a scenario tree in which each node represents a situation in a stage, where a decision can be taken and, after that, various possible situations may occur. In our example there are two situations in stage $t = 2$. This information is generally presented in the form of a tree in which each path from the root to a leaf represents a scenario and corresponds to the realization of the entire set of uncertain parameters. For example, path $\{1, 3, 6, 12\}$ represents one scenario, and it is customary to call it scenario 12. In what follows, we do not distinguish between a scenario (or a group) and the corresponding node on the tree (with the same number). Each node in the tree must be associated with a scenario group in such a manner that any two scenarios belong to the same group (i.e., they have the same partial scenario) in a given stage if they include the same occurrences of uncertain parameters up to that stage. In this case, the well known nonanticipativity principle applies. It was stated in [61] and restated in [53]; see also [13], among others. This principle requires that the decisions pertaining to scenarios in the same group (i.e., partial scenarios with the same value in the parameters) be the same. For example, for stage 3, scenarios 12 and 13 belong to the same group associated with path $\{1, 3, 6\}$, i.e., with group $g = 6$. Notice the difference between a scenario (a path from the root node to a leaf node) and a partial scenario (a path from the root to an intermediate node).

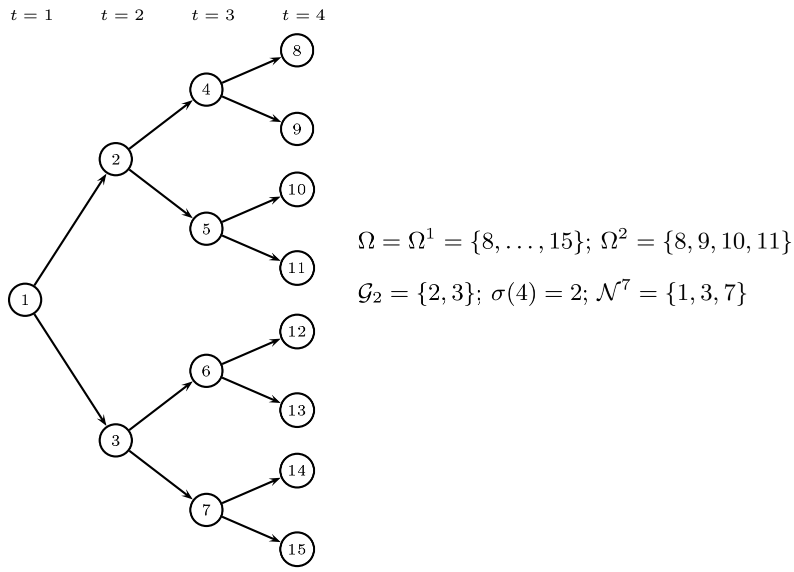

Figure 1: A multistage scenario tree

Definition 5 *A* ***symmetric tree*** *is a tree where the number of branches is the same for all conditional distributions in the same stage, that is, the number of branches arising from any scenario group at each stage t to the next one is the same for all groups in the stage.*

Definition 6 *A* ***nonsymmetric tree*** *is a tree where the number of branches is not the same for all conditional distributions in one stage, at least.*

See e.g., [6, 7, 26] for symmetric scenario trees, among many others, and [29] for nonsymmetric ones. Note: Fig. 1 depicts a symmetric tree, and the tree that results from taking out any arc is a nonsymmetric tree.

It is out of the scope of this work to present a methodology for multistage scenario tree generation and reduction; see e.g., [20, 37, 38] and references therein.

The notation for the scenario tree to be used in the paper is as follows:

$\mathcal{T}$, set of stages $\{1, 2, ..., T\}$ in the time horizon with $T = |\mathcal{T}|$.

$\mathcal{T}^-$, set of all stages except the last one.

Ω, set of scenarios.

$\mathcal{G}$, set of scenario groups.

$\mathcal{G}_t$, set of scenario groups in stage t ($\mathcal{G}_t \subseteq \mathcal{G}$), for $t \in \mathcal{T}$.

$t(g)$, stage to whom group g belongs to, such that $g \in \mathcal{G}_{t(g)}$.

Ω^g, set of scenarios in group g ($\Omega^g \subseteq \Omega$), for $g \in \mathcal{G}$.

$\sigma(g)$, immediate ancestor node of node g, for $g \in \mathcal{G}$.

$\mathcal{N}^g$, set of ancestor groups (i.e, nodes) to group g, including itself.

d, any scenario group that belongs to the last stage, i.e., $g \in \mathcal{G}_T$. Note: Ω^d is a singleton set.

Let us assume that all or some of the parameters in problem (1) are random ones to be presented by a set of discrete occurrences, say, a_t^ω and b_t^ω for the objective function vectors a_t and b_t, respectively, $A_t^{'\omega}$, A_t^ω, $B_t^{'\omega}$ and B_t^ω for the A_t', A_t, B_t' and B_t constraint matrices, respectively, and h_t^ω for the rhs h_t, for scenario $\omega \in \Omega$. So, the model for maximizing the expected objective value over the scenarios can be expressed

$$
\begin{array}{lll}
\max & \sum_{\omega\in\Omega}\sum_{t\in\mathcal{T}} w^\omega(a_t^\omega x_t^\omega + b_t^\omega y_t^\omega) & \\
\text{s.t.} & A_t^{'\omega} x_{t-1}^\omega + A_t^\omega x_t^\omega + B_t^{'\omega} y_{t-1} + B_t^\omega y_t^\omega = h_t^\omega & \forall t \in \mathcal{T}, \omega \in \Omega \\
& (x,y) \in NAC & \\
& x_t^\omega \in \{0,1\}^{nx_t^\omega}, y_t^\omega \in I\!R^{+ny_t^\omega} & \forall t \in \mathcal{T}, \omega \in \Omega,
\end{array}
\tag{2}
$$

where w^ω is a positive weight assigned to scenario ω, for instance its probability such that $\sum_{\omega\in\Omega} w^\omega = 1$, x_t^ω and y_t^ω represent the replicas of x_t and y_t variables for scenario ω, respectively, $x = (x_t^\omega \ \forall t \in \mathcal{T}, \omega \in \Omega)$ and $y = (y_t^\omega \ \forall t \in \mathcal{T}, \omega \in \Omega)$. The nonanticipativity set is defined by

$$
NAC = \{x_t^\omega = x_t^{\omega'}; y_t^\omega = y_t^{\omega'} \quad \forall \omega, \omega' \in \Omega_g, g \in \mathcal{G}_t, t \in \mathcal{T}^-\}. \tag{3}
$$

The nonanticipativity principle ensures that the solution for stage t in the model does not depend on information that is yet unavailable. For modeling the set (3) in model (2), two different approaches can be used, namely, the compact representation and the splitting variable representation, see [23], among others. For the purpose of presenting the risk measures we will only consider the first representation. However we notice that the algorithm BFC-MS [29] uses a mixture of the compact and splitting variable representations.

Upon incorporating the set (3) in model (2), we can obtain the related multistage mixed 0-1 Deterministic Equivalent Model (DEM) in its compact representation. The new model can be expressed

$$
\begin{array}{lll}
Q_E = \max & \sum_{g\in\mathcal{G}} \overline{w}^g(a^g x^g + b^g y^g) & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G},
\end{array}
\tag{4}
$$

where $\overline{w}^g = \sum_{\omega\in\Omega_g} w^\omega$ gives the weight assigned to scenario group g, and a^g and b^g are the counterparts of parameters a_t and b_t related to scenario group g, for $g \in \mathcal{G}_t, t \in \mathcal{T}$, such that the values of the parameters for each scenario in the group are identical. Additionally, x^g and y^g represent the replicas of x and y variables for scenario group g, respectively, A'^g and B'^g are the constraint matrices in scenario group g for the x and y variables related to the immediate ancestor of group g and, similarly, we have the matrices A^g and B^g and the rhs h^g, such that we will consider $t \equiv t(g)$ through Section 3.

3 Risk aversion management

The model (4) aims to maximize the objective function expected value (i..e., mean) alone, then, the so named risk neutral strategy is considered. The main criticism that can be made to this very popular strategy is that, as we noted above, it ignores the variance on the objective function value over the scenarios and, in particular, the "left" tail of the non-wanted scenarios. However, there are some risk averse approaches that additionally deal with risk management by considering, e.g., the following coherent [9] measures: scenario immunization, see [17] and its treatment in [22], semi-deviations [2, 49], Value-and-Risk [15, 32, 33], Conditional Value-at-Risk [2, 10, 12, 45, 50, 52, 56, 57], excess probabilities [55], and first- and second-order Stochastic Dominance Constraints (SDC) strategies, see [34, 35, 36] and the references therein, among others. See also [5, 8, 14, 19, 30, 31, 58] for applications of SDC, specifically, in energy, finance and mining, among others, particularly for second-order SDC for the two-stage environment by using Lagrangean and cutting plane approaches.

Let us consider the following risk averse measures, that take into account the bad tail of the objective value distribution over the scenarios:

- Scenario Immunization (SI): Minimizing a norm of the expected deviation of the objective function value over the scenarios given by the solution offered by the model while jointly satisfying the constraints for all escenarios from the optimal objective function value obtained by considering each scenario alone.

- Value-at-Risk (VaR): Well known theoretical research subject in finance suggests that the measures based on quantiles are good functions for risk management. Among them, the Value-at-Risk (VaR) has also turned into a reference to many applications in other sectors such as transportation, production planning, etc. That approach is very attractive since it is easy to interpret. By definition, the β-VaR of an objective function over a set of scenarios is its lowest value, say α, such that the objective function value of the scenario to occur is over α with β probability. The strategy consists of maximizing VaR. Note: The β probability is provided by the modeler.

- Conditional Value-at-Risk (CVaR): The advantage of the VaR strategy over the traditional maxmin strategy is obvious, since it takes into account an upper bound β on the probability of the occurrence of a scenario whose objective value is not below α. However, it does not consider how bad the scenarios with an objective value below VaR can be. The β-Conditional Value-at-Risk (CVaR) strategy takes into account the objective value of the bad scenarios for a $\frac{1}{\beta}$ weighting parameter, where CVaR is the conditional expectation of the objective value below α.

- Deficit Probability (DP): As an alternative to the VaR and CVaR strategies, DP is a risk measure for weighting the probability that a non-desired scenario will occur, that is, the scenario where the objective value is below a given threshold, say ϕ. This parameter is provided by the modeler.

- Stochastic Dominance Constraints strategies (SDC), where the objective function expected value is maximized, such that a set of thresholds of the objective function value for each scenario is to be satisfied with either a given failure's probability on each threshold (the so named first-order SDC) or a bound on the expected objective function shortfall on reaching it (the so named second-order SDC). Note 1: A profile is said to be included by the pair given by a threshold and a bound on either its failure probability or its expected shortfall. Note 2: The set of profiles is provided by the modeler.

Some of these risk measures and other approaches in the literature try to reduce either the probability of the occurrence of non-wanted scenarios or the maximization of the objective value for the worst scenario with a given failure's probability. However, they do not pay attention to the good scenarios (except the last strategy depending on the set of profiles to consider). On the contrary, decision makers usually look for a trade-off between the risk minimization and the objective value maximization. For this reason, the above cited risk measures are usually combined with the optimization of the objective function, leading to strategies as the combination of the Expected Value and Deficit Probability [55] and the combination of Expected Value and CVaR [56], among others.

We present in this work the modeling of the above risk averse strategies for multistage stochastic mixed 0-1 programs by including some new variables and constraints, where the new variables are 0-1 ones but for the CVaR and second-order SDC strategies. Additionally, some strategies require constraints with variables from different scenarios, such as VaR, SDC and other strategies, see below.

3.1 Two step Scenario Immunization strategy

The model for Scenario Immunization minimizes a given norm, say ℓ, of the expected deviation of the scenario objective function value given by the model over the scenarios (where the constraints for all the scenarios are jointly satisfied) from the optimal objective value of each scenario individually considered. It can be represented as follows,

$$\begin{array}{lll} D = \min & \sum\limits_{\omega\in\Omega} w^\omega (Q^{\omega^*} - Q^\omega)^\ell & \\ \text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\ & x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G}, \end{array} \qquad (5)$$

where

$$Q^\omega = \sum_{t\in\mathcal{T}}(a_t^\omega \hat{x}_t^\omega + b_t^\omega \hat{y}_t^\omega) = \sum_{g\in\mathcal{N}^d}(a^g\hat{x}^g + b^g\hat{y}^g)$$

is the objective function value of the DEM problem ((2) or (4)) for the solution values of the x and y variables under each scenario ω. Notice that $x^\omega = \{x^g\,\forall g \in \mathcal{N}^d\}$ and $y^\omega = \{y^g\ \forall g \in \mathcal{N}^d\}$, where $\hat{x}^g$ and $\hat{y}^g$ are the solution of the variables x^g and y^g, respectively, and ω is the unique scenario in set Ω^d, for $d \in \mathcal{G}_T$. Remember that $\mathcal{N}^d$ gives the set of ancestor nodes (i.e., scenario groups) in the back path from leaf node d to root node 1.

Additionally, the objective function value Q^{ω^*} for the individual model related to each scenario ω can be expressed

$$\begin{array}{lll} Q^{\omega^*} = \max & \sum_{g\in\mathcal{N}^d}(a^g x^g + b^g y^g) & \\ \text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{N}^d \\ & x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{N}^d. \end{array} \tag{6}$$

Notice that $Q^{\omega^*} - Q^\omega \geq 0$.

In order to make computationally manageable the above approach and at the same time risk effective, the following two-step approach is introduced in [22], where the minimization of the norm $\ell = \infty$ is performed in the first stage, let $\overline{D}$ denote the value. Additionally, the minimization of the norm $\ell = 1$ is performed in the second stage, subject to the constraint that force the deviation of the objective function value given by the model for each scenario from its optimal objective value in its individual model be not greater than $\overline{D}$, the minimum greatest deviation by considering jointly all scenarios. The two step strategy is as follows.

Step 1 in strategy $(\infty, 1)$

$$\begin{array}{lll} \overline{D} = \min & \max_{\omega\in\Omega} w^\omega(Q^{\omega^*} - Q^\omega) & \\ \text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\ & x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G}. \end{array} \tag{7}$$

Step 2 in strategy $(\infty, 1)$

$$\begin{array}{lll} D = \min & \sum_{\omega\in\Omega} w^\omega(Q^{\omega^*} - Q^\omega) & \\ \text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\ & w^\omega(Q^{\omega^*} - Q^\omega) \leq \overline{D} & \forall \omega \in \Omega \\ & x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G}. \end{array} \tag{8}$$

Notice the relationship between D (8) and Q_E (4), since the objective function of model (8) can be expressed

$$\sum_{\omega\in\Omega} Q^{\omega^*} + \max \sum_{g\in\mathcal{G}} \overline{w}^g(a^g x^g + b^g y^g).$$

3.2 Value-at-Risk strategy

The model that maximizes a combination of the objective function expected value and the β-VaR can be expressed as follows,

$$
\begin{array}{lll}
\max & \gamma \sum\limits_{g\in\mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) + \rho\alpha & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum\limits_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) + M^\omega \nu^\omega \geq \alpha & \forall \omega \in \Omega \\
& \sum\limits_{\omega\in\Omega} w^\omega \nu^\omega \leq 1 - \beta & \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& \nu^\omega \in \{0,1\} & \forall \omega \in \Omega \\
& \alpha \in \mathbb{R}, &
\end{array}
\tag{9}
$$

where α is a rational variable expressing the VaR to maximize, ν^ω is a 0-1 variable such that its value is 1 if the objective value for scenario ω is smaller than α and otherwise, 0, M^ω is the "big M" parameter, preferably to be the smallest one, which does not eliminate any feasible solution for scenario ω in the original stochastic model, ρ is a weighting parameter, and $\gamma \in \{0,1\}$ is another parameter such that for $\gamma = 0$ and $\rho = 1$ it results the classical VaR objective function. Note: The $1-\beta$ failure's probability of not satisfying a given constraint may have its roots in the concept of Chance Constraints introduced in [16].

3.3 Conditional expectation below VaR strategy

As stated above, the advantage of the VaR strategy over the traditional maxmin strategy is obvious since it takes into account the probability of the occurrence of scenarios whose objective value is below VaR. However, it does not consider how bad the scenarios with a objective value *below* VaR can be. On the contrary, the model that maximizes a combination of the objective function expected value and the β-CVaR can be expressed as follows,

$$
\begin{array}{ll}
\max & \gamma \sum\limits_{g\in\mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) + \rho\Big(\alpha - \dfrac{1}{\beta} \sum\limits_{d\in\mathcal{G}_T} \overline{w}^d \Big(\alpha - \sum\limits_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g)\Big)_+\Big) \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g \quad \forall g \in \mathcal{G} \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} \quad \forall g \in \mathcal{G} \\
& \alpha \in \mathbb{R},
\end{array}
\tag{10}
$$

where $\overline{w}^d = w^\omega$ being $\omega \in \Omega^d$, and $z_+ = \max\{0, z\}$. Recall that $d \in \mathcal{G}_T$. Note: For $\gamma = 0$ and $\rho = 1$, it results the CVaR strategy introduced in [52]. See also [46, 47].

Also denoted as CVaR$^-$, a more amenable representation of model (10), see [56], is given as follows

$$
\begin{array}{lll}
\max & \gamma \sum_{g\in\mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) + \rho\Big(\alpha - \frac{1}{\beta}\sum_{\omega\in\Omega} w^\omega v^\omega\Big) & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \alpha - \sum_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) \le v^\omega & \forall \omega \in \Omega \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& v^\omega \ge 0 & \forall \omega \in \Omega \\
& \alpha \in \mathbb{R}, &
\end{array}
\tag{11}
$$

such that v^ω is a non-negative variable equal to the difference (if its positive) between α and the objective value for scenario ω, the so named objective function shortfall on reaching VaR α. For $\gamma = 1$ and $\rho = 1$,it results the maximization of the so named average VaR deviation (also known as the average CVaR deviation), see in [39] an interesting application for selecting the optimal pension plan fund.

3.4 Conditional expectation above VaR strategy

As an alternative to the Conditional expectation below VaR strategy, see model (11), the so named CVaR$^+$ strategy maximizes a combination of VaR and the weighted Conditional expectation *above* VaR, such that the model is as follows,

$$
\begin{array}{lll}
\max & \alpha + \rho \sum_{d\in\mathcal{G}_T} \overline{w}^d \Big(\sum_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) - \alpha \Big)_+ & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) + M^\omega \nu^\omega \ge \alpha & \forall \omega \in \Omega \\
& \sum_{\omega\in\Omega} w^\omega \nu^\omega \le 1 - \beta & \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& \nu^\omega \in \{0,1\} & \forall \omega \in \Omega.
\end{array}
\tag{12}
$$

A more amenable representation of model (12) is as follows,

$$
\begin{array}{lll}
\max & \alpha + \rho \sum_{\omega\in\Omega} w^\omega v^\omega & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) - \alpha + M^\omega \nu^\omega \ge v^\omega & \forall \omega \in \Omega \\
& v^\omega \le M^\omega (1 - \nu^\omega) & \forall \omega \in \Omega \\
& \sum_{\omega\in\Omega} w^\omega \nu^\omega \le 1 - \beta & \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& \nu^\omega \in \{0,1\},\ v^\omega \ge 0 & \forall \omega \in \Omega,
\end{array}
\tag{13}
$$

such that v^ω is a non-negative variable equal to the difference between the objective value for scenario ω and α in case the difference is positive and otherwise, 0.

3.5 Conditional expectation of the objective function shortfall on reaching a threshold

$$
\begin{array}{lll}
\max & \sum_{g\in\mathcal{G}} \overline{w}^g(a^g x^g + b^g y^g) - \rho \sum_{\omega\in\Omega} w^\omega v^\omega & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum_{g\in\mathcal{N}^d} \overline{w}^g(a^g x^g + b^g y^g) + v^\omega \geq \phi & \forall \omega \in \Omega \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& v^\omega \geq 0 & \forall \omega \in \Omega,
\end{array}
\qquad (14)
$$

where v^ω is a nonnegative variable equal to the difference between a given threshold, say, ϕ and the objective value for scenario ω and otherwise, 0. The model for the maximization of the objective function expected value minus the weighted expected objective function shortfall on reaching threshold ϕ is inspired in [21], and related to the Integrated Chance Constraints concept due to [42], see also [43, 44]. In a different context see [11, 59].

3.6 Deficit probability

The strategy that maximizes the objective function expected value minus the weighted probability of having the objective value for the scenario to occur below a given threshold is modeled below, see [55]. As in the VaR strategy, a new 0-1 variable per scenario, say ν^ω is needed, such that its value is 1 if the objective value for scenario ω is smaller than threshold ϕ and otherwise, 0.

$$
\begin{array}{lll}
\max & \sum_{g\in\mathcal{G}} \overline{w}^g(a^g x^g + b^g y^g) - \rho \sum_{\omega\in\Omega} w^\omega \nu^\omega & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) + M^\omega \nu^\omega \geq \phi & \forall \omega \in \Omega \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& \nu^\omega \in \{0,1\} & \forall \omega \in \Omega.
\end{array}
\qquad (15)
$$

Notice that the threshold satisfaction constraints allow to impose a lower limit, say L^ω, in the objective value for scenario ω, just by fixing $M^\omega = \phi - L^\omega$. For example, $M^\omega = \phi$ means that a negative objective value is not allowed for scenario $\omega \in \Omega$.

3.7 Stochastic dominance constraint strategies SDC

As an alternative to the above strategies, let the recent approaches based on the first-order and second-order stochastic dominance constraints (SDC) for mixed-integer linear recourse introduced in [34] and [35], respectively. See also [18] for the case of continuous variables where the problems is considered as a semi-infinite one.

The first-order so named SDC-1 requires a set of profiles, say $\mathcal{P}$, given by the pairs (ϕ^p, β^p) $\forall p \in \mathcal{P}$, where ϕ^p is the threshold on the objective function value to be satisfied by scenario ω with a success probability β^p. Let us implement this strategy by proposing model (16), such that the function value $\sum_{g\in\mathcal{N}^d}(a^g x^g + b^g y^g)$ (i.e., the objective value of the unique scenario ω in set Ω^d where $d \in \mathcal{G}_T$) is not below threshold ϕ^p with probability β^p.

$$
\begin{array}{lll}
\max & \sum\limits_{g\in\mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum\limits_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) + M^\omega \nu^{\omega p} \geq \phi^p & \forall \omega \in \Omega, p \in \mathcal{P} \\
& \sum\limits_{\omega\in\Omega} w^\omega \nu^{\omega p} \leq 1 - \beta^p & \forall p \in \mathcal{P} \\
& x^g \in \{0,1\}^{nx_t}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& \nu^{\omega p} \in \{0,1\} & \forall \omega \in \Omega, p \in \mathcal{P},
\end{array}
\tag{16}
$$

where $\nu^{\omega p}$ is a 0-1 variable such that its value is 1 if the objective value for scenario ω is smaller than threshold ϕ^p and otherwise, 0.

The second-order stochastic dominance constraints strategy (the so named SDC-2) requires a set of profiles given by the pairs (ϕ^p, e^p) $\forall p \in \mathcal{P}$, where e^p is the upper bound of the expected objective function shortfall on reaching threshold ϕ^p. It can be implemented as follows,

$$
\begin{array}{lll}
\max & \sum\limits_{g\in\mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \phi^p - \sum\limits_{g\in\mathcal{N}^d} (a^g x^g + b^g y^g) \leq v^{\omega p} & \forall \omega \in \Omega, p \in \mathcal{P} \\
& \sum\limits_{\omega\in\Omega} w^\omega v^{\omega p} \leq e^p & \forall p \in \mathcal{P} \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& v^{\omega p} \geq 0 & \forall \omega \in \Omega, p \in \mathcal{P},
\end{array}
\tag{17}
$$

such that $v^{\omega p}$ is a non-negative variable equal to the difference (if it is positive) between threshold ϕ^p and the objective value for scenario ω. Notice that this strategy does not require additional 0-1 variables. The concept of the expected objective function shortfall on reaching a given threshold may have its roots in the Integrated Chance Constraints concept introduced in [42] see also [44].

3.8 VaR & Stochastic dominance constraint strategies

As an alternative to the SDC-1 and SDC-2 strategies, let a mixture of the VaR (9) & SDC strategies (16) and (17), such that the new models are (18) for SDC-1 and (19) for SDC-2.

$$
\begin{array}{lll}
\max & \gamma \sum_{g \in \mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) + \rho \sum_{p \in \mathcal{P}} \alpha^p & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \sum_{g \in \mathcal{N}^d} (a^g x^g + b^g y^g) + M^\omega \nu^{\omega p} \geq \alpha^p & \forall \omega \in \Omega, p \in \mathcal{P} \\
& \sum_{\omega \in \Omega} w^\omega \nu^{\omega p} \leq 1 - \beta^p & \forall p \in \mathcal{P} \\
& \alpha^p \geq \phi^p & \forall p \in \mathcal{P} \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& \nu^{\omega p} \in \{0,1\} & \forall \omega \in \Omega, p \in \mathcal{P}.
\end{array}
\tag{18}
$$

$$
\begin{array}{lll}
\max & \gamma \sum_{g \in \mathcal{G}} \overline{w}^g (a^g x^g + b^g y^g) + \rho \sum_{p \in \mathcal{P}} \alpha^p & \\
\text{s.t.} & A'^g x^{\sigma(g)} + A^g x^g + B'^g y^{\sigma(g)} + B^g y^g = h^g & \forall g \in \mathcal{G} \\
& \alpha^p - \sum_{g \in \mathcal{N}^d} (a^g x^g + b^g y^g) \leq v^{\omega p} & \forall \omega \in \Omega, p \in \mathcal{P} \\
& \sum_{\omega \in \Omega} w^\omega v^{\omega p} \leq e^p & \forall p \in \mathcal{P} \\
& \alpha^p \geq \phi^p & \forall p \in \mathcal{P} \\
& x^g \in \{0,1\}^{nx^g}, y^g \in I\!R^{+ny^g} & \forall g \in \mathcal{G} \\
& v^{\omega p} \geq 0 & \forall \omega \in \Omega, p \in \mathcal{P}.
\end{array}
\tag{19}
$$

Note: In some applications, mainly in the financial sector, see [8], the set of profiles $\mathcal{P}$ refers to a selected subset of scenario groups in $\mathcal{G}$, mainly for the groups related to non-wanted scenarios.

Solution considerations: We must point out that the models (9), (13), (16), (17), (18) and (19) have a computational disadvantage when comparing them with the models (11), (14) and (15), since they have constraints linking variables from different scenarios. Notice that the disadvantage is stronger for the models (16) and (17) with $|\mathcal{P}| > 1$ than for the models (9) and (13), and it is stronger for the models (18) and (19) than for the models (16) and (17). In any case, a decomposition approach must be used for problem solving of huge instances. A Lagrange relaxation can be considered for dualizing those linking constraints as done in the strategy presented in [35] for the second-order SDC, see [28], or an extension of the BFC-MS algorithm presented in [29] in order to obtain an exact optimal solution in an affordable computing effort, see [27].

4 Conclusions

Several risk averse strategies existing in the literature for two-stage stochastic programs have been proposed in this work for risk management in multistage stochastic mixed 0-1 programs. Those strategies are the two step scenario immunization (7)-(8), VaR (9), CVaR (11), mean-risk (here named Deficit probability) (15), and the first- and second-order stochastic dominance constraints (SDC-1 and SDC-2, respectively) (16) and (17). Some other risk averse strategies have been proposed, such as the conditional expectation above VaR (13), the minimization of the objective function expected shortfall on reaching a given threshold (14), and the mixture of VaR & SDC (18) and (19).

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