eman ta zabal zazu

Universidad del País Vasco — Euskal Herriko Unibertsitatea

On the controllability of Partial Differential Equations involving non-local terms and singular potentials

DOCTORAL THESIS/TESIS DOCTORAL

Author/Autor:

Umberto BICCARI

Advisor/Director:

Enrique ZUAZUA IRIONDO

Universidad Autónoma de Madrid

Bilbao, 2016

DOCTORAL THESIS

On the controllability of Partial Differential Equations involving non-local terms and singular potentials

TESIS DOCTORAL

Sobre la controlabilidad de Ecuaciones en Derivadas Parciales con términos no-locales y potenciales singulares

Author/Autor: Umberto BICCARI

Advisor/Director: Enrique ZUAZUA IRIONDO

Universidad Autónoma de Madrid

Bilbao, 2016

“There is no subject so old that something new cannot be said about it.”

Fëdor Dostoevskij

This thesis is supported by the Advanced Grants NUMERIWAVES/FP7-246775 of the European Research Council Executive Agency, FA9550-15-1-0027 of the AFOSR, PI2010-04 and the BERC 2014-2017 program of the Basque Government, the MTM2011-29306-C02-00, MTM2014-52347 and SEV-2013-0323 Grants of the MINECO and Enrique Zuazua's Chair in Control, PDEs, Numerics and Applications in CIMI (Centre International de Mathématiques et Informatique) of Toulouse, France

Acknowledgements

Firstly, I would like to express my most sincere gratitude to my advisor, Enrique Zuazua, for his continuous support during the years of my Ph.D. studies, for his patience and motivation. I am truly pleased to have been his student. His guidance has been a precious help in all the time of research and writing of this thesis. Eskerrik asko!

I am deeply grateful to Mahamadi Warma, who has been a great host during my two months stay in Puerto Rico. His availability, kindness and permanent interest in my research have significantly contributed to the development of the results that I present.

Thank you to Sylvain Ervedoza, for having kindly accepted to be a reporter of this thesis. His valuable comments have been fundamental for the improvement and the correction of several parts of this work. Thank you also to Piermarco Cannarsa, for having been him too a reporter of my work and for being part of the thesis committee. Besides, I would like to thank the rest of the committee, Jean-Bernard Bru and Ireneo Peral, as well as my UPV/EHU tutor, Miguel Escobedo.

I thank CIMI and its staff for the hospitality during my two months visit in Toulouse. In particular, thank you to Jean-Pierre Raymond, Sylvain Ervedoza and also to Isabelle Guichard, for her precious help in all the administrative issues related to my visit.

Thank you to all the professors and researcher that I met during these years of PhD, and that contributed to my work with many hours of interesting discussions. Among them, Carlos Castro, Cristian Cazacu, Felipe Chaves, Denys Dutykh, Genni Fragnelli, Mamadou Gueye, Valentin Keyantuo, Qi Lü, Aurora Marica, Denis Matignon, Enrique Otárola and Xavier Ros-Oton.

My sincere thanks goes to BCAM people, for these three and a half year that we spent together. In particular, I would like to mention Carmen Alonso, Sébastien Breteaux, Biagio Cassano, Roberto Castelli, Francesco Fanelli, Daniel García, Imanol García, Inderpreet Kaur, Alessia Kogoj, Julia Kroos, Ariel Lozano, Thomas Ourmières-Bonafos, Tijana Radivojevic, Antsa Ratsimanetrimanana, Simone Rusconi and Goran Stipcich.

I thank my two officemates Fabio Pizzichillo and Julia Sánchez for the good ambience we had inside and outside the office. A special thank also goes to the fourth Y19 component, Aitziber Ibañez, and to Mario Fernández, my direct line with the RAE, for their help with the revision and corrections on the Spanish in some parts of the thesis.

I am grateful to all the people who made my venue and stay in the Basque Country so pleasant. I cannot name all of them, but I want to mention above all my team-mates from Diskolaris Ultimate Bilbao, who introduced me to the wonderful sport of the Ultimate Frisbee and who shared with me wonderful moments during our trips all around Spain.

A special thanks goes to my dear friend Leonardo Falco, whose friendship accompanied me throughout these years, even after my moving abroad, and who always greets me with great enthusiasm each time that I came back to Italy.

Thank you to my brother Edoardo Biccari who, like me, had to leave Italy for pursuing his future. We now live at the two extrema of Europe, and I am glad that we still manage to meet from time to time and to exchange our mutual experiences.

Last but not least, I would like to sincerely thank my parents Davide Biccari and Fiorenza Nigro, who have been extremely understanding and encouraging during all my life. I can always count on their support and approval. Grazie mille!

Agradecimientos

En primer lugar, quiero agradecer a mi director, Enrique Zuazua, por su apoyo constante en los años de mis estudios de doctorado, por su paciencia y motivación. Estoy muy contento de haber sido su estudiante. Su guía ha sido una ayuda preciosa a lo largo de todo este tiempo en que me he dedicado a la investigación y a la escritura de la tesis. Eskerrik asko!

Estoy profundamente agradecido con Mahamadi Warma, que ha sido un anfitrión excelente durante mi estancia de dos meses en Puerto Rico. Su disponibilidad, generosidad y su interés constante en mi trabajo han contribuido considerablemente al desarrollo de los resultados que presento.

Gracias a Sylvain Ervedoza, por haber aceptado amablemente ser revisor de esta tesis. Sus valiosos comentarios han sido fundamentales para la mejora y la corrección de ciertas partes de este trabajo. Gracias a Piermarco Cannarsa, por haber sido él también revisor de mi trabajo, y por haber aceptado formar parte del tribunal. Además, querría agradecer tanto al resto del tribunal, Jean-Bernard Bru y Ireneo Peral, como al poniente de mi tesis en la UPV-EHU, Miguel Escobedo.

Agradezco a CIMI y a su personal para la hospitalidad durante mi visita de dos meses en Toulouse. En particular, gracias a Jean-Pierre Raymond, Sylvain Ervedoza y también a Isabelle Guichard, para su ayuda con todas la cuestiones administrativas relacionadas con mi estancia.

Gracias a todos los profesores y investigadores que he encontrado durante estos años de doctorado y que han contribuido a mi trabajo con muchas horas de discusiones interesantes. Entre ellos, Carlos Castro, Cristian Cazacu, Felipe Chaves, Denys Dutykh, Genni Fragnelli, Mamadou Gueye, Valentin Keyantuo, Qi Lü, Aurora Marica, Denis Matignon, Enrique Otárola y Xavier Ros-Oton.

Mis agradecimientos más sinceros van a todas las personas de BCAM, por estos tres años y medio que hemos compartido. En particular, querría mencionar a Carmen Alonso, Sébastien Breteaux, Biagio Cassano, Roberto Castelli, Francesco Fanelli, Daniel García, Imanol García, Inderpreet Kaur, Alessia Kogoj, Julia Kroos, Ariel Lozano, Thomas Ourmières-Bonafos, Alejandro Pozo, Tijana Radivojevic, Antsa Ratsimanetrimanana, Simone Rusconi y Goran Stipcich.

Gracias a mis dos compañeros de despacho, Fabio Pizzichillo y Julia Sánchez, por el buen ambiente dentro y fuera de la oficina. Un gracias especial lo reservo para la cuarta componente del Y19, Aitziber Ibañez, y para Mario Fernández, mi línea directa con la RAE, por sus ayuda en la revisión y la corrección del español en algunas partes de la tesis.

Agradezco a todas las personas que contribuyeron a que mi venida y mi estancia en el País Vasco fueran tan agradables. No puedo nombrarlos todos, pero querría mencionar en particular a mis compañeros del equipo Diskolaris Ultimate Bilbao, que me introdujeron al maravilloso deporte del Ultimate Frisbee y que han compartido conmigo momentos preciosos durante todos nuestros viajes alrededor de España.

Un gracias particular va a mi amigo Leonardo Falco, cuya amistad me ha acompañado a lo largo de todos estos años, también después de mi mudanza en Bilbao, y que me acoge con gran entusiasmo cada vez que vuelvo a Italia.

Gracias a mi hermano Edoardo Biccari, que como yo ha tenido que dejar Italia para seguir con su futuro. Ahora vivimos a los dos extremos de Europa, y soy muy contento que todavía conseguimos reencontrarnos de vez en cuando e intercambiarnos nuestras reciprocas experiencias.

Por último, pero no por ello menos importante, quiero agradecer sinceramente a mis padres Davide Biccari y Fiorenza Nigro, que han sido extraordinariamente comprensivos y alentadores en toda mi vida. Siempre pude contar con su apoyo y aprobación. Grazie mille!

Ringraziamenti

Vorrei innanzitutto esprimere la mia piú sincera gratitudine a Enrique Zuazua, per il suo supporto costante durante gli anni dei miei studi di dottorato, per la sua pazienza e la sua motivazione. Sono estremamente contento di essere stato suo studente. La sua guida ha rappresentato un aiuto prezioso durante tutto il mio periodo di ricercatore e durante la scrittura di questa tesi. Eskerrik asko!

Sono profondamente riconoscente a Mahamadi Warma, che é stato un anfitrione eccellente durante la mia visita di due mesi a Puerto Rico. La sua disponibilitá, la sua gentilezza ed il suo costante interesse per la mia ricerca hanno contribuito significativamente allo sviluppo dei risultati che presento.

Grazie a Sylvain Ervedoza, per aver gentilmente accettato di essere revisore di questa tesi. I suoi preziosi commenti sono stati fondamentali per correggere e migliorare alcune parti di questo lavoro. Grazie a Piermarco Cannarsa, per essere stato anche lui revisore del mio lavoro, e per aver accettato di far parte del tribunale. Inoltre, vorrei ringraziare il resto del tribunale, Jean-bernard Bru e Ireneo Peral, cosí come il mio tutore all'UPV-EHU, Miguel Escobedo.

Ringrazio il CIMI e tutto il suo personale per l'ospitalitá mostratami durante la mia visita di due mesi a Toulouse. In particolare, grazie a Jean-Pierre Raymond, e Sylvain Ervedoza e anche a Isabelle Guichard, per il suo aiuto in tutte le questioni amministrative relazionate al mio soggiorno.

Grazie a tutti i professori e ricercatori che ho incontrato durante questi anni di dottorato, e che hanno contribuito al mio lavoro con lunghe ore di interessanti discussioni. Tra questi, Carlos Castro, Cristian Cazacu, Felipe Chaves, Denys Dutykh, Genni Fragnelli, Mamadou Gueye, Valentin Keyantuo, Qi Lü, Aurora Marica, Denis Matignon, Enrique Otárola e Xavier Ros-Oton.

I miei piú sinceri ringraziamenti vanno alle persone del BCAM, per questi tre anni e mezzo che abbiamo passato insieme. In particolare, vorrei ricordare Carmen Alonso, Sébastien Breteaux, Biagio Cassano, Roberto Castelli, Francesco Fanelli, Daniel García, Imanol García, Inderpreet Kaur, Alessia Kogoj, Julia Kroos, Ariel Lozano, Thomas Ourmières-Bonafos, Alejandro Pozo, Tijana Radivojevic, Antsa Ratsimanetrimanana, Simone Rusconi e Goran Stipcich.

Grazie ai miei due compagni di ufficio Fabio Pizzichillo e Julia Sánchez, per l'ottimo ambiente dentro e fuori del luogo di lavoro. Inoltre, un riconoscimento speciale va alla quarta componente del Y19, Aitziber Ibañez, e a Mario Fernández, la mia linea diretta con la RAE, per il loro aiuto nella revisione e correzione delle parti della mia tesi scritte in spagnolo.

Ringrazio tutte le persone che hanno reso il mio arrivo e la mia permanenza nei Paesi Baschi cosí gradevoli. Non mi é possibile nominarli uno per uno, ma vorrei almemo menzionare i miei compagni della squadra Diskolaris Ultimate Bilbao, che mi hanno introdotto a quel meraviglioso sport che é l'Ultimate Frisbee e che hanno condiviso con me momenti indimenticabili durante i nostri viaggi a giro per la Spagna.

Un grazie sincero al mio caro amico Leonardo Falco, la cui amicizia mi ha accompagnato durante tutti questi anni, anche dopo essermi trasferito a Bilbao, e che mi accoglie con gran entusiasmo tutte le volte che torno in Italia.

Grazie a mio fratello Edoardo Biccari, che come me ha dovuto lasciare l'Italia per inseguire il suo futuro. Ora viviamo a due estremi di uno stesso continente, e sono felice che nonostante tutto riusciamo ancora ad incontraci ogni tanto, scambiandoci le nostre reciproche esperienze.

Ultimi, ma non per questo meno importanti, vorrei ringraziare profondamente i miei genitori Davide Biccari e Fiorenza Nigro, che sono sempre stati straordinariamente comprensivi e incoraggianti durante tutta la mia vita. Ho sempre potuto contare sul loro appoggio e sulla loro approvazione. Grazie mille!

Abstract

In this thesis, we investigate controllability and observability properties of Partial Differential Equations describing various phenomena appearing in several fields of the applied sciences such as elasticity theory, ecology, anomalous transport and diffusion, material science, porous media flow and quantum mechanics. In particular, we focus on evolution Partial Differential Equations with non-local and singular terms.

Concerning non-local problems, we analyse the interior controllability of a Schrödinger and a wave-type equation in which the Laplace operator is replaced by the fractional Laplacian $(-\Delta)^s$. Under appropriate assumptions on the order s of the fractional Laplace operator involved, we prove the exact null controllability of both equations, employing a L^2 control supported in a neighbourhood ω of the boundary of a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$. More precisely, we show that both the Schrödinger and the wave equation are null-controllable, for $s \geq 1/2$ and for $s \geq 1$ respectively. Furthermore, these exponents are sharp and controllability fails for $s < 1/2$ (resp. $s < 1$) for the Schrödinger (resp. wave) equation. Our proof is based on multiplier techniques and the very classical Hilbert Uniqueness Method.

For models involving singular terms, we firstly address the boundary controllability problem for a one-dimensional heat equation with the singular inverse-square potential $V(x) := \mu/x^2$, whose singularity is localised at one extreme of the space interval $(0, 1)$ in which the PDE is defined. For all $0 < \mu < 1/4$, we obtain the null controllability of the equation, acting with a L^2 control located at $x = 0$, which is both a boundary point and the pole of the potential. This result follows from analogous ones presented in [76] for parabolic equations with variable degenerate coefficients.

Finally, we study the interior controllability of a heat equation with the singular inverse-square potential $\Lambda(x) := \mu/\delta^2$, involving the distance δ to the boundary of a bounded and C^2 domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. For all $\mu \leq 1/4$ (the critical Hardy constant associated to the potential Λ), we obtain the null controllability employing a L^2 control supported in an open subset $\omega \subset \Omega$. Moreover, we show that the upper bound $\mu = 1/4$ is sharp. Our proof relies on a new Carleman estimate, obtained employing a weight properly designed for compensating the singularities of the potential.

Resumen

En esta tesis analizamos la controlabilidad y observabilidad de ciertos tipos de Ecuaciones en Derivadas Parciales que describen varios fenómenos que se presentan en muchos campos de las ciencias aplicadas, como por ejemplo la teoría de la elasticidad, ecología, transporte y difusión anómalos, ciencia de los materiales, filtración en medios porosos y mecánica cuántica. En particular, nos centramos en EDPs de evolución con términos no-locales o singulares.

Con respecto a los problemas no-locales, analizamos la controlabilidad interior de ecuaciones de tipo Schrödinger y ondas, donde el operador de Laplace es sustituido por el Laplaciano fraccionario $(-\Delta)^s$. Bajo hipótesis adecuadas sobre el orden s del operador de Laplace fraccionario involucrado, probamos la controlabilidad exacta a cero de ambas ecuaciones, a través de un control de clase L^2 que actúa desde un conjunto ω de la frontera de un dominio $\Omega \subset \mathbb{R}^N$, acotado y de clase $C^{1,1}$. Con más detalles, mostramos que tanto la ecuación de Scrödinger como la de ondas se pueden controlar a cero, para $s \geq 1/2$ y para $s \geq 1$ respectivamente. En cambio, probamos que, fuera de este rango de valores para el exponente s, las ecuaciones no son controlables. Nuestros resultados se basan en el técnicas de multiplicadores y en el famoso Método de Unicidad de Hilbert.

Para modelos que involucran a términos singulares, en primer lugar tratamos el problema de la controlabilidad de frontera para una ecuación del calor unidimensional con el potencial singular cuadrático-inverso $V(x) := \mu/x^2$, cuya singularidad surge en uno de los extremos del intervalo $(0, 1)$ donde está definida la EDP. Para todo $0 < \mu < 1/4$, obtenemos la controlabilidad a cero de la ecuación, empleando un control de clase L^2 posicionado en $x = 0$, que es a la vez un punto de frontera y el polo del potencial singular. Este resultado es consecuencia de resultados análogos presentados en [76] para ecuaciones parabólicas con coeficientes degenerados.

Por último, nos interesamos en la controlabilidad interior de una ecuación del calor con el potencial singular cuadrático-inverso $\Lambda(x) := \mu/\delta^2$, donde δ^2 es la distancia desde el borde de un dominio $\Omega \subset \mathbb{R}^N$, $N \geq 3$, acotado y de clase C^2. Para cada $\mu \leq 1/4$ (el valor crítico de la constante de Hardy asociada al potencial Λ), obtenemos la controlabilidad exacta a cero de la ecuación estudiada, por medio de un control de clase L^2 con soporte en un subconjunto abierto $\omega \subset \Omega$. Además, mostramos que el valor $\mu = 1/4$ es óptimo para la controlabilidad. Nuestros resultados se basan sobre una nueva estimación de Carleman, obtenida empleando un peso que consigue compensar las singularidades del potencial, que esta vez se distribuyen en toda la frontera.

Contents

List of Figures

Chapter 1

Introduction

Control theory is the branch of mathematics that studies the possibility of modifying the behaviour of a dynamical system employing one or more controls applied through actuators.

Very early examples of controlled systems can be traced back, for instance, to the ancient Romans, who developed smart devices of regulating valves for keeping the water level in their aqueducts constant. Furthermore, according to some scholars, we can find even earlier applications of control theory in the irrigation systems employed in the ancient Mesopotamia, more than 2000 years B.C.

Nevertheless, for having a first mathematical approach to control theory we have to wait until the 19$^{\text{th}}$ century, during the industrial revolution. In particular, we can mention the work of the British astronomer G. Airy (1801-1892), who analysed mathematically the operating principles regulating the well-known steam engine invented by J. Watt (1736-1819). Finally, the first definitive mathematical description of control theory is dated 1868, and it is due to J.C. Maxwell (1831-1879), who encountered some erratic behaviours in Watt's device and proposed some control mechanisms to correct them.

Since its origin, control theory has captured the interest of many mathematicians and engineers, who contributed to its extensive development. Nowadays, this is a very prosperous field, with many different practical applications in areas such as engineering, biology, economics and medicine. For more details see, for instance, [61] and the rich references therein.

Mathematically speaking, a very general and abstract way of writing a control problem is through the following dynamical system

$$\begin{cases} \dfrac{dy}{dt} = A(y,u), \quad t > 0,\, y \in Y,\, u \in U_{\text{ad}} \\ y(0) = y_0 \end{cases} \tag{1.0.1}$$

in which y represents the state that we want to control, y_0 is the initial state and u is the control function. Y and U_{ad} are the state space and the set of admissible controls, respectively.

Given a control system in the form (1.0.1), the main purpose is to find u such that the corresponding state y behaves in an appropriate manner in a given final time. This is the so-called controllability problem.

It is possible to identify several notions of controllability, depending on whether it is possible or not to achieve the objective described above. We say that the system is *exactly controllable* if any initial state y_0 can be driven to any desired final state y_T in a finite time T. If, in addition, it is possible to reach the zero state (i.e. $y(T) = 0$), then the system is said to be *null-controllable*. On the other hand, if we can only reach a state arbitrarily close (in some topology) to the target y_T, then we speak of *approximate controllability*. Finally, if one can show that there is no way to find a function u allowing to drive the solution of (1.0.1) to the desired state (or arbitrarily close to it), then this means that the system is not controllable.

In this thesis, we are interested in the analysis of exact controllability properties for some given type of PDEs, describing several physical phenomena. We devote the next sections to a complete description of the kind of problems treated in this work, providing a general overview of the existing literature and briefly introducing the main results that we achieved.

1.1 Main topics and motivation

This thesis is concerned with the analysis of controllability properties for some complex PDE problems, whose study is motivated by many real world applications. In particular, we focus on two very general families of models, that have largely interested the applied mathematical research in the last decades: *non-local PDEs* and *PDEs involving singular inverse-square potentials*.

The problems that we are going to treat are, in our opinion, very fascinating and challenging. Due to their difficulties, in many cases they require the development of new mathematical techniques and, also when classical results can be applied, their adaptation to the systems under consideration is not trivial.

We devote this section to a very general presentation of the motivations on the basis of the growing interest in the PDEs models subject of our work, with particular attention to their employment to several fields of applied sciences, engineering and finance.

For the sake of a more clear and neat presentation, and for providing a better understanding, we are going to consider separately the two main categories of equations that we analyse.

1.1.1 Partial Differential Equations involving non-local terms

A non-local PDE is a particular type of differential equation in which either some or all the components involve non-local terms. As the name suggests, the first and main difference with respect to classical PDEs is that, in order to check whether a non-local equation holds at a point, it is necessary to have information also about the values of the function far away from that point; most often, this is because the equation involves integral terms. For this reason, in the literature these problems are often referred as *integro-differential* or *pseudo-differential*.

The analysis of non-local operators and non-local PDEs is a topic in continuous development. In the last decades, many researchers have started devoting their attention to this branch of the mathematics, motivated in particular by a large number of possible applications in the modelling of several complex phenomena for which a local approach turns up to be inappropriate or limiting.

Indeed, there is an ample spectrum of situations in which a non-local equation gives a significantly better description than a PDE of the problem one wants to analyse.

In elasticity, for instance, many models involve non-local terms; an important example is certainly the Peierls-Nabarro equation, which arises in the description of phenomena of dislocation dynamics in crystals ([49, 101]).

Further, in material sciences non-local models take into account that in many materials the stress at a point depends on the strains in a region near that point ([85]).

Integro-differential equations also appear in ecology. For instance, in population dynamics, non-local reaction-diffusion equations arise in models for ecosystems structure that analyse the interplay between food-dependent growth and size-dependent mortality in certain predator-prey systems ([46]).

In finance, the prices of assets can have frequent and unexpected changes. Therefore, models involving jump processes turn out to be particularly appropriate for describing, for instance, the pricing of American options ([96, 112]).

Finally, other examples in which integro-differential equations appear are models for turbulence ([3]), anomalous transport and diffusion ([14, 105]), porous media flow ([15]), image processing ([71]), wave propagation in heterogeneous high contrast media ([146]).

1.1.2 Partial Differential Equations involving inverse-square potentials

The second part of this thesis is devoted to the study of evolution PDEs containing singular inverse-square potentials. In this framework, for the analysis of these equations a fundamental tool is the very famous Hardy inequality, which takes its name from the British mathematician G.H. Hardy (1877-1947). In 1925, he proved in [77] that for any $u \in H_0^1(0, +\infty)$, it holds

$$\int_0^{+\infty} |u'(x)|^2 \, dx \geq \frac{1}{4} \int_0^{+\infty} \left(\frac{u(x)}{x} \right)^2 dx. \tag{1.1.1}$$

This inequality was the conclusion of twenty years of investigation, starting from a closely related result obtained by D. Hilbert in 1904 ([79]). For its development, we need to remind the fundamental contributions of many famous mathematicians, other than Hardy, such as E.

Landau, G. Pólya, M. Riesz and I. Schur. The interested reader may refer to [90] and to the bibliography therein for a complete survey of the history of (1.1.1).

Nine years after the 1925 paper by Hardy, inequality (1.1.1) was used in [95] for the study of the existence of regular solutions for the viscous Navier's equation.

Later on, it was again Hardy, in collaboration with J.E. Littlewood and G. Pólya, who generalised (1.1.1) to the multi-dimensional case and, in [78], it was firstly introduced the Hardy inequality in its more classical and known version. Namely, the authors proved that, for any open domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, containing the origin, and for any $u \in H_0^1(\Omega)$, then $u/|x| \in L^2(\Omega)$ and the following estimate holds:

$$\int_\Omega |\nabla u(x)|^2 \, dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2(x)}{|x|^2} \, dx. \tag{1.1.2}$$

The constant $(N-2)^2/4$ in (1.1.2) is optimal and it is not attained in $H_0^1(\Omega)$, meaning that the continuous embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2})$ is not compact.

The study of the Hardy inequality and of related integral-differential inequalities is motivated by applications in several fields.

In Quantum Physics, for instance, there are close relations between (1.1.2) and the Uncertainty Principle of Heisenberg (see e.g. [59]), while in Quantum Mechanics, (1.1.2) is fundamental when studying the non-relativistic Schrödinger equation for a single particle moving in an electric field ([58]).

In the theory of ordinary differential equations, Hardy type inequalities are applied to the study of oscillations of solutions ([82, 109]) or in approximation problems ([86]).

Furthermore, from a mathematical perspective, we can mention several applications also in Sturm-Liouville problems ([8, 116]), in the theory of Fourier series ([38]), in the spectral analysis of differential operators ([56, 114]), in differential geometry ([65, 134]), in functional analysis, for obtaining embedding theorems for weighted Sobolev spaces ([87, 88]), and in complex functions theory ([110]).

In the theory of singular PDEs, the Hardy inequality has a crucial role in the analysis of qualitative properties of (generalised) Schrödinger operators of the form $-\Delta - V(x)$, with inverse-square potentials. This kind of operators arises, for instance, in quantum cosmological models, as emphasized by the Wheeler-de-Witt equation ([9]), or in electron capture problems ([72]), but also in the linearisation of non-linear reaction-diffusion problems involving the heat equation with supercritical reaction terms, with application in thermodynamics ([39]) and in combustion theory ([69, 70]).

There is nowadays a well established literature on the Hardy inequality and on many different extensions of this important result. The interested reader may refer, for instance, to the following papers and to the references therein: [5, 21, 24, 44, 62, 63, 64, 69, 84, 125, 133, 141]. Furthermore, it is also worth to cite the articles [16, 60], regarding inequalities with multipolar singularities. Finally, for some of the results presented in this thesis we mention the works [21, 22], that concern singular potentials involving the distance to the boundary.

1.2 Contents of the Thesis

In this work, we are mainly interested in obtaining control properties for the two classes of problems that we mentioned in the previous section. We are therefore considering some explicit examples of evolution PDEs involving non-local terms or singular potentials and, for each of them, we are going to study the possibility of obtaining controllability results, both from the interior and from the boundary of the domains in which we define our equations.
In more detail, the main body of this thesis is composed of the following Chapters:

- **Chapter 3: Internal control for non-local Schrödinger and wave equations involving the fractional Laplace operator**. In this Chapter, we study a non-local version of the classical Schrödinger equation, where the Laplace operator is replaced by the fractional Laplacian $(-\Delta)^s$. We show that for $s \in [1/2, 1)$ null controllability holds, acting from a neighbourhood ω of the boundary of a bounded domain $\Omega \in C^{1,1}$. On the other hand, we also show that this result is sharp, i.e. it is not achievable for exponents $s < 1/2$. In our analysis, we use multiplier techniques ([83]) and the Pohozaev identity for the fractional Laplacian ([119]) for obtaining the observability inequality that we need for applying the Hilbert Uniqueness Method ([97, 98]). As a consequence of the controllability for the fractional Schrödinger equation, an analogous property for a non-local wave equation with fractional Laplacian is obtained. The results of this Chapter are contained in the research article [11].

- **Chapter 4: Boundary controllability for a one-dimensional heat equation with a singular inverse-square potential**. This Chapter is concerned with the analysis of the parabolic problem for the generalised one-dimensional Schrödinger operator $\mathcal{A} = -d^2_{xx} - V(x)$ where, for all $\mu \in \mathbb{R}$, $V(x)$ is the inverse-square potential defined as

  $$V(x) := \frac{\mu}{x^2}.$$

  For any time $T > 0$, we assume the domain of definition for our equation to be the set $Q := \{(x,t) \in (0,1) \times (0,T)\}$; this means that the singularity of the potential V arises at a boundary point. For all $0 < \mu < 1/4$, we prove the null controllability acting from the point $x = 0$ as a consequence of analogous results presented in [76].

- **Chapter 5: Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function**. In this Chapter, we consider a parabolic equation with singular potential

$$\Lambda(x) := \frac{\mu}{\delta(x)^2},$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$ is the distance between a point x and the boundary of a bounded C^2 domain $\Omega \subset \mathbb{R}^N$. The coefficient μ is assumed to be lower or equal to $\mu^* = 1/4$, which is the critical value for the generalised Hardy inequality involving the function δ ([21]). As a consequence of a new Carleman estimate, we obtain the null controllability acting from a subset ω of our domain of definition. Moreover, we show that in the supercritical case, i.e. for $\mu > 1/4$, there is no way of preventing the solutions of the equation from blowing-up, obtaining thus the impossibility of controlling the system. These properties are obtained adapting analogous proofs in [35, 53]. The results of this Chapter are contained in the research article [12], in collaboration with E. Zuazua.

- **Chapter 6: Open problems**. In this Chapter, we present some open problems related to the results obtained in the thesis, discussing their motivation and interest and briefly introducing the difficulties that they hide.

We give now a preliminary survey of the contents of each chapter, introducing the main results that we obtained with more details.

1.2.1 Chapter 3: Internal control for non-local Schrödinger and wave equations involving the fractional Laplace operator

In this Chapter, we are concerned with the null controllability problem for the following Schrödinger-type equation involving the fractional Laplace operator

$$\begin{cases} iu_t + (-\Delta)^s u = h\chi_{\{\omega\times(0,T)\}}, & (x,t) \in \Omega \times (0,T) \\ u \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2.1}$$

defined on a bounded and $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$. In (1.2.1), the control region ω is a neighbourhood of the boundary of Ω.

The study of evolution equations involving the fractional Laplacian is a quite new topic and at the moment there is not a very extended literature. To the best of our knowledge, the results that we are going to present are among the first available in control theory for non-local PDEs. In fact, the main result that we are going to employ ([119]) has been obtained very recently.

We are going to show that in the range of exponents $s \in [1/2, 1)$, there exists a L^2-control function h, supported in ω, such that the unique solution u of (1.2.1) satisfies

$$u(x,T) = 0. \tag{1.2.2}$$

Besides, we will also show that the lower bound $s = 1/2$ is sharp, meaning that, whenever $s < 1/2$, there is no possibility of controlling the equation. Therefore, the main result of Chapter 3 will be the following:

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain and let $s \in [1/2, 1)$. Moreover, let us define $\Gamma_0 := \{x \in \partial\Omega \,|\, (x \cdot \nu) > 0\}$, where ν is the unit normal vector to $\partial\Omega$ at x pointing towards the exterior of Ω, and $\omega = \mathcal{O}_\varepsilon \cap \Omega$, where $\mathcal{O}_\varepsilon$ is a neighbourhood of Γ_0 in $\mathbb{R}^N$.*

(i) If $s \in (1/2, 1)$, for any $T > 0$ and for any $u_0 \in L^2(\Omega)$ there exists a control function $h \in L^2(\omega \times [0, T])$ such that the solution u of (1.2.1) satisfies $u(x, T) = 0$;

(ii) if $s = 1/2$, there exists a minimal time $T_0 > 0$ such that the same controllability result as in (i) holds for any $T > T_0$.

Besides, in both cases there exists a positive constant C_T such that

$$\|h\|_{L^2(\omega\times[0,T])} \leq C_T \|u_0\|_{L^2(\Omega)}.$$

Theorem 1.2.1 will be obtained applying the classical technique that combines multiplier methods and the Hilbert Uniqueness Method ([83, 97]), and it will be a consequence of an observability inequality for the adjoint system associated to (1.2.1), namely

$$\begin{cases} iv_t + (-\Delta)^s v = 0, & (x,t) \in \Omega \times (0,T) \\ v \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2.3}$$

In particular, employing the regularity theory for fractional elliptic problems developed in [117, 118], and by means of a new Pohozaev identity for the fractional Laplacian ([119]), we are going to prove that, under the conditions on the time T imposed in Theorem (1.2.1), there exists a positive constant $C > 0$ such that the solution of (1.2.3) satisfies:

$$\|v_0\|^2_{L^2(\Omega)} \leq C \int_0^T \|v\|^2_{L^2(\omega)}\, dt; \tag{1.2.4}$$

this immediately implies (1.2.2) by means of a duality argument.

Regarding the impossibility of controlling the equation for $s < 1/2$, this fact will be justified through a Fourier analysis of the following one-dimensional problem

$$\begin{cases} iu_t + (-d_x^2)^s u = g\chi_{\{\omega\times(0,T)\}}, & (x,t) \in (-1,1) \times (0,T) \\ u \equiv 0, & (x,t) \in (-1,1)^c \times (0,T), \\ u(x,0) = u_0(x), & x \in (-1,1), \end{cases} \tag{1.2.5}$$

where $\omega \subset (-1, 1)$ is the subset of the domain from which we aim to control.

Our analysis for (1.2.5) will be based on some results presented in [91, 92] on the asymptotic behaviour of the spectrum of the one-dimensional fractional Laplacian on the interval $(-1,1)$. In particular, in [92] it is shown that for the eigenvalues associated to the problem

$$\begin{cases} (-d_x^2)^s\phi_k(x) = \lambda_k\phi_k(x), & x\in(-1,1) \\ \phi_k(x)\equiv 0, & x\in(-1,1)^c \end{cases} \tag{1.2.6}$$

it holds

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(2-2s)\pi}{8}\right)^{2s} + O\left(\frac{1}{k}\right), \quad \text{as} \;\; k\to+\infty. \tag{1.2.7}$$

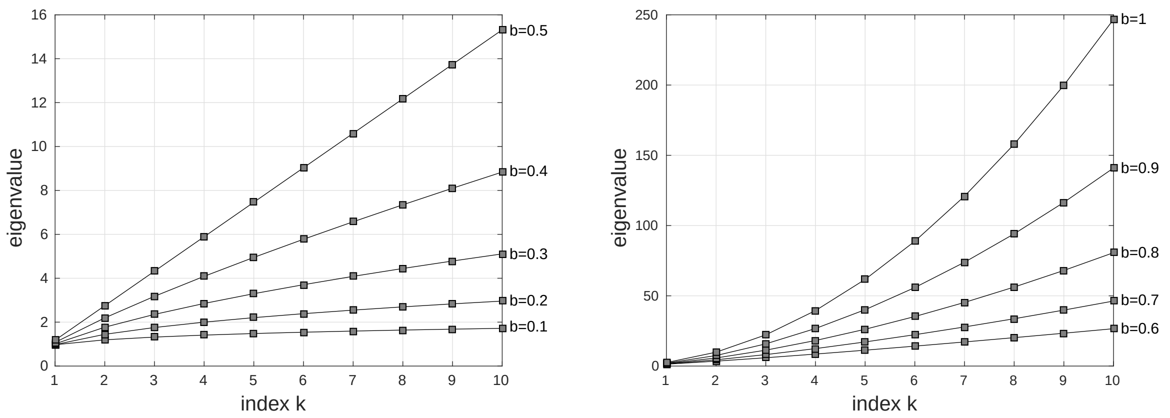

Figure 1.1: First 10 eigenvalues of the fractional Laplacian $(-d_x^2)^\beta$ on $(-1,1)$ for $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (left) and for $\beta = 0.6, 0.7, 0.8, 0.9, 1$ (right).

Employing (1.2.7) it is possible to show that, for $s < 1/2$, the asymptotic gap between the eigenvalues goes to zero with k, i.e. that (see also Figure 1.2 below)

$$\liminf_{k\to+\infty}(\lambda_{k+1}-\lambda_k) = 0. \tag{1.2.8}$$

Referring to Ingham theory ([107]), (1.2.8) implies that, in this case, the observability inequality fails, which means that equation (1.2.1) fails to be controllable.

Finally, it is worth to spend some additional words on the controllability Theorem 1.2.1, in particular on the introduction of a minimal time $T_0 > 0$ when $s = 1/2$. As it will be explained in details in Section 3.3, this minimal time will appear naturally during the proof of our result. It will be needed for obtaining the observability of (1.2.3) due to the fact that, when $s = 1/2$, we will encounter terms which are not compact with respect to the quantity that we want to observe and that will need a time T large enough in order to be absorbed.

In addition, we point out that the introduction of T_0 has not only technical motivations but, in our opinion, it is really related to the structure of our problem. Indeed, when $s = 1/2$, the

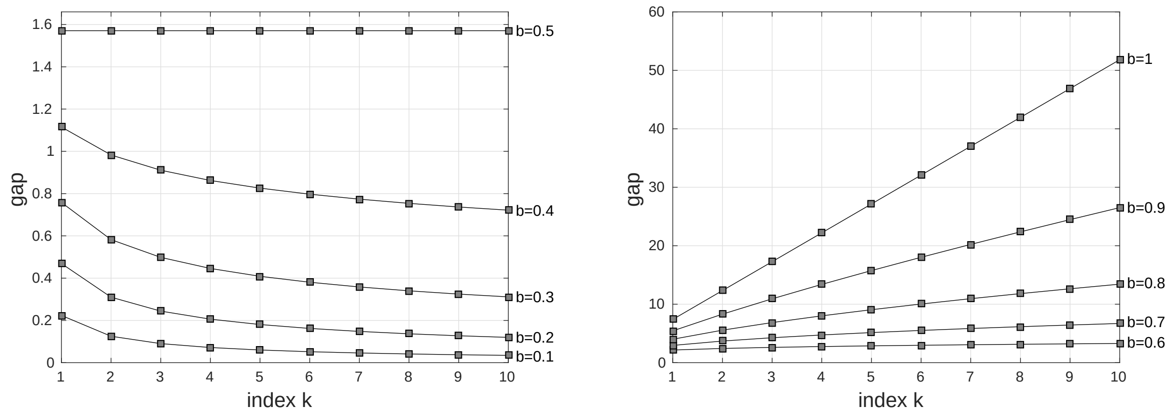

Figure 1.2: Gap between the first 10 eigenvalues of the fractional Laplacian $(-d_x^2)^\beta$ on $(-1,1)$ for $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (left) and for $\beta = 0.6, 0.7, 0.8, 0.9, 1$ (right). At any index k corresponds the gap $\lambda_{k+1} - \lambda_k$.

solutions of our equation have a uniform velocity of propagation and this implies that we need a time interval sufficiently large in order to observe them. A justification to this fact is provided by formula (1.2.7) for the behaviour of the eigenvalues of the one-dimensional fractional Laplacian that, in this limit case, gives us a constant gap (see also Figure 1.2)

$$\lambda_{k+1} - \lambda_k = \frac{\pi}{2}, \quad \text{for all } \; k > 0.$$

Referring again to Ingham theory ([107]), this last condition automatically yields to the introduction of T_0, since we know that this is the case when we have a uniform asymptotic gap. On the other hand, when the asymptotic gap is $\gamma_\infty = \infty$, as in the case $s > 1/2$, observation is expected for all time $T > 0$.

The last part of the Chapter will be devoted to the study of the wave-type equation

$$\begin{cases} u_{tt} + (-\Delta)^{2s} u = h\chi_{\{\omega\times(0,T)\}}, & (x,t) \in \Omega \times (0,T), \\ u \equiv (-\Delta)^s u \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.2.9}$$

where the higher order fractional Laplace operator $(-\Delta)^{2s}$ is defined simply as the square of the fractional Laplacian $(-\Delta)^s$, as follows

$$(-\Delta)^{2s} u(x) := (-\Delta)^s (-\Delta)^s u(x), \quad s \in [1/2, 1),$$

$$\mathcal{D}((-\Delta)^{2s}) = \Big\{ u \in H_0^s(\Omega) \,\Big|\, (-\Delta)^s u|_{\Omega^c} \equiv 0, (-\Delta)^{2s} u \in L^2(\Omega) \Big\}.$$

Notice that $(-\Delta)^{2s}$ is symmetric, positive and self-adjoint, since it is the double composition of the symmetric, positive and self-adjoint operator $(-\Delta)^s$

As a consequence of Theorem 1.2.1, and applying an abstract machinery presented in [135], we will be able to obtain an observability inequality for the solution v of the adjoint system

$$\begin{cases} v_{tt} + (-\Delta)^{2s} v = 0, & (x,t) \in \Omega \times (0,T), \\ v \equiv (-\Delta)^s v \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), & x \in \Omega. \end{cases} \tag{1.2.10}$$

In more detail, let T_0 be the observation time introduced in Theorem 1.2.1. Then, applying [135, Proposition 6.8.2], from (1.2.4) we will obtain that, for $s \in [1/2, 1)$ and for any $T > T_0$, there exists a positive constant C such that it holds

$$\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-2s}(\Omega)} \leq C \int_0^T \|v\|^2_{L^2(\omega)}\, dt. \tag{1.2.11}$$

From (1.2.11), we will deduce that also equation (1.2.9) is null-controllable with a L^2-control h distributed in a neighbourhood ω of the boundary of the domain.

1.2.2 Chapter 4: Boundary controllability for a one-dimensional heat equation with a singular inverse-square potential

This Chapter is devoted to the analysis of the following one-dimensional heat equation with a singular inverse-square potential

$$\begin{cases} u_t - u_{xx} - \dfrac{\mu}{x^2} u = 0, & (x,t) \in (0,1) \times (0,T), \\ x^{-\lambda} u(x,t)\big|_{x=0} = f(t), \ \ u(1,t) = 0, & t \in (0,T) \\ u(x,0) = u_0(x), & x \in (0,1). \end{cases} \tag{1.2.12}$$

Once again, we will focus on the study of controllability properties. In particular, we are interested in solving the following problem.

Problem 1.2.1. *Given u_0 in an appropriate functional space X on $(0,1)$, find f in a functional space Y on $(0,T)$, such that the corresponding solution u of (1.2.12) satisfies $u(x,T) = 0$ for all $T > 0$.*

The strategy that we will apply for obtaining this result consists in showing that, by means of the change of variables

$$u(x,t) := x^{\frac{\alpha}{2(2-\alpha)}}\, \psi(x,t), \quad x(\xi) := \left(\frac{2}{2-\alpha}\right) \xi^{\frac{2-\alpha}{2}},$$

with

$$\alpha = \frac{2 + 8\mu - 2\sqrt{1-4\mu}}{3+4\mu},$$

we can transform our original equation (1.2.12) in the following one with variable degenerate coefficients

$$\psi_t - (\xi^\alpha \psi_\xi)_\xi = 0, \tag{1.2.13}$$

for which there are already results of boundary controllability (see [76]).

Evolution equations with singular inverse-square potentials have already attracted the interest of the control community in the past years. Among other works, we recall here [35, 53, 137], regarding the heat equation, and [34, 138], for the wave equation; in all these papers, the authors are able to obtain controllability properties, acting from the interior of the domain where the equation is defined.

However, to the best of our knowledge, the issue of boundary controllability for these equations was not addressed before. Moreover, another main novelty of our research is that, for the first time, we are able to control from a point where the singularity arises.

In the analysis of our problem, a first important aspect that we want to underline is the fact that, due to the presence of the singularity at $x = 0$, it turns out that in (1.2.12) we cannot impose a boundary condition of the type $u(0,t) = f(t) \neq 0$; instead, we need to introduce the "weighted" boundary condition

$$x^{-\lambda} u(x,t)\Big|_{x=0} = f(t),$$

with

$$\lambda := \frac{1}{2}\left(1 - \sqrt{1-4\mu}\,\right).$$

This issue will be carefully justified throughout the Chapter.

As usual, by means of the classical Hilbert Uniqueness Method, Problem 1.2.1 will be equivalent to the proof of a suitable observability estimate for the adjoint system associated to (1.2.12), namely

$$\begin{cases} v_t + v_{xx} + \dfrac{\mu}{x^2} v = 0, & (x,t) \in (0,1) \times (0,T) \\ v(0,t) = v(1,t) = 0, & t \in (0,T) \\ v(x,T) = v_T(x), & x \in (0,1). \end{cases} \tag{1.2.14}$$

This estimate, in turn, will be obtained from the corresponding inequality presented in [76] for equation (1.2.13), passing through the change of variables mentioned above.

Nevertheless, this approach provides limitations on the values that can be assumed by the coefficient μ. Indeed, while by means of transposition techniques ([99]) equation (1.2.14) turns out to be well-posed for all $\mu \leq 1/4$, our proof of its observability will be valid only for $0 < \mu < 1/4$. We will present more details on this fundamental issue throughout the Chapter.

Finally, we want to stress the fact that in the adjoint system (1.2.14) we are imposing classical Dirichlet boundary conditions, that is, without introducing any weight. Indeed, in equation (1.2.12) the weight at $x = 0$ is needed if we want to detect a non-zero boundary data; on the contrary, when considering a problem with homogeneous boundary conditions the polynomial behaviour of the solution ensures the well-posedness in the classical framework.

1.2.3 Chapter 5: Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary

In this Chapter, we consider the following heat equation with singular potential

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2}u = f, & (x,t) \in \Omega \times (0,T) \\ u = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2.15}$$

defined on a bounded and C^2 domain $\Omega \subset \mathbb{R}^N$, where $\delta(x) := \text{dist}(x, \partial\Omega)$ is the distance to the boundary function. Again, we aim to obtain controllability results.

Also in this case, we will show that (1.2.15) is null-controllable with a $L^2(\omega)$-control f distributed in an open subset $\omega \subset \Omega$. In particular, the main result of this Chapter will be the following:

Theorem 1.2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain and assume $\mu \leq 1/4$. Given any non-empty open set $\omega \subset \Omega$, for any time $T > 0$ and any initial datum $u_0 \in L^2(\Omega)$ there exists a control function $f \in L^2(\omega \times (0,T))$ such that the solution of* (1.2.15) *satisfies $u(x,T) = 0$.*

The upper bound for the coefficient μ plays a fundamental role in our analysis and it is related to the following generalised Hardy inequality involving the potential μ/δ^2, presented in [21]

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{\delta^2} \, dx.$$

Problems of the type of (1.2.15) have been widely studied in the last decades; in [25], for instance, it is shown that the value $\mu = 1/4$ is critical for the well-posedness of (1.2.15), meaning that for $\mu > 1/4$ the equation admits no positive weak solution for any u_0 positive and $f = 0$. Moreover, there is instantaneous and complete blow-up of approximate solutions.

Again by means of Hilbert Uniqueness Method ([97]), Theorem 1.2.2 will be a consequence of the possibility of observing the solution of the adjoint system associated to (1.2.15), namely

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{\delta^2} v = 0, & (x,t) \in Q \\ v = 0, & (x,t) \in \Gamma \times (0,T) \\ v(x,T) = v_T(x), & x \in \Omega. \end{cases} \tag{1.2.16}$$

More precisely, for any $\mu \leq 1/4$ we are going to prove that, for any time $T > 0$, there exists a positive constant C_T such that, for all $v_T \in L^2(\Omega)$, the solution of (1.2.16) satisfies

$$\int_\Omega v(x,0)^2 \, dx \leq C_T \int_{\omega \times (0,T)} v(x,t)^2 \, dxdt. \tag{1.2.17}$$

The inequality above, in turn, will be obtained as a consequence of a new Carleman estimate for the solution of (1.2.16), where the weight employed is chosen in such a way to permit us to deal with the blowing-up of the potential on the boundary. We remark that this Carleman estimate cannot be trivially derived from the ones already available in the literature for equations with singular inverse-square potentials ([35, 53]), since in our case the singularity considered is of a different nature.

Finally, adapting the argument presented in [53] we will show that the bound $\mu \leq 1/4$ is sharp for our controllability result.

Capítulo 1

Introducción

La teoría del control es la rama de las matemáticas que estudia la posibilidad de modificar el comportamiento de un sistema dinámico utilizando uno o más controles aplicados a través de activadores.

Ejemplos precoces de sistemas controlados se pueden encontrar ya en la época de los Romanos, que supieron desarrollar mecanismos de válvulas para mantener constante el nivel del agua en sus acueductos. Además, según algunos investigadores, es posible encontrar aplicaciones de la teoría del control aún más antiguas en los sistemas de irrigación empleados en Mesopotamia, que datan aproximadamente del 2000 A.C.

Sin embargo, no es hasta el siglo XIX, durante la revolución industrial, cuando se aborda la teoría del control desde un punto de vista matemático. En particular, merece especial mención el trabajo del astrónomo inglés G. Airy (1801-1892), que analizó matemáticamente los principios que regulaban el funcionamiento de la máquina de vapor inventada por J. Watt (1736-1819). Por último, la primera descripción matemática completa de la teoría del control data de 1868 y se debe a J.C. Maxwell (1831-1879), quien encontró algunos comportamientos erráticos en el aparato de Watt y propuso mecanismos de control para corregirlos.

Desde su origen, la teoría del control ha captado el interés de un gran número de matemáticos e ingenieros, que contribuyeron a su vasto desarrollo. Hoy en día, este es un campo muy próspero, con una gran cantidad de aplicaciones prácticas en áreas como la ingeniería, la biología, la economía y la medicina. Más detalles se pueden encontrar, por ejemplo, en [61] y en la amplia bibliografía allí contenida.

En lenguage matemático, una manera general y abstracta para escribir un problema de control es mediante el siguiente sistema dinámico

$$\begin{cases} \dfrac{dy}{dt} = A(y,u), \quad t > 0,\, y \in Y,\, u \in U_{\text{ad}} \\ y(0) = y_0 \end{cases} \tag{1.0.1}$$

en el cual y representa el estado que queremos controlar, y_0 es el estado inicial y u es el control. Y y U_{ad} son el espacio de los estados y el conjunto de los controles admisibles, respectivamente.

Dato un sistema de control en la forma (1.0.1), el objetivo principal es buscar u tal que el estado correspondiente y se comporte de una manera establecida en un tiempo final fijado.

Se pueden definir varias nociones de controlabilidad, dependiendo de si es posible o no conseguir el propósito descrito antes. Decimos que el sistema es *exactamente controlable* si cualquier estado inicial y_0 puede ser conducido en un tiempo T finito a cualquier estado final y_T, previamente elegido. Si, además, es posible llegar al estado cero (es decir $y(T) = 0$), entonces el sistema se dice *controlable a cero*. Por otra parte, si solo es posible acercarse arbitrariamente (en alguna topología) al objetivo y_T, entonces se habla de *controlabilidad aproximada*. Por último, si se puede mostrar que no hay manera alguna de encontrar una función u que permita conducir la solución de (1.0.1) al estado deseado (o arbitrariamente cerca de él), eso significa que el sistema no es controlable.

En esta tesis, estamos interesados en el análisis de las propiedades de controlabilidad para determinados tipos de EDP que describen varios fenómenos físicos. Dedicamos las secciones siguientes a una descripción mas detallada de las clases de problemas que trataremos en este trabajo, dando un resumen general de la literatura existente y presentando brevemente los resultados logrados.

1.1 Temas principales y motivación

En esta tesis se desarrolla al análisis de propiedades de controlabilidad para algunos problemas de EDP, cuyo estudio está motivado por muchas aplicaciones en el mundo real. En particular, el trabajo se centra en dos familias de modelos muy generales, que han interesado ampliamente la investigación en matemática aplicada en las últimas décadas: *EDPs no-locales* y *EDPs con potenciales singulares cuadráticos-inversos*.

Los problemas que vamos a tratar son, en nuestra opinión, muy interesantes y desafiantes. Debido a sus dificultades, muchas veces requieren el desarrollo de nuevas técnicas matemáticas e, incluso en los casos en que se pueden aplicar resultados clásicos, su adaptación para enfrentarse a las particulares características de los sistemas que nos proponemos investigar no suele ser elemental.

Dedicamos esta sección a presentar de manera muy general las razones de la base del creciente interés en los modelos de EDPs objeto en el presente trabajo. Pondremos especial atención en sus empleo en distintos campos de las ciencias aplicadas, de la ingeniería y de las finanzas. A fin de dar una presentación lo más clara posible, y para favorecer una mayor comprensión, consideraremos las dos categorías de ecuaciones por separado.

1.1.1 EDPs con términos no-locales

Una EDP no-local es un tipo particular de ecuación diferencial donde una o todas las componentes involucran a términos no-locales. Como sugiere el nombre, la primera y mayor diferencia respecto a una EDP clásica es que, para comprobar si una ecuación no-local se satisface en un punto, se necesita información también de los valores de la función lejos de éste punto. En la mayoría de los casos, esto ocurre debido a que la ecuación contiene términos integrales. Esta es también la razón por la cual, en la literatura, se pueden encontrar las denominaciones de ecuaciones *integro-diferenciales* o *pseudo-diferenciales* en referencia a estos problemas.

El análisis de operadores y EDP no-locales es un tema en continuo desarrollo. En las últimas décadas, muchos investigadores empezaron a dedicarse a esta rama de las matemáticas, motivados en particular por el gran número de posibles aplicaciones en modelos para varios fenómenos complejos, para los que un enfoque local resulta ser inadecuado o restrictivo.

De hecho, hay un amplio espectro de situaciones en las cuales una ecuación no-local da una descripción considerablemente mejor, con respecto a una EDP, del problema que se quiere analizar.

En elasticidad, por ejemplo, hay muchos modelos que involucran a términos no-locales; uno muy importante es sin duda la ecuación de Peierls-Nabarro, que describe fenómenos de dinámica de dislocación en cristales ([49, 101]).

Por otro lado, en la ciencia de los materiales modelos no-locales tienen en cuenta la propiedad de que en muchos materiales el estrés en un punto depende del esfuerzo en una región al rededor del mismo ([85]).

Ecuaciones no-locales se pueden encontrar también en ecología. Por ejemplo, en dinámica de poblaciones surgen ecuaciones de reacción-difusión no-locales en modelos para estructuras de ecosistemas, a la hora de analizar la dependencia del crecimiento respecto al alimento y de la mortalidad respecto al tamaño en ciertos sistemas predador-presa ([46]).

En finanzas, los precios de las acciones pueden tener cambios frecuentes e imprevistos. Por tanto, modelos que involucran a procesos de salto resultan ser particularmente adecuados para describir, por ejemplo, la tarificación de las opciones americanas ([96, 112]).

Por último, otros ejemplos donde aparecen ecuaciones integro-diferenciales son modelos para turbulencias ([3]), transporte y difusión anómalos ([14, 105]), filtración en medios porosos ([15]), proceso de imágenes ([71]), propagación de ondas en medios heterogéneos de alto contraste ([146]).

1.1.2 EDPs con potenciales cuadráticos-inversos

La segunda parte de esta tesis se dedica al estudio de EDPs de evolución que contienen potenciales singulares cuadráticos-inversos. En este ámbito, por el análisis de estas ecuaciones será

fundamental la famosa desigualdad de Hardy, que toma su nombre del matemático británico G.H. Hardy (1877-1947).
En 1925, este demostró en [77] que para cada $u \in H_0^1(0, +\infty)$, se verifica

$$\int_0^{+\infty} |u'(x)|^2 \, dx \geq \frac{1}{4} \int_0^{+\infty} \left(\frac{u(x)}{x} \right)^2 dx. \tag{1.1.1}$$

Esta desigualdad fue el fruto de veinte años de investigación, a partir de un resultado relacionado estrechamente y obtenido por D. Hilbert en 1904 ([79]); en su desarrollo, hay que recordar las contribuciones fundamentales de muchos insignes matemáticos, además de Hardy, como E. Landau, G. Pólya, M. Riesz and I. Schur. El lector interesado puede consultar [90] y las referencias allí incluidas para una panorámica completa sobre la historia de (1.1.1).

Nueve años después del artículo de Hardy de 1925, la desigualdad (1.1.1) aparece en [95], para el estudio de la existencia de soluciones regulares para la ecuación de Navier viscosa.

Posteriormente, fue otra vez Hardy, en colaboración con J.E. Littlewood y G. Pólya, quien generalizó (1.1.1) al caso multi-dimensional y, en [78], se introdujo por primera vez la desigualdad de Hardy en su forma más clásica y conocida; concretamente, los autores demostraron que, para cualquier dominio abierto $\Omega \subset \mathbb{R}^N$, $N \geq 3$, que contenga al origen, y para cada $u \in H_0^1(\Omega)$, $u/|x| \in L^2(\Omega)$ y se verifica:

$$\int_\Omega |\nabla u(x)|^2 \, dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2(x)}{|x|^2} \, dx. \tag{1.1.2}$$

La constante $(N-2)^2/4$ en (1.1.2) es óptima y no se puede alcanzar en $H_0^1(\Omega)$, en el sentido de que la inmersión continua $H_0^1(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2})$ no es compacta.

El estudio de la desigualdad de Hardy y de otras desigualdades integro-diferenciales relacionadas está motivado por aplicaciones en varios campos.

En física cuántica, por ejemplo, hay relaciones estrechas entre (1.1.2) y el principio de incertidumbre de Heisenberg (véase, por ejemplo, [59]), mientras que en mecánica cuántica, (1.1.2) es fundamental para el estudio de la ecuación de Schrödinger no-relativista de una única partícula que se mueve en un campo eléctrico ([58]).

En la teoría de ecuaciones diferenciales ordinarias, desigualdades de tipo Hardy pueden encontrarse aplicadas al estudio de las oscilaciones de las soluciones ([82, 109]) o en problemas de aproximación ([86]).

Además, desde el punto de vista matemático, se pueden mencionar varias aplicaciones en problemas de Sturm-Liouville ([8, 116]), en la teoría de series de Fourier ([38]), en el análisis espectral para operadores diferenciales ([56, 114]), en la geometría diferencial ([65, 134]), en el análisis funcional, para obtener teoremas de inmersión para espacios de Sobolev con pesos ([87, 88]), y en la teoría de las funciones de variable compleja ([110]).

En la teoría de las EDPs singulares, la desigualdad de Hardy tiene un rol crucial en el análisis de propiedades cualitativas de operadores de Schrödinger (generalizados) en la forma $-\Delta - V(x)$, con potenciales singulares cuadráticos-inversos. Este tipo de operadores se encuentra, por ejemplo, en modelos cosmológicos cuánticos, como está enfatizado por la ecuación de Wheeler-de-Witt ([9]), en problemas de captura de electrones ([72]), o incluso en la linearización de problemas de reacción-difusión no-lineal que involucran a la ecuación del calor con términos de reacción supercríticos, que tienen aplicaciones en termodinámica ([39]) y en la teoría de la combustión ([69, 70]).

Hoy en día existe una literatura muy consolidada sobre la desigualdad de Hardy y sobre varias extensiones de este resultado. El lector interesado puede consultar, por ejemplo, los artículos siguientes y las referencias allí contenidas: [5, 21, 24, 44, 62, 63, 64, 69, 84, 125, 133, 141]. Además, merece la pena mencionar los artículos [16, 60] sobre desigualdades con singularidades multipolares. Por último, para algunos de los resultados presentados en esta tesis, recordamos los trabajos [21, 22] relativos a potenciales singulares que involucran a la función distancia al borde.

1.2 Contenidos de la tesis

En este trabajo estamos interesados en el análisis de las propiedades de control de las dos clases de problemas presentados en la Sección anterior. Por lo tanto, consideraremos algunos ejemplos explícitos de EDPs de evolución con términos no-locales o potenciales singulares y, para cada una de ellas, estudiaremos la posibilidad de obtener resultados de controlabilidad, tanto desde el interior como desde el borde de los dominios donde dichas ecuaciones serán definidas.

Más detalladamente, el cuerpo principal de esta tesis está compuesto por los Capítulos siguientes:

- **Capítulo 3: Control interno de ecuaciones de Schrödinger y ondas no-locales que involucran al operador de Laplace fraccionario**. En este Capítulo estudiamos una versión no-local de la ecuación de Schrödinger clásica, donde al operador de Laplace se sustituye el Laplaciano fraccionario $(-\Delta)^s$. Mostramos que para cada $s \in [1/2, 1)$ es posible obtener la controlabilidad a cero actuando desde un conjunto ω de la frontera de un dominio acotado $\Omega \in C^{1,1}$. Por otro lado, también probamos que no se puede lograr este resultado en el caso de exponentes $s < 1/2$. En nuestro análisis, utilizamos técnicas de multiplicadores ([83]) y la identidad de Pohozaev para el Laplaciano fraccionario ([119]), a fin de obtener la desigualdad de observabilidad que necesitamos para aplicar el Método de Unicidad de Hilbert ([97, 98]). Como consecuencia de la controlabilidad para la ecuación de Schrödinger fraccionaria, obtenemos una propiedad análoga para una ecuación de ondas con Laplaciano fraccionario. Los resultados de este Capítulo están contenidos en el artículo científico [11].

- **Capítulo 4: Controlabilidad al borde de una ecuación del calor unidimensional con un potenciales singular cuadrático-inverso**. En este Capítulo se investiga el problema parabólico para el operador de Schrödinger unidimensional $\mathcal{A} = -d^2_{xx} - V(x)$ donde, para todo $\mu \in \mathbb{R}$, $V(x)$ es el potencial cuadrático inverso definido como

$$V(x) := \frac{\mu}{x^2}.$$

Para cada tiempo $T > 0$, asumimos que el dominio de definición de nuestra ecuación sea el conjunto $Q := \{(x,t) \in (0,1) \times (0,T)\}$; esto significa que la singularidad del potencial V surge en un punto de frontera. Para cualquier $0 < \mu < 1/4$, obtenemos la controlabilidad a cero actuando desde el punto $x = 0$, como consecuencia de resultados análogos obtenidos en [76].

- **Capítulo 5: Controlabilidad a cero de una ecuación del calor con un potencial singular cuadrático-inverso que involucra a la función distancia**. En este Capítulo consideramos una ecuación parabólica con potencial singular

$$\Lambda(x) := \frac{\mu}{\delta(x)^2},$$

donde $\delta(x) := \text{dist}(x, \partial\Omega)$ es la distancia entre un punto x y el borde de un dominio $\Omega \subset \mathbb{R}^N$ acotado y de clase C^2. El coeficiente μ se asume menor o igual que $\mu^* = 1/4$, siendo éste el valor crítico para la desigualdad de Hardy generalizada que involucra a la función δ ([21]). Como consecuencia de una nueva estimación de Carleman, obtenemos la controlabilidad a cero actuando desde un conjunto ω de nuestro dominio de definición. Además, demostraremos que en el caso supercrítico, es decir para $\mu > 1/4$, no hay manera alguna de prevenir que las soluciones de la ecuación exploten, obteniendo así la imposibilidad de controlar el sistema. Estas propiedades se encuentran adaptando pruebas análogas presentadas en [35, 53]. Los resultados de este Capítulo están contenidos en el artículo científico [12], en colaboración con E. Zuazua.

- **Capítulo 6: Problemas abiertos**. En este Capítulo presentamos algunos problemas abiertos relacionados con los resultados obtenidos en la tesis, motivando las razones por las que encontramos estas cuestiones interesantes y discutiendo brevemente las dificultades que esconden.

Damos ahora un resumen preliminar de los contenidos de cada capítulo, introduciendo con más detalles los resultados principales que obtenemos.

1.2.1 Capítulo 3: Control interno de ecuaciones de Schrödinger y ondas no-locales que involucran al operador de Laplace fraccionario

En este Capítulo tratamos el problema de controlabilidad a cero para la ecuación de tipo Schrödinger que involucra al operador de Laplace fraccionario

$$\begin{cases} iu_t + (-\Delta)^s u = h\chi_{\{\omega\times(0,T)\}}, & (x,t) \in \Omega \times (0,T) \\ u \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2.1}$$

definida en un dominio $\Omega \subset \mathbb{R}^N$, acotado y de clase $C^{1,1}$. En (1.2.1), la región de control ω es un conjunto de la frontera de Ω.

El estudio de ecuaciones de evolución que involucran al Laplaciano fraccionario es un tema bastante nuevo y, desde el punto de vista de la investigación en las matemáticas puras, actualmente no existe una literatura muy extendida. Hasta donde llega nuestro conocimiento, los resultados que vamos a presentar están entre los primeros disponibles en la teoría del control para EDPs no-locales. De hecho, el resultado principal que utilizaremos ([119]) se ha obtenido muy recientemente.

Enseñaremos que, por valores de $s \in [1/2, 1)$, existe una función de control h de clase L^2 y con soporte en ω, tal que la única solución u de (1.2.1) satisface

$$u(x,T) = 0. \tag{1.2.2}$$

Además, mostraremos también que el límite inferior $s = 1/2$ es óptimo en el sentido de que, cuando $s < 1/2$, no hay posibilidad alguna de controlar la ecuación. Por lo tanto, el resultado principal del Capítulo 3 será el siguiente:

Teorema 1.2.1. *Sea $\Omega \subset \mathbb{R}^N$ un dominio acotado y de clase $C^{1,1}$, y sea $s \in [1/2, 1)$. Definimos también $\Gamma_0 := \{x \in \partial\Omega \,|\, (x\cdot\nu) > 0\}$, donde ν es el vector normal unitario en $x \in \partial\Omega$ que apunta hacia el exterior de Ω, y $\omega = \mathcal{O}_\varepsilon \cap \Omega$, donde $\mathcal{O}_\varepsilon$ es un conjunto de Γ_0 en $\mathbb{R}^N$.*

(i) Si $s \in (1/2, 1)$, para todos $T > 0$ y para cada $u_0 \in L^2(\Omega)$ existe una función de control $h \in L^2(\omega \times [0,T])$ tal que la solución u de (1.2.1) satisface $u(x,T) = 0$;

(ii) si $s = 1/2$, existe un tiempo mínimo $T_0 > 0$ tal que el mismo resultado de controlabilidad que en (i) vale para cada $T > T_0$.

Además, en ambos casos existe una constante positiva C_T tal que

$$\|h\|_{L^2(\omega\times[0,T])} \leq C_T \|u_0\|_{L^2(\Omega)}.$$

El Teorema 1.2.1 se obtendrá aplicando la técnica clásica que combina el método de los multiplicadores y el Método de Unicidad de Hilbert ([83, 97]), y será consecuencia de una desigualdad de observabilidad para el siguiente sistema adjunto asociado a (1.2.1)

$$\begin{cases} iv_t + (-\Delta)^s v = 0, & (x,t) \in \Omega \times (0,T) \\ v \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2.3}$$

En particular, utilizando la teoría de regularidad para el problema elíptico fraccionario desarrollada en [117, 118], y gracias a una nueva identidad de Pohozaev para el Laplaciano fraccionario ([119]), enseñaremos que, bajo las condiciones sobre el tiempo T impuestas en el Teorema 1.2.1, existe una constante positiva $C > 0$ tal que la solución de (1.2.3) satisface:

$$\|v_0\|^2_{L^2(\Omega)} \leq C \int_0^T \|v\|^2_{L^2(\omega)}\, dt; \tag{1.2.4}$$

esto implica inmediatamente (1.2.2), por medio de un argumento de dualidad.

La imposibilidad de controlar la ecuación cuando $s < 1/2$ será justificada a través de un análisis de Fourier para el siguiente problema unidimensional

$$\begin{cases} iu_t + (-d_x^2)^s u = g\chi_{\{\omega\times(0,T)\}}, & (x,t) \in (-1,1) \times (0,T) \\ u(-1,t) = u(1,t) = 0, & (x,t) \in (-1,1)^c \times (0,T), \\ u(x,0) = u_0(x), & x \in (-1,1), \end{cases} \tag{1.2.5}$$

donde $\omega \subset (-1,1)$ es el subconjunto del dominio desde el cual queremos controlar.

Nuestro análisis para (1.2.5) se basará en unos resultados presentados en [91, 92] sobre el comportamiento asintótico del espectro del Laplaciano fraccionario en dimensión uno en el intervalo $(-1,1)$. En particular, en [92] se muestra que los valores propios asociados al problema

$$\begin{cases} (-d_x^2)^s \phi_k(x) = \lambda_k \phi_k(x), & x \in (-1,1) \\ \phi_k(x) \equiv 0, & x \in (-1,1)^c \end{cases} \tag{1.2.6}$$

valen

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(2-2s)\pi}{8}\right)^{2s} + O\left(\frac{1}{k}\right), \quad \text{cuando } k \to +\infty. \tag{1.2.7}$$

Por medio de (1.2.7), se puede mostrar que, para cada $s < 1/2$, el salto asintótico entre los valores propios se acerca a cero cuando k tiende a infinito, es decir

$$\liminf_{k\to+\infty} (\lambda_{k+1} - \lambda_k) = 0. \tag{1.2.8}$$

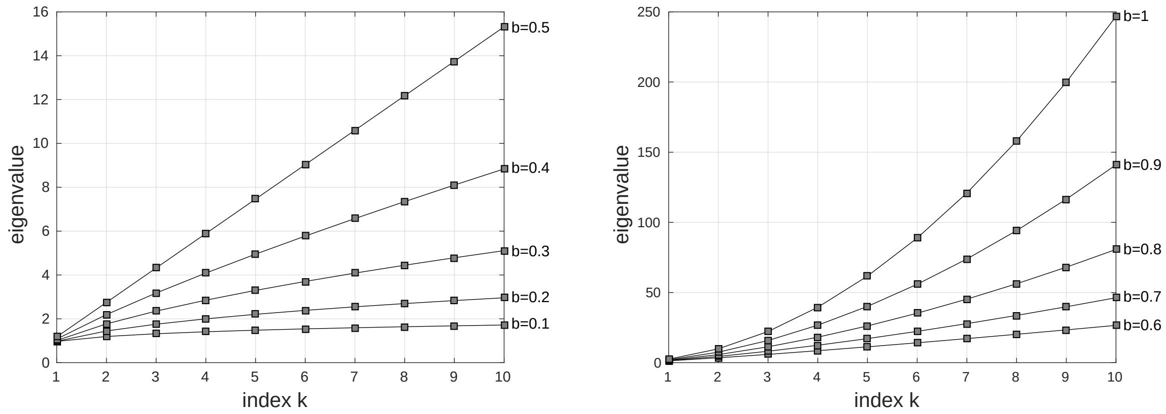

Figure 1.1: Primeros 10 valores propios del Laplaciano fraccionario $(-d_x^2)^\beta$ en $(-1,1)$ para $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (izquierda) y para $\beta = 0.6, 0.7, 0.8, 0.9, 1$ (derecha).

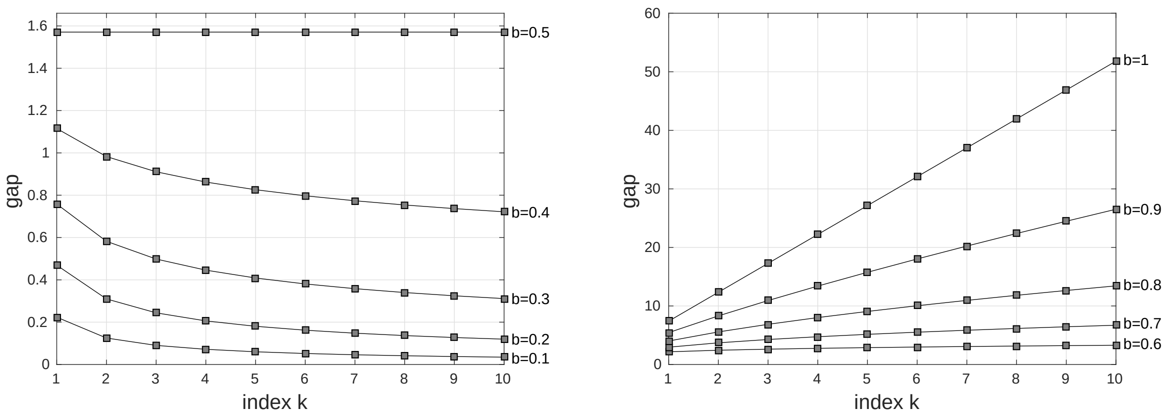

Figure 1.2: Salto asintótico entre los 10 primeros valores propios del Laplaciano fraccionario $(-d_x^2)^\beta$ en $(-1,1)$ para $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (izquierda) y para $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (derecha). A cada índice k le corresponde el salto $\lambda_{k+1} - \lambda_k$.

Haciendo referencia a la teoría de Ingham ([107]), (1.2.8) implica que, en este caso, la desigualdad de observabilidad falla, por lo que no conseguimos probar la controlabilidad a cero de la ecuación (1.2.1).

Por último, merece la pena dedicar algunas palabras más al Teorema 1.2.1 sobre la controlabilidad de nuestra ecuación, en particular respecto a la introducción de un tiempo mínimo $T_0 > 0$ cuando $s = 1/2$. Como se explicará en detalles en la Sección 3.3, este tiempo mínimo aparece de manera natural a lo largo de la prueba de nuestro resultado. Será necesario para obtener la observabilidad de (1.2.3), debido al hecho que, para $s = 1/2$, vamos a encontrar algunos términos que no son compactos respecto a la cantidad que queremos observar y que van a necesitar un tiempo T suficientemente largo para poder ser absorbidos.

Además, remarcamos que la introducción de T_0 no tiene sólo motivaciones técnicas sino que, en nuestra opinión, está relacionada estrictamente con la estructura de nuestro problema. De hecho, cuando $s = 1/2$, las soluciones de nuestra ecuación tienen una velocidad de propagación uniforme, y esto implica que vamos a necesitar un intervalo de tiempo suficientemente grande para poder observarlas. Una justificación de este hecho está en la formula (1.2.7) para el comportamiento asintótico de los valores propios del Laplaciano fraccionario en dimensión uno que, en este caso límite, nos da un salto constante (véase también la Imagen 1.2)

$$\lambda_{k+1} - \lambda_k = \frac{\pi}{2}, \quad \text{para todo } \; k > 0.$$

Haciendo otra vez referencia a la teoría de Ingham ([107]), esta última condición nos lleva automáticamente a la introducción de T_0, pues sabemos que ésto es lo que pasa en el caso de un salto uniforme. Por otro lado, cuando el salto asintótico es $\gamma_\infty = \infty$, como en el caso $s > 1/2$, se espera la observabilidad para cualquier tiempo $T > 0$.

La última parte del Capítulo está dedicada al estudio de la ecuación de tipo ondas

$$\begin{cases} u_{tt} + (-\Delta)^{2s} u = h\chi_{\{\omega\times(0,T)\}}, & (x,t) \in \Omega \times (0,T), \\ u \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.2.9}$$

donde el operador de Laplace de orden mayor $(-\Delta)^{2s}$ se define simplemente como el cuadrado del Laplaciano fraccionario clásico $(-\Delta)^s$, como sigue

$$(-\Delta)^{2s} u(x) := (-\Delta)^s (-\Delta)^s u(x), \quad s \in [1/2, 1),$$

$$\mathcal{D}((-\Delta)^{2s}) = \left\{ u \in H_0^s(\Omega) \,\middle|\, (-\Delta)^s u\big|_{\Omega^c} \equiv 0, (-\Delta)^{2s} u \in L^2(\Omega) \right\}.$$

Remarcamos que, $(-\Delta)^{2s}$ es un operador simétrico, positivo y auto-adjunto, siendo definido como la doble composición del operador simétrico, positivo y auto-adjunto $(-\Delta)^s$.

Como consecuencia del Teorema 1.2.1, y aplicando un argumento general presentado en [135], seremos capaces de obtener una desigualdad de observabilidad para la solución v del sistema adjunto

$$\begin{cases} v_{tt} + (-\Delta)^{2s} v = 0, & (x,t) \in \Omega \times (0,T), \\ v \equiv 0, & (x,t) \in \Omega^c \times (0,T), \\ v(x,0) = v_0(x), \; v_t(x,0) = v_1(x), & x \in \Omega. \end{cases} \tag{1.2.10}$$

Con más detalles, sea T_0 el tiempo de observación introducido en el Teorema 1.2.1. Aplicando [135, Proposición 6.8.2] obtenemos que, para todos $s \in [1/2, 1)$ y para cada $T > T_0$, existe una constante positiva $C > 0$ tal que

$$\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-2s}(\Omega)} \leq C \int_0^T \|v\|^2_{L^2(\omega)}\, dt. \tag{1.2.11}$$

Desde (1.2.11), deducimos que la ecuación (1.2.9) es controlable a cero, a través de un control h de clase L^2, con soporte en un subconjunto ω de la frontera del dominio.

1.2.2 Capítulo 4: Controlabilidad de frontera para una ecuación del calor unidimensional con un potencial singular cuadrático-inverso

Este Capítulo está dedicado al análisis de la siguiente ecuación del calor unidimensional con un potencial singular cuadrático-inverso

$$\begin{cases} u_t - u_{xx} - \dfrac{\mu}{x^2}u = 0, & (x,t) \in (0,1) \times (0,T), \\ x^{-\lambda}u(x,t)\big|_{x=0} = f(t), \quad u(1,t) = 0, & t \in (0,T) \\ u(x,0) = u_0(x), & x \in (0,1). \end{cases} \tag{1.2.12}$$

Otra vez, nos centraremos en particular en el estudio de las propiedades de controlabilidad. En concreto, estamos interesados en resolver el siguiente problema.

Problema 1.2.1. *Dado u_0 en un espacio funcional X sobre $(0,1)$, buscar f en un espacio funcional Y sobre $(0,T)$ tal que la solución correspondiente u de* (1.2.12) *satisface $u(x,T) = 0$ para todo $T > 0$.*

La estrategia que vamos a aplicar consiste en mostrar que, por medio del cambio de variables

$$u(x,t) := x^{\frac{\alpha}{2(2-\alpha)}}\, \psi(x,t), \quad x(\xi) := \left(\frac{2}{2-\alpha}\right) \xi^{\frac{2-\alpha}{2}},$$

con

$$\alpha = \frac{2 + 8\mu - 2\sqrt{1-4\mu}}{3+4\mu},$$

podemos trasformar nuestra ecuación original (1.2.12) en la siguiente ecuación con coeficientes degenerados

$$\psi_t - (\xi^\alpha \psi_\xi)_\xi = 0, \tag{1.2.13}$$

que ya sabemos que es controlable a cero desde la frontera (véase [76]).

En los últimos años, la comunidad del control ya se ha interesado en EDPs de evolución con potenciales singulares. Entre otros trabajos, recordamos aquí [35, 53, 137], sobre la ecuación del calor, y [34, 138], para la ecuación de ondas; en todos estos artículos, los autores son capaces de controlar la ecuación que estudian, actuando desde el interior del dominio donde está definida.

Sin embargo, hasta donde alcanza nuestro conocimiento, la controlabilidad de frontera para estas ecuaciones es una cuestión que nadie ha tratado anteriormente. Además, una novedad importante de nuestra investigación es que, por primera vez, somos capaces de controlar desde el punto donde surge la singularidad.

En el análisis de nuestro problema, un primer aspecto que queremos enfatizar es el hecho que, debido a la presencia de la singularidad en $x = 0$, resulta que en (1.2.12) no podemos imponer una condición de frontera del tipo $u(0,t) = f(t) \neq 0$; en cambio, tenemos que introducir la condición de frontera "pesada"

$$x^{-\lambda} u(x,t)\Big|_{x=0} = f(t),$$

con

$$\lambda := \frac{1}{2}\left(1 - \sqrt{1-4\mu}\right).$$

Este hecho se justificará en detalles a lo largo del Capítulo.

Como es habitual, gracias al clásico Método de Unicidad de Hilbert el Problema 1.2.1 será equivalente a probar una desigualdad de observabilidad para el sistema adjunto asociado a (1.2.12)

$$\begin{cases} v_t + v_{xx} + \dfrac{\mu}{x^2} v = 0, & (x,t) \in (0,1) \times (0,T) \\ v(0,t) = v(1,t) = 0, & t \in (0,T) \\ v(x,T) = v_T(x), & x \in (0,1). \end{cases} \tag{1.2.14}$$

Esta desigualdad, en cambio, será obtenida desde la desigualdad correspondiente presentada en [76] para la ecuación (1.2.13), aplicando el cambio de variables mencionado anteriormente.

Sin embargo, este método genera limitaciones para los valores del coeficiente μ. Desde luego, mientras que a través de técnicas de transposición ([99]) la ecuación (1.2.12) resulta estar bien definida para todo $\mu \leq 1/4$, nuestra prueba de su observabilidad será valida solo para $0 < \mu < 1/4$. Daremos más detalles sobre este hecho fundamental a lo largo del Capítulo.

Por último, remarcamos que en el sistema adjunto (1.2.14) estamos imponiendo condiciones de contorno de Dirichlet clásicas, es decir sin introducir pesos. De hecho, en la ecuación (1.2.12) el peso en $x = 0$ es necesario si queremos detectar un dato de borde que no es cero; sin embargo, cuando consideramos un problema con condiciones de contorno homogéneas, el comportamiento polinomial de las soluciones garantiza la buena definición en el sentido clásico.

1.2.3 Capítulo 5: Controlabilidad a cero para una ecuación del calor con un potencial singular cuadrático-inverso que involucra a la función distancia

En este capítulo consideramos la siguiente ecuación del calor con potencial singular

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2} u = f, & (x,t) \in \Omega \times (0,T) \\ u = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2.15}$$

definida en un dominio $\Omega \subset \mathbb{R}^N$ acotado y de clase C^2, donde $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ es la función distancia al borde. De nuevo, nuestro objetivo es obtener resultados de controlabilidad.

También en este caso, enseñaremos que (1.2.15) es exactamente controlable a cero a través de una función de control f de clase L^2 y localizada en un subconjunto abierto $\omega \subset \Omega$. En particular, el resultado principal de este Capítulo será el Teorema siguiente:

Teorema 1.2.2. *Sea $\Omega \subset \mathbb{R}^N$ un dominio acotado y de clase C^2 y sea $\mu \leq 1/4$. Para cada subconjunto $\omega \subset \Omega$, abierto y no vacío, para cada tiempo $T > 0$ y para cada dato inicial $u_0 \in L^2(\Omega)$, existe una función de control $f \in L^2(\omega \times (0,T))$ tal que la solución de* (1.2.15) *satisface $u(x,T) = 0$.*

La acotación superior para el coeficiente μ juega un rol fundamental en nuestro análisis y está relacionada con la siguiente desigualdad de Hardy generalizada, que involucra al potencial μ/δ^2, presentada en [21]

$$\int_\Omega |\nabla u|^2\, dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{\delta^2}\, dx.$$

Problemas del tipo de (1.2.15) han sido estudiados con intensidad en las últimas décadas; en [25], por ejemplo, se muestra que el valor $\mu = 1/4$ es crítico para que el problema (1.2.15) esté bien definido, en el sentido que para cada $\mu > 1/4$ la ecuación no admite ninguna solución débil positiva para todos $u_0 > 0$ y $f = 0$. Además, hay una explosión instantánea y completa de las soluciones aproximadas.

Otra vez más, por medio del Método de Unicidad de Hilbert ([97]), el Teorema 1.2.2 será consecuencia de la posibilidad de observar la solución del siguiente sistema adjunto asociado a (1.2.15)

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{\delta^2} v = 0, & (x,t) \in Q \\ v = 0, & (x,t) \in \Gamma \times (0,T) \\ v(x,T) = v_T(x), & x \in \Omega. \end{cases} \tag{1.2.16}$$

Con más detalles, para cada $\mu \leq 1/4$ y cada $T > 0$ probaremos la existencia de una constante positiva C_T tal que, para cada $v_T \in L^2(\Omega)$, la solución de (1.2.16) satisface

$$\int_\Omega v(x,0)^2\, dx \leq C_T \int_{\omega\times(0,T)} v(x,t)^2\, dxdt. \tag{1.2.17}$$

La desigualdad anterior, sin embargo, será obtenida a través de una nueva estimación de Carleman para la solución de (1.2.16), donde el peso utilizado está elegido de manera que nos permita compensar la explosión del potencial en la frontera. Remarcamos que esta desigualdad de Carleman no puede deducirse de manera trivial desde las que ya están disponibles en la literatura para ecuaciones con potenciales singulares cuadráticos-inversos ([35, 53]), pues en nuestro caso la singularidad del potencial es de natura diferente.

Por último, adaptando el argumento presentado en [53] demostraremos que la acotación $\mu \leq 1/4$ es óptima para nuestro resultado de controlabilidad.

Chapter 2

Preliminaries

2.1 Controllability and observability

The notions introduced in this section rely on the presentations given in [42, 107, 120, 135].

As we were mentioning in Chapter 1, roughly speaking the exact controllability problem may be formulated as follows. Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations). Starting from a given initial state at time $t = 0$, we want to act on the trajectories of the system through a suitable control in order to match a desired final state in a finite time $T > 0$. Most of the time, but not always, this control is the right-hand side of the system or a boundary condition.

This is a very classical problem in control theory and there is by now an extended literature on the topic. Research in this field has been very intensive in the last decades and it touches nowadays a huge spectrum of PDEs models. To present a complete survey of the progress achieved in this area of mathematics would be, of course, impossible and is not in the interest of this thesis; the interested reader can refer to some of the titles included in the references ([97, 107, 123, 135, 148])

When treating control problems, there is a first very general classification which has to be done: one has to distinguish between finite-dimensional systems (modelled by ODEs) and infinite-dimensional distributed systems (described by PDEs). This distinction is necessary since finite-dimensional and infinite-dimensional systems have, in general, quite different properties from the point of view of control ([147]).

For linear finite-dimensional systems, there is by now a completely developed theory based on the famous Kalman rank condition ([94, 130]). Moreover, also in the case of non-linear finite-dimensional systems the problem is quite well understood, and there are nowadays many powerful tools for investigating local and global controllability ([42]).

For PDEs the situation is a bit more delicate, even in the linear framework, one main reason being the fact that a linear PDE may be of many different types such as:

- hyperbolic type (wave equation, Maxwell equations);
- parabolic type (heat or Stokes equation);
- dispersive type (Schrödinger, Korteweg-de Vries or Boussinesq equation).

Each one of these equations is characterised by very specific properties on the flow. For instance, it is classical that parabolic equations are time irreversible and that they have a strong smoothing effect. For these reasons, it is well known that one cannot expect an exact controllability result to hold with a control localised in some small part of the domain, meaning that one cannot reach an arbitrary final state; therefore, it is instead natural to look at the properties of approximate or null controllability. On the other hand, for hyperbolic equations we have the Huygens principle and the property of propagation of singularities with finite velocity; moreover, these problems are time reversible and this makes natural to seek for the property of exact controllability.

Let us now go into more details, describing a general mathematical framework for controllability. We will follow here the presentation given in [40].

Consider two (real or complex) Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H)$ and $(U, \langle\cdot,\cdot\rangle_U)$, a time $T > 0$, an initial datum $y_0 \in H$ and a closed unbounded operator $A : \mathcal{D}(A) \to H$ which generates a strongly continuous semi-group $S(t)_{t\geq 0}$. We are interested in the following class of linear control problems

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bu, \quad t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{2.1.1}$$

where $B \in \mathcal{L}(U; \mathcal{D}(A))$ is the operator describing the way the control u acts on the system. Moreover, for the operator B we assume to hold the following admissibility condition

$$\forall T > 0,\ \exists C_T > 0 \quad \text{such that} \quad \int_0^T \|B^* S(t)^* z\|_U \, dt \leq C_T \|z\|_H^2,\ \forall z \in \mathcal{D}(A^*), \tag{2.1.2}$$

where B^*, $S(t)^*$ and A^* are the adjoint operators of B, $S(t)$ and A, respectively.

First of all, it is possible to show that, under the admissibility condition (2.1.2), the Cauchy problem (2.1.1) is well-posed in the sense of Hadarmad, i.e. that, for every $y_0 \in H$ and $u \in L^2(0,T;U)$ there exists a unique $y \in C([0,T];H)$ satisfying (2.1.1). Moreover,

$$\|y\|_{C([0,T];H)} \leq C \left(\|y_0\|_H + \|u\|_{L^2(0,T;U)} \right), \tag{2.1.3}$$

for a positive constant C depending on T, A and B. Let us now introduce a first notion of controllability

Definition 2.1.1. *System* (2.1.1) *is* ***exactly controllable*** *at time T if, for any $y_0, y_T \in H$, there exists $u \in L^2(0,T;U)$ such that the solution y of* (2.1.1) *fulfills $y(T) = y_T$.*

As we said at the very beginning, according to this definition the aim of the control process consists in driving the solution y of (2.1.1) from the initial state y_0 to the final one y_T in time T by acting on the system through the control u.

Remark 2.1.1 ([107], Remark 1.1)**.** *In the view of the linearity of the system, without any loss of generality, we may suppose that $y_T = 0$. Indeed, if $y_T \neq 0$ we may solve*

$$\begin{cases} \dfrac{dz}{dt} = Az, \quad t \in [0,T] \\ z(T) = y_T \end{cases}$$

backward in time and define a new state $w = y - z$ which verifies

$$\begin{cases} \dfrac{dw}{dt} = Aw + Bu, \quad t \in [0,T] \\ w(0) = y_0 - z(0) \end{cases} \tag{2.1.4}$$

Notice that $y(T) = y_T$ if and only if $w(T) = 0$. Hence, driving the solution y of (2.1.1) *from y_0 to y_T is equivalent to leading the solution w of* (2.1.4) *from the initial data $w_0 = y_0 - z(0)$ to the zero state.*

It is therefore justified the following definition of *null controllability*

Definition 2.1.2. *System* (2.1.1) *is* ***exactly null-controllable*** *at time T if, for any $y_0 \in H$, there exists $u \in L^2(0,T;U)$ such that the solution y of* (2.1.1) *fulfills $y(T) = 0$.*

Moreover, according to Remark 2.1.1, the properties of exact and null controllability are equivalent for finite-dimensional linear systems. However, this is not necessarily true in the case nonlinear systems or of systems with a strongly time irreversibility. For instance, the heat equation is a well known example of null-controllable system that is not exactly controllable. For the sake of completeness, we present here also the notion of *approximate controllability*

Definition 2.1.3. *System* (2.1.1) *is* ***approximately controllable*** *at time T if, for any $y_0, y_T \in H$ and any $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the solution y of* (2.1.1) *fulfills $\|y(T) - y_T\|_H < \varepsilon$.*

It is well known that in the linear finite dimensional case (i.e., for $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times M}$, $N, M \in \mathbb{N}$), the three definitions we gave above are all equivalent to a purely algebraic condition, the so-called Kalman condition:

$$rank(B, AB, A^2B, \ldots, A^{N-1}B) = N. \tag{2.1.5}$$

As a consequence, for finite dimensional systems, controllability at a time $T_0 > 0$ implies controllability at any time $T > 0$. This may no longer be true in the context of PDEs. A typical example here is the wave equation, that is a model in which propagation occurs with finite velocity; due to this fact, for controllability properties to hold, the control time needs to be large enough so that the effect of the control may spread everywhere.

As noticed by D. Russell in [122], and then formalised by J. L. Lions in the famous Hilbert Uniqueness Method (HUM, [97, 98]), the properties of controllability for system (2.1.1) are equivalent to certain measurements (observabilities) of its adjoint (dual problem). Indeed, let us consider the adjoint system of (2.1.1):

$$\begin{cases} -\dfrac{dz}{dt} = A^* z, & t \in [0,T] \\ z(T) = z_T \in H. \end{cases} \tag{2.1.6}$$

The following results hold.

Theorem 2.1.1. *System* (2.1.1) *is exactly controllable at time T if and only if there exists a constant $C > 0$ such that*

$$\|z_T\|_H^2 \leq C \int_0^T \|B^* z(t)\|_U^2 \, dt, \quad \textit{for all} \quad z_t \in H. \tag{2.1.7}$$

Inequality (2.1.7) is the so-called *strong observability inequality.* Roughly speaking, it permits to recover a complete information about the initial state z_T simply from a measurement on $[0,T]$ of the output $B^* z(t)$.

Theorem 2.1.2. *System* (2.1.1) *is null-controllable at time T if and only if there exists a constant $C > 0$ such that*

$$\|z(0)\|_H^2 \leq C \int_0^T \|B^* z(t)\|_U^2 \, dt, \quad \textit{for all} \quad z_t \in H. \tag{2.1.8}$$

Inequality (2.1.8), instead, is called *weak observability inequality.* In this case, only $z(0)$ is recovered; notice, however, that when system (2.1.1) is reversible then null and exact controllability are equivalent, which is not the case if the system is not reversible.

Besides, we point out that the proof of an observability inequality is not straightforward and it requires tools adapted to the PDE under investigation; e.g. *multiplier methods*, *Carleman inequalities*, *Ingham inequalities* or *microlocal analysis* ([6, 42, 68, 83, 97, 107, 135]).

We remark that a control driving an initial state y_0 to a final state y_T is not necessary unique. However, for the exact and null controllability problem, it is possible to identify in a natural way a distinguished control, the one of $L^2(0,T;U)$ minimal norm. This issue is related to the concept of the *cost of controllability*

Let us assume that (2.1.1) is exactly controllable at time T. Then, for every $y_T \in H$, the set

$$U^T(y_T) := \{u \in L^2(0,T;U) \text{ such that } [y_t = Ay + Bu,\, y(0) = 0] \Rightarrow y(T) = y_T\}$$

is a nonempty close and affine subspace of $L^2(0,T;U)$. Let us now indicate with $\mathcal{U}^T(y_T)$ the elements of $U^T(y_T)$ of smallest $L^2(0,T;U)$-norm. It immediately follows that the map

$$\begin{aligned} \mathcal{U}^T(y_T): \quad H &\longrightarrow L^2(0,T;U) \\ y_T &\longmapsto \mathcal{U}^T(y_T) \end{aligned}$$

is linear. Moreover, through the closed graph theorem, it can be shown that this map is also continuous. The norm of $\mathcal{U}^T(y_T)$, that we will denote by $C^E_{\text{opt}}(T)$, is called the *cost of the exact controllability* of system (2.1.1). Moreover, the following result holds.

Proposition 2.1.1. *$C^E_{opt}(T)$ is the infimum of the constants $C > 0$ for which the strong observability* (2.1.7) *holds, i.e.,*

$$C^E_{opt}(T) = \left\|\mathcal{U}^T(y_T)\right\|_{\mathcal{L}(H;L^2(0,T;U))} = \inf_{C>0}\left\{\|z_T\|^2_H \leq C\int_0^T \|B^*z(t)\|^2_U\,dt,\ \forall z_T \in H\right\}$$

Therefore, Proposition 2.1.1 tells us that the cost of the exact controllability of (2.1.1) is the optimal constant for which the strong observability for the adjoint system (2.1.6) holds.

Furthermore, if system (2.1.1) is exactly controllable we can describe a constructive way to build the controls $\mathcal{U}^T(y_T)$ of $L^2(0,T;U)$ minimal norm. For any $y_0 \in H$, by duality between (2.1.1) and (2.1.6) we obtain

$$\langle y(T), z_T\rangle_H = \int_0^T \langle u(t), B^*z(t)\rangle_U\,dt + \langle y_0, z(0)\rangle_H.$$

Now, let us introduce the following functional $J : H \to \mathbb{R}$

$$J(z_T) = \frac{1}{2}\int_0^T \|B^*z(t)\|^2_U\,dt + \langle y_0, z(0)\rangle_H - \langle y_T, z_T\rangle_H. \tag{2.1.9}$$

If J has a minimum $\hat{z}_T$, then one can easily show that the solution y of (2.1.1) with control $\hat{u} = B^*\hat{z}$, where $\hat{z}$ is the solution of (2.1.6) associated to $\hat{z}_T$, satisfies $y(T) = y_T$.

Indeed, the functional J is clearly strictly convex, while the admissibility condition (2.1.2) ensures its continuity. Finally, the strong observability inequality (2.1.7) easily implies also the coercivity, telling us that J has a unique minimizer $\hat{z}_T$ and that the control $\hat{u} = B^*\hat{z}$ is the one of $L^2(0,T;U)$ minimal norm. Moreover, the following estimate holds:

$$\|\hat{u}\|_{L^2(0,T;U)} \leq C^E_{\text{opt}}(T)\|y_T\|_H.$$

A similar argument can be repeated also in the case where (2.1.1) is null controllable at time T, leading to the concept of the *cost of the null controllability* $C^N_{\text{opt}}(T)$. Moreover, in this case, one obtains the control $\hat{u}_N$ of $L^2(0,T;U)$ minimal norm as the minimiser of the following functional

$$J_N(z_T) = \frac{1}{2}\int_0^T \|B^* z(t)\|_U^2\,dt + \langle y_0, z(0)\rangle_H.$$

While J_N is clearly strictly convex and continuous, its coercivity is not straightforward as in the exact controllability case. Nevertheless, it can be shown that J_N is coercive in the space $\overline{H}$, the completion of H with respect to the norm given by the weak observability (2.1.8)

$$\|z_T\|_* = \left(\int_0^T \|B^* z(t)\|_U^2\,dt\right)^{\frac{1}{2}}.$$

Therefore, the control $\hat{u}_N$ obtained as the minimizer of J_N satisfies

$$\|\hat{u}_N\|_{L^2(0,T;U)} \leq C_{\text{opt}}^E(T)\|y_0\|_H.$$

We conclude this section pointing out that, in this thesis, the analysis of the cost of the controllability for the PDE problems that we study is not approached; however, we retained that this was a concept worth to be mentioned for giving a complete survey on controllability theory.

2.2 State of the art

We exhibit here a very general survey of what we believe are the most relevant theoretical results available in the literature for the two main topics addressed in this thesis, namely non-local PDEs and PDEs with singular potentials.

As we already did in Chapter 1, for the sake of a more clear and neat presentation we are going to consider separately these two classes of problems.

2.2.1 Partial Differential Equations involving the fractional Laplace operator

In the wide family of non-local operators, a relevant role is surely taken by the fractional Laplacian; its analysis, with significant applications in many kinds of different models, is a topic relatively new, that has been particularly developed in the last years.

From a mathematical perspective, there is nowadays a well established and rich literature on the fractional Laplacian, concerning both the study of the properties of this operator and its applications in PDEs models. Among many others contributions, we remind here some works of L. Caffarelli and L. Sylvestre ([27], [28]), of R. Servadei and E. Valdinoci ([126], [127]), of J-L. Vázquez ([139]) and of X. Ros-Oton and J. Serra ([117], [119]).

Let us now recall the definition of the fractional Laplacian. For any function u sufficiently regular and for any $s \in (0,1)$, the s-th power of the Laplace operator is given by ([117], [119], [124])

$$(-\Delta)^s u(x) = c_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = c_{N,s} \lim_{\varepsilon\to 0^+} \int_{\{y\in\mathbb{R}^N:\,|x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \tag{2.2.1}$$

provided that the limit exists. We notice that, for $0 < s < 1/2$ and u sufficiently smooth, for instance Lipschitz continuous, then the integral in (2.2.1) is not really singular near x (see e.g. [48, Remark 3.1]).
In (2.2.1), $c_{N,s}$ is a normalisation constant with value ([119])

$$c_{N,s} = \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)}, \tag{2.2.2}$$

where Γ is the classical Euler Gamma Function; this constant is in fact chosen so that the fractional Laplacian is a pseudo-differential operator with symbol $|\xi|^{2s}$ ([48]). Moreover, the terminology "*fractional Laplacian*" is justified by the observation that, in the limit $s \to 1$, it is possible to recover the standard Laplace operator $-\Delta$ ([17, 48, 104, 140]).

The fractional Laplacian $(-\Delta)^s$ can also be defined through the method of bilinear Dirichlet forms, that is, $(-\Delta)^s$ is the close self-adjoint operator on $L^2(\mathbb{R}^N)$ associated with the bilinear symmetric closed form

$$\mathcal{E}(u,v) = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy, \quad u,v \in H^s(\mathbb{R}^N),$$

in the sense that

$$\mathcal{D}((-\Delta)^s) = \left\{u \in H^s(\mathbb{R}^N) \,\middle|\, (-\Delta)^s u \in L^2(\mathbb{R}^N)\right\}$$

and

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^N} v(-\Delta)^s u\, dx, \quad \text{for all } u \in \mathcal{D}((-\Delta)^s), v \in H^s(\mathbb{R}^N).$$

We remark that on $\mathbb{R}^N$ the three definitions we gave for the fractional Laplacian (as a singular integral, through the Fourier transform or through the a bilinear form) are all equivalent; this, however, is not true anymore when working on open subsets of $\mathbb{R}^N$, the main reason being the non-locality of the operator.

Therefore, for using this operator on domains, one has to proceed as follows ([143, 144]). Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set and, for $0 < s < 1$, let us introduce the space

$$\mathcal{L}^1(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \int_\Omega \frac{|u(x)|}{(1+|x|)^{N+2s}}\, dx \leq \infty \right\}.$$

Then, for $u \in \mathcal{L}^1(\Omega)$ we restrict the kernel of the fractional Laplacian to Ω and we define the operator

$$A^s_\Omega u = c_{N,s} P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = c_{N,s} \lim_{\varepsilon\to 0^+} \int_{\{y\in\Omega\colon |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega, \quad (2.2.3)$$

provided that the limit exists. As for the operator defined on the whole $\mathbb{R}^N$, for $s < 1/2$ and u sufficiently smooth the integral in (2.2.3) is not really singular near x and it is not necessary to consider it in principal value.

In the literature, the operator A^s_Ω is usually called *regional fractional Laplacian* ([73, 74, 75]). Now, for functions $u \in \mathcal{D}(\Omega)$, hence vanishing in Ω^c, straightforward computations yield

$$\begin{aligned} A^s_\Omega u(x) &= c_{N,s} P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy \\ &= c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy - c_{N,s} P.V. \int_{\Omega^c} \frac{u(x)}{|x-y|^{N+2s}}\, dy \\ &= (-\Delta)^s u(x) - V_\Omega(x)u(x), \end{aligned}$$

that is,

$$(-\Delta)^s u(x) = A^s_\Omega u(x) + V_\Omega(x)u(x), \quad \text{for all } u \in \mathcal{D}(\Omega),$$

where the potential V_Ω is given by

$$V_\Omega(x) := c_{N,s} \int_{\Omega^c} \frac{dy}{|x-y|^{N+2s}}, \quad x \in \Omega.$$

With this construction in mind, in [143, 144] it is defined a realisation of the operator A^s_Ω, i.e. it is given a sense to the elliptic problem, with Dirichlet, Neumann and Robin-type boundary conditions. In particular, it is shown that, in the Dirichlet case, $(-\Delta)^s$ and A^s_Ω coincide from the point of view of elliptic theory.

Finally, we have to mention that it is possible to characterise the fractional Laplacian also employing the heat semi-group in the following way: for any function u sufficiently smooth and for all $s \in (0, 1)$,

$$(-\Delta)^s u = \frac{1}{\Gamma(-s)} \int_0^{+\infty} \left(e^{t\Delta}u - u\right) \frac{dt}{t^{1+s}}, \quad (2.2.4)$$

where $v := e^{t\Delta}u$ is the solution of the following heat equation on $\mathbb{R}^N$

$$v_t - \Delta v = 0, \quad v(0) = u.$$

This characterisation is equivalent to the definition given in (2.2.1) (see, e.g., [131, Section 2.1]); sometimes, it permits to obtain regularity properties whose proof is far from being trivial when considering the operator defined through a singular integral.

We have to remark that in the literature it is possible to find also a different notion of the fractional Laplacian, apart for the one defined as in (2.2.1), which is usually known as *spectral fractional Laplacian*([18, 128]) and which is sometimes denoted by A_s. This operator consists in the s-th power of the Laplacian $-\Delta$, obtained by using its spectral decomposition.

Namely, let Ω be a smooth bounded domain of $\mathbb{R}^N$, and let λ_k and ϕ_k, $k \in \mathbb{N}$, be the eigenvalues and the corresponding eigenfunctions of the Laplace operator $-\Delta$ in Ω with zero Dirichlet boundary data, that is

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k, & x \in \Omega \\ \phi_k = 0, & x \in \partial\Omega. \end{cases}$$

Moreover, without loss of generality let us assume that the functions ϕ_k are normalized in such a way that they form an orthonormal basis of $L^2(\Omega)$, i.e. $\langle\phi_k, \phi_\ell\rangle_{L^2(\Omega)} = \delta_{k,\ell}$. For any $s \in (0,1)$ and any $u \in H_0^1(\Omega)$ with

$$u(x) = \sum_{j\in\mathbb{N}} a_j\phi_j(x),$$

the spectral fractional Laplacian A_s is then defined as

$$A_s u(x) = \sum_{j\in\mathbb{N}} a_j\lambda_j^s\phi_j(x).$$

It is important to note that these two fractional operators, the *integral* one and the *spectral* one, are different. For instance, the spectral operator A_s depends on the domain Ω considered, while the integral one $(-\Delta)^s$ is independent on the domain in which the equation is set. Furthermore, while it is easily seen that the eigenvalues and the eigenfunctions of A_s are respectively λ_k^s and ϕ_k, that is the s-power of the eigenvalues of the Laplacian and the very same eigenfunctions, the spectrum of $(-\Delta)^s$ may be less explicit to describe. More details on this specific topic can be found in [128].

One of the main difficulties when treating problems involving the fractional Laplacian is the non-locality of the operator. For dealing with this inconvenience, a well celebrate result of L. Caffarelli and L. Silvestre ([27]) introduces a localisation procedure, showing that any power of the fractional Laplacian in $\mathbb{R}^N$ can be realised as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space $\mathbb{R}^{N+1}$. For a bounded domain , the result by Caffarelli and Silvestre has been adapted in [18] and [33], where it is shown that this extension argument gives an alternative definition of the spectral fractional Laplacian.

The main ideas of this extension procedure are the following: given $x \in \mathbb{R}^N$ and a function $f : \mathbb{R}^N \to \mathbb{R}$, we consider $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ that satisfies the equation

$$\begin{cases} \operatorname{div}\big(y^\alpha \nabla u(x,y)\big) = 0, & (x,y) \in \mathbb{R}^{N+1} \\ u(x,0) = f(x); \end{cases} \tag{2.2.5}$$

in (2.2.5), y is the extended variable. Then, we have

$$d_s(-\Delta)^s f(x) = -\lim_{y \to 0^+} y^\alpha \partial_y u, \tag{2.2.6}$$

with d_s a positive normalization constant which depends only on s. The parameters α and s are linked by the relation $\alpha = 1 - 2s$; we notice that, for $s \in (0,1)$, we have $\alpha \in (-1,1)$. Finally, the limit in (2.2.6) must be understood in the distributional sense; see [18, 26, 27] for more details.

Paying the price of increasing by one the dimension of the problem analysed, this extension procedure has instead the advantage of allowing to work in a local framework; since its first introduction, it has been employed for several different applications, such as the proof of Carleman estimates for the fractional Laplacian ([121]), or for the built of algorithms for the finite element discretisation of PDEs problems involving this operator ([41], [111]).

The results presented in this thesis, however, are not based on the extension of Caffarelli and Sylvestre; this because, as we mentioned above, when working on bounded domains this extension gives the spectral fractional Laplacian instead of the integral operator (2.2.1).

Instead, we will rely mostly on some recent paper of X. Ros-Oton and J. Serra ([117, 118, 119]). In these articles, the authors study the elliptic problem for the fractional Laplacian on a bounded $C^{1,1}$ domain Ω

$$\begin{cases} (-\Delta)^s u = f, & x \in \Omega \\ u = 0, & x \in \Omega^c, \end{cases} \tag{2.2.7}$$

analysing the well-posedness and the regularity of the solutions up to $\partial\Omega$. Furthermore, the main novelty of [119] is the following Pohozaev identity

Proposition 2.2.1. *Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^N$ and $s \in (0,1)$; moreover, for any $x \in \Omega$ let $\delta(x) = dist(x, \partial\Omega)$ be the distance of x from $\partial\Omega$. Let $u \in H^s(\mathbb{R}^N)$ be a function vanishing in Ω^c and satisfying the following:*

(i) $u \in C^s(\mathbb{R}^N)$ and, for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C\rho^{s-\beta} \quad \textit{for all } \rho \in (0,1);$$

(ii) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\gamma \in (0,1)$ such that $u/\delta^s \in C^\gamma(\overline{\Omega})$. In addition, for all $\beta \in [\gamma, s+\gamma]$ it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x\in\Omega|\delta(x)\geq\rho\})} \leq C\rho^{\gamma-\beta} \quad \textit{for all } \rho \in (0,1);$$

(iii) $(-\Delta)^s u$ is pointwise bounded in Ω.

Then, the following identity holds

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-N}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)\,d\sigma \qquad (2.2.8)$$

where ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

In the proposition above, following the notation introduced in [117, 119], $C^\beta(\Omega)$ with $\beta > 0$ indicates the space $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and $\beta' = \beta - k$.

Identity (2.2.8) extends to the non-local case the by now well known result proved by S.I. Pohozaev for the classical Dirichlet Laplacian ([113]). In it, $u/\delta^s|_{\partial\Omega}$ plays the role that the normal derivative $\partial_\nu u$ plays in the classical Pohozaev identity. Moreover, we want to remark here that the boundary term u/δ^s is completely local. As also the authors underline in [119], this is a very surprising fact, since the original problem is non-local; it means that, although the function u has to be defined in all $\mathbb{R}^N$ for computing its fractional Laplacian at a given point, knowing u only in a neighbourhood of the boundary we can already compute $\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)d\sigma$.

In addition, we notice that, setting $s = 1$ in (2.2.8), one trivially recovers the classical identity since $u/\delta|_{\partial\Omega} = \partial u/\partial\nu$ and $\Gamma(2) = 1$.

Finally, we recall that, as in the classical local theory, (2.2.8) has many consequences, such as the non-existence of non-trivial bounded solutions to (2.2.7) for supercritical non-linearities f, but also monotonicity formulas, energy estimates or unique continuation properties.

2.2.2 Hardy-type inequalities and Partial Differential Equations involving inverse-square potentials

The singular potential $V(x) = |x|^{-2}$, with its homogeneity equal to -2, is critical both from the mathematical and the physical point of view. Mainly motivated by the analysis of PDEs models involving this potential, in the recent past many researchers approached the subject of Hardy inequalities, obtaining many interesting improved version of the classical result proved by Hardy, Littlewood an Pólya that we mentioned in Chapter 1.

In [24], for instance, it is shown that for a bounded domain $\Omega \subset \mathbb{R}^N$ and for any function $u \in C_0^\infty(\Omega)$ it holds

$$\int_\Omega |\nabla u|^2\,dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}\,dx \geq \Lambda^2\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2\,dx,$$

where ω_N indicates the measure of the unit sphere in $\mathbb{R}^N$ and Λ^2 is the square of the first zero of the Bessel function J_0.

Hardy inequalities with multi-polar singularities were introduced e.g. in [60], where the authors proved that, under the condition $\sum_{i=1}^{k} \mu_i \leq (N-2)^2/4$, the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \sum_{i=1}^{k} \mu_i \int_{\mathbb{R}^N} \frac{u^2}{|x-a_i|^2} \, dx$$

holds for any function $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, where the space $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

This result was later improved by R. Bosi, J. Dolbeault and M.J. Esteban ([16]) who showed that, for any $\mu \in (0, (N-2)^2/4]$, and for any $(a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$, $N \geq 2$, there exists a positive constant $K_N < \pi^2$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{K_N + (N+1)\mu}{\rho^2} \int_{\mathbb{R}^N} u^2 \, dx \geq \mu \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{u^2}{|x-a_i|^2} \, dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N),$$

where $\rho := \min_{i \neq j} |a_i - a_j|/2$.

Further extensions of (1.1.2) involving the distance function δ have been obtained, for instance, in [21], with the following inequality

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{u^2}{\delta^2} \, dx \geq \lambda \int_\Omega u^2 \, dx,$$

valid for any function $u \in H_0^1(\Omega)$, with Ω a bounded and smooth domain.

We remind that in the literature can be found also examples of Hardy-type inequalities for the fractional Laplacian. Indeed, in [100] it is proved

$$\frac{1}{2} \int_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x-y|^{N+\alpha}} \, dxdy \geq k_{N,\alpha} \int_\Omega \frac{u^2}{\operatorname{dist}(x, \Omega^c)^\alpha} \, dx, \quad \forall u \in C_0(\Omega),$$

for $\Omega \subset \mathbb{R}^N$ convex, $\alpha \in (1,2)$, and where the constant

$$k_{N,\alpha} := \pi^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \frac{B\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right) - 2^\alpha}{\alpha 2^\alpha}$$

is optimal. Here B is the Euler beta function, while $C_0(\Omega)$ is the space of the continuous functions with compact support contained in Ω. Finally, a stronger version of this inequality is proved in [52], but only on an interval

$$\frac{1}{2} \int_{(x_0,x_1)\times(x_0,x_1)} \frac{(u(x) - u(y))^2}{|x-y|^{1+\alpha}} \, dxdy \geq k_{1,\alpha} \int_{x_0}^{x_1} u^2 \left(\frac{1}{x - x_0} + \frac{1}{x - x_1} \right)^\alpha dx$$
$$+ \frac{4 - 2^{3-\alpha}}{\alpha(x_1 - x_0)} \int_{x_0}^{x_1} u^2 \left(\frac{1}{x - x_0} + \frac{1}{x - x_1} \right)^{\alpha - 1} dx,$$

for all $u \in C_0(x_0, x_1)$.

A first immediate application of Hardy inequalities is in the analysis of the well-posedness of certain types of PDEs. For better contextualise this fact, let us consider the following semi-linear elliptic equation

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2.2.9}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is an open bounded domain. The nonlinearity f is assumed to be a continuous, positive, increasing and convex function, satisfying

$$f(0) = 0, \quad \lim_{s \to +\infty} \frac{f(s)}{s} = \infty.$$

In [20, 23], it has been shown the existence a positive number

$$\lambda^* = \lambda^*(\Omega) < +\infty,$$

called the *extremal value*, that defines whether (2.2.9) is well or ill-posed. Indeed, for any $0 \leq \lambda < \lambda^*$ the problem admits a classical solution $u_\lambda \in C^2(\overline{\Omega})$ which has the further property of being minimal among all possible solutions; on the other hand, if $\lambda > \lambda^*$, (2.2.9) has no weak solutions. Moreover, H. Brezis and J.L. Vázquez proved in [24] some sort of "continuous dependence" of u_λ with respect to the parameter λ, showing the existence a.e. of the following limit

$$u^*(x) := \lim_{\lambda \to \lambda^*} u_\lambda(x)$$

and that $u^* \in L^1(\Omega)$ is a weak solution of problem (2.2.9); u^* is the so-called *extremal solution* corresponding to λ^*. Furthermore, in this work the authors gave a characterization of the unbounded extremal solutions u^* (in the space $H_0^1(\Omega)$) and of the extremal value λ^*. Indeed, in [24] it was shown that $u^* \in H_0^1(\Omega)$ is an unbounded extremal solution for (2.2.9) corresponding to $\lambda = \lambda^*$ if and only if the first eigenvalue of the linearised operator $-\Delta - \lambda^* f'(u^*)$ is non-negative; recalling the definition through a Rayleigh quotient, this means that

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda^* \int_\Omega f'(u^*) u^2 \, dx, \quad \text{for all } \ u \in H_0^1(\Omega).$$

Just for giving an example, if $f(u) = e^u$ and $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$, for any $N \geq 10$ we can explicitly compute the extremal value and the extremal solution of (2.2.9), that are given by $(\lambda^*, u^*) = (2(N-2), -2\log(|x|))$, while if $N \leq 9$ it was shown in [24] that there are not extremal solutions ([70]); this because the inequality

$$\int_\Omega |\nabla u|^2 \, dx \geq 2(N-2) \int_\Omega \frac{u^2}{|x|^2} \, dx$$

is true for $N \geq 10$, due to the Hardy inequality (1.1.2).

Hardy inequalities have a fundamental role also when dealing with evolution equations involving the Schrödinger operator $\mathcal{A} = -\Delta - \mu/|x|^2 \mathfrak{I}$.

In their pioneering paper [4], P. Baras and J.A. Goldstein considered a heat equation with potential $-\mu/|x|^2$, defined in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. Assuming positive initial data, they proved that the Cauchy problem is well-posed in the case $\mu \leq \mu^* := (N-2)^2/4$, while if $\mu > \mu^*$ the solution presents an instantaneous blow-up.

Later on, this result has been improved by X. Cabre and Y. Martel ([25]) and by J. L. Vazquez and E. Zuazua ([141]); in particular, in [141] the authors were able to drop the hypothesis of positivity for the initial data and, for the first time, they gave a complete description of the functional framework in which the singular heat equation that they analysed is well-posed.

Finally, there are already several results in the literature on control theory for evolution equations with singular potentials.

In [138], the authors obtained the null controllability of the wave equation with inverse-square potential and, for this result, a fundamental tool is a new sharp Hardy-type inequality

$$\int_\Omega |x|^2 |\nabla u|^2 \, dx \leq R_\Omega^2 \int_\Omega \left(|\nabla u|^2 - \mu^* \frac{u^2}{|x|^2} \right) dx + \frac{N^2-4}{4} \int_\Omega u^2 \, dx, \quad \forall u \in H_0^1(\Omega),$$

where $\mu^* := (N-2)^2/4$ and $R_\Omega := \max_{x \in \Omega} |x|$.

Concerning heat-type equations, instead, in [137] it has been obtained the null controllability by means of a L^2 control distributed in an annular set surrounding the singularity. This result has later been generalised in [53], where any geometrical constraint of the control region was removed. Finally, [35] addresses the case of boundary singularities; in particular, for obtaining the null controllability the author has to rely also on some new weighted Hardy inequalities (see [35, Proposition 1.2, 1.3]).

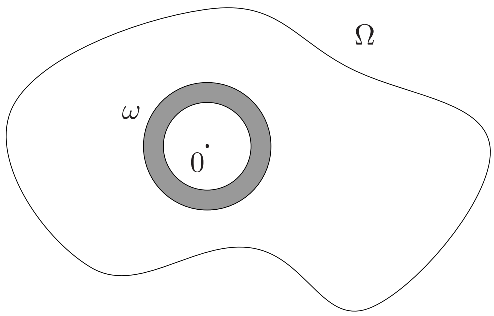

Figure 2.1: In [137], the control region is an annular set around the singularity.

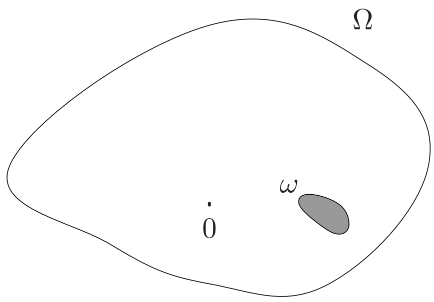

Figure 2.2: In [53], the control region is any open subset $\omega \subset \Omega$.

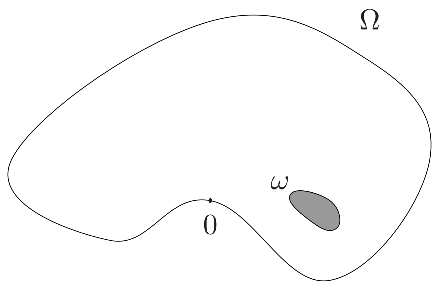

Figure 2.3: In [35], the singularity is on the boundary of Ω.

Chapter 3

Internal control for non-local Schrödinger and wave equations involving the fractional Laplace operator

Abstract.
We analyse the interior controllability problem for a non-local Schrödinger equation involving the fractional Laplace operator $(-\Delta)^s$, $s \in (0,1)$, on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$. The controllability from a neighbourhood of the boundary of the domain is obtained for exponents s in the interval $[1/2,1)$, while for $s < 1/2$ the equation is shown to be not controllable. As a consequence of that, we obtain the controllability for a non-local wave equation involving the higher order fractional Laplace operator $(-\Delta)^{2s} = (-\Delta)^s(-\Delta)^s$, $s \in [1/2,1)$. The results follow from a new Pohozaev-type identity for the fractional laplacian recently proved by X. Ros-Oton and J. Serra and from an explicit computation of the spectrum of the operator in the one-dimensional case. The results obtained in this Chapter are presented in the research article [11].

3.1 Introduction and main results

This Chapter is devoted to the analysis of a non-local Schrödinger equation, involving the fractional Laplace operator, defined on a bounded $C^{1,1}$ domain Ω of the Euclidean space $\mathbb{R}^N$. Our main purpose will be to address the interior controllability problem with a single control located in a neighbourhood of the boundary of the domain.

In the last years many attention has been given to the analysis of non-local operators and many interesting results have been proved. Indeed, concerning practical applications, these operators have shown to be particularly appropriate for the study of a huge spectrum of phenomena, arising in several areas of geophysics, physics, finance, biology, and many others, such as dislocation dynamics in crystals ([49]), anomalous transport and diffusion ([105]), market fluctuations ([106]), population dynamics ([142]), wave propagation in heterogeneous high contrast media ([146]).

The complete problem that we are considering for our fractional Schrödinger equation is the following:

$$\begin{cases} iu_t + (-\Delta)^s u = h\chi_{\{\omega\times[0,T]\}}, & (x,t)\in\Omega\times[0,T] := Q, \\ u\equiv 0, & (x,t)\in\Omega^c\times[0,T], \\ u(x,0) = u_0(x), & x\in\Omega,. \end{cases} \tag{3.1.1}$$

In (3.1.1), ω is a neighbourhood of the boundary of the domain Ω, $h\in L^2(\omega\times[0,T])$ is the control function and the fractional Laplacian $(-\Delta)^s$ is the operator defined as ([117, 119, 124])

$$(-\Delta)^s u(x) := c_{N,s} P.V.\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad s\in(0,1), \tag{3.1.2}$$

with $c_{N,s}$ a normalization constant given by ([119])

$$c_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{N/2}\Gamma(1-s)},$$

where Γ is the Gamma function.

A first important aspect that we want to underline is the particular formulation for the boundary conditions which, due to the non-local nature of the operator, are imposed not only on the boundary but everywhere outside of the domain Ω; moreover, we are imposing boundary conditions of Dirichlet type, meaning that we are asking the solution u to vanish everywhere in Ω^c.

Let us now formulate precisely the interior controllability problem for the fractional evolution equation that we are considering. Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^N$; we introduce a partition (Γ_0,Γ_1) of $\partial\Omega$ given by

$$\Gamma_0 = \{x\in\partial\Omega \,|\, (x\cdot\nu)>0\},\;\; \Gamma_1 = \{x\in\partial\Omega \,|\, (x\cdot\nu)\leq 0\}, \tag{3.1.3}$$

where ν is the unit normal vector to $\partial\Omega$ at x pointing towards the exterior of Ω. Moreover, for a given $\varepsilon>0$ let us consider the sets

$$\mathcal{O}_\varepsilon := \bigcup_{x\in\Gamma_0} B(x,\varepsilon), \quad \omega := \mathcal{O}_\varepsilon\cap\Omega. \tag{3.1.4}$$

The main result of this work will be

Theorem 3.1.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded $C^{1,1}$ domain and $s\in[1/2,1)$. Moreover, let $\omega\subset\Omega$ be a neighbourhood of Γ_0, defined as in* (3.1.4).

(i) If $s \in (1/2, 1)$, for any $T > 0$ and for any $u_0 \in L^2(\Omega)$ there exists a control function $h \in L^2(\omega \times [0, T])$ such that the solution u of (3.1.1) satisfies $u(x, T) = 0$;

(ii) if $s = 1/2$, there exists a minimal time $T_0 > 0$ such that the same controllability result as in (i) holds for any $T > T_0$.

Besides, in both cases there exists a positive constant C_T such that

$$\|h\|_{L^2(\omega\times[0,T])} \leq C_T \|u_0\|_{L^2(\Omega)}.$$

The range of the exponent of the fractional Laplace operator is fundamental for the positivity of the controllability result; indeed, although the fractional Laplacian is well defined for any s in the interval $(0, 1)$, we can show that the sharp power when dealing with the control problem for our fractional Schrödinger equation is $s = 1/2$, meaning that below this critical value the equation becomes non-controllable. This fact is proved in one space dimension by developing a Fourier analysis for our equation based on the results contained in [91, 92], where the authors compute an explicit approximation of the eigenvalues of the fractional Laplacian with Dirichlet boundary conditions on the half-line $(0, +\infty)$ and on the interval $(-1, 1)$.

For proving the controllability Theorem 3.1.1, we are going to apply the very classical technique combining the multiplier method ([83]) and the Hilbert Uniqueness Method (HUM, [42, 97]). Thus, we are reduced to derive an observability inequality for the adjoint problem associated to (3.1.1), and then argue by duality. In particular, we are going to prove that any solution v of the adjoint system

$$\begin{cases} iv_t + (-\Delta)^s v = 0, & (x,t) \in Q, \\ v \equiv 0, & (x,t) \in \Omega^c \times [0,T], \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1.5}$$

satisfies

$$\|v_0\|^2_{L^2(\Omega)} \leq C \int_0^T \|v(t)\|^2_{L^2(\omega)}\, dt. \tag{3.1.6}$$

This inequality will be, in turn, a consequence of a Pohozaev-type identity for the solution of the equation considered, obtained applying the multiplier method and a new Pohozaev identity for the fractional Laplacian, which has been recently proved by Ros-Oton and Serra in [119] and which extends to the fractional case the by now well-known identity presented by Pohozaev in [113].

However, the identity by Ros-Oton and Serra holds under very strict regularity assumptions for the functions involved (see [119, Proposition 1.6]), which are not automatically guaranteed for the solution of our fractional Schrödinger equation. Therefore, for bypassing this regularity issue, we are going to divide the proof of this result into two steps: firstly we will prove the identity for solutions of (3.1.5) involving a finite number of eigenfunctions of the fractional Laplacian on Ω with Dirichlet boundary conditions; then, we are going to recover the result for general finite energy solutions by employing a density argument.

We are allowed to follow this path because the fractional Laplacian, being a positive and self-adjoint operator, possesses a basis of eigenfunctions which forms a dense subspace of $L^2(\Omega)$; moreover, as we are going to show in the appendix to this work, these eigenfunctions are bounded on Ω, and this is enough to recover the regularity that we need, according to [119, Theorem 1.4].

The Chapter is organised as follows. Section 3.2 is devoted to the presentation of the functional setting in which we will work; moreover, we will recall some very classical results ([48]) related to the fractional Laplace operator, as well as the recent ones of Ros-Oton and Serra concerning the regularity of the fractional Dirichlet problem and the Pohozaev-type identity ([117, 119]). In Section 3.3, we analyse the fractional Schrödinger equation (3.1.1). We first check its well-posedness applying Hille-Yosida theorem. Then, we derive the Pohozaev identity and we apply it for proving the observability inequality (3.1.6). Our main result, Theorem 3.1.1, will then be a consequence of this inequality. In Section 3.4, we present a spectral analysis for our equation, which will allow us to identify the sharp exponent needed for the fractional Laplace operator in order to get a positive control result. In Section 3.5, we briefly present an abstract argument, due to Tucsnak and Weiss ([135]), which will permit us to employ the observability results for our fractional Schrödinger equation in order to obtain the observability for a fractional wave equation involving the higher order operator $(-\Delta)^{2s} := (-\Delta)^s(-\Delta)^s$. Section 3.6 is devoted to the proof of the L^∞ regularity of the eigenfunctions of the fractional Laplacian with Dirichlet boundary conditions, following a bootstrap argument presented in [36]. Finally, in Section 3.7 we present a technical Lemma, which is needed in the proof of the observability inequality.

3.2 Fractional Laplacian: definition, Dirichlet problem and Pohozaev-type identity

We present here some preliminary results about the fractional Laplacian, which we are going to use throughout this Chapter.

We start by introducing the fractional order Sobolev space $H^s(\Omega)$. Since we are dealing with smooth domains, say of class $C^{1,1}$, we introduce this space by assuming that our open set $\Omega \subset \mathbb{R}^N$ is smooth. For $s \in (0,1)$, we denote by

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \;\middle|\; \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx < \infty \right\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{H^s(\Omega)} = \left(\int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

Moreover, referring to [127] let us introduce the space

$$H_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) \,\middle|\, u = 0 \text{ on } \Omega^c \right\}; \tag{3.2.1}$$

since Ω is supposed to be smooth, then we have that $\mathcal{D}(\Omega)$ (the space of the test functions) is dense in $H_0^s(\Omega)$. Finally, we mention that $H_0^s(\Omega)$ is a Hilbert space, endowed with a norm equivalent to the $H^s(\Omega)$-norm (see [127, Lemmas 6, 7]), and we denote its dual by $H^{-s}(\Omega)$.

Let $u \in H^s(\mathbb{R}^N)$, $s \in (0,1)$, and let us consider the fractional Laplace operator $(-\Delta)^s$ as defined in (3.1.2). The following result, (see e.g. [48, Proposition 3.3]), tells us that the fractional Laplacian is, in fact, the pseudo-differential operator associated to the symbol $|\xi|^{2s}$.

Proposition 3.2.1. *Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplace operator defined in* (3.1.2)*. Then, for any $u \in H^s(\mathbb{R}^N)$*

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u) \qquad \forall \xi \in \mathbb{R}^N.$$

Proposition 3.2.1 can be used, joint with the Plancherel theorem, to prove many other results such as the following.

Proposition 3.2.2. *Let u,v be two functions in $H_0^s(\Omega)$; then, it holds the following integration formula*

$$\int_\Omega v(-\Delta)^s u dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx = \int_\Omega u(-\Delta)^s v dx. \tag{3.2.2}$$

Our work principally uses the results by Ros-Oton and Serra contained in [117, 118, 119]; we present here the most important ones. Let us consider the Dirichlet problem associated to the fractional Laplace operator

$$\begin{cases} (-\Delta)^s u = g, & x \in \Omega, \\ \qquad u \equiv 0, & x \in \Omega^c. \end{cases} \tag{3.2.3}$$

In [117, Proposition 1.1] and in [119, Proposition 1.6] respectively, the following results have been proved.

Proposition 3.2.3. *Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^N$ and $s \in (0,1)$. For every $g \in L^\infty(\Omega)$, let $u \in H^s(\mathbb{R}^N)$ satisfy* (3.2.3)*. Then $u \in C^s(\mathbb{R}^N)$ and $\|u\|_{C^s(\mathbb{R}^N)} \le C(s,\Omega)\|g\|_{L^\infty(\Omega)}$, where C is a constant depending only on Ω and s.*

Proposition 3.2.4. *Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^N$, $s \in (0,1)$ and $\delta(x) = dist(x, \partial\Omega)$, with $x \in \Omega$, be the distance of a point x from $\partial\Omega$. Let $u \in H_0^s(\Omega)$ satisfy the following:*

(i) $u \in C^s(\mathbb{R}^N)$ *and, for every* $\beta \in [s, 1+2s)$, u *is of class* $C^\beta(\Omega)$ *and*

$$[u]_{C^\beta(\{x\in\Omega|\delta(x)\geq\rho\})} \leq C\rho^{s-\beta}, \qquad \text{for all } \rho \in (0,1);$$

(ii) *The function* $u/\delta^s|_\Omega$ *can be continuously extended to* $\overline{\Omega}$. *Moreover, there exists* $\gamma \in (0,1)$ *such that* $u/\delta^s \in C^\gamma(\overline{\Omega})$. *In addition, for all* $\beta \in [\gamma, s+\gamma]$ *it holds the estimate*

$$[u/\delta^s]_{C^\beta(\{x\in\Omega|\delta(x)\geq\rho\})} \leq C\rho^{\gamma-\beta} \qquad \text{for all } \rho \in (0,1);$$

(iii) $(-\Delta)^s u$ *is pointwise bounded in* Ω.

Then, the following identity holds

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-N}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)\,d\sigma, \qquad (3.2.4)$$

where ν *is the unit outward normal to* $\partial\Omega$ *at* x *and* Γ *is the Gamma function.*

In the two propositions above, following the notation introduced by Ros-Oton and Serra in [117, 119], $C^\beta(\Omega)$ with $\beta > 0$ indicates the space $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and $\beta' = \beta - k$.

Identity (3.2.4) is the Pohozaev identity for the fractional Laplacian and it will be the starting point for our control problem. In it, $u/\delta^s|_{\partial\Omega}$ plays the role that the normal derivative $\partial_\nu u$ plays in the classical Pohozaev identity. Moreover, we want to remark here that the boundary term u/δ^s is completely local. As also the authors underline in [119], this is a very surprising fact, since the original problem is non-local; it means that, although the function u has to be defined in all $\mathbb{R}^N$ for computing its fractional Laplacian at a given point, knowing u only in a neighbourhood of the boundary we can already compute $\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)d\sigma$.

3.3 Fractional Schrödinger equation

We analyse here the fractional Schrödinger equation (3.1.1). As already written before, our principal aim will be to show that the problem is exactly controllable from a neighbourhood of the boundary of the domain. However, the first issue we have to deal with is, of course, the one of the well-posedness.

3.3.1 Well-posedness

We apply Hille-Yosida theorem to obtain the existence and uniqueness of the solution of the following problem

$$\begin{cases} iu_t + (-\Delta)^s u = -f, & (x,t) \in Q, \\ u \equiv 0, & (x,t) \in \Omega^c \times [0,T], \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.3.1}$$

Therefore, let us consider the operator $A : \mathcal{D}(A) \to L^2(\Omega)$ defined as

$$\mathcal{D}(A) = \Big\{ u \in H_0^s(\Omega) \,\Big|\, (-\Delta)^s u \in L^2(\Omega) \Big\}, \quad Au := -(-\Delta)^s u.$$

It is straightforward to check, using (3.2.2), that the operator A is self-adjoint and negative. Therefore, thanks to the classical Stone's theorem ([145, Chapter XI, Section 13, Theorem 1]), iA is the generator of a one parameter C_0 group of unitary operators and we have the following well-posedness result (see,e.g., [37, Chapter 4])

Theorem 3.3.1. *Given $u_0 \in L^2(\Omega)$ and $f \in C([0,T]; L^2(\Omega))$, the system* (3.3.1) *admits a unique solution*

$$u \in C\big([0,T]; L^2(\Omega)\big).$$

Moreover, if $u_0 \in \mathcal{D}(A)$ then

$$u \in C\big([0,T]; \mathcal{D}(A)\big) \cap C^1\big([0,T]; L^2(\Omega)\big).$$

3.3.2 Pohozaev-type identity

In this Section, we introduce one of the main tools that we need in order to obtain the controllability Theorem 3.1.1, a Pohozaev-type identity for the solution of our fractional Schrödinger equation. In particular, we are going to prove the following result.

Proposition 3.3.1. *Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^N$, $s \in [1/2, 1)$ and $\delta(x)$ be the distance of a point x from $\partial\Omega$. For any $f \in C([0,T]; L^2(\Omega))$ and for any initial datum $u_0 \in L^2(\Omega)$, let u be the corresponding solution of* (3.3.1)*. Then, the following identity holds*

$$\begin{aligned} \Gamma(1+s)^2 \int_\Sigma \left(\frac{|u|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma dt = 2s \int_0^T \left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|^2_{L^2(\mathbb{R}^N)} dt + \Im \int_\Omega \bar{u}(x\cdot\nabla u)\, dx \Big|_0^T \\ + \Re \int_Q f\left(N\bar{u} + 2x\cdot\nabla\bar{u}\right) dx dt, \end{aligned} \tag{3.3.2}$$

where ν is the unit outward normal to $\partial\Omega$ at x, Γ is the Gamma function and $\Sigma := \partial\Omega \times [0,T]$.

For proving Proposition 3.3.1, we are going to apply the classical method of multipliers ([83]), joint with the Pohozaev identity proved by Ros-Oton and Serra in [119].

However, as we were mentioning in the introduction to this Chapter, the identity by Ros-Oton and Serra holds under some very strict regularity assumptions, which are not necessarily satisfied by the solution u of (3.3.1). Therefore, we are going to bypass this regularity issue, proving our result in two steps: firstly, we are going to derive the identity for solutions of the equation corresponding to an initial datum $u_{k,0}$ given as a linear combination of a finite number of eigenfunctions of the fractional Laplacian on Ω, taken with Dirichlet boundary conditions; then, we will recover the result for any finite energy solution u by applying a density argument. We are allowed to follow this path since in Section 3.6 we will show that these eigenfunctions are bounded on Ω, and we know from [117, Theorem 1.4] that this is enough to guarantee the regularity we need to apply (3.2.4).

Proof of Proposition 3.3.1.

Step 1: Let us consider an initial datum $u_{k,0} \in span(\phi_1, \dots, \phi_k)$, where $\phi_1, \dots, \phi_k$ are the first k eigenfunctions of the fractional Laplacian on Ω with Dirichlet boundary conditions, and let u_k be the corresponding solution of (3.3.1).

Since, as we are going to show in Section 3.6, the eigenfunctions of the fractional Laplacian with Dirichlet boundary conditions are bounded, by means of [117, Theorem 1.4] this implies that we have enough regularity in order to apply the result of Ros-Oton and Serra. Indeed, with some abuse of notation, let us firstly introduce

$$u_k(x,t) = \sum_{j=1}^{k} \beta_j a_j(t)\phi_j(x) \tag{3.3.3}$$

as the solution of (3.3.1) with $f = 0$, where, for every $j = 1 \dots k$, $a_j(t) := e^{i\lambda_j t}$ while β_j and λ_j are respectively the Fourier coefficient of u_0 and the eigenvalue associated to ϕ_j. We have

$$(-\Delta)^s u_k = \sum_{j=1}^{k} \beta_j a_j (-\Delta)^s \phi_j = \sum_{j=1}^{k} \beta_j \lambda_j a_j \phi_j$$

and

$$x \cdot \nabla u_k = \sum_{l=1}^{n} x_l \partial_{x_l} u_k = \sum_{l=1}^{n} x_l \sum_{j=1}^{k} \beta_j a_j \partial_{x_l} \phi_j = \sum_{j=1}^{k} \beta_j a_j (x \cdot \nabla \phi_j).$$

Thus,

$$(-\Delta)^s u_k (x\cdot\nabla u_k) = \left[\sum_{l=1}^k \lambda_l \beta_l a_l \phi_l\right]\cdot\left[\sum_{j=1}^k \beta_j a_j (x\cdot\nabla\phi_j)\right]$$
$$= \sum_{j=1}^k \beta_j a_j \left[\sum_{l=1}^k \beta_l\lambda_l a_l\phi_l\right](x\cdot\nabla\phi_j) = \sum_{j=1}^k\sum_{l=1}^k \beta_j\beta_l\lambda_l a_l a_j \phi_l (x\cdot\nabla\phi_j)$$

and

$$\int_\Omega (-\Delta)^s u_k (x\cdot\nabla u_k)\,dx = \sum_{j,l=1}^k \beta_j a_j \beta_l a_l \int_\Omega (-\Delta)^s\phi_l (x\cdot\nabla\phi_j)\,dx.$$

Since in the previous equality we have to deal also with cross terms, appearing each time that $j\neq l$, we use the identity

$$\int_\Omega (-\Delta)^s\phi_l(x\cdot\nabla\phi_j)\,dx + \int_\Omega(-\Delta)^s\phi_j(x\cdot\nabla\phi_l)\,dx$$
$$= \frac{2s-N}{2}\int_\Omega \phi_l(-\Delta)^s\phi_j\,dx + \frac{2s-N}{2}\int_\Omega\phi_j(-\Delta)^s\phi_l\,dx - \Gamma(1+s)^2\int_{\partial\Omega}\frac{\phi_l}{\delta^s}\frac{\phi_j}{\delta^s}(x\cdot\nu)\,d\sigma,$$

which follows from [119, Lemma 5.1, 5.2] and holds for functions satisfying the same hypothesis of Proposition 3.2.4; after some simple technical computation we get

$$\sum_{j,l=1}^k \beta_j a_j\beta_l a_l\int_\Omega(-\Delta)^s\phi_l(x\cdot\nabla\phi_j)\,dx$$
$$= (2s-N)\int_\Omega u_k(-\Delta)^s u_k\,dx - \Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u_k}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma - \int_\Omega(-\Delta)^s u_k(x\cdot\nabla u_k)\,dx.$$

Summarising,

$$\int_\Omega(-\Delta)^s u_k(x\cdot\nabla u_k)dx = (2s-N)\int_\Omega u_k(-\Delta)^s u_k dx$$
$$-\Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u_k}{\delta^s}\right)^2(x\cdot\nu)d\sigma - \int_\Omega(-\Delta)^s u_k(x\cdot\nabla u_k)dx,$$

and from here we finally recover the Pohozaev identity for the fractional Laplacian (3.2.4) applied to the function u_k. Coming back to the non-homogeneous case, we can now use this identity in order to prove (3.3.2).

At this purpose, we multiply our equation by $x\cdot\nabla\bar u_k + (n/2)\bar u_k$, we take the real part and we integrate over Q, obtaining

$$-\Re\int_Q f\left(x\cdot\nabla\bar u_k + \frac{N}{2}\bar u_k\right)\,dxdt = \underbrace{\Re\int_Q(-\Delta)^s u_k(x\cdot\nabla\bar u_k)\,dxdt}_{A_1}\ \underbrace{\Re\int_Q\frac{N}{2}\bar u_k(-\Delta)^s u_k\,dxdt}_{A_2}$$
$$+\underbrace{\Re\int_Q i(u_k)_t\left(\frac{N}{2}\bar u_k + x\cdot\nabla\bar u_k\right)\,dxdt}_{A_3} \tag{3.3.4}$$

We now compute the three contributions on the right hand side separately. For the first integral, we have

$$\begin{aligned}A_1 =& \int_Q \Big\{\big[(-\Delta)^s \Re(u_k)\big]\big(x\cdot\nabla\Re(u_k)\big) + \big[(-\Delta)^s \Im(u_k)\big]\big(x\cdot\nabla\Im(u_k)\big)\Big\}\, dxdt \\ =& \frac{2s-N}{2}\int_Q \big\{\Re(u_k)(-\Delta)^s\Re(u_k) + \Im(u_k)(-\Delta)^s\Im(u_k)\big\}\, dxdt \\ &- \frac{\Gamma(1+s)^2}{2}\int_\Sigma \left[\left(\frac{\Re(u_k)}{\delta^s}\right)^2 + \left(\frac{\Im(u_k)}{\delta^s}\right)^2\right](x\cdot\nu)\, d\sigma dt \\ =& \frac{2s-N}{2}\int_Q u_k(-\Delta)^s\bar{u}_k dxdt - \frac{\Gamma(1+s)^2}{2}\int_\Sigma \left(\frac{|u_k|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma dt \\ =& \frac{2s-N}{2}\int_0^T \left\|(-\Delta)^{\frac{s}{2}}u_k(t)\right\|^2_{L^2(\mathbb{R}^N)} dt - \frac{\Gamma(1+s)^2}{2}\int_\Sigma \left(\frac{|u_k|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma dt,\end{aligned}$$

while, for the second one,

$$A_2 = \frac{N}{2}\Re\int_Q \bar{u}_k(-\Delta)^s u_k\, dxdt = \frac{N}{2}\int_0^T \left\|(-\Delta)^{\frac{s}{2}}u_k(t)\right\|^2_{L^2(\mathbb{R}^N)} dt;$$

thus,

$$A_1 + A_2 = s\int_0^T \left\|(-\Delta)^{\frac{s}{2}}u_k(t)\right\|_{L^2(\mathbb{R}^N)} dt - \frac{\Gamma(1+s)^2}{2}\int_\Sigma \left(\frac{|u_k|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma dt.$$

Finally, let us compute the integral A_3; we observe that, by considering the function $\psi(x) := |x|^2/4$ we have

$$\nabla\psi = \frac{x}{2}, \quad \Delta\psi = \frac{N}{2}.$$

Thus

$$\begin{aligned}A_3 =& \Re\int_Q i(u_k)_t\left(\bar{u}_k\Delta\psi + 2\nabla\psi\cdot\nabla\bar{u}_k\right)\, dxdt = -\Im\int_Q (u_k)_t\left(\bar{u}_k\Delta\psi + 2\nabla\psi\cdot\nabla\bar{u}_k\right)\, dxdt \\ =& -\Im\int_Q \Big\{-\nabla\big[(u_k)_t\bar{u}_k\big]\cdot\nabla\psi + 2(u_k)_t\nabla\bar{u}_k\cdot\nabla\psi\Big\}\, dxdt \\ =& -\Im\int_Q \Big\{-\bar{u}_k\nabla(u_k)_t\cdot\nabla\psi - (u_k)_t\nabla\bar{u}_k\cdot\nabla\psi + 2(u_k)_t\nabla\bar{u}_k\cdot\nabla\psi\Big\}\, dxdt \\ =& \Im\int_Q \big[\bar{u}_k\nabla(u_k)_t\cdot\nabla\psi - (u_k)_t\nabla\bar{u}_k\cdot\nabla\psi\big] dxdt = \Im\int_Q \partial_t\big[\bar{u}_k\nabla u_k\cdot\nabla\psi\big]\, dxdt \\ =& \Im\int_Q \partial_t\left[\frac{\bar{u}_k}{2}(x\cdot\nabla u_k)\right]\, dxdt = \Im\int_\Omega \frac{\bar{u}_k}{2}(x\cdot\nabla u_k)dx\Big|_0^T.\end{aligned}$$

Adding now the components just obtained we finally get

$$\begin{aligned}\Gamma(1+s)^2\int_\Sigma \left(\frac{|u_k|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma dt =& 2s\int_0^T \left\|(-\Delta)^{\frac{s}{2}}u_k(t)\right\|^2_{L^2(\mathbb{R}^N)} dt + \Im\int_\Omega \bar{u}_k(x\cdot\nabla u_k)\, dx\Big|_0^T \\ &+ \Re\int_Q f\left(N\bar{u}_k + 2x\cdot\nabla\bar{u}_k\right) dxdt. \qquad (3.3.5)\end{aligned}$$

Step 2: Since the constants appearing in (3.3.5) do not depend on the frequency k, we can now take the limit as $k \to +\infty$ for recovering (3.3.2) for any u finite energy solution of (3.3.1). $\square$

3.3.3 Boundary observability

We now use (3.3.2) applied to the solution v of the adjoint equation (3.1.5), to obtain upper and lower estimates for the $H^s(\Omega)$ norm of the initial datum v_0 with respect to the boundary term appearing in the identity. In order to do that, we will firstly need the following result.

Proposition 3.3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For all $g \in H_0^s(\Omega)$ and $h \in H_0^1(\Omega)$, let us define*

$$T(g,h) := \int_\Omega \bar{g}(x \cdot \nabla h)\, dx. \tag{3.3.6}$$

Then, for all $s \in [1/2, 1)$ there exist two positive constants N_1 and N_2, depending only on N, s and Ω, such that

$$|T(g,h)| \leq N_1 \|g\|_{H_0^{1-s}(\Omega)} \|h\|_{H_0^s(\Omega)} \tag{3.3.7}$$

and

$$|T(g,h)| \leq N_2 \|g\|_{H_0^s(\Omega)} \|h\|_{H_0^s(\Omega)}. \tag{3.3.8}$$

Proof. Let us consider a sequence of test functions $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $g_k \to g$ in $H_0^s(\Omega)$ as $k \to +\infty$; since Ω is bounded, we have

$$\left| \int_\Omega \bar{g}_k(x \cdot \nabla h)\, dx \right| \leq d(\Omega) \|h\|_{H_0^1(\Omega)} \|g_k\|_{L^2(\Omega)}, \tag{3.3.9}$$

where $d(\Omega)$ is the diameter of Ω. Moreover, integrating by parts

$$\begin{aligned} \left| \int_\Omega \bar{g}_k(x \cdot \nabla h) dx \right| = \left| \int_\Omega (\nabla \bar{g}_k \cdot xh + N\bar{g}_k h)\, dx \right| &\leq \big(d(\Omega) \|g_k\|_{H_0^1(\Omega)} + N \|g_k\|_{L^2(\Omega)} \big) \|h\|_{L^2(\Omega)} \\ &\leq \big(d(\Omega) + PN \big) \|g_k\|_{H_0^1(\Omega)} \|h\|_{L^2(\Omega)}, \end{aligned} \tag{3.3.10}$$

where P is the Poincaré constant associated to the domain Ω.

Now, since the constants in (3.3.9) and in (3.3.10) do not depend on k, we can take the limit as $k \to +\infty$, obtaining

$$\left| \int_\Omega \bar{g}(x \cdot \nabla h)\, dx \right| \leq d(\Omega) \|h\|_{H_0^1(\Omega)} \|g\|_{L^2(\Omega)}, \tag{3.3.11}$$

and

$$\left| \int_\Omega \bar{g}(x \cdot \nabla h) dx \right| \leq (d(\Omega) + PN) \|g\|_{H_0^1(\Omega)} \|h\|_{L^2(\Omega)}. \tag{3.3.12}$$

From (3.3.11) we have that $T \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$; on the other hand, (3.3.12) implies $T \in \mathcal{L}(H_0^1(\Omega), L^2(\Omega))$. Therefore, applying [99, Theorem 5.1] we have $T \in \mathcal{L}(H_0^s(\Omega), H_0^{1-s}(\Omega))$ and, consequently,

$$|T(g,h)| \leq N_1 \|h\|_{H_0^s(\Omega)} \|g\|_{H_0^{1-s}(\Omega)},$$

with $N_1 = N_1(N, s, \Omega)$. Finally, the second inequality

$$N_1 \|h\|_{H_0^s(\Omega)} \|g\|_{H_0^{1-s}(\Omega)} \leq N_2 \|h\|_{H_0^s(\Omega)} \|g\|_{H_0^s(\Omega)},$$

is trivial since, for $s \geq 1/2$, we have $H_0^s(\Omega) \hookrightarrow H_0^{1-s}(\Omega)$ with continuous injection ([48]). □

We now have all we need in order to prove the following result.

Proposition 3.3.3. *There exist two positive constants A_1 and A_2, depending only on s, T, N and Ω, such that*

(i) if $s \in (1/2, 1)$, then for any $T > 0$ and for all v finite energy solution of (3.1.5) it holds

$$A_1 \|v_0\|^2_{H_0^s(\Omega)} \leq \int_\Sigma \left(\frac{|v|}{\delta^s}\right)^2 (x \cdot \nu)\, d\sigma dt \leq A_2 \|v_0\|^2_{H_0^s(\Omega)}; \tag{3.3.13}$$

(ii) if $s = 1/2$, there exists a minimal time $T_0 > 0$ such that (3.3.13) holds for any $T > T_0$.

Proof. First of all, without loss of generality, we will assume that the function v is smooth enough for our computations; as we did before, this fact can be justified passing through the decomposition of v in the basis of the eigenfunctions ϕ_k and then arguing by density.
Moreover, since $i(-\Delta)^s$ is a skew-adjoint operator, for all $t \in [0, T]$ it holds

$$\|v(x,t)\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega)}, \quad \|v(x,t)\|_{H_0^s(\Omega)} = \|v_0\|_{H_0^s(\Omega)}. \tag{3.3.14}$$

Furthermore, by the regularity obtained in the well-posedness Theorem 3.3.1, we have that $(-\Delta)^s v = -v_t \in L^2(\Omega)$ and this fact immediately implies $v \in H_0^{2s}(\Omega)$, due to the elliptic regularity results contained in [118]. In particular, since $s \geq 1/2$ we also have $v \in H_0^1(\Omega)$.
Now, considering (3.3.2) with $f = 0$ we obtain

$$\Gamma(1+s)^2 \int_\Sigma \left(\frac{|v|}{\delta^s}\right)^2 (x \cdot \nu)\, d\sigma dt = 2s \int_0^T \left\|(-\Delta)^{\frac{s}{2}} v(t)\right\|^2_{L^2(\mathbb{R}^N)} dt + \Im \int_\Omega \bar{v}(x \cdot \nabla v)\, dx \Bigg|_0^T. \tag{3.3.15}$$

For proving our result, we will apply Proposition 3.3.2 to the last term of the identity above, obtaining in this way the following estimate

$$\left| \int_\Omega \bar{v}(x \cdot \nabla v)\, dx \right| \leq N_1 \|v(t)\|_{H_0^s(\Omega)} \|v(t)\|^2_{H_0^{1-s}(\Omega)}.$$

Therefore, it will be necessary to distinguish the two cases $s > 1/2$ and $s = 1/2$. Indeed, for $s > 1/2$, since the H_0^{1-s} terms are lower order with respect to the H_0^s ones, we can deal with them by applying a compactness-uniqueness argument. However for $s = 1/2$, since of course H_0^{1-s} and H_0^s coincide, we have to proceed in a different way.

Step 1: $s = 1/2$**.** Employing 3.3.8, we obtain

$$\left| \int_\Omega \bar{v}(x \cdot \nabla v)\, dx \right| \le N_2 \|v(t)\|^2_{H^{1/2}(\Omega)},$$

Hence, from (3.3.15) we get

$$\frac{4(T - 2N_2)}{\pi} \|v_0\|^2_{H^{1/2}(\Omega)} \le \int_\Sigma \left(\frac{|v|}{\delta^{1/2}} \right)^2 (x \cdot \nu)\, d\sigma dt \le \frac{4(T + 2N_2)}{\pi} \|v_0\|^2_{H^{1/2}(\Omega)}.$$

Thus, finally, if $T > 2N_2 := T_0$,

$$A_1 \|v_0\|^2_{H^{1/2}(\Omega)} \le \int_\Sigma \left(\frac{|v|}{\delta^s} \right)^2 (x \cdot \nu)\, d\sigma dt \le A_2 \|v_0\|^2_{H^{1/2}(\Omega)}$$

holds with A_1, $A_2 > 0$. Moreover, this minimal time T_0 is the optimal one we can obtain following the path we chose for our proof.

Step 2: $s > 1/2$**.** First of all, we have

$$\Gamma(1+s)^2 \int_\Sigma \left(\frac{|v|}{\delta^s} \right)^2 (x \cdot \nu)\, d\sigma dt \le 2sT \|v_0\|^2_{H^s_0(\Omega)} + 2 \left| \int_\Omega \bar{v}(x \cdot \nabla v)\, dx \right| \le A_2 \|v_0\|^2_{H^s_0(\Omega)},$$

where we used (3.3.8) with $g = h := v$, (3.3.14) and the fact that

$$\left\| (-\Delta)^{\frac{s}{2}} v(t) \right\|_{L^2(\mathbb{R}^N)} \le \varpi \|v(t)\|_{H^s_0(\Omega)},$$

for some positive constant ϖ.

Let us now prove the other estimate. By using (3.3.7) and (3.3.14), and applying Young's inequality, we have

$$\left| \int_\Omega \bar{v}(x \cdot \nabla v)\, dx \right| \le N_1 \varepsilon \, \|v_0\|^2_{H^s_0(\Omega)} + \frac{N_1}{4\varepsilon} \, \|v_0\|^2_{H^{1-s}_0(\Omega)}.$$

Thus, choosing $\varepsilon < 2sT/N_1$, we get that

$$(2sT - N_1 \varepsilon) \|v_0\|^2_{H^s_0(\Omega)} \le \Gamma(1+s)^2 \int_\Sigma \left(\frac{|v|}{\delta^s} \right)^2 (x \cdot \nu)\, d\sigma dt + \frac{N_1}{4\varepsilon} \, \|v_0\|^2_{H^{1-s}_0(\Omega)}. \tag{3.3.16}$$

We conclude now by observing that, thanks to a compactness-uniqueness argument we can prove that there exists a positive constant M, not depending on v, such that

$$\|v_0\|^2_{H^{1-s}_0(\Omega)} \le M \int_\Sigma \left(\frac{|v|}{\delta^s} \right)^2 (x \cdot \nu)\, d\sigma dt. \tag{3.3.17}$$

Indeed, let us assume that the previous inequality does not hold; then, there exists a sequence $\{v^j\}_{j \in \mathbb{N}} \subset H^{1-s}_0(\Omega)$ of solutions of (3.1.5) such that

$$\left\| v^j(0) \right\|_{H^{1-s}_0(\Omega)} = 1, \quad \text{for all } j \in \mathbb{N} \tag{3.3.18}$$

and

$$\lim_{j\to+\infty}\int_\Sigma\left(\frac{|v^j|}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma dt=0. \tag{3.3.19}$$

From (3.3.18) we deduce that $\{v^j(0)\}_{j\in\mathbb{N}}$ is bounded in $H_0^s(\Omega)$ and then, from (3.1.5) and (3.3.14), $\{v^j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H_0^s(\Omega))\cap W^{1,\infty}(0,T,H^{-s}(\Omega))$. Therefore, by extracting a subsequence, that we will still note by $\{v^j\}$, we have

$$\begin{cases} v^j\rightharpoonup v & \text{in } L^\infty(0,T;H_0^s(\Omega)),\\ \partial_t v^j\rightharpoonup\partial_t v & \text{in } L^\infty(0,T;H^{-s}(\Omega)).\end{cases}$$

The function $v\in L^\infty(0,T;H_0^s(\Omega))\cap W^{1,\infty}(0,T,H^{-s}(\Omega))$ is a solution of the equation and, from the compactness of the embedding (see [129])

$$L^\infty(0,T;H_0^s(\Omega))\cap W^{1,\infty}(0,T,H^{-s}(\Omega))\hookrightarrow C(0,T;H_0^{1-s}(\Omega))$$

and (3.3.18) we deduce that $\|v_0\|_{H_0^{1-s}(\Omega)}=1$; on the other hand, (3.3.19) implies $|v|/\delta^s=0$ on Σ. We now claim that it holds the following result, which proof will be given later at the end of this section.

Lemma 3.3.1. *Let $v\in L^\infty(0,T;H_0^s(\Omega))\cap W^{1,\infty}(0,T,H^{-s}(\Omega))$ be a solution of the adjoint equation* (3.1.5) *such that*

$$\frac{|v|}{\delta^s}=0\quad\textit{on }\Sigma.$$

Then, $v\equiv 0$.

Applying the Lemma just stated, we immediately have $v\equiv 0$ and this, of course, is a contradiction. Hence (3.3.17) holds and the proof for $s>1/2$ is concluded. □

Proof of Lemma 3.3.1. For simplicity of notation, let us define

$$X:=L^\infty(0,T;H_0^s(\Omega))\cap W^{1,\infty}(0,T,H^{-s}(\Omega))$$

and, for every $v\in X$, let us consider the space

$$\mathcal{V}:=\left\{v\in X\;\middle|\;v\text{ solves (3.1.5) and }\frac{|u|}{\delta^s}=0\text{ on }\Sigma\right\}\subset X, \tag{3.3.20}$$

equipped with the norm endowed by X. Clearly it is enough to prove that $\mathcal{V}=\{0\}$.
We are going to proceed in two steps.

Step 1: We firstly show that $\dim(\mathcal{V}) < \infty$. At this purpose, let us define

$$z := iv_t.$$

With the same argument as the one employed in the proof of [97, Appendix I, Lemma 2.1], we can immediately show that $z \in X$; moreover, it is straightforward to check that z is also a solution (3.1.5) and that the condition $|z|/\delta^s = 0$ on Σ is satisfied. Therefore, $z \in \mathcal{V}$ and, using the results of [129], we have that the injection

$$\big\{v \in \mathcal{V}\,;\, iv_t \in \mathcal{V}\big\} \hookrightarrow \mathcal{V}$$

is continuous and compact. This, in particular, implies that the dimension of $\mathcal{V}$ is finite.

Step 2: We argue now by contradiction, assuming that $\mathcal{V} \neq \{0\}$. Since the map $\Phi : \mathcal{V} \to \mathcal{V}$ introduced before is antisymmetric, there exists $\lambda \in \mathbb{C}$ and $\psi \in \mathcal{V} \setminus \{0\}$ such that

$$i\psi_t = \lambda\psi. \tag{3.3.21}$$

First of all, we observe that we can assume $\lambda \neq 0$. Indeed, if $\lambda = 0$ we have $\psi_t = 0$ and, since by definition ψ is a solution of (3.1.5), this implies that it solves also

$$\begin{cases} (-\Delta)^s\psi = 0, & x \in \Omega \\ \psi \equiv 0, & x \in \Omega^c, \end{cases}$$

i.e. $\psi \equiv 0$, which is contradictory.
Now, for $\lambda \neq 0$ using the Pohozaev identity (3.2.4) and (3.3.21) we have that

$$\begin{aligned} \frac{\Gamma(1+s)^2}{2}\int_\Sigma \left(\frac{|\psi|}{\delta^s}\right)^2 (x\cdot\nu)\,d\sigma dt &= \frac{2s-N}{2}\Re\int_Q \bar{\psi}(-\Delta)^s\psi\,dxdt - \int_Q (x\cdot\nabla\bar{\psi})(-\Delta)^s\psi\,dxdt \\ &= -\frac{2s-N}{2}\Re\int_Q \bar{\psi}(i\psi_t)\,dxdt + \Re\int_Q (x\cdot\nabla\bar{\psi})(i\psi_t)\,dxdt \\ &= -\lambda\frac{2s-N}{2}\Re\int_Q \psi\bar{\psi}\,dxdt + \lambda\Re\int_Q (x\cdot\nabla\bar{\psi})\psi\,dxdt \\ &= -\lambda\frac{2s-N}{2}\Re\int_Q \psi\bar{\psi}\,dxdt - \lambda\frac{N}{2}\Re\int_Q \psi\bar{\psi}\,dxdt \\ &= -s\lambda\|\psi\|^2_{L^2(Q)}. \end{aligned}$$

However, since $|\psi|/\delta^s = 0$, from the computations above we immediately have that also in this case $\psi \equiv 0$. This concludes the proof. □

3.3.4 Observability from a neighbourhood of the boundary and controllability

This section is dedicated to the proof of the observability inequality (3.1.6) and of the main result of this Chapter, Theorem 3.1.1.

Theorem 3.3.2. *Let $s \in [1/2, 1)$ and let Ω and ω be as in the statement of Theorem 3.1.1. For any $v_0 \in L^2(\Omega)$, let $v = v(x,t)$ be the corresponding solution of* (3.1.5)*.*

(i) If $s \in (1/2, 1)$, then for every $T > 0$ there exists a positive constant C, depending only on s, T, N Ω and ω, such that

$$\|v_0\|^2_{L^2(\Omega)} \leq C \int_0^T \|v(t)\|^2_{L^2(\omega)}\, dt. \tag{3.3.22}$$

(ii) If $s = 1/2$, then (3.3.22) *holds for any $T > T_0$, where T_0 is the minimal time introduced in Proposition 3.3.3.*

Proof. First of all, we notice that in the statement of the Theorem, as we already did in Proposition 3.3.3, we are distinguishing two cases: $s = 1/2$ and $s \in (1/2, 1)$. The main difference between this two cases is the need of a minimal time for the observability when $s = 1/2$, this fact being a consequence of the employing of (3.3.13) when deriving the observability inequality.

On the other hand, the procedure for proving (3.3.22) follows essentially the same path, both for $s > 1/2$ and for $s = 1/2$; therefore, we are going to present here only the first case, $s > 1/2$, leaving to the reader the proof for $s = 1/2$.

Thus, until the end of this Section let us assume $s > 1/2$. Moreover, we proceed in several steps passing through some preliminary Lemmas.

Step 1: We firstly establish the H^s version of (3.3.22).

Lemma 3.3.2. *Let us assume that the hypothesis of Theorem 3.3.2 hold. Then, for any $T > 0$ there exists a positive constant C_1, depending only on s, T, N Ω and ω, such that for all v finite energy solution of* (3.1.5) *it holds*

$$\|v_0\|^2_{H_0^s(\Omega)} \leq C_1 \int_0^T \|v(t)\|^2_{H^s(\omega)}\, dt. \tag{3.3.23}$$

Proof. Without loss of generality, we will assume that the function v is smooth enough for our computations; as we did before, this fact can be justified passing through the decomposition of v in the basis of the eigenfunctions ϕ_k and then arguing by density.

Moreover, we point out that 3.3.23 will be a consequence of our previous result of boundary observability, Proposition 3.3.3.

First of all, let us recall the definition of the neighbourhood of the boundary ω that we introduced in (3.1.4), which is

$$\omega := \Omega \cap \mathcal{O}_\varepsilon, \quad \mathcal{O}_\varepsilon := \bigcup_{x\in\Gamma_0} B(x,\varepsilon),$$

with Γ_0 as in (3.1.3). Then, let us consider the cut-off function $\eta \in C^\infty(\mathbb{R}^N)$ defined as follows

$$\begin{cases} \eta(x) \equiv 1, & x \in \hat{\omega}, \\ 0 \leq \eta(x) \leq 1, & x \in \omega \setminus \hat{\omega}, \\ \eta(x) \equiv 0, & x \in \Omega \setminus \omega, \end{cases} \tag{3.3.24}$$

where $\hat{\omega} := \Omega \cap \mathcal{O}_{\varepsilon_1}$, with $\varepsilon_1 < \varepsilon$, is another neighbourhood of the boundary, thinner than ω (see Figure 3.1 below).

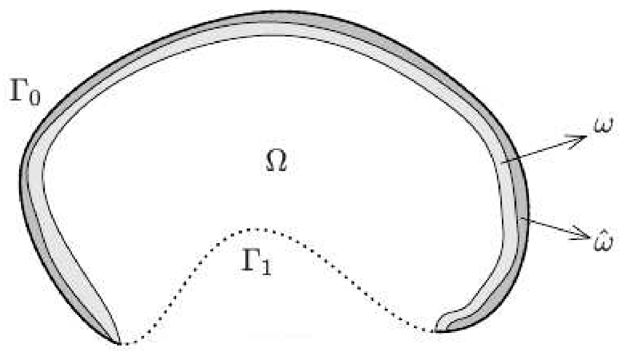

Figure 3.1: Example of the domain Ω with the partition of the boundary (Γ_0, Γ_1) and the two neighbourhood of the boundary $\hat{\omega}$ and ω.

Moreover, let us define $w(x,t) := \eta(x)v(x,t)$. It can be easily checked through the definition (see, e.g., [119, Section 3]) that the fractional Laplacian of w is given by

$$(-\Delta)^s w = (-\Delta)^s(\eta v) = \eta(-\Delta)^s v + R \tag{3.3.25}$$

where R is a reminder term. Therefore, this new function w satisfies the equation

$$\begin{cases} iw_t + (-\Delta)^s w = R, & (x,t) \in Q \\ w \equiv 0, & (x,t) \in \Omega^c \times [0,T] \\ w(x,0) = w_0, & x \in \Omega. \end{cases}$$

Now, starting from (3.3.2) applied to w, we have

$$\underbrace{\Gamma(1+s)^2 \int_\Sigma \left(\frac{|w|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma dt}_{J} = s\int_0^T \left\|(-\Delta)^{\frac{s}{2}} w(t)\right\|^2_{L^2(\omega)} dt + \Im \int_\omega \bar{w}(x\cdot\nabla w)\, dx\,\Big|_0^T$$
$$+ \Re \int_0^T\int_\omega R\Big(N\bar{w} + 2x\cdot\nabla\bar{w}\Big)\, dxdt.$$

Hence, applying (3.3.8) we have

$$\begin{aligned} J \leq \alpha_1 \int_0^T \|w(t)\|^2_{H^s(\omega)}\, dt + \alpha_2 \int_0^T \|w(t)\|_{L^2(\omega)} \|R(t)\|_{L^2(\omega)}\, dt \\ + \alpha_3 \int_0^T \|w(t)\|_{H^s(\omega)} \|R(t)\|_{H^{1-s}(\omega)}\, dt. \end{aligned} \tag{3.3.26}$$

From (3.3.26), by means of Young's inequality, we get

$$\begin{aligned} J \le \alpha_1 \int_0^T \|w(t)\|^2_{H^s(\omega)}\,dt + \frac{\alpha_2}{2}\int_0^T \|w(t)\|^2_{L^2(\omega)}\,dt + \frac{\alpha_2}{2}\int_0^T \|R(t)\|^2_{L^2(\omega)}\,dt \\ + \frac{\alpha_3}{2}\int_0^T \|w(t)\|^2_{H^s(\omega)}\,dt + \frac{\alpha_3}{2}\int_0^T \|R(t)\|^2_{H^{1-s}(\omega)}\,dt, \end{aligned}$$

from which it is straightforward to obtain

$$J \le \alpha_4 \int_0^T \|v(t)\|^2_{H^s(\omega)}\,dt + \frac{\alpha_2}{2}\int_0^T \|R(t)\|^2_{L^2(\omega)}\,dt + \frac{\alpha_3}{2}\int_0^T \|R(t)\|^2_{H^{1-s}(\omega)}\,dt. \tag{3.3.27}$$

We now claim that there exists a constant $B_1 > 0$, not depending on v, such that

$$\|R(t)\|_{L^2(\omega)} \le B_1 \left[\|v(t)\|_{H^s(\omega)} + \|v(t)\|_{L^2(\omega^c)}\right]. \tag{3.3.28}$$

The proof of (3.3.28) is quite technical and it will be given later, in Section 3.7. As a consequence, through a compactness-uniqueness argument it is easy to show that there exist another constant $B_2 > 0$ such that it also holds

$$\|R(t)\|_{H^{1-s}(\omega)} \le B_2 \left[\|v(t)\|_{H^s(\omega)} + \|v(t)\|_{L^2(\omega^c)}\right]. \tag{3.3.29}$$

Therefore, using (3.3.28) and (3.3.29) in the right hand side of (3.3.27), we have the estimate

$$J \le \alpha_4 \int_0^T \|v(t)\|^2_{H^s(\omega)}\,dt + \alpha_5 \int_0^T \|v(t)\|^2_{L^2(\omega^c)}\,dt. \tag{3.3.30}$$

Moreover, we notice that the last term on the right hand side of (3.3.30) is lower order, and it can be absorbed again by compactness-uniqueness. Therefore, by means of this last observation, and applying (3.3.13), we finally get (3.3.23). □

Step 2: In what follows, we will need the following result.

Lemma 3.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain, $f \in H^{-s}(\Omega)$ and let $v \in H^s_0(\Omega)$ be the solution of*

$$\begin{cases} (-\Delta)^s v = f, & x \in \Omega, \\ v \equiv 0, & x \in \Omega^c. \end{cases}$$

Then, there exists a constant $\gamma > 0$ such that

$$\|v\|^2_{H^s(\hat{\omega})} \le \gamma \left[\|f\|^2_{H^{-s}(\omega)} + \|v\|^2_{L^2(\omega)}\right]. \tag{3.3.31}$$

Proof. Let us consider again the function $\eta(x)$ defined in (3.3.24) and let $w(x,t) = \eta(x)v(x,t)$. Thus, w satisfies

$$\begin{cases} (-\Delta)^s w = \eta f + R := g, & x \in \omega, \\ w \in H^s_0(\omega), \end{cases}$$

where R is the reminder term introduced in (3.3.25).

We already proved before that the reminder term $R := v(-\Delta)^s\eta - I_s(\eta, v)$ is L^2 regular; therefore, since η is a smooth function, we have that $g \in H^{-s}(\omega)$. Thus, by classical elliptic regularity we can conclude that $w \in H^s(\omega)$ and

$$\|w\|^2_{H^s(\omega)} \leq \gamma \|g\|^2_{H^{-s}(\omega)},$$

for some positive constant γ independent of g.

Expanding this last expression we easily obtain the existence of another positive constant, that we will still note by γ, such that

$$\|w\|^2_{H^s(\omega)} \leq \gamma \left[\|f\|^2_{H^{-s}(\omega)} + \|v\|^2_{L^2(\omega)} \right].$$

Hence, since

$$\|v\|^2_{H^s(\hat{\omega})} = \|w\|^2_{H^s(\hat{\omega})} \leq \|w\|^2_{H^s(\omega)},$$

we finally obtain the estimate (3.3.31). □

We now establish

Lemma 3.3.4. *For any $T > 0$ there exists a positive constant C_2, depending only on s, T, N Ω and ω, such that for all v finite energy solution of* (3.1.5) *it holds*

$$\|v_0\|^2_{H^{-s}(\Omega)} \leq C_2 \int_0^T \|v(t)\|^2_{H^{-s}(\omega)}\, dt. \tag{3.3.32}$$

Proof. Let us define

$$\psi(x,t) := \int_0^t v(x,s)ds + \Theta(x),$$

where

$$\begin{cases} (-\Delta)^s\Theta = -iv_0, & x \in \Omega, \\ \Theta \in H^s_0(\Omega). \end{cases}$$

Thus, ψ is a solution of (3.1.5) with initial datum $\psi(x,0) = \Theta(x)$. Applying (3.3.23) to ϕ we have

$$\|\Theta\|^2_{H^s(\Omega)} \leq C_1 \int_0^T \|\psi(t)\|^2_{H^s(\omega)}\, dt$$

which, by elliptic regularity, and using (3.3.31), becomes

$$\|v_0\|^2_{H^{-s}(\Omega)} \leq \gamma\, C_1 \int_0^T \left(\|\psi_t(t)\|^2_{H^{-s}(\omega)} + \|\psi(t)\|^2_{L^2(\omega)} \right) dt. \tag{3.3.33}$$

We observe that $\psi_t = v$ and that the last term on the right hand side of (3.3.33) is lower order and can be absorbed applying a compactness-uniqueness argument. Therefore we finally obtain

$$\|v_0\|^2_{H^{-s}(\Omega)} \leq C_2 \int_0^T \|v(t)\|^2_{H^{-s}(\omega)}\, dt. \tag{3.3.34}$$

□

Step 3: From (3.3.23) and (3.3.32) we have

$$\|v_0\|^2_{H^s_0(\Omega)} \leq C_1 \int_0^T \|v(t)\|^2_{H^s(\omega)}\, dt = C_1 \|v\|^2_{L^2(0,T;H^s(\omega))}, \tag{3.3.35}$$

$$\|v_0\|^2_{H^{-s}(\Omega)} \leq C_2 \int_0^T \|v(t)\|^2_{H^{-s}(\omega)}\, dt = C_2 \|v\|^2_{L^2(0,T;H^{-s}(\omega))}. \tag{3.3.36}$$

We are finally going to prove (3.3.22) by interpolation. Let us consider the linear operator

$$\Lambda : H^{-s}(\Omega) \to L^2(0,T;H^{-s}(\omega))$$

defined by

$$\Lambda v_0 := \left(e^{it(-\Delta)^s} v \right)\Big|_{\omega}.$$

Clearly,

$$\|\Lambda v_0\|_{L^2(0,T;H^{-s}(\omega))} \leq c_1 \|v_0\|_{H^{-s}(\Omega)}.$$

Furthermore, from (3.3.36) it follows that

$$\|\Lambda v_0\|_{L^2(0,T;H^{-s}(\omega))} \geq c_2 \|v_0\|_{H^{-s}(\Omega)}.$$

Therefore, we can consider the closed subspace $X_0 := \Lambda(H^{-s}(\Omega))$ of $L^2(0,T;H^{-s}(\omega))$ and the linear operator $\Pi := \Lambda^{-1}$ (since Λ is an isomorphism between $H^{-s}(\Omega)$ and X_0). Thus,

$$\Pi \in \mathcal{L}(X_0, Y_0), \tag{3.3.37}$$

with $Y_0 := H^{-s}(\Omega)$. If now we set $X_1 := X_0 \cap L^2(0,T;H^s(\omega))$, it follows from (3.3.35) that

$$\Pi \in \mathcal{L}(X_1, Y_1), \tag{3.3.38}$$

with $Y_1 := H^s(\Omega)$. From (3.3.37), (3.3.38) and [99, Theorem 5.1], we have

$$\Pi \in \mathcal{L}([X_0, X_1]_{1/2}, [Y_0, Y_1]_{1/2}).$$

Moreover, from [99, Lemma 12.1] we have $[Y_0, Y_1]_{1/2} = L^2(\Omega)$ and from [10, Theorem 5.1.2] we have that

$$[L^2(0,T;H^s(\omega)), L^2(0,T;H^{-s}(\omega))]_{1/2} = L^2(0,T;[H^s(\omega);H^{-s}(\omega)]_{1/2}) = L^2(0,T;L^2(\omega)).$$

Hence, since X_0 and X_1 are closed subspaces of $L^2(0,T;H^{-s}(\omega))$ and $L^2(0,T;H^s(\omega))$ respectively, using [99, Theorem 15.1] we can verify that the norm of the space $[X_0, X_1]_{1/2}$ is equivalent to the norm of $L^2(0,T;L^2(\omega))$ and, since $\Pi \in \mathcal{L}([X_0,X_1]_{1/2};L^2(\Omega))$, we finally have 3.3.22. □

Having proved the observability of the problem that we are considering from a neighbourhood of the boundary of the domain, our controllability theorem is now a direct consequence of a duality argument.

Proof of Theorem 3.1.1. Let us introduce the linear continuous operator $\Phi : L^2(\Omega) \to L^2(\Omega)$ defined as

$$\Phi v_0 = -iu(0),$$

where $u = u(x,t)$ is the solution of the problem

$$\begin{cases} iu_t + (-\Delta)^s u = v\chi_\omega, & (x,t) \in Q, \\ u \equiv 0, & (x,t) \in \Omega^c \times [0,T], \\ u(x,T) = 0, & x \in \Omega, \end{cases} \tag{3.3.39}$$

and v is the solution of (3.1.5) with initial datum $v_0 \in L^2(\Omega)$.

By multiplying (3.3.39) by $\bar{v}$, taking the real part and integrating over Q, it is straightforward to see that for all $v_0 \in L^2(\Omega)$ the following identity is satisfied

$$\langle \Phi v_0, v_0 \rangle_{L^2(\Omega)} = \int_0^T \|v(t)\|^2_{L^2(\omega)}\, dt.$$

By combining it with the observability inequality (3.3.22), we deduce that Φ is an isomorphism from $L^2(\Omega)$ to $L^2(\Omega)$. Therefore, given $u_0 \in L^2(\Omega)$, in (3.1.1) we can choose the control $h = v$, with v the solution of (3.1.5) corresponding to the initial datum $v_0 = \Phi^{-1}(-iu_0)$ and our proof is concluded. □

3.4 Fourier analysis for the one dimensional problem

We show here that, if we want to prove a positive control result, we need to consider a Schrödinger equation with a fractional Laplacian of order $s \geq 1/2$. At this purpose, we analyse our evolution problem in one space dimension and we show that, when the exponent of the fractional Laplace operator is below the critical value written above, we are not able to prove the observability inequality. In this way, we immediately obtain the sharpness of the exponents $s = 1/2$. Thus, the main result of this section will be the following Theorem.

Theorem 3.4.1. *Let us consider the following one-dimensional problem for the fractional Schrödinger equation on the interval* $(-1, 1)$

$$\begin{cases} iu_t + (-d_x^2)^\beta u = g\chi_{\{\omega\times[0,T]\}}, & (x,t) \in (-1,1)\times[0,T], \\ u \equiv 0, & (x,t) \in (-1,1)^c \times [0,T], \\ u(x,0) = u_0(x), & x \in (-1,1). \end{cases} \tag{3.4.1}$$

with $\beta \in (0, 1)$ *and* $\omega \subset (-1, 1)$*. Then,* (3.4.1) *is controllable if and only if* $\beta \geq 1/2$*.*

For the proof Theorem 3.4.1, we will use the results contained in [91, 92]. In this two works, the authors have studied the eigenvalue problem for the fractional Laplacian both on the half line $(0, +\infty)$ and on the interval $(-1, 1)$. In particular, [91] is devoted only to the analysis of the square root of the Laplacian. The main result we will apply is the following, taken from [92, Theorem 1].

Theorem 3.4.2. *Let* $\beta \in (0, 1)$*. For the eigenvalues associated to the problem*

$$\begin{cases} (-d_x^2)^\beta \phi_k(x) = \lambda_k\phi_k(x), & x \in (-1,1), \\ \phi_k(x) \equiv 0, & x \in (-1,1)^c, \end{cases}$$

it holds

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(2-2\beta)\pi}{8}\right)^{2\beta} + O\left(\frac{1}{k}\right) \qquad \text{as } k \to +\infty. \tag{3.4.2}$$

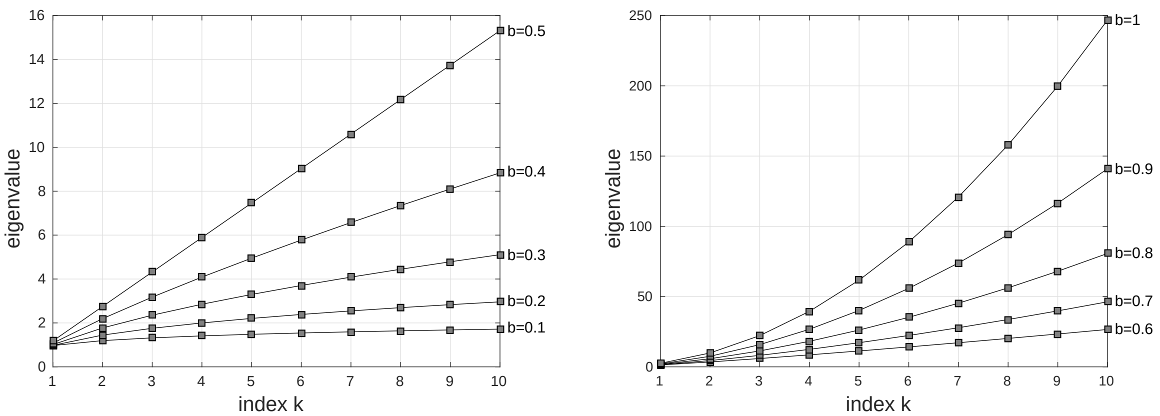

Figure 3.2: First 10 eigenvalues of $(-d_x^2)^\beta$ on $(-1, 1)$ for $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (left) and $\beta = 0.6, 0.7, 0.8, 0.9, 1$ (right).

Proof of Theorem 3.4.1. We are interested in getting a control result by applying HUM. This is equivalent to the proof of an observability inequality for the solution of the adjoint system

$$\begin{cases} iv_t + (-d_x^2)^\beta v = 0, & (x,t) \in (-1,1) \times [0,T], \\ v \equiv 0, & (x,t) \in (-1,1)^c \times [0,T], \\ v(x,0) = v_0(x), & x \in (-1,1). \end{cases} \tag{3.4.3}$$

Following the same path as in Section 3.3 before, a preliminary step for obtaining this inequality will be a boundary observation as in (3.3.13). In our case, $s = \beta$ and $N = 1$, the boundary integral in (3.3.13) simply reduces to computing the value of the integrand in the extremal points of the interval considered, $x = \pm 1$; therefore, the inequality that we get is

$$C\|v_0\|^2_{H_0^\beta(-1,1)} \leq \int_0^T \left(\frac{|v|}{(1-|x|)^\beta} \right)^2 \Bigg|_{x=-1}^{x=1} dt. \tag{3.4.4}$$

Moreover, since (3.4.4) involves the H_0^β norm of the initial datum, the natural space in which to analyse the problem is $H_0^\beta(-1,1)$; we remind that this is an Hilbert space, naturally endowed with the inner product

$$\langle v_1, v_2 \rangle_{H_0^\beta(-1,1)} = \int_{-1}^1 v_1 v_2 \, dx + \int_{-1}^1 (-d_x^2)^{\beta/2} v_1 (-d_x^2)^{\beta/2} v_2 \, dx. \tag{3.4.5}$$

The solution of (3.4.3) will be given spectrally, i.e in terms of the eigenvalues and eigenfunctions of the operator $(-d_x^2)^\beta$ with Dirichlet boundary conditions, which are the solutions of the problem

$$\begin{cases} (-d_x^2)^\beta \phi_k = \lambda_k \phi_k, & x \in (-1,1), \\ \phi_k \equiv 0, & x \in (-1,1)^c. \end{cases}$$

Now, it is classical that the eigenfunctions ϕ_k form an orthonormal basis of $L^2(-1,1)$, i.e.

$$\langle \phi_k, \phi_j \rangle_{L^2(-1,1)} = \delta_{kj}.$$

If, instead, we compute $\langle \phi_k, \phi_j \rangle_{H_0^\beta(-1,1)}$ we have

$$\begin{aligned} \langle \phi_k, \phi_j \rangle_{H_0^\beta(-1,1)} &= \int_{-1}^1 \phi_k(x)\phi_j(x) \, dx + \int_{-1}^1 (-d_x^2)^{\beta/2} \phi_k(x) (-d_x^2)^{\beta/2} \phi_j(x) \, dx \\ &= \langle \phi_k, \phi_j \rangle_{L^2(-1,1)} + \int_{-1}^1 \phi_k(x) (-d_x^2)^\beta \phi_j(x) \, dx \\ &= \delta_{kj} + \int_{-1}^1 \lambda_j \phi_k(x)\phi_j(x) \, dx = \delta_{kj} + \lambda_j \langle \phi_k, \phi_j \rangle_{L^2(-1,1)} = (1+\lambda_j)\delta_{kj}. \end{aligned}$$

This fact tells us that if we introduce the following normalization for the eigenfunctions ϕ_k

$$\{\theta_k\}_{k\geq 1} = \left\{ \frac{\phi_k}{\sqrt{1+\lambda_k}} \right\}_{k\geq 1}$$

we get an orthonormal basis for the space $H^\beta(-1,1)$; this is the basis that we are going to use for the representation of the solution of the problem; we remark here that for the $\{\theta_k\}_{k\geq 1}$ clearly holds

$$(-d_x^2)^\beta \theta_k(x) = \lambda_k \theta_k(x).$$

Formally, (3.4.3) has a solution of the form

$$v(x,t) = \sum_{k\geq 1} a_k \theta_k(x) e^{i\lambda_k t},$$

where a_k are the Fourier coefficients of the function $v_0(x)$ with respect to the basis of the eigenfunctions and are the ones which guarantee that the solution v satisfies the initial condition. Since $\{\theta_k\}_{k\geq 1}$ is an orthonormal basis, they are given by

$$a_k = \frac{1}{2}\int_{-1}^{1} v_0(x)\theta_k(x)\,dx. \tag{3.4.6}$$

Now, coming back to (3.4.4), we have

$$\|v_0\|^2_{H_0^\beta(-1,1)} = \left\langle \sum_{k\geq 1} a_k\theta_k, \sum_{k\geq 1} a_k\theta_k \right\rangle_{H^\beta(-1,1)} = \sum_{k\geq 1} |a_k|^2 (\theta_k,\theta_k)_{H^\beta(-1,1)} = \sum_{k\geq 1} |a_k|^2;$$

thus, the inequality becomes

$$C_1 \sum_{k\geq 1} |a_k|^2 \leq \int_0^T \left(\sum_{k\geq 1} a_k \frac{\theta_k(x)}{(1-|x|)^\beta} e^{i\lambda_k t} \right)^2 \Bigg|_{x=-1}^{x=1} dt. \tag{3.4.7}$$

As we already stated before, and as it is proved in [117], the function $\theta_k(x)/(1-|x|)^\beta$ is continuous up to the boundary. In our case, this means that, in the limit for $x \to \pm 1$, even if either the numerator and the denominator separately goes to zero, we get a constant value. Hence (3.4.7) becomes

$$C_2 \sum_{k\geq 1} |a_k|^2 \leq \int_0^T \left| \sum_{k\geq 1} a_k e^{i\lambda_k t} \right|^2 dt. \tag{3.4.8}$$

Now, thanks to a very classical result due to A.E. Ingham (see [107, Section 4] and the references therein) we know that (3.4.8) holds if there is a positive gap between the eigenvalues, namely

$$\liminf_{k\to+\infty} (\lambda_{k+1} - \lambda_k) = \gamma_\infty > 0. \tag{3.4.9}$$

Moreover, in this case the observability result will hold in a time $T > 2/\gamma_\infty$.

Since we know from (3.4.2) the behaviour of the eigenvalues of $(-d_x^2)^\beta$, we can immediately check that (3.4.9) holds only for $\beta \geq 1/2$ while for $\beta < 1/2$ we have

$$\liminf_{k\to+\infty}(\lambda_{k+1} - \lambda_k) = 0.$$

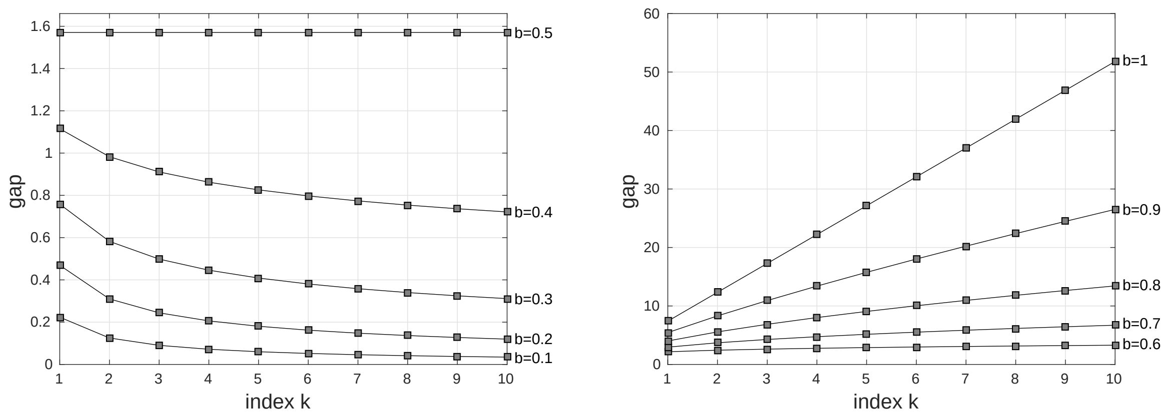

Figure 3.3: Gap between the first 10 eigenvalues of $(-d_x^2)^\beta$ on $(-1, 1)$ for $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ (left) and $\beta = 0.6, 0.7, 0.8, 0.9, 1$ (right). At any index k corresponds the gap $\lambda_{k+1} - \lambda_k$.

This means that we are able to prove the observability inequality, i.e. we can control the equation (3.4.1), only for $\beta \geq 1/2$. $\square$

Remark 3.4.1. *As a final remark, we would like to stress the fact that, in the limit case $s = 1/2$, formula* (3.4.2) *for the behaviour of the eigenvalues of the one-dimensional fractional Laplacian gives us a constant gap (see also Figure 3.3)*

$$\lambda_{k+1} - \lambda_k = \frac{\pi}{2}, \quad \textit{for all} \;\; k > 0.$$

Referring again to Ingham theory ([107]), this condition justifies the introduction of the minimal time T_0 needed for obtain the observability of our equation. On the other hand, when the asymptotic gap is $\gamma_\infty = \infty$, as in the case $s > 1/2$, observation is expected for all time $T > 0$.

3.5 Application to the observability of a fractional wave equation

As an immediate consequence of the null controllability result obtained in Section 3.3 for the fractional Schrödinger equation (3.1.1), we derive here the null controllability for the following fractional wave equation

$$\begin{cases} u_{tt} + (-\Delta)^{2s} u = h\chi_{\{\omega\times[0,T]\}}, & (x,t)\in Q, \\ u \equiv (-\Delta)^s u \equiv 0, & (x,t)\in \Omega^c\times[0,T], \\ u(x,0) = u_0(x) & \\ u_t(x,0) = u_1(x) & x\in\Omega. \end{cases} \tag{3.5.1}$$

In (3.5.1), the operator $(-\Delta)^{2s}$ is an higher order fractional Laplacian, which is defined by composition between two lower order operators as follows.

$$(-\Delta)^{2s}u(x) := (-\Delta)^s(-\Delta)^s u(x), \quad s\in[1/2,1), \tag{3.5.2}$$

$$\mathcal{D}((-\Delta)^{2s}) = \Big\{ u\in H_0^s(\Omega) \,\Big|\, (-\Delta)^s u|_{\Omega^c} \equiv 0, (-\Delta)^{2s}u \in L^2(\Omega)\Big\}. \tag{3.5.3}$$

The reason why we are introducing it is that, with an analysis similar to the one presented in Section 3.4, we can show that a wave equation involving the fractional Laplacian is controllable if and only if we consider an operator of order $s\geq 1$; otherwise, we are not able to prove any observability inequality. Moreover, we are defining the operator as in (3.5.2) because this choice allows us to preserve the regularity properties that $(-\Delta)^s$ possesses. In particular, $(-\Delta)^{2s}$ is symmetric, positive and self-adjoint on the domain Ω, simply because it is defined applying twice the same symmetric, positive and self-adjoint operator. Of course, we can admit other definition of an higher order fractional Laplacian on a regular domain by composition, but we do not always obtain a suitable operator; for instance

$$(-\Delta)^{s+1}u(x) := (-\Delta)^s(-\Delta u)(x) = c_{N,s}\ P.V.\int_{\mathbb{R}^N} \frac{-\Delta u(x) + \Delta u(y)}{|x-y|^{N+2s}}\,dy, \quad s\in(0,1)$$

is a well defined higher order fractional Laplacian, meaning that we can identify its domain and the way it operates but, in this case, it is easy to see through the definition that the operator is not self-adjoint.

Finally, we notice that in the boundary condition in (3.5.1) we are imposing that both the function u and its fractional Laplacian $(-\Delta)^s u$ have to vanish outside the domain Ω. This assumption, which is of course related to the definition given for the operator $(-\Delta)^{2s}$ (in particular to its domain), is needed for the well-posedness of the problem according to the classical semi-group theory. Therefore, we remark that, in the limit $s\to 1/2$, (3.5.1) does not coincide with the usual wave equation.

The null controllability for (3.5.1) will be obtained, again, applying the Hilbert Uniqueness Method. Therefore, we need an observability inequality for the solution of the adjoint equation associated to (3.5.1), namely

$$\begin{cases} v_{tt} + (-\Delta)^{2s} v = 0, & (x,t) \in Q, \\ v \equiv (-\Delta)^s v \equiv 0, & (x,t) \in \Omega^c \times [0,T], \\ v(x,0) = v_0(x) & \\ v_t(x,0) = v_1(x) & x \in \Omega. \end{cases} \tag{3.5.4}$$

For obtaining this inequality, we are going to apply an abstract argument introduced by M. Tucsnak and G. Weiss in [135]. Let A_0 be a linear, self-adjoint operator such that A_0^{-1} is compact, H be an Hilbert space and $H_1 := \mathcal{D}(A_0)$; moreover, let us denote $X := H_1 \times H$, which is an Hilbert space with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle A_0 f_1, A_0 f_2 \rangle_H + \langle g_1, g_2 \rangle_H = \int_\Omega A_0 f_1 A_0 f_2 \, dx + \int_\Omega g_1 g_2 \, dx.$$

We define $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X$ by $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A_0^2) \times H$ and

$$\mathcal{A} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}.$$

Now, let Y be another Hilbert space and let $C_0 \in \mathcal{L}(H_1, Y)$ be such that the pair (iA_0, C_0) is exactly observable in some time T_0. From [135, Proposition 6.8.2] we have that, if the eigenvalues of the operator A_0 satisfy

$$\sum_{k \in \mathbb{N}} \lambda_k^{-d} < +\infty \tag{3.5.5}$$

for some $d \in \mathbb{N}$, then the pair $(\mathcal{A}, C)$, with $C \in \mathcal{L}(\mathcal{D}(\mathcal{A}), Y)$ given by $C = [0 \;\, C_0]$, is exactly observable in any time $T > T_0$.

In our case, we have $A_0 := (-\Delta)^s$, $A_0^2 := (-\Delta)^{2s}$, $H = Y := L^2(\Omega)$ and

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \int_\Omega (-\Delta)^s f_1 (-\Delta)^s f_2 \, dx + \int_\Omega g_1 g_2 \, dx;$$

moreover, the eigenvalues condition (3.5.5) is satisfied with $d = N$ (see e.g. [13, 67]).

Thus, we can apply [135, Proposition 6.8.2] and, from the observability of the fractional Schrödinger equation we immediately get the following inequality for the the fractional wave equation (3.5.4)

$$\|v_0\|^2_{H_0^{2s}(\Omega)} + \|v_1\|^2_{L^2(\Omega)} \leq C \int_0^T \|v_t(t)\|^2_{L^2(\omega)} \, dt, \tag{3.5.6}$$

which holds for any $T > T_0$ with $T_0 = 0$, when $s \in (1/2, 1)$, or for $T_0 > 0$, when $s = 1/2$. Now, let us define

$$\phi(x,t) := \int_0^t u(x,\tau) d\tau - \Theta(x),$$

with $\Theta(x)$ such that $(-\Delta)^{2s}\Theta(x) = u_1(x)$; thus, the function ϕ satisfies

$$\begin{cases} \phi_{tt} + (-\Delta)^{2s}\phi = 0, & (x,t) \in Q, \\ \phi \equiv (-\Delta)^{s}\phi \equiv 0, & (x,t) \in \Omega^c \times [0,T], \\ \phi(x,0) = -\Theta(x) & \\ \phi_t(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

By applying (3.5.6) to the solution of this last equation, we finally obtain

$$\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-2s}(\Omega)} \leq C \int_0^T \|v\|^2_{L^2(\omega)}\, dt. \tag{3.5.7}$$

Therefore, employing (3.5.7) with a duality argument analogous to the one that we developed for the proof of Theorem 3.1.1, for all $T > T_0$ we obtain the existence of a control function $h \in L^2(\omega \times [0,T])$ such that the solution u of (3.5.1) satisfies $u(x,T) = u_t(x,T) = 0$.

3.6 L^∞-regularity of the eigenfunctions of the fractional Laplacian

In order to bypass the regularity issue for the solution of our fractional Schrödinger equation, and to be allowed to apply the Pohozaev identity for the fractional Laplacian in the proof of Proposition 3.3.1, we firstly dealt with solutions given as a linear combination of a finite number of eigenfunctions and, in a second moment, we recovered the result we needed for general finite energy solutions by density. To justify this procedure, we show here that the eigenfunctions of the fractional Laplacian on a bounded, regular domain Ω possess the regularity required in the hypothesis of Proposition 3.2.4. We are going to proceed in two steps. First of all, we show L^p regularity for the eigenfunctions for any $p \in [2, +\infty)$; then, we show that we can reach L^∞ regularity and, according to [119, Theorem 1.4], this will imply enough regularity to apply the Pohozaev identity.

3.6.1 Step 1: L^p-regularity of the eigenfunctions

Let us consider the eigenvalues problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda u, & x \in \Omega, \\ u \equiv 0, & x \in \Omega^c. \end{cases}$$

We multiply the equation for $\phi := |u|^{p+1}\mathrm{sgn}(u)$ and we integrate over Ω. First of all, we notice that the function ϕ vanishes outside the domain, thus we can consider the integrals over

Ω as integrals over the whole space $\mathbb{R}^N$. Therefore, we get

$$
\begin{aligned}
\lambda \int_{\mathbb{R}^N} u(x)|u(x)|^{p+1}\operatorname{sgn}(u(x))\,dx &= \lambda \int_{\mathbb{R}^N} |u(x)|^{p+2}dx = \int_{\mathbb{R}^N} |u(x)|^{p+1}\operatorname{sgn}(u(x))(-\Delta)^s u(x)dx \\
&= \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}|u(x)|^{p+1}\operatorname{sgn}(u(x))(-\Delta)^{\frac{s}{2}}u(x)\,dx \\
&= \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\Big[|u(x)|^{p+1}\operatorname{sgn}(u(x)) - |u(y)|^{p+1}\operatorname{sgn}(u(y))\Big]\,dxdy \\
&\geq c_{N,s}\frac{2(p+1)}{(p+2)^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big||u(x)|^{\frac{p+2}{2}} - |u(y)|^{\frac{p+2}{2}}\Big|^2}{|x-y|^{N+2s}}\,dxdy.
\end{aligned}
$$

In the previous computations, we used the inequality

$$
\Big||\alpha|^{\frac{p}{2}} - |\beta|^{\frac{p}{2}}\Big|^2 \leq \frac{p^2}{4(p-1)}(\alpha-\beta)\big(|\alpha|^{p-1}\operatorname{sgn}(\alpha) - |\beta|^{p-1}\operatorname{sgn}(\beta)\big) \quad \forall \alpha,\beta\in\mathbb{R}, \quad \forall p\geq 2
$$

presented in [2, Chapter 4]. Thus, at the end we have

$$
\lambda\int_\Omega |u(x)|^{p+2}dx \geq c_{N,s}\frac{2(p+1)}{(p+2)^2}\int_\Omega\int_\Omega \frac{\Big||u(x)|^{\frac{p+2}{2}} - |u(y)|^{\frac{p+2}{2}}\Big|^2}{|x-y|^{N+2s}}\,dxdy.
$$

Using the embedding theorems for the fractional Sobolev spaces (see e.g. [48, Theorem 6.5]), we finally get

$$
\lambda\int_\Omega |u|^{p+2}\,dx \geq Ac_{N,s}\frac{2(p+1)}{(p+2)^2}\Big\||u|^{\frac{p+2}{2}}\Big\|^2_{L^{\frac{2N}{N-2s}}(\Omega)},
$$

which is , of course, the same as

$$
\lambda\Big\||u|^{\frac{p+2}{2}}\Big\|^2_{L^2(\Omega)} \geq Ac_{N,s}\frac{2(p+1)}{(p+2)^2}\Big\||u|^{\frac{p+2}{2}}\Big\|^2_{L^{\frac{2N}{N-2s}}(\Omega)}.
$$

Since $N/(N-2s) > 1$, this argument allows us to gain regularity for the function u as follows

$$
p+2 \mapsto (p+2)\frac{N}{N-2s}.
$$

Coming back now to our original problem, since u is an eigenfunction for the fractional Laplacian, we know that it is, at least, L^2 regular. Thus, by applying the procedure above for $p=0$ we can increase its regularity up to $L^{\frac{2N}{N-2s}}$.

If now we iterate the same argument we see that, in a finite number of steps, we can get L^p regularity for any $p\in[2,+\infty)$.

3.6.2 Step 2: L^∞-regularity of the eigenfunctions

We prove here the L^∞-regularity for the eigenfunctions of the fractional Laplacian, as an immediate consequence of the following result.

Theorem 3.6.1. *Let $u \in H_0^s(\Omega)$ be the solution of*

$$\begin{cases} (-\Delta)^s u - \lambda u = f, & x \in \Omega, \\ u \equiv 0, & x \in \Omega^c. \end{cases} \tag{3.6.1}$$

If $f \in L^p(\Omega) + L^\infty(\Omega)$ for some $p > 1$, $p > N/2s$, i.e. $f = f_1 + f_2$ with $f_1 \in L^p(\Omega)$ and $f_2 \in L^\infty(\Omega)$, then $u \in L^\infty(\Omega)$.

Proof. This proof is an adaptation of an analogous result from [36].

First of all we observe that, since $-u$ solves the same equation as u with f replaced by its opposite $-f$, which clearly satisfies the same assumptions, it is enough to estimate $\|u^+\|_{L^\infty(\Omega)}$, where

$$u^+ = \begin{cases} u, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0. \end{cases}$$

At this purpose, set $T := \|u^+\|_{L^\infty(\Omega)} \in [0, +\infty]$; without loss of generality, we can assume $T > 0$, since $T = 0$ only for $u = 0$, in which case the Theorem is trivially satisfied. Now, for any $t \in (0, T)$, set $v(t) := (u - t)^+$ and define

$$\alpha(t) := \left| \{ x \in \Omega |\, u(x) > t \} \right|$$

for all $t > 0$ (note that $\alpha(t)$ is always finite).

Since $v(t) \in L^2(\Omega)$ is supported in the set $\{ x \in \Omega |\, u(x) > t \}$, we have $v(t) \in L^1(\Omega)$. Therefore, it is well defined the function

$$\beta(t) := \int_\Omega v(t)\, dx;$$

moreover, integrating the characteristic function $\chi_{\{u>s\}}$ on $(t, +\infty) \times \Omega$ and applying Fubini's theorem we obtain

$$\beta(t) := \int_t^{+\infty} \alpha(s)\, ds,$$

so that $\beta \in W^{1,1}_{\text{loc}}(0, +\infty)$ and $\beta'(t) = -\alpha(t)$ for a.e. $t > 0$. Now, from (3.6.1) we obtain

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u dx - \lambda \int_{\mathbb{R}^N} uv dx = \int_{\mathbb{R}^N} f v\, dx,$$

which yields to

$$\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} v \right|^2 dx - \lambda \int_{\mathbb{R}^N} |v|^2 dx = \int_{\mathbb{R}^N} (f + \lambda t) v\, dx.$$

From this last identity and from the fact that u vanishes outside Ω, if follows immediately

$$|1 - \lambda|\, \|u\|^2_{H^s(\Omega)} \leq \int_\Omega \big(|f| + t|\lambda| \big) v\, dx. \tag{3.6.2}$$

We now observe that, thanks to the Hölder inequality,

$$\int_\Omega |f| v\,dx \le \int_\Omega \big(|f_1| + |f_2|\big) v\,dx$$
$$\le \|f_1\|_{L^p(\Omega)} \|v\|_{L^{\frac{p}{p-1}}(\Omega)} + \|f_2\|_{L^\infty(\Omega)} \|v\|_{L^1(\Omega)} \le C_1 \|v\|_{L^{\frac{p}{p-1}}(\Omega)} + C_2 \|v\|_{L^1(\Omega)}$$

and we deduce from (3.6.2) that

$$\|v\|_{H^s(\Omega)} \le C_3(1+t)\big(\|v\|_{L^{\frac{p}{1-p}}(\Omega)} + \|v\|_{L^1(\Omega)}\big). \tag{3.6.3}$$

Fix now $\rho > 2p/(p-1)$ such that $\rho < 2N/(N-2s)$. From the embedding theorems for the fractional Sobolev spaces ([47, 48]) we have $H^s(\Omega) \hookrightarrow L^\rho(\Omega)$. Moreover, it follows from the Hölder inequality that

$$\|v\|_{L^1(\Omega)} \le \alpha(t)^{1-\frac{1}{\rho}} \|v\|_{L^\rho(\Omega)}$$

and

$$\|v\|_{L^{\frac{p}{p-1}}(\Omega)} \le \alpha(t)^{1-\frac{1}{p}-\frac{1}{\rho}} \|v\|_{L^\rho(\Omega)}.$$

Thus, we deduce from (3.6.3) that

$$\|v\|^2_{L^\rho(\Omega)} \le C_3(1+t)\Big[\alpha(t)^{1-\frac{1}{p}-\frac{1}{\rho}} + \alpha(t)^{1-\frac{1}{\rho}}\Big] \|v\|_{L^\rho(\Omega)}.$$

Since $\beta(t) = \|v\|_{L^1(\Omega)} \le \alpha(t)^{1-\frac{1}{\rho}} \|v\|_{L^\rho(\Omega)}$, we obtain

$$\beta(t) \le C_3(1+t)\Big[\alpha(t)^{2-\frac{1}{p}-\frac{2}{\rho}} + \alpha(t)^{2-\frac{2}{\rho}}\Big]$$

which can be written as

$$\beta(t) \le C_3(1+t) F(\alpha(t)),$$

with $F(s) = s^{2-\frac{1}{p}-\frac{2}{\rho}} + s^{2-\frac{2}{\rho}}$. It follows that

$$-\alpha(t) + F^{-1}\left(\frac{\beta(t)}{C_3(1+t)}\right) \le 0.$$

Setting now $z(t) = \beta(t)/C_3(1+t)$, and remembering that $\beta'(t) = -\alpha(t)$, we deduce

$$z'(t) + \frac{\psi(z(t))}{C_3(1+t)} \le 0,$$

with $\psi(s) = F^{-1}(s) + C_3 s$. Integrating the above differential inequality we get

$$\int_s^t \frac{d\sigma}{C_3(1+\sigma)} \le \int_{z(t)}^{z(s)} \frac{d\sigma}{\psi(\sigma)}$$

for all $0 < s < t < T$. Now, if $T \leq 1$, then $\|u^+\|_{L^\infty} \leq 1$ by definition. Otherwise, we obtain

$$\int_1^t \frac{d\sigma}{C_3(1+\sigma)} \leq \int_{z(t)}^{z(1)} \frac{d\sigma}{\psi(\sigma)}$$

for all $1 < t < T$, which implies in particular that

$$\int_1^T \frac{d\sigma}{C_3(1+\sigma)} \leq \int_0^{z(1)} \frac{d\sigma}{\psi(\sigma)}.$$

Note now that $F(s) \approx s^{2-\frac{1}{p}-\frac{2}{\rho}}$ as $s \downarrow 0$ and $2 - 1/p - 2/\rho > 1$, so that $1/\psi$ is integrable near zero. Since, instead, the function $1/(1+\sigma)$ is not integrable at $+\infty$, this finally implies that $T = \|u^+\|_{L^\infty(\Omega)} < +\infty$. □

Since, of course, the theorem we just proved can be applied to the function $f \equiv 0$, this automatically implies the L^∞-regularity for the eigenfunctions of the fractional Laplacian. Now, this is enough to allow us to apply the Pohozaev identity for the fractional Laplacian to the solution u of our fractional Schrödinger equation. Indeed, [119, Theorem 1.4] states that any bounded solution of

$$\begin{cases} (-\Delta)^s u = f(x,u), & x \in \Omega, \\ u \equiv 0, & x \in \Omega^c. \end{cases} \tag{3.6.4}$$

with $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$, i.e. Lipschitz, satisfies the hypothesis (i) and (ii) of Proposition 3.2.4. But this is exactly our case, since, by definition any eigenfunction of the fractional Laplacian satisfies the problem

$$\begin{cases} (-\Delta)^s \phi_k = \lambda_k \phi_k, & x \in \Omega, \\ \phi_k \equiv 0, & x \in \Omega^c. \end{cases}$$

which is in the form of (3.6.4) with f clearly Lipschitz, and since we just showed that all the eigenfunctions are bounded. Moreover, we can conclude by observing that, always from the definition of eigenfunction, also hypothesis (iii) is clearly satisfied.

3.7 A technical Lemma

One of the main ingredients for obtaining the observability inequality (3.3.22), is the estimate (3.3.28), which is needed for controlling some reminder terms arising during our computations. Being quite long and technical, the proof of this estimate had been postponed, in order not to extend excessively the proof of Lemma 3.3.2.

Instead, we are going to present this proof in the present Section. In particular, (3.3.28) will be a trivial consequence of the following more general result.

Lemma 3.7.1. *Let* $1/2 < s < 1$ *and* $\psi \in H_0^s(\Omega)$. *Moreover, let* η *be the cut-off function introduced in* (3.3.24) *and let* R *be the reminder term in the expression*

$$(-\Delta)^s(\eta\psi) = \eta(-\Delta)^s\psi + R.$$

Then, there exists a constant $C > 0$, *not depending on* ψ, *such that*

$$\|R\|_{L^2(\mathbb{R}^N)} \leq C\left[\|\psi\|_{H^s(\omega)} + \|\psi\|_{L^2(\omega^c)}\right]. \tag{3.7.1}$$

Proof. We are going to use the characterisation of the fractional Laplacian through the heat kernel, that is,

$$(-\Delta)^s(\eta\psi) := \frac{1}{\Gamma(-s)}\int_0^{+\infty}\left(e^{t\Delta}(\eta\psi) - (\eta\psi)\right)\frac{dt}{t^{1+s}}, \tag{3.7.2}$$

where Γ is the Euler Gamma function.

We remark that this characterisation is equivalent to the one given through a singular integral (see, e.g., [131, Section 2.1]). Moreover, for simplicity of notation let us define

$$\varrho := e^{t\Delta}(\eta\psi). \tag{3.7.3}$$

Then, by definition we have that ϱ satisfies the following heat equation on $\mathbb{R}^N$

$$\varrho_t - \Delta\varrho = 0, \quad \varrho(0) = \eta\psi. \tag{3.7.4}$$

Furthermore, the solution of (3.7.4) can be written in the form $\varrho = \phi\eta + z$ with

$$\phi_t - \Delta\phi = 0, \quad \phi(0) = \psi \tag{3.7.5}$$

and

$$z_t - \Delta z = 2\nabla\phi\cdot\nabla\eta + \phi\Delta\eta, \quad z(0) = 0. \tag{3.7.6}$$

Finally, it is simply a matter of computations to show that, from (3.7.2) we obtain the following expression for the reminder term R

$$R := \frac{1}{\Gamma(-s)}\int_0^{+\infty}\frac{z(x,t)}{t^{1+s}}\,dt. \tag{3.7.7}$$

Therefore, for estimating the L^2-norm of R it will be enough to obtain suitable bounds of the L^2-norm of z. Furthermore, we know that the solution of (3.7.6) can be computed explicitly as

$$z(x,t) = \int_0^t\int_{\mathbb{R}^N} G(x-y,t-\tau)h(y,\tau)\,dyd\tau = \int_0^t [G(\cdot,t-\tau)*h(\cdot,\tau)](x)\,d\tau, \tag{3.7.8}$$

where G is the Gaussian kernel

$$G(x,t) := (4\pi t)^{-\frac{N}{2}}\exp\left(-\frac{|x|^2}{4t}\right),$$

while with h we indicated the non-homogeneous right hand side $h := 2\nabla\phi\cdot\nabla\eta+\phi\Delta\eta$. We have

$$\begin{aligned}\left\|\frac{1}{\Gamma(-s)}\int_0^{+\infty}\frac{z(x,t)}{t^{1+s}}\,dt\right\|_{L^2(\mathbb{R}^N)} &\leq \frac{1}{\Gamma(-s)}\int_0^{+\infty}\frac{\|z(x,t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}}\,dt\\ &= \frac{1}{\Gamma(-s)}\left(\int_0^1\frac{\|z(x,t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}}\,dt+\int_1^{+\infty}\frac{\|z(x,t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}}\,dt\right)\\ &:= A_1+A_2. \end{aligned} \tag{3.7.9}$$

We proceed now estimating the terms A_1 and A_2 separately.

Step 1. Preliminary estimates.

First of all, we observe that by classical energy estimates for the heat equation we have

$$\begin{aligned}&\frac{d}{dt}\|\phi(x,t)\|^2_{L^2(\mathbb{R}^N)} = -2\|\nabla\phi(x,t)\|^2_{L^2(\mathbb{R}^N)}\leq 0 &&\Rightarrow\ \|\phi(x,t)\|_{L^2(\mathbb{R}^N)}\leq\|\psi\|_{L^2(\Omega)},\\ &\frac{d}{dt}\|\nabla\phi(x,t)\|^2_{L^2(\mathbb{R}^N)} = -2\|\Delta\phi(x,t)\|^2_{L^2(\mathbb{R}^N)}\leq 0 &&\Rightarrow\ \|\phi(x,t)\|_{H^1(\mathbb{R}^N)}\leq\|\psi\|_{H^1(\Omega)}.\end{aligned} \tag{3.7.10}$$

These inequalities are trivial, multiplying (3.7.5) by ϕ and $\Delta\phi$ respectively and integrating by parts. In particular, from (3.7.10) it follows by interpolation

$$\|\phi(x,t)\|_{H^s(\mathbb{R}^N)}\leq\|\psi\|_{H^s(\Omega)},\quad \text{for all }\ s\in(0,1). \tag{3.7.11}$$

In our proof, we will also need the following classical property of the convolution ([66, Proposition 8.9])

$$\|\varphi_1*\varphi_2\|_{L^r(\mathbb{R}^N)}\leq\|\varphi_1\|_{L^p(\mathbb{R}^N)}\|\varphi_2\|_{L^q(\mathbb{R}^N)}, \tag{3.7.12}$$

which is a straightforward consequence of Young inequality and holds for all $\varphi_1\in L^p(\mathbb{R}^N)$, $\varphi_2\in L^q(\mathbb{R}^N)$ and for all p,q and r satisfying

$$1\leq p,q,r<+\infty,\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1. \tag{3.7.13}$$

Finally, we recall that for all $1<p\leq q<+\infty$ and $k\geq 0$ the function G satisfies the following decay properties (see, e.g., [80])

$$\left\|D^kG(x,t)\right\|_{L^p(\mathbb{R}^N)}\leq\beta_1\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{k}{2}},\quad \left\|(D^kG*h)(x,t)\right\|_{L^q(\mathbb{R}^N)}\leq\beta_2\,t^{-\frac{1}{2}\left(k+\frac{N}{p}-\frac{N}{q}\right)}\|h\|_{L^p(\omega)}. \tag{3.7.14}$$

Here, $k=(k_1,k_2\ldots,k_N)$ is a multi-index with modulus $|k|=k_1+k_2+\cdots+k_N$ and we used the classical Schwartz notation

$$D^k\phi(x)=\frac{\partial^{|k|}\phi(x)}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_N^{k_N}}.$$

In particular

$$\begin{aligned} &\|G(x,t)\|_{L^2(\mathbb{R}^N)} \le \beta_1 t^{-\frac{N}{4}}, && \|\nabla G(x,t)\|_{L^2(\mathbb{R}^N)} \le \beta_1\, t^{-\frac{N}{4}-\frac{1}{2}} \\ &\|(G*h)(x,t)\|_{L^2(\mathbb{R}^N)} \le \beta_2 \|h\|_{L^2(\omega)}, && \|(\nabla G*h)(x,t)\|_{L^2(\mathbb{R}^N)} \le \beta_2\, t^{-\frac{1}{2}} \|h\|_{L^2(\omega)}. \end{aligned} \tag{3.7.15}$$

Step 2. Upper bound of A_2.

First of all, from now on, for keeping the notation lighter we will omit the dependence on the space variables in the functions involved in our computations. Moreover, we observe that

$$\nabla\phi(\tau)\cdot\nabla\eta = \operatorname{div}(\phi(\tau)\nabla\eta) - \phi(\tau)\Delta\eta;$$

therefore, starting from (3.7.8) we have

$$z(t) = 2\int_0^t G(t-\tau)*\operatorname{div}(\phi(\tau)\nabla\eta)\,d\tau - \int_0^t G(t-\tau)*(\phi(\tau)\Delta\eta)\,d\tau := z_1(t) - z_2(t), \tag{3.7.16}$$

and, clearly, $\|z(t)\|_{L^2(\mathbb{R}^N)} \le \|z_1(t)\|_{L^2(\mathbb{R}^N)} + \|z_2(t)\|_{L^2(\mathbb{R}^N)}$. Now, using (3.7.15),

$$\begin{aligned} \|z_1(t)\|_{L^2(\mathbb{R}^N)} &\le \int_0^t \|G(t-\tau)*\operatorname{div}(\phi(\tau)\nabla\eta))\|_{L^2(\mathbb{R}^N)}\,d\tau \\ &= \int_0^t \|\nabla G(t-\tau)*(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)}\,d\tau \\ &\le \gamma_1 \int_0^t (t-\tau)^{-\frac{1}{2}} \|\phi(\tau)\nabla\eta\|_{L^2(\mathbb{R}^N)}\,d\tau \le \gamma_2 t^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

The estimate for $z_2(t)$ is more delicate and we need to distinguish three cases: $N=1$, $N=2$ and $N\ge 3$.

Let us consider firstly $N=2$; using (3.7.15) we have

$$\|z_2(t)\|_{L^2(\mathbb{R}^2)} \le \gamma_3 \int_0^t (t-\tau)^{-\frac{1}{2}} \|\phi(\tau)\Delta\eta\|_{L^1(\mathbb{R}^2)}\,d\tau \le \gamma_4\, t^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)}.$$

Therefore, since $s > 1/2$, from the definition of A_2 we obtain the estimate

$$A_2 \le (\gamma_2+\gamma_4)\|\psi\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{1}{2}}} \le \gamma_5 \|\psi\|_{L^2(\Omega)}. \tag{3.7.17}$$

Let us now assume $N \ge 3$; in this case, we are going to use (3.7.12) with

$$p = \frac{N-1}{N-2}, \quad q = \frac{2N-2}{N+1} \quad \text{and} \quad r = 2; \tag{3.7.18}$$

it is straightforward to check that this choice of the parameters p, q and r satisfies (3.7.13). Now, since η is compactly supported in ω and $q < 2$, using (3.7.15) we have

$$\begin{aligned}\|z_2(t)\|_{L^2(\mathbb{R}^N)} &\leq \int_0^t \|G(t-\tau) * (w(\tau)\Delta\eta)\|_{L^2(\mathbb{R}^N)}\,d\tau\\ &\leq \int_0^t \|G(t-\tau)\|_{L^{\frac{N-1}{N-2}}(\mathbb{R}^N)} \|w(\tau)\Delta\eta\|_{L^{\frac{2N-2}{N+1}}(\mathbb{R}^N)}\,d\tau\\ &\leq \kappa_1 \int_0^t (t-\tau)^{-\frac{N}{2N-2}} \|w(\tau)\Delta\eta\|_{L^2(\mathbb{R}^N)}\,d\tau \leq \kappa_2 \|\psi\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{N}{2N-2}}\,d\tau\\ &= \kappa_3\, t^{\frac{N-2}{2N-2}} \|\psi\|_{L^2(\Omega)}.\end{aligned}$$

Hence, by definition of A_2 we obtain the estimate

$$A_2 \leq \gamma_2 \|\psi\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{1}{2}}} + \kappa_3 \|\psi\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{N}{2N-2}}} \leq \kappa_4 \|\psi\|_{L^2(\Omega)}, \tag{3.7.19}$$

since both s and $N/(2N-2)$ are greater than $1/2$.

Therefore, it only remains to analyse the case $N = 1$. First of all, since $\psi \in L^2(\Omega)$ and Ω is bounded, we also have $\psi \in L^1(\Omega)$. Hence, it is well defined the quantity

$$m := \int_{\mathbb{R}^N} \psi\,dx = \int_\Omega \psi\,dx.$$

Let us now rewrite $\psi = (\psi - m\delta_0) + m\delta_0$, where δ_0 is the Dirac delta at $x = 0$. With this splitting in mind, we have that the function ϕ solution of (3.7.5) can be seen as the sum $\phi = p + mG$, with p solving

$$p_t - p_{xx} = 0, \quad p(0) = \psi - m\delta_0. \tag{3.7.20}$$

Therefore, we obtain

$$z_2(t) = \int_0^t G(t-\tau) * (p(\tau)\eta_{xx})\,d\tau + \int_0^t G(t-\tau) * (mG(\tau)\eta_{xx})\,d\tau := z_{2,p}(t) + z_{2,G}(t).$$

Let us analyse firstly the term $z_{2,G}$. First of all, we have

$$z_{2,G}(t) = m\int_0^t G(t-\tau) * (G(\tau)\eta_{xx})\,d\tau = m\int_0^t G(t-\tau) * [(G(\tau)\eta_x)_x - G_x(\tau)\eta_x]\,d\tau,$$

and

$$\begin{aligned}\|z_{2,G}(t)\|_{L^2(\mathbb{R})} &\leq m\int_0^t \|G(t-\tau) * (G(\tau)\eta_x)_x\|_{L^2(\mathbb{R})}\,d\tau\\ &\quad + m\int_0^t \|G(t-\tau) * (G_x(\tau)\eta_x)\|_{L^2(\mathbb{R})}\,d\tau = J_1 + J_2.\end{aligned}$$

Now, since ψ is compactly supported in Ω, using Cauchy-Schwarz inequality we have $m \leq \|\psi\|_{L^1(\Omega)} \leq \sqrt{|\Omega|}\|\psi\|_{L^2(\Omega)}$, where $|\Omega|$ is the measure of Ω; hence

$$\begin{aligned}
J_1 &\leq m \int_0^t \|G_x(t-\tau) * (G(\tau)\eta_x)\|_{L^2(\mathbb{R})}\, d\tau \leq \rho_1 \|\psi\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{1}{2}} \|G(\tau)\eta_x\|_{L^2(\mathbb{R})}\, d\tau \\
&\leq \rho_2 \|\psi\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{4}}\, d\tau = \rho_3 \|\psi\|_{L^2(\Omega)}\, t^{\frac{1}{4}}. \\
J_2 &\leq \rho_4 \|\psi\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{1}{4}} \|G_x(\tau)\eta_x\|_{L^1(\mathbb{R})}\, d\tau \leq \rho_5 \|\psi\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{1}{4}} \tau^{-\frac{1}{2}}\, d\tau \\
&= \rho_6 \|\psi\|_{L^2(\Omega)}\, t^{\frac{1}{4}}.
\end{aligned}$$

Therefore

$$\int_1^{+\infty} \frac{\|z_{2,G}\|_{L^2(\mathbb{R})}}{t^{1+s}}\, dt \leq (\rho_3 + \rho_6) \|\psi\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{3}{4}}}\, dt \leq \rho_7 \|\psi\|_{L^2(\Omega)}.$$

Finally, let us consider the term $z_{2,p}$. First of all, we notice that $p = q_x$ with

$$q_t - q_{xx} = 0, \quad q(0) = \int_{-\infty}^x (\psi - m\delta_0)\, d\xi, \tag{3.7.21}$$

and, therefore,

$$z_{2,p}(t) = \int_0^t G(t-\tau) * (q_x(\tau)\eta_{xx})\, d\tau.$$

Now

$$\|z_{2,p}(t)\|_{L^2(\mathbb{R})} \leq \int_0^t \|G(t-\tau) * (q_x(\tau)\eta_{xx})\|_{L^2(\mathbb{R})}\, d\tau \leq \int_0^t (t-\tau)^{-\frac{1}{4}} \|q_x(\tau)\eta_{xx}\|_{L^1(\mathbb{R})}\, d\tau.$$

Moreover, we have

$$\|q_x(t)\eta_{xx}\|_{L^1(\mathbb{R})} = \|q_x(t)\eta_{xx}\|_{L^1(\Omega)} \leq \sigma_1 \|q_x(t)\|_{L^1(\Omega)} \leq \sigma_2\, \|q(0)\|_{L^1(\Omega)}\, t^{-\frac{1}{2}} \leq \sigma_3 \|\psi\|_{L^2(\Omega)}\, t^{-\frac{1}{2}},$$

where the last inequality is justified by the fact that the initial datum $q(0)$ is well defined as a L^1 function compactly supported in Ω and there exists a constant $M > 0$, such that

$$\|q(0)\|_{L^1(\Omega)} \leq M \|\psi\|_{L^2(\Omega)}.$$

See [51, Theorem 1] for more details. Hence,

$$\|z_{2,p}(t)\|_{L^2(\mathbb{R})} \leq \sigma_4 \|\psi\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{1}{4}} \tau^{-\frac{1}{2}}\, d\tau = \sigma_5 \|\psi\|_{L^2(\Omega)}\, t^{\frac{1}{4}}.$$

and

$$\int_1^{+\infty} \frac{\|z_{2,p}(t)\|_{L^2(\mathbb{R})}}{t^{1+s}}\, dt \leq \sigma_5 \|\psi\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{3}{4}}} \leq \sigma_6 \|\psi\|_{L^2(\Omega)}.$$

Recollecting all the contributions that we have calculated, for $N = 1$ we obtained the following estimate

$$\int_1^{+\infty} \frac{\|z_2(t)\|_{L^2(\mathbb{R})}}{t^{1+s}}\, dt \le (\rho_7 + \sigma_6)\|\psi\|_{L^2(\Omega)}.$$

Therefore, by definition of A_2 we get

$$A_2 \le (\gamma_2 + \rho_7 + \sigma_6)\|\psi\|_{L^2(\Omega)}. \tag{3.7.22}$$

Summarising, from (3.7.17), (3.7.19) and (3.7.22) we can conclude that, for all $N \ge 1$ there exists a constant $P > 0$ such that

$$A_2 \le P\|\psi\|_{L^2(\Omega)}. \tag{3.7.23}$$

Step 3. Upper bound of A_1.

Let us now analyse the term A_1. At this purpose, we recall that

$$\|z(t)\|_{L^2(\mathbb{R}^N)} \le \|z_1(t)\|_{L^2(\mathbb{R}^N)} + \|z_2(t)\|_{L^2(\mathbb{R}^N)},$$

with z_1 and z_2 as in (3.7.16). Let us firstly analyse the contribution of z_1; we get

$$\begin{aligned}
\|z_1(t)\|_{L^2(\mathbb{R}^N)} &\le \int_0^t \|G(t-\tau) * \operatorname{div}(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)}\, d\tau \\
&\le \varrho_1 \int_0^t \left\|D^{1-s}G(t-\tau) * D^s(\phi(\tau)\nabla\eta)\right\|_{L^2(\mathbb{R}^N)} d\tau \\
&\le \varrho_2 \int_0^t (t-\tau)^{-\frac{1-s}{2}} \|\phi(\tau)\nabla\eta\|_{H^s(\mathbb{R}^N)}\, d\tau \le \varrho_3\|\psi\|_{H^s(\Omega)} \int_0^t (t-\tau)^{-\frac{1-s}{2}}\, d\tau \\
&= \varrho_4\|\psi\|_{H^s(\Omega)}\, t^{\frac{1+s}{2}}.
\end{aligned}$$

In the previous computations, we indicated with D^s the differential operator with Fourier symbol $|\xi|^s$, that is $\mathcal{F}(D^s\zeta)(\xi) = |\xi|^s\mathcal{F}\zeta(\xi)$ for all functions ζ sufficiently smooth. Concerning the contribution of z_2, instead, we have

$$\begin{aligned}
\|z_2(t)\|_{L^2(\mathbb{R}^N)} &\le \int_0^t \|G(t-\tau) * (\phi(\tau)\Delta\eta)\|_{L^2(\mathbb{R}^N)}\, d\tau \\
&\le \varrho_5 \int_0^t \|\phi(\tau)\Delta\eta\|_{L^2(\mathbb{R}^N)}\, d\tau \le \varrho_6\|\psi\|_{L^2(\Omega)} \int_0^t d\tau = \varrho_6\|\psi\|_{L^2(\Omega)}\, t.
\end{aligned}$$

Therefore, finally, since $s < 1$,

$$A_1 \le \varrho_4\|\psi\|_{H^s(\Omega)} \int_0^1 \frac{dt}{t^{\frac{1+s}{2}}} + \varrho_6\|\psi\|_{L^2(\Omega)} \int_0^1 \frac{dt}{t^s} \le \varrho_7\|\psi\|_{H^s(\Omega)} + \varrho_8\|\psi\|_{L^2(\Omega)}. \tag{3.7.24}$$

Summarising, we can conclude that there exist two constants C_1, $C_2 > 0$, not depending on f, such that

$$\|R\|_{L^2(\mathbb{R}^N)} = \left\| \int_0^{+\infty} \frac{z(t)}{t^{1+s}}\, dt \right\|_{L^2(\mathbb{R}^N)} \le C_1 \|\psi\|_{H^s(\Omega)} + C_2 \|\psi\|_{L^2(\Omega)}$$

and, by definition of the $H^S(\Omega)$-norm, we have

$$\|R\|_{L^2(\mathbb{R}^N)} \le C_3 \|\psi\|_{H^s(\Omega)}. \tag{3.7.25}$$

Step 4. Conclusion.

Let us now conclude our proof, deriving (3.7.1) from (3.7.25). First of all, we have

$$\|R\|_{L^2(\mathbb{R}^N)} \le C_3 \|\psi\|_{H^s(\Omega)} = \|\phi(0)\|_{H^s(\Omega)} \le \sup_{t\in[0,T]} \|\phi(t)\|_{H^s(\Omega)};$$

moreover, we know that the function ϕ solution of (3.7.5) is given by

$$\begin{aligned}\phi(x,t) &= [G(\cdot,t) * \psi(\cdot)]\,(x) = \int_{\mathbb{R}^N} G(x-y,t)\psi(y)\, dy \\ &= \int_{\omega} G(x-y,t)\psi(y)\, dy + \int_{\omega^c} G(x-y,t)\psi(y)\, dy := \phi_I(x,t) + \phi_E(x,t).\end{aligned}$$

Since we are interested in obtaining an estimate involving the norm of v in a neighbourhood of the boundary of Ω, let us assume form now on that $x \in \hat{\omega}$. Moreover, it is straightforward that we can see the integral defining ϕ_E as computed on the whole $\mathbb{R}^N$ in the following way

$$\phi_E(x,t) = \int_{\mathbb{R}^N} G(x-y,t)\psi(y)\chi_{\omega^c}(y)\, dy,$$

where χ_{ω^c} is the characteristic function of the set ω^c.

Now, since $x \in \hat{\omega}$ while $y \in \omega^c$ due to the presence of the function χ_{ω^c} in the integrand, we know that the heat kernel an all its derivative are uniformly bounded. This, in particular, implies

$$|D^s \phi_E|^2 \le P_1 \int_{\mathbb{R}^N} |\psi(y)\chi_{\omega^c}(y)|^2\, dy = P_1 \|\psi\|^2_{L^2(\omega^c)}.$$

Therefore,

$$\|\phi_E\|_{H^s(\Omega)} \le P_2 \|\psi\|_{L^2(\omega^c)}.$$

Hence, it only remains to treat the component $\phi_I(x,t)$; at this purpose, let us rewrite the function ψ as

$$\psi = \eta\psi + (1-\eta)\psi := \psi_1 + \psi_2,$$

where η is the same cut-off function that we introduced before in (3.3.24). Thus,

$$\phi_I(x,t) = \int_{\mathbb{R}^N} G(x-y,t)\psi_1(y)\,dy + \int_{\omega\setminus\hat{\omega}} G(x-y,t)\psi_2(y)\,dy$$

since, by definition, $\text{supp}(\psi_1) = \omega$ and $\text{supp}(\psi_2) = \mathbb{R}^N \setminus \hat{\omega}$. Therefore, we have

$$\begin{aligned} D^s\phi_I(x,t) &= \int_{\mathbb{R}^N} D^sG(x-y,t)\psi(y)\eta(y)\,dy - \int_{\omega\setminus\hat{\omega}} D^sG(x-y,t)\psi(y)\eta(y)\,dy \\ &= \int_{\mathbb{R}^N} G(x-y,t)D^s(\psi(y)\eta(y))\,dy - \int_{\omega\setminus\hat{\omega}} D^sG(x-y,t)\psi(y)\eta(y)\,dy. \end{aligned}$$

In particular,

$$\begin{aligned} |D^s\phi_I(x,t)|^2 &\leq P_3\int_{\mathbb{R}^N} |D^s(\psi(y)\eta(y))|^2\,dy + P_4\int_{\omega\setminus\hat{\omega}} |\psi(y)\eta(y)|^2\,dy \\ &\leq P_5\|\psi\|^2_{H^s(\omega)} + P_4\|\psi\|^2_{L^2(\omega\setminus\hat{\omega})}, \end{aligned}$$

and this gives us the estimate

$$\|\phi_I\|_{H^s(\Omega)} \leq P_6\|\psi\|_{H^s(\omega)} + P_7\|\psi\|_{L^2(\tilde{\omega}\setminus\omega)} \leq P_6\|\psi\|_{H^s(\omega)} + P_7\|\psi\|_{L^2(\omega)} \leq P_8\|\psi\|_{H^s(\omega)}.$$

Therefore, recollecting all the contributes we obtained, we can finally conclude that there exists a constant C, not depending on v, such that

$$\|R\|_{L^2(\mathbb{R}^N)} \leq C\left[\|\psi\|_{H^s(\omega)} + \|\psi\|_{L^2(\omega^c)}\right].$$

□

Chapter 4

Boundary controllability for a one-dimensional heat equation with a singular inverse-square potential

Abstract.
This Chapter is devoted to the analysis of the boundary controllability for a one-dimensional heat equation, defined on the domain $(x,t) \in (0,1) \times (0,T)$, involving the singular inverse-square potential μ/x^2, whose singularity arises at the boundary of the domain. For any $0 < \mu < 1/4$, we show that we can lead the system to the zero state using a control $f \in L^2(0,T)$ located at $x = 0$. The result is obtained through an appropriate change of variables that transforms our problem in a parabolic equation with variable degenerate coefficients, for which boundary controllability properties are already known to hold ([76]).

4.1 Introduction and main results

Let $T > 0$ and set $Q := (0,1) \times (0,T)$. We are interested in proving boundary controllability for a one-dimensional heat equation on the domain Q, presenting a singular inverse-square potential with singularity located on the boundary that is, given the operator

$$\mathcal{A} = \mathcal{A}(\mu) := -\frac{d^2}{dx^2} - \frac{\mu}{x^2}\mathcal{I}, \quad \mu \leq 1/4, \tag{4.1.1}$$

we are going to consider the following parabolic equation

$$\begin{cases} u_t - u_{xx} - \dfrac{\mu}{x^2} u = 0, & (x,t) \in Q \\ x^{-\lambda} u(x,t)\big|_{x=0} = f(t), \quad u(1,t) = 0, & t \in (0,T) \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases} \tag{4.1.2}$$

with the intent of proving that it is possible to choose the control function f in an appropriate functional space Y such that the corresponding solution of (4.1.2) satisfies

$$u(x,T) = 0, \quad \text{for all } x \in (0,1). \tag{4.1.3}$$

Moreover, we recall that $1/4$ is the critical value for the constant in the one-dimensional Hardy inequality, guaranteeing that for any function $z \in H_0^1(0,1)$ we have $z/x \in L^2(0,1)$ and it holds (see, e.g., [45, Chapter 5, Section 3] or [90, Theorem 6])

$$\int_0^1 z_x^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x^2} \, dx. \tag{4.1.4}$$

A first important aspect that we want to underline is the non standard formulation of the boundary conditions in (4.1.2). Indeed, due to the presence of the singularity at $x = 0$ it turns out that it is not possible to impose a boundary condition of the type $u(0,t) = f(t) \neq 0$; instead, we need to introduce the "weighted" boundary condition

$$x^{-\lambda} u(x,t)\Big|_{x=0} = f(t), \tag{4.1.5}$$

with

$$\lambda := \frac{1}{2}\left(1 - \sqrt{1-4\mu}\,\right). \tag{4.1.6}$$

This fact is justified by the observation that the general solution of the second order elliptic equation $u_{xx} + (\mu/x^2)u = 0$ may be calculated explicitly and it is given by

$$u(x) = C_1 x^{\frac{1}{2} - \frac{1}{2}\sqrt{1-4\mu}} + C_2 x^{\frac{1}{2} + \frac{1}{2}\sqrt{1-4\mu}}, \tag{4.1.7}$$

with $(C_1, C_2) \neq (0,0)$; therefore,

$$\begin{cases} u(0) = 0, & \text{for } \mu > 0, \\ u(0) = \pm\infty, & \text{for } \mu < 0, \end{cases} \tag{4.1.8}$$

where the sign of $u(0)$ for $\mu < 0$ is given by the sign of the constant C_1. On the other hand, we have

$$\lim_{x \to 0^+} x^{-\frac{1}{2} + \frac{1}{2}\sqrt{1-4\mu}} u(x) = \lim_{x \to 0^+} x^{-\lambda} u(x) = C_1.$$

We remark that in (4.1.8) we are not considering the case $\mu = 0$; this case, indeed, corresponds simply to a one-dimensional Laplace equation for which, of course, we do not need any further analysis. Moreover, we notice that for $\mu = 0$ we have also $\lambda = 0$; therefore, the boundary condition (4.1.5) becomes $u(0,t) = f(t)$, which is consistent with the classical theory. Finally, it is evident from the argument above that $x^{-\lambda}$ is the sharp weight for defining a non-homogeneous boundary condition at $x = 0$. As we shall see with more details later, the parameter λ has a fundamental role in our analysis.

As we are going to show in Section 4.2 by means of transposition techniques ([99]), equation (4.1.2) is well posed for all $\mu \leq 1/4$.

Concerning instead control properties, in this Chapter we are interested in solving the following problem.

Problem 4.1.1. *Given u_0 in an appropriate functional space X on $(0,1)$, find f in a functional space Y on $(0,T)$, such that the corresponding solution u of (4.1.2) satisfies (4.1.3).*

Due to technical reasons that we will underline later, for obtaining the controllability of (4.1.2) we will need to impose further restrictions on the values that can be assumed by the coefficient μ; in particular, we have to assume μ to be positive and non-critical (i.e. $0 < \mu < 1/4$). This restriction will be justified with more details in Section 4.4.

Moreover, at this stage we do not specify the functional setting in which the controllability result will hold, since it is not the standard one. Its detailed description will instead be postponed to Section 4.3.

As it is by now classical, for proving Theorem 4.4.1 we will apply the Hilbert Uniqueness Method (HUM, [42, 97]); hence the controllability property will be equivalent to the observability of the adjoint system associated to (4.1.2), namely

$$\begin{cases} v_t + v_{xx} + \dfrac{\mu}{x^2}v = 0, & (x,t) \in Q \\ v(0,t) = v(1,t) = 0, & t \in (0,T) \\ v(x,T) = v_T(x), & x \in (0,1). \end{cases} \tag{4.1.9}$$

Finally, we want to stress the fact that in the adjoint system (4.1.9) we are imposing classical Dirichlet boundary conditions, that is, without any weight. Indeed, in equation (4.1.2) the weight at $x = 0$ is needed since we want to detect a non-zero boundary data; on the contrary, when considering a problem with homogeneous boundary conditions the polynomial behaviour of the solution (see (4.1.7)) ensures the well-posedness in the classical framework.

Singular inverse-square potentials arise in quantum cosmology ([9]), in electron capture problems ([72]), but also in the linearisation of reaction-diffusion problems involving the heat equation with supercritical reaction term ([69]); also for these reasons, starting from the pioneering work [4] evolution problems involving this kind of potentials have been intensively studied in the last decades.

Moreover, it is by now well known that equations of the type of (4.1.2) are closely related, through an appropriate change of variables (see, for instance, [103, Chapter 4]), to another class of PDE problems with variable degenerate coefficients, i.e. in the form

$$u_t - (a(x)u_x)_x = 0, \quad \alpha \in (0,1), \quad (x,t) \in Q, \tag{4.1.10}$$

$$u_{tt} - (a(x)u_x)_x = 0, \quad \alpha \in (0,1), \quad (x,t) \in Q, \tag{4.1.11}$$

with a coefficient $a(x)$ that vanishes at a certain $x_0 \in (0,1)$.

In the recent past, it has been given many attention to this kind of equations; in particular, they have been obtained several controllability results.

In [29, 30, 103], the authors obtained the null-controllability for (4.1.10) by means of a distributed control supported in a non-empty subset $\omega \subset (0,1)$. Furthermore, an analogous result has been recently proved in [1] for a wave equation of the type of (4.1.11), with coefficient $a(x)$ vanishing at $x = 0$ and control at $x = 1$.

In [32], instead, the authors considered the equation (4.1.10) in the case $a(x) = x^\alpha$, $\alpha \in (0,1)$ and they proved approximate controllability from $x = 0$.

In all the works mentioned above, the main tool for obtaining the controllability results presented is an appropriate Carleman estimate.

Finally, in [76] it is considered again the case $a(x) = x^\alpha$, $\alpha \in (0,1)$, and it is proved the null controllability both for (4.1.11) and (4.1.10), again from $x = 0$. In this case, the result is obtained implementing a spectral analysis of the equation under consideration.

Also for evolution equations with singular inverse-square potentials the controllability problem has already been addressed in the past; among other works, we recall here [34, 35, 53, 137, 138].

In all these articles, the authors analysed heat and wave equations involving a potential of the type $\mu/|x|^2$ on a bounded regular domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and proved null controllability choosing a control region inside of the domain, away from the singularity point $x = 0$.

However, to the best of our knowledge, there are no results on boundary controllability, or of controllability acting from the singularity point. The analysis of problem (4.1.2) that we are presenting is a first step in this direction, in which the two issues mentioned above appear together. Indeed, we are going to prove that it is possible to control the equation from the boundary, and in particular from the extrema where the singularity of the potential arises.

For doing that, it will be fundamental to understand the level of degeneracy of the solution of the equation at the singularity point, in order to be able to compensate it properly. We believe that this is one of the main novelties of our work.

The strategy that we will follow for obtaining our result consists in showing that, applying the following change of variables

$$v(x,t) := x^{\frac{\alpha}{2(2-\alpha)}}\,\psi(x,t), \quad x(\xi) := \left(\frac{2}{2-\alpha}\right)\xi^{\frac{2-\alpha}{2}},$$

with

$$\alpha = \frac{2+8\mu-2\sqrt{1-4\mu}}{3+4\mu}, \tag{4.1.12}$$

we can transform our original adjoint equation (4.1.9) in the following one with variable degenerate coefficients

$$\psi_t + (\xi^\alpha \psi_\xi)_\xi = 0. \tag{4.1.13}$$

In [76] it is proved that, for $0 < \alpha < 1$, (4.1.13) is null-controllable with a control $f \in L^2(0,T)$ located at $x = 0$. This result is obtained as a consequence of an observability inequality for the adjoint equation associated. From this inequality, applying the inverse change of variable we can recover the observability of (4.1.9). The controllability of (4.1.2) will then be consequence of a duality argument.

Nevertheless, this approach provides limitations on the values that can be assumed by the coefficient μ. In particular, our proof will be valid only for $0 < \mu < 1/4$, which corresponds to imposing that α defined as in (4.4.2) satisfies $0 < \alpha < 1$. We will present more details on this issue in the following Sections.

This Chapter is organized as follows. In Section 4.2, we analyse the existence and uniqueness of solutions for (4.1.2), applying classical semi-group theory and transposition techniques ([99]); moreover, passing through the decomposition of the solution of the equation in the basis of the eigenfunctions of the corresponding elliptic operator (that can be computed explicitly), we derive the sharp weight needed for compensating the singularity of the normal derivative approaching the boundary. In Section 4.3, we introduce some existing results obtained in [76] for parabolic equations with degenerate coefficients. In particular, we will present the functional setting in which the results of [76] are stated, as well as the observability inequality employed for obtaining the boundary controllability of (4.1.13). Finally, Section 4.4 is devoted to the proof of the observability inequality and of the controllability of equation (4.1.2) acting from $x = 0$.

4.2 Well-posedness and reguality

We analyse here existence and uniqueness of solutions of the heat equation (4.1.2). As it is classical, the question of the well-posedness of this non-homogeneous boundary problem will be treated employing transposition techniques ([99]); at this purpose, we firstly need to state the existence and uniqueness of solutions for heat equations of the type

$$\begin{cases} w_t - w_{xx} - \dfrac{\mu}{x^2}w = h, & (x,t) \in Q \\ w(0,t) = w(1,t) = 0, & t \in (0,T) \\ w(x,0) = w_0(x), & x \in (0,1). \end{cases} \tag{4.2.1}$$

Therefore, let us introduce the Hilbert space H defined as the closure of $C_0^\infty(0,1)$ with respect to the norm

$$\forall w \in H_0^1(0,1), \quad \|w\|_H = \left[\int_0^1 \left(w_x^2 - \frac{\mu}{x^2}w^2\right) dx\right]^{\frac{1}{2}}.$$

It is simply a matter of computations to show that, for all $\mu \leq 1/4$, there exist two positive constants M_1 and M_2, depending on μ, such that it holds the following inequality

$$(1-4\mu)\int_0^1 w_x^2 + M_1\int_0^1 w^2\,dx \leq \|w\|_H^2 \leq (1+4\mu)\int_0^1 w_x^2 + M_2\int_0^1 w^2\,dx. \tag{4.2.2}$$

Therefore, it is evident that, in the sub-critical case $\mu < 1/4$, from (4.2.2) it follows the identification $H = H_0^1(0,1)$ with equivalent norms. On the contrary, for $\mu = 1/4$ this identification does not hold anymore and the space H is slightly (but strictly) larger than $H_0^1(0,1)$. For a complete and sharp description of the space H in this case, we refer to [141].

Let us now consider the unbounded operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$, defined for all $\mu \leq 1/4$ as

$$\begin{gathered} \mathcal{D}(\mathcal{A}) := \left\{ w \in H \,\middle|\, w_{xx} + \frac{\mu}{x^2}w \in L^2(0,1)\right\}, \\ \mathcal{A}w := -w_{xx} - \frac{\mu}{x^2}w, \end{gathered} \tag{4.2.3}$$

whose norm is given by

$$\|w\|_{\mathcal{A}} = \|w\|_{L^2(0,1)} + \|\mathcal{A}w\|_{L^2(0,1)}.$$

With the definitions we just gave, by standard semi-group theory we have that for any $\mu \leq 1/4$ the operator (4.2.3) generates an analytic semi-group in the pivot space $L^2(0,1)$ for the equation (4.2.1).

Therefore, referring to [136, Theorem II.1], we immediately have the following well-posedness result

Theorem 4.2.1. *Let $\mu \leq 1/4$. Given $w_0 \in L^2(0,1)$ and $h \in L^2(0,T;L^2(0,1))$, the problem* (4.2.1) *admits a unique weak solution*

$$w \in L^2(0,T;\mathcal{D}(\mathcal{A})) \cap H^1(0,T;L^2(0,1))$$

satisfying the following estimate

$$\|w\|_{L^2(0,T;\mathcal{D}(\mathcal{A}))} + \|w\|_{H^1(0,T;L^2(0,1))} \leq C\left(\|h\|_{L^2(0,T;L^2(0,1))} + \|w_0\|_{L^2(0,1)}\right).$$

Finally, coming back to the non-homogeneous boundary value problem (4.1.2), we can now introduce the notion of a weak solution defined by transposition in the spirit of [99].

Definition 4.2.1. *Let $T > 0$ and $\mu \leq 1/4$. For any $u_0 \in L^2(0,1)$ and $f \in L^2(0,T)$, $u \in L^2(0,T;L^2(0,1))$ is a solution of* (4.1.2) *defined by transposition if it satisfies the identity*

$$\int_0^T f(t)\left[x^\lambda \phi_x(x,t)\right]\Big|_{x=0} dt + \int_0^1 \phi(x,0)u_0(x)\,dx = \int_Q uh\,dxdt \tag{4.2.4}$$

where, for any $h \in L^2(0,T;L^2(0,1))$, ϕ is the solution of the adjoint system

$$\begin{cases} \phi_t + \phi_{xx} + \dfrac{\mu}{x^2}\phi = -h, & (x,t) \in Q \\ \phi(0,t) = \phi(1,t) = 0, & t \in (0,T) \\ \phi(x,T) = 0, & x \in (0,1). \end{cases} \tag{4.2.5}$$

Theorem 4.2.2. *Let $T > 0$ and $\mu \leq 1/4$. Given $u_0 \in L^2(0,1)$ and $f \in L^2(0,T)$, the problem* (4.1.2) *admits a unique weak solution $u \in L^2(0,T;L^2(0,1))$ defined by transposition in the sense of Definition 4.2.1. Moreover, there exists a constant C independent of u_0 and f such that*

$$\|u\|_{L^2(0,T;L^2(0,1))} \leq C\left(\|u_0\|_{L^2(0,1)} + \|f\|_{L^2(0,T)}\right). \tag{4.2.6}$$

For proving Theorem 4.2.2, we will need the following result on the regularity of the normal derivative approaching the singularity point.

Lemma 4.2.1. *Let $\mu \leq 1/4$. For any $h \in L^2(0,T;L^2(0,1))$, let ϕ be the corresponding solution of the adjoint problem* (4.2.5)*. Then, there exists a positive constant B, not depending on h, such that*

$$\int_0^T \left|\left[x^\lambda \phi_x(x,t)\right]\Big|_{x=0}\right|^2 dt \leq B\|h\|^2_{L^2(0,T;L^2(0,1))}, \tag{4.2.7}$$

where λ is the constant introduced in (4.1.6)*.*

Moreover, if $h \neq 0$, then there exists a function $g \in L^2_{loc}(0,T)$, still not identically zero, such that it holds

$$\left[x^\lambda \phi_x(x,t)\right]\Big|_{x=0} = g(t). \tag{4.2.8}$$

Proof. First of all we notice that, reversing the time in the adjoint equation (4.2.5), we obtain an equation of the type of (4.2.1); in more details, applying the change of variables $t \mapsto T - t$ in (4.2.5), we get

$$\begin{cases} \phi_t - \phi_{xx} - \dfrac{\mu}{x^2}\phi = h, & (x,t) \in Q \\ \phi(0,t) = \phi(1,t) = 0, & t \in (0,T) \\ \phi(x,0) = 0, & x \in (0,1). \end{cases} \tag{4.2.9}$$

Therefore, we are going to proof the Lemma for the solution of (4.2.9), instead of for the one of (4.2.5).

The solution of (4.2.9) can be expressed in terms of the eigenfunctions of the operator $-d_{xx} - \mu/x^2$ with Dirichlet boundary conditions, that is

$$\phi(x,t) = \sum_{k\geq 1} \phi_k(t)\varrho_k(x) \tag{4.2.10}$$

where, for any $k \geq 1$, $\varrho_k(x)$ is the unique solution of the second order ODE

$$\begin{cases} -\varrho_k''(x) - \dfrac{\mu}{x^2}\varrho_k(x) = \lambda_k\varrho_k(x), & x \in (0,1) \\ \varrho_k(0) = \varrho_k(1) = 0. \end{cases} \tag{4.2.11}$$

We notice that (4.2.11) is a Bessel equation, therefore its solution can be computed explicitly; in particular, we have

$$\varrho_k(x) = x^{\frac{1}{2}} J_\nu(j_{\nu,k}x), \quad \lambda_k = j_{\nu,k}^2, \quad \nu := \frac{1}{2}\sqrt{1-4\mu},$$

where J_ν is the Bessel function of first kind of order ν and $j_{\nu,k}$ are the zeros of J_ν.

Moreover, using classical properties of the Bessel's functions ([93, Chapter 5, Section 3]), we can easily show that there exists a constant $C(\nu) > 0$, depending only on ν, such that

$$x^\lambda \phi_x(x,t)\Big|_{x=0} = C(\nu)\sum_{k\geq 1}\phi_k(t) j_{\nu,k}^{\nu} e^{-j_{\nu,k}^2 t}.$$

Let us now calculate the coefficients $\phi_k(t)$; plugging (4.2.10) in (4.2.9), for any $k \geq 0$ we obtain the following equation

$$\begin{cases} \phi_{k,t} + \lambda_k\phi_k = h_k, & t \in (0,T) \\ \phi_k(0) = 0, \end{cases} \tag{4.2.12}$$

where $h_k = h_k(t) = \langle h, \varrho_k\rangle_{L^2(0,1)}$ is the Fourier coefficient of h corresponding to the eigenfunction ϱ_k.

Also the solution of (4.2.12) can be computed explicitly, using the variation of constants formula, and it takes the form

$$\phi_k(t) = \phi_k(0)e^{-\lambda_k t} + \int_0^t h_k(s)e^{-\lambda_k(t-s)}\,ds = e^{-\lambda_k t}\int_0^t h_k(s)e^{\lambda_k s}\,ds.$$

By means of this expression, we finally have

$$x^\lambda \phi_x(x,t)\Big|_{x=0} = C(\nu)\sum_{k\geq 1}\int_0^t h_k(s)e^{-j_{\nu,k}^2(2t-s)} j_{\nu,k}^{\,\nu}\,ds \leq C(\nu)\sum_{k\geq 1} e^{-j_{\nu,k}^2 t} j_{\nu,k}^{\,\nu}\int_0^t h_k(s)\,ds. \tag{4.2.13}$$

First of all we have that, due to the presence of the exponential factor with negative argument, for all $t > 0$ the sum

$$\sum_{k\geq 1} j_{\nu,k}^{\,\nu} e^{-j_{\nu,k}^2 t}$$

is convergent applying classical summation criteria (see, for instance, [89, Theorem 1.5]); on the other hand, for $t = 0$ this sum becomes

$$\sum_{k\geq 1} j_{\nu,k}^{\,\nu},$$

which is clearly divergent. Therefore, we can conclude that

$$x^\lambda \phi_x(x,t)\Big|_{x=0} = C(\nu)\sum_{k\geq 1} j_{\nu,k}^{\,\nu} e^{-j_{\nu,k}^2 t} = g \in L^2_{\text{loc}}(0,T). \tag{4.2.14}$$

Finally, using the expression (4.2.13), the Cauchy-Schwarz inequality and the Bessel inequality

$$\sum_{k\geq 1} |h_k|^2 \leq \|h\|^2_{L^2(0,1)}$$

it is now straightforward to check that

$$\begin{aligned}&\int_0^T \left| \left[x^{\,\lambda}\phi_x(x,t)\right]\Big|_{x=0}\right|^2 dt \\ &\qquad = C(\nu)^2\int_0^T\left(\sum_{k\geq 1}\int_0^t h_k(s)e^{-j_{\nu,k}^2(2t-s)} j_{\nu,k}^{\,\nu}\,ds\right)^2 dt \leq B\|h\|^2_{L^2(0,T;L^2(0,1))}.\end{aligned}$$

□

Proof of Theorem 4.2.2. Let $h \in L^2(0,T;L^2(0,1))$. Then, applying Theorem 4.2.1, there exists a unique solution $\phi \in L^2(0,T;\mathcal{D}(\mathcal{A})) \cap H^1(0,T;L^2(0,1))$ of (4.2.5); moreover,

$$\|\phi\|_{L^2(0,T;\mathcal{D}(\mathcal{A}))} + \|\phi\|_{H^1(0,T;L^2(0,1))} \leq C\|h\|_{L^2(0,T;L^2(0,1))}. \tag{4.2.15}$$

Therefore, thanks also to Lemma 4.2.1 the transposition identity (4.2.4) makes sense for all $f \in L^2(0,T)$ and it uniquely determines $u \in L^2(0,T;L^2(0,1))$ satisfying (4.2.6). □

Remark 4.2.1. *We point out that the results presented in this Section are valid for all* $\mu \leq 1/4$*. Indeed, for obtaining them we are only employing the classical Hardy inequality and the spectral decomposition of the operator involved in our equation, and this can be done for all the values of* μ *below the critical Hardy constant.*

Therefore, we stress the fact that the further limitation $0 < \mu < 1/4$ *is not required at the level of the well-posedness and regularity analysis. As we will justify in details in Section 4.4, this condition will therefore appear when dealing with the problem of boundary controllability, and it is strictly related with the change of variables that we will employ.*

4.3 Existing results for parabolic equations with degenerate coefficients

As we mentioned in the introduction, our approach for obtaining the boundary controllability of (4.1.2) will rely on an analogous result that has been recently proved for a one-dimensional parabolic equation with degenerate coefficients.

In particular, we will apply the results of [76], where the author has analysed the one-dimensional heat equation

$$u_t - (x^\alpha u_x)_x = 0, \quad \alpha \in (0,1), \tag{4.3.1}$$

obtaining the null controllability from $x = 0$ by means of a L^2 control.

Before going into more details, let us introduce the particular functional setting in which it is developed the analysis of [76]; in what follows, we will always assume $\alpha \in [0,1)$. First of all, let us define the space

$$H^1_\alpha(0,1) := \left\{ f \in L^2(0,1) \,\Big|\, x^{\alpha/2} f' \in L^2(0,1) \right\} \tag{4.3.2}$$

Note that $H^1_\alpha(0,1)$ is a Hilber space for the scalar product

$$(f,g)_{H^1_\alpha} := \int_0^1 \left(fg + x^\alpha f'g' \right) dx, \quad \text{for all} \quad f,g \in H^1_\alpha(0,1). \tag{4.3.3}$$

Besides, $H^1_\alpha(0,1)$ is continuously embedded in $C([0,1])$ (see, for instance, [30]), which means that the functions in this space have a trace both at $x = 0$ and at $x = 1$. Thus, we can define

$$H^1_{\alpha,0}(0,1) := \left\{ f \in H^1_\alpha(0,1) \,\Big|\, f(0) = f(1) = 0 \right\}. \tag{4.3.4}$$

Moreover, again in [30] it is presented the following Hardy-Poincaré inequality, that plays a similar role as the classical Poincaré inequality for standard Sobolev spaces:

$$\forall\, f \in H^1_{\alpha,0}(0,1), \quad \int_0^1 f^2\, dx \leq C_\alpha \int_0^1 \left(x^{\alpha/2} f' \right)^2 dx; \tag{4.3.5}$$

therefore, we have that

$$\|f\|_{H^1_{\alpha,0}} := \left[\int_0^1 \left(x^{\alpha/2} f'\right)^2 dx\right]^{\frac{1}{2}} \tag{4.3.6}$$

defines a norm on $H^1_{\alpha,0}(0,1)$ which is equivalent to the one induced by (4.3.3).

Let $H^{-1}_\alpha(0,1)$ be the dual space of $H^1_{\alpha,0}(0,1)$ with respect to the pivot space $L^2(0,1)$, endowed with the natural norm

$$\|f\|_{H^{-1}_\alpha} := \sup_{\|g\|_{H^1_{\alpha,0}}=1} \langle f, g\rangle_{H^{-1}_\alpha, H^1_{\alpha,0}}. \tag{4.3.7}$$

We introduce now the unbounded operator $A : \mathcal{D}(A) \subset L^2(0,1) \to L^2(0,1)$ defined by

$$\begin{cases} \mathcal{D}(A) := \left\{ u \in H^1_{\alpha,0}(0,1) \middle|\, x^\alpha u_x \in H^1(0,1)\right\}, \\ \forall u \in \mathcal{D}(A),\ Au := -(x^\alpha u_x)_x \end{cases} \tag{4.3.8}$$

It is not difficult to see that A is a self-adjoint, positive operator, with compact resolvent. Thus, there exists a Hilbertian basis $(\Phi_n)_{n\in\mathbb{N}^*}$ of $L^2(0,1)$ and a sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ of real, positive numbers, with $\lambda_n \to \infty$, such that

$$A\Phi_n = \lambda_n \Phi_n, \quad \text{for all} \quad n \in \mathbb{N}^*.$$

This enables us to introduce the following weighted space

$$\mathbf{H}^s_\alpha(0,1) := \mathcal{D}\left(A^{\frac{s}{2}}\right) = \left\{ u = \sum_{n\in\mathbb{N}^*} a_n\Phi_n \,\middle|\, \|u\|^2_s := \sum_{n\in\mathbb{N}^*} |a_n|^2 \lambda_n^s < \infty \right\}; \tag{4.3.9}$$

notice that

$$\mathbf{H}^2_\alpha(0,1) = \mathcal{D}(A), \ \ \mathbf{H}^1_\alpha(0,1) = H^1_{\alpha,0}(0,1) \quad \text{and} \quad \mathbf{H}^{-1}_\alpha(0,1) = H^{-1}_\alpha(0,1).$$

This weighted spaces just defined are the ones in which it is possible to prove boundary controllability for the degenerate parabolic equation (4.3.1); in particular, one of the main results of [76] is the following.

Theorem 4.3.1 (Theorem 3.4 of [76]). *Let $0 \le \alpha < 1$, $\beta = (1-\alpha)/(2-\alpha)$, $w_0 \in \mathbf{H}_\alpha^{\frac{1}{2}(1-2\beta)}$ and $T > 0$. Then, there exists a control $\varrho \in L^2(0,T)$ such that the corresponding solution of*

$$\begin{cases} w_t - (x^\alpha w_x)_x = 0, & (x,t) \in Q \\ w(0,t) = \varrho(t),\ w(1,t) = 0, & t \in (0,T) \\ w(x,0) = w_0(x), & x \in (0,1) \end{cases} \tag{4.3.10}$$

satisfies $u(x,T) \equiv 0$. Moreover, there exists a constant C (independent of w_0) such that

$$\|\varrho\|_{L^2(0,T)} \le C\|w_0\|_{\mathbf{H}_\alpha^{\frac{1}{2}(1-2\beta)}}. \tag{4.3.11}$$

Theorem 4.3.1, in turn, is a consequence of the following observability result for the adjoint system associated to (4.3.10)

Theorem 4.3.2 (Theorem 3.3 of [76])**.** *Let* $0 \le \alpha < 1$, $\beta = (1-\alpha)/(2-\alpha)$ *and* $T > 0$*. For all* $z_T \in \mathbf{H}_\alpha^{\frac{1}{2}(2\beta-1)}$*, let* z *be the solution of the adjoint equation*

$$\begin{cases} z_t + (x^\alpha z_x)_x = 0, & (x,t) \in Q \\ z(0,t) = z(1,t) = 0, & t \in (0,T) \\ z(x,T) = z_T(x), & x \in (0,1). \end{cases} \tag{4.3.12}$$

Then, there exist two constants C_0 *and* C_1*, independent of* z_T *and* T*, such that the solution of (4.3.12) satisfies*

$$\|z(x,0)\|^2_{\mathbf{H}_\alpha^{\frac{1}{2}(2\beta-1)}} \le \frac{C_0}{T^2} \exp\left(\frac{C_1}{T}\right) \int_0^T \left[x^{2\alpha} z_x^2(x,t)\right]\Big|_{x=0} dt. \tag{4.3.13}$$

We are going to show that, through an appropriate change of variables, it is possible to reduce our equation (4.1.2) with singular potential in the form of a degenerate problem and that from (4.3.13) we can prove the observability for the adjoint system (4.1.9); as a consequence of that, we will have our controllability result.

4.4 Boundary controllability

Now that we have defined in details the functional setting in which we will work, we can present the main result of this Chapter.

Theorem 4.4.1. *Let* $0 < \mu < 1/4$, $T > 0$ *and* $u_0 \in \mathbf{H}_\alpha^\lambda$*, with* λ *and* α *as in* (4.1.6) *and* (4.1.12)*, respectively. Then, there exists a control function* $f \in L^2(0,T)$ *such that the solution of* (4.1.2) *satisfies* (4.1.3)*.*

Applying HUM, Theorem 4.4.1 will be a consequence of the following observability inequality for the solution of the adjoint system (4.1.9).

Theorem 4.4.2. *Let* $0 < \mu < 1/4$, $T > 0$ *and* $v_T \in \mathbf{H}_\alpha^{-\lambda}$*, with* λ *and* α *as in* (4.1.6) *and* (4.1.12)*, respectively. Then, there exist two constants* C_0 *and* C_1*, independent of* v_T *and* T*, such that, for all solution* v *of* (4.1.9) *it holds*

$$\|v(x,0)\|^2_{\mathbf{H}_\alpha^{-\lambda}} \le \frac{C_0}{T^2} \exp\left(\frac{C_1}{T}\right) \int_0^T \left[x^{2\lambda} v_x^2(x,t)\right]\Big|_{x=0} dt. \tag{4.4.1}$$

Proof. We are going to obtain (4.4.1) as a consequence of the results presented in [76] for equations with variable degenerate coefficients. In particular, we will mostly rely on Theorem 4.3.2. At this purpose, let us introduce the following change of variables

$$v(x,t) := x^{\frac{\alpha}{2(2-\alpha)}}\,\psi(x,t), \quad x(\xi) := \left(\frac{2}{2-\alpha}\right)\xi^{\frac{2-\alpha}{2}};$$

with

$$\alpha = \frac{2+8\mu-2\sqrt{1-4\mu}}{3+4\mu}. \tag{4.4.2}$$

Then, (4.1.9) is transformed in the following equation with variable degenerate coefficients

$$\begin{cases} \psi_t + (\xi^\alpha \psi_\xi)_\xi = 0, & (x,t)\in Q \\ \psi(0,t) = \psi(\xi_0,t) = 0, & t\in(0,T) \\ \psi(\xi,T) = \psi_T(\xi), & x\in(0,\xi_0), \end{cases} \tag{4.4.3}$$

where

$$\xi_0 := \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}.$$

We remind that Theorem 4.3.2 holds for values of the parameter α satisfying $0<\alpha<1$. By means of (4.4.2), this give us the condition $0<\mu<1/4$.

Therefore, for values of the parameter μ in this interval, we can apply Theorem 4.3.2, obtaining the following inequality

$$\|\psi(\xi,0)\|^2_{\mathbf{H}_\alpha^{\frac{1}{2}(2\beta-1)}} \leq \frac{C_0}{T^2}\exp\left(\frac{C_1}{T}\right)\int_0^T \left[\xi^{2\alpha}\psi_\xi^2(\xi,t)\right]\Big|_{\xi=0} dt, \tag{4.4.4}$$

where, we remind, $\beta = (1-\alpha)/(2-\alpha)$.

Now, applying the inverse change of variables, it is simply a matter of computations to show that

$$\lim_{\xi\to 0^+}\xi^\alpha\phi_\xi = A(\mu)\lim_{x\to 0^+} x^\lambda v_x,$$

where

$$A(\mu) := \frac{1}{2}\left(1-\sqrt{1-4\mu}\right) - \frac{1+4\mu-\sqrt{1-4\mu}}{4+4\mu} > 0,$$

and (4.4.4) becomes

$$\left\|x^{-\lambda}v(x,0)\right\|^2_{\mathbf{H}_\alpha^{-\lambda}} \leq \frac{A(\mu)C_0}{T^2}\exp\left(\frac{C_1}{T}\right)\int_0^T\left[x^{2\lambda}v_x^2(x,t)\right]\Big|_{x=0} dt. \tag{4.4.5}$$

Finally, it is straightforward that it holds

$$\left\|x^{-\lambda}v(x,0)\right\|^2_{\mathbf{H}_\alpha^{-\lambda}} \gtrsim \|v(x,0)\|_{\mathbf{H}_\alpha^{-\lambda}}$$

and, from (4.4.5) we finally recover (4.4.1). □

Proof of Theorem 4.4.1. Once the observability inequality (4.4.1) is known to hold, we can immediately obtain the controllability of our original equation through a $L^2(0,T)$ control f. To do that it is sufficient to minimize the functional

$$J(v_T) := \frac{1}{2}\int_0^T \left[x^{2\lambda}v_x^2(x,t)\right]\Big|_{x=0} dt + \langle v(\cdot,0), u_0\rangle_{\mathbf{H}_\alpha^{-\lambda},\mathbf{H}_\alpha^{\lambda}} \tag{4.4.6}$$

over the Hilbert space

$$H := \left\{ v_T \;\middle|\; \text{the solution } v \text{ of (4.1.9) satisfies } \int_0^T \left[x^{2\lambda}v_x^2(x,t)\right]\Big|_{x=0} dt \le +\infty \right\}. \tag{4.4.7}$$

To be more precise, H is the completion of $L^2(0,1)$ with respect to the norm

$$\left(\int_0^T \left[x^{2\lambda}v_x^2(x,t)\right]\Big|_{x=0} dt\right)^{1/2}.$$

Now, observe that J is convex and, according to (4.4.1), it is also continuous in H; on the other hand, again (4.4.1) gives us also the coercivity of J. Therefore, there exists $v^* \in H$ minimizing J. The corresponding Euler-Lagrange equation is

$$\forall v \in H, \quad \int_0^T \left[x^{\lambda}v_x(x,t)\right]\Big|_{x=0} F(t)\,dt + \langle v(\cdot,0), u_0\rangle_{\mathbf{H}_\alpha^{-\lambda},\mathbf{H}_\alpha^{\lambda}} = 0 \tag{4.4.8}$$

where

$$F(t) := \left[x^{\lambda}v_x^*(x,t)\right]\Big|_{x=0}.$$

This F will be our control function; we observe that, by definition $F \in L^2(0,T)$. Now, considering equation (4.1.2) with $f = F$, multiplying it by v and integrating by parts, we get

$$\langle v_T, u(\cdot,T)\rangle_{\mathbf{H}_\alpha^{-\lambda},\mathbf{H}_\alpha^{\lambda}} = \int_0^T \left[x^{\lambda}v_x(x,t)\right]\Big|_{x=0} F(t)\,dt + \langle v(\cdot,0), u_0\rangle_{\mathbf{H}_\alpha^{-\lambda},\mathbf{H}_\alpha^{\lambda}}$$

for any $v_T \in \mathbf{H}_\alpha^{-\lambda}$. Hence, using (4.4.8) we immediately conclude $u(x,T) = 0$. □

Remark 4.4.1. *We conclude this Chapter pointing out that our main result, Theorem 4.4.1, is only partial, in the sense that it is not valid for all the values of the parameter μ for which equation (4.1.2) is well posed. This fact is due to the technique that we used in the proof of the observability inequality, that required us to impose the restriction $0 < \mu < 1/4$. On the other hand, we do not exclude that it is possible to obtain the null controllability of the equation also for negative or critical values of μ.*

A good approach to this problem would certainly be the proof of an appropriate Carleman estimate for the solution of the adjoint equation, which is one of the most classical techniques in control theory for parabolic equations. However, as we are going to present with more details in the last Section of this thesis, to obtain such an inequality is a very tricky issue, this being mostly related with the fact that we aim to control from the singularity point and with the non-standard behaviour of the normal derivative of the solution of our equation when approaching $x = 0$.

Chapter 5

Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary

Abstract.

This Chapter is devoted to the analysis of control properties for a heat equation with a singular potential μ/δ^2, defined on a bounded C^2 domain $\Omega \subset \mathbb{R}^N$, where δ is the distance to the boundary function. More precisely, we show that for any $\mu \leq 1/4$ the system is exactly null controllable using a distributed control located in any open subset of Ω, while for $\mu > 1/4$ there is no way of preventing the solutions of the equation from blowing-up. The main tool that we employ is a new Carleman estimate, which is able to deal with the specificity of the singularity that we are considering. The results obtained in this Chapter are presented in the research article [12], in collaboration with E. Zuazua.

5.1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded and C^2 domain such that $0 \in \Omega$ and with boundary $\Gamma := \partial\Omega$. For any $T > 0$, set $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$. Moreover, let $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ be the distance to the boundary function. We are interested in proving the exact null controllability for a heat equation with singular inverse-square potential of the type $-\mu/\delta^2$, that is, given the generalised Schrödinger operator

$$\mathcal{A} = \mathcal{A}(\mu) := -\Delta - \frac{\mu}{\delta^2}\mathcal{I}, \quad \mu \in \mathbb{R}, \tag{5.1.1}$$

we are going to consider the following parabolic equation

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2} u = f, & (x,t) \in Q, \\ u = 0, & (x,t) \in \Sigma, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.1.2}$$

with the intent of proving that it is possible to choose the control function f in an appropriate functional space X such that the corresponding solution of (5.1.2) satisfies

$$u(x,T) = 0, \quad \text{for all } x \in \Omega. \tag{5.1.3}$$

In particular, the main result of this paper will be the following.

Theorem 5.1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and C^2 domain and assume $\mu \leq 1/4$. Given any non-empty open set $\omega \subset \Omega$, for any time $T > 0$ and any initial datum $u_0 \in L^2(\Omega)$, there exists a control function $f \in L^2(\omega \times (0,T))$ such that the solution of* (5.1.2) *satisfies* (5.1.3).

The upper bound for the coefficient μ, which is related to a Hardy-Poincaré inequality involving the potential μ/δ^2 presented in [21], plays a fundamental role in our analysis. Indeed, in [25] is shown that, for $\mu > 1/4$, (5.1.2) admits no positive weak solution for any u_0 positive and $f = 0$. Moreover, there is instantaneous and complete blow-up of approximate solutions.

As it is by now classical, for proving Theorem 5.1.1 we will apply the Hilbert Uniqueness Method (HUM, [97]); hence the controllability property will be equivalent to the observability of the adjoint system associated to (5.1.2), namely

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{\delta^2} v = 0, & (x,t) \in Q, \\ v = 0, & (x,t) \in \Sigma, \\ v(x,T) = v_T(x), & x \in \Omega. \end{cases} \tag{5.1.4}$$

In more details, for any $\mu \leq 1/4$ we are going to prove that there exists a positive constant C_T such that, for all $v_T \in L^2(\Omega)$, the solution of (5.1.4) satisfies

$$\int_\Omega v(x,0)^2 \, dx \leq C_T \int_{\omega \times (0,T)} v(x,t)^2 \, dxdt. \tag{5.1.5}$$

The inequality above, in turn, will be obtained as a consequence of a Carleman estimate for the solution of (5.1.4), which is derived taking inspiration from the works [35, 53].

Finally, adapting an argument developed in [53] we will also show that the bound $\mu \leq 1/4$ is sharp for controllability, meaning that this result cannot be achieved for $\mu > 1/4$.

As we extensively debated in Chapter 2, singular inverse-square potentials arise in several areas of pure and applied mathematics, being this one of the main reasons justifying the growing interest of the recent years for this class of PDEs.

Regarding controllability problems for evolution equations involving singular inverse-square potentials, among other works it is worth to mention the ones by S. Ervedoza ([53]), J. Vancostenoble and E. Zuazua ([137, 138]) and C. Cazacu ([34, 35]).

Both in [34] and in [138] it is analysed the case of a wave and a Schrödinger equation with potential $\mu/|x|^2$ and it is proved exact boundary controllability for sub-critical and critical values of the coefficient μ.

Regarding instead heat-type equations, in [137] the null controllability is obtained choosing a control region containing an annular set around the singularity and using appropriate cut-off functions in order to split the problem in two:

- in a region of the domain away from the singularity, in which it is possible to employ classical Carleman estimates;
- in the remaining part of the domain, a ball centred in the singularity, in which the authors can apply polar coordinates and reduce themselves to a one-dimensional equation, which is easier to handle.

This result was then generalised in [53], where the author was able to remove any geometrical constraint on the control region and proved exact controllability from any open subset of Ω that does not contains the singularity.

Finally, in [35] is treated the case of a potential with singularity located on the boundary of the domain and is proved again null controllability with an internal control. Moreover, the author shows that the presence of the singularity on the boundary of the domain allows to slightly enlarge the critical value for the constant μ, up to $\mu^* := N^2/4$.

In our work we consider the more general case of a heat equation with a potential whose singularity is distributed all over the boundary of the domain. To the best of our knowledge, this is a problem that has never been treated in precedence, although it is a natural extension of the results achieved in the articles presented above.

This Chapter is organized as follows: in Section 5.2 we present a generalisation of the classical Hardy-Poincaré inequality, introduced by H. Brezis and M. Marcus in [21], which will then be applied for obtaining well-posedness of the equation that we consider; we also give some extensions of this inequality, needed for obtaining the Carleman estimate. In Section 5.3 we present the Carleman estimate, showing what are the main differences between our result and the ones obtained in previous papers. In Section 5.4 we derive the observability inequality

(5.1.5) and we apply it in the proof of Theorem 5.1.1. In Section 5.5 we prove that the bound 1/4 for the Hardy constant μ is sharp for control, showing the impossibility of preventing the solutions of the equation from blowing-up in the case of supercritical potentials. The Carleman estimates is proved in Section 5.6. Finally, Section 5.7 is dedicated to the proof of the Hardy-Poincaré inequalities of Section 5.2 and of other technical Lemmas.

5.2 Hardy-Poincaré inequalities and well-posedness

When dealing with equations involving singular inverse-square potentials, it is by now classical that of great importance is an Hardy-type inequality. This kind of inequalities has been proved to hold also in the more general case of the potential μ/δ^2 (see, for instance [21, 102]); in particular, we have

Proposition 5.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain; then, for any $u \in H_0^1(\Omega)$, and for any $\mu \leq 1/4$, it holds*

$$\mu \int_\Omega \frac{u^2}{\delta^2}\,dx \leq \int_\Omega |\nabla u|^2\,dx. \tag{5.2.1}$$

Inequality (5.2.1) will be applied for obtaining the well-posedness of (5.1.2), as well as the observability inequality (5.1.5). For obtaining the Carleman estimate, instead, we are going to need the following Propositions.

Proposition 5.2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. For any $\mu \leq 1/4$ and any $\gamma \in (0,2)$ there exist two positive constants A_1 and A_2, depending on γ and Ω, such that for all $u \in H_0^1(\Omega)$ the following inequality holds*

$$A_1 \int_\Omega \frac{u^2}{\delta^\gamma}\,dx + \mu \int_\Omega \frac{u^2}{\delta^2}\,dx \leq \int_\Omega |\nabla u|^2\,dx + A_2 \int_\Omega u^2\,dx. \tag{5.2.2}$$

Proposition 5.2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. For any $\mu \leq 1/4$ and any $\gamma \in (0,2)$ there exists a positive constant A_3 depending on γ, μ and Ω, such that for all $u \in H_0^1(\Omega)$ the following inequality holds*

$$\int_\Omega \delta^{2-\gamma}|\nabla u|^2\,dx \leq R_\Omega^{2-\gamma} \int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{\delta^2}\right)dx + A_3 \int_\Omega u^2\,dx. \tag{5.2.3}$$

Proposition 5.2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. For any $\mu \leq 1/4$ and any $\gamma \in (0,2)$ there exist two positive constants A_4 and A_5 depending on γ, μ and Ω, such that for all $u \in H_0^1(\Omega)$ the following inequality holds*

$$\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{\delta^2}\right)dx + A_4 \int_\Omega u^2\,dx \geq A_5 \int_\Omega \left(\delta^{2-\gamma}|\nabla u|^2 + A_1\frac{u^2}{\delta^\gamma}\right)dx, \tag{5.2.4}$$

where A_1 is the positive constant introduced in Proposition 5.2.2.

The proofs of Propositions 5.2.2, 5.2.3 and 5.2.4 will be presented in Section 5.7.

We conclude this Section analysing existence and uniqueness of solutions of the heat equation (5.1.2), applying classical semi-group theory. At this purpose, for any fixed $\gamma \in [0,2)$ let us define the set

$$\mathcal{L}^\gamma := \left\{ A>0 \,\middle|\, \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega \left(|\nabla u|^2 - \mu^* u^2/\delta^2 + Au^2\right)\,dx}{A_1\int_\Omega u^2/\delta^\gamma\,dx} \geq 1 \right\}. \tag{5.2.5}$$

We remind here that μ^* is the critical Hardy constant and that in our case we have $\mu^* = 1/4$. Moreover, the set (5.2.5) is clearly non empty since it contains the constant A_2 in the inequality (5.2.2). Now, we define

$$A_0^\gamma := \inf_{A\in\mathcal{L}^\gamma} A \tag{5.2.6}$$

and, for any $\mu \leq \mu^*$, we introduce the functional

$$\Phi_\mu^\gamma(u) := \int_\Omega |\nabla u|^2\,dx - \mu\int_\Omega \frac{u^2}{\delta^2}\,dx + A_0^\gamma\int_\Omega u^2\,dx;$$

we remark that this functional is positive for any test function, due to (5.2.2) and to the particular choice of the constant A_0^γ.

Next, let us define the Hilbert space H_μ^γ as the closure of $C_0^\infty(\Omega)$ with respect to the norm induced by Φ_μ^γ; if $\mu \leq \mu^*$ we obtain

$$\left(1-\frac{\mu^+}{\mu^*}\right)\int_\Omega \left(|\nabla u|^2 + A_0^\gamma u^2\right)\,dx + \frac{\mu^+}{\mu^*}\int_\Omega \frac{u^2}{\delta^\gamma}\,dx \leq \|u\|_H^2 \leq \left(1+\frac{\mu^-}{\mu^*}\right)\int_\Omega \left(|\nabla u|^2 + A_0^\gamma u^2\right)\,dx, \tag{5.2.7}$$

where $\mu^+ := \max\{0,\mu\}$ and $\mu^- := \max\{0,-\mu\}$.

From the norm equivalence (5.2.7), in the sub-critical case $\mu < \mu^*$ it follows the identification $H_\mu^\gamma = H_0^1(\Omega)$; in the critical case $\mu = \mu^*$, instead, this identification does not hold anymore and the space H_μ^γ is slightly larger than $H_0^1(\Omega)$. For more details on the characterisation of these kind of spaces, we refer to [141].

Let us now consider the unbounded operator $\mathcal{B}_\mu^\gamma : \mathcal{D}(\mathcal{B}_\mu^\gamma) \subset L^2(\Omega) \to L^2(\Omega)$ defined as

$$\begin{aligned} \mathcal{D}(\mathcal{B}_\mu^\gamma) &:= \left\{ u\in H_\mu^\gamma \,\middle|\, -\Delta u - \frac{\mu}{\delta^2}u + A_0^\gamma u \in L^2(\Omega)\right\}, \\ \mathcal{B}_\mu^\gamma u &:= -\Delta u - \frac{\mu}{\delta^2}u + A_0^\gamma u, \end{aligned} \tag{5.2.8}$$

whose norm is given by

$$\|u\|_{\mathcal{B}_\mu^\gamma} = \|u\|_{L^2(\Omega)} + \left\|\mathcal{B}_\mu^\gamma u\right\|_{L^2(\Omega)}.$$

With the definitions we just gave, by standard semi-group theory we have that for any $\mu \leq \mu^*$ the operator $(\mathcal{B}_\mu^\gamma, \mathcal{D}(\mathcal{B}_\mu^\gamma))$ generates an analytic semi-group in the pivot space $L^2(\Omega)$ for the equation (5.1.2). For more details we refer to the Hille-Yosida theory, presented in [19, Chapter 7], which can be adapted in the context of the space H_μ^γ introduced above.

Therefore, from the construction we just presented we immediately have the following well-posedness result

Theorem 5.2.1. *Given $u_0 \in L^2(\Omega)$ and $f \in C([0,T]; L^2(\Omega))$, for any $\mu \leq 1/4$ the problem* (5.1.2) *admits a unique weak solution*

$$u \in \mathrm{C}^0([0,T]; L^2(\Omega)) \cap \mathrm{L}^2((0,T); H_\mu^\gamma).$$

5.3 Carleman estimate

The observability inequality (5.1.5) will be proved, as it is classical in controllability problems for parabolic equations, applying a Carleman estimate.

First of all, throughout the Chapter, for a given function f we apply the formal notations

$$\begin{aligned} &|f|_\infty := \|f\|_{L^\infty(\Omega)}, && |Df|_\infty := \|\nabla f\|_{L^\infty(\Omega)}, \\ &D^2 f(\xi,\xi) := \sum_{i,j=1}^N \partial^2_{x_i x_j} f \xi_i \xi_j, \quad \forall \xi \in \mathbb{R}^N, && |D^2 f|_\infty := \sum_{i,j=1}^N \left\|\partial^2_{x_i x_j} f\right\|_{L^\infty(\Omega)}. \end{aligned} \tag{5.3.1}$$

Moreover, for a given open set $\omega_0 \subset \omega$, whose closure is contained in ω, and for any $\beta_0 > 0$, we denote

$$\begin{aligned} &\Omega_{\beta_0} := \{x \in \Omega \,|\, \delta(x) < \beta_0\}, \quad \Sigma_{\beta_0} := \{x \in \Omega \,|\, \delta(x) = \beta_0\}, \\ &\mathcal{O} := \Omega \setminus \left(\overline{\omega_0} \cup \overline{\Omega_{\beta_0}}\right), \quad \tilde{\mathcal{O}} := \Omega \setminus \overline{\Omega_{\beta_0}}. \end{aligned} \tag{5.3.2}$$

As in Theorem 5.1.1, $\omega \subset \Omega$ is the non-empty open subset where the control is implemented. Finally, we introduce a smooth function $\psi_1 \in C^4(\overline{\Omega})$ satisfying the conditions

$$\begin{cases} \psi_1(x) = \delta(x), & x \in \Omega_{\beta_0}, \\ \psi_1(x) > \beta_0, & x \in \Omega \setminus \overline{\Omega_{\beta_0}}, \\ \psi_1(x) = \beta_0, & x \in \Sigma_{\beta_0}, \\ |\nabla \psi_1(x)| \geq \rho_0 > 0, & x \in \Omega \setminus \overline{\omega_0}, \end{cases} \tag{5.3.3}$$

for a given $\rho_0 > 0$.

We remark that such a function ψ_1 exists, but its construction is not trivial. See [35, Section 2.1.1] for more details.

Now, the main problem when designing a Carleman estimate is the choice of a proper weight function $\sigma(x,t)$. In general, σ has to be smooth, positive and has to blow up as t goes to 0 and T; in our case, this weight σ will be an adaptation of the one used in [35], that we conveniently modify in order to deal with the presence of the singularities distributed all over the boundary. In particular, the weight that we propose is the following

$$\sigma(x,t) = \theta(t)\left(C_\lambda - \delta^2\psi - \left(\frac{\delta}{r_0}\right)^\lambda \phi\right), \quad \phi = e^{\lambda\psi}, \tag{5.3.4}$$

where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^3. \tag{5.3.5}$$

Here, C_λ is a positive constant large enough as to ensure the positivity of σ, λ is a positive parameter aimed to be large, while r_0 is another positive parameter aimed to be small. Besides, ψ is a bounded regular function defined as

$$\psi = \rho(\psi_1 + 1), \tag{5.3.6}$$

where ρ is a positive constant such that $\rho > 2C_\Omega/\rho_0$. Referring to [35, Section 2], C_Ω is a positive constant for which it holds $|x \cdot \nu(x)| \leq C_\Omega |x|^2$ for any point $x \in \Gamma$, with $\nu(x)$ the outward unit normal vector at x; this estimate is valid due to the C^2 regularity of Ω. In particular, under the conditions (5.3.3), ψ satisfies the following useful properties

$$\begin{cases} \psi(x) = 1 & \forall x \in \Gamma, \\ \psi(x) > 1 & \forall x \in \Omega, \\ |\nabla\psi(x)| \geq 2C_\Omega & \forall x \in \Omega \setminus \overline{\omega_0}. \end{cases} \tag{5.3.7}$$

Due to technical computations, we fix ρ such that

$$\rho \geq \max\left\{1, \frac{1}{\rho_0^2}\left(1 + 2D_{\psi_1} + |D^2\psi_1|_\infty\right), \frac{2}{\rho_0^2}\left(1 + 2D_{\psi_1}\right), \frac{4D_{\psi_1}}{\rho_0^2}, \frac{24D_{\psi_1}R_\Omega}{\rho_0^2}, \frac{2}{\rho_0}\right\}, \tag{5.3.8}$$

where R_Ω is the diameter of the domain Ω, while D_{ψ_1} is a positive constant that will be introduced later, in Lemma 5.7.1. Finally, again for technical reasons, we will assume that r_0 satisfies

$$\begin{aligned} r_0 \leq \min\Bigg\{ & 1, \frac{\beta_0}{2}, \frac{2}{4|D\psi|_\infty + |D^2\psi|_\infty}, \frac{1}{R_\Omega\sqrt{4|D\psi|_\infty^2 + 2|D^2\psi|_\infty}}, \frac{1}{2(2-\gamma)|D\psi|_\infty}, \frac{3}{4|D\psi|_\infty}, \\ & \left(\frac{M_2}{4|\mu||D\psi|_\infty}\right)^{1/(\gamma-1)}, \frac{1}{\sqrt{8D_{\psi_1}|D\psi|_\infty/\rho_0 + 3|D^2\psi|_\infty}}, \frac{1}{|D\psi|_\infty^2 + 2|D\psi|_\infty}, \\ & \frac{2}{|D\psi|_\infty^2 + (1 + 2|\psi|_\infty)|D\psi|_\infty}, \frac{1}{|D\psi|_\infty\sqrt{D_3|\psi|_\infty^2 + D_4}}\Bigg\}, \end{aligned} \tag{5.3.9}$$

where γ is the parameter appearing in the Hardy inequalities presented in Section 5.2, with the particular choice $\gamma \in (1,2)$, while M_2, D_3 and D_4 are positive constant, not depending on r_0, that will be introduced in (5.7.40) and in Proposition 5.7.4, respectively.

Remark 5.3.1. *In the previous construction the set ω_0 is not allowed to be reduced to a single point. When doing that the weight function would develop a singularity and, on the other hand, the problem under consideration would then be that of pointwise controllability, i. e. the control would only be acting effectively in a single point.*

Pointwise control is a delicate topic. Even in the one-dimensional case (see, for instance, [50, 81]) the quality of the control results one may get by means of pointwise control depends on irrationality and diophantine properties of the point where the control is supported with respect to the extremes of the interval.

This is an evidence of the fact that the Carleman approach cannot be pushed to handle the pointwise control problem.

Motivation for the choice of σ

The weigh σ that we propose for our Carleman estimates is not the standard one; we had to modify it in order to deal with some critical terms that emerge in our computations due to the presence of the singular potential. We justify here our choice, highlighting the reasons why the weights presented in previous works ([35, 53, 68]) are not suitable for the problem that we consider.

In general, the weight used to obtain Carleman estimates for parabolic equations is assumed to be positive and to blow-up at the extrema of the time interval; besides, this weight has to be taken in separated variables. Therefore, we are looking for a functions $\sigma(x,t)$ satisfying

$$\begin{cases} \sigma(x,t) = \theta(t)p(x), & (x,t) \in Q, \quad (5.3.10a) \\ \sigma(x,t) > 0, & (x,t) \in Q, \quad (5.3.10b) \\ \lim\limits_{t\to 0^+} \sigma(x,t) = \lim\limits_{t\to T^-} \sigma(x,t) = +\infty, & x \in \Omega. \quad (5.3.10c) \end{cases}$$

The function θ is usually chosen in the form

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^k$$

for $k \geq 1$, and this choice in particular ensures the validity of (5.3.10c); in our case we assume $k = 3$ which, as we will remark later, is the minimum value for obtaining some important estimates that we need in the proof of the Carleman inequality.

While the choice of θ is standard, the main difficulty when building a proper σ is to identify a suitable $p(x)$ which is able to deal with the specificity of the equation that we are analysing.

In [68], A.V. Fursikov and O.Y. Imanuvilov obtained the controllability of the standard heat equation employing a positive weight in the form

$$\sigma_1 = \theta(t)\left(C_\lambda - e^{\lambda\psi}\right),$$

with a function $\psi \in C^2(\overline{\Omega})$ satisfying

$$\begin{cases} \psi(x) > 0, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \\ |\nabla\psi(x)| > 0, & x \in \overline{\Omega} \setminus \omega_0. \end{cases}$$

An example of a ψ with this behaviour is shown in Figure 5.1 below; in particular, we notice that this function is required to be always strictly monotone outside of the control region.

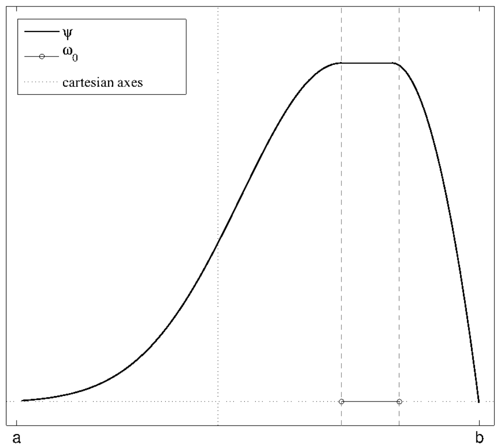

Figure 5.1: Function ψ of Fursikov and Imanuvilov in one space dimension on the interval (a, b)

This standard weight was later modified by S. Ervedoza in [53], for dealing with problems with interior quadratic singularities; in this case, the author applies the weight

$$\sigma_2 = \theta(t)\left(C_\lambda - \frac{1}{2}|x|^2 - e^{\lambda\psi(x)}\right),$$

with a function ψ such that (see Figure 5.2 below)

$$\begin{cases} \psi(x) = \ln(|x|), & x \in B(0,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \setminus \overline{B}(0,1), \\ |\nabla\psi(x)| \geq \gamma > 0, & x \in \overline{\Omega} \setminus \omega_0. \end{cases}$$

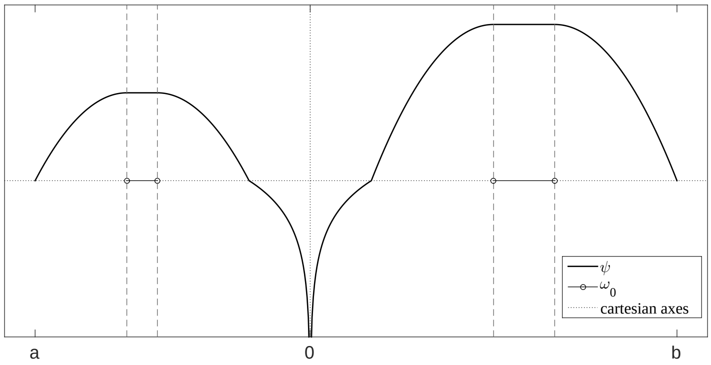

Figure 5.2: Function ψ of Ervedoza in one space dimension on the interval (a, b)

This choice of the weight is motivated by the observation that, near the singularity, when λ is large enough σ_2 behaves like

$$\sigma_2 \sim \theta(t)\left(C_\lambda - \frac{1}{2}|x|^2\right),$$

which is the weight employed by J. Vancostenoble and E. Zuazua in [138] for their proof of the controllability of the hat equation with a singular potential; on the other hand, away from the origin σ_2 maintains the behaviour of the classical weight σ_1. The modification near the origin is motivated by some critical terms which must be absorbed outside ω in the Carleman estimate (see [53, Equation 2.7]). In particular, in order to take advantage of the Hardy inequality, the author needed to get rid of singular terms in the form $(x \cdot \nabla\sigma)/|x|^4$, imposing the degeneracy $\nabla\sigma \sim x$ as $x \to 0$.

A further modification is proposed by C. Cazacu in [35], in the case of an equation with boundary singularity. In this case, indeed, the weight employed by Ervedoza is not suitable anymore since the move of the singularity up to the boundary produces a loss of regularity for σ_2 that, in particular, does not allow to absorb some boundary terms in a neighbourhood of the origin. Hence, the author proposes the weight

$$\sigma_3 = \theta(t)\left(C_\lambda - |x|^2\psi - \left(\frac{|x|}{r_0}\right)^\lambda e^{\lambda\psi}\right),$$

where the function ψ is chosen as in (5.3.6), with the fundamental property of being constant and non-zero on the boundary (see Figure 5.3 below).

Finally, when dealing as in our case with a singularity distributed all over the boundary the weights presented above do not allow anymore to manage properly the terms containing the singularities, since they now have a different nature. Therefore, we need to introduce further modifications in the weight that we want to employ, designing it in a way that could compensate this kind of degeneracies. At this purpose, it is sufficient to modify the weight σ_3 of

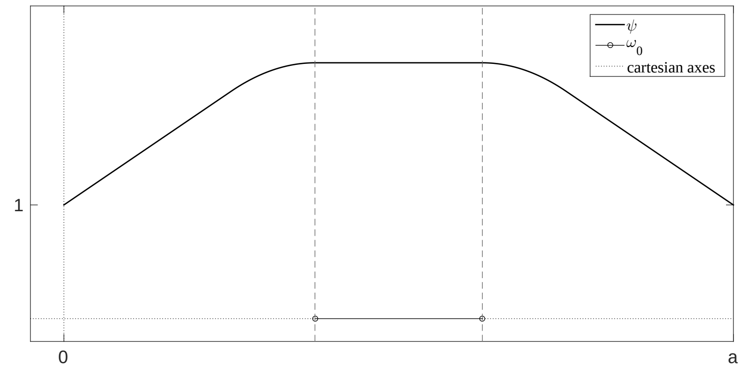

Figure 5.3: Function ψ of Cazacu in one space dimension on the interval $(0, a)$

Cazacu replacing the terms of the form $|x|$ with the distance function δ; being still in the case of boundary singularities the function ψ introduced in [35] (see (5.3.6) above) turns out to be a suitable one also in our case.

We now have all we need for introducing the Carleman estimate.

Theorem 5.3.1. *Let* σ *be the weight defined in* (5.3.4)*. There exist two positive constants* λ_0 *and* $\mathcal{M}$ *such that for any* $\lambda \geq \lambda_0$ *there exists* $R_0 = R_0(\lambda)$ *such that for any* $R \geq R_0$ *and for any solution* v *of* (5.1.4) *it holds*

$$\begin{aligned}
&R\int_Q \theta e^{-2R\sigma}\left(\delta^{2-\gamma}|\nabla v|^2 + A_1\frac{v^2}{\delta^\gamma}\right)dxdt + \lambda R \int\limits_{\Omega_{r_0}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^{\lambda-2} e^{-2R\sigma}|\nabla v|^2\,dxdt\\
&+\lambda^2 R \int\limits_{\mathcal{O}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^{\lambda}\phi e^{-2R\sigma}|\nabla v|^2\,dxdt + R^3\int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 e^{-2R\sigma}v^2\,dxdt\\
&+\lambda^4 R^3\int\limits_{\mathcal{O}\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3 e^{-2R\sigma}v^2\,dxdt\\
&\qquad\leq \mathcal{M}\int\limits_{\omega_0\times(0,T)}\left[\lambda^4R^3\theta^3\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3v^2 + \lambda^2 R\theta\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla v|^2\right]e^{-2R\sigma}\,dxdt
\end{aligned} \tag{5.3.11}$$

The proof of Theorem 5.3.1 is very technical and will be presented in Section 5.6. It relies on several technical Lemmas that we are going to prove in Section 5.7.

5.4 Proof of the observability inequality and of the controllability Theorem

We now apply the Carleman estimate that we just obtained for proving the observability inequality (5.1.5). This inequality will then be employed in the proof of our main result, the controllability Theorem 5.1.1.

Prooof of the observability inequality (5.1.5). Let us fix $\lambda \geq \lambda_0$ and $R \geq R_0(\lambda)$ such that (5.3.11) holds. These parameters now enter in the constant $\mathcal{M}$; in particular, we have

$$\int_Q \theta e^{-2R\sigma} \frac{v^2}{\delta^\gamma}\, dxdt \leq \mathcal{M}_1 \left(\int\limits_{\omega_0\times(0,T)} \theta^3\phi^3 e^{-2R\sigma} v^2\, dxdt + \int\limits_{\omega_0\times(0,T)} \theta\phi e^{-2R\sigma} |\nabla v|^2\, dxdt \right).$$

Now, it is straightforward to check that there exist four positive constants $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ such that

$$\begin{array}{ll} \theta e^{-2R\sigma}\frac{1}{\delta^\gamma} \geq \mathcal{P}_1 e^{-\mathcal{P}_2/t^3}, & (x,t) \in \Omega \times \left[\frac{T}{4}, \frac{3T}{4}\right], \\ \theta^3\phi^3 e^{-2R\sigma} \leq \mathcal{P}_3, & (x,t) \in \omega_0 \times (0,T), \\ \theta\phi e^{-2R\sigma} \leq \mathcal{P}_4 e^{-R\sigma}, & (x,t) \in \omega_0 \times (0,T). \end{array}$$

Thus the inequality above becomes

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega v^2\, dxdt \leq \mathcal{M}_1 \exp\left(\frac{\mathcal{P}_2}{T^3}\right) \left(\int\limits_{\omega_0\times(0,T)} v^2\, dxdt + \int\limits_{\omega_0\times(0,T)} e^{-R\sigma} |\nabla v|^2\, dxdt \right).$$

Moreover, multiplying equation (5.1.4) by v and integrating over Ω and applying (5.2.1) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\, dx = \int_\Omega |\nabla v|^2\, dx - \mu \int_\Omega \frac{v^2}{\delta^2}\, dx \geq 0$$

Hence, the function $t \mapsto \|v(\cdot,t)\|_{L^2(\Omega)}$ is increasing, that is

$$\int_\Omega v(x,0)^2\, dx \leq \int_\Omega v(x,t)^2\, dx,$$

and, integrating in time between $T/4$ and $3T/4$ we have

$$\frac{T}{2}\int_\Omega v(x,0)^2\, dx \leq \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega v(x,t)^2\, dx.$$

Thus, we obtain the inequality

$$\int_\Omega v(x,0)^2\, dxdt \leq \frac{2\mathcal{M}_1}{T} \exp\left(\frac{\mathcal{P}_2}{T^3}\right) \left(\int\limits_{\omega_0\times(0,T)} v^2\, dxdt + \int\limits_{\omega_0\times(0,T)} e^{-R\sigma} |\nabla v|^2\, dxdt \right).$$

Therefore to conclude the proof of (5.1.5), it is sufficient to apply the following Lemma.

Lemma 5.4.1 (Caccioppoli's inequality)**.** *Let $\bar{\sigma} : (0,T) \times \omega_0 \to \mathbb{R}^*_+$ be a smooth non-negative function such that*

$$\bar{\sigma}(x,t) \to +\infty, \quad \text{as } t \to 0^+ \quad \text{and as } t \to T^-,$$

and let $\mu \leq \mu^*$. Then, there exists a constant M independent of μ such that any solution v of (5.1.4) *satisfies*

$$\int\limits_{\omega_0 \times (0,T)} e^{-R\bar{\sigma}} |\nabla v|^2 \, dxdt \leq M \int\limits_{\omega \times (0,T)} v^2 \, dxdt. \tag{5.4.1}$$

Lemma 5.4.1 is a trivial adaptation of an analogous result, [137, Lemma 3.3], and its proof is left to the reader. It is now straightforward that, applying (5.4.1) for σ as in (5.3.4) we finally get

$$\int_\Omega v(x,0)^2 \, dxdt \leq \frac{\mathcal{B}_1}{T} \exp\left(\frac{\mathcal{B}_2}{T^3}\right) \int\limits_{\omega_0 \times (0,T)} v^2 \, dxdt,$$

that clearly implies (5.1.5), due to the definition of ω_0. □

Proof of Theorem (5.1.1)*.* Once the observability inequality (5.1.5) is known to hold, we can immediately obtain the controllability of our equation through a control $f \in L^2(\omega \times (0,T))$. To do that, we are going to introduce the functional

$$J(v_T) := \frac{1}{2} \int_{\omega \times (0,T)} v^2 \, dxdt + \int_\Omega v(x,0) u_0(x) \, dx, \tag{5.4.2}$$

defined over the Hilbert space

$$H := \left\{ v_T \in L^2(\Omega) \;\middle|\; \text{the solution } v \text{ of (5.1.4) satisfies } \int_{\omega \times (0,T)} v^2 \, dxdt \leq +\infty \right\}. \tag{5.4.3}$$

To be more precise, H is the completion of $L^2(\Omega)$ with respect to the norm

$$\left(\int_0^T \int_\omega v^2 \, dxdt \right)^{1/2}.$$

Observe that J is convex and, according to (5.1.5), it is also continuous on H; on the other hand, again (5.1.5) gives us also the coercivity of J. Therefore, there exists $v^* \in H$ minimizing J. The corresponding Euler-Lagrange equation is

$$\int_{\omega \times (0,T)} v(x,t) F(x,t) \, dxdt + \int_\Omega u_0(x) v(x,0) \, dx = 0, \tag{5.4.4}$$

where $F(x,t) := v^*(x,t)\chi_\omega$. F will be our control function; we observe that, by definition $F \in L^2(\omega \times (0,T))$. Now, considering equation (5.1.2) with $f = F$, multiplying it by v and integrating by parts, we get

$$\int_\Omega u(x,T) v_T(x) \, dx = \int_{\omega \times (0,T)} v(x,t) F(x,t) \, dxdt + \int_\Omega u_0(x) v(x,0) \, dx,$$

for any $v_T \in L^2(\Omega)$. Hence, from (5.4.4) we immediately conclude $u(x,T) = 0$. □

5.5 Non existence of a control in the supercritical case

As we mentioned before, in [25] is proved that in the super-critical case, i.e. for $\mu > 1/4$, the Cauchy problem for our singular heat equation is severely ill-posed. However, a priori this fact does not exclude that, given $u_0 \in L^2(\Omega)$, it is possible to find a control $f \in L^2((0,T); L^2(\Omega))$ localised in ω such that there exists a solution of (5.1.2). If this fact occurs, it would mean that we can prevent blow-up phenomena by acting on a subset of the domain.

However, as we are going to show in this Section, this control function f turns out to be impossible to find for $\mu > 1/4$ and, in this case, we cannot prevent the system from blowing up. Therefore, the upper bound $1/4$ for the Hardy constant μ shows up to be sharp for control.

The proof of this fact will rely on an analogous result presented in [53]. Following the ideas of optimal control, for any $u_0 \in L^2(\Omega)$ we consider the functional

$$J_{u_0}(u, f) := \frac{1}{2}\int_Q |u(x,t)|^2\, dxdt + \frac{1}{2}\int_0^T \|f(t)\|^2_{L^2(\Omega)}\, dt,$$

defined on the set

$$\mathcal{C}(u_0) := \left\{ (u, f) \in L^2((0,T), H_0^1(\Omega)) \times L^2((0,T), L^2(\Omega)) \,\middle|\, u \text{ satisfies } (5.1.2)\right\}.$$

We say that it is possible to stabilise system (5.1.2) if we can find a constant A such that

$$\inf_{(u,f)\in\mathcal{C}(u_0)} J_{u_0}(u, f) \leq A\|u_0\|^2_{L^2(\Omega)}.$$

Now, for $\varepsilon > 0$, we approximate (5.1.2) by the system

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2+\varepsilon^2} u = f, & (x,t) \in Q \\ u = 0, & (x,t) \in \Gamma \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.5.1}$$

Due to the boundedness of the potential, (5.5.1) is well-posed; therefore, we can define the functional

$$J^\varepsilon_{u_0}(f) := \frac{1}{2}\int_Q |u(x,t)|^2\, dxdt + \frac{1}{2}\int_0^T \|f(t)\|^2_{L^2(\Omega)}\, dt,$$

where $f \in L^2((0,T); L^2(\Omega))$ is localised in ω and u is the corresponding solution of (5.5.1). We are going to prove the following.

Theorem 5.5.1. *Assume that $\mu > 1/4$. There is no constant A such that, for all $\varepsilon > 0$ and all $u_0 \in L^2(\Omega)$,*

$$\inf_{f\in L^2((0,T);L^2(\Omega))} J^\varepsilon_{u_0}(f) \leq A\|u_0\|^2_{L^2(\Omega)}.$$

We are going to prove Theorem 5.5.1 in two steps: firstly, we give some basic estimates on the spectrum of the operator

$$\mathcal{L}^\varepsilon := -\Delta - \frac{\mu}{\delta^2 + \varepsilon^2}\,\mathcal{I} \tag{5.5.2}$$

on Ω with Dirichlet boundary conditions; secondly, we will apply these estimates for proving the main result of this Section, Theorem 5.5.1.

Spectral estimates

Since the function $1/(\delta^2 + \varepsilon^2)$ is smooth and bounded in Ω for any $\varepsilon > 0$, the spectrum of $\mathcal{L}^\varepsilon$ is given by a sequence of real eigenvalues $\lambda_0^\varepsilon \le \lambda_1^\varepsilon \le \ldots \le \lambda_k^\varepsilon \le \ldots$, with $\lambda_k^\varepsilon \to +\infty$ as $k \to +\infty$, with corresponding eigenfunctions ϕ_k^ε that form an orthonormal basis of $L^2(\Omega)$.

Proposition 5.5.1. *Assume $\mu > 1/4$ and let Ω_β be as in* (5.3.2). *Then we have*

$$\lim_{\varepsilon \to 0^+} \lambda_0^\varepsilon = -\infty \tag{5.5.3}$$

and, for all $\beta > 0$,

$$\lim_{\varepsilon \to 0^+} \|\phi_0^\varepsilon\|_{H^1(\Omega \setminus \overline{\Omega}_\beta)} = 0. \tag{5.5.4}$$

Proof. We argue by contradiction and we assume that λ_0^ε is bounded from below by some constant M. From the Rayleigh formula we have

$$\mu \int_\Omega \frac{u^2}{\delta^2 + \varepsilon^2}\, dx \le \int_\Omega |\nabla u|^2\, dx - M \int_\Omega u^2\, dx,$$

for all $\varepsilon > 0$ and any $u \in H_0^1(\Omega)$. Taking now $u \in \mathcal{D}(\Omega)$, we pass to the limit as $\varepsilon \to 0^+$ in the inequality above and we get

$$\mu \int_\Omega \frac{u^2}{\delta^2}\, dx \le \int_\Omega |\nabla u|^2\, dx - M \int_\Omega u^2\, dx, \tag{5.5.5}$$

that holds for any $u \in H_0^1(\Omega)$ by a density argument. Therefore, we should have $\mu \le 1/4$, since this is the Hardy-Poincaré inequality in the set Ω_{β_1} ([21]); then, we have a contradiction.
Now, consider the first eigenfunction $\phi_0^\varepsilon \in H_0^1(\Omega)$ of $\mathcal{L}^\varepsilon$, that by definition satisfies

$$-\Delta\phi_0^\varepsilon - \mu\frac{\phi_0^\varepsilon}{\delta^2 + \varepsilon^2} = \lambda_0^\varepsilon \phi_0^\varepsilon, \tag{5.5.6}$$

in Ω. Observe that, since the potential is smooth in Ω, also the function ϕ_0^ε is smooth by classical elliptic regularity.

Set $\beta > 0$ and let ξ_β be a non-negative smooth function, vanishing in $\Omega_{\beta/2}$ and equals to 1 in $\mathbb{R}^N \setminus \Omega_\beta$, with $\|\xi_\beta\|_\infty \le 1$. Multiplying 5.5.6 by $\xi_\beta \phi_0^\varepsilon$ and integrating by parts we obtain

$$\int_\Omega \xi_\beta\, |\nabla \phi_0^\varepsilon|^2\; dx + |\lambda_0^\varepsilon| \int_\Omega \xi_\beta\, (\phi_0^\varepsilon)^2\; dx = \mu \int_\Omega \xi_\beta \frac{(\phi_0^\varepsilon)^2}{\delta^2 + \varepsilon^2}\, dx + \frac{1}{2}\int_\Omega \Delta\xi_\beta\, (\phi_0^\varepsilon)^2\; dx. \tag{5.5.7}$$

Therefore, since ϕ_0^ε is of unit L^2-norm, and due to the definition of ξ_β, we get

$$|\lambda_0^\varepsilon| \int\limits_{\Omega\setminus\Omega_\beta} (\phi_0^\varepsilon)^2 \, dx \leq \frac{4\mu}{\beta^2} + \frac{1}{2} \|\Delta\xi_\beta\|_{L^\infty(\Omega)}.$$

Since $|\lambda_0^\varepsilon| \to \infty$ as $\varepsilon \to 0^+$, we obtain that for any $\beta > 0$

$$\lim_{\varepsilon\to 0^+} \int\limits_{\Omega\setminus\Omega_\beta} (\phi_0^\varepsilon)^2 \, dx = 0. \tag{5.5.8}$$

Furthermore, using again (5.5.7) and the definition of ξ_β

$$\int\limits_{\Omega\setminus\Omega_\beta} |\nabla\phi_0^\varepsilon|^2 \, dx \leq \left(\frac{4\mu}{\beta^2} + \frac{1}{2} \|\Delta\xi_\beta\|_{L^\infty(\Omega)}\right) \int\limits_{\Omega\setminus\Omega_{\beta/2}} (\phi_0^\varepsilon)^2 \, dx.$$

Hence, the proof of (5.5.4) is completed by using (5.5.8) for $\beta/2$. □

Proof of Theorem 5.5.1. Fix $\varepsilon > 0$ and choose $u_0^\varepsilon = \phi_0^\varepsilon$, that by definition is of unit L^2-norm. We want to show that

$$\inf_{f\in L^2((0,T);L^2(\Omega))} J^\varepsilon_{u_0^\varepsilon}(f) \to +\infty$$

as $\varepsilon \to 0^+$.

At this purpose, let $f \in L^2((0,T); L^2(\Omega))$ and consider the corresponding solution u of (5.1.2) with initial data $u_0^\varepsilon = \phi_0^\varepsilon$. Set

$$h(t) = \int_\Omega u(x,t)\phi_0^\varepsilon(x) \, dx, \quad \text{and} \quad \zeta(t) = \langle f(t), \phi_0^\varepsilon\rangle_{L^2(\Omega)};$$

then, $h(t)$ satisfies the first order differential equation

$$\begin{cases} h'(t) + \lambda_0^\varepsilon \, h(t) = \zeta(t), \\ h(0) = 1. \end{cases}$$

By the Duhamel's formula we obtain

$$h(t) = e^{-\lambda_0^\varepsilon t} + \int_0^t e^{-\lambda_0^\varepsilon(t-s)}\zeta(s) \, ds.$$

Therefore,

$$\int_Q u^2 \, dxdt \geq \int_0^T h(t)^2 \, dt \geq \frac{1}{2}\int_0^T e^{-\lambda_0^\varepsilon t} \, dt - \int_0^T \left(\int_0^t e^{-\lambda_0^\varepsilon(t-s)}\zeta(s) \, ds\right)^2 dt. \tag{5.5.9}$$

Of course

$$\frac{1}{2}\int_0^T e^{-\lambda_0^\varepsilon t} \, dt = \frac{1}{4\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right);$$

on the other hand, by trivial computations we have

$$\int_0^T \left(\int_0^t e^{-\lambda_0^\varepsilon (t-s)} \zeta(s)\, ds \right)^2 dt \le \frac{1}{4\,(\lambda_0^\varepsilon)^2} e^{2\lambda_0^\varepsilon T} \int_0^T \zeta(s)^2\, ds.$$

Besides, from the definition of $\zeta(t)$, and since f is localized in ω, it immediately follows

$$|\zeta(t)|^2 \le \|f(t)\|^2_{L^2(\Omega)} \|\phi_0^\varepsilon\|^2_{L^2(\omega)}.$$

Hence, we deduce from (5.5.9) that

$$\frac{1}{4\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right) \le \int_Q u^2\, dxdt + \frac{\|\phi_0^\varepsilon\|^2_{L^2(\omega)}}{4\,(\lambda_0^\varepsilon)^2} e^{2\lambda_0^\varepsilon T} \int_0^T \|f(t)\|^2_{L^2(\Omega)}\, dt,$$

that implies either

$$\frac{1}{8\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right) \le \int_Q u^2\, dxdt$$

or

$$\frac{1}{8\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right) \le \frac{\|\phi_0^\varepsilon\|^2_{L^2(\omega)}}{4\,(\lambda_0^\varepsilon)^2} e^{2\lambda_0^\varepsilon T} \int_0^T \|f(t)\|^2_{L^2(\Omega)}\, dt.$$

In any case, for any $f \in L^2((0,T); L^2(\Omega))$ with support in ω we get

$$J_{u_0^\varepsilon}^\varepsilon(f) \ge \inf \left\{ \frac{e^{2\lambda_0^\varepsilon T} - 1}{16\lambda_0^\varepsilon},\ \frac{\lambda_0^\varepsilon}{4\|\phi_0^\varepsilon\|^2_{L^2(\omega)}} \left(1 - e^{2\lambda_0^\varepsilon T}\right) \right\}.$$

This last bound blows up as $\varepsilon \to 0^+$, due to the estimates (5.5.3) and (5.5.4). Indeed, by definition of ω, we can find $\beta > 0$ such that $\omega \subset \Omega \setminus \Omega_\beta$ and therefore

$$\|\phi_0^\varepsilon\|_{L^2(\omega)} \le \|\phi_0^\varepsilon\|_{L^2(\Omega\setminus\Omega_\beta)} \le \|\phi_0^\varepsilon\|_{H^1(\Omega\setminus\Omega_\beta)} \to 0,$$

as $\varepsilon \to 0^+$. This concludes the proof. □

5.6 Proof of the Carleman estimate

Before giving the proof of the Carleman estimate (5.3.11), it is important to remark that, in principle, the solutions of (5.1.4) do not have enough regularity to justify the computations; in particular, the H^2 regularity in the space variable that would be required for applying standard integration by parts may not be guaranteed. For this reason, we need to add some regularisation argument.

In our case, this can be done by regularising the potential, i.e. by considering, instead of the operator $\mathcal{A}$ defined in (5.1.1), the following

$$\mathcal{A}_n v := \Delta v + \frac{\mu_1}{(\delta + 1/n)^2} v, \quad n > 0. \tag{5.6.1}$$

The domain of this new operator is $\mathcal{D}(\mathcal{A}_n) = \mathcal{D}(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$, due to the fact that now our potential is bounded on Ω, and the solution v_n of the corresponding parabolic equation possess all the regularity needed to justify the computations. Passing to the limit as $n \to +\infty$, we can then recover our result for the solution v of (5.1.4).

In order to simplify our presentation, we will skip this regularisation process and we will write directly the formal computations for the solution of (5.1.4). Moreover, we are going to present here the main ideas of the proof of the inequality, using some technical Lemmas which will be proved in Section 5.7.

Step 1. Notation and rewriting of the problem

For any solution v of the adjoint problem (5.1.4), and for any $R > 0$, we define

$$z(x,t) := v(x,t)e^{-R\sigma(x,t)}, \tag{5.6.2}$$

which satisfies

$$z(x,0) = z(x,T) = 0 \tag{5.6.3}$$

in $H_0^1(\Omega)$, due to the definition of σ. The positive parameter R is meant to be large. Plugging $v(x,t) = z(x,t)e^{R\sigma(x,t)}$ in (5.1.4), we obtain that z satisfies

$$z_t + \Delta z + \frac{\mu}{\delta^2} z + 2R\nabla z \cdot \nabla\sigma + Rz\Delta\sigma + z\left(R\sigma_t + R^2|\nabla\sigma|^2\right) = 0, \quad (x,t) \in \Omega \times (0,T) \tag{5.6.4}$$

with boundary conditions

$$z(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T). \tag{5.6.5}$$

Next, we define a smooth positive function $\alpha(x)$ such that

$$\alpha(x) = \begin{cases} 0, & x \in \Omega_{r0/2} \\ 1, & x \in \Omega \setminus \Omega_{r_0} \end{cases} \tag{5.6.6}$$

where Ω_{r_0} has been introduced in (5.3.2). Setting

$$\begin{aligned} \mathrm{S}z &:= \Delta z + \frac{\mu}{\delta^2} z + z\left(R\sigma_t + R^2|\nabla\sigma|^2\right), \\ \mathrm{A}z &:= z_t + 2R\nabla z \cdot \nabla\sigma + Rz\Delta\sigma(1+\alpha), \\ \mathrm{P}z &:= -R\alpha z\Delta\sigma, \end{aligned}$$

one easily deduce from (5.6.4) that

$$\mathrm{S}z + \mathrm{A}z + \mathrm{P}z = 0, \quad \|\mathrm{S}z\|^2_{L^2(Q)} + \|\mathrm{A}z\|^2_{L^2(Q)} + 2\langle \mathrm{S}z, \mathrm{A}z\rangle_{L^2(Q)} = \|\mathrm{P}z\|^2_{L^2(Q)}.$$

In particular, we obtain that the quantity

$$I = \langle \mathrm{S}z, \mathrm{A}z\rangle_{L^2(Q)} - \frac{1}{2}\|R\alpha z\Delta\sigma\|^2_{L^2(Q)} \tag{5.6.7}$$

is not positive.

Step 2. Computation of the scalar product

Lemma 5.6.1. *The following identity holds:*

$$
\begin{aligned}
I = {} & R\int_\Sigma |\partial_n z|^2 \partial_n\sigma \, dsdt - 2R\int_Q D^2\sigma(\nabla z, \nabla z)\, dxdt - R\int_Q \alpha\Delta\sigma|\nabla z|^2\, dxdt \\
& + R\int_Q (\nabla(\Delta\sigma)\cdot\nabla\alpha)\, z^2\, dxdt + \frac{R}{2}\int_Q \Delta\sigma\Delta\alpha\, z^2\, dxdt + R\mu\int_Q \alpha\Delta\sigma\frac{z^2}{\delta^2}\, dxdt \\
& + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma)\,\frac{z^2}{\delta^3} dxdt + \frac{R}{2}\int_Q \Delta^2\sigma(1+\alpha)z^2 dxdt - 2R^3\int_Q D^2\sigma(\nabla\sigma,\nabla\sigma)z^2 dxdt \\
& + R^3\int_Q \alpha\Delta\sigma|\nabla\sigma|^2 z^2\, dxdt - \frac{R^2}{2}\int_Q \alpha^2|\Delta\sigma|^2 z^2\, dxdt - \frac{1}{2}\int_Q \left(R\sigma_{tt} + 2R^2(|\nabla\sigma|^2)_t\right) z^2\, dxdt \\
& + R^2\int_Q \alpha\sigma_t\Delta\sigma\, z^2\, dxdt.
\end{aligned} \tag{5.6.8}
$$

The proof of Lemma 5.6.1 will be presented in Section 5.7. Moreover, in what follows we will split (5.6.8) in four parts; first of all, let us define the boundary term

$$
I_{bd} = R\int_\Sigma |\partial_n z|^2\partial_n\sigma\, dsdt. \tag{5.6.9}
$$

Secondly, we define I_l as the sum of the integrals linear in σ which do not involve any time derivative

$$
\begin{aligned}
I_l = {} & -2R\int_Q D^2\sigma(\nabla z,\nabla z)\, dxdt - R\int_Q \alpha\Delta\sigma|\nabla z|^2\, dxdt + R\int_Q (\nabla(\Delta\sigma)\cdot\nabla\alpha)\, z^2\, dxdt \\
& + \frac{R}{2}\int_Q \Delta\sigma\Delta\alpha\, z^2\, dxdt + R\mu\int_Q \alpha\Delta\sigma\frac{z^2}{\delta^2}\, dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma)\,\frac{z^2}{\delta^3}\, dxdt \\
& + \frac{R}{2}\int_Q \Delta^2\sigma(1+\alpha)z^2\, dxdt.
\end{aligned} \tag{5.6.10}
$$

Then, we consider the sum of the integrals involving non-linear terms in σ and without any time derivative, that is

$$
I_{nl} = -2R^3\int_Q D^2\sigma(\nabla\sigma,\nabla\sigma)z^2\, dxdt + R^3\int_Q \alpha\Delta\sigma|\nabla\sigma|^2 z^2\, dxdt - \frac{R^2}{2}\int_Q \alpha^2|\Delta\sigma|^2 z^2\, dxdt. \tag{5.6.11}
$$

Finally, we define the terms involving the time derivative in σ as

$$
I_t = -\frac{1}{2}\int_Q \left(R\sigma_{tt} + 2R^2(|\nabla\sigma|^2)_t\right) z^2\, dxdt + R^2\int_Q \alpha\sigma_t\Delta\sigma\, z^2\, dxdt. \tag{5.6.12}
$$

Step 3. Bounds for the quantities I_{bd}, I_l, I_{nl} and I_t

We now estimates the four quantities (5.6.9), (5.6.10), (5.6.11) and (5.6.12) separately.

Lemma 5.6.2. *It holds that* $I_{bd} > 0$ *for any* $\lambda > 0$

Lemma 5.6.3. *There exists λ_0 such that for any $\lambda \geq \lambda_0$ and any $R > 0$, and for any r_0 as in* (5.3.9), *it holds*

$$I_l \geq B_1 R \int_Q \theta \left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma}\right) dxdt + \frac{\lambda R}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta \left(\frac{\delta}{r_0}\right)^{\lambda-2} |\nabla z|^2 \, dxdt$$
$$- B_2 \lambda^2 R \int\limits_{\omega_0\times(0,T)} \theta \left(\frac{\delta}{r_0}\right)^{\lambda} \phi |\nabla z|^2 \, dxdt + B_3 \lambda^2 R \int\limits_{\mathcal{O}\times(0,T)} \theta \left(\frac{\delta}{r_0}\right)^{\lambda} \phi |\nabla z|^2 \, dxdt$$
$$- B_\lambda R \int_Q \theta z^2 \, dxdt, \tag{5.6.13}$$

where B_1, B_2 and B_3 are positive constants independent on R and λ, and B_λ is a positive constant independent on R.

Lemma 5.6.4. *There exists λ_0 such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for and any $R \geq R_0$ and for any r_0 as in* (5.3.9) *it holds*

$$I_{nl} \geq \frac{R^3}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3 \delta^2 z^2 \, dxdt + B_5 \lambda^4 R^3 \int\limits_{\mathcal{O}\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt$$
$$- B_6 \lambda^4 R^3 \int\limits_{\omega_0\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt, \tag{5.6.14}$$

for some positive constants B_5 and B_6 uniform in R and λ.

Taking into account the negative terms in the expression of I_l that we want to get rid of, we define

$$I_r = I_t - B_\lambda R \int_Q \theta z^2 \, dxdt. \tag{5.6.15}$$

Lemma 5.6.5. *There exists λ_0 such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for and any $R \geq R_0$ and for any r_0 as in* (5.3.9) *it holds*

$$|I_r| \leq \frac{B_1}{2} R \int_Q \theta \frac{z^2}{\delta^\gamma} dxdt + \frac{B_5}{2} \lambda^4 R^3 \int\limits_{\mathcal{O}\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 dxdt + \frac{R^3}{4} \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3 \delta^2 z^2 dxdt, \tag{5.6.16}$$

where B_1 and B_5 are the positive constants introduced in Lemmas 5.6.3 and 5.6.4, respectively.

The proofs of Lemmas 5.6.2, 5.6.3, 5.6.4 and 5.6.5 will be presented again in Section 5.7.

Step 4. Conclusion

From the Lemmas above, we obtain the Carleman estimates in the variable z as follows

Theorem 5.6.1. *There exist two positive constants λ_0 and $\mathcal{L}$ such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for any $R \geq R_0$ it holds*

$$R \int_Q \theta \left(\delta^{2-\gamma} |\nabla z|^2 + \frac{1}{2} \frac{z^2}{\delta^\gamma} \right) dxdt + \lambda R \int_{\Omega_{r_0} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^{\lambda-2} |\nabla z|^2 dxdt + R^3 \int_{\Omega_{r_0} \times (0,T)} \theta^3 \delta^2 z^2 dxdt$$
$$+ \lambda^2 R \int_{\mathcal{O} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^{\lambda} \phi |\nabla z|^2 \, dxdt + \lambda^4 R^3 \int_{\mathcal{O} \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 z^2 \, dxdt$$
$$\leq \mathcal{L} \left(\lambda^4 R^3 \int_{\omega_0 \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 z^2 \, dxdt + \lambda^2 R \int_{\omega_0 \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^{\lambda} \phi |\nabla z|^2 \, dxdt \right) \tag{5.6.17}$$

Coming back from the variable z to the solution v of (5.1.4), we finally obtain Theorem 5.3.1.

Remark 5.6.1. *We observe that the distance function δ is only $C^{1,0}$ and, in principle, a weight defined as in* (5.3.4) *does not have enough regularity for proving the Carleman estimate* (5.3.11)*. On the other hand, during our computations this lack of regularity will be compensated by means of the cut-off function α and of the Hardy inequality* (5.2.1)*. This will therefore justify our proof.*

5.7 Proof of technical Lemmas

We present now the proof of the technical Lemmas 5.6.1 to 5.6.5, introduced in Section 5.6. At this purpose, we remind that the distance function δ satisfies the following classical properties

$$\delta \in C^{0,1}(\overline{\Omega}), \tag{5.7.1a}$$
$$|\nabla u| = 1, \text{ a.e. in } \Omega, \tag{5.7.1b}$$
$$\text{there exists a constant } P > 0 \text{ such that } |\Delta\delta| \leq P/\delta, \text{ a.e. in } \Omega. \tag{5.7.1c}$$

Furthermore, we are going to need the following result

Lemma 5.7.1. *Assume that ψ is the function defined in* (5.3.6) *by means of ψ_1 and ρ. Then, there exists a constant $D_{\psi_1} > 0$, which depends only on ψ_1, such that*

$$|\nabla\delta \cdot \nabla\psi(x) - \rho\psi_1(x)| \leq \rho D_{\psi_1} \delta, \quad \textit{for all } x \in \mathcal{O}. \tag{5.7.2}$$

Proof. By definition of ψ and Cauchy-Scwarz inequality, using (5.7.1b) and since ψ_1 is bounded, we immediately have

$$|\nabla\delta \cdot \nabla\psi(x) - \rho\psi_1(x)| = \rho|\nabla\delta \cdot \nabla\psi_1(x) - \psi_1(x)| \leq \rho|\nabla\psi_1 - \psi_1| \leq \rho\hat{D}_{\psi_1} \leq \rho D_{\psi_1}\delta.$$

Furthermore, we emphasise that the constant D_{ψ_1} does not depend on r_0. □

Now, for σ as in (5.3.4) we introduce the notations

$$\sigma_\delta = -\theta\tau_\delta = -\theta\delta^2\psi, \qquad \sigma_\phi = -\theta\tau_\phi = -\theta\left(\frac{\delta}{r_0}\right)^\lambda \phi, \qquad \tau = \tau_\delta + \tau_\phi,$$

so that $\sigma(x,t) = C_\lambda\theta(t) + \sigma_\delta(x,t) + \sigma_\phi(x,t)$. Next, we deduce some formulas for τ_δ and τ_ϕ that we are going to use later in our computations. More precisely, for all $x, \xi \in \mathbb{R}^N$ and any $i, j \in \{1, \dots, N\}$ we have

$$\partial_{x_i}\tau_\delta = 2\psi\delta\delta_{x_i} + \delta^2\psi_{x_i}, \tag{5.7.3}$$

$$\partial^2_{x_ix_j}\tau_\delta = 2\psi\delta_{x_i}\delta_{x_j} + 2\delta(\psi_{x_j}\delta_{x_i} + \psi\delta_{x_ix_j}) + 2\delta\psi_{x_i}\delta_{x_j} + \delta^2\psi_{x_ix_j} \tag{5.7.4}$$

and

$$\Delta\tau_\delta = 2\psi + 4\delta(\nabla\delta\cdot\nabla\psi) + 2\delta\psi\Delta\delta + \delta^2\Delta\psi, \tag{5.7.5}$$

$$D^2\tau_\delta(\xi,\xi) = 2\psi(\xi\cdot\nabla\delta)^2 + 2\delta\psi D^2\delta(\xi,\xi) + 4\delta(\xi\cdot\nabla\delta)(\xi\cdot\nabla\psi) + \delta^2 D^2\psi(\xi,\xi). \tag{5.7.6}$$

On the other hand

$$\partial_{x_i}\tau_\phi = \frac{\phi}{r_0^\lambda}(\lambda\delta^{\lambda-1}\delta_{x_i} + \lambda\delta^\lambda\psi_{x_i}), \tag{5.7.7}$$

$$\begin{aligned}\partial^2_{x_ix_j}\tau_\phi = \frac{\phi}{r_0^\lambda}\Big(&\lambda(\lambda-1)\delta^{\lambda-2}\delta_{x_i}\delta_{x_j} + \lambda\delta^{\lambda-1}\delta_{x_ix_j} + \lambda^2\delta^{\lambda-1}(\psi_{x_j}\delta_{x_i} + \psi_{x_i}\delta_{x_j}) + \lambda\delta^\lambda\psi_{x_ix_j}\\ &+ \lambda^2\delta^\lambda\psi_{x_i}\psi_{x_j}\Big)\end{aligned} \tag{5.7.8}$$

and

$$\Delta\tau_\phi = \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2} + \lambda\delta^{\lambda-1}\Delta\delta + 2\lambda^2\delta^{\lambda-1}(\nabla\delta\cdot\nabla\psi) + \lambda\delta^\lambda\Delta\psi + \lambda^2\delta^\lambda|\nabla\psi|^2\Big), \tag{5.7.9}$$

$$\begin{aligned}D^2\tau_\phi(\xi,\xi) = \frac{\phi}{r_0^\lambda}\Big(&\lambda(\lambda-1)\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + 2\lambda^2\delta^{\lambda-1}(\xi\cdot\nabla\delta)(\xi\cdot\nabla\psi)\\ &+ \lambda\delta^\lambda D^2\psi(\xi,\xi) + \lambda^2\delta^\lambda(\xi\cdot\nabla\psi)^2\Big).\end{aligned} \tag{5.7.10}$$

Upper and lower bounds for $\Delta\tau_\delta$, $\Delta\tau_\phi$, $D^2\tau_\delta(\xi,\xi)$ and $D^2\tau_\phi(\xi,\xi)$

Proposition 5.7.1. *For r_0 as in (5.3.9) we have*

$$\Delta\tau_\delta \geq 0, D^2\tau_\delta(\xi,\xi) \geq 0, \qquad \forall x \in \Omega_{r_0}, \forall\xi \in \mathbb{R}^N, \tag{5.7.11}$$

$$|D^2\tau_\delta(\xi,\xi)| \leq C_1|\xi|^2, \qquad \forall x \in \Omega, \forall\xi \in \mathbb{R}^N, \tag{5.7.12}$$

$$|\Delta\tau_\delta| \leq C_2, \qquad \forall x \in \Omega_{r_0}, \tag{5.7.13}$$

where C_1 and C_2 are constants depending on Ω and ψ.

Proposition 5.7.2. *For ρ and r_0 as in* (5.3.8) *and* (5.3.9) *we have*

$$D^2\tau_\phi \geq \frac{\lambda}{2}\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi|\xi|^2, \qquad \forall x \in \Omega_{r_0}, \forall \xi \in \mathbb{R}^N, \tag{5.7.14}$$

$$\Delta\tau_\phi \geq \lambda^2\left(\frac{\delta}{r_0}\right)^{\lambda}\phi, \qquad \forall x \in \mathcal{O}, \tag{5.7.15}$$

$$D^2\tau_\phi \geq -\lambda C_3\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi|\xi|^2, \qquad \forall x \in \Omega, \forall \xi \in \mathbb{R}^N, \tag{5.7.16}$$

for λ large enough, where C_3 is a constant depending on Ω, r_0 and ψ.

Proof of Proposition 5.7.1. Observe that the proofs of (5.7.12) and (5.7.13) are trivial. To prove (5.7.11), instead, it is enough to show that $D^2\tau_\delta(\xi,\xi) \geq 0$ in Ω_{r_0} since this also implies that $\Delta\tau_\delta \geq 0$ in Ω_{r_0}, simply choosing $\xi = e_i$ for all $i \in \{1,\dots,N\}$. Now, we have that, for $x \in \Omega_{r_0}$

$$\delta(x) = |x - \mathrm{pr}(x)| \tag{5.7.17}$$

where $\mathrm{pr}(x)$ is the projection of x on Γ. Hence (5.7.6) becomes

$$D^2\tau_\delta(\xi,\xi) = 2\psi|\xi|^2 + 4\Big(\xi\cdot\Big(x-\mathrm{pr}(x)\Big)\Big)(\xi\cdot\nabla\psi) + \delta^2 D^2\psi(\xi,\xi), \quad \forall \xi \in \mathbb{R}^N.$$

Now, using Cauchy-Scwarz inequality, and since $\psi > 1$, we obtain

$$D^2\tau_\delta(\xi,\xi) \geq (2\psi - 4\delta|D\psi|_\infty - \delta^2|D^2\psi|_\infty)|\xi|^2 \geq (2 - r_0(4|D\psi|_\infty + |D^2\psi|_\infty))|\xi|^2 \geq 0,$$

since r_0 satisfies (5.3.9). ☐

Proof of Proposition 5.7.2. Let us rewrite (5.7.10) as $D^2\tau_\phi(\xi,\xi) = \phi(1/r_0)^\lambda\,\mathcal{S}_\phi$, where

$$\begin{aligned}\mathcal{S}_\phi = {} & \lambda(\lambda-1)\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + 2\lambda^2\delta^{\lambda-1}(\xi\cdot\nabla\delta)(\xi\cdot\nabla\psi) + \lambda\delta^\lambda D^2\psi(\xi,\xi) \\ & + \lambda^2\delta^\lambda(\xi\cdot\nabla\psi)^2.\end{aligned} \tag{5.7.18}$$

Next, we have

$$|2\lambda^2\delta^{\lambda-1}(\xi\cdot\nabla\delta)(\xi\cdot\nabla\psi)| \leq a\lambda^2\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \frac{\lambda^2}{a}\delta^\lambda(\xi\cdot\nabla\psi)^2, \quad \forall a > 0,$$

which combined with (5.7.18) leads to

$$\mathcal{S}_\phi \geq (\lambda^2 - \lambda - a\lambda^2)\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + \lambda\delta^\lambda D^2\psi(\xi,\xi) + \left(\lambda^2 - \frac{\lambda^2}{a}\right)\delta^\lambda(\xi\cdot\nabla\psi)^2.$$

Choosing now a such that $\lambda^2(1-a) - \lambda = 0$, i.e. $a = (\lambda-1)/\lambda$, we have

$$\mathcal{S}_\phi \geq \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + \lambda\delta^\lambda D^2\psi(\xi,\xi) - \frac{\lambda^2}{\lambda-1}\delta^\lambda|\nabla\psi|^2|\xi|^2. \tag{5.7.19}$$

Applying (5.7.19) for $x \in \Omega_{r_0}$ we deduce

$$\begin{aligned}\mathcal{S}_\phi &\geq \frac{\lambda}{2}\delta^{\lambda-2}|\xi|^2 + \lambda\delta^{\lambda-2}|\xi|^2\left(\frac{1}{2} - \frac{\lambda}{\lambda-1}\delta^2|D\psi|_\infty^2 - \delta^2|D^2\psi|_\infty\right)\\ &\geq \frac{\lambda}{2}\delta^{\lambda-2}|\xi|^2 + \lambda\delta^{\lambda-2}|\xi|^2\left(\frac{1}{2} - r_0^2\Big(2|D\psi|_\infty^2 + |D^2\psi|_\infty\Big)\right) \geq \frac{\lambda}{2}\delta^{\lambda-2}|\xi|^2,\end{aligned}$$

for r_0 as in (5.3.9). This immediately yields the proof of (5.7.14).

Let us now prove (5.7.15). According to Lemma 5.7.1, to the definition of ψ and to (5.7.1c) and (5.7.9) we get

$$\begin{aligned}\Delta\tau_\phi &\geq \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1-P)\delta^{\lambda-2} + 2\lambda^2\delta^{\lambda-1}(\rho\psi_1 - \rho D_{\psi_1}\delta) + \lambda\delta^\lambda\Delta\psi + \lambda^2\delta^\lambda|\nabla\psi|^2\Big)\\ &\geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi\left(|\nabla\psi|^2 - 2\rho D_{\psi_1} - \frac{|\Delta\psi|}{\lambda}\right) \geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi\left(\rho^2\rho_0^2 - 2\rho D_{\psi_1} - \rho|D^2\psi|_\infty\right)\\ &\geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi\end{aligned}$$

for all $x \in \mathcal{O}$, if we take ρ as in (5.3.8) and $\lambda > 1$.

We conclude with the proof of (5.7.16). From (5.7.10) for any $x \in \Omega$ we have

$$\begin{aligned}D^2\tau_\phi(\xi,\xi) &= \frac{\phi}{r_0^\lambda}\left(\lambda^2\Big(\delta^{\frac{\lambda}{2}-1}(\xi\cdot\nabla\delta) + \delta^{\frac{\lambda}{2}}(\xi\cdot\nabla\psi)\Big)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + \lambda\delta^\lambda D^2\psi(\xi,\xi)\right.\\ &\qquad\left. - \lambda\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2\right)\\ &\geq \lambda\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi\left(\frac{1}{r_0^2}\left(\delta D^2\delta(\xi,\xi) + \delta^2 D^2\psi(\xi,\xi) - (\xi\cdot\nabla\delta)^2\right)\right)\\ &\geq -\lambda\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi\left(\frac{1}{r_0^2}\left(|D^2\delta|_\infty + R_\Omega^2|D^2\psi|_\infty + 1\right)\right)|\xi|^2,\end{aligned}$$

which gives us the validity of (5.7.16) for $C_3 = \left(|D^2\delta|_\infty + R_\Omega^2|D^2\psi|_\infty + 1\right)/r_0^2$. □

Bounds for $2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2$

We provide here pointwise estimates for the quantity $2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2$, which appears in the identity in Lemma 5.6.1. First of all, we have

$$\begin{aligned}\partial_{x_i}\tau &= 2\psi\delta\delta_{x_i} + \delta^2\psi_{x_i} + \frac{\phi}{r_0^\lambda}(\lambda\delta^{\lambda-1}\delta_{x_i} + \lambda\delta^\lambda\psi_{x_i}),\\ \partial^2_{x_ix_j}\tau &= 2\psi\delta_{x_i}\delta_{x_j} + 2\delta(\psi_{x_j}\delta_{x_i} + \psi\delta_{x_ix_j}) + 2\delta\psi_{x_i}\delta_{x_j} + \delta^2\psi_{x_ix_j}\\ &\quad + \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2}\delta_{x_i}\delta_{x_j} + \lambda\delta^{\lambda-1}\delta_{x_ix_j} + \lambda^2\delta^{\lambda-1}(\psi_{x_j}\delta_{x_i} + \psi_{x_i}\delta_{x_j}) + \lambda\delta^\lambda\psi_{x_ix_j}\\ &\quad + \lambda^2\delta^\lambda\psi_{x_i}\psi_{x_j}\Big),\end{aligned}$$

and, in consequence,

$$\begin{aligned}\Delta\tau &= 2\psi + 4\delta(\nabla\delta\cdot\nabla\psi) + 2\psi\Delta\delta + \delta^2\Delta\psi \\ &\quad + \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2} + \lambda\delta^{\lambda-1}\Delta\delta + 2\lambda^2\delta^{\lambda-1}(\nabla\delta\cdot\nabla\psi) + \lambda\delta^\lambda\Delta\psi + \lambda^2\delta^\lambda|\nabla\psi|^2\Big),\end{aligned} \tag{5.7.20}$$

$$\begin{aligned}D^2\tau(\nabla\tau,\nabla\tau) &= 2\psi(\nabla\tau\cdot\nabla\delta)^2 + 2\delta\psi D^2\delta(\nabla\tau,\nabla\tau) + 4\delta(\nabla\tau\cdot\nabla\delta)(\nabla\tau\cdot\nabla\psi) \\ &\quad + \delta^2 D^2\psi(\nabla\tau,\nabla\tau) + \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2}(\nabla\tau\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\nabla\tau,\nabla\tau) \\ &\quad + 2\lambda^2\delta^{\lambda-1}(\nabla\tau\cdot\nabla\delta)(\nabla\tau\cdot\nabla\psi) + \lambda\delta^\lambda D^2\psi(\nabla\tau,\nabla\tau) + \lambda^2\delta^\lambda(\nabla\tau\cdot\nabla\psi)^2\Big).\end{aligned} \tag{5.7.21}$$

Using the expressions above we obtain the following useful formulas

$$\begin{aligned}(\nabla\delta\cdot\nabla\tau)^2 &= |\nabla\tau|^2 + \left((\nabla\delta\cdot\nabla\psi)^2 - |\nabla\psi|^2\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right)^2, \\ (\nabla\delta\cdot\nabla\tau)(\nabla\psi\cdot\nabla\tau) &= |\nabla\tau|^2(\nabla\delta\cdot\nabla\psi) \\ &\quad + \left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(2\delta\psi + \lambda\frac{\phi}{r_0^\lambda}\delta^{\lambda-1}\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right), \\ (\nabla\psi\cdot\nabla\tau)^2 &= |\nabla\psi|^2|\nabla\tau|^2 + \left((\nabla\delta\cdot\nabla\psi)^2 - |\nabla\psi|^2\right)\left(2\delta\psi + \lambda\frac{\phi}{r_0^\lambda}\delta^{\lambda-1}\right)^2,\end{aligned}$$

and we finally conclude

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2 = T_1 + T_2 + T_3,$$

where

$$\begin{aligned}T_1 &= 2\psi(2-\alpha)|\nabla\tau|^2 + 4\delta\psi D^2\delta(\nabla\tau,\nabla\tau) + 2\delta^2 D^2\psi(\nabla\tau,\nabla\tau) + 4(2-\alpha)\delta(\nabla\delta\cdot\nabla\psi)|\nabla\tau|^2 \\ &\quad - 2\delta\psi\alpha\Delta\delta|\nabla\tau|^2 - \delta^2\alpha\Delta\psi|\nabla\tau|^2,\end{aligned} \tag{5.7.22}$$

$$\begin{aligned}T_2 &= 4\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right)\left(5\delta^2\psi + \lambda(2-\psi)\frac{\phi}{r_0^\lambda}\delta^\lambda\right) \\ &\quad + \frac{\phi}{r_0^\lambda}\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(2\lambda^3\delta^{3\lambda-2}\left(\frac{\phi}{r_0^\lambda}\right)^2 + 4\lambda^2\frac{\phi}{r_0^\lambda}\delta^{2\lambda} + 2\lambda\delta^{\lambda+2}\right) \\ &\quad + \frac{\phi}{r_0^\lambda}\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(\lambda^2(8\psi(1-\psi)-2)\delta^{\lambda+2}\right),\end{aligned} \tag{5.7.23}$$

$$\begin{aligned}T_3 &= \frac{\phi}{r_0^\lambda}\Big\{\Big[(\lambda^2(2-\alpha) - \lambda(2-\alpha+\alpha\delta\Delta\delta))\delta^{\lambda-2} + 2\lambda^2\delta^{\lambda-1}(2-\alpha)(\nabla\delta\cdot\nabla\psi) \\ &\quad + \lambda^2\delta^\lambda(2-\alpha)|\nabla\psi|^2 - \lambda\alpha\delta^\lambda\Delta\psi\Big]|\nabla\tau|^2 + 2\lambda\delta^{\lambda-1}D^2\delta(\nabla\tau,\nabla\tau) + 2\lambda\delta^\lambda D^2\psi(\nabla\tau,\nabla\tau)\Big\}.\end{aligned} \tag{5.7.24}$$

Proposition 5.7.3. *For r_0 as in (5.3.9), there exist two positive constants D_1 and D_2 depending on Ω and ψ such that the term T_1 in (5.7.22) satisfies*

$$T_1 \geq |\nabla\tau|^2, \qquad \forall x \in \Omega_{r_0}, \tag{5.7.25}$$

$$T_1 \geq -D_1|\nabla\tau|^2, \qquad \forall x \in \mathcal{O}, \tag{5.7.26}$$

$$|T_1| \leq D_2|\nabla\tau|^2, \qquad \forall x \in \omega_0. \tag{5.7.27}$$

Proposition 5.7.4. *There exists λ_0 large enough such that, for any $\lambda \geq \lambda_0$ and r_0 as in (5.3.9), the term T_2 in (5.7.23) satisfies*

$$T_2 \geq -\frac{\phi}{r_0^\lambda}|D\psi|_\infty^2 \left(D_3\lambda^2\psi^2 + D_4\lambda^2\right)\delta^{\lambda+2}, \qquad \forall x \in \Omega_{r_0}, \tag{5.7.28}$$

$$T_2 \geq 0, \qquad \forall x \in \tilde{\mathcal{O}}, \tag{5.7.29}$$

for some positive constants D_3 and D_4 not depending on r_0.

Proposition 5.7.5. *There exists λ_0 large enough such that, for any $\lambda \geq \lambda_0$ and ρ and r_0 as in (5.3.8) and (5.3.9), the term T_3 in (5.7.24) satisfies*

$$T_3 \geq \lambda^2\left(\frac{\phi}{r_0^\lambda}\delta^{\lambda-2} + \left(\frac{\delta}{r_0}\right)^\lambda \phi\right)|\nabla\tau|^2, \qquad \forall x \in \Omega \setminus \overline{\omega_0}, \tag{5.7.30}$$

$$T_3 \leq \lambda^2 D_5 \frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2, \qquad \forall x \in \Omega, \tag{5.7.31}$$

for some positive constant D_5, not depending on λ.

Proposition 5.7.6. *For any r_0 and ρ as in (5.3.9) and (5.3.8) it holds*

$$|\nabla\tau|^2 \geq \delta^2, \qquad \forall x \in \Omega_{r_0}, \tag{5.7.32}$$

$$|\nabla\tau|^2 \geq \lambda^2\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2, \qquad \forall x \in \mathcal{O}, \tag{5.7.33}$$

$$|\nabla\tau|^2 \leq \lambda^2 D_6\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2, \qquad \forall x \in \omega_0, \tag{5.7.34}$$

where D_6 is a positive constant depending only on Ω and ψ.

Proof of Proposition 5.7.3. The inequalities (5.7.26) and (5.7.27) are obvious. Hence, we only need to prove (5.7.25). Due to the definition of α, to the properties of ψ and to Lemma 5.7.1, and using (5.7.17), we have (see also [35, Proposition 3.4])

$$T_1 \geq \left(2 - r_0^2(8\rho D_{\psi_1} + 3|D^2\psi|_\infty)\right)|\nabla\tau|^2 \geq \left(2 - r_0^2\left(8\frac{D_{\psi_1}}{\rho_0}|D\psi|_\infty + 3|D^2\psi|_\infty\right)\right)|\nabla\tau|^2 \geq |\nabla\tau|^2,$$

in Ω_{r_0}, for r_0 as in (5.3.9). □

Proof of Proposition 5.7.4. Due to Cuachy-Scwarz inequality, the term $|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2$ in (5.7.23) is positive; hence

$$4\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(\delta^2+\lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right)\left(5\delta^2\psi+\lambda(2-\psi)\frac{\phi}{r_0^\lambda}\delta^\lambda\right)$$
$$\geq 4D_7\delta^2\left(5\delta^2\psi+\lambda(2-\psi)\frac{\phi}{r_0^\lambda}\delta^\lambda\right)\geq -4D_7\lambda\psi\frac{\phi}{r_0^\lambda}\delta^{\lambda+2}\geq -D_8\lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda+2}$$

for λ large enough. From this (5.7.28) follows trivially.

Concerning (5.7.29), it is straightforward to check that the inequality holds for λ large enough, since the term in λ^3 is positive and it dominates all the other terms far away from the boundary. □

Proof of Proposition 5.7.5. For $x\in\Omega_{r_0}$, due to (5.7.17), the proof is analogous to the one of [35, Proposition 3.6] and we omit it here. Therefore, let us assume now $x\in\tilde{\mathcal{O}}$. Due to the definition of α, for λ large enough we have

$$\lambda^2(2-\alpha)-\lambda(2-\alpha-\alpha\delta\Delta\delta)\geq\lambda^2.$$

Hence, from Lemma 5.7.1 and from the properties of ψ, for $x\in\Omega\setminus\overline{\omega_0}$ we have

$$T_3\geq\frac{\phi}{r_0^\lambda}\Big(\lambda^2\delta^{\lambda-2}+2\lambda^2\delta^{\lambda-1}(2-\alpha)(\rho\psi_1-\rho D_{\psi_1}\delta)+\lambda^2\delta^\lambda(2-\alpha)|\nabla\psi|^2-\lambda\alpha\delta^\lambda|D^2\psi|_\infty$$
$$-2\lambda\delta^{\lambda-2}|D^2\delta|_\infty-2\lambda\delta^\lambda|D^2\psi|_\infty\Big)|\nabla\tau|^2$$
$$\geq\lambda^2\frac{\phi}{r_0^\lambda}\left[\delta^{\lambda-2}+\delta^\lambda\left(\rho^2|\nabla\psi_1|^2-2\rho D_{\psi_1}-\frac{2+\alpha}{\lambda}|D^2\psi|_\infty-2\frac{|D^2\delta|_\infty}{\delta^2\lambda}\right)\right]|\nabla\tau|^2$$
$$\geq\lambda^2\frac{\phi}{r_0^\lambda}\left[\delta^{\lambda-2}+\delta^\lambda\left(\rho^2\rho_0^2-2\rho D_{\psi_1}\right)\right]|\nabla\tau|^2\geq\lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2+\lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda|\nabla\tau|^2,$$

for λ large enough and ρ as in (5.3.8). Concerning (5.7.31), once again the proof is trivial and we omit it here. □

Proof of Proposition 5.7.6. We have

$$|\nabla\tau|^2=4\delta^2\psi^2+\delta^4|\nabla\psi|^2+\lambda^2\left(\frac{\phi}{r_0^\lambda}\right)^2\left(\delta^{2\lambda-2}+\delta^{2\lambda}|\nabla\psi|^2+2\delta^{2\lambda-1}(\nabla\delta\cdot\nabla\psi)\right)+4\delta^3(\nabla\delta\cdot\nabla\psi)$$
$$+\lambda\frac{\phi}{r_0^\lambda}\left(2\delta^{2+\lambda}|\nabla\psi|^2+4\delta^\lambda\psi+2(1+2\psi)\delta^{1+\lambda}(\nabla\delta\cdot\nabla\psi)\right).\tag{5.7.35}$$

Now we observe that, for r_0 as in (5.3.9), and since $\psi>1$, we have

$$3\delta^2\psi^2+4\delta^3(\nabla\delta\cdot\nabla\psi)\geq\delta^2(3\psi^2-4\delta|\nabla\psi|)\geq\delta^2(3-4r_0|\nabla\psi|)\geq 0,$$

$$2\delta^{2+\lambda}|\nabla\psi|^2+4\delta^\lambda\psi+2(1+2\psi)\delta^{1+\lambda}(\nabla\delta\cdot\nabla\psi)\geq 2\delta^\lambda\left(2\psi-\delta^2|\nabla\psi|^2-(1+2\psi)\delta(\nabla\delta\cdot\nabla\psi)\right)$$
$$\geq 2\delta^\lambda\left(2-r_0\left(|\nabla\psi|^2+(1+2\psi)|\nabla\psi|\right)\right)\geq 0,$$

and

$$\delta^{2\lambda-2}+\delta^{2\lambda}|\nabla\psi|^2+2\delta^{2\lambda-1}(\nabla\delta\cdot\nabla\psi)=\delta^{2\lambda-2}\left(1+\delta^2|\nabla\psi|^2+2\delta(\nabla\delta\cdot\nabla\psi)\right)$$
$$\geq\delta^{2\lambda-2}\left(1-\delta^2|\nabla\psi|^2-2\delta|\nabla\psi|\right)\geq\delta^{2\lambda-2}\left(1-r_0\left(|\nabla\psi|^2+2|\nabla\psi|\right)\right)\geq 0.$$

Therefore, (5.7.32) immediately follows.

Let us now prove (5.7.33). Firstly, we observe that, thanks to Lemma 5.7.1 and to the properties of ψ, we get

$$\delta^{2\lambda-2}+\delta^{2\lambda}|\nabla\psi|^2+2\delta^{2\lambda-1}(\nabla\delta\cdot\nabla\psi)\geq\delta^{2\lambda}\left(|\nabla\psi|^2+\frac{2}{\delta}(\nabla\delta\cdot\nabla\psi)\right)$$
$$\geq\delta^{2\lambda}\left(\rho^2\rho_0^2-\frac{2\rho D_{\psi_1}}{r_0}\right)\geq\frac{\rho^2\rho_0^2}{2}\delta^{2\lambda},$$

for all $x\in\tilde{\mathcal{O}}$ and for ρ as in (5.3.8). Moreover,

$$2\delta^{2+\lambda}|\nabla\psi|^2+4\delta^{\lambda}\psi+2(1+2\psi)\delta^{1+\lambda}(\nabla\delta\cdot\nabla\psi)\geq-2(1+2\psi)\rho D_{\psi_1}\delta^{\lambda+1};$$

hence

$$|\nabla\tau|^2\geq\lambda^2\frac{\rho^2\rho_0^2}{2}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2-2(1+2\psi)\rho D_{\psi_1}R_\Omega\left(\frac{\delta}{r_0}\right)^{\lambda}\phi.$$

Now, since by definition $\lambda\psi\leq\phi$,

$$\lambda^2\frac{\rho^2\rho_0^2}{4}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2-2(1+2\psi)\rho D_{\psi_1}R_\Omega\left(\frac{\delta}{r_0}\right)^{\lambda}\phi$$
$$=\frac{\rho^2\rho_0^2}{4}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2\left(\lambda^2-\lambda\frac{8(1+2\psi)\rho D_{\psi_1}R_\Omega}{\rho^2\rho_0^2}\left(\frac{r_0}{\delta}\right)^{\lambda}\frac{1}{\phi}\right)$$
$$\geq\frac{\rho^2\rho_0^2}{4}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2\left(\lambda^2-\lambda\frac{24\psi\rho D_{\psi_1}R_\Omega}{\rho^2\rho_0^2}\frac{1}{\phi}\right)\geq\frac{\rho^2\rho_0^2}{2}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2\left(\lambda^2-\frac{\lambda\psi}{\phi}\right)$$
$$\geq\frac{\rho^2\rho_0^2}{2}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2\left(\lambda^2-1\right),$$

for ρ as in (5.3.8). Therefore we can conclude

$$|\nabla\tau|^2\geq\lambda^2\frac{\rho^2\rho_0^2}{4}\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2,$$

which implies (5.7.34), again for ρ as in (5.3.8). □

5.7.1 Proof of the Lemmas of Section 5.6

Proof of Lemma 5.6.1. To simplify the presentation, we define

$$\mathrm{S}_1:=\Delta z,\qquad \mathrm{S}_2:=\frac{\mu}{\delta^2}z,\qquad \mathrm{S}_3:=\left(R\sigma_t+R^2|\nabla\sigma|^2\right)z,$$
$$\mathrm{A}_1:=z_t,\qquad \mathrm{A}_2:=2R\,\nabla\sigma\cdot\nabla z,\qquad \mathrm{A}_3:=R\Delta\sigma(1+\alpha)z,$$

and we denote by $I_{i,j}$, $i,j=1,2,3$, the scalar product $\langle \mathrm{S}_i, \mathrm{A}_j\rangle_{L^2(Q)}$. We compute each term separately. Moreover, the computations for $I_{1,j}$ and $I_{3,j}$, $j=1,2,3$, are the same as in [53, Lemma 2.4] and we will omit them here.

Computations for $I_{2,1}$: Due to the boundary conditions (5.6.3), we immediately have

$$I_{2,1} = \frac{\mu}{2}\int_Q \frac{\partial_t(z^2)}{\delta^2}\,dxdt = \frac{\mu}{2}\int_\Omega \frac{z^2}{\delta^2}\Big|_0^T\,dx - \frac{\mu}{2}\int_Q z^2\partial_t\left(\frac{1}{\delta^2}\right)\,dxdt = 0.$$

Computations for $I_{2,2}$: Applying integration by parts and (5.6.5) we have

$$I_{2,2} = R\mu\int_Q \frac{1}{\delta^2}\left(\nabla\sigma\cdot\nabla(z^2)\right)\,dxdt = -R\mu\int_Q \Delta\sigma\frac{z^2}{\delta^2}\,dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma)\frac{z^2}{\delta^3}\,dxdt.$$

Computations for $I_{2,3}$:

$$I_{2,3} = R\mu\int_Q \Delta\sigma(1+\alpha)\frac{z^2}{\delta^2}\,dxdt.$$

Identity (5.6.8) follows immediately. □

Proof of Lemma 5.6.2. It is sufficient to prove that $\nabla\sigma\cdot n = 0$ for all $(x,t)\in\Gamma\times(0,T)$ and $\lambda>1$. First of all, we have

$$\nabla\sigma = \theta\left(-2\delta\psi\nabla\delta - \delta^2\nabla\psi - \frac{\lambda}{r_0^\lambda}\left(\delta^{\lambda-1}\nabla\delta + \delta^\lambda\nabla\psi\right)\phi\right).$$

Moreover, because of the assumptions that we made on the function ψ, for any $x\in\Gamma$ we have $\nabla\psi\cdot n = -|\nabla\psi|$; furthermore, it is a classical property of the distance function that $\nabla\delta\cdot n = -1$. Therefore,

$$\begin{aligned}\nabla\sigma\cdot n &= \theta\left(-2\delta\psi(\nabla\delta\cdot n) + \delta^2|\nabla\psi| - \frac{\lambda}{r_0^\lambda}\left(\delta^{\lambda-1}\nabla\delta\cdot n - \delta^\lambda|\nabla\psi|\right)\phi\right)\\ &= \theta\left(2\delta\psi + \delta^2|\nabla\psi| + \frac{\lambda}{r_0^\lambda}\delta^{\lambda-1}\left(1+\delta|\nabla\psi|\right)\phi\right).\end{aligned}$$

It is thus evident that, for any $\lambda>1$, $\nabla\sigma\cdot n = 0$ on $\Gamma\times(0,T)$. □

Proof of Lemma 5.6.3. We split I_l in two parts, $I_l = I_l^1 + I_l^2$, where

$$I_l^1 = -2R\int_Q D^2\sigma(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma|\nabla z|^2\,dxdt + 2R\mu\int_Q(\nabla\delta\cdot\nabla\sigma)\frac{z^2}{\delta^3}\,dxdt, \quad (5.7.36)$$

$$\begin{aligned}I_l^2 = &-\frac{R}{2}\int_Q \Delta^2\sigma(1+\alpha)z^2\,dxdt + R\int_Q\left(\nabla(\Delta\sigma)\cdot\nabla\alpha\right)z^2\,dxdt + \frac{R}{2}\int_Q\Delta\sigma\Delta\alpha z^2\,dxdt\\ &+ R\mu\int_Q \alpha\Delta\sigma\frac{z^2}{\delta^2}\,dxdt.\end{aligned} \quad (5.7.37)$$

Moreover, we also split $I_l^1 = I_{l,\delta}^1 + I_{l,\phi}^1$ where

$$I_{l,\delta}^1 = -2R\int_Q D^2\sigma_\delta(\nabla z, \nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma_\delta)\frac{z^2}{\delta^3}\,dxdt, \tag{5.7.38}$$

$$I_{l,\phi}^1 = -2R\int_Q D^2\sigma_\phi(\nabla z, \nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\phi|\nabla z|^2\,dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma_\phi)\frac{z^2}{\delta^3}\,dxdt. \tag{5.7.39}$$

Estimates for $I_{l,\delta}^1$: From (5.7.5) and (5.7.6) we have

$$\begin{aligned}
I_{l,\delta}^1 &= 4R\int_Q \theta\psi(\nabla\delta\cdot\nabla z)^2\,dxdt + 4R\int_Q \theta\psi\delta D^2\delta(\nabla z,\nabla z)\,dxdt + R\int_Q \theta\delta^2 D^2\psi(\nabla z,\nabla z)\,dxdt \\
&\quad + 8R\int_Q \theta\delta(\nabla\delta\cdot\nabla z)(\nabla\psi\cdot\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt - 4R\mu\int_Q \theta\psi\frac{z^2}{\delta^2}\,dxdt \\
&\quad - 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_{l,\delta}^1 &\geq 4R\int_Q \theta\psi\left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2}\right)dxdt - 8R\int_Q \theta\psi|\nabla z|^2\,dxdt \\
&\quad + 8R\int_Q \theta\delta(\nabla\delta\cdot\nabla z)(\nabla\psi\cdot\nabla z)\,dxdt + R\int_Q \theta\delta^2 D^2\psi(\nabla z,\nabla z)\,dxdt \\
&\quad - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt - 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt + 4R\int_Q \theta\psi\delta D^2\delta(\nabla z,\nabla z)\,dxdt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_{l,\delta}^1 &\geq 4R\int_Q \theta\psi\left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2}\right)dxdt - 8R\int_Q \theta\psi|\nabla z|^2\,dxdt - 4R|D^2\delta|_\infty\int_Q \theta\psi|\nabla z|^2\,dxdt \\
&\quad - 8R|D\psi|_\infty R_\Omega\int_Q \theta|\nabla z|^2\,dxdt - R|D^2\psi|_\infty R_\Omega^2\int_Q \theta|\nabla z|^2\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt \\
&\quad - 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt \\
&\geq 4R\int_Q \theta\psi\left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2}\right)dxdt - RM_1\int_Q \theta|\nabla z|^2\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt \\
&\quad - 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt.
\end{aligned}$$

where $M_1 = M_1(\mu,\psi,\Omega)$ is a positive constant.

Next, we estimate the first term in the expression above applying the Hardy-Poincaré inequality (5.2.4). First of all, by integration by parts we obtain the identities

$$\begin{aligned}
&\int_\Omega z(\nabla\psi\cdot\nabla z)\,dx = -\frac{1}{2}\int_\Omega z^2\Delta\psi\,dx, \\
&\int_\Omega \delta^{2-\gamma}z(\nabla\psi\cdot\nabla z)\,dx = -\frac{1}{2}\int_\Omega \delta^{2-\gamma}\Delta\psi z^2\,dx - \frac{2-\gamma}{2}\int_\Omega \delta^{1-\gamma}(\nabla\delta\cdot\nabla\psi)\,dx.
\end{aligned}$$

Secondly, we apply (5.2.4) for $u := z\sqrt{\psi}$ and, after integrating in time, we get

$$
\begin{aligned}
&A_4\int_Q \theta\psi z^2 dxdx + \int_Q \theta\psi\left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2}\right)dxdt + \frac{1}{4}\int_Q \theta\frac{|\nabla\psi|^2}{\psi}z^2 dxdt - \frac{1}{2}\int_Q \theta z^2\Delta\psi dxdt\\
&\geq A_5\int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + A_1\frac{z^2}{\delta^\gamma}\right)dxdt + \frac{A_5}{4}\int_Q \theta\delta^{2-\gamma}\frac{|\nabla\psi|^2}{\psi}z^2\,dxdt\\
&\quad - \frac{A_5}{2}\int_Q \theta\delta^{2-\gamma}z^2\Delta\psi\,dxdt - A_5\frac{2-\gamma}{2}\int_Q \theta\delta^{1-\gamma}(\nabla\delta\cdot\nabla\psi)z^2\,dxdt,
\end{aligned}
$$

where A_1, A_4 and A_5 are the constants of Proposition 5.2.4. Now, since $\psi > 1$, for r_0 as in (5.3.9) we have

$$
\frac{A_5\psi}{4\delta^\gamma} \geq \frac{A_5}{4\delta^\gamma} \geq \frac{A_5}{2}(2-\gamma)\delta^{1-\gamma}|D\psi|_\infty, \quad \forall x \in \Omega_{r_0};
$$

therefore,

$$
\begin{aligned}
&\frac{A_5}{2}\int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + \frac{1}{2}\frac{z^2}{\delta^\gamma}\right)dxdt - A_5\frac{2-\gamma}{2}\int_Q \theta\delta^{1-\gamma}(\nabla\delta\cdot\nabla\psi)z^2\,dxdt\\
&\qquad\geq -\frac{A_5}{2}(2-\gamma)|D\psi|_\infty\left|\sup_{\delta>r_0}\delta^{1-\gamma}\right|\int_{\tilde{\mathcal{O}}\times(0,T)} \theta z^2\,dxdt.
\end{aligned}
$$

Combing the two expressions above, we finally obtain

$$
\int_Q \theta\psi\left(|\nabla z|^2 - \mu^*\frac{z^2}{\delta^2}\right)dxdt \geq \frac{A_5}{2}\int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + \frac{1}{2}\frac{z^2}{\delta^\gamma}\right)dxdt - A_6\int_Q \theta z^2\,dxdx,
$$

where

$$
A_6 := \frac{A_5}{4}\left(R_\Omega^{2-\gamma}|D\psi|_\infty^2 + 2R_\Omega^{2-\gamma} + 2(2-\gamma)|D\psi|_\infty\left|\sup_{\delta>r_0}\delta^{1-\gamma}\right|\right).
$$

Therefore

$$
\begin{aligned}
I^1_{l,\delta} \geq{}& M_2 R\int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma}\right)dxdt - RM_1\int_Q \theta|\nabla z|^2\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt\\
&- 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt - A_6 R\int_Q \theta z^2\,dxdt.
\end{aligned}
$$

Since $\gamma > 1$, there exists a constant $M_2 > 0$, not depending on r_0, such that for r_0 as in (5.3.9) we have

$$
\frac{2|\mu||D\psi|_\infty}{\delta} \leq \frac{M_2}{2\delta^\gamma}, \quad \forall x \in \Omega_{r_0}; \tag{5.7.40}
$$

knowing this, we can finally conclude

$$
\begin{aligned}
I^1_{l,\delta} \geq{}& B_1 R\int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma}\right)dxdt - RM_1\int_Q \theta|\nabla z|^2 dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2 dxdt\\
&- A_6 R\int_Q \theta z^2\,dxdx,
\end{aligned} \tag{5.7.41}
$$

where $B_1 := M_2/2$.

Estimates for $I^1_{l,\phi}$: In order to get rid of the gradient terms with negative signs in (5.7.41), we introduce the quantity

$$\begin{aligned}\mathcal{T} =& I^1_{l,\phi} - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt - RM_1\int_Q \theta|\nabla z|^2\,dxdt\\ =& -2R\int_Q D^2\sigma_\phi(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\phi|\nabla z|^2\,dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma_\phi)\frac{z^2}{\delta^3}\,dxdt\\ &- R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt - RM_1\int_Q \theta|\nabla z|^2\,dxdt, \qquad (5.7.42)\end{aligned}$$

and we need to estimate it from below. At this purpose we notice that, according to Propositions 5.7.1 and 5.7.2 we remark that

$$2D^2\tau_\phi(\nabla z,\nabla z) + \alpha\Delta\tau_\phi|\nabla z|^2 + \alpha\Delta\tau_\delta|\nabla z|^2 \geq \lambda\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi|\nabla z|^2, \qquad \forall x\in\Omega_{r_0},$$

$$\left|2D^2\tau_\phi(\nabla z,\nabla z) + \alpha\Delta\tau_\phi|\nabla z|^2 + (\alpha\Delta\tau_\delta - M_1)|\nabla z|^2\right| \leq M_2\lambda^2\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla z|^2, \qquad \forall x\in\omega_0,$$

$$2D^2\tau_\phi(\nabla z,\nabla z) + \alpha\Delta\tau_\phi|\nabla z|^2 + (\alpha\Delta\tau_\delta - M_1)|\nabla z|^2 \geq M_3\lambda^2\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla z|^2, \qquad \forall x\in\mathcal{O},$$

for λ large enough and for some positive constants M_2 and M_3 not depending on λ. On the other hand, there exists a positive constant M_4, again not depending on λ, such that it holds

$$\left|\frac{2|\mu||(\nabla\delta\cdot\nabla\tau_\phi)|}{\delta^3}\right| \leq M_4\lambda\left(\frac{\delta}{r_0}\right)^{\lambda-4}\phi, \quad \forall x\in\Omega.$$

Therefore it follows

$$\begin{aligned}\mathcal{T} \geq& \frac{\lambda R}{2}\int_{\Omega_{r_0}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^{\lambda-2}|\nabla z|^2\,dxdt - M_2\lambda^2 R\int_{\omega_0\times(0,T)}\theta\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla z|^2\,dxdt\\ &+ M_3\lambda^2 R\int_{\mathcal{O}\times(0,T)}\theta\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla z|^2\,dxdt - M_4\lambda R\int_Q \theta\left(\frac{\delta}{r_0}\right)^{\lambda-4}\phi z^2\,dxdt,\end{aligned}$$

for λ large enough. Joining the two expression obtained for $I^1_{l,\delta}$ and $\mathcal{T}$ we finally have

$$\begin{aligned}I^1_l \geq& B_1R\int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma}\right)dxdt - A_6R\int_Q \theta z^2\,dxdx\\ &+ \frac{\lambda R}{2}\int_{\Omega_{r_0}\times(0,T)}\theta\left(\frac{\delta}{r_0}\right)^{\lambda-2}|\nabla z|^2\,dxdt - B_2\lambda^2R\int_{\omega_0\times(0,T)}\theta\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla z|^2\,dxdt\\ &+ B_3\lambda^2R\int_{\mathcal{O}\times(0,T)}\theta\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla z|^2\,dxdt - M_5\lambda R\int_Q\theta\left(\frac{\delta}{r_0}\right)^{\lambda-4}\phi z^2\,dxdt. \qquad (5.7.43)\end{aligned}$$

Estimates for I_l^2: Using the fact that the support of α is located away from the origin, we note that there exists a positive constant A_λ such that, for all $x \in \Omega$,

$$\left|\alpha \frac{\Delta \tau_\delta}{\delta^2}\right|, \left|\alpha \frac{\Delta \tau_\phi}{\delta^2}\right|, |\Delta\alpha\Delta\tau_\delta|, |\Delta\alpha\Delta\tau_\phi|, |\nabla(\Delta\tau_\delta)\cdot\nabla\alpha|, |\nabla(\Delta\tau_\phi)\cdot\nabla\alpha|, |\Delta^2\tau_\phi(1+\alpha)| \leq A_\lambda.$$

Moreover, there exists another positive constant Υ such that

$$|\Delta^2\tau_\delta(1+\alpha)| \leq \frac{2\Upsilon}{\delta^2}, \quad \forall x \in \Omega.$$

Hence

$$I_l^2 \geq -A_\lambda R \int_Q \theta z^2 \, dxdt - \Upsilon R \int_Q \theta |\nabla z|^2 \, dxdt$$

and, for λ large enough, we finally have (5.6.13) with $B_\lambda := A_\lambda + A_6 + M_5 \lambda \sup_{x\in\Omega}\{(\delta/r_0)^{\lambda-4}\phi\}$.

□

Proof of Lemma 5.6.4. We split $I_{nl} = I_{nl,1} + I_{nl,2}$, where $I_{nl,1}$ indicates the integrals in I_{nl} restricted to Ω_{r_0}, while $I_{nl,2}$ are the terms in I_{nl} restricted to $\tilde{\mathcal{O}}$. Moreover, if we put $\sigma = -\theta\tau$, then I_{nl} can be rewritten as

$$I_{nl} = 2R^3 \int_Q \theta^3 D^2\tau(\nabla\tau,\nabla\tau) z^2 \, dxdt - R^3 \int_Q \theta^3 \alpha \Delta\tau |\nabla\tau|^2 z^2 \, dxdt - \frac{R^2}{2}\int_Q \theta^2\alpha^2|\Delta\tau|^2 z^2 \, dxdt.$$

Computations for $I_{nl,1}$: From (5.7.28), (5.7.30) and (5.7.32), for any $x \in \Omega_{r_0}$ we have

$$\begin{aligned}
T_2 + T_3 &\geq \lambda^2 \left(\frac{\phi}{r_0^\lambda}\delta^{\lambda-2} + \left(\frac{\delta}{r_0}\right)^\lambda \phi\right)|\nabla\tau|^2 - \lambda^2 \frac{\phi}{r_0^\lambda}|D\psi|_\infty^2 \left(D_3\psi^2 + D_4\right)\delta^{\lambda+2} \\
&= \lambda^2 \frac{\phi}{r_0^\lambda}\delta^{\lambda-2}\left(|\nabla\tau|^2 + \delta^2|\nabla\tau|^2 - |D\psi|_\infty^2\left(D_3\psi^2 + D_4\right)\delta^4\right) \\
&\geq \lambda^2 \frac{\phi}{r_0^\lambda}\delta^\lambda\left(1 - |D\psi|_\infty^2\left(D_3\psi^2+D_4\right)\delta^2\right) \geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda\left(1 - |D\psi|_\infty^2\left(D_3\psi^2+D_4\right)r_0^2\right) \geq 0,
\end{aligned}$$

for r_0 as in (5.3.9). Hence, using (5.7.25) and (5.7.32) we conclude

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2 \geq \delta^2, \quad \forall x \in \Omega_{r_0};$$

as a consequence,

$$I_{nl,1} \geq R^3 \int_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt - \frac{R^2}{2}\int_{\Omega_{r_0}\times(0,T)} \theta^2\alpha^2|\Delta\tau|^2 z^2 \, dxdt.$$

Moreover, since α is supported away from the boundary we also have

$$\alpha^2|\Delta\tau|^2 \leq A'_\lambda \delta^2, \quad \forall x \in \Omega_{r_0};$$

hence, finally, there exists $R_0 = R_0(\lambda)$ large enough such that, for any $R \geq R_0$

$$I_{nl,1} \geq \frac{R^3}{2}\int_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt.$$

Computations for $I_{nl,2}$: According to Propositions 5.7.3, 5.7.4 and 5.7.5 and to (5.7.33), for all $x \in \mathcal{O}$ we have

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2 \geq G_1\lambda^2\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla\tau|^2 \geq G_1\lambda^4\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3.$$

In addition, it holds

$$\alpha^2|\Delta\tau|^2 \leq G_2\lambda^4\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2, \qquad \forall x \in \tilde{\mathcal{O}},$$
$$\left|2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2\right| \leq G_3\lambda^2\left(\frac{\delta}{r_0}\right)^{\lambda}\phi|\nabla\tau|^2 \leq G_4\lambda^4\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3, \qquad \forall x \in \omega_0.$$

The previous inequalities follows from (5.7.20), (5.7.21) and (5.7.34); the constants G_1, G_2, G_3 and G_4 are all positive and independent on λ. Therefore we obtain

$$I_{nl,2} \geq G_1\lambda^4R^3 \int\limits_{\mathcal{O}\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3z^2\,dxdt - G_4\lambda^4R^3 \int\limits_{\omega_0\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3z^2\,dxdt$$
$$- \frac{G_2}{2}\lambda^4R^2 \int\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^2\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2\,dxdt.$$

Joining now the two expressions we get for $I_{nl.1}$ and $I_{nl,2}$, we finally obtain that there exists $R_0 = R_0(\lambda)$ large enough such that for $R \geq R_0$

$$I_{nl} \geq \frac{R^3}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2z^2\,dxdt + G_5\lambda^4R^3 \int\limits_{\mathcal{O}\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3z^2\,dxdt$$
$$- G_6\lambda^4R^3 \int\limits_{\omega_0\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{3\lambda}\phi^3z^2\,dxdt,$$

where $G_5 := G_1/2$ and $G_6 := G_2/2 + G_4$. ☐

Proof of Lemma 5.6.5. According to the expression of θ, there exists a constant $\varsigma > 0$ such that

$$|\theta\theta_t| \leq \varsigma\theta^3, \quad |\theta_{tt}| \leq \varsigma\theta^{5/3};$$

on the other hand, from the definition of σ we obtain

$$|\Delta\sigma| \leq E_\lambda\theta, \quad |\sigma_t| \leq E_\lambda\theta_t, \qquad \forall x \in \Omega,$$
$$\partial_t\left(|\nabla\sigma|^2\right) \leq E_\lambda\theta\theta_t\delta^2, \qquad \forall x \in \Omega_{r_0},$$
$$\partial_t\left(|\nabla\sigma|^2\right) \leq E_\lambda\theta\theta_t\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2 \qquad \forall x \in \tilde{\mathcal{O}}, \tag{5.7.44}$$

for some positive constant E_λ large enough.

Since α is supported away from the boundary, we can write

$$R^2 \int_Q \left|\alpha\sigma_t\Delta\sigma z^2\right| dxdt \leq \frac{\varsigma E_\lambda^2}{r_0^2} R^2 \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2\, dxdt + \varsigma E_\lambda^2 R^2 \int\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\delta^2 z^2\, dxdt.$$

Furthermore, from (5.7.44) we obtain

$$R^2 \left|\int_Q \partial_t\left(|\nabla\sigma|^2\right) z^2\, dxdt\right| \leq \varsigma E_\lambda R^2 \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2\, dxdt + \varsigma E_\lambda R^2 \int\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{2\lambda} \phi^2 z^2\, dxdt.$$

Now we define

$$\Theta := -\frac{R}{2}\int_Q \sigma_{tt} z^2\, dxdt - B_\lambda R \int_Q \theta z^2\, dxdt,$$

where B_λ is the same introduced in Lemma 5.6.3. It is straightforward that there exists a positive constant F_λ such that

$$|\Theta| \leq 2F_\lambda R \int_Q \theta^{5/3} z^2\, dxdt.$$

Next, for $1 < q$, $q' < \infty$ such that $1/q + 1/q' = 1$ and $\ell > 0$ we can write

$$\int_Q \theta^{5/3} z^2\, dxdt = \int_Q \left(\ell\theta^{5/3-1/q'}\delta^{1/q'} z^{2/q}\right)\left(\frac{1}{\ell}\theta^{1/q'}\delta^{-1/q'} z^{2/q'}\right) dxdt;$$

choosing $q = 3$ and $q' = 3/2$ in the previous expression, and using Young's inequality, we obtain

$$\int_Q \theta^{5/3} z^2\, dxdt \leq \frac{\ell^3}{3}\int_Q \theta^3\delta^2 z^2\, dxdt + \frac{2R_\Omega^{\gamma-1}}{3\ell^{3/2}}\int_Q \theta\frac{z^2}{\delta^\gamma}\, dxdt,$$

for some positive parameter $\gamma \in (1,2)$. Therefore we have

$$|\Theta| \leq 2F_\lambda R\left(\frac{\ell^3}{3}\int_Q \theta^3\delta^2 z^2\, dxdt + \frac{2R_\Omega^{\gamma-1}}{3\ell^{3/2}}\int_Q \theta\frac{z^2}{\delta^\gamma}\, dxdt\right).$$

Consequently, it follows that

$$|I_r| \leq G_\lambda\Bigg(R^2\int_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2\, dxdt + \ell^3 R\int_Q \theta^3\delta^2 z^2\, dxdt + \frac{R}{\ell^{3/2}}\int_Q \theta\frac{z^2}{\delta^\gamma}\, dxdt + R^2\int_{\tilde{\mathcal{O}}} \theta^3\left(\frac{\delta}{r_0}\right)^{2\lambda} z^2\, dxdt\Bigg),$$

for some new constant $G_\lambda > 0$. Take now ℓ such that $G_\lambda/\ell^{3/2} = B_1/2$; then there exists $R_0 = R_0(\lambda)$ such that for any $R \geq R_0$ (5.6.16) holds.

We conclude pointing out that, if we choose an exponent $k < 3$ for the function θ in the definition of our weight σ (see Section 5.3), it is straightforward to check that some of the passages in the computations above are not true anymore and there are terms in the expression I_r that we are not able to handle. Therefore, the value $k = 3$ turns out to be sharp for obtaining our Carleman inequality. □

5.7.2 Proof of the Propositions of Section 5.2

Proof of Proposition 5.2.3. We split the proof in two parts: firstly, we derive (5.2.3) in Ω_{r_0} and, in a second moment, we extend the result to the whole Ω.

Step 1. inequality on Ω_{r_0}: Let us consider a smooth function $\phi > 0$ that satisfies

$$-\Delta\phi \geq \mu\frac{\phi}{\delta^2} + \phi^p, \quad \forall p \in \left[1, \frac{N-k+2}{N-k-2}\right), \tag{5.7.45}$$

for $k \in (1, N-2)$. According to [57], for $\delta < 1$ the function

$$\delta^{-A_k^{1/2}(1-\delta^{1/2})}\left(1 + \frac{1}{\log\delta}\right), \quad A_k := \left(\frac{N-k-2}{2}\right)^2 \tag{5.7.46}$$

has this property. Hence, for any $x \in \Omega_{r_0}$ with $r_0 \leq 1$ we define $v := \phi z$ for $z \in C_0^\infty(\Omega_{r_0})$; in particular, $v \in C_0^\infty(\Omega_{r_0})$ and

$$|\nabla v|^2 = \phi^2|\nabla z|^2 + z^2|\nabla\phi|^2 + \frac{1}{2}\nabla(\phi^2)\cdot\nabla(z^2).$$

By applying integration by parts, it is simply a matter of computations to show

$$\int_{\Omega_{r_0}} |\nabla v|^2\,dx = \int_{\Omega_{r_0}} \phi^2|\nabla z|^2\,dx - \int_{\Omega_{r_0}} \frac{\Delta\phi}{\phi}v^2\,dx$$

and

$$\begin{aligned}\frac{1}{2}\int_{\Omega_{r_0}} \delta^{2-\gamma}\nabla(\phi^2)\cdot\nabla(z^2)\,dx = &-(2-\gamma)\int_{\Omega_{r_0}} \delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx - \int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\Delta\phi}{\phi}v^2\,dx\\ &- \int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla\phi|^2z^2\,dx.\end{aligned}$$

The two identities above implies

$$\begin{aligned}\int_{\Omega_{r_0}} \delta^{2-\gamma}\phi^2|\nabla z|^2\,dx &\leq R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \phi^2|\nabla z|^2\,dx = R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2 + \frac{\Delta\phi}{\phi}v^2\right)dx\\ &\leq R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx - R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\phi^{p-1}v^2\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2\,dx = &\int_{\Omega_{r_0}} \delta^{2-\gamma}\phi^2|\nabla z|^2\,dx - (2-\gamma)\int_{\Omega_{r_0}} \delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx\\ &- \int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\Delta\phi}{\phi}v^2\,dx;\end{aligned}$$

hence

$$\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2\,dx \le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx - R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\phi^{p-1}v^2\,dx$$
$$+\mu\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{v^2}{\delta^2}\,dx+\int_{\Omega_{r_0}}\delta^{2-\gamma}\phi^{p-1}v^2\,dx$$
$$-(2-\gamma)\int_{\Omega_{r_0}}\delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx.$$

Now, again by integration by parts we have

$$-(2-\gamma)\int_{\Omega_{r_0}}\delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx$$
$$=\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\Delta\phi}{\phi}v^2\,dx-\int_{\Omega_{r_0}}\frac{\delta^{2-\gamma}}{\phi^2}|\nabla\phi|^2v^2\,dx+2\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx$$
$$\le-\mu\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{v^2}{\delta^2}\,dx-\int_{\Omega_{r_0}}\delta^{2-\gamma}\phi^{p-1}v^2\,dx+2\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx;$$

therefore

$$\int_{\Omega_{r_0}}\delta^{2-\gamma}|\nabla v|^2\,dx$$
$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx-R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\phi^{p-1}v^2\,dx+2\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx$$
$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx+P_1\int_{\Omega_{r_0}}\phi^{p-1}v^2\,dx+2\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx$$
$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx+P_2\int_{\Omega_{r_0}}v^2\,dx+2\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx.$$

By definition of ϕ we have

$$\frac{\nabla\phi\cdot\nabla v}{\phi}=\left(1+\frac{1}{\log\delta}\right)^{-1}\left(\frac{A_k^{1/2}}{2}\frac{\log\delta}{\delta^{1/2}}-A_k^{1/2}\frac{1-\delta^{1/2}}{\delta}-\frac{1}{\delta\log^2\delta}\right)(\nabla\delta\cdot\nabla v);$$

plugging this expression in the inequality above we immediately get

$$\int_{\Omega_{r_0}}\delta^{2-\gamma}|\nabla v|^2dx\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx+P_2\int_{\Omega_{r_0}}v^2dx$$
$$+P_3\int_{\Omega_{r_0}}\delta^{2-\gamma}\frac{\log\delta}{\delta^{1/2}}(\nabla\delta\cdot\nabla v)v\,dx$$

with

$$P_3:=A_k^{1/2}\left|\sup_{x\in\Omega_{r_0}}\left(1+\frac{1}{\log\delta}\right)^{-1}\right|.$$

Now, using another time integration by parts, and since $\log\delta < \delta^{3/2}$, we finally obtain

$$\begin{aligned}\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2\,dx &\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx + P_2\int_{\Omega_{r_0}} v^2\,dx\\ &\quad + P_3\int_{\Omega_{r_0}}\delta^{3-\gamma}(\nabla\delta\cdot\nabla(v^2))\,dx\\ &\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}}\left(|\nabla v|^2-\mu\frac{v^2}{\delta^2}\right)dx + A_2\int_{\Omega_{r_0}} v^2\,dx.\end{aligned}$$

Step 2. inequality on Ω: We apply a cut-off argument to recover the validity of the inequality on the whole Ω. More in details, we consider a function $\psi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi(x)=\begin{cases}1, & \forall x\in\Omega_{r_0/2},\\ 0, & \forall x\in\Omega\setminus\Omega_{r_0},\end{cases}$$

and we split $v\in C_0^\infty(\Omega)$ as $v=\psi v+(1-\psi)v := v_1+v_2$. Thus, we get

$$\int_\Omega \delta^{2-\gamma}|\nabla v|^2\,dx = \int_{\Omega_{r_0}}\delta^{2-\gamma}|\nabla v_1|^2\,dx + \int\limits_{\Omega\setminus\Omega_{r_0/2}}\delta^{2-\gamma}|\nabla v_2|^2\,dx + 2\int\limits_{\Omega_{r_0}\setminus\Omega_{r_0/2}}\delta^{2-\gamma}(\nabla v_1\cdot\nabla v_2)\,dx.$$

Applying (5.2.3) to the previous identity we obtain

$$\begin{aligned}\int_\Omega \delta^{2-\gamma}|\nabla v|^2\,dx &\le R_\Omega^{2-\gamma}\left(\int_\Omega|\nabla v|^2\,dx-\mu\int_{\Omega_{r_0}}\frac{v^2}{\delta^2}\,dx\right)\\ &\quad-\int\limits_{\Omega_{r_0}\setminus\Omega_{r_0/2}} 2\left(R_\Omega^{2-\gamma}-\delta^{2-\gamma}\right)(\nabla v_1\cdot\nabla v_2)\,dx + J_1\int_\Omega v^2\,dx.\end{aligned}$$

As shown in [34, Lemma 5.1], for a smooth function $q: C^\infty(\Omega)\to\mathbb{R}$ which is bounded and non-negative, there exists a constant $C>0$ depending on Ω and q such that it holds

$$\int_\Omega q(x)(\nabla v_1\cdot\nabla v_2)\,dx \ge -C\int_\Omega v^2\,dx; \tag{5.7.47}$$

hence, considering (5.7.47) with

$$q = 2\left(R_\Omega^{2-\gamma}-\delta^{2-\gamma}\right)\Big|_{\Omega_{r_0}\setminus\Omega_{r_0/2}},$$

we get

$$\int_\Omega \delta^{2-\gamma}|\nabla v|^2\,dx \le R_\Omega^{2-\gamma}\left(\int_\Omega|\nabla v|^2\,dx-\mu\int_{\Omega_{r_0}}\frac{v^2}{\delta^2}\,dx\right)+J_2\int_\Omega v^2\,dx. \tag{5.7.48}$$

On the other hand we have

$$\int_{\Omega_{r_0}}\frac{v^2}{\delta^2}\,dx \ge \int_\Omega\frac{v^2}{\delta^2}\,dx - J_3\int_\Omega v^2\,dx.$$

Plugging this last inequality in (5.7.48), we finally obtain (5.2.3). □

Chapter 6

Conclusions and open problems

In this thesis, we have treated the following problems:

- In Chapter 3, we analysed the interior controllability problem for non-local Schrödinger and wave equations in which the classical Laplace operator has been substituted by the fractional Laplacian $(-\Delta)^s$. We employed a L^2 control supported in a neighbourhood ω of the boundary of a bounded and $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$ and, using the Hilbert Uniqueness Method we obtained the following results:
  - null controllability of the Schrödinger equation, for $s \geq 1/2$;
  - null controllability of the wave equation, for $s \geq 1$.
- In Chapter 4, we addressed the boundary controllability for a one-dimensional heat equation involving a singular inverse-square potential, defined on the space interval $(0,1)$. Applying analogous results obtained in [76] for parabolic equations with variable degenerate coefficients, we obtained the null controllability of the equation by means of a L^2 control acting from the boundary point $x = 0$, which is also one of the singularity points for the potential.
- In Chapter 5, we treated the interior controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function. By means of a new Carleman inequality for the problem under analysis, we obtained the null controllability employing a L^2 control supported in a generic open subset ω of a bounded and C^2 domain $\Omega \subset \mathbb{R}^N$.

Related with the topics addressed in this thesis and with the results obtained, in what follows we present in a systematic way several open problems that, in our opinion, are of great interest.

6.1 Equations involving the fractional Laplacian with non-homogeneous boundary conditions

In Chapter 3, we considered evolution equations involving the fractional Laplacian with homogeneous boundary conditions. The main reason of this choice was that, for obtaining the controllability properties that we were seeking, we relied mostly on the theory developed by X. Ros-Oton and J. Serra ([117, 118, 119]), whose results hold for functions vanishing outside the domain of definition of the problems analysed. Moreover, we have to mention that, when we first started approaching this topic, the Pohozaev identity obtained in [119] was a very recent result and also the only one of this type available for non-local operators.

As we mentioned in Chapter 2, in a couple of very recent works ([143, 144]) M. Warma started analysing the elliptic problem for the regional fractional Laplacian on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$, with Neumann and Robin boundary conditions, developing a theory of existence and regularity of solutions. Moreover, he obtained a new Pohozaev identity which generalises the one of Ros-Oton and Serra.

In more detail, he proved that for functions u sufficiently smooth it holds the identity

$$\begin{aligned}\int_\Omega (x\cdot\nabla u)\,A^s_\Omega u\,dx = \frac{2s-N}{2}\int_\Omega uA^s_\Omega u\,dx + \frac{c_{N,s}}{2}\int_{\partial\Omega}\left(\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\right)(y\cdot\nu)\,d\sigma \\ - \frac{B_{N,s}}{2}\int_{\partial\Omega}(x\cdot\nu)\frac{\partial u}{\partial\nu}\mathcal{N}^{2-2s}u\,d\sigma + \frac{B_{N,s}(2s-N)}{2}\int_{\partial\Omega} u\mathcal{N}^{2-2s}u\,d\sigma,\end{aligned} \tag{6.1.1}$$

where $B_{N,s}$ is an explicit constant depending only on N and s while $\mathcal{N}^{2-2s}$ is a fractional version of the classical normal derivative defined as

$$\mathcal{N}^{2-2s}u(z) := -\lim_{t\to 0}\frac{du(z+\nu(z)t)}{dt}\,t^{2-2s}, \quad z\in\partial\Omega,$$

whenever this limit exists.

It would be therefore natural to apply these results for analysing, for instance, controllability properties for fractional Schrödinger and wave equations of the type of the ones presented in Chapter 3, but this time with non-homogeneous boundary conditions; in particular, the study of boundary controllability would be a very interesting problem.

6.2 Asymptotic analysis for the solutions of evolution equations with the fractional Laplacian

Geometric Optics expansion for the solutions of an evolution PDE is a very powerful tool that, if well developed, can provide relevant informations on propagation and dispersion properties and on the way in which these solutions interact with the boundaries of the domains one can consider, or with eventual interfaces (see, e.g. [55, 115]).

With the intent of better justifying the impossibility of controlling the fractional wave equation analysed in Chapter 3 when $s < 1$, with M. Warma we started approaching the problem from the point of view of asymptotic analysis, taking inspiration from the results presented in [115] for the local case.

Just for giving a preliminary clue of how this machinery works, let us consider the following one-dimensional wave equation involving the fractional Laplacian on $\mathbb{R}$

$$\Box^s u = u_{tt} + (-d_x^2)^s u = 0, \tag{6.2.1}$$

and let us look for approximate solutions with an ansatz of the type

$$u^\varepsilon(x,t) = e^{i[(\xi/\varepsilon)x + (\xi^s/\varepsilon^s)t]}\phi^\varepsilon(x,t), \quad \phi^\varepsilon(x,t) = \sum_{j=0}^{\infty} \varepsilon^j \phi_j(x,t), \tag{6.2.2}$$

with $\xi \in \mathbb{R}$ and where the functions ϕ_j have to be determined.

Therefore, we need to compute $\Box^s u^\varepsilon$, identifying the order, with respect to ε, of each one of the terms that we obtain. First of all, we can easily show that, for any $\alpha \in \mathbb{R}$, we have

$$(-d_x^2)^s e^{i\alpha x} = \alpha^{2s} e^{i\alpha x}; \tag{6.2.3}$$

indeed, by definition of fractional Laplacian

$$(-d_x^2)^s e^{i\alpha x} = c_{1,s}\, P.V. \int_{\mathbb{R}} \frac{e^{i\alpha x} - e^{i\alpha y}}{|x-y|^{1+2s}}\, dy = c_{1,s}\, e^{i\alpha x} P.V. \int_{\mathbb{R}} \frac{1 - e^{i\alpha(y-x)}}{|x-y|^{1+2s}}\, dy.$$

Now, applying the change of variables $z = \alpha(y-x)$, and using the definition of principal value and the expression for the constant $c_{1,s}$ given in [48, Section 3], we get

$$\begin{aligned}
(-d_x^2)^s e^{i\alpha x} &= c_{1,s}\alpha^{2s}\, e^{i\alpha x} P.V. \int_{\mathbb{R}} \frac{1-e^{iz}}{|z|^{1+2s}}\, dz = c_{1,s}\alpha^{2s}\, e^{i\alpha x} \lim_{\varepsilon\to 0^+} \int_{|z|>\varepsilon} \frac{1-e^{iz}}{|z|^{1+2s}}\, dz \\
&= c_{1,s}\alpha^{2s}\, e^{i\alpha x} \lim_{\varepsilon\to 0^+} \left(\int_{\varepsilon}^{+\infty} \frac{1-e^{iz}}{z^{1+2s}}\, dz + \int_{-\infty}^{\varepsilon} \frac{1-e^{iz}}{(-z)^{1+2s}}\, dz \right) \\
&= c_{1,s}\alpha^{2s}\, e^{i\alpha x} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{+\infty} \frac{2-2cos(z)}{z^{1+2s}}\, dz = c_{1,s}\alpha^{2s}\, e^{i\alpha x} \int_{\mathbb{R}} \frac{1-cos(z)}{|z|^{1+2s}}\, dz \\
&= c_{1,s}\alpha^{2s}\, e^{i\alpha x} c_{1,s}^{-1} = \alpha^{2s}\, e^{i\alpha x}.
\end{aligned}$$

Further, employing the formula (3.3.25) that we derived in Chapter 3 for the fractional Laplacian of the product of two functions, we can derive the following useful expressions

$$
\begin{aligned}
1.)\ (-d_x^2)^s(fg) &= (-d_x^2)^{\frac{s}{2}}(-d_x^2)^{\frac{s}{2}}(fg) = (-d_x^2)^{\frac{s}{2}}\Big[f(-d_x^2)^{\frac{s}{2}}g + R_1\Big] \\
&= f(-d_x^2)^{\frac{s}{2}}(-d_x^2)^{\frac{s}{2}}g + (-d_x^2)^{\frac{s}{2}}R_1 + R_2 = f(-d_x^2)^s g + (-d_x^2)^{\frac{s}{2}}R_1 + R_2
\end{aligned}
$$

$$
2.)\ (-d_x^2)^s(fg) = (-d_x^2)^{\frac{s}{2}}\Big[f(-d_x^2)^{\frac{s}{2}}g + R_1\Big] = (-d_x^2)^{\frac{s}{2}}f(-d_x^2)^{\frac{s}{2}}g + (-d_x^2)^{\frac{s}{2}}R_1 + R_3
$$

$$
3.)\ (-d_x^2)^s(fg) = (-d_x^2)^{\frac{s}{2}}\Big[g(-d_x^2)^{\frac{s}{2}}f + R_4\Big] = g(-d_x^2)^s f + (-d_x^2)^{\frac{s}{2}}R_4 + R_5. \tag{6.2.4}
$$

Summing the first an the third expression in (6.2.4) and subtracting from the result the second one we get

$$
(-d_x^2)^s(fg) = f(-d_x^2)^s g + g(-d_x^2)^s f - (-d_x^2)^{\frac{s}{2}}f(-d_x^2)^{\frac{s}{2}}g + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5). \tag{6.2.5}
$$

Now, using (6.2.5) with $f = \phi^\varepsilon$ and $g = u := e^{i(\xi x/\varepsilon + \xi^s t/\varepsilon^s)}$, and thanks to (6.2.3), we find

$$
(-d_x^2)^s u^\varepsilon = \frac{\xi^{2s}}{\varepsilon^{2s}}u\phi^\varepsilon + u(-d_x^2)^s\phi^\varepsilon - \frac{\xi^s}{\varepsilon^s}u(-d_x^2)^{\frac{s}{2}}\phi^\varepsilon + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5).
$$

Hence

$$
\begin{aligned}
\Box^s u^\varepsilon &= u\left[\frac{\xi^s}{\varepsilon^s}\Big(2i\phi_t^\varepsilon - (-d_x^2)^{\frac{s}{2}}\phi^\varepsilon\Big) + \Box^s\phi^\varepsilon\right] + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5) \\
&= \varepsilon^{-s}u\left[\xi^s\Big(2i\phi_t^\varepsilon - (-d_x^2)^{\frac{s}{2}}\phi^\varepsilon\Big) + \varepsilon^s\Box^s\phi^\varepsilon\right] + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5).
\end{aligned} \tag{6.2.6}
$$

The idea would now be to identify the order of each term appearing in (6.2.6), to find which are the equations satisfied by the ones of leading order and to properly compensate the lower order components. In that way, from (6.2.2) one can build quasi-solutions localised along rays, and employ them for studying, for instance, propagation and reflection properties.

6.3 Extension of the results of Chapter 4

In Chapter 4, we have been able to obtain the null controllability from $x = 0$ for the following one-dimensional heat equation

$$
u_t - u_{xx} - \frac{\mu}{x^2}u = 0, \quad (x,t) \in (0,1) \times (0,T),
$$

which involves a singular inverse-square potential whose singularity arises exactly at the boundary point in which the control is located.

However, the controllability result that we presented is not complete, in the sense that we were able to achieve it only for coefficients μ satisfying the condition $0 \leq \mu < 1/4$, while the equation is well posed for all $\mu \leq 1/4$.

The reason of this incompleteness is in the technique that we employed for obtaining the observability inequality for the adjoint system associated to our equation, that is essentially based on a observability result recently obtained in [76] for parabolic equations with variable degenerate coefficients of the following type

$$u_t - (x^\alpha u_x)_x = 0 \quad (x,t) \in (0,1) \times (0,T),\ \alpha \in (0,1).$$

Therefore, a first extension of the results presented in Chapter 4 would be the obtaining of the boundary controllability for the equation considered also in the two cases $\mu < 0$ and $\mu = 1/4$.

An approach that can be successful would be to derive an appropriate Carleman estimate for the adjoint problem associated. Of course, since our intention would be to obtain boundary controllability, this estimate would need to take into account the degeneracy of the normal derivative of the solution of the equation approaching the point $x = 0$.

However, this is not an easy problem. Since we showed that the first derivative of the function v solution of (4.1.9) has the following behaviour

$$v_x^2(x,t) \sim x^{-2\lambda}, \quad \text{as } x \to 0^+,$$

with λ as in (4.1.6), we believe that the weight to employ for obtaining the Carleman estimate should be in the form $\sigma(x,t) = \theta(t)p(x)$, with a function p involving the term $x^{2\lambda+1}$.

Nevertheless, this choice appears not to be a suitable one, since the quantity $2\lambda + 1$ becomes negative for $\mu < -3/4$, hence producing a weight σ which is not bounded approaching the boundary. On the other hand, to understand which function could allow to obtain the right boundary term in the inequality, without generating singularities, is not an elementary issue.

Finally, we remark that throughout the Chapter we had to work with initial data belonging to specific fractional Sobolev spaces, even if for the controllability of our equation we can employ a L^2 control. Once again, this fact is due to the technique that we used in our proof, which strongly depends on the results of [76]. On the other hand, it has been recently brought to our attention an interesting new work ([31]) in which the same problem as in [76] is addressed, obtaining new improved results. In particular, the authors managed to deal with L^2 initial data and to construct H^1 controls. Therefore, it would be worth to adapt our analysis to these new contribution, trying to extend our result to the more natural case of an initial data in L^2.

The problem treated in Chapter 4, apart from being interesting by itself, is also a preliminary step for the analysis of a more general issue, the one of the boundary controllability of the following heat equation

$$u_t - u_{xx} - \frac{\mu_1}{x^2}u - \frac{\mu_2}{(1-x)^2}u = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{6.3.1}$$

involving a singular inverse-square potential whose singularities arise all over the boundary of the space domain $(0,1)$.

Our main interest for equations like (6.3.1) originates from the fact that this is a one-dimensional version of equations with a potential that blows-up all over the boundary of the domain of definition, whose analysis has been addressed in Chapter 5 in the case of interior controllability.

First of all, we have to point out that the homogeneous Dirichlet boundary problem for an equation of the type of (6.3.1) is well-posed thanks to the multi-polar Hardy inequality

$$\int_0^1 z_x^2\,dx + M\int_0^1 z^2\,dx \geq \frac{1}{4}\int_0^1 \frac{z^2}{x^2}\,dx + \frac{1}{4}\int_0^1 \frac{z^2}{(1-x)^2}\,dx, \tag{6.3.2}$$

that can be proved starting from (4.1.4) and applying a C^∞ partition of the unity defined as follows (see also Figure 6.1 below)

$$\begin{cases} \phi_1 \equiv 0, & x \in (0,1/2] \\ \phi_1 \in (0,1), & x \in (1/2,3/4] \\ \phi_1 \equiv 1, & x \in (3/4,1) \end{cases}, \quad \begin{cases} \phi_2 \equiv 1, & x \in (0,1/4] \\ \phi_2 \in (0,1), & x \in (1/4,1/2] \\ \phi_2 \equiv 0, & x \in (1/2,1) \end{cases}, \quad \phi_3 := 1 - \phi_1 - \phi_2,$$

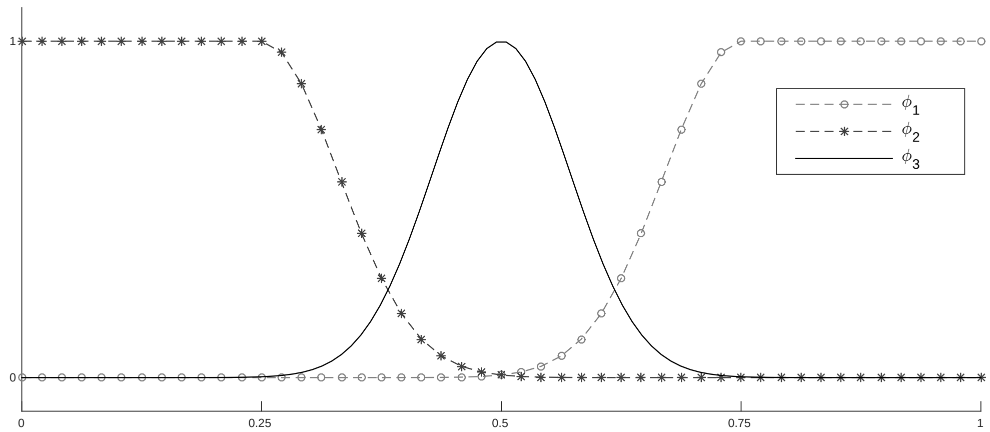

Figure 6.1: Graph of the partition of the unity employed for the proof of (6.3.2).

For more details see, for instance, [16].

The boundary controllability of (6.3.1), instead, is a very tricky issue, which is not trivial to address directly through a Carleman approach. Therefore, a good strategy would be to split the problem into two more simple ones.

At this purpose, we believe that all the analysis developed in the Chapter 4 can be adapted to the case of an equation with singular inverse-square potential arising at $x = 1$. In more detail, given the following parabolic equation

$$\begin{cases} u_t - u_{xx} - \dfrac{\mu}{(1-x)^2}u = 0, & (x,t) \in Q \\ u(0,t) = 0, \quad x^{-\lambda}u(x,t)\big|_{x=1} = f(t), & t \in (0,T) \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases} \tag{6.3.3}$$

we retain that, with the same kind of arguments employed for studying the equation (4.1.2), for all $0 \leq \mu < 1/4$ it would be possible to obtain an observability inequality in the form

$$\|v(x,0)\|^2_{\mathbf{H}^\beta_\alpha} \leq C_T \int_0^T \left[(1-x)^{2\lambda}v_x^2\right]\Big|_{x=0} dt,$$

where the space $\mathbf{H}^\beta_\alpha$ is defined as in (4.3.9), with α and β to be determined, and v is the solution of the adjoint system

$$\begin{cases} v_t + v_{xx} + \dfrac{\mu}{(1-x)^2}v = 0, & (x,t) \in Q \\ v(0,t) = v(1,t) = 0, & t \in (0,T) \\ u(x,T) = v_T(x), & x \in (0,1). \end{cases}$$

As a consequence, one would have the following boundary controllability result

Theorem 6.3.1. *Let $0 \leq \mu < 1/4$. For any $T > 0$ and any initial datum $u_0 \in L^2(0,1)$, there exists a control function $f \in L^2(0,T)$ such that the solution of* (6.3.3) *satisfies $u(x,T) = 0$.*

Finally, knowing that both (4.1.2) and (6.3.3) are null controllable acting from the boundary, the boundary controllability of (6.3.1) could then be obtained employing a splitting argument, as the one presented in the proof of [137, Lemma 3.2].

6.4 Boundary controllability for the heat equation with singular inverse-square potential involving the distance to the boundary

In Chapter 5, we analysed the control problem for the heat equation

$$u_t - \Delta u - \frac{\mu}{\delta^2}u = 0, \quad (x,t) \in \Omega \times (0,T), \tag{6.4.1}$$

obtaining null controllability with a distributed control located in an open set $\omega \subset \Omega$.

An immediate and interesting extension, would be to investigate boundary controllability properties. In this framework, the problem addressed in Chapter 4 can be seen as a first approach, in one space dimension, to this challenging issue. As it is explained in that Chapter, one of the main difficulties when aiming to obtain boundary controllability for equations with singular potentials, whose singularities are located precisely on the boundary, is to understand the degeneracy of the normal derivative of the solution when approaching the set of the singularities. Then, this degeneracy would need to be properly compensated, in order to build the control for our equation.

For the case of equation (6.4.1), in analogy with what we obtained for the one-dimensional case, we believe that we need to introduce a weighted normal derivative in the form $\delta^\alpha \partial_\nu u$, with a coefficient α which has to be identified.

This claim is justified by a very simple analysis of the problem on the unit sphere. Indeed, let $T > 0$, $\mu \leq 1/4$, and let $B^N(1)$ be the unit ball in $\mathbb{R}^N$; we consider the system

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2} u = 0, & (x,t) \in B^N(1) \times (0,T) := Q \\ \\ u = f, & (x,t) \in \partial B^N(1) \times (0,T) := \Sigma \\ u(x,0) = u_0(x), & x \in B^N(1). \end{cases} \tag{6.4.2}$$

Our main goal is to obtain a boundary controllability result for (6.4.2); therefore, we need to prove the observability from the boundary of the adjoint problem

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{\delta^2} v = 0, & (x,t) \in B^N(1) \times (0,T) := Q \\ \\ v = 0, & (x,t) \in \partial B^N(1) \times (0,T) := \Sigma \\ v(x,T) = v_T(x), & x \in B^N(1). \end{cases} \tag{6.4.3}$$

We notice now that it is possible to simplify our problem, decomposing (6.4.3) in spherical coordinates. Indeed, let us introduce the change of variables

$$\begin{aligned} \Phi : \ \mathbb{R}^N \setminus \{0\} &\longrightarrow (0,+\infty) \times \mathbb{S}^{N-1} \\ x &\longmapsto (r,\phi) := \left(|x|, \frac{x}{|x|} \right), \end{aligned}$$

and let us denote $w(r,\sigma,t) := v(r\phi,t)$; then, (6.4.3) becomes

$$\begin{cases} w_t + w_{rr} + \dfrac{N-1}{r} w_r + \dfrac{1}{r^2} \Delta_\phi w + \dfrac{\mu}{(1-r)^2} w = 0, & (r,\phi,t) \in (0,1) \times \mathbb{S}^{N-1} \times (0,T) \\ \\ w(1,\phi,t) = 0, & (\phi,t) \in \mathbb{S}^{N-1} \times (0,T) \\ w(r,\phi,T) = w_T(r,\phi), & (r,\phi) \in (0,1) \times \mathbb{S}^{N-1}, \end{cases}$$

where Δ_ϕ is the Laplace-Beltrami operator, defined by (see [43, Chapter 2, Section 1.4])

$$\Delta_\phi w := \Delta\left(w\left(\frac{x}{|x|}\right)\right)\Big|_{|x|=1}.$$

We recall that the eigenvalues of Δ_ϕ associated to Dirichlet boundary conditions are given by (see [43, Chapter 8, Section 8.1.4] for the case $N = 3$ or [7, 132] for the general case)

$$\lambda_k = k(N + k - 2), \quad k \geq 0,$$

and that the Hilbert space $L^2\left(\mathbb{S}^{N-1}\right)$ can be decomposed as

$$L^2\left(\mathbb{S}^{N-1}\right) = \bigoplus_{k\geq 0} \Lambda_k,$$

with Λ_k the eigenspaces associated to λ_k.

Let us denote $\ell_k := \dim(\Lambda_k)$; then, there exists an orthonormal basis of $L^2\left(\mathbb{S}^{N-1}\right)$, that we will indicate with $\left\{f^{k\ell}\right\}_{1\leq\ell\leq\ell_k,\, k\geq 0}$, such that

$$\begin{cases} -\Delta_\sigma f^{k\ell} = \lambda_k f^{k\ell}, & x \in B^N(1), \\ f^{k\ell} = 0, & x \in \partial B^N(1). \end{cases}$$

Therefore, if we decompose w with respect to this basis as follows

$$w(r,\sigma,t) = \sum_{k,\ell} \psi^{k\ell}(r,t) f^{k\ell}(\sigma),$$

for any $k \geq 0$, and for any $1 \leq \ell \leq \ell_k$ we obtain the following equation

$$\begin{cases} \psi^{k\ell}_t + \psi^{k\ell}_{rr} + \dfrac{N-1}{r}\psi^{k\ell}_r - \dfrac{\lambda_k}{r^2}\psi^{k\ell} + \dfrac{\mu}{(1-r)^2}\psi^{k\ell} = 0, & (r,t) \in (0,1)\times(0,T), \\ \\ \psi^{k\ell}(1,t) = 0, & t \in (0,T) \\ \psi^{k\ell}(r,T) = \psi^{k\ell}_T(r), & r \in (0,1). \end{cases}$$

Moreover, we can get rid of the first order term in the equation above by introducing a last change of variables

$$\phi^{k\ell}(r,t) = r^{\frac{N-1}{2}}\psi^{k\ell}(r,t),$$

from which we get the equation

$$\begin{cases} \phi^{k\ell}_t + \phi^{k\ell}_{rr} + \dfrac{\lambda_{kN}}{r^2}\phi^{k\ell} + \dfrac{\mu}{(1-r)^2}\phi^{k\ell} = 0, & (r,t) \in (0,1)\times(0,T) \\ \\ \phi^{k\ell}(0,t) = \phi^{k\ell}(1,t) = 0, & t \in (0,T) \\ \phi^{k\ell}(r,T) = \phi^{k\ell}_T(r), & r \in (0,1). \end{cases} \tag{6.4.4}$$

with

$$\lambda_{kN} := \frac{(1-N)(N-3)}{4} - \lambda_k.$$

Finally, by definition of λ_k, it is straightforward to check that for any $N \geq 1$ we have $\lambda_{kN} \leq 1/4$. Therefore, we obtain the same one-dimensional problem that we introduced at the end of the previous Section. In particular, to prove boundary controllability for (6.4.2) would be equivalent to obtain controllability from $r = 1$ for (6.4.4). At this purpose, we would need an observability inequality involving the weighted normal derivative $(1-r)^{\alpha}\phi_r^{k\ell}$, with $\alpha = 1/2\left(1-\sqrt{1-4\mu}\right)$.

Applying the inverse change of variables, we would get an observability inequality for the original problem involving the term $\delta^{\alpha}(\partial v/\partial\nu)$.

With the intent of recovering this weighted normal derivative, the weight σ that we employed in Chapter 5 has to be modified accordingly. We propose

$$\tilde{\sigma}(x,t) = \left(\frac{1}{t(T-t)}\right)^3 \left(C_\lambda + \delta^{1+2\alpha}\psi - \left(\frac{\delta}{r_0}\right)^{\lambda}\phi\right),$$

with the same function ψ that we introduced before.

The main difficulty would then be to show that, with this choice of the weight, it is possible to obtain suitable bounds for the distributed terms that shall lead to the inequality that we seek.

6.5 Control properties for wave equations with singular potentials

It would be interesting to investigate controllability properties for wave equations with singular inverse-square potentials of the type μ/δ^2 since, in our knowledge, at the present time this is a problem which has not been addressed yet.

Concerning the more classical case of problems involving potentials like $\mu/|x|^2$, there exist already results in the literature concerning internal controllability (see, for instance [34, 138]).

Regarding boundary controllability, instead, we can refer to [76], where this issue is analysed for a one-dimensional wave equation with variable degenerate coefficients in the form

$$u_{tt} - (x^{\alpha}u_x)_x = 0, \quad (x,t) \in (0,1)\times(0,T), \tag{6.5.1}$$

remembering that (6.5.1) is equivalent, through a change of variables, to a wave equation with one singular potential arising at $x = 0$.

To extend these results to the case of a potential involving the distance function is a very challenging issue; indeed, already in the one dimensional case, the presence of the singularity all over the boundary makes the multiplier approach extremely tricky, in the sense that is very difficult to identify, if possible, the correct multiplier for obtaining a Pohozaev identity.

For better justifying this fact, we can for instance consider the following one-dimensional wave equation with two singular inverse-square potentials arising at the boundary points of the space interval $(0,1)$

$$\begin{cases} u_{tt} - u_{xx} - \dfrac{\mu_1}{x^2}u - \dfrac{\mu_2}{(1-x)^2}u = 0, & (x,t) \in (0,1)\times(0,T), \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1). \end{cases} \tag{6.5.2}$$

If we multiply (6.5.2) by $f(x)u_x$ and we integrate over $(0,1)\times(0,T)$, after several computations we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^T\int_0^1 f'(x)\left(u_t^2 + u_x^2 + \frac{\mu_1}{x^2}u^2 + \frac{\mu_2}{(1-x)^2}u^2\right)dxdt + \int_0^1 f(x)u_tu_x\Big|_0^T dx \\ &-\frac{1}{2}\int_0^T f(x)u_x^2\Big|_0^1 dt - 2\mu_1\int_0^T\int_0^1 f(x)\frac{u^2}{x^3}\,dxdt + 2\mu_2\int_0^T\int_0^1 f(x)\frac{u^2}{(1-x)^3}\,dxdt = 0. \end{aligned} \tag{6.5.3}$$

Now, we have to choose properly the function f in the multiplier, in order to recover from (6.5.3) an identity which could be suitable for deriving an observability inequality. In this process, there are two main aspects that need to be taken into account:

- the function f has to compensate the super-critical singularities x^{-3} and $(1-x)^{-3}$ in the last two terms of (6.5.3);
- the first derivative of f has to be positive in the interval $(0,1)$, guaranteeing the positivity of the first term of (6.5.3), that can be correlated with the energy associated to (6.5.2).

However, this two conditions are incompatibles; indeed, the first one would require the function f to vanish both at $x = 0$ and $x = 1$, and this is, of course, impossible without allowing a change of monotonicity, i.e. a change of sign for the first derivative.

Therefore, multiplier techniques do not seem to be a proper way to address the problem. An alternative approach that is, instead, worth to try, is to derive also in this case a Carleman estimate in the spirit of what we did in Chapter 5 for the heat equation.

6.6 Optimality of the results of Chapter 5

The main result of Chapter 5 has been obtained as a consequence of a specific Carleman estimate for the problem under consideration. For obtaining this estimate we employed a weight $\sigma(x,t)$, that we chose in the classical form in separated variables

$$\sigma(x,t) = \theta(t)p(x) = \left(\frac{1}{t(T-t)}\right)^k p(x). \tag{6.6.1}$$

In our particular case, we consider an exponent $k = 3$ for the function θ, the motivation of this choice being the fact that in our computations appears some terms that we are not able to bound for lower exponents. However, this choice has consequences on the cost of the control as the time tends to zero (see, for instance, [54, 108]), which is not of the order of $\exp(C/T)$, as expected for the heat equation, but rather of $\exp(C/T^3)$. Therefore, it would be interesting to reduce the exponent in the definition of θ up to $k = 1$ and try to obtain a Carleman estimate with this new choice for the weight.

Capítulo 6

Conclusiones y problemas abiertos

En esta tesis, se han tratado los siguientes problemas:

- En el Capítulo 3, ha sido analizado el problema de la controlabilidad interior para ecuaciones de tipo Schrödinger y ondas no-locales, en que al operador de Laplace clásico ha sido sustituido el Laplaciano fraccionario $(-\Delta)^s$. Hemos empleado un control h de clase L^2 con soporte en un conjunto ω de la frontera de un dominio $\Omega \subset \mathbb{R}^N$, acotado y de clase $C^{1,1}$ y, a través del Método de Unicidad de Hilbert, hemos obtenido los resultado siguientes:
    - controlabilidad a cero de la ecuación de Schrödinger, para cualquier $s \in [1/2, 1)$;
    - controlabilidad a cero de la ecuación de ondas, para cualquier $s \in (1, 2)$.
- En el Capítulo 4, hemos tratado la controlabilidad de frontera para una ecuación del calor unidimensional, definida sobre el intervalo $x \in (0, 1)$, que involucra a un potencial singular cuadrático-inverso. Aplicando resultados análogos contenidos en [76] para ecuaciones parabólicas con coeficientes degenerados, hemos obtenido la controlabilidad exacta a cero de la ecuación, a través de un control f de clase L^2 localizado en $x = 0$, que es a la vez un punto de frontera y un polo de singularidad para el potencial.
- En el Capítulo 5, hemos estudiado la controlabilidad interior para una ecuación del calor con un potencial singular cuadrático inverso que involucra a la función distancia al borde. Por medio de una nueva estimación de Carleman, hemos obtenido la controlabilidad exacta a cero gracias a un control f de clase L^2, localizado en un conjunto abierto ω de un dominio $\Omega \subset \mathbb{R}^N$ acotado y de clase C^2.

Relacionados con los temas abordados en esta tesis y con los resultados que se han obtenido, presentamos ahora de manera sistemática distintos problemas abiertos que, en nuestra opinión, pueden ser de gran interés.

6.1 Ecuaciones que involucran al Laplaciano fraccionario con condiciones de contorno no homogéneas

En el Capítulo 3 hemos considerado ecuaciones de evolución que involucran al Laplaciano fraccionario con condiciones de borde homogéneas. La razón principal de esta elección ha sido que, para obtener la propiedad de controlabilidad que estábamos buscando, nos hemos basado principalmente en la teoría desarrollada por X. Ros-Oton y J. Serra ([117, 118, 119]), cuyos resultados se satisfacen para funciones que se anulan fuera del dominio de definición de los problemas analizados. Además, es necesario mencionar que, cuando nos acercamos por primera vez a este tema, la identidad de Pohozoaev obtenida en [119] era un resultado muy reciente y, al mismo tiempo, el único disponible para operadores no-locales.

Como hemos mencionado en el Capítulo 2, en dos trabajos muy recientes ([144, 143]) M. Warma ha empezado el análisis del problema elíptico para el Laplaciano fraccionario regional en un dominio $\Omega \subset \mathbb{R}^N$ acotado y de clase $C^{1,1}$, con condiciones de borde de tipo Neumann o Robin, desarrollando una teoría de existencia y regularidad de soluciones. Además, ha obtenido una nueva identidad de Pohozaev, que generaliza el resultado de Ros-Oton y Serra.

En concreto, ha probado que para funciones u suficientemente regulares se satisface la identidad

$$\begin{aligned}\int_\Omega (x\cdot\nabla u)\, A_\Omega^s u\, dx = \frac{2s-N}{2}\int_\Omega u A_\Omega^s u\, dx + \frac{c_{N,s}}{2}\int_{\partial\Omega}\left(\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dx\right)(y\cdot\nu)\, d\sigma \\ - \frac{B_{N,s}}{2}\int_{\partial\Omega}(x\cdot\nu)\frac{\partial u}{\partial\nu}\mathcal{N}^{2-2s}u\, d\sigma + \frac{B_{N,s}(2s-N)}{2}\int_{\partial\Omega} u\mathcal{N}^{2-2s}u\, d\sigma,\end{aligned} \tag{6.1.1}$$

donde $B_{N,s}$ es una constante explícita dependiente exclusivamente de N y s, mientras que $\mathcal{N}^{2-2s}$ es una versión fraccionaria de la clásica derivada normal, y está definida como

$$\mathcal{N}^{2-2s}u(z) := -\lim_{t\to 0}\frac{du(z+\nu(z)t)}{dt}\, t^{2-2s}, \quad z \in \partial\Omega,$$

cuando este límite existe.

Sería entonces natural aplicar estos resultados para analizar, por ejemplo, propiedades de controlabilidad para ecuaciones de Schrödinger y de ondas fraccionarias del tipo de las presentadas en el Capítulo 3, pero esta vez con condiciones de borde no homogéneas; en particular, el estudio de la controlabilidad de borde sería un problema muy interesante.

6.2 Análisis asintótico para las soluciones de ecuaciones de evolución con el Laplaciano fraccionario

La expansión en geometría óptica de las soluciones de una EDP de evolución es una técnica muy eficaz que, cuando se desarrolla correctamente, puede facilitar información relevante sobre propiedades de propagación y de dispersión, y sobre cómo se comportan estas soluciones al encontrarse con la frontera del dominio donde la ecuación está definida o en presencia de eventuales interfases ([55, 115]).

Con la intención de justificar de manera más rigurosa la imposibilidad de controlar la ecuación de ondas fraccionaria analizada en el Capítulo 3 para $s < 1$, con M. Warma hemos empezado a estudiar el problema del punto de vista del análisis asintótico, tomando como inspiración los resultados presentados en [115] para el caso local.

Simplemente para dar una idea preliminar de como se desarrolla esta técnica, consideramos la siguiente ecuación de ondas unidimensional, que involucra al Laplaciano fraccionario en $\mathbb{R}$

$$\Box^s u = u_{tt} + (-d_x^2)^s u = 0, \tag{6.2.1}$$

y buscamos soluciones aproximadas con un ansatz del tipo

$$u^\varepsilon(x,t) = e^{i[(\xi/\varepsilon)x + (\xi^s/\varepsilon^s)t]}\phi^\varepsilon(x,t), \quad \phi^\varepsilon(x,t) = \sum_{j=0}^{\infty} \varepsilon^j \phi_j(x,t), \tag{6.2.2}$$

con $\xi \in \mathbb{R}$ y donde las funciones ϕ_j serán determinadas en una segunda fase.

Por lo tanto, necesitamos calcular $\Box^s u^\varepsilon$, determinando el orden de todos los términos que obtenemos con respecto al parámetro ε.

En primer lugar, es muy sencillo mostrar que, para todo $\alpha \in \mathbb{R}$, tenemos

$$(-d_x^2)^s e^{i\alpha x} = \alpha^{2s} e^{i\alpha x}; \tag{6.2.3}$$

en efecto, de la definición que dimos del Laplaciano fraccionario se deduce que

$$(-d_x^2)^s e^{i\alpha x} = c_{1,s}\, P.V. \int_{\mathbb{R}} \frac{e^{i\alpha x} - e^{i\alpha y}}{|x-y|^{1+2s}}\, dy = c_{1,s}\, e^{i\alpha x} P.V. \int_{\mathbb{R}} \frac{1 - e^{i\alpha(y-x)}}{|x-y|^{1+2s}}\, dy.$$

Aplicando ahora el cambio de variable $z = \alpha(y-x)$, y utilizando la definición de valor principal y la expresión de la constante $c_{1,s}$ presentada en [48, Sección 3], obtenemos

$$\begin{aligned}
(-d_x^2)^s e^{i\alpha x} &= c_{1,s}\alpha^{2s}\, e^{i\alpha x} P.V. \int_{\mathbb{R}} \frac{1-e^{iz}}{|z|^{1+2s}}\, dz = c_{1,s}\alpha^{2s}\, e^{i\alpha x} \lim_{\varepsilon\to 0^+} \int_{|z|>\varepsilon} \frac{1-e^{iz}}{|z|^{1+2s}}\, dz \\
&= c_{1,s}\alpha^{2s}\, e^{i\alpha x} \lim_{\varepsilon\to 0^+} \left(\int_{\varepsilon}^{+\infty} \frac{1-e^{iz}}{z^{1+2s}}\, dz + \int_{-\infty}^{\varepsilon} \frac{1-e^{iz}}{(-z)^{1+2s}}\, dz \right) \\
&= c_{1,s}\alpha^{2s}\, e^{i\alpha x} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{+\infty} \frac{2-2cos(z)}{z^{1+2s}}\, dz = c_{1,s}\alpha^{2s}\, e^{i\alpha x} \int_{\mathbb{R}} \frac{1-cos(z)}{|z|^{1+2s}}\, dz \\
&= c_{1,s}\alpha^{2s}\, e^{i\alpha x} c_{1,s}^{-1} = \alpha^{2s}\, e^{i\alpha x}.
\end{aligned}$$

Además, por medio del la formula (3.3.25) que obtuvimos en el Capítulo 3 para el Laplaciano fraccionario del producto de dos funciones, se pueden derivar las siguientes expresiónes

$$1.)\ (-d_x^2)^s(fg) = (-d_x^2)^{\frac{s}{2}}(-d_x^2)^{\frac{s}{2}}(fg) = (-d_x^2)^{\frac{s}{2}}\left[f(-d_x^2)^{\frac{s}{2}}g + R_1\right]$$
$$= f(-d_x^2)^{\frac{s}{2}}(-d_x^2)^{\frac{s}{2}}g + (-d_x^2)^{\frac{s}{2}}R_1 + R_2 = f(-d_x^2)^s g + (-d_x^2)^{\frac{s}{2}}R_1 + R_2$$

$$2.)\ (-d_x^2)^s(fg) = (-d_x^2)^{\frac{s}{2}}\left[f(-d_x^2)^{\frac{s}{2}}g + R_1\right] = (-d_x^2)^{\frac{s}{2}}f(-d_x^2)^{\frac{s}{2}}g + (-d_x^2)^{\frac{s}{2}}R_1 + R_3$$

$$3.)\ (-d_x^2)^s(fg) = (-d_x^2)^{\frac{s}{2}}\left[g(-d_x^2)^{\frac{s}{2}}f + R_4\right] = g(-d_x^2)^s f + (-d_x^2)^{\frac{s}{2}}R_4 + R_5. \qquad (6.2.4)$$

Sumando la primera expresión en (6.2.4) con la tercera y sustrayendo al resultado la segunda expresión, obtenemos

$$(-d_x^2)^s(fg) = f(-d_x^2)^s g + g(-d_x^2)^s f - (-d_x^2)^{\frac{s}{2}}f(-d_x^2)^{\frac{s}{2}}g + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5). \qquad (6.2.5)$$

Utilizando (6.2.5) con $f = \phi^\varepsilon$ y $g = u := e^{i(\xi x/\varepsilon + \xi^s t/\varepsilon^s)}$, y gracias a (6.2.3), encontramos

$$(-d_x^2)^s u^\varepsilon = \frac{\xi^{2s}}{\varepsilon^{2s}}u\phi^\varepsilon + u(-d_x^2)^s\phi^\varepsilon - \frac{\xi^s}{\varepsilon^s}u(-d_x^2)^{\frac{s}{2}}\phi^\varepsilon + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5).$$

Por lo tanto

$$\Box^s u^\varepsilon = u\left[\frac{\xi^s}{\varepsilon^s}\left(2i\phi_t^\varepsilon - (-d_x^2)^{\frac{s}{2}}\phi^\varepsilon\right) + \Box^s\phi^\varepsilon\right] + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5)$$
$$= \varepsilon^{-s}u\left[\xi^s\left(2i\phi_t^\varepsilon - (-d_x^2)^{\frac{s}{2}}\phi^\varepsilon\right) + \varepsilon^s\Box^s\phi^\varepsilon\right] + (-d_x^2)^{\frac{s}{2}}R_4 + (R_2 - R_3 + R_5). \qquad (6.2.6)$$

El problema se reduciría entonces a identificar el orden de cada término que aparece en (6.2.6), encontrar cuáles son las ecuaciones que se satisfacen para los de orden principal y estimar de una manera adecuada las componentes de orden menor. En esta manera, desde (6.2.2) se podrían construir casi-soluciones localizadas sobre rayos, y emplearlas en el estudio, por ejemplo, de propiedades de propagación y reflexión.

6.3 Extensión de los resultados del Capítulo 4

En el Capítulo 4, hemos conseguido obtener la controlabilidad a cero desde $x = 0$ para la siguiente ecuación del calor unidimensional

$$u_t - u_{xx} - \frac{\mu}{x^2}u = 0, \quad (x,t) \in (0,1) \times (0,T),$$

que involucra a un potencial singular cuadrático-inverso cuya singularidad surge exactamente en el punto de la frontera donde está localizado el control.

Sin embrago, el resultado de controlabilidad que presentamos no está completo, en el sentido de que logramos probarlo solo para coeficientes μ que satisfacen la condición $0 \leq \mu < 1/4$, mientras que la ecuación tiene solución para todo $\mu \leq 1/4$.

La razón de esta inconclusión está en la técnica que empleamos para obtener la desigualdad de observabilidad para la solución del sistema adjunto asociado a nuestra ecuación, que se basa esencialmente en un resultado de observabilidad obtenido en [76] para ecuaciones parabólicas con coeficientes degenerados del tipo

$$u_t - (x^\alpha u_x)_x = 0 \quad (x,t) \in (0,1) \times (0,T),\ \alpha \in (0,1).$$

Por lo tanto, una primera extensión del resultado presentado en el Capítulo 4 sería obtener la controlabilidad de frontera para la ecuación considerada, también en los casos $\mu < 0$ y $\mu = 1/4$.

Una manera de tratar la cuestión que podría resultar exitosa sería probar una desigualdad de Carleman para el problem adjunto. Por supuesto, puesto que nuestra intención es obtener controlabilidad de frontera, ésta estimación necesitaría tener en cuenta la degeneración de la derivada normal de la solución de la ecuación acercándose al punto $x = 0$.

Desde luego, este problema no es elemental. Habiendo mostrado que la derivada de la función v solución de (4.1.9) tiene el siguiente comportamiento

$$v_x^2(x,t) \sim x^{-2\lambda}, \quad \text{cuando } \ x \to 0^+,$$

con λ como en (4.1.6), creemos que el peso que se debe emplear para obtener la desigualdad de Carleman tendría que ser de la forma $\sigma(x,t) = \theta(t)p(x)$, con una función p que contenga el término $x^{2\lambda+1}$.

Por otro lado, esta elección no parece ser apropiada, pues la cantidad $2\lambda + 1$ se convierte en negativa para $\mu < -3/4$, generando así un peso σ que no está acotado en la frontera. Sin embargo, entender qué función podría permitirnos obtener el término de borde correcto, sin introducir singularidades, no es trivial.

Por último, remarcamos que a lo largo del Capítulo hemos tenido que trabajar con datos iniciales en ciertos espacios particulares de Sobolev fraccionarios, aunque para la controllabilidad de nuestra ecuación podemos utilizar un control de clase L^2. Otra vez, este hecho es consecuencia de la técnica que empleamos en nuestra demostración, que se basa ampliamente en los resultados de [76]. Por otro lado, se nos ha dado a conocer recientemente un trabajo nuevo muy interesante ([31]), donde está tratado el mismo problema que [76] y se han obtenidos nuevos y mejores resultados. En particular, los autores consiguen abordar el caso de datos iniciales en L^2, construyendo controles en H^1. Por lo tanto, merecería la pena adaptar nuestro análisis a estas nuevas contribuciones, intentando extender nuestros resultados al caso (más natural) de datos iniciales de clase L^2.

El problema tratado en el Capítulo 4, además de ser interesante por sí mismo, es también un primer paso hacia el análisis de una cuestión más general, la de la controlabilidad de frontera de la ecuación del calor siguiente

$$u_t - u_{xx} - \frac{\mu_1}{x^2}u - \frac{\mu_2}{(1-x)^2}u = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{6.3.1}$$

que involucra a un potencial singular cuadrático-inverso cuyas singularidades aparecen en toda la frontera del dominio espacial $(0,1)$.

Nuestro interés en ecuaciones como (6.3.1) se origina principalmente en el hecho de que esta es una versión unidimensional de problemas con potenciales que explotan en toda la frontera del dominio de definición, cuyo análisis ha sido abordado en el Capítulo 5, en el caso de controlabilidad interior.

En primer lugar, remarcamos que el problema de Dirichlet homogéneo para una ecuación como (6.3.1) admite una solución gracias a la desigualdad de Hardy multi-polar

$$\int_0^1 z_x^2\,dx + M\int_0^1 z^2\,dx \geq \frac{1}{4}\int_0^1 \frac{z^2}{x^2}\,dx + \frac{1}{4}\int_0^1 \frac{z^2}{(1-x)^2}\,dx, \tag{6.3.2}$$

que se puede probar a través de (4.1.4) empleando una partición de la unidad de clase C^∞, definida como sigue (véase también la Imagen 6.1 abajo)

$$\begin{cases} \phi_1 \equiv 0, & x \in (0,1/2] \\ \phi_1 \in (0,1), & x \in (1/2,3/4] \\ \phi_1 \equiv 1, & x \in (3/4,1) \end{cases}, \quad \begin{cases} \phi_2 \equiv 1, & x \in (0,1/4] \\ \phi_2 \in (0,1), & x \in (1/4,1/2] \\ \phi_2 \equiv 0, & x \in (1/2,1) \end{cases}, \quad \phi_3 := 1 - \phi_1 - \phi_2,$$

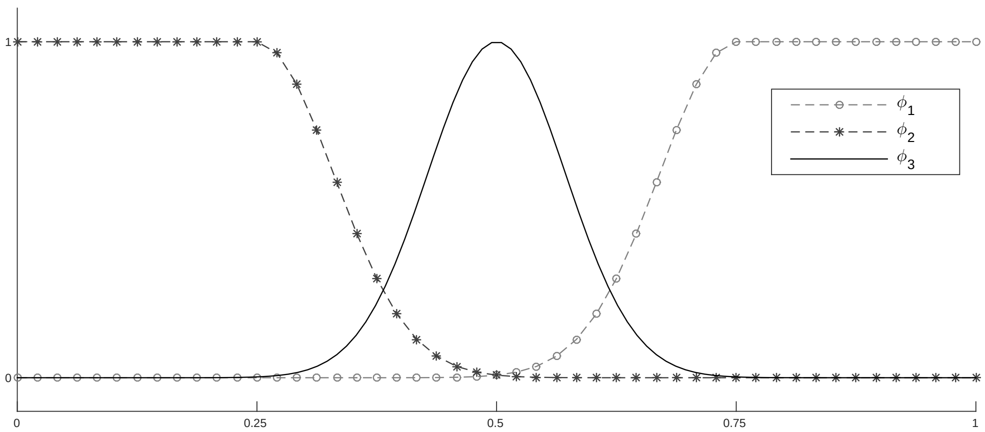

Imagen 6.1: Gráfico de la partición de la unidad empleada en la prueba de (6.3.2).

Más detalles se pueden encontrar, por ejemplo, en [16].

Por otro lado, la controlabilidad de frontera de (6.3.1) es una cuestión muy delicada, y no es trivial abordarla directamente con una estimación de Carleman. Por lo tanto, una buena estrategia sería dividir el problema en dos más sencillos.

Desde luego, creemos que todo el análisis desarrollado en el Capítulo 4 se puede adaptar al caso de una ecuación con un potencial singular cuadrático-inverso localizado en $x = 1$. Más detalladamente, dada la ecuación parabólica

$$\begin{cases} u_t - u_{xx} - \dfrac{\mu}{(1-x)^2} u = 0, & (x,t) \in Q \\ u(0,t) = 0, \quad x^{-\lambda} u(x,t)\big|_{x=1} = f(t), & t \in (0,T) \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases} \tag{6.3.3}$$

creemos que con los mismos argumentos empleados para estudiar la ecuación (4.1.2), para todo $0 \leq \mu < 1/4$ sería posible obtener la desigualdad de observabilidad

$$\|v(x,0)\|^2_{\mathbf{H}^\beta_\alpha} \leq C_T \int_0^T \Big[(1-x)^{2\lambda} v_x^2\Big]\Big|_{x=0} \, dt,$$

en que el espacio $\mathbf{H}^\beta_\alpha$ está definido como en (4.3.9), con α and β por determinar, y v es la solución del sistema adjunto

$$\begin{cases} v_t + v_{xx} + \dfrac{\mu}{(1-x)^2} v = 0, & (x,t) \in Q \\ v(0,t) = v(1,t) = 0, & t \in (0,T) \\ u(x,T) = v_T(x), & x \in (0,1). \end{cases}$$

Como consecuencia, obtendríamos el siguiente resultado de controlabilidad de frontera

Teorema 6.3.1. *Sea* $0 \leq \mu < 1/4$*. Para cada* $T > 0$ *y cada dato inicial* $u_0 \in L^2(0,1)$*, existe una función de control* $f \in L^2(0,T)$ *tal que la solución de* (6.3.3) *satisface* $u(x,T) = 0$*.*

Por último, sabiendo que tanto (4.1.2) como (6.3.3) son controlables a cero desde la frontera, se podría probar la controlabilidad de frontera de (6.3.1) empleando un argumento análogo al que se utiliza en la prueba de [137, Lemma 3.2].

6.4 Controlabilidad de frontera para la ecuación del calor con potencial singular cuadrático-inverso que involucra a la función distancia al borde

En el Capítulo 5, hemos analizado el problema de control para la ecuación del calor

$$u_t - \Delta u - \frac{\mu}{\delta^2} u = 0, \quad (x,t) \in \Omega \times (0,T), \tag{6.4.1}$$

obteniendo la controlabilidad exacta a cero con un control localizado en un conjunto abierto $\omega \subset \Omega$. Una extensión inmediata y seguramente muy interesante de este resultato, sería la investigación de propiedades de controlabilidad de borde.

En este contexto, el problema abordado en el Capítulo 4 puede ser interpretado como un primer intento de responder a la cuestión en dimensión uno. Como se explica en ese Capítulo, cuando queremos estudiar la controlabilidad de borde de ecuaciones con potenciales singulares, cuyas singularidades surgen exactamente en la frontera, una de las mayores dificultades está en entender la degeneración de la derivada normal de la solución al acercarse al conjunto de las singularidades. Una vez entendida esta degeneración, es necesario compensarla adecuadamente, de modo que se pueda construir el control para la ecuación.

En el caso de (6.4.1), análogamente a lo que se obtuvo para el caso unidimensional, creemos que se requiere la introducción de una derivada normal pesada, en la forma $\delta^\alpha \partial_\nu u$, con un coeficiente α que se debe identificar.

Este hecho está justificado por un análisis muy sencillo del problema en la esfera unitaria. Sean $T > 0$, $\mu \leq 1/4$, y llamamos $B^N(1)$ a la esfera unitaria en $\mathbb{R}^N$; consideramos el sistema

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2} u = 0, & (x,t) \in B^N(1) \times (0,T) := Q \\ \\ u = f, & (x,t) \in \partial B^N(1) \times (0,T) := \Sigma \\ u(x,0) = u_0(x) \end{cases} \tag{6.4.2}$$

Nuestro objetivo principal es probar un resultado de controlabilidad de borde para (6.4.2); por ello, nos hace falta deducir una desigualdad de observabilidad para el problema adjunto

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{\delta^2} v = 0, & (x,t) \in B^N(1) \times (0,T) := Q \\ \\ v = 0, & (x,t) \in \partial B^N(1) \times (0,T) := \Sigma \\ v(x,T) = v_T(x) \end{cases} \tag{6.4.3}$$

Nótese que nuestro problema puede ser simplificado, descomponiéndolo en armónicas esféricas. Entonces, introducimos al cambio de variables

$$\begin{array}{rccl} \Phi: & \mathbb{R}^N \setminus \{0\} & \longrightarrow & (0,+\infty) \times \mathbb{S}^{N-1} \\ & x & \longmapsto & (r,\phi) := \left(|x|, \dfrac{x}{|x|} \right), \end{array}$$

y denotamos $w(r,\sigma,t) := v(r\phi,t)$; (6.4.3) se convierte en

$$\begin{cases} w_t + w_{rr} + \dfrac{N-1}{r} w_r + \dfrac{1}{r^2} \Delta_\phi w + \dfrac{\mu}{(1-r)^2} w = 0, & (r,\phi,t) \in (0,1) \times \mathbb{S}^{N-1} \times (0,T) \\ \\ w(1,\phi,t) = 0, & (\phi,t) \in \mathbb{S}^{N-1} \times (0,T) \\ w(r,\phi,T) = w_T(r,\phi), & (r,\phi) \in (0,1) \times \mathbb{S}^{N-1}, \end{cases}$$

donde Δ_ϕ es el operador de Laplace-Beltrami, definido como ([43, Capítulo 2, Sección 1.4])

$$\Delta_\phi w := \Delta \left(w \left(\frac{x}{|x|} \right) \right) \Big|_{|x|=1}.$$

Recordemos que los valores propios de Δ_ϕ asociados con condiciones de borde de Dirichlet son (véase [43, Chapter 8, Section 8.1.4] para el caso $N = 3$ o [7, 132] para el caso general)

$$\lambda_k = k(N + k - 2), \quad k \geq 0$$

y que el espacio de Hilbert $L^2\left(\mathbb{S}^{N-1}\right)$ puede descomponerse como

$$L^2\left(\mathbb{S}^{N-1}\right) = \bigoplus_{k\geq 0} \Lambda_k,$$

con Λ_k espacios propios asociados con λ_k.

Denotamos con $\ell_k := \dim(\Lambda_k)$; existe una base ortonormal de $L^2\left(\mathbb{S}^{N-1}\right)$, que indicaremos con $\left\{f^{k\ell}\right\}_{1\leq \ell \leq \ell_k,\, k\geq 0}$, tal que

$$\begin{cases} -\Delta_\sigma f^{k\ell} = \lambda_k f^{k\ell}, & x \in B^N(1), \\ f^{k\ell} = 0, & x \in \partial B^N(1). \end{cases}$$

Entonces, si volvemos a escribir w en esta base, es decir

$$w(r,\sigma,t) = \sum_{k,\ell} \psi^{k\ell}(r,t) f^{k\ell}(\sigma),$$

para todo $k \geq 0$, y para todo $1 \leq \ell \leq \ell_k$, obtenemos la ecuación siguiente

$$\begin{cases} \psi_t^{k\ell} + \psi_{rr}^{k\ell} + \dfrac{N-1}{r}\psi_r^{k\ell} - \dfrac{\lambda_k}{r^2}\psi^{k\ell} + \dfrac{\mu}{(1-r)^2}\psi^{k\ell} = 0, & (r,t) \in (0,1) \times (0,T), \\ \\ \psi^{k\ell}(1,,t) = 0, \\ \psi^{k\ell}(r,T) = \psi_T^{k\ell}(r). \end{cases}$$

Por último, podemos eliminar el término de orden uno en la ecuación anterior introduciendo otro cambio de variables

$$\phi^{k\ell}(r,t) = r^{\frac{N-1}{2}} \psi^{k\ell}(r,t),$$

obteniendo así

$$\begin{cases} \phi_t^{k\ell} + \phi_{rr}^{k\ell} + \dfrac{\lambda_{kN}}{r^2}\phi^{k\ell} + \dfrac{\mu}{(1-r)^2}\phi^{k\ell} = 0, & (r,t) \in (0,1) \times (0,T) \\ \\ \phi^{k\ell}(0,t) = \phi^{k\ell}(1,t) = 0 \\ \phi^{k\ell}(r,T) = \phi_T^{k\ell}(r) \end{cases} \tag{6.4.4}$$

con

$$\lambda_{kN} := \frac{(1-N)(N-3)}{4} - \lambda_k.$$

Gracias a la definición de λ_k, es ahora elemental comprobar que, para cada $N \geq 1$, $\lambda_{kN} \leq 1/4$. Entonces, llegamos al mismo problema unidimensional que introdujimos al final de la Sección anterior. En particular, probar la controlabilidad de frontera para (6.4.2) sería equivalente a obtener controlabilidad desde $r = 1$ para (6.4.4). Por esto, necesitaríamos una desigualdad de observabilidad que involucre a la derivada normal pesada $(1-r)^\alpha \phi_r^{k\ell}$, con $\alpha = 1/2\left(1-\sqrt{1-4\mu}\right)$.

Aplicando el cambio de variable inverso, se obtendría así una desigualdad de observabilidad para el problema original en la que aparece el término $\delta^\alpha(\partial v/\partial \nu)$.

Con la intención final de recuperar esta derivada normal pesada, el peso σ que empleamos en la estimación de Carleman del Capítulo 5 debe ser modificado en conformidad. Proponemos

$$\tilde{\sigma}(x,t) = \left(\frac{1}{t(T-t)}\right)^3 \left(C_\lambda + \delta^{1+2\alpha}\psi - \left(\frac{\delta}{r_0}\right)^\lambda \phi\right),$$

con la misma función ψ que introducimos anteriormente.

Esta nueva función $\tilde{\sigma}$ nos permitiría de obtener la derivada normal pesada que hemos mencionado antes en el término de borde de la desigualdad de Carleman. La dificultad mayor entonces sería demostrar que, con la elección de este peso, se pueden obtener acotaciones apropiadas para los términos distribuidos, que tendrían que llevarnos a la estimación que buscamos.

6.5 Propiedades de controlabilidad para ecuaciones de ondas con potenciales singulares

Sería interesante investigar propiedades de controlabilidad para ecuaciones de ondas con potenciales singulares cuadráticos-inversos del tipo μ/δ^2, puesto que, hasta donde llega nuestro conocimiento, por el momento este es un problema que nadie ha tratado todavía.

Con respecto al caso más clásico de problemas que involucran al potencial $\mu/|x|^2$, en la literatura ya existen resultados de control interior ([34, 138]). Por el contrario, por lo que concierne a la controlabilidad de borde podemos hacer referencia a [76], donde se ha analizado este tema para ecuaciones de ondas unidimensionales con coeficientes variables y degenerados, en la forma

$$u_{tt} - (x^\alpha u_x)_x = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{6.5.1}$$

teniendo en cuenta que (6.5.1) es equivalente, a través de un cambio de variables, a una ecuación de ondas con un potencial singular que surge en $x = 0$.

Extender estos resultados al caso de un potencial que involucre a la función distancia al borde no es una cuestión sencilla; de hecho, ya en el caso unidimensional, la presencia de la singularidad en toda la frontera hace que la técnica de los multiplicadores sea extremamente complicada, en el sentido de que es muy difícil identificar, si es posible, el multiplicador correcto para obtener una identidad de Pohozaev.

Para justificar este hecho con más precisión, podemos considerar, por ejemplo, la siguiente ecuación de ondas unidimensional con dos potenciales singulares cuadráticos-inversos que surgen en los dos puntos de la frontera del intervalo espacial $(0,1)$

$$\begin{cases} u_{tt} - u_{xx} - \dfrac{\mu_1}{x^2}u - \dfrac{\mu_2}{(1-x)^2}u = 0, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1). \end{cases} \tag{6.5.2}$$

Si multiplicamos (6.5.2) por $f(x)u_x$ e integramos sobre $(0,1) \times (0,T)$, después de algunos cálculos obtenemos

$$\frac{1}{2}\int_0^T \int_0^1 f'(x)\left(u_t^2 + u_x^2 + \frac{\mu_1}{x^2}u^2 + \frac{\mu_2}{(1-x)^2}u^2\right) dxdt + \int_0^1 f(x)u_t u_x\Big|_0^T dx$$
$$-\frac{1}{2}\int_0^T f(x)u_x^2\Big|_0^1 dt - 2\mu_1\int_0^T \int_0^1 f(x)\frac{u^2}{x^3}\,dxdt + 2\mu_2\int_0^T \int_0^1 f(x)\frac{u^2}{(1-x)^3}\,dxdt = 0. \tag{6.5.3}$$

Ahora tenemos que elegir la función f en el multiplicador de una manera adecuada, para que de (6.5.3) se pueda recuperar una identidad que sea apta para obtener una desigualdad de observabilidad. En el proceso, hay dos aspectos principales a tener en cuenta:

- la función f tiene que compensar las singularidades supercríticas x^{-3} y $(1-x)^{-3}$ en los últimos dos términos de (6.5.3);
- la derivada primera de f tiene que ser positiva en el intervalo $(0,1)$, asegurando la positividad del primer término de (6.5.3), que puede estar relacionado con la energía asociada a (6.5.2).

Sin embargo, estas dos condiciones son incompatibles; de hecho, la primera requeriría que la función f se anulara tanto en $x = 0$ como en $x = 1$, y esto no puede ser posible sin un cambio de monotonía, es decir sin que la derivada primera cambie de signo.

En consecuencia, la técnica de los multiplicadores parece no ser una manera apropiada para enfrentarse con el problema. Por el contrario, una opción alternativa que merece la pena intentar es obtener, también en este caso, una desigualdad de Carleman, siguiendo el espíritu de lo que hicimos en el Capítulo 5 para la ecuación del calor.

6.6 Optimalidad de los resultados del Capítulo 5

El resultado principal del Capítulo 5 ha sido obtenido como consecuencia de una desigualdad de Carleman específica para el problema que estábamos considerando. Para probar esta desigualdad, empleamos un peso $\sigma(x,t)$ que ha sido elegido en la forma clásica en variables separadas

$$\sigma(x,t) = \theta(t)p(x) = \left(\frac{1}{t(T-t)}\right)^k p(x). \tag{6.6.1}$$

En nuestro caso, consideramos un exponente $k = 3$ para la función θ, estando esta elección motivada por el hecho que, a lo largo de nuestros cálculos, aparecen términos que no sabemos acotar si tomamos exponentes menores. Sin embargo, esta elección tiene consecuencias en el coste del control cuando el tiempo tiende a cero (véase, por ejemplo, [54, 108]), que no va a ser del orden de $\exp(C/T)$, como nos esperaríamos para la ecuación del calor, si no de $\exp(C/T^3)$. Por lo tanto, sería interesante reducir el exponente en la definición de θ a $k = 1$ e intentar obtener una desigualdad de Carleman con esta nueva elección del peso.

Bibliography

[1] Alabau-Boussouira, F., Cannarsa, P., and Leugering, G. Control and stabilization of degenerate wave equations. *arXiv preprint arXiv:1505.05720* (2015).

[2] Andreu-Vaillo, F., Mazón, J. M., Rossi, J. D., and Toledo-Melero, J. J. *Nonlocal diffusion problems*, vol. 165. American Mathematical Society, 2010.

[3] Bakunin, O. G. *Turbulence and diffusion: scaling versus equations.* Springer Science & Business Media, 2008.

[4] Baras, P., and Goldstein, J. A. The heat equation with a singular potential. *Trans. Amer. Math. Soc. 284*, 1 (1984), 121--139.

[5] Barbatis, G., Filippas, S., and Tertikas, A. A unified approach to improved L^p Hardy inequalities with best constants. *Trans. Amer. Math. Soc. 356*, 6 (2004), 2169--2196.

[6] Bardos, C., Lebeau, G., and Rauch, J. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim. 30*, 5 (1992), 1024--1065.

[7] Bebernes, J., and Eberly, D. *Mathematical problems from combustion theory*, vol. 83. Springer Science & Business Media, 2013.

[8] Beesack, P. R. Minimum properties of eigenvalues - Elementary proofs. In *General Inequalities 2*. Springer, 1980, pp. 109--120.

[9] Berestycki, H., and Esteban, M. J. Existence and bifurcation of solutions for an elliptic degenerate problem. *J. Differential Equations 134*, 1 (1997), 1--25.

[10] Bergh, J., and Löfström, J. *Interpolation spaces. An introduction.* Berlin, 1976.

[11] BICCARI, U. Internal control for non-local Schrödinger and wave equations involving the fractional laplace operator. *arXiv preprint arXiv:1411.7800* (2014).

[12] BICCARI, U., AND ZUAZUA, E. Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function. *J. Differential Equations 261*, 5 (2016), 2809--2853.

[13] BLUMENTHAL, R. M. C., AND GETOOR, R. K. The asymptotic distribution of the eigenvalues for a class of Markov operators. *Pacific J. Math 9*, 2 (1959), 399--408.

[14] BOLOGNA, M., TSALLIS, C., AND GRIGOLINI, P. Anomalous diffusion associated with nonlinear fractional derivative Fokker-Planck-like equation: Exact time-dependent solutions. *Phys. Rev. E 62*, 2 (2000), 2213.

[15] BONFORTE, M., AND VÁZQUEZ, J. L. Quantitative local and global a priori estimates for fractional nonlinear diffusion equations. *Adv. Math. 250* (2014), 242--284.

[16] BOSI, R., DOLBEAULT, J., AND ESTEBAN, M. J. Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators. *Comm. Pure Appl. Math. 7* (2008), 533--562.

[17] BOURGAIN, J., BREZIS, H., AND MIRONESCU, P. Limiting embedding theorems for $W^{s,p}$ when $s \to 1$ and applications. *J. Anal. Math. 87*, 1 (2002), 77--101.

[18] BRÄNDLE, C., COLORADO, E., DE PABLO, A., AND SÁNCHEZ, U. A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A 143*, 01 (2013), 39--71.

[19] BREZIS, H. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

[20] BREZIS, H., CAZENAVE, T., MARTEL, Y., AND RAMIANDRISOA, A. Blow up for $u_t - \delta u = g(u)$ revisited. *Adv. Differential Equations 1*, 1 (1996), 73--90.

[21] BREZIS, H., AND MARCUS, M. Hardy's inequalities revisited. *Ann. Sc. Nor. Sup. Pisa Cl. Sci. (4) 25*, 1-2 (1997), 217--237.

[22] BREZIS, H., MARCUS, M., AND SHAFRIR, I. Extremal functions for Hardy's inequality with weight. *J. .Funct. Anal. 171*, 1 (2000), 177--191.

[23] BRÉZIS, H., AND NIRENBERG, L. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math. 36*, 4 (1983), 437--477.

[24] Brezis, H., and Vázquez, J. L. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Complut. 10*, 2 (1997), 443.

[25] Cabré, X., and Martel, Y. Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier. *C. R. Math. Acad. Sci. 329*, 11 (1999), 973--978.

[26] Cabré, X., and Sire, Y. Nonlinear equations for fractional Laplacians ii: existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc. 367*, 2 (2015), 911--941.

[27] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Part. Diff. Eq. 32*, 8 (2007), 1245--1260.

[28] Caffarelli, L., and Silvestre, L. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math. 62*, 5 (2009), 597--638.

[29] Cannarsa, P., Martinez, P., and Vancostenoble, J. Null controllability of degenerate heat equations. *Adv. Diff. Eq. 10*, 2 (2005), 153--190.

[30] Cannarsa, P., Martinez, P., and Vancostenoble, J. Carleman estimates for a class of degenerate parabolic operators. *SIAM J. Control Optim. 47*, 1 (2008), 1--19.

[31] Cannarsa, P., Martinez, P., and Vancostenoble, J. The cost of controlling degenerate parabolic equations by boundary controls. *arXiv preprint arXiv:1511.06857* (2015).

[32] Cannarsa, P., Tort, J., and Yamamoto, M. Unique continuation and approximate controllability for a degenerate parabolic equation. *Appl. Anal. 91*, 8 (2012), 1409--1425.

[33] Capella, A., Dávila, J., Dupaigne, L., and Sire, Y. Regularity of radial extremal solutions for some non-local semilinear equations. *Comm. Part. Diff. Eq. 36*, 8 (2011), 1353--1384.

[34] Cazacu, C. Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results. *J. Funct. Anal. 263*, 12 (2012), 3741--3783.

[35] Cazacu, C. Controllability of the heat equation with an inverse-square potential localized on the boundary. *SIAM J. Control Optim. 52*, 4 (2014), 2055--2089.

[36] Cazenave, T. An introduction to semilinear elliptic equations. *Editora do Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro* (2006).

[37] Cazenave, T., and Haraux, A. *An introduction to semilinear evolution equations*, vol. 13. Oxford University Press, 1998.

[38] Chan, L.-Y. Some further extensions of Hardy's inequality. *Canad. Math. Bull. 24*, 4 (1981), 393--400.

[39] Chandrasekhar, S., and Chandrasekhar, S. *An introduction to the study of stellar structure*, vol. 2. Courier Corporation, 1958.

[40] Chaves-Silva, F. W. *About the controlability of some equations in Cardiology, Biology, Fluid Mechanics and Viscoelasticity.* PhD thesis, Universidad del País Vasco, 2014.

[41] Chen, L., Nochetto, R. H., Otárola, E., and Salgado, A. J. A pde approach to fractional diffusion: a posteriori error analysis. *J. Comput. Phys. 293* (2015), 339--358.

[42] Coron, J.-M. *Control and nonlinearity.* No. 136. American Mathematical Soc., 2009.

[43] Dautray, R., and Lions, J.-L. *Mathematical analysis and numerical methods for science and technology: Volume 1 - Physical origins and classical methods.* Springer Science & Business Media, 2012.

[44] Davies, E. A review of Hardy inequalities. In *The Maz'ya anniversary collection.* Springer, 1999, pp. 55--67.

[45] Davies, E. B. *Spectral theory and differential operators*, vol. 42. Cambridge University Press, 1996.

[46] De Roos, A. M., and Persson, L. Size-dependent life-history traits promote catastrophic collapses of top predators. *Proceedings of the National Academy of Sciences 99*, 20 (2002), 12907--12912.

[47] Demengel, F., Demengel, G., and Erné, R. *Functional spaces for the theory of elliptic partial differential equations.* Springer, 2012.

[48] Di Nezza, E., Palatucci, G., and Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*.

[49] Dipierro, S., Palatucci, G., and Valdinoci, E. Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting. *Commun. Math. Phys. 333*, 2 (2015), 1061--1105.

[50] Dolecki, S. Observability for the one-dimensional heat equation. *Studia Math. 48*, 3 (1973), 291--305.

[51] Duoandikoetxea, J., and Zuazua, E. Moments, masses de dirac et décomposition de fonctions. *C. R. Acad. Sci. Paris, Série 1, Mathématique 315*, 6 (1992), 693--698.

[52] Dyda, B. The fractional Hardy inequality with a remainder term. *arXiv preprint arXiv:0907.4448* (2009).

[53] Ervedoza, S. Control and stabilization properties for a singular heat equation with an inverse-square potential. *Comm. Part. Diff. Eq. 33*, 11 (2008), 1996--2019.

[54] Ervedoza, S., and Zuazua, E. A systematic method for building smooth controls for smooth data. *Discrete Contin. Dyn. Syst. Ser. B 14*, 4 (2010), 1375--1401.

[55] Evans, L. C. *Partial differential equations*. American Mathematical Society, 2010.

[56] Everitt, W. Integral inequalities and spectral theory. In *Spectral Theory and Differential Equations*. Springer, 1975, pp. 148--166.

[57] Fall, M. M. Nonexistence of distributional supersolutions of a semilinear elliptic equation with hardy potential. *J. Funct. Anal. 264*, 3 (2013), 661--690.

[58] Fanelli, L., Felli, V., Fontelos, M. A., and Primo, A. Time decay of scaling critical electromagnetic schrödinger flows. *Comm. Math. Phys. 324*, 3 (2013), 1033--1067.

[59] Fefferman, C. L. The uncertainty principle. *Bull. American Math. Soc. 9*, 2 (1983), 129--206.

[60] Felli, V., and Terracini, S. Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity. *Comm. Part. Diff. Eq. 31*, 3 (2006), 469--495.

[61] Fernández-Cara, E., and Zuazua, E. Control Theory: History, mathematical achievements and perspectives. *Bol. Soc. Esp. Mat. Aplic. 26*, 12 (2003), 79--140.

[62] Filippas, S., Maz'ya, V., and Tertikas, A. On a question of Brezis and Marcus. *Calc. Var. Partial Differential Equations 25*, 4 (2006), 491--501.

[63] Filippas, S., Maz'ya, V. G., and Tertikas, A. Sharp Hardy-Sobolev inequalities. *C. R. Math. Acad. Sci. Paris 339*, 7 (2004), 483--486.

[64] Filippas, S., Tertikas, A., et al. On the best constant of Hardy-Sobolev inequalities. *Nonlinear Anal. 70*, 8 (2009), 2826--2833.

[65] Flanders, H., et al. Remarks on Almgren's interior regularity theorems. *Illinois J. Math. 13*, 4 (1969), 707--716.

[66] Folland, G. B. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

[67] Frank, R. L., and Geisinger, L. Refined semiclassical asymptotics for fractional powers of the Laplace operator. *J. Reine Angew. Math.* (2014).

[68] Fursikov, A. V., and Imanuvilov, O. Y. Controllability of evolution equations, Lect. Note Series 34. *Res. Inst. Math., GARC, Seoul National University* (1996).

[69] García Azorero, J., and Peral Alonso, I. Hardy inequalities and some critical elliptic and parabolic problems. *J. Differential Equations 144*, 2 (1998), 441--476.

[70] Gel'fand, I. M. Some problems in the theory of quasi-linear equations. *Uspekhi Mat. Nauk 14*, 2 (1959), 87--158.

[71] Gilboa, G., and Osher, S. Nonlocal operators with applications to image processing. *Multiscale Model. Simul. 7*, 3 (2008), 1005--1028.

[72] Giri, P. R., Gupta, K. S., Meljanac, S., and Samsarov, A. Electron capture and scaling anomaly in polar molecules. *Phys. Lett. A 372*, 17 (2008), 2967--2970.

[73] Guan, Q.-Y. Integration by parts formula for regional fractional Laplacian. *Comm. Math. Phys. 266*, 2 (2006), 289--329.

[74] Guan, Q.-Y., and Ma, Z.-M. Boundary problems for fractional Laplacians. *Stoch. Dyn. 5*, 03 (2005), 385--424.

[75] Guan, Q.-Y., and Ma, Z.-M. Reflected symmetric α-stable processes and regional fractional Laplacian. *Probab. Theory Relat. Fields 134*, 4 (2006), 649--694.

[76] Gueye, M. Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations. *SIAM J. Control Optim. 52*, 4 (2014), 2037--2054.

[77] Hardy, G. An inequality between integrals. *Messenger of Math 54*, 1925 (1925), 150--156.

[78] Hardy, G. H., Littlewood, J. E., and Pólya, G. *Inequalities.* Cambridge university press, 1952.

[79] Hilbert, D. Grundzüge einer allgeminen theorie der linaren integralrechnungen.(erste mitteilung). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1904* (1904), 49--91.

[80] KATO, T. Strong L^p-solutions of the navier-stokes equation inr m, with applications to weak solutions. *Mathematische Zeitschrift 187*, 4 (1984), 471--480.

[81] KHODJA, F. A., BENABDALLAH, A., GONZALEZ-BURGOS, M., AND DE TERESA, L. Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences. *J. Funct. Anal. 267*, 7 (2014), 2077--2151.

[82] KIM, W. On the zeros of solutions of $y(n) + py = 0$. *J. Math. Anal. Appl. 25*, 1 (1969), 189--208.

[83] KOMORNIK, V. *Exact controllability and stabilization: the multiplier method*, vol. 36. Masson, 1994.

[84] KREJČIŘÍK, D., AND ZUAZUA, E. The Hardy inequality and the heat equation in twisted tubes. *J. Math. Pures Appl. 94*, 3 (2010), 277--303.

[85] KRÖNER, E. Elasticity theory of materials with long range cohesive forces. *Int. J. Solids and Structures 3*, 5 (1967), 731--742.

[86] KRZYWICKI, A., AND RYBARSKI, A. On linearization of an equation of an elastic rod. *Appl. Math. 3*, 6 (1962), 321--332.

[87] KUFNER, A. Imbedding theorems for general Sobolev weight spaces. *Ann. Sc. Norm. Sup. Pisa Cl. Sci. 23*, 2 (1969), 373--386.

[88] KUFNER, A. *Weighted Sobolev spaces*, vol. 31. John Wiley & Sons Incorporated, 1985.

[89] KUFNER, A., AND KADLEC, J. *Fourier series*. Iliffe, 1971.

[90] KUFNER, A., MALIGRANDA, L., AND PERSSON, L.-E. The prehistory of the Hardy inequality. *Amer. Math. Monthly 113*, 8 (2006), 715--732.

[91] KULCZYCKI, T., KWAŚNICKI, M., MALECKI, J., AND STOS, A. Spectral properties of the Cauchy process on half-line and interval. *Proc. Lond. Math. Soc.* (2010), pdq010.

[92] KWAŚNICKI, M. Eigenvalues of the fractional Laplace operator in the interval. *J. Funct. Anal. 262*, 5 (2012), 2379--2402.

[93] LEBEDEV, N. N., AND SILVERMAN, R. A. *Special functions and their applications*. Courier Corporation, 1972.

[94] LEE, E. B., AND MARKUS, L. Foundations of optimal control theory. Tech. rep., DTIC Document, 1967.

[95] Leray, J. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math. 63*, 1 (1934), 193--248.

[96] Levendorski, S. Pricing of the American put under Lévy processes. *Int. J. Theor. Appl. Finance 7*, 03 (2004), 303--335.

[97] Lions, J. L. *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués(Tome 1, Contrôlabilité exacte. Tome 2, Perturbations).* Recherches en mathematiques appliquées, Masson, 1988.

[98] Lions, J.-L. Exact controllability, stabilization and perturbations for distributed systems. *SIAM review 30*, 1 (1988), 1--68.

[99] Lions, J. L., and Magenes, E. *Problèmes aux limites non homogènes et applications.* Dunod, 1968.

[100] Loss, M., and Sloane, C. Hardy inequalities for fractional integrals on general domains. *J. Funct. Anal. 259*, 6 (2010), 1369--1379.

[101] Lu, G. The Peierls—Nabarro model of dislocations: a venerable theory and its current development. In *Handbook of Materials Modeling.* Springer, 2005, pp. 793--811.

[102] Marcus, M., Mizel, V., and Pinchover, Y. On the best constant for hardy's inequality in $\mathbb{R}^n$. *Trans. Amer. Math. Soc. 350*, 8 (1998), 3237--3255.

[103] Martinez, P., and Vancostenoble, J. Carleman estimates for one-dimensional degenerate heat equations. *J. Evol. Equ. 6*, 2 (2006), 325--362.

[104] Maz'ya, V., and Shaposhnikova, T. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal. 195*, 2 (2002), 230--238.

[105] Meerschaert, M. M. Fractional calculus, anomalous diffusion, and probability. *Fractional Dynamics* (2012), 265--284.

[106] Merton, R. C. Option pricing when underlying stock returns are discontinuous. *J. Finan. Econ. 3*, 1-2 (1976), 125--144.

[107] Micu, S., and Zuazua, E. An introduction to the controllability of partial differential equations. *Quelques questions de théorie du contrôle. Sari, T., ed., Collection Travaux en Cours Hermann, to appear* (2004).

[108] Miller, L. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim. 45*, 2 (2006), 762--772.

[109] Moore, R. A., and Nehari, Z. Nonoscillation theorems for a class of nonlinear differential equations. *Trans. Amer. Math. Soc. 93*, 1 (1959), 30--52.

[110] Nehari, Z. The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc. 55*, 6 (1949), 545--551.

[111] Nochetto, R. H., Otárola, E., and Salgado, A. J. A PDE approach to fractional diffusion in general domains: a priori error analysis. *Found. Comput. Math. 15*, 3 (2014), 733--791.

[112] Pham, H. Optimal stopping, free boundary, and American option in a jump-diffusion model. *Appl. Math. Optim. 35*, 2 (1997), 145--164.

[113] Pohozaev, S. I. Eigenfunctions of the equation Δ u+ λf (u)= 0. In *Soviet Math. Dokl* (1965), vol. 6, pp. 1408--1411.

[114] Putnam, C. R. An application of spectral theory to a singular calculus of variations problem. *Amer. J. Math. 70*, 4 (1948), 780--803.

[115] Rauch, J., Zhang, X., and Zuazua, E. Polynomial decay for a hyperbolic-parabolic coupled system. *J. Math. Pures Appl. 84*, 4 (2005), 407--470.

[116] Redheffer, R. Inequalities with three functions. *J. Math. Anal. Appl. 16*, 2 (1966), 219--242.

[117] Ros-Oton, X., and Serra, J. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. 101*, 3 (2014), 275--302.

[118] Ros-Oton, X., and Serra, J. The extremal solution for the fractional Laplacian. *Calc. Var. Partial Differential Equations 50*, 3-4 (2014), 723--750.

[119] Ros-Oton, X., and Serra, J. The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal. 213*, 2 (2014), 587--628.

[120] Rosier, L. A survey of controllability and stabilization results for partial differential equations. *Journal européen des systèmes automatisés 41*, 3/4 (2007), 365.

[121] Rüland, A. Unique continuation for fractional Schrödinger equations with rough potentials. *Comm. Part. Diff. Eq. 40*, 1 (2015), 77--114.

[122] Russell, D. L. A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Stud. Appl. Math. 52*, 3 (1973), 189--211.

[123] Russell, D. L. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Rev. 20*, 4 (1978), 639--739.

[124] Samko, S. G., Kilbas, A. A., and Marichev, O. I. Fractional integrals and derivatives. *Theory and Applications, Gordon and Breach, Yverdon 1993* (1993).

[125] Sandeep, K. Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator. *Proc. Roy. Soc. Edinburgh Sect. A 132*, 05 (2002), 1021--1043.

[126] Servadei, R., and Valdinoci, E. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl. 389*, 2 (2012), 887--898.

[127] Servadei, R., and Valdinoci, E. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst 33*, 5 (2013), 2105--2137.

[128] Servadei, R., and Valdinoci, E. On the spectrum of two different fractional operators. *Proc. Roy. Soc. Edinburgh Sect. A 144*, 04 (2014), 831--855.

[129] Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. 146*, 1 (1986), 65--96.

[130] Sontag, E. D. *Mathematical control theory: deterministic finite dimensional systems*, vol. 6. Springer Science & Business Media, 2013.

[131] Stinga, P. R. *Fractional powers of second order partial differential operators: extension problem and regularity theory.* PhD thesis.

[132] Strauss, W. A. *Partial differential equations*, vol. 92. Wiley New York, 1992.

[133] Tertikas, A. Critical phenomena in linear elliptic problems. *J. Funct. Anal. 154*, 1 (1998), 42--66.

[134] Troesch, B. A. An isoperimetric sloshing problem. *Comm. Pure Appl. Math. 18*, 1-2 (1965), 319--338.

[135] Tucsnak, M., and Weiss, G. *Observation and control for operator semigroups.* Springer Science & Business Media, 2009.

[136] Vancostenoble, J. Lipschitz stability in inverse source problems for singular parabolic equations. *Comm. Part. Diff. Eq. 36*, 8 (2011), 1287--1317.

[137] Vancostenoble, J., and Zuazua, E. Null controllability for the heat equation with singular inverse-square potentials. *J. Funct. Anal. 254*, 7 (2008), 1864--1902.

[138] Vancostenoble, J., and Zuazua, E. Hardy inequalities, observability, and control for the wave and Schrödinger equations with singular potentials. *SIAM J. Math. Anal. 41*, 4 (2009), 1508--1532.

[139] Vázquez, J. L. Nonlinear diffusion with fractional Laplacian operators. In *Proc. Abel. Symp.* Springer, 2012, pp. 271--298.

[140] Vázquez, J. L. Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators. *Discrete Contin. Dyn. Syst. 7*, 4 (2014), 857--885.

[141] Vázquez, J. L., and Zuazua, E. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal. 173*, 1 (2000), 103--153.

[142] Viswanathan, G. M., Afanasyev, V., Buldyrev, S., Murphy, E., Prince, P., Stanley, H. E., et al. Lévy flight search patterns of wandering albatrosses. *Nature 381*, 6581 (1996), 413--415.

[143] Warma, M. The Pohozaev identity for the regional fractional Laplacian with general boundary conditions, Submitted.

[144] Warma, M. The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets. *Potential Anal. 42*, 2 (2015), 499--547.

[145] Yosida, K. *Functional analysis.* Springer-Verlag, 1980.

[146] Zhu, T., and Harris, J. M. Modeling acoustic wave propagation in heterogeneous attenuating media using decoupled fractional Laplacians. *Geophysics 79*, 3 (2014), T105--T116.

[147] Zuazua, E. Propagation, observation, control and numerical approximation of waves. *SIAM Rev. 47*, 2 (2005), 197--243.

[148] Zuazua, E. Controllability and observability of partial differential equations: some results and open problems. *Handb. Differ. Equ. 3* (2007), 527--621.