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Research Article

About the Stability and Positivity of a Class of Discrete Nonlinear Systems of Difference Equations

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This paper investigates stability conditions and positivity of the solutions of a coupled set of nonlinear difference equations under very generic conditions of the nonlinear real functions which are assumed to be bounded from below and nondecreasing. Furthermore, they are assumed to be linearly upper bounded for sufficiently large values of their arguments. These hypotheses have been stated in 2007 to study the conditions permanence.

1. Introduction

There is a wide scientific literature devoted to investigate the properties of the solutions of nonlinear difference equations of several types [1–9]. Other equations of increasing interest are as follows:

(1) stochastic difference equations and systems (see, e.g., [10] and references therein);

(2) nonstandard linear difference equations like, for instance, the case of time-varying coefficients possessing asymptotic limits and that when there are contributions of unmodeled terms to the difference equation (see, e.g., [11, 12]);

(3) coupled differential and difference systems (e.g., the so-called hybrid systems of increasing interest in control theory and mathematical modeling of dynamic systems, [13–16] and the study of discretized models of differential systems which are computationally easier to deal with than differential systems; see, e.g., [17, 18]).

In particular, the stability, positivity, and permanence of such equations are of increasing interest. In this paper, the following system of difference equations is considered [1]:

$$x_{n+1}^{(i)} = \lambda_i x_n^{(i)} + f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\right), \quad \forall i \in \overline{k} := \{1, 2, \ldots, k\}, \tag{1.1}$$

with $x_n^{(k+1)} \equiv x_n^{(1)}$, for all $n \in \mathbf{N}$; $\lambda_i \in \mathbf{R}$, $\alpha_i \in \mathbf{R}$, $\beta_i \in \mathbf{R}$; and $f_i : \mathbf{R} \to \mathbf{R}$, for all $i \in \overline{k}$, under arbitrary initial conditions $x_0^{(i)}$, $x_{-1}^{(i)}$, for all $i \in \overline{k}$. The identity $x_n^{(k+1)} \equiv x_n^{(1)}$ allows the inclusion in a unified shortened notation via (1.1) of the dynamics:

$$x_{n+1}^{(k)} = \lambda_i x_n^{(k)} + f_i\left(\alpha_i x_n^{(1)} - \beta_i x_{n-1}^{(1)}\right), \quad \forall i \in \overline{k}, \tag{1.2}$$

as it follows by comparing (1.1) for $i = k$ with (1.2). The solution vector sequence of (1.1) will be denoted as $x_n := (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)})^T \in \mathbf{R}^k$, for all $n \in \mathbf{N}$, under initial conditions $x_j := (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(k)})^T \in \mathbf{R}^k$, $j = -1, 0$. The above difference system is very useful for modeling discrete neural networks which are very useful to describe certain engineering, computation, economics, robotics, and biological processes of populations evolution or genetics [1]. The study in [1] about the permanence of the above system is performed under very generic conditions on the functions $f_i : \mathbf{R} \to \mathbf{R}$, for all $i \in \overline{k}$. It is only requested that the functions be bounded from below, nondecreasing, and linearly upper bounded for large values, exceeding a prescribed threshold, of their real arguments. In this paper, general conditions for the global stability and positivity of the solutions are investigated.

1.1. Notation

$\mathbf{R}_+ := \{z \in \mathbf{R} : z > 0\}$, $\mathbf{R}_{0+} := \{z \in \mathbf{R} : z \geq 0\}$, $\mathbf{R}_{0-} := \{z \in \mathbf{R} : z \leq 0\}$. "$\wedge$" is the logic conjunction symbol. $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. If $P \in \mathbf{R}^{n \times n}$, then P^T is the transpose of P.

$P \succ 0$, $P \succeq 0$, $P \prec 0$, $P \preceq 0$ denote, respectively, P positive definite, semidefinite positive, negative definite, and negative semidefinite. $P \geq 0$, $P > 0$, $P \gg 0$ denote, respectively, P nonnegative (i.e., none of its entries is negative, also denoted as $P \in \mathbf{R}_{0+}^{n \times n}$), P positive (i.e., $P \geq 0$ with at least one of its entries being positive), and P strictly positive (i.e., all of its entries are positive). Thus, $P > 0 \Rightarrow P \geq 0$ and $P \gg 0 \Rightarrow P > 0 \Rightarrow P \geq 0$, but the converses are not generically true. The same concepts and notation of nonnegativity, positivity, and strict positivity will be used for real vectors. Then, the solution vector sequence in $\mathbf{R}^k$ of (1.1) will be nonnegative in some interval S, denoted by $x_n \geq 0$ (identical to $x_n \in \mathbf{R}_{0+}^k$), for all $n \in S \subset \mathbf{N}$, if all the components are nonnegative for $n \in S \subset \mathbf{N}$. If, in addition, at least one component is positive, then the solution vector is said to be positive, denoted by $x_n > 0$ (implying that $x_n \in \mathbf{R}_{0+}^k$), for all $n \in S \subset \mathbf{N}$. If all of them are positive in S, then the solution vector is said to be strictly on a discrete interval, denoted by $x_n \gg 0$ (identical to $x_n \in \mathbf{R}_+^k$ and implying that $x_n > 0$ and $x_n \in \mathbf{R}_{0+}^k$), for all $n \in S \subset \mathbf{N}$.

$\|\ \|_2$ and $\|\ \|_1$ are the ℓ_2 and ℓ_1 norms of vectors and induced norms of matrices, respectively. I_n is the nth identity matrix.

2. Preliminaries

In order to characterize the properties of system (1.1), firstly define sets of nondecreasing and bounded-from-below functions $f_i : \mathbf{R} \to \mathbf{R}$ in system (1.1) as follows irrespective of the initial conditions:

$$B(K_i) := \{f_i : \mathbf{R} \longrightarrow \mathbf{R} : f_i(y) \geq f_i(x) \geq K_i,\ \forall x, y(>x) \in \mathbf{R},\ K_i \in \mathbf{R}\}, \quad \forall\, i \in \overline{k}, \tag{2.1}$$

and sets of linearly upper bounded real functions:

$$C(\gamma_i, \delta_i, M_i) := \left\{ f_i : \mathbf{R} \longrightarrow \mathbf{R} : f_i(x) \leq \frac{\delta_i}{\gamma_i} x,\ \forall x > M_i \in \mathbf{R}_+,\ \delta_i \in (0,1) \right\}, \quad \forall i \in \overline{k}, \tag{2.2}$$

for $\gamma_i \neq 0$ irrespective of the initial conditions as well. In a natural form, define also sets of nondecreasing, bounded-from-below, and linearly upper bounded real functions, again independent of the initial conditions, $BC(K_i, \gamma_i, \delta_i, M_i) := B(K_i) \cap C(\gamma_i, \delta_i, M_i)$, that is,

$$\begin{aligned} BC(K_i, \gamma_i, \delta_i, M_i) := \Bigg\{ f_i : \mathbf{R} \longrightarrow \mathbf{R} : f_i(y) \geq f_i(x) \geq K_i \wedge f_i(x) \leq \frac{\delta_i}{\gamma_i} x, \\ \forall x, y(>x) \in \mathbf{R},\ K_i \in \mathbf{R},\ \delta_i \in (0,1) \Bigg\}, \quad \forall i \in \overline{k}, \end{aligned} \tag{2.3}$$

for $\gamma_i \neq 0$. The above definitions facilitate the potential restrictions on the functions $f_i : \mathbf{R} \to \mathbf{R}$, $i \in \overline{k}$, required to derive the various results of the paper. The constraints on the functions $f_i : \mathbf{R} \to \mathbf{R}$, for all $i \in \overline{k}$, used in the above definitions of sets, have been proposed by Stević for $f_i \in BC(K_i, \gamma_i, \delta_i, M_i)$ and then used to prove the conditions of permanence of (1.1) in [1] for some $K_i = K$, $M_i = M > 0$, and $\delta_i \in (0,1)$, for all $i \in \overline{k}$, subject to $\lambda_i \in [0, \beta_i/\alpha_i]$, $\alpha_i > \beta_i \geq 0$, for all $i \in \overline{k}$. The subsequent technical assumption will be then used in some of the forthcoming results.

Assumption 2.1. $\alpha_i > 0$ and $0 < \delta_i < \min(1, \alpha_i^{-1})$.

The following two assertions are useful for the analysis of the difference system (1.1).

Assertion 2.2. For any given $i \in \overline{k}$, $f_i \in B(K_i) \Rightarrow f_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}) \geq K_i$, for all $n \in \mathbf{N} \cup \{0, -1\}$.

Assertion 2.3. (i) For any given $i \in \overline{k}$, $f_i \in C(\gamma_i, \delta_i, M_i) \Leftrightarrow f_i(\gamma_i((\alpha_i/\gamma_i) x_n^{(i+1)} - (\beta_i/\gamma_i) x_{n-1}^{(i+1)})) \leq (\delta_i/\gamma_i)(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)})$ if $x_n^{(i+1)} > (\beta_i/\alpha_i) x_{n-1}^{(i+1)} + (\gamma_i/\alpha_i) M_i$, for all $n \in \mathbf{N} \cup \{0, -1\}$, for any real constants $\beta_i, \alpha_i > 0$.

(ii) $f_i \in C(\alpha_i, \delta_i, M_i) \Leftrightarrow f_i(\alpha_i(x_n^{(i+1)} - (\beta_i/\alpha_i) x_{n-1}^{(i+1)})) \leq (\delta_i/\alpha_i)(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)})$ if $x_n^{(i+1)} > (\beta_i/\alpha_i) x_{n-1}^{(i+1)} + M_i$, for all $n \in N \cup \{0, -1\}$, for any real constants $\beta_i, \alpha_i > 0$.

(iii) $f_i \in C(1, \delta_i, M_i) \Leftrightarrow f_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}) \leq \delta_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)})$ if $x_n^{(i+1)} > (\beta_i/\alpha_i) x_{n-1}^{(i+1)} + M_i/\alpha_i$, for all $n \in N \cup \{0, -1\}$, for any real constants $\beta_i, \alpha_i > 0$.

(iv) $C(1, \delta_i, M_i) = C(\alpha_i, \alpha_i\delta_i, M_i/\alpha_i)$ if Assumption 2.1 holds.

Proof. Assertion 2.3(i)–(iii) follow directly from the definitions of $B(K_i)$ and $C(\gamma_i, \delta_i, M_i)$, for all $i \in \overline{k}$.

Assertion 2.3(iv) The proof is split into proving the two claims below.

Claim 1. $C(1, \delta_i, M_i) \subset C(\alpha_i, \alpha_i\delta_i, M_i/\alpha_i)$.

Proof of Claim 1. $f_i \in C(1, \delta_i, M_i) \Leftrightarrow f_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}) = f_i(\alpha_i(x_n^{(i+1)} - (\beta_i/\alpha_i) x_{n-1}^{(i+1)})) \leq \delta_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}) = \delta_i\alpha_i(x_n^{(i+1)} - (\beta_i/\alpha_i) x_{n-1}^{(i+1)})$ if $\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)} > M_i \Rightarrow f_i \in C(\alpha_i, \alpha_i\delta_i, M_i/\alpha_i)$ if Assumption 2.1 holds. □

Claim 2. $C(\alpha_i, \alpha_i\delta_i, M_i/\alpha_i) \subset C(1, \delta_i, M_i)$.

Proof of Claim 2. $f_i \in C(\alpha_i, \alpha_i\delta_i, M_i/\alpha_i) \Rightarrow f_i(\alpha_i(x_n^{(i+1)} - (\beta_i/\alpha_i)x_{n-1}^{(i+1)})) \le \alpha_i\delta_i(x_n^{(i+1)} - (\beta_i/\alpha_i)x_{n-1}^{(i+1)}) = \delta_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)})$ if $\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)} > M_i \Rightarrow f_i \in C(1, \delta_i, M_i)$ if Assumption 2.1 holds. □

Then, Assertion 2.3(iv) has been proved from Claims 1-2. □

The following result establishes that it is not possible to obtain equivalence classes from any collection of parts of the sets of functions in the definitions of $B(K_i)$, $C(\gamma_i, \delta_i, M_i)$, and $BC(K_i, \gamma_i, \delta_i, M_i)$.

Assertion 2.4. For any $i \in \overline{k}$, consider $C(\gamma_i, \delta_i, M_i)$ for some given 3-tuple $(\gamma_i, \delta_i, M_i)$ in $\mathbf{R} \times (0,1) \times \mathbf{R}$, and consider any discrete collection of distinct admissible triples $(\gamma_{ij_{i\gamma}}, \delta_{ij_{i\delta}}, M_{ij_{iM}}) \in \mathbf{R} \times (0,1) \times \mathbf{R}$ $(j_{i\gamma} \in \overline{J}_{i\gamma}, j_{i\delta} \in \overline{J}, j_{iM} \in \overline{J}_{iM})$ subject to the constraints $\delta_{ij_{i\delta}} \le \delta_i$ and $M_{ij_{iM}} \ge M_i$, for all $(j_{i\delta}, j_{iM}) \in \overline{J}_{i\delta} \times \overline{J}_{iM}$, leading to the associated $C(\gamma_{ij_{i\gamma}}, \delta_{ij_{i\delta}}, M_{ij_{iM}})$. Define the binary relation R_i in $C(\gamma_i, \delta_i, M_i)$ as $f_i R_i g_i \Leftrightarrow f_i, g_i \in C(\gamma_{ij_{i\gamma}}, \delta_{ij_{i\delta}}, M_{ij_{iM}})$. Then, R_i is not an equivalence relation so that $C(\gamma_{ij_{i\gamma}}, \delta_{ij_{i\delta}}, M_{ij_{iM}})$ are not equivalence classes in $C(\gamma_i, \delta_i, M_i)$ with respect to R_i. Also, the sets $B(K_{ij_{iK}})$ and $BC(K_{ij_{iK}}, \gamma_{ij_{i\gamma}}, \delta_{ij_{i\delta}}, M_{ij_{iM}})$ for any given respective collections $K_{ij_{iK}} \le K_i$, $\delta_{ij_{i\delta}} \le \delta_i$, $M_{ij_{iM}} \ge M_i$, for all $(j_{iK}, j_{i\delta}, j_{iM}) \in \overline{J}_{iK} \times \overline{J}_{i\delta} \times \overline{J}_{iM}$, are not equivalence classes, respectively, in $B(K_i)$ and $BC(K_i, \gamma_i, \delta_i, M_i)$.

Proof. In view of Assertion 2.3(iv), $\gamma_{ij_{i\gamma}}$ can be all set equal to unity with no loss of generality, which is done to simplify the notation in the proof. Note that $f_i R_i g_i \Leftrightarrow f_i, g_i \in C(1, \delta_{ij_{i\delta}}, M_{ij_{iM}}) \Rightarrow f_i, g_i \in C(1, \delta_{ij_{i\delta}}, M_{ij_{iM}})$ for some $(\delta_{ij_{i\delta}}, M_{ij_{iM}}) \in (0,1) \times \mathbf{R}$. Now, consider $C(1, \delta'_{ij_{i\delta}}, M_{ij_{iM}})$ with $\delta'_{ij_{i\delta}} > \delta_i$ such that $\delta_i \ge \delta'_{ij_{i\delta}}(> \delta_{ij_{i\delta}}) \in \{\delta_{ij} : j \in J_{i\delta}\}$. Then, $C(1, \delta_{ij_{i\delta}}, M_{ij_{iM}}) \subset C(1, \delta'_{ij_{i\delta}}, M_{ij_{iM}})$. Since the equivalence classes with respect to any equivalence relation are disjoint, $C(1, \delta_{ij_{i\delta}}, M_{ij_{iM}})$ in $C(1, \delta_i, M_i)$ with respect to R_i is not an equivalence class unless $C(1, \delta_{ij_{i\delta}}, M_{ij_{iM}}) = C(1, \delta'_{ij_{i\delta}}, M_{ij_{iM}})$. Now, consider the linear function $h_i : \mathbf{R} \to \mathbf{R}$ defined by $h_i(x) := \delta'_{ij_{i\delta}} x > \delta_{ij_{i\delta}} x$ so that $C(1, \delta'_{ij_{i\delta}}, M_{ij_{i\delta}}) \ni h_i \notin C(1, \delta_{ij_{i\delta}}, M_{ij_{i\delta}})$. Thus, $C(1, \delta_{ij_{i\delta_i}}, M_i) \ne C(1, \delta'_{ij_{i\delta}}, M_{ij_{i\delta}})$. Then, R_i $(i \in \overline{k})$ are not equivalence relations, and there are no equivalence classes in $C(\gamma_i, \delta_i, M_i)$ $(i \in \overline{k})$ with respect to R_i $(i \in \overline{k})$. The remaining part of the proof follows in a similar way by using the definitions of the sets $B(K_i)$ and $BC(K_i, \gamma_i, \delta_i, M_i)$, and it is omitted. □

3. Necessary conditions for stability and positivity

Now, linear systems for system (1.1) with all the nonlinear functions in some specified class are investigated. Those auxiliary systems become relevant to derive necessary conditions for a given property to hold for all possible systems (1.1), whose functions are in some appropriate set $B(K_i)$, $C(\gamma_i, \delta_i, M_i)$, or $BC(K_i, \gamma_i, \delta_i, M_i)$. This allows the characterization of the above properties under few sets of conditions on the nonlinear functions in the difference system (1.1). If $f_i \in C(1, \delta_i, M_i)$, for all $i \in \overline{k}$, then the auxiliary linear system to (1.1) is

$$x_{n+1}^{(i)} = \lambda_i x_n^{(i)} + \delta_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\right), \quad \forall i \in \overline{k}. \tag{3.1}$$

If $f_i \in C(\alpha_i, \delta_i, M_i)$, for all $i \in \overline{k}$, then the auxiliary linear system to (1.1) is

$$x_{n+1}^{(i)} = \lambda_i x_n^{(i)} + \delta_i \left(x_n^{(i+1)} - \frac{\beta_i}{\alpha_i} x_{n-1}^{(i+1)} \right), \quad \forall i \in \overline{k}. \tag{3.2}$$

System (3.1) may be equivalently rewritten as follows by defining the state vector sequence $x_n := (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})^T \in \mathbf{R}^k$, for all $n \in \mathbf{N}$, as the kth-order difference system:

$$x_{n+1} = Ax_n + Bx_{n-1} = (\Lambda + C)x_n + Bx_{n-1} = \Lambda x_n + \overline{B}\overline{x}_{n-1}, \quad \forall n \in \mathbf{N}, \tag{3.3}$$

with initial conditions $x_i := (x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(k)})^T \in \mathbf{R}^k$ for $i = 0, -1$, where $\overline{x}_n := (x_n^T \vdots x_{n-1}^T)^T \in \mathbf{R}^{2k}$ and

$$A = \begin{bmatrix} \lambda_1 & \delta_1\alpha_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & \delta_2\alpha_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \delta_{k-1} & \alpha_{k-1} \\ \delta_k\alpha_k & 0 & \cdots & 0 & & \lambda_k \end{bmatrix} \in \mathbf{R}^{k\times k}, \tag{3.4}$$

$$B = \begin{bmatrix} 0 & -\delta_1\beta_1 & 0 & \cdots & & 0 \\ 0 & 0 & -\delta_2\beta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & -\delta_{k-1} & \beta_{k-1} \\ -\delta_k\beta_k & 0 & \cdots & 0 & & 0 \end{bmatrix} \in \mathbf{R}^{k\times k}, \tag{3.5}$$

$$\Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_k), \quad C = \begin{bmatrix} 0 & \delta_1\alpha_1 & 0 & \cdots & & 0 \\ 0 & 0 & \delta_2\alpha_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \delta_{k-1} & \alpha_{k-1} \\ \delta_k\alpha_k & 0 & \cdots & 0 & & 0 \end{bmatrix}, \tag{3.6}$$

$$\overline{B} = (B \vdots C) \in \mathbf{R}^{k\times 2k}. \tag{3.7}$$

The one-step delay may be removed by defining the following extended $2k$th-order system of state vector $\overline{x}_n := (x_n^T \vdots x_{n-1}^T)^T \in \mathbf{R}^{2k}$ satisfying

$$\overline{x}_{n+1} = \overline{A}\overline{x}_n, \quad \forall n \in \mathbf{N}, \tag{3.8}$$

with $\overline{x}_0 := (x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(k)}, x_{-1}^{(1)}, x_{-1}^{(2)}, \dots, x_{-1}^{(k)})^T \in \mathbf{R}^{2k}$ and

$$\overline{A} = \begin{bmatrix} A & \vdots & B \\ & \cdots & \\ I_k & \vdots & 0 \end{bmatrix} \in \mathbf{R}^{2k\times 2k}. \tag{3.9}$$

Note that the extended system (3.8)-(3.9) is fully equivalent to system (3.3)–(3.7) since both have identical solutions for each given common set of initial conditions. Now, let $\|(\cdot)\|_2$ be the ℓ_2-norm of real vectors of any order and associated induced norms of matrices (i.e., spectral norms of vectors and matrices). The following definitions are useful to investigate (1.1).

Definition 3.1. System (1.1) is said to be globally Lyapunov stable (or simply globally stable) if any solution is bounded for any finite initial conditions.

Definition 3.2. System (1.1) is said to be permanent if any solution enters a compact set K for $n \geq n_0$ for any bounded initial conditions with n_0 depending on the initial conditions.

Definition 3.3. System (1.1) is said to be positive if any solution is nonnegative for any finite nonnegative initial conditions.

The system is locally stable around an equilibrium point if any solution with initial conditions in a neighborhood of such an equilibrium point remains bounded. Local or global asymptotic stability to the equilibrium point occurs, respectively, under local or global stability around a unique equilibrium point if, furthermore, any solution tends asymptotically to such an equilibrium point as $n \to \infty$. Definition 3.2 is the definition of permanence in the sense used in [1], which is compatible with global and local stability and with global or local asymptotic stability according to Definition 3.1 and the above comments if $0 \in K$. However, it has to be pointed out that there are different definitions of permanence, like, for instance, in [2], where vanishing solutions (related to asymptotic stability to the equilibrium) or, even, negative solutions at certain intervals are not allowed for permanence. On the other hand, note that a continuous-time nonlinear differential system may be permanent without being globally stable in the case that finite escape times t of the solution exist, implying that because of unbounded discontinuities of the solution at finite time t, that solution is unbounded in $[t, t+\varepsilon)$ for some finite $\varepsilon \in \mathbf{R}_+$. This phenomenon cannot occur for system (1.1) under the requirement $f_i \in BC(K_i, \gamma_i, \delta_i, M_i)$, for all $i \in \overline{k}$, which avoids the solution being infinity at finite values of the discrete index n for any finite initial conditions. The following result is concerned with necessary conditions of global Lyapunov stability of system (1.1) for all the sets of functions $f_i \in BC(K_i, 1, \delta_i, M_i)$, for all $i \in \overline{k}$, since the linear system defined with $f_i(x) = \delta_i x$, for all $i \in \overline{k}$, in (1.1) has to be globally stable in order to keep global stability for any $f_i \in BC(K_i, 1, \delta_i, M_i)$, for all $i \in \overline{k}$.

Theorem 3.4. *System* (1.1) *is globally stable and permanent for any given set of functions* $f_i \in BC(K_i, 1, \delta_i, M_i)$ *for any given* $K_i \in \mathbf{R}$ *and any given* $M_i \in \mathbf{R}$, *for all* $i \in \overline{k}$, *only if the subsequent properties hold.*

(i) $|\lambda_i| \leq 1$, *for all* $i \in \overline{k}$.

(ii) $\|\overline{A}\|_2 \leq 1$, *equivalently,*

$$\|W\|_2 = \left\|\overline{A}^T \overline{A}\right\|_2 \leq 1, \quad \text{where } W := \overline{A}^T \overline{A} = \begin{bmatrix} W_{11} & \vdots & W_{12} \\ & \cdots & \\ W_{12}^T & \vdots & W_{22} \end{bmatrix} \in \mathbf{R}^{2k \times 2k}, \tag{3.10}$$

where

$$W_{11} := A^T A + I_k = \begin{bmatrix} 1+\lambda_1^2+\delta_k^2\alpha_k^2 & \lambda_1\delta_1\alpha_1 & \lambda_3\delta_3\alpha_3 & \cdots & \lambda_k\delta_k\alpha_k \\ \lambda_1\delta_1\alpha_1 & 1+\lambda_2^2+\delta_1^2\alpha_1^2 & \lambda_2\delta_2\alpha_2 & \cdots & \lambda_{k-1}\delta_{k-1}\alpha_{k-1} \\ \vdots & \vdots & \vdots & & \\ \lambda_{k-2}\delta_{k-2}\alpha_{k-2} & \lambda_{k-1}\delta_{k-1}\alpha_{k-1} & \cdots & \lambda_{k-1}^2+\delta_{k-2}^2\alpha_{k-2}^2 & \lambda_2\delta_2\alpha_2 \\ \lambda_k\delta_k\alpha_k & \lambda_{k-1}\delta_{k-1}\alpha_{k-1} & \cdots & \lambda_2\delta_2\alpha_2 & 1+\lambda_k^2+\delta_{k-1}^2\alpha_{k-1}^2 \end{bmatrix}, \tag{3.11}$$

$W_{12} := A^T B$, *and* $W_{22} := B^T B = \mathrm{Diag}(\delta_k^2\beta_k, \delta_1^2\beta_1, \ldots, \delta_{k-1}^2\beta_{k-1})$, *with* I_k *being the kth identity matrix. A necessary condition is* $\sum_{i=1}^{k}(\lambda_i^2 + \delta_i^2(\alpha_i^2 + \beta_i^2)) \le k$.

(iii) *There exists*

$$\overline{P} = \overline{P}^T := \begin{bmatrix} \overline{P}_{11} & \vdots & \overline{P}_{12} \\ & \cdots & \\ P_{12}^T & \vdots & \overline{P}_{22} \end{bmatrix} \succ 0 \quad in \quad \mathbf{R}^{2k\times 2k}, \tag{3.12}$$

where $\overline{P}_{ij} \in \mathbf{R}^{k\times k}$ $(i,j = 1,2)$, *which is a solution to the matrix identity*

$$\begin{bmatrix} (A^T\overline{P}_{11} + \overline{P}_{12}^T)A + A^T\overline{P}_{12} + \overline{P}_{22} - \overline{P}_{11} & \vdots & (A^T\overline{P}_{11} + \overline{P}_{12}^T)B - \overline{P}_{12} \\ & \cdots & \\ B^T(\overline{P}_{11}A + \overline{P}_{12}) - \overline{P}_{12}^T & \vdots & B^T\overline{P}_{11}B - \overline{P}_{22} \end{bmatrix} = -Q \tag{3.13}$$

for any given

$$\overline{Q} = \overline{Q}^T := \begin{bmatrix} \overline{Q}_{11} & \vdots & \overline{Q}_{12} \\ & \cdots & \\ \overline{Q}_{12}^T & \vdots & \overline{Q}_{22} \end{bmatrix} \succeq 0 \quad in \ \mathbf{R}^{2k\times 2k}. \tag{3.14}$$

Proof. (i) Note that the identically zero functions $f_i : \mathbf{R} \to 0$, for all $i \in \overline{k}$, are all in $BC(K_i, 1, \delta_i, M_i)$ for any $K_i \le 0$, $\delta_i \in (0,1)$, $M_i > 0$, for all $i \in \overline{k}$. Proceed by contradiction by assuming that $|\lambda_i| > 1$ and $f_i \equiv 0$ for some $i \in \overline{k} := \{1, 2, \ldots, k\}$, with the system being globally stable. Thus, $|x_{n+1}^{(i)}| > |x_n^{(i)}|$ if $x_0^{(i)} \neq 0$ so that $|x_n^{(i)}| \to \infty$ as $n \to \infty$, and then the system is unstable for some function $f_i \in BC(K_i, 1, \delta_i, M_i)$. Thus, the necessary condition for global stability has been proved, implying also the permanence of all the solutions in some compact real interval K.

(ii) Assume $f_i(x) = \delta_i x$ with $\delta_i \in (0,1)$ everywhere in $\mathbf{R}$ so that $f_i \in C(1, \delta_i, M_i)$, $M_i > 0$. Let the spectrum of W be $\sigma(W) := \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$, with each eigenvalue being repeated as many times as its multiplicity. Then, $\|\overline{A}\|_2 = \max_{1\le i\le k} \sigma_i^{1/2}$. It is first proved by complete induction that if $\overline{x}_0 \neq 0$ is an eigenvector of $\overline{A}$, then $\overline{x}_k$ is an eigenvector of $\overline{A}$ for any $k \ge 1$. Assume that $\overline{x}_k$ is an eigenvector of $\overline{A}$ for some arbitrary $k \ge 1$ and some eigenvalue ρ_i. Then, $\overline{A}\overline{x}_{k+1} = \overline{A}(\overline{A}\overline{x}_k) = \overline{A}(\rho_i\overline{x}_k) = \rho_i(\overline{A}\overline{x}_k) = \rho_i\overline{x}_{k+1}$ so that $\overline{x}_{k+1}$ is also an eigenvector of $\overline{A}$ for the same eigenvalue ρ_i. This property leads to

$$\|\overline{x}_{k+1}\|_2^2 = \|\overline{A}\,\overline{x}_k\|_2^2 = \overline{x}_k^T\,\overline{A}^T\,\overline{A}\,\overline{x}_k = \rho_i^2\|\overline{x}_k\|_2^2 = \sigma_i\|\overline{x}_k\|_2^2 = \rho_i^{2k}\|\overline{x}_0\|_2^2 = \sigma_i^k\|\overline{x}_0\|_2^2. \tag{3.15}$$

Proceed by contradiction by assuming that system (1.1) is stable, for all $f_i \in C(1, \delta_i, M_i)$, with $|\rho_i| = \sigma_i^{1/2} > 1$. From (3.15), $|x_n^{(i)}| \to \infty$ as $n \to \infty$, and then the system is unstable for a function $f_i \in C(1, \delta_i, M_i)$ for any real constant K_i since it possesses an unbounded solution for some finite initial conditions. Now, redefine the functions $\overline{f}_i(x)$ from the above $f_i(x)$, $i \in \overline{k}$, as follows:

$$\overline{f}_i(x) = \begin{cases} f_i(x) = \delta_i x & \text{if } x \geq 0, \\ \mathbf{R} \ni \lambda < -\left(1 + \max\limits_{1 \leq i \leq k}\left(|\lambda_i|\right)\right) < 0 & \text{if } x < 0. \end{cases} \tag{3.16}$$

It is clear by construction that if $\overline{f}_i(x) = f_i(x) = \delta_i x$ on an interval of infinite measure and if $0 > \lambda = \overline{f}_i(x) \neq f_i(x)$ occurs on a real interval of finite measure, then the above contradiction obtained for $f_i \in C(1, \delta_i, M_i)$ still applies for $\overline{f}_i \in BC(K_i, 1, \delta_i, M_i)$ for any finite negative $K_i < -\lambda$. If $\overline{f}_i(x) = f_i(x)$ occurs on an interval of finite measure and if $\overline{f}_i(x) \neq f_i(x)$ occurs on an interval of infinite measure, then the linear system resulting from (1.1) with the replacement $f_i(x) \to \overline{f}_i(x)$ is unstable so that any nontrivial solution is unbounded. Furthermore, since $\overline{f}_i(x) \to -\infty$ as $x \to \infty$ (function diverging to $-\infty$) and $\overline{f}_i(x)$ being unbounded on $\mathbf{R}$ (implying that $\overline{f}_i(x_k) \to -\infty$ for $\{x_k\}_0^\infty$ being some monotonically increasing sequence of real numbers) are both impossible situations for some $i \in \overline{k}$ since $\overline{f}_i : \mathbf{R} \to \mathbf{R}\,(i \in \overline{k})$ are all nondecreasing, it follows again that the functions are bounded from below so that $\overline{f}_i \in BC(K_i, 1, \delta_i, M_i)$ for some finite $K_i < 0$. If the real subintervals within which $\overline{f}_i(x)$ equalizes $f_i(x)$ or differs from $f_i(x)$ are both of infinite measure, the result $\overline{f}_i \in BC(K_i, 1, \delta_i, M_i)$ with some unbounded solution still applies trivially for some finite $K_i < 0$. Thus, system (1.1) is globally stable for any given set of functions $f_i \in BC(K_i, 1, \delta_i, M_i)$ for any $K_i \in \mathbf{R}$ and any $M_i \in \mathbf{R}$, for all $i \in \overline{k}$, only if the subsequent equivalent properties hold: $\|\overline{A}\|_2 \leq 1$, $\|W\|_2 \leq 1$. The necessary condition $\sum_{i=1}^{k}(\lambda_i^2 + \delta_i^2(\alpha_i^2 + \beta_i^2)) \leq k$ follows by inspecting the sum of entries of the main diagonal of W which equalizes the sum of nonnegative real eigenvalues of W (which are also the squares of the modules of the eigenvalues of $\overline{A}$, i.e., the squares of the singular values of $\overline{A}$) which have to be all of modules not greater than unity to guarantee global stability.

(iii) The property derives directly from discrete Lyapunov global stability theorem and its associate discrete Lyapunov matrix equation $\overline{A}^T \overline{P}\,\overline{A} - \overline{P} = -\overline{Q}$ which has to possess a solution $\overline{P} \succ 0$ for any given $\overline{Q} \succeq 0$. This property is a necessary and sufficient condition for the global stability of the extended linear system (3.8)-(3.9), and then for that of system (3.3)–(3.7). The proof ends by noting that system (3.8)-(3.9) has to be stable in order to guarantee the global stability of system (1.1) for any set $f_i \in BC(K_i, 1, \delta_i, M_i)$, for all $i \in \overline{k}$, according to Property (ii). □

Concerning positivity (Definition 3.3), it is well known that in the continuous-time and discrete-time linear and time-invariant cases, the positivity property may be established via a full characterization of the parameters (see, e.g., [2, 13, 17] as well as references therein). In particular, for a continuous-time linear time-invariant dynamic system to be positive, the matrix of dynamics has to be a Meztler matrix, while in a discrete-time one it has to be positive, where the control, output, and input-output interconnection matrices have to be positive in both (continuous-time and discrete-time) cases [2]. Under these conditions, each solution is always nonnegative all the time provided that all the components of the control and initial condition vectors are nonnegative [2, 13]. In general, in the nonlinear case, it is necessary to characterize the nonnegativity of the solutions over certain intervals and for certain values of initial conditions and parameters; that is, the positivity is not a general property associated with the differential system itself all the time but with some particular solutions on certain time intervals associated with certain constraints on the corresponding initial conditions. The positivity of (1.1) for linear functions $f_i(x) = \delta_i x$ is now invoked (in terms of necessary conditions) to guarantee the positivity of all the solutions of (1.1) for any set of nonnegative initial conditions and any potential set $f_i : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ with $f_i \in BC(K_i, 1, \delta_i, M_i)$ for any given $K_i \in \mathbf{R}$ and any given $M_i \in \mathbf{R}$, for all $i \in \overline{k}$. This is formally addressed in the subsequent result.

Theorem 3.5. *System* (1.1) *is positive for any given set of nonnegative functions $f_i : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ with $f_i \in BC(K_i, 1, \delta_i, M_i)$ for any given $K_i \in \mathbf{R}$ and any given $M_i \in \mathbf{R}$, for all $i \in \overline{k}$, only if $\lambda_i \in \mathbf{R}_{0+}$, $\alpha_i \in \mathbf{R}_{0+}$, $\beta_i \in \mathbf{R}_{0-}$, for all $i \in \overline{k}$.*

Outline of proof

As argued in the proof of Theorem 3.4 for stability, the linear system has to be positive in order to guarantee that it is positive for any set $f_i : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ with $f_i \in BC(K_i, 1, \delta_i, M_i)$ for any given $K_i \in \mathbf{R}$ and $M_i \in \mathbf{R}$, for all $i \in \overline{k}$. The linear system (3.8)-(3.9) for $f_i(x) = \delta_i x$ is positive if and only if $\overline{A} \in \mathbf{R}_{0+}^{n\times n}$ [3] since, in addition, this implies $f_i \in BC(K_i, 1, \delta_i, M_i)$. The proof follows since $\overline{A} \in \mathbf{R}_{0+}^{n\times n}$ by direct inspection if and only if $\lambda_i \in \mathbf{R}_{0+}$, $\alpha_i \in \mathbf{R}_{0+}$, $\beta_i \in \mathbf{R}_{0-}$, for all $i \in \overline{k}$.

Necessary joint conditions for stability, permanence, and positivity of (1.1) for any set $f_i : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ with $f_i \in BC(K_i, 1, \delta_i, M_i)$ for any given $K_i \in \mathbf{R}$ and $M_i \in \mathbf{R}$, for all $i \in \overline{k}$, follow directly by combining Theorems 3.4 and 3.5.

4. Main stability results

This section derives sufficiency-type conditions (easy to test) for global stability of the linear system (3.3)–(3.7) independently of the signs of the parameters α_i, β_i, and δ_i, $i \in \overline{k}$ (which are also allowed to take values out of the interval $(0,1)$, but on their maximum sizes). It is allowed that λ_i be independent of the above parameters and negative, but fulfilling that their modules are less than unity. The mechanism of proof for the linear case is then extended directly to the general nonlinear system (1.1). The α_i, β_i, and λ_i, $i \in \overline{k}$, are allowed to be negative but $\delta_i \in (0,1)$, $i \in \overline{k}$, is required to formulate an auxiliary result for the main proof.

Theorem 4.1. *Assume that $|\lambda_i| < 1$, for all $i \in \overline{k}$, and*

$$\max\left(\max_{1\le i\le k}|\alpha_i|, \max_{1\le i\le k}|\beta_i|\right) < \frac{1 - \max_{1\le i\le k}|\lambda_i|}{2\sqrt{k}\max_{1\le i\le k}|\delta_i|}. \tag{4.1}$$

Then, the linear system (3.3)–(3.7), *equivalently system* (3.8)-(3.9), *is globally Lyapunov stable for any finite arbitrary initial conditions. It is also permanent for any initial conditions:*

$$\begin{aligned} \overline{x}_0 &\in K_0(a_1, \dots, a_{2k}, b_1, \dots, b_{2k}) \\ &:= \left\{x = (x_1, x_2, \dots, x_{2k})^T \in \mathbf{R}^{2k} : x_i \in [a_i, b_i],\ \infty > b_i > a_i > -\infty,\ \forall i \in \overline{2k}\right\} \subset \mathbf{R}^{2k}. \end{aligned} \tag{4.2}$$

Proof. The successive use of the recursive second identity in (3.3) with initial condition $\overline{x}_0 = (x_0^T, x_{-1}^T)^T$ leads to

$$x_{n+k} = \Lambda^{n+k} x_0 + \sum_{i=0}^{n+k-1} \Lambda^{n+k-i-1} \overline{B}\overline{x}_i, \quad \forall n \in \mathbf{N},\ \forall k \in \overline{k}, \tag{4.3}$$

and taking ℓ_2-norms in (4.3) with $\lambda := \max_{1\le i\le k}|\lambda_i| < 1$, we get

$$\begin{aligned}
\left\|x_{n+k}\right\|_2 &= \left\|\Lambda^{n+k}\right\|_2 \left\|x_0\right\|_2 + \sum_{i=0}^{n+k-1} \left\|\Lambda^{n+k-i-1}\right\|_2 \|\overline{B}\|_2 \left\|\overline{x}_i\right\|_2 \\
&\le \lambda^{n+k}\left\|x_0\right\|_2 + \frac{1-\lambda^{n+k}}{1-\lambda}\|\overline{B}\|_2 \max_{0\le i\le n+k-1}\left\|\overline{x}_i\right\|_2 \\
&\le \lambda^{n}\left\|x_0\right\|_2 + \frac{1-\lambda^{n}}{1-\lambda}\|\overline{B}\|_2 \max_{0\le i\le n+k-1}\left\|\overline{x}_i\right\|_2 \\
&\le \lambda^{n}\left\|x_0\right\|_2 + \frac{\delta\max(\alpha,\beta)\left(1-\lambda^{n}\right)}{1-\lambda}\sqrt{k} \max_{0\le i\le n+k-1}\left\|\overline{x}_i\right\|_2 \\
&\le \lambda^{n}\left\|x_0\right\|_2 + \frac{2\delta\max(\alpha,\beta)\left(1-\lambda^{n}\right)}{1-\lambda}\sqrt{k} \max_{-1\le i\le n+k-1}\left\|x_i\right\|_2, \quad \forall n \in \mathbf{N},\ \forall k \in \overline{k},
\end{aligned} \tag{4.4}$$

where $\delta := \max_{1\le i\le k}(|\delta_i|)$, $\alpha := \max_{1\le i\le k}(|\alpha_i|)$, and $\beta := \max_{1\le i\le k}(|\beta_i|)$ since $\lambda < 1$ and

$$\begin{aligned}
\|\overline{B}\|_2 = \sqrt{\lambda_{\max}\left(\overline{B}^T\overline{B}\right)} &\le \sqrt{k}\|\overline{B}\|_1 \le \sqrt{k}\delta\max(\alpha,\beta) \quad \text{for any } x \in \mathbf{R}^k, \\
\left\|\Lambda^j\right\|_2^2 &= \max_{1\le i\le k}\left(\left|\lambda_i\right|^{2j}\right) = \lambda^{2j} \le \lambda < 1, \quad \forall j \in \mathbf{N}, \\
\max_{0\le i\le n+k-1}\left(\left\|\overline{x}_i\right\|_2\right) &= \max_{0\le i\le n+k-1}\left(x_i^T x_i + x_{i-1}^T x_{i-1}\right)^{1/2} \\
&\le \max_{0\le i\le n+k-1}\left(\left\|x_i\right\|_2 + \left\|x_{i-1}\right\|_2\right) \\
&\le 2\max_{-1\le i\le n+k-1}\left\|x_i\right\|_2.
\end{aligned} \tag{4.5}$$

Note that (4.4) is still valid if the term preceding the equality is any $\|x_{n+\ell}\|_2$, for all $\ell \in \mathbf{N} \setminus \overline{n+k}$, since they are all upper bounded by all the right-hand side upper bounds. Then,

$$\left\|x_{n+\ell}\right\|_2 \le \lambda^{n}\left\|x_0\right\|_2 + \frac{2\delta\max(\alpha,\beta)\left(1-\lambda^{n}\right)}{1-\lambda}\sqrt{k} \max_{-1\le i\le n+k-1}\left\|x_i\right\|_2, \tag{4.6}$$

for all $n \in \mathbf{N}$, for all $k \in \overline{k}$, for all $\ell \in \mathbf{N} \setminus \overline{n+k}$, which implies directly that

$$\begin{aligned}
&\max_{-1\le i\le n+k-1}\left(\left\|x_{n+i}\right\|_2\right) \\
&\quad\le \lambda^{n}\left\|x_0\right\|_2 + \frac{2\delta\max(\alpha,\beta)}{1-\lambda}\sqrt{k} \max_{-1\le i\le n+k-1}\left\|x_i\right\|_2 + \left(\left\|x_0\right\|_2 + \left\|x_{-1}\right\|_2\right), \\
&\qquad\qquad \forall n \in \mathbf{N},\ \forall k \in \overline{k}.
\end{aligned} \tag{4.7}$$

If the condition $\max(\alpha,\beta) < (1-\lambda)/2\delta\sqrt{k}$ with $\lambda \in [0,1)$ holds, then the second term of the right-hand side of (4.7) may be combined with the left-hand-side term to yield

$$\begin{aligned}
\|x_n\|_2 &\le \max_{-1\le i\le n+k-1} \|x_{n+i}\|_2 \\
&\le \left(\frac{1-\lambda}{1-\lambda-2\sqrt{k}\delta\max(\alpha,\beta)}\right)\left((1+\lambda^n)\|x_0\|_2 + \|x_{-1}\|_2\right) \\
&\le \left(\frac{1-\lambda}{1-\lambda-2\sqrt{k}\delta\max(\alpha,\beta)}\right)\left((1+\lambda^{n_0})\|x_0\|_2 + \|x_{-1}\|_2\right) \\
&\le (1+\varepsilon)\left(\frac{1-\lambda}{1-\lambda-2\sqrt{k}\delta\max(\alpha,\beta)}\right)\left(\|x_0\|_2 + \|x_{-1}\|_2\right) \qquad (4.8)\\
&\le (1+\varepsilon)\left(\frac{1-\lambda}{1-\lambda-2\sqrt{k}\delta\max(\alpha,\beta)}\right)\left(\sum_{i=1}^{2k}\max\left(a_i^2, b_i^2\right)\right)^{1/2} \\
&\le (1+\varepsilon)\left(\frac{1-\lambda}{1-\lambda-2\sqrt{k}\delta\max(\alpha,\beta)}\right)\left(\sum_{i=1}^{2k}\max\left(|a_i|, |b_i|\right)\right) \\
&\le 3\left(\frac{1-\lambda}{1-\lambda-2\sqrt{k}\delta\max(\alpha,\beta)}\right)\max\left(\|x_0\|_2, \|x_{-1}\|_2\right), \qquad (4.9)
\end{aligned}$$

for all $\varepsilon \in \mathbf{R}_+$, for all $n(\ge n_0) \in \mathbf{N}$, depending on n_0, which depends on ε, for any $\mathbf{N} \ni n_0 \ge \ln\varepsilon/\ln\lambda$, for all $\overline{x}_0 \in K_0$. Since K_0 is compact, it follows from (4.9) that any solution sequence is bounded for any $n \in \mathbf{N}$ and any finite initial conditions. Thus, the linear system (3.3)–(3.7) is globally Lyapunov stable. Also, since K_0 is compact, it follows from (4.8) that any solution sequence is permanent since it enters the prefixed compact set

$$K := \left\{x \in \mathbf{R}^k : |x_i| \le \frac{(1+\varepsilon)}{k}\left(\frac{1-\lambda}{1-\lambda-2\delta\sqrt{k}\max(\alpha,\beta)}\right)\left(\sum_{i=1}^{2k}\max\left(|a_i|, |a_i|\right)\right),\ \forall i \in \overline{k}\right\} \qquad (4.10)$$

for any $n(\ge n_0) \in \mathbf{N}$ and any finite initial conditions $(x_0 \vdots x_{-1})^T$ in K_0. Furthermore, K is independent of each particular set of initial conditions in K_0. Thus, the linear system (3.3)–(3.7) is permanent. □

The following technical result will be then useful as an auxiliary one to prove the stability of (1.1) under a set of sufficiency-type conditions based on extending the proof mechanism of Theorem 4.1 to the nonlinear case. Basically, it is proved that the functions $f_i : \mathbf{R} \to \mathbf{R}$, $i \in \overline{k}$, grow at most linearly with their argument.

Lemma 4.2. *$f_i \in C(\alpha_i, \delta_i, M_i) \Rightarrow f_i(x) = O(x)$, for all $i \in \overline{k}$. In addition, $f_i(x)$ is bounded for all $x \ge M_i$. The result also holds if $f_i \in BC(K_i, \alpha_i, \delta_i, M_i)$, for all $K_i \in \mathbf{R}$, for all $i \in \overline{k}$.*

Proof. Now, it is proved that $f_i(x) = O(x)$ (notation of "big Landau O" of x) for any $f_i \in C(\alpha_i, \delta_i, M_i)$, for all $i \in \overline{k}$. First, note that for all $i \in \overline{k}$ for some $\varepsilon_i : [M, \infty) \to \mathbf{R}_{0+}$, $f_i \in C(\alpha_i, \delta_i, M_i) \wedge x \geq M_i \in \mathbf{R}_+ \Rightarrow f_i(x) = \delta_i x - \varepsilon_i(x) \leq \delta_i x \Rightarrow f_i(x) = O(x)$ since $f_i(x) \leq \delta_i x + K$ for any $K \in \mathbf{R}_{0+}$, for all $x \geq M_i$. The result also holds if $f_i \in BC(K_i, \alpha_i, \delta_i, M_i)$, for all $K_i \in \mathbf{R}$, since $BC(K_i, \alpha_i, \delta_i, M_i) \subset C(K_i, \alpha_i, \delta_i, M_i)$. It is now proved by a contradiction argument that if $f_i \in C(K_i, \alpha_i, \delta_i, M_i)$, then it is bounded, for all $x < M_i$. Assume $x < M_i \in \mathbf{R}_+$ with $f_i(x_1)$ being arbitrarily large for some $x_{1i} < M_i$. Thus, there exists $M_{2i} \in \mathbf{R}_+$ being arbitrarily large so that $M_{2i} \leq f_i(x_{1i}) \leq f_i(M_i) \leq \delta_i M_i < \infty$ for $x_{1i} = \alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)} < M_i$ since $f_i \in C(\alpha_i, \delta_i, M_i)$ so that it is monotonically nondecreasing. This is a contradiction since M_{2i} is arbitrarily large. Thus, $f_i \in C(\alpha_i, \delta_i, M_i)$ is bounded, for all $x < M_i$. Since it is bounded, then $f_i(x) = O(x) \leq |f_i(x)| \leq \delta|x| + C_1$ for some finite $C_1 \in \mathbf{R}_+$ for $x < M_i$ as a result, so that $f_i(x) = O(x)$ on $\mathbf{R}$. Again, the result still holds if $f_i \in BC(K_i, \alpha_i, \delta_i, M_i)$, for all $K_i \in \mathbf{R}$. □

Theorem 4.3. *If* $\lambda := \max_{1\leq i\leq k}|\lambda_i| < 1 - \delta$, $\delta := \max_{1\leq i\leq k}\delta_i \in (0,1)$, $f_i \in BC(K_i, \alpha_i, \delta_i, M_i)$, *for all* $K_i \in \mathbf{R}$, *for all* $i \in \overline{k}$, *and* $\max(\max_{1\leq i\leq k}|\alpha_i|, \max_{1\leq i\leq k}|\beta_i|) < (1 - \max_{1\leq i\leq k}|\lambda_i| - \delta)/4\sqrt{k}\max_{1\leq i\leq k}\delta_i$, *then system* (1.1) *is globally Lyapunov stable for any finite arbitrary initial conditions. It is also permanent for any initial conditions* $\overline{x}_0 \in K_0(a_1, \ldots, a_{2k}, b_1, \ldots, b_{2k})$ *with the compact set* K_0 *defined in Theorem 4.1.*

Proof. If system (1.1) is taken, then (4.4) is replaced with

$$x_{n+1} = \Lambda x_n + \overline{B}\,\overline{x}_{n-1} + \left(\overline{f}(\overline{x}_{n-1}) - \overline{B}\,\overline{x}_{n-1}\right), \quad \forall n, j \in \mathbf{N}, \tag{4.11}$$

where

$$\overline{f}(\overline{x}_{n-1}) = \left(f_1\left(\alpha_2 x_n^{(2)} - \beta_{i+1}x_{n-1}^{(2)}\right), \ldots, f_k\left(\alpha_1 x_n^{(1)} - \beta_{i+1}x_{n-1}^{(1)}\right)\right)^T. \tag{4.12}$$

The description (4.6) is similar to (1.1) via an unforced linear system (3.3)–(3.7) with a forcing sequence $\{(\overline{f}(\overline{x}_{n-1}) - \overline{B}\,\overline{x}_{n-1})\}_0^\infty$ so that both solution sequences are identical under identical initial conditions. One gets directly from (4.11) that

$$x_{n+k} = \Lambda^{n+k}x_0 + \sum_{i=0}^{n+k-1}\Lambda^{n+k-i-1}\left(\overline{B}\,\overline{x}_i + \left(\overline{f}(\overline{x}_{n-1}) - \overline{B}\,\overline{x}_{n-1}\right)\right), \quad \forall n \in \mathbf{N},\ \forall k \in \overline{k}, \tag{4.13}$$

so that

$$\begin{aligned} \left\|x_{n+k}\right\|_2 &\leq \lambda^n\left\|x_0\right\|_2 + \frac{1-\lambda^n}{1-\lambda}\left(\|\overline{B}\|_2 \max_{0\leq i\leq n+k-1}\left\|\overline{x}_i\right\|_2 + \max_{0\leq i\leq n+k-1}\left\|\overline{f}(\overline{x}_i) - \overline{B}\overline{x}_i\right\|_2\right) \\ &\leq \lambda^n\left\|x_0\right\|_2 + \frac{1-\lambda^n}{1-\lambda}\left(2\|\overline{B}\|_2 + \delta\right)\max_{0\leq i\leq n+k-1}\left(\left\|\overline{x}_i\right\|_2\right) + \frac{(1-\lambda^n)C_1}{1-\lambda}. \end{aligned} \tag{4.14}$$

Then by direct extension of (4.7) when using (4.14),

$$\begin{aligned} \max_{-1\leq i\leq n+k-1}\left(\left\|x_{n+i}\right\|_2\right) \leq \lambda^n\left\|x_0\right\|_2 &+ \frac{1}{1-\lambda}\left(\left(4\delta\ \max(\alpha,\beta)\sqrt{k} + \delta\right)\max_{-1\leq i\leq n+k-1}\left\|x_i\right\|_2 + C_1\right) \\ &+ \left(\left\|x_0\right\|_2 + \left\|x_{-1}\right\|_2\right), \quad \forall n \in \mathbf{N},\ \forall k \in \overline{k}, \end{aligned} \tag{4.15}$$

with $\delta \in (0,1)$ for some finite $C_1 \in \mathbf{R}_+$ since $|f_i(x)| \le \delta|x| + C_1$, for all $i \in \overline{k}$, from Lemma 4.2. Thus, $\max_{-1\le i\le n+k-1}(\|x_{n+i}\|_2)$ may be regrouped in the left-hand side provided that

$$1 > \frac{1}{1-\lambda}\left(4\delta\ \max(\alpha,\beta)\sqrt{k}+\delta\right) \Longleftrightarrow \max(\alpha,\beta) < \frac{1-\lambda-\delta}{4\delta\sqrt{k}}. \tag{4.16}$$

Then, under similar reasoning as that used to derive (4.8)-(4.9), one gets from (4.15) that

$$\begin{aligned}
\|x_n\|_2 &\le \max_{-1\le i\le n+k-1}\|x_{n+i}\|_2 \\
&\le \left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right)\left((1-\lambda)\left((1+\lambda^{n})\|x_0\|_2+\|x_{-1}\|_2+C_1\right)\right) \\
&\le \left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right)\left((1-\lambda)\left((1+\lambda^{n_0})\|x_0\|_2+\|x_{-1}\|_2+C_1\right)\right) \\
&\le (1+\varepsilon)\left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right)\left((1-\lambda)\left(\|x_0\|_2+\|x_{-1}\|_2\right)+C_1\right) \\
&\le (1+\varepsilon)\left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right)\left((1-\lambda)\sum_{i=1}^{2k}\max\left(a_i^2,b_i^2\right)+C_1\right)^{1/2} \\
&\le (1+\varepsilon)\left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right)\left((1-\lambda)\max\left(\|x_0\|_2,\|x_{-1}\|_2\right)+C_1\right) \\
&\le 3\left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right)\left((1-\lambda)\max\left(\|x_0\|_2,\|x_{-1}\|_2\right)+C_1\right),
\end{aligned} \tag{4.17}$$

for all $\varepsilon \in \mathbf{R}_+$, for all $n(\ge n_0) \in \mathbf{N}$, depending on n_0, which depends on ε, for any $\mathbf{N} \ni n_0 \ge \ln\varepsilon/\ln\lambda$. The solution sequences are all bounded under any finite initial conditions and enter the compact set K defined by

$$\begin{aligned}
\Bigg\{x \in \mathbf{R}^k : |x_i| \le \frac{(1+\varepsilon)}{k}\Bigg(&\left(\frac{1}{1-\lambda-\delta\left(1+4\sqrt{k}\max(\alpha,\beta)\right)}\right) \\
&\times\left((1-\lambda)\sum_{i=1}^{2k}\max\left(|a_i|,|a_i|\right)+C_1\right)\Bigg),\ \forall i \in \overline{k}\Bigg\},
\end{aligned} \tag{4.18}$$

for all $n(\ge n_0) \in \mathbf{N}$, for any set of initial conditions in the compact set K_0. Furthermore, K is independent of each particular set of initial conditions in K_0. Then, system (1.1) is globally Lyapunov stable and permanent. □

Some simple properties concerning the instability of (1.1) based on simple constraints on the nonlinear functions, such as the stated boundedness from below of the strongest one of boundedness from above and below, are now established in the subsequent result.

Theorem 4.4. *The following properties hold.*

(i) *If* $|\lambda_i| \leq 1$ *and* $f_i : \mathbf{R} \to \mathbf{R}$ *is bounded from above and below, then* $|x_n^{(i)}|$ *is bounded, for all* $n \in \mathbf{N}$. *If* $|\lambda_i| > 1$ *and* $f_i : \mathbf{R} \to \mathbf{R}$ *is bounded from above and below, then almost all solution sequences* $\{x_n^{(i)}\}_0^\infty$ *for sufficiently large finite absolute values of the initial conditions are unbounded. Thus, system* (1.1) *is unstable under sufficiently large absolute values of the initial conditions for some* $i \in \overline{k}$.

(ii) *Assume that* $f_i \in B(K_i)$ *and* $|\lambda_i| > 1$ *for some* $i \in \overline{k}$. *Then* $|x_{n+1}^{(i)}| > |x_n^{(i)}|$, *for all* $n \in \mathbf{N}$, *and* $|x_n^{(i)}| \to \infty$ *as* $n \to \infty$ *if* $|x_0^{(i)}| > |K_i|/(|\lambda_i| - 1)$ ($|x_0^{(i)}| \geq |K_i|/(|\lambda_i| - 1)$ *if* $K_i \neq 0$). *Thus, system* (1.1) *is unstable under such sufficiently large absolute values of the initial conditions for some* $i \in \overline{k}$.

Proof. (i) If $-\infty < M_{1i} \leq f_i(x) \leq M_{2i} < \infty$, for all $x \in \mathbf{R}$, for some M_{ji}, $j = 1,2$, and some $i \in \overline{k}$, then

$$\begin{aligned}
&|\lambda_i^n|\left(|x_0^{(i)}| + \left|\sum_{j=0}^{\infty}\lambda_i^{-j-1}\right|\max\left(|M_{1i}|, |M_{2i}|\right)\right) \\
&\geq |x_n^{(i)}| \geq |\lambda_i^n|\left(|x_0^{(i)}| - \left|\sum_{j=0}^{n-1}\lambda_i^{-j-1}\right|\max_{0\leq j\leq i}\left|f_i\left(\alpha_i x_j^{(i+1)} - \beta_i x_{j-1}^{(i+1)}\right)\right|\right) \\
&\geq |\lambda_i^n|\left(|x_0^{(i)}| - \left|\sum_{j=0}^{n-1}\lambda_i^{-j-1}\right|\max\left(|M_{1i}|, |M_{2i}|\right)\right) \\
&\geq |\lambda_i^n|\left||x_0^{(i)}| - \left|\sum_{j=0}^{\infty}\lambda_i^{-j-1}\right|\max\left(|M_{1i}|, |M_{2i}|\right)\right|.
\end{aligned} \tag{4.19}$$

If $|\lambda_i| \leq 1$, then the sequence $\{|x_n^{(i)}|\}_0^\infty$ is bounded so that the sequence $\{|x_n^{(i)}|\}_0^\infty$ may be unbounded only if $|\lambda_i| > 1$. If $|\lambda_i| > 1$, then $0 \leq ||x_0^{(i)}| - |\sum_{j=0}^{\infty}\lambda_i^{-j-1}|\max(|M_{1i}|, |M_{2i}|)| < \infty$, and, furthermore, if $|x_0^{(i)}| > (|\lambda_i|/(|\lambda_i| - 1))\max(|M_{1i}|, |M_{2i}|) \geq |\sum_{j=0}^{\infty}\lambda_i^{-j-1}|\max(|M_{1i}|, |M_{2i}|)$, then there is a strictly monotonically increasing subsequence $\{|x_n^{(i)}|\}_{n\in S}$ of $\{|x_n^{(i)}|\}_0^\infty$, where $S := \{n_1, n_2, \ldots\}$ is a countable subset of $\mathbf{N}$, so that $|x_{n_{j+1}}^{(i)}| > |x_{n_j}^{(i)}|$, for all $n_j \in S$, and $|x_{n_j}^{(i)}| \to \infty$ as $S \ni n_j \to \infty$ (i.e., it diverges).

If $f_i \in BC(K_i, 1, \delta_i, M_i)$, then $-\infty < -|K_i'| \leq f_i(x) \leq \delta x$, for all $x(\geq M_i) \in \mathbf{R}$ and all K_i', such that $K_i + |K_i'| \geq 0$.

(ii) From (1.1), $f_i \in B(K_i)$, and $|\lambda_i| > 1$, it follows that

$$\begin{aligned}
\left(x_{n+1}^{(i)}\right)^2 - \left(x_n^{(i)}\right)^2 &= \left(\lambda_i^2 - 1\right)\left(x_n^{(i)}\right)^2 + f_i^2\left(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\right) + 2\lambda_i f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\right)x_n^{(i)} \\
&\geq g_n^{(i)}\left(|x_n^{(i)}|\right) := K_i^2 - \left(2\lambda_i|K_i| - \left(\lambda_i^2 - 1\right)|x_n^{(i)}|\right)|x_n^{(i)}| > 0
\end{aligned} \tag{4.20}$$

if $|x_0^{(i)}| > 2|\lambda_i||K_i|/(\lambda_i^2 - 1)$ ($|x_0^{(i)}| \geq |K_i|/(|\lambda_i| - 1)$ if $K_i \neq 0$) $\Rightarrow |x_{n+1}^{(i)}| > |x_n^{(i)}|$, for all $n \in \mathbf{N}$, so that the absolute value of the solution sequence is monotonically increasing so that it diverges. Less stringent condition for the initial conditions follows by calculating the zeros of the convex function $g_n^{(i)}(|x_n^{(i)}|)g_n^{(i)}(|x_n^{(i)}|)$ which are $g_{2n}^{(i)} = |K_i|/(|\lambda_i| - 1) \geq g_{1n}^{(i)} = |K_i|/(|\lambda_i| + 1)$, which implies that $g_n^{(i)}(|x_n^{(i)}|) \leq 0$ if $|x_n^{(i)}| \in [g_{1n}^{(i)}, g_{2n}^{(i)}]$, and $(x_{n+1}^{(i)})^2 - (x_n^{(i)})^2 \geq g_n^{(i)}(|x_n^{(i)}|) > 0$ if $|x_n^{(i)}| \in (-\infty, g_{2n}^{(i)}) \cup (g_{2n}^{(i)}, \infty)$. This directly completes the proof. □

5. Positivity results

Some positivity properties of the solution sequences of system (1.1) are now formulated in the subsequent formal result.

Theorem 5.1. *The following properties hold.*

(i) *Any solution vector sequence* $x_n := (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)})^T$ *of* (1.1) *is nonnegative, for all* $n \in \mathbf{N}$, *and any finite nonnegative* $x_0^{(i)} \geq 0$, *for all* $i \in \overline{k}$, *if* $f_i(\alpha_i x_0^{(i+1)} - \beta_i x_{-1}^{(i+1)}) \geq -\lambda_i x_0^{(i)}$, *for all* $i \in \overline{k}$, *and*

$$f_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}) \geq -\lambda_i x_n^{(i)} = -\left(\lambda_i^{n+1} x_0^{(i)} + \sum_{j=0}^{n-1} \lambda_i^{n-j} f_i(\alpha_i x_j^{(i+1)} - \beta_i x_{j-1}^{(i+1)})\right), \tag{5.1}$$

for all $i \in \overline{k}$, *for all* $n \in \mathbf{N}$. *Then, system* (1.1) *is positive.*

(ii) *Any solution vector sequence of* (1.1) *is nonnegative, for all* $n \in \mathbf{N}$, *and any finite nonnegative* $x_0^{(i)} \geq 0$, *for all* $i \in \overline{k}$, *if* $\lambda_i \in \mathbf{R}_{0+}$ *and* $f_i : \mathbf{R} \to \mathbf{R}_{0+}$, *for all* $i \in \overline{k}$. *Then, system* (1.1) *is positive.*

(iii) *Assume that* $\lambda_i \in \mathbf{R}_{0+}$, *for all* $i \in \overline{k}$, *and that there exist* $2k$ *real constants* $C_j^{(i)} \in \mathbf{R}_0^+$, $i \in \overline{k}$, $j = 1, 2$, *independent of* n, *such that*

$$\begin{aligned} -\infty < -C_1^{(i)} \leq f_i\left(\alpha_i x_0^{(i+1)} - \beta_i x_{-1}^{(i+1)}\right) \leq C_2^{(i)} < \infty, \quad \forall i \in \overline{k}, \\ -\infty < -C_1^{(i)} \leq \sum_{j=0}^{n-1} \lambda_i^{n-j-1} f_i\left(\alpha_i x_j^{(i+1)} - \beta_i x_{j-1}^{(i+1)}\right) \leq C_2^{(i)} < \infty, \quad \forall i \in \overline{k},\ \forall n \in \mathbf{N}. \end{aligned} \tag{5.2}$$

Then, the solution vector sequence is nonnegative, for all $n \in N_0 \setminus \overline{n}_0$, *for some finite* $n_0 \in \mathbf{N}_0$, *depending on* $x_j^{(i)}$ ($j = 0, -1$, *for all* $i \in \overline{k}$), *for any given finite* $x_0^{(i)} > 0$, *for all* $i \in \overline{k}$.

(iv) *Assume that* $f_i \in B(K_i)$ *and* $\lambda_i > 1$, *for all* $i \in \overline{k}$. *Then, any solution vector sequence of* (1.1) *is nonnegative; that is,* $x_n \in \mathbf{R}_{0+}^n$, *for all* $n \in \mathbf{N}$, *for any given finite* $x_{-1} \in \mathbf{R}^k$ *and some* $\mathbf{R}^k \ni x_0 \gg 0$ *of sufficiently large components (i.e.,* $x_0 \in \mathbf{R}_+^k$ *and* $x_0^{(i)} \geq \upsilon^{(i)} > 0$, *for some positive lower bound, with* $\upsilon^{(i)}$ *being sufficiently large, for all* $i \in \overline{k}$). *The solution vector sequence is positive by increasing the size of the initial condition of at least one component, and strictly positive by increasing simultaneously the sizes of the initial conditions of all the components. If* $f_i \in B(K_i)$ *with* $K_i \in \mathbf{R}_{0+}$, *for all* $i \in \overline{k}$, *then the constraints* $\lambda_i > 1$ *are weakened to* $\lambda_i \in \mathbf{R}_{0+}$, *for all* $i \in \overline{k}$ *(Property (ii)).*

(v) *Assume that* $[A \vdots B] > 0$ *with at least a positive entry per row, with the matrices* A *and* B *defined in* (3.4), *and that* $\lambda_i > 1$ *and* $f_i \in BC(K_i, 1, \delta_i, M_i)$, *for all* $i \in \overline{k}$. *Thus, there exists* $x_0 \gg 0$ *of sufficiently large finite components so that any solution is strictly positive, that is,* $x_n \gg 0$, *for all* $n \in \mathbf{N}$, *under initial condition* $x_0 \gg 0$. *The sizes are quantifiable from* *the knowledge of the scalars K_i, δ_i, M_i ($i \in \overline{k}$) and upper bounds of the nonzero entries of A and B.*

Proof. (i) The recursive use of (1.1) yields

$$x_n^{(i)} = \lambda_i^n x_0^{(i)} + \sum_{j=0}^{n-1} \lambda_i^{n-j-1} f_i\left(\alpha_i x_j^{(i+1)} - \beta_i x_{j-1}^{(i+1)}\right), \quad \forall i \in \overline{k},\ \forall n \in \mathbf{N}, \tag{5.3}$$

for any given $x_i^{(i)}$ ($i = 0, -1$), for all $i \in \overline{k}$. Then,

$$f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\right) \geq -\lambda_i x_n^{(i)} = -\left(\lambda_i^{n+1} x_0^{(i)} + \sum_{j=0}^{n-1} \lambda_i^{n-j} f_i\left(\alpha_i x_j^{(i+1)} - \beta_i x_{j-1}^{(i+1)}\right)\right), \quad \forall n \in \mathbf{N}_0$$
$$\Longrightarrow x_{n+1}^{(i)} = \lambda_i x_n^{(i)} + f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\right) \geq 0, \quad \forall n \in \mathbf{N}_0. \tag{5.4}$$

(ii) $x_n^{(i)} = \lambda_i x_{n-1}^{(i)} + f_i(\alpha_i x_{n-1}^{(i+1)} - \beta_i x_{n-2}^{(i+1)}) \geq 0$, for all $n \in \mathbf{N}$, if $x_0^{(i)} \geq 0$, $\lambda_i \in \mathbf{R}_{0+}$, and $f_i : \mathbf{R} \to \mathbf{R}_{0+}$, for all $i \in \overline{k}$.

(iii) $x_n^{(i)} = \lambda_i^n x_0^{(i)} + \sum_{j=0}^{n-1} \lambda_i^{n-j-1} f_i(\alpha_i x_j^{(i+1)} - \beta_i x_{j-1}^{(i+1)}) \geq \lambda_i^n x_0^{(i)} - C_1^{(i)} \geq 0$, for all $n \geq n_0 := \max_{1 \leq i \leq k}(\ln(C_1^{(i)}/x_0^{(i)})/\ln \lambda_i) - 1$, for all $i \in \overline{k}$. Such an n_0, being dependent on $x_0^{(i)}$, always exists for $\lambda_i > 1$ since $C_1^{(i)} < \infty$ and $\lambda_i^n x_0^{(i)} \to \infty$ as $n \to \infty$ for any $x_0^{(i)} > 0$, for all $i \in \overline{k}$.

(iv) Since $f_i : \mathbf{R} \to \mathbf{R}$, for all $i \in \overline{k}$, are bounded from below on $\mathbf{R}$, then $\max_{n \in \mathbf{N}} f_i(\alpha_i x_n^{(i+1)} - \beta_i x_n^{(i+1)}) \geq K_i > -\infty$, $\liminf_{n \to \infty} f_i(\alpha_i x_n^{(i+1)} - \beta_i x_n^{(i+1)}) \geq K_i > -\infty$ for some finite $K_i \in \mathbf{R}$, for all $i \in \overline{k}$. Irrespective of the value of K_i, since it is finite, there always exists a finite constant $K_i' \in \mathbf{R}_+$ fulfilling $K_i \geq -K_i'$ such that

$$\max_{n \in \mathbf{N}} f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_n^{(i+1)}\right) \geq -K_i' = -\left|K_i'\right| > -\infty,$$
$$\liminf_{n \to \infty} f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_n^{(i+1)}\right) \geq -\left|K_i'\right| > -\infty, \tag{5.5}$$

for all $i \in \overline{k}$. Since $\lambda_i > 1$, the series $\sum_{j=0}^{\infty} \lambda_i^{-j}$ converges so that

$$\sum_{i=0}^{n} \lambda_i^{-j} = \frac{1 - \lambda_i^{-(n+1)}}{1 - \lambda_i^{-1}} = \frac{\lambda_i^{n+1} - 1}{\lambda_i^n(\lambda_i - 1)} \leq \sum_{j=0}^{\infty} \lambda_i^{-j} = \frac{1}{1 - \lambda_i^{-1}} = \frac{\lambda_i}{\lambda_i - 1}, \quad \forall i \in \overline{k},\ \forall n \in \mathbf{N}. \tag{5.6}$$

Then,

$$\begin{aligned} x_n^{(i)} &= \lambda_i^n \left(x_0^{(i)} - \sum_{j=0}^{n-1} \lambda_i^{-j-1} \left|f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_n^{(i+1)}\right)\right|\right) \\ &\geq \lambda_i^n \left(x_0^{(i)} - \sum_{j=0}^{n-1} \lambda_i^{-j-1} \max_{n \in \mathbf{N}} \left|f_i\left(\alpha_i x_n^{(i+1)} - \beta_i x_n^{(i+1)}\right)\right|\right) \\ &\geq \lambda_i^n \left(x_0^{(i)} - \frac{(\lambda_i^{n+1} - 1)\left|K_i'\right|}{\lambda_i^n(\lambda_i - 1)}\right) \\ &\geq \lambda_i^n \left(x_0^{(i)} - \frac{\lambda_i \left|K_i'\right|}{\lambda_i - 1}\right), \end{aligned} \tag{5.7}$$

for all $i \in \overline{k}$, for all $n \in \mathbf{N}$. As a result, $x_n^{(i)} \in \mathbf{R}_{0+}$, for all $i \in \overline{k}$, for all $n \in \mathbf{N}$, if $x_0^{(i)} \geq \lambda |K_i'|/(\lambda - 1) > 0$, for all $i \in \overline{k}$. Then, $x_n \geq 0$, for all $n \in \mathbf{N}$. If $x_0^{(i)} > \lambda |K_i'|/(\lambda - 1)$ for at least one $i \in \overline{k}$, then $x_n > 0$, for all $n \in \mathbf{N}$. If $x_0^{(i)} > \lambda |K_i'|/(\lambda - 1)$, for all $i \in \overline{k}$, then $x_n \gg 0$, for all $n \in \mathbf{N}$.

(v) Define $M := (M_1, M_2, \ldots, M_k)^T \gg 0$ with the constants M_i of the sets $BC(K_i, 1, \delta_i, M_i)$, for all $i \in \overline{k}$. Since $f_i(x) = \delta_i x - \widetilde{f}_i(x)$ for some $\widetilde{f}_i : [M, \infty) \to \mathbf{R}_{0+}$, for all $i \in \overline{k}$, for all $x \geq M_i$, from the definition of the sets $C(1, \delta_i, M_i)$, it follows from (3.9) that

$$\overline{x}_n \geq \overline{A}^n \overline{x}_0 - \sum_{i=0}^{n-1} \overline{A}^{n-1-i} \overline{B} \widetilde{K}' \tag{5.8}$$

for any $\widetilde{K}' := (\widetilde{K}_1', \widetilde{K}_2', \ldots, \widetilde{K}_k')^T \gg 0$ such that $K_i \geq -|\widetilde{K}_i'|$, for all $i \in \overline{k}$. Since $\lambda_i > 1$, for all $i \in \overline{k}$, then from the structure of the matrix $\overline{A}$ in (3.9),

$$\begin{aligned} x_n^{(i)} &\geq \lambda_i^n x_0^{(i)} + \sum_{i=0}^{n-1} e_i^T \Lambda^{n-1-i} \overline{B}\, \overline{x}_i - \sum_{i=0}^{n-1} e_i^T \Lambda^{n-1-i} \overline{B} \widetilde{K}' \\ &\geq \lambda_i^n x_0^{(i)} + \sum_{i=0}^{n-1} e_i^T \Lambda^{n-1-i} \overline{B} \left(\overline{A}^i \overline{x}_0 - \sum_{j=0}^{i-1} \overline{A}^{i-1-j} \overline{B} \widetilde{K}' \right) - \sum_{i=0}^{n-1} e_i^T \Lambda^{n-1-i} \overline{B} \widetilde{K}' \\ &= \lambda_i^n x_0^{(i)} + \sum_{i=0}^{n-1} e_i^T \Lambda^{n-1-i} \overline{B}\, \overline{A}^i \overline{x}_0 - e_i^T \left(\Lambda^{n-1-i} + \sum_{j=0}^{i-1} \overline{A}^{i-1-j} \right) \overline{B} \widetilde{K}' \\ &= e_i^T \left(\lambda_i^n I_n + \sum_{i=0}^{n-1} \Lambda^{n-1-i} \overline{B}\, \overline{A}^i \right) \overline{x}_0 - e_i^T \left(\Lambda^{n-1-i} + \sum_{j=0}^{i-1} \overline{A}^{i-1-j} \right) \overline{B} \widetilde{K}' \gg M_i, \end{aligned} \tag{5.9}$$

since $\lambda_i > 1$, for all $i \in \overline{k}$, provided that it is sufficiently large, $x_0^{(i)} \geq \max(M_i, \upsilon_i) > 0$ (i.e., $x_0 \gg 0$ has sufficiently large positive components), for all $i \in \overline{k}$, for all $n \in \mathbf{N}$, where e_i^T is the ith unity vector in $\mathbf{R}^k$ of components $e_{ij} = \delta_{ij}$ (the Kronecker delta), for all $i, j \in \overline{k}$. □

Note that the properties associated with $f_i \in BC(K_i, 1, \delta_i, M_i)$, for all $i \in \overline{k}$, have not been invoked in Theorem 5.1(i)–(iii). Theorem 5.1(ii) implicitly assumes $f_i \in B(K_i)$, since they are assumed to be nonnegative, for all $i \in \overline{k}$.

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