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Research Article

Some Combined Relations between Contractive Mappings, Kannan Mappings, Reasonable Expansive Mappings, and T-Stability

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In recent literature concerning fixed point theory for self-mappings $T : X \to X$ in metric spaces (X, d), there are some new concepts which can be mutually related so that the inherent properties of each one might be combined for such self-mappings. Self-mappings $T : X \to X$ can be referred to, for instance, as Kannan-mappings, reasonable expansive mappings, and Picard T-stable mappings. Some relations between such concepts subject either to sufficient, necessary, or necessary and sufficient conditions are obtained so that in certain self-mappings can exhibit combined properties being inherent to each of its various characterizations.

1. Introduction

As it is wellknown fixed point theory and related techniques are of increasing interest for solving a wide class of mathematical problems where convergence of a trajectory or sequence to some equilibrium set is essential, (see, e.g., [1–7]). Some of the specific topics recently covered in the field of fixed point theory are, for instance as follows.

(1) The properties of the so-called n-times reasonably expansive mapping are investigated in [1] in complete metric spaces (X, d) as those fulfilling the property that $d(x, T^n x) \geq \beta d(x, Tx)$ for some real constant $\beta > 1$. The conditions for the existence of fixed points in such mappings are investigated.

(2) Strong convergence of the wellknown Halpern's iteration and variants is investigated in [2, 8] and several the references therein.

(3) Fixed point techniques have been recently used in [4] for the investigation of global stability of a wide class of time-delay dynamic systems which are modeled by functional equations.

(4) Generalized contractive mappings have been investigated in [5] and references therein, weakly contractive and nonexpansive mappings are investigated in [6] and references therein.

(5) The existence of fixed points of Liptchitzian semigroups has been investigated, for instance, in [3].

(6) Picard's T-stability is discussed in [9] related to the convergence of perturbed iterations to the same fixed points as the nominal iteration under certain conditions in a complete metric space.

(7) The so-called Kannan mappings in [10] are recently investigated in [11, 12] and references therein.

Let (X,d) be a metric space. Consider a self-mapping $T : X \to X$. The basic concepts used through the manuscript are the subsequent ones:

(1) $T : X \to X$ is k-contractive, following the contraction Banach's principle, if there exists a real constant $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y); \quad \forall x,y \in X, \tag{1.1}$$

(2) $T : X \to X$ is α-Kannan, [10–12], if there exists a real constant $\alpha \in [0,1/2)$ such that

$$d(Tx,Ty) \leq \alpha(d(x,Tx) + d(y,Ty)); \quad \forall x,y \in X, \tag{1.2}$$

(3) $T : X \to X$ is n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mapping if there exists a real constant $\beta > 1$ such that $d(x,T^n x) \geq \beta d(x,Tx)$; $\forall x \in X$, $\mathbf{Z}_+ \ni n \geq 2$, [1],

(4) Picard's T-stability means that if (X,d) is a complete metric space and Picard's iteration $x_{k+1} = Tx_k$ satisfies $d(y_{k+1},Ty_k) \to 0$ as $k \to \infty$ for $\{y_k\} \subset X$ then $\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_k = q \in F(T)$, that is, q is a fixed point of T, [9]. It is proven in [9] that, if the self-mapping T satisfies a property, referred to through this manuscript as the (L,m) property for some real constants $L \in \mathbf{R}_{0+}$ and $m \in [0,1) \in \mathbf{R}_{0+}$ (see Definition 1.2 in what follows), then Picard's iteration is T-stable if $\lim_{k\to\infty} d(y_{k+1},Ty_k) = 0$.

The following result is direct.

Proposition 1.1. *If a self-mapping $T : X \to X$ is k-contractive, then it is also k'-contractive; $\forall k' \in [k,1)$.*

If a self-mapping $T : X \to X$ is α-Kannan, then it is also α'-Kannan; $\forall \alpha' \in [\alpha,1/2)$.

The so- called the (L,m)-property is defined as follows.

Definition 1.2. *A self-mapping $T : X \to X$ with $F(T) \neq \emptyset$ possesses the (L,m)-property for some real constants $L \in \mathbf{R}_{0+}$ and $m \in [0,1) \in \mathbf{R}_{0+}$ if $d(Tx,q) \leq Ld(x,Tx) + md(x,q)$; $\forall q \in F(T)$, $\forall x \in X$.*

The above property has been introduced in [9] to discuss the T-stability of Picard's iteration. If the (L,m)-property is fulfilled in a complete metric space and, furthermore, $\lim_{k\to\infty} d(y_{k+1},Ty_k) = 0$, then Picard's iteration $x_{k+1} = Tx_k$ is T-stable defined as $d(x_{k+1},Ty_k) \to 0$ as $k \to \infty \Rightarrow x_k \to q \in F(T)$ as $k \to \infty$. The main results obtained in this paper rely on the following features.

(1) *In fact k-contractive mappings $T : X \to X$ are α-Kannan self-mappings and vice-versa under certain mutual constraints between the constants k and α, [10–12]. A necessary and sufficient condition for both properties to hold is given. Some of such constraints are obtained in the manuscript. The existence of fixed points and their potential uniqueness is discussed accordingly under completeness of the metric space, [1–4, 8–10, 13].*

(2) *If $T : X \to X$ is n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mapping then it cannot be contractive as expected but it is α-Kannan under certain constraints. The converse is also true under certain constraints. Some of such constraints referred to are obtained explicitly in the manuscript. The existence of fixed points is also discussed for two types of n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mappings proposed in [1].*

(3) *The (L, m)-property guaranteeing Picard's T-stability of iterative schemes, under the added condition $\lim_{k\to\infty} d(y_{k+1}, Ty_k) = 0$, is compatible with both contractive self-mappings and α-Kannan ones under certain constraints. A sufficient condition that as self-mapping possessing the (L, m)-property is α-Kannan is also given. It may be also fulfilled by n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mappings.*

1.1. Notation

Assume that $\mathbf{Z}$ and $\mathbf{R}$ are the sets of integer and real numbers, $\mathbf{Z}_+ := \{z \in \mathbf{Z} : z > 0\}$, $\mathbf{Z}_{0+} := \{z \in \mathbf{Z} : z \geq 0\}$, $\mathbf{R}_+ := \{r \in \mathbf{R} : r > 0\}$, $\mathbf{R}_{0+} := \{r \in \mathbf{R} : r \geq 0\}$.

If $T : X \to X$ is a self mapping in a metric space (X, d), then $F(T)$ denotes the set of fixed points of T.

2. Combined Compatible Relations of k-Contractive Mappings, α-Kannan Mappings, and the $(L - m)$-Property

It is of interest to establish when a k-contractive mapping is also α-Kannan and viceversa.

Theorem 2.1. *The following properties hold:*

(i) *if $T : X \to X$ is k-contractive with $k \in [0, 1/3)$ then it is α-Kannan with $\alpha = k/(1 - k)$,*

(ii) *$T : X \to X$ is k-contractive and α-Kannan if and only if*

$$\begin{aligned} d(Tx, Ty) &\leq \min(kd(x, y), \alpha(d(x, Tx) + d(y, Ty))) \\ &= k \min\left(d(x, y), \frac{\alpha}{k}(d(x, Tx) + d(y, Ty))\right) \\ &= \alpha \min\left(\frac{k}{\alpha} d(x, y), (d(x, Tx) + d(y, Ty))\right), \end{aligned} \tag{2.1}$$

(iii) *if $T : X \to X$ is k-contractive and α-Kannan with $k \neq 0$ and $\alpha \neq 0$ then the inequality*

$$\alpha(d(x, Tx) + d(y, Ty)) \leq kd(x, y) \tag{2.2}$$

cannot hold for all x, y in X,

(iv) *if* $T : X \to X$ *is* k*-contractive and* α*-Kannan with* $k \neq 0$*, and* $0 \neq \alpha < k$ *then the inequalities:*

$$
\begin{aligned}
d(x,y) &\leq \frac{\alpha}{k}\left(d(x,Tx)+d(y,Ty)\right),\\
d(x,y) &\leq \frac{\alpha}{k-\alpha}\min\left(d(x,Tx)+d(x,Ty),d(y,Tx)+d(y,Ty)\right),\\
d(Tx,Ty) &\leq \min\Bigg(\frac{k\alpha}{k-\alpha}\min\left(d(x,Tx)+d(x,Ty),d(y,Tx)+d(y,Ty)\right),\\
&\qquad \frac{k}{1-k}\left(d(x,Tx)+d(Ty,y)\right)\Bigg)
\end{aligned}
\tag{2.3}
$$

are feasible for all x, y *in* X.

Proof. (i) Since $T : X \to X$ is k-contractive, then

$$
d(Tx,Ty) \leq kd(x,y) \leq k\left(d(x,Tx)+d(Tx,Ty)+d(Ty,y)\right); \quad \forall x,y \in X, \tag{2.4}
$$

from the triangle inequality property of the distance in metric spaces. Since $k \in [0,1)$, then

$$
d(Tx,Ty) \leq \frac{k}{1-k}\left(d(x,Tx)+d(Ty,y)\right); \quad \forall x,y \in X, \tag{2.5}
$$

so that $T : X \to X$ is α-Kannan with $\alpha = k/(1-k)$ provided that $k/(1-k) < 1/2 \Leftrightarrow k < 1/3$. As a result, if $T : X \to X$ is k-contractive with $k \in [0,1/3)$, then it is also $k/(1-k)$-Kannan.

(ii) It is direct if $T : X \to X$ is k-contractive and α-Kannan with $k \neq 0$ and $\alpha \neq 0$. For $\alpha = k = 0$, the result holds trivially.

(iii) Proceed by contradiction. Assume that the inequality holds for $x, y \in X \cap \overline{F}(T)$ with $x = y$ where $F(T)$ is the (empty or nonempty) set of fixed points of T. Since $x = y$, the inequality leads to $2\alpha d(x,Tx) = 0$. This implies that $d(x,Tx) = 0$ since $\alpha \neq 0$. However, $d(x,Tx) > 0; \forall x \notin F(T)$, what is a contradiction. Therefore, the inequality cannot cold in X.

(iv) The first inequality can potentially hold even for the set of fixed points. Furthermore, one gets from the triangle inequality for the distance $d(x,Tx) \leq d(x,y) + d(y,Tx), \forall x,y \in X$:

$$
\begin{aligned}
kd(x,y) &\leq \alpha\left(d(x,Tx)+d(y,Ty)\right) \leq \alpha d(x,y) + \alpha\left(d(y,Tx)+d(y,Ty)\right)\\
&\Longrightarrow d(x,y) \leq \frac{\alpha}{k-\alpha}\left(d(y,Tx)+d(y,Ty)\right)
\end{aligned}
\tag{2.6}
$$

for all $x, y \in X$ since $\alpha < k$. Also, by using $d(y,Ty) \leq d(x,y) + d(x,Tx)$, one gets $d(x,y) \leq \alpha/(k-\alpha)(d(x,Tx)+d(x,Ty))$. As a result, the second inequality follows by combining both partial results. The third inequality follows from the second one and Property (i). Property (iv) has been proven. □

Theorem 2.1(ii) leads to the subsequent result.

Corollary 2.2. *If $T : X \to X$ is k-contractive and α-Kannan, then*

$$d\left(Tx, T^2x\right) \leq \min\left(k, \frac{\alpha}{1-\alpha}\right) d(x, Tx); \quad \forall x \in X. \tag{2.7}$$

Proof. One gets from Theorem 2.1(ii) for $y = Tx$ that $d(Tx, T^2x) \leq kd(x, Tx)$; $\forall x \in X$ and $d(Tx, T^2x) \leq \alpha(d(x, Tx) + d(Tx, T^2x)) \Rightarrow d(Tx, T^2x) \leq (\alpha/(1-\alpha))d(x, Tx)$; $\forall x \in X$. Both inequalities together yield the result. □

The following two results follows directly from Theorem 2.1(iii) for $y = Tx$.

Corollary 2.3. *If $T : X \to X$ is k-contractive and α-Kannan with $k > \alpha \neq 0$, then the inequality $d(Tx, T^2x) \leq ((k-\alpha)/\alpha)d(x, Tx)$ cannot hold $\forall x \in X$.*

Corollary 2.4. *If $T : X \to X$ is k-contractive and α-Kannan with $\alpha > k \neq 0$, then the inequality $(\alpha - k)d(x, Tx) + \alpha d(Tx, T^2x) \leq 0$ cannot hold for $x \in X \cap \overline{F}(T)$.*

The following three results follows directly from Theorem 2.1(iv) for $y = Tx$.

Corollary 2.5. *If $T : X \to X$ is k-contractive and α-Kannan with $k > \alpha \neq 0$, then the inequality $d(x, Tx) \leq (\alpha/(k-\alpha))d(Tx, T^2x)$ is feasible $\forall x \in X$.*

Proof. The proof follows since

$$d(x, Tx) \leq \frac{\alpha}{k}\left(d(x, Tx) + d\left(Tx, T^2x\right)\right) \Longrightarrow d\left(Tx, T^2x\right) \leq \left(1 - \frac{\alpha}{k}\right)^{-1} \frac{\alpha}{k} d\left(Tx, T^2x\right) \tag{2.8}$$

is feasible from the first feasible inequality in Theorem 2.1(ii) $\forall x \in X$ and $y = Tx$. □

Corollary 2.6. *If $T : X \to X$ is k-contractive and α-Kannan with $k > 2\alpha \neq 0$, then the inequality $d(x, Tx) \leq (\alpha/(k-\alpha)(k-2\alpha))d(x, T^2x)$ is feasible $\forall x \in X$.*

Proof. The proof follows since

$$d(x, Tx) \leq \frac{\alpha}{k-\alpha}\left(d(x, Tx) + d\left(x, T^2x\right)\right) \leq \left(1 - \frac{\alpha}{k-\alpha}\right)^{-1} \frac{\alpha}{k-\alpha} d\left(Tx, T^2x\right) \tag{2.9}$$

is feasible from the second feasible inequality in Theorem 2.1(ii) $\forall x \in X$ and $y = Tx$. □

Corollary 2.7. *If $T : X \to X$ is k-contractive and α-Kannan with $k > 2\alpha \neq 0$, then the inequality $d(x, Tx) \leq (\alpha/(k-\alpha)(k-2\alpha))d(x, T^2x)$ is feasible $\forall x \in X$.*

Proof. The proof follows directly since

$$\begin{aligned} d\left(Tx, T^2x\right) &\leq \frac{k\alpha}{k-\alpha}\left(d(x, Tx) + d\left(x, T^2x\right)\right), \\ d\left(Tx, T^2x\right) &\leq \left(1 - \frac{k^2\alpha}{(1-k)(k-\alpha)}\right)^{-1}\left(d(x, Tx) + d\left(x, T^2x\right)\right) \end{aligned} \tag{2.10}$$

are feasible from the third feasible inequality in Theorem 2.1(ii) $\forall x \in X$ and $y = Tx$. □

Remark 2.8. *It turns out from Definition 1.2 that if $T : X \to X$ has the (L, m) property for some real constants $L \in \mathbf{R}_{0+}$ and $m \in [0, 1) \in \mathbf{R}_{0+}$, then it has also the (L_0, m_0); $\forall L_0 \in [L, \infty)$, $\forall m_0 \in [m, 1)$. The subsequent result is concerned with some joint (L, m), α-Kannan and k-contractiveness of a self-mapping $T : X \to X$.*

Theorem 2.9. *The following properties hold:*

(i) *$T : X \to X$ is α-Kannan if it has the (L, m)-property for any real constants L and m which satisfy the constraints $\alpha = (L + m)/(1 - m)$, $0 \le L < (1 - 3m)/2, 0 \le m < 1/3$,*

(ii) *assume that $T : X \to X$ is k-contractive. Then, it is also $(k/(1 - k))$-Kannan and it possesses the $((k - m)/(1 - k), m)$-property for any real constant m which satisfies $0 \le m \le k < 1/3$,*

(iii) *assume that $T : X \to X$ is α-Kannan and $F(T) \neq \emptyset$. Then $T : X \to X$ has the (L, m)-property with $L = \alpha + 2/(1 - \alpha)$ and $\forall m \in (0, 1) \cap \mathbf{R}$,*

(iv) *assume that $T : X \to X$ is k-contractive with $k \in [0, 1/3) \cap \mathbf{R}$ and $F(T) \neq \emptyset$. Then $T : X \to X$ is $(k/(1-k))$-Kannan and it has the (L, m)-property with $L = (2-3k)/((1-k)(1-2k))$ and $\forall m \in (0, 1) \cap \mathbf{R}$.*

Proof. (i) If $T : X \to X$ has the (L, m)-property, one has from the triangle inequality for distances

$$d(Tx, q) \le (L + m)d(x, Tx) + md(Tx, q) \Longrightarrow d(Tx, q) \le \frac{L + m}{1 - m}d(x, Tx); \quad \forall q \in F(T),\ \forall x \in X, \tag{2.11}$$

since $m < 1$. The above inequality together with the triangle inequality leads to

$$d(Tx, Ty) \le (Tx, q) + d(Ty, q) \le \frac{L + m}{1 - m}(d(x, Tx) + d(y, Ty)); \quad \forall q \in F(T),\ \forall x \in X. \tag{2.12}$$

Thus, $T : X \to X$ is α-Kannan with $\alpha := (L + m)/(1 - m) < 1/2$ which holds if $0 \le L < (1-3m)/2$ and $0 \le m < 1/3$. Property (i) is proven. Furthermore, if $T : X \to X$ is k-contractive then it is also α-Kannan if $\alpha = k/(1 - k)$ with $k < 1/3$ from Theorem 2.1(ii). Then, $T : X \to X$ is k-contractive, α-Kannan, and it has the (L, m)-property if $\alpha := (L+m)/(1-m) = k/(1-k) < 1/2$ which holds for $k := (L+m)/(L+1) = \alpha/(1+\alpha) < 1/3$ if $0 \le L := (k-m)/(1-k) < (1-3m)/2$ and $0 \le m \le k < 1/3$ which is already fulfilled since $T : X \to X$ is α-Kannan with the $((k - m)/(1 - k), m)$-property. Property (ii) has been proven.

(iii) By using the triangle inequality for distances and taking $x \in X$ and $q \in F(T)$, one gets

$$\begin{aligned} d(Tx, q) &\le d\left(Tx, T^2x\right) + d\left(x, T^2x\right) + d(x, q) \\ &\le 2d\left(Tx, T^2x\right) + d(x, Tx) + d(x, q) \\ &\le \frac{1 + \alpha}{1 - \alpha}d(x, Tx) + d(x, q), \end{aligned} \tag{2.13}$$

for any real constant $m \in [0,1)$ after using the subsequent relation:

$$d\left(Tx,T^2x\right) \le \alpha\left(d(x,Tx) + d\left(Tx,T^2x\right)\right) \Longrightarrow d\left(Tx,T^2x\right) \le \frac{\alpha}{1-\alpha} d(x,Tx); \quad \forall x \in X, \tag{2.14}$$

which follows directly from the α-Kannan property. Furthermore, since $q = T^2q \in F(T)$, the relation (2.14) leads to

$$d(Tx,q) = d\left(Tx,T^2q\right) \Longrightarrow d(Tx,q) \le \alpha\left(d(x,Tx) + d\left(Tq,T^2q\right)\right) \le \alpha d(x,Tx); \quad \forall x \in X, \tag{2.15}$$

$$d(x,q) \le d(x,Tx) + d(Tx,q) \le (1+\alpha)d(x,Tx) + md(x,q); \quad \forall x \in X,\ \forall m \in [0,1) \cap \mathbf{R}. \tag{2.16}$$

Then, the substitution of (2.16) into (2.13) yields

$$d(Tx,q) \le \left(\alpha + \frac{2}{1-\alpha}\right) d(x,Tx) + md(x,q); \quad \forall x \in X,\ \forall m \in [\,0,1) \cap \mathbf{R} \tag{2.17}$$

which proves Property (iii). Property (iv) is a direct consequence of Properties (ii)-(iii) since $T : X \to X$ is α-Kannan with $\alpha = k/(1-k)$. □

Further results concerning α-Kannan mappings follow below.

Theorem 2.10. *Assume that $T : X \to X$ is α-Kannan. Then, the following properties hold:*

(i) $d(Tx,T^{n+1}x) \le \sum_{i=1}^{n} (\alpha/(1-\alpha))^i d(x,Tx) \le ((1-\alpha)/(1-2\alpha))d(x,Tx)$; $\forall x \in X$, $\forall n \in \mathbf{Z}_+$,

(ii) *if $T : X \to X$ is α-Kannan and k-contractive, then*

(ii.1) $d(Tx,T^2x) \le \min(k, \alpha/(1-\alpha))d(x,Tx)$; $\forall x \in X$,

(ii.2) $d(T^jx,T^{n+j+1}x) \le k^{m-1}d(Tx,T^{n+1}x) \le \sum_{i=1}^{n} (\alpha/(1-\alpha))^i k^{j-1} d(x,Tx) \le ((k^{j-1}(1-\alpha)/(1-2\alpha))d(x,Tx)$ $\forall x \in X$, $\forall n \in \mathbf{Z}_+$, $\forall j\ (\ge 2) \in \mathbf{Z}_+$,

(ii.3) $\lim_{j\to\infty} T^{n+j}x = z = z(x) \in \operatorname{cl} X$; $\forall x \in X$, $\forall n \in \mathbf{Z}_+$,

(iii) *if $T : X \to X$ is k-contractive for some $k \in [0,1/3)$, then*

$$d\left(T^jx,T^{n+j+1}x\right) \le k^{m-1}d\left(Tx,T^{n+1}x\right) \le \sum_{i=1}^{n}\left(\frac{\alpha}{1-\alpha}\right)^i k^{m-1}d(x,Tx)$$
$$\le \frac{k^{m-1}(1-\ 2k)}{1-3k} d(x,Tx) \tag{2.18}$$

$\forall x \in X$, $\forall n \in \mathbf{Z}_+$, $\forall m(\ge 2) \in \mathbf{Z}_+$, *also*, $\lim_{j\to\infty} T^{n+j}x = z = z(x) \in \operatorname{cl} X$; $\forall x \in X$, $\forall n \in \mathbf{Z}_+$,

(iv) *if (X,d) is a complete metric space and $T : X \to X$ is k-contractive for some $k \in [0,1/3)$ or if it is α-Kannan and k-contractive, then $z = \lim_{j\to\infty} T^{n+j}x \in X$ is independent of x; $\forall x \in X$, $\forall n \in \mathbf{Z}_+$ so that $F(T) = \{z\}$ consists of a unique fixed point.*

Proof. Proceed by complete induction by assuming that $d(Tx, T^{j+1}x) \leq \sum_{i=1}^{j} (\alpha/(1-\alpha))^i d(x, Tx)$; $\forall x \in X, \forall j \in \overline{n-1} := \{1, 2, \ldots, n-1\}$. Since $T : X \to X$ is α-Kannan, take $y = T^n x$ so that one gets from the triangle inequality for distances and the above assumption for $j \in \overline{n-1}$ that

$$
\begin{aligned}
d\left(Tx, T^{n+1}x\right) &\leq \alpha\left(d(x, Tx) + d\left(T^n x, T^{n+1}x\right)\right) \\
&\leq \alpha\left(d(x, Tx) + d(T^n x, Tx) + d\left(Tx, T^{n+1}x\right)\right) \\
&\Longrightarrow d\left(Tx, T^{n+1}x\right) \leq \frac{\alpha}{1-\alpha}(d(x, Tx) + d(T^n x, Tx)) \\
&\leq \frac{\alpha}{1-\alpha} d(x, Tx) + \left(\frac{\alpha}{1-\alpha}\right) \sum_{i=1}^{n-1} \left(\frac{\alpha}{1-\alpha}\right)^i d(x, Tx) \\
&= \sum_{i=1}^{n} \left(\frac{\alpha}{1-\alpha}\right)^i d(x, Tx); \quad \forall x \in X, \forall n \in \mathbf{Z}_+.
\end{aligned}
\tag{2.19}
$$

Since $\alpha/(1-\alpha) < 1$; $\forall \alpha \in [0, 1/2)$, then $\sum_{i=1}^{n} (\alpha/(1-\alpha))^i \leq \sum_{i=1}^{\infty} (\alpha/(1-\alpha))^i = 1/(1-\alpha/(1-\alpha)) = (1-\alpha)/(1-2\alpha)$ so that $d(Tx, T^{n+1}x) \leq ((1-\alpha)/(1-2\alpha)) d(x, Tx)$; $\forall x \in X$ and the proof of Property (i) is complete.

Property (ii.1) follows from Property (i), since T is α-Kannan, by taking into account that it is k-contractive Property (ii.2) follows directly from Property (i) and Theorem 2.1(i). Property (ii.3) follows from

$$
\begin{aligned}
&0 \leq d\left(T^j x, T^{n+j+1}x\right) \leq \frac{1-\alpha}{1-2\alpha} d(x, Tx) \left(\limsup_{j \to \infty} k^{j-1}\right) = 0 \\
&\Longrightarrow \lim_{j \to \infty} T^j x = \lim_{j \to \infty} T^{n+j} x, \quad \forall x \in X, \forall n \in \mathbf{Z}_+.
\end{aligned}
\tag{2.20}
$$

Property (iii) follows again directly from Property (i) and Theorem 2.1(i) and the first part of Property (ii) for $m \to \infty$.

Property (iv) follows directly from Properties (ii) and (iii) from the uniqueness of the fixed point Banach's contraction mapping principle since T is a strict contraction. □

Proposition 2.11. *If $T : X \to X$ is α-Kannan, then $d(Tx, x) \leq ((1-\alpha)/(1-2\alpha)) d(Tx, T^2 x)$; $\forall x \in X$. If, in addition, $T : X \to X$ is k-contractive, then $(1-2\alpha)/(1-\alpha) \leq k < 1$.*

Proof. It holds that

$$
\begin{aligned}
&d(Tx, x) \leq d\left(Tx, T^2 x\right) + d\left(T^2 x, x\right) \leq \frac{\alpha}{1-\alpha} d(Tx, x) + d\left(T^2 x, x\right); \quad \forall x \in X \\
&\Longrightarrow d(Tx, x) \leq \frac{1-\alpha}{1-2\alpha} d\left(T^2 x, x\right); \quad \forall x \in X,
\end{aligned}
\tag{2.21}
$$

for all $x \in X$ by using the triangle property of distances and Theorem 2.10(i). The first part of the result has been proven. The second part of the result follows since

$$d(Tx, x) \leq \frac{1-\alpha}{1-2\alpha} d\left(Tx, T^2 x\right) \leq \frac{(1-\alpha)k}{1-2\alpha} d(Tx, x); \quad \forall x \in X \text{ so that } k \geq \frac{(1-2\alpha)}{1-\alpha}, \tag{2.22}$$

if $T : X \to X$ is k-contractive. □

Remark 2.12. *If $T : X \to X$ is k-contractive and α-Kannan, it follows from Corollary 2.2 and Proposition 2.11 that* $1 > \min(k, \alpha/(1-\alpha)) \geq (1-\alpha/(1-\alpha), \alpha/(1-\alpha)); \forall x \in X$.

Proposition 2.13. *If $T : X \to X$ is α-Kannan then $d(Tx, T^{n+1}x) \leq ((1-\alpha)/(1-2\alpha))^2 d(Tx, T^2 x)$; $\forall x \in X$.*

Proof. It follows from Proposition 2.11 and Theorem 2.10(i) since

$$d\left(Tx, T^{n+1}x\right) \leq \frac{1-\alpha}{1-2\alpha} d(x, Tx) \leq \left(\frac{1-\alpha}{1-2\alpha}\right)^2 d\left(x, T^2 x\right); \quad \forall x \in X, \forall n \in \mathbf{Z}_+. \tag{2.23}$$

□

Proposition 2.14. *If $T : X \to X$ is α-Kannan for some $\alpha \in [1/3, 1/2)$, then*

$$\frac{1-\alpha}{\alpha} d\left(Tx, T^2 x\right) \leq d(Tx, x) \leq \frac{1-\alpha}{1-2\alpha} d\left(Tx, T^2 x\right); \quad \forall x \in X. \tag{2.24}$$

Proof. The upper-bound for $d(Tx, x)$ has been obtained in Proposition 2.11. Its lower-bound $((1-\alpha)/\alpha) d(Tx, T^2 x)$ follows from Theorem 2.10(i) subject to $(1-\alpha)/\alpha \leq (1-\alpha)/(1-2\alpha)$ which holds $\forall x \in X$ if and only if $\alpha \geq 1/3$. The proof is complete. □

3. Combined Compatible Results about the (L, m)-Property, α-Kannan-Mappings, and a Class of Expansive Mappings

Definition 3.1 (see [1]). *Let (X, d) be a complete metric space. Also, $T : X \to X$ is said to be an n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mapping if there exists a real constant $\beta > 1$ such that*

$$d(x, T^n x) \geq \beta d(x, Tx); \quad \forall x \in X, \mathbf{Z}_+ \ni n \geq 2. \tag{3.1}$$

Theorem 3.2. *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a continuous surjective self-mapping which is continuous everywhere in X and α-Kannan while it also satisfies $d(T^{n-1}x, T^n x) \geq \beta d(x, Tx)$ for some real constant $\beta > 1$, some n $(\geq 2) \in \mathbf{Z}_+$, $\forall x \in X$ (i.e., $T : X \to X$ is n ($\mathbf{Z}_+ \ni n \geq 2$) times reasonable expansive self-mapping). Then, the following properties hold if*

$\beta > 1/(1-\alpha)$:

(i) $d(x,Tx) \leq ((\beta(1-\alpha)-\alpha)/\beta(1-\alpha))d(T^{n-1}x,T^n x)$; $\forall x \in X$,

(ii) $T : X \to X$ *has a unique fixed point in* X,

(iii) $T : X \to X$ *has a fixed point in* X *even if it is not* α*-Kannan.*

Proof. Since $T : X \to X$ is α-Kannan and it satisfies $d(T^{n-1}x,T^n x) \geq \beta d(x,Tx)$; some real constant $\beta > 1$, some n $(\geq 2) \in \mathbf{Z}_+$, $\forall x \in X$, then

$$\begin{aligned}\alpha\Big(d\Big(T^{n-2}x,T^{n-1}x\Big)+d\Big(T^{n-1}x,T^n x\Big)\Big)-d(x,Tx) &\geq d\Big(T^{n-1}x,T^n x\Big)-d(x,Tx)\\ \geq (\beta-1)d(x,Tx) \Longrightarrow d(x,Tx) &\leq \frac{1}{\beta-1}\Big(d\Big(T^{n-1}x,T^n x\Big)-d(x,Tx)\Big); \quad \forall x \in X.\end{aligned} \tag{3.2}$$

Since $\alpha \in [0,1/2\ )$ and $\beta > 1$, then

$$d(x,Tx) \leq \frac{\beta(1-\alpha)-\alpha}{\beta(1-\alpha)}d\Big(T^{n-1}x,T^n x\Big); \quad \forall x \in X, \tag{3.3}$$

and Property (i) has been proven. Also,

$$\begin{aligned}d(x,Tx) &\leq \min\left(\frac{1}{\beta-1},\frac{\beta(1-\alpha)-\alpha}{\beta(1-\alpha)}\right)\Big(d\Big(T^{n-1}x,T^n x\Big)-d(x,Tx)\Big)\\ &= \frac{\beta(1-\alpha)-\alpha}{\beta(1-\alpha)}\Big(d\Big(T^{n-1}x,T^n x\Big)-d(x,Tx)\Big).\end{aligned} \tag{3.4}$$

The last expression can be rewritten as

$$d(f(x),g(x)) \leq \varphi(f(x))-\varphi(g(x)); \quad \forall x \in X, \tag{3.5}$$

where $g : X \to X$ is the identity mapping on X; that is, $g(x) = x$; $\forall x \in X$, $f : X \to X$ is defined by $f(x) = Tx = T(g(x))$; $\forall x \in X$ (and then it is a surjective mapping since T is surjective) and the functional $\varphi : \mathrm{Im}\ (T) \subset X \to \mathbf{R}_{0+}$ is defined as $\varphi(x) = (\beta(1-\alpha)-\alpha)/(\beta\ (1-\alpha))(\sum_{j=0}^{n-2} d(T^j x,T^{j+1}x))$. It turns out that $\varphi : \mathrm{Im}(T) \subset X \to \mathbf{R}_{0+}$ is continuous everywhere on its definition domain (and then lower semicontinuous bounded from below as a result) since the distance mapping $d : X\times X \to \mathbf{R}_{0+}$ is continuous on X. Then, $T : X \to X$ has a fixed point in X in [1, Lemma 2.4], even if $T : X \to X$ is not α-Kannan, since f is surjective on X, g is the identity mapping on X, and φ is lower semicontinuous bounded from below. The fixed point is unique since (X,d) is a complete metric space. Properties (ii)-(iii) have been proven. □

The subsequent result gives necessary conditions for Theorem 3.2 to hold as well as a sufficient condition for such a necessary condition to hold.

Theorem 3.3. *Let (X,d) be a complete metric space. Assume that $T : X \to X$ is a surjective self-mapping which is continuous everywhere in X which satisfies $d(T^{n-1}x,T^nx) \geq \beta d(x,Tx)$ for some real constant $\beta > 1$, some $n(\geq 2) \in \mathbf{Z}_+$, $\forall x \in X$. The following holds.* (i) *The following zero limit exists*

$$\lim_{j\to\infty} \left| d\left(T^{j+n}x, T^{jn+1}x\right) - d\left(T^{j+n-1}x, T^{j+n}x\right)\right| = 0; \quad \forall x \in X. \tag{3.6}$$

(ii) *If $T : X \to X$ is α-Kannan then a sufficient condition for Property (i) to hold is:*

$$\beta d(x,Tx) \geq \alpha\left(d\left(T^{n-1}x,T^nx\right) + d\left(T^nx,T^{n+1}x\right)\right); \quad \forall x \in X, \tag{3.7}$$

and a necessary condition for the above sufficient condition to hold is:

$$d\left(T^{n-1}x,T^nx\right) \leq \frac{\beta}{\alpha} d(x,Tx); \quad \forall x \in X. \tag{3.8}$$

(iii) *If $T : X \to X$ is α-Kannan then two joint necessary conditions for Property (i) to hold are:*

$$\begin{gathered}\limsup_{j\to\infty}\left(\beta d\left(T^{j+1}x,T^{j+2}x\right) - \alpha\left(d\left(T^{j+n-2}x,T^{j+n-1}x\right) + d\left(T^{j+n-1}x,T^{j+n}x\right)\right)\right) \leq 0,\\ \liminf_{j\to\infty}\left(\alpha\left(d\left(T^{j+n-1}x,T^{j+n}x\right) + d\left(T^{j+n}x,T^{j+n+1}x\right)\right) - \beta d\left(T^jx,T^{j+1}x\right)\right) \geq 0\end{gathered} \tag{3.9}$$

and such limits superior and inferior coincide as existing limits and are zero.

Proof. (i) Assume that Property (i) does not hold. Then, T has not a fixed point in X what contradicts Theorem 3.2(iii). Thus, Property (i) holds.

(ii) The condition $d(T^{n-1}x,T^nx) \geq \beta d(x,Tx)$; $\forall x \in X$ together with the α-Kannan property yield:

$$\begin{gathered}-\beta d(x,Tx) + \alpha\left(d\left(T^{n-1}x,T^nx\right) + d\left(T^nx,T^{n+1}x\right)\right),\\ d\left(T^nx,T^{n+1}x\right) - d\left(T^{n-1}x,T^{\,n}x\right) \geq \beta d\left(Tx,T^2x\right) - \alpha\left(d\left(T^{n-2}x,T^{n-1}x\right) + d\left(T^{n-1}x,T^nx\right)\right)\end{gathered} \tag{3.10}$$

for all $x \in X$. If

$$\beta d(x,Tx) \geq \alpha\ \left(d\left(T^{n-1}x,T^nx\right) + d\left(T^nx,T^{n+1}x\right)\right) \geq \alpha d\left(T^{n-1}x,T^{n+1}x\right); \quad \forall x \in X \tag{3.11}$$

then

$$\begin{aligned}d\left(T^nx,T^{n+1}x\right) \leq d\left(T^{n-1}x,T^nx\right); \quad \forall x \in X \Longrightarrow d\left(T^{j+n}x,T^{j+n+1}x\right)\\ \equiv d\left(T^nz_j,T^{n+1}z_j\right) \leq d\left(T^{j+n-1}x,T^{j+n}x\right) \equiv d\left(T^{n-1}z_j,T^nz_j\right)\end{aligned} \tag{3.12}$$

with $z_j = T^j x \in X, \forall j \in \mathbf{Z}_+$ since

$$T^{j+n+1}x = T^{n+1}\left(T^j x\right) = T^{n+1}z_j; \qquad T^{j+n-1}x = T^{n-1}\left(T^j x\right) = T^{n-1}z_j \tag{3.13}$$

for all $x \in X, \forall j \in \mathbf{Z}_+$ so that $d(T^{j+n}x, T^{j+n+1}x) - d(T^{j+n-1}x, T^{j+n}x) \to 0$ as $\mathbf{Z}_+ \ni j \to \infty$ is a a sufficient condition for Property (i) to hold. The necessary condition for the above sufficient to hold follows directly from the constraint $\beta d(x, Tx) \geq \alpha d(T^{n-1}x, T^{n+1}x); \forall x \in X$.

(iii) It follows since the subsequent constraints follow directly from the hypotheses and $T : X \to X$ has a fixed point

$$\begin{aligned}
&\beta d\left(T^{j+1}x, T^{j+2}x\right) - \alpha\left(d\left(T^{j+n-2}x, T^{j+n-1}x\right) + d\left(T^{j+n-1}x, T^{j+n}x\right)\right)\\
&\quad \leq d\left(T^{j+n}x, T^{j+n+1}x\right) - d\left(T^{j+n-1}x, T^{j+n}x\right)\\
&\quad \leq \alpha\left(d\left(T^{j+n-1}x, T^{j+n}x\right) + d\left(T^{j+n}x, T^{j+n+1}x\right)\right) - \beta d\left(T^j x, T^{j+1}x\right); \quad \forall x \in X
\end{aligned} \tag{3.14}$$

□

Theorem 3.2 may be generalized by generalizing the inequality $d(T^{n-1}x, T^n x) \geq \beta d(x, Tx)$ to eventually involve other powers of T, not necessarily being respectively identical to $(n-1)$ and n, as follows.

Theorem 3.4. *Let (X, d) be a complete metric space. Then, the following properties hold.*

(i) *assume that $T : X \to X$ is a surjective self-mapping which is continuous everywhere in X and satisfies:*

$$\begin{aligned}
&d\left(T^{n-j}x, T^{n-j-1}x\right)\\
&\quad \geq \beta_{n-j-1} d\left(T^j x, T^{j+1}x\right); \quad \forall j \in J\ (\neq \emptyset) \subset \overline{n-1} \cup \{0\},\ \text{with } \mathbf{Z}_+ \supset \overline{j} := \{1, 2, \ldots, j\}
\end{aligned} \tag{3.15}$$

for some real constants $\beta_{n-j-1} > 1$; $\forall j \in J$, some $n\ (\geq 2) \in \mathbf{Z}_+$, $\forall x \in X$, then, $T : X \to X$ has at least a fixed point in X and it may eventually possess $\delta = \operatorname{card} J \geq 1$ fixed points in X.

(ii) *if Property (i) holds for $J = \overline{n-1} \cup \{0\}$ then $T : X \to X$ has at least a fixed point in X and, furthermore,*

$$d(T^n x, x) \leq \sum_{i=0}^{n-1} \frac{1}{\beta_{n-i-1} - 1}\left(d\left(T^{n-i}x, T^{n-i-1}x\right) - d\left(T^i x, T^{i+1}x\right)\right); \quad \forall x \in X. \tag{3.16}$$

Proof. (i) From the statement constraints, it follows that

$$d\left(T^j x, T^{j+1}x\right) \leq \frac{1}{\beta_{n-j-1} - 1}\left(d\left(T^{n-j}x, T^{n-j-1}x\right) - d\left(T^j x, T^{j+1}x\right)\right); \quad \forall x \in X,\ \forall j \in J \tag{3.17}$$

so that

$$d\left(T^j x, T^{j+1} x\right) \leq \frac{1}{\beta_{n-j-1}-1}\left(d\left(T^{n-j} x, T^{n-j-1} x\right)-d\left(T^j x, T^{j+1} x\right)\right)=\varphi_j\left(f_j(x)\right)-\varphi_j\left(g_j(x)\right) \tag{3.18}$$

$\forall x \in X, \forall j \in J$ where each functional $\varphi_j : X \to \mathbf{R}_0^+; \forall j \in J$ is defined by

$$\varphi_j\left(g_j(x)\right)=\varphi_j\left(T^j x\right):=\sum_{i=j}^{n-2} \frac{d\left(T^{i+1} x, T^i x\right)}{\beta_i-1} \tag{3.19}$$

and the functions $\varphi_j : X \to X$ and $f_j : X \to X$ are defined, respectively, as $g_j(Tx) = T^j x$, $f_j(x) = g_j(Tx) = T^{j+1}x; \forall x \in X, \forall j \in J$. Note that $\varphi_j : X \to \mathbf{R}_0^+, \forall j \in J$ is continuous, and then lower semicontinuous, on $X; \forall j \in J$ since $g_j : X \to X$ and $f_j : X \to X$ are both continuous in X. Since $T : X \to X$ is surjective then $T^j : X \to X$ is also surjective $\forall j \in \mathbf{Z}_+$ so that $g_j : X \to X$ and $f_j : X \to X$ are also surjective $\forall j \in \mathbf{Z}_+$. From [1, Lemma 2.4], they have a coincidence point since (3.18) holds and $\varphi_j : X \to \mathbf{R}_0^+, \forall j \in J$ is continuous. Then, there exists $X \ni q_j = Tq_j = T^j z_j = T(T^j z_j)$ for some $z_j \in X$ for each $j \in J$ so that $q_j \in F(T)$ so that $F(T) \neq \emptyset$ with card $F(T) \geq 1$ provided that $\emptyset \neq J = \overline{n-1} \cup \{0\}$.

(ii) It follows directly from Property (i), (3.18) and $d(T^n x, x) \leq \sum_{i=0}^{n-1}(d(T^i x, T^{i+1}x) - d(T^i x, T^{i+1} x)); \forall x \in X$. □

Remark 3.5. *Note that although card $F(T) \geq 1$ if $J \neq \emptyset$, it is not proven that card $F(T) \geq$ card J since some of the existing fixed points for $j \in J$ can mutually coincide or even more than one fixed point can eventually exist for each $j \in J$.*

It is wellknown that nonexpansive and asymptotically non-expansive mappings can have fixed points as contractions have. See, for instance, [1, 2, 6, 14–18]. However, and generally speaking, n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mappings do not necessarily have a fixed point, although they might have them, [1]. It has been proven in [1] that continuous and surjective n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mappings $T : X \to X$ in complete metric spaces (X, d) have a fixed pointing X if they fulfil the property:

$$d\left(T^n x, T^n y\right) \geq \gamma \min\ \left(d(x, y), d\left(y, T^n y\right)\right); \quad \forall x, y \in X, \mathbf{Z}_+ \ni n \geq 2, \text{ some real constant } \gamma>1. \tag{3.20}$$

Proposition 3.6. *Assume that $d(T^n x, T^n y) \geq \gamma \min(d(x, y), d(y, T^n y)); \forall x, y \in X$, $\mathbf{Z}_+ \ni n \geq 2$, some real constant $\gamma > 1$. Then,*

$$\begin{aligned} d\left(T^n x, T^n y\right) & \geq \gamma \min \left(d(x, y), d\left(x, T^n x\right)\right) \wedge d\left(T^n x, T^n y\right) \\ & \geq \gamma \min \left(d(x, y), d\left(y, T^n y\right), d\left(x, T^n x\right)\right) \wedge d\left(T^n x, T^n y\right) \\ & \geq \gamma \min \left(d(x, y), \min \left(d\left(y, T^n y\right), d\left(x, T^n x\right)\right)\right) \end{aligned} \tag{3.21}$$

for all $x, y \in X$, $\mathbf{Z}_+ \ni n \geq 2$, some real constant $\gamma > 1$.

Proof. It follows directly from (3.20) by interchanging $x \in X$ and $y \in X$ in (3.20). □

Proposition 3.7. *If* (3.20) *holds* $\forall x, y \in X$, $\mathbf{Z}_+ \ni n \geq 2$, *then*

$$d\left(T^n x, T^{n+1} x\right) \geq \gamma \min\left(d(x, Tx), d\left(Tx, T^{n+1} x\right)\right) \tag{3.22}$$

for all $x, y \in X$, $\mathbf{Z}_+ \ni n \geq 2$.

Proof. Take $X \ni y = Tx$; $\forall x \in X$ in (3.20). □

Proposition 3.8. *Assume that* $T : X \to X$ *is an* ($\mathbf{Z}_+ \ni n \geq 2$)*-times reasonable expansive self-mapping which satisfies* (3.20) *and* (X, d) *is a complete metric space. Then*

$$d\left(T^n x, T^{n+1} x\right) \geq \gamma d(x, T^n x) \geq \gamma \beta d(x, Tx); \quad \forall x \in X,\ \mathbf{Z}_+ \ni n \geq 2,\ \text{some real constants } \beta > 1,\ \gamma > 1. \tag{3.23}$$

Proof. If follows from (3.20) and Proposition 3.6 that

$$d\left(T^n x, T^{n+1} x\right) \geq \gamma \min(d(x, Tx), d(x, T^n x)) = \gamma d(x, Tx); \quad \forall x \in X,\ \mathbf{Z}_+ \ni n \geq 2, \tag{3.24}$$

for $X \ni y = Tx$; $\forall x \in X$ provided that $d(x, Tx) \geq d(x, T^n x)$, and

$$d\left(T^n x, T^{n+1} x\right) \geq \gamma \min(d(x, Tx), d(x, T^n x)) = \gamma d(x, T^n x); \quad \forall x \in X,\ \mathbf{Z}_+ \ni n \geq 2, \tag{3.25}$$

for $X \ni y = Tx$; $\forall x \in X$ provided that $d(x, Tx) \geq d(x, T^n x)$.

Assume that (3.23) holds. Since $T : X \to X$ is an n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mapping, there exists a real constant $\beta > 1$ such that $d(x, Tx) \geq d\ (x, T^n x) \geq \beta d(x, Tx)$; $\forall x \in X$ which is impossible since $\beta > 1$. Instead of (3.24) one can have:

$$d\left(T^n x, T^{n+1} x\right) \geq \gamma \min(d(x, Tx), d(x, T^n x)) = \gamma \mathrm{d}(x, Tx); \quad \forall x \in X,\ \mathbf{Z}_+ \ni n \geq 2 \tag{3.26}$$

for $X \ni y = Tx$; $\forall x \in X$ provided that $d(x, Tx) \leq d(x, T^n x)$. Since $T : X \to X$ is an n ($\mathbf{Z}_+ \ni n \geq 2$) times reasonable expansive self-mapping, there exists a real constant $\beta > 1$ such that

$$\min\left(d(x, Tx), \gamma^{-1} d\left(T^n x, T^{n+1} x\right)\right) \geq d(x, T^n x) \geq \max(d(x, Tx), \beta d(x, Tx)); \quad \forall x \in X. \tag{3.27}$$

Then, either $d(x, T^n x) = d(x, Tx) = 0$ so that $x \in F(T)$, or

$$\gamma^{-1} d\left(T^n x, T^{n+1} x\right) \geq d(x, T^n x) \geq \beta d(x, Tx) \quad \text{with } d(x, Tx) \neq 0 \tag{3.28}$$

and the proof is complete. □

Proposition 3.8 may be rewritten in a more clear equivalent form as follows:

Proposition 3.9. *A necessary condition for a self-mapping* $T : X \to X$ *in complete metric space* (X, d) *to be an* n *(*$\mathbf{Z}_+ \ni n \geq 2$*) times reasonable expansive self-mapping which satisfies Property* (3.20) *is that* (3.23) *holds.*

Theorem 2.10 of [1] may be reformulated subject to the above necessary condition as follows.

Theorem 3.10. *Assume that* (X, d) *is a complete metric space and that* $T : X \to X$ *is a continuous surjective* n *(*$\mathbf{Z}_+ \ni n \geq 2$*)-times reasonable expansive self-mapping which satisfies the constraint* (3.20) *and the necessary condition of Proposition 3.9. Then* T *has a fixed point in* X*.*

If the self-mapping $T : X \to X$ satisfies Theorem 3.10 and it is also α-Kannan, then the subsequent result holds:

Theorem 3.11. *Assume that Theorem 3.10 holds. Then,* $T : X \to X$ *is in addition* α*-Kannan if and only if*

$$\gamma\beta d(x, Tx) \leq \gamma d(x, T^n x) \leq d\left(T^n x, T^{n+1} x\right) \leq \frac{\alpha}{1-\alpha} d\left(T^{n-1} x, T^n x\right); \quad \forall x \in X \tag{3.29}$$

and the existing fixed point is unique.

(ii) *The following inequalities also hold:*

$$d(x, Tx) \leq \frac{1}{\beta} d(x, T^n x) \leq \frac{\alpha}{\gamma\beta(1-\alpha)} d\left(T^{n-1} x, T^n x\right); \quad \forall x \in X. \tag{3.30}$$

Proof. The proof follows from (3.28) and the α-Kannan-property.

$$\gamma^{-1}\alpha\left(d\left(T^{n-1} x, T^n x\right) + d\left(T^n x, T^{n+1} x\right)\right) \geq \gamma^{-1} d\left(T^n x, T^{n+1} x\right); \quad \forall x \in X \tag{3.31}$$

which, together with (3.28), yields (3.29) since $\alpha \in [0, 1/2)$. The fixed point of $T : X \to X$ (Theorem 3.10) is unique since (X, d) is a complete metric space. Property (i) has been proven. Property (ii) is a direct result from Property (i) and (3.28). □

Remark 3.12. *It is interesting to compare Theorem 3.2 with Theorem 3.11, subject to Proposition 3.9, and their respective guaranteed inequalities for distances in* X *for the case when* $T : X \to X$ *is simultaneously* α*-Kannan and* n *(*$\mathbf{Z}_+ \ni n \geq 2$*)-times reasonable expansive self-mapping. Note that Theorem 3.2 is based on the fulfilment of the inequality* $d(T^{n-1}x, T^n x) \geq \beta d(x, Tx); \forall x \in X$*, for some* $\beta > 1$ *for some real constant* $\beta > 1$ *while Theorem 3.11 is based on* $d(T^n x, T^{n+1} x) \geq \gamma d(x, T^n x) \geq \gamma\beta d(x, Tx); \forall x \in X$ *for some real constants* $\beta > 1, \gamma > 1$*.*

It is also of interest to investigate when $T : X \to X$ being a continuous surjective n ($\mathbf{Z}_+ \ni n \geq 2$)-times reasonable expansive self-mapping (Definition 3.1) satisfying either Theorem 3.10 or Theorem 3.2 has also the (L, m)-property for some real constants $L \in \mathbf{R}_{0+}$ and $m \in [0, 1) \in \mathbf{R}_{0+}$ (Definition 1.2). Note that if either Theorem 3.10 or Theorem 3.2 are fulfilled then $F(T) \neq \emptyset$ so that Definition 1.2 is well-posed.

Theorem 3.13. *The following properties hold:*

(i) *assume that* (X, d) *is a nonempty complete metric space and that* $T : X \to X$ *is a continuous surjective* n $(\mathbf{Z}_+ \ni n \geq 2)$*-times reasonable expansive self-mapping according to Theorem 3.10 so that it has a fixed point in* X*. Then,* $T : X \to X$ *also possesses the* (L, m)*-property for some real constants* $L \in \mathbf{R}_{0+}$ *and* $m \in [0, 1) \in \mathbf{R}_{0+}$ *if*

$$L\left(d\left(T^{n-1}x, T^n x\right) + d\left(T^n x, T^{n+1}x\right)\right) + m\left(d\left(T^{n-1}x, q\right) + d(T^n x, q)\right)$$
$$\geq d(T^n x, q) + d\left(T^{n+1}x, q\right) \geq d\left(T^n x, T^{n+1}x\right) \geq \gamma d(x, T^n x) \geq \beta d(x, T^n x) \qquad (3.32)$$
$$\forall q \in F(T), \quad \forall x \in X.$$

Two necessary conditions for the above condition to hold are:

$$d(T^n x, q) \leq \frac{1}{1-m}\left(L\left[d\left(T^{n-1}x, T^n x\right) + d\left(T^n x, T^{n+1}x\right)\right] + md\left(T^{n-1}x, q\right) - d\left(T^{n+1}x, q\right)\right)$$
$$\forall q \in F(T), \quad \forall x \in X, \qquad (3.33)$$

provided that $d(T^{n+1}x, q) \leq L[d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x)] + md(T^{n-1}x, q)$*; for all* $q \in F(T)$*, for all* $x \in X$ *and*

$$d\left(T^n x, T^{n+1}x\right) \geq \gamma d(x, T^n x) \geq \beta\gamma d(x, Tx); \quad \forall x \in X, \text{ some real constants } \beta, \gamma > 1 \qquad (3.34)$$

(ii) *assume that* (X, d) *is a nonempty complete metric space and that* $T : X \to X$ *is a continuous surjective* n $(\mathbf{Z}_+ \ni n \geq 2)$ *times reasonable expansive self-mapping which satisfies Theorem 3.2. Then,* $T : X \to X$ *also possesses the* (L, m)*-property for some real constants* $L \in \mathbf{R}_{0+}$ *and* $m \in [0, 1) \in \mathbf{R}_{0+}$ *if and only if*

$$d\left(T^j x, q\right) \leq Ld\left(T^{j-1}x, T^j x\right) + md\left(T^{j-1}x, q\right); \quad j = n-1, n, \quad \forall q \in F(T), \forall x \in X. \qquad (3.35)$$

Two necessary conditions for the above necessary and sufficient condition to hold are,

$$d\left(T^{n-1}x, q\right) \leq \frac{1}{1-m}\left(L\left[d\left(T^{n-2}x, T^{n-1}x\right) + d\left(T^{n-1}x, T^n x\right)\right] + md\left(T^{n-2}x, q\right) - d(T^n x, q)\right);$$
$$\forall q \in F(T), \quad \forall x \in X, \qquad (3.36)$$

provided that $d(T^n x, q) \leq L[d(T^{n-2}x, T^{n-1}x) + d(T^{n-1}x, T^n x)] + md(T^{n-2}x, q)$*;* $\forall q \in F(T), \forall x \in X$ *and* $d(T^{n-1}x, T^n x) \geq \beta d(x, Tx)$*;* $\forall x \in X$*, some real constant* $\beta > 1$*.*

Proof. It follows from (3.28) and the (L, m)-property under direct calculations. □

Remark 3.14. *Note from direct inspection of Definition 1.2 and the triangle property of distances that if $T : X \to X$ has the (L, m)-property then for any $x \in X$.*

$$d(Tx, q) \le \frac{L+m}{1-m} d(x, Tx); \quad \forall L \in \mathbf{R}_{0+},$$
$$d(Tx, q) \le \min\left(\frac{L+m}{1-m} d(x, Tx), \frac{L+m}{1-L} d(x, q)\right); \quad \forall L \in [0,1) \cap \mathbf{R}_{0+},\ \forall q \in F(T). \tag{3.37}$$

The subsequent results are focused on the combinations of one of the two conditions below, compatible under extra conditions with the existence of fixed points, with the (L, m)-property in a metric space (X, d) for some real constant $\beta > 1$:

(a)

$$d\left(T^{n-1}x, T^n x\right) \ge \beta d(x, Tx); \quad \forall x \in X, \tag{3.38}$$

where $T : X \to X$ (see Theorem 3.2)

(b)

$$d(T^n x, T^n y) \ge \beta \min(d(x, y), d(y, T^n y)); \quad \forall x \in X, \tag{3.39}$$

where $T : X \to X$ being a surjective n ($\mathbf{Z}_+ \ni n \ge 2$)-times reasonable expansive self-mapping (see Propositions 3.7 and 3.8). Note by direct inspection that (3.40) is equivalent to

$$d(x, y) \le \min\left(d(y, T^n y), \beta^{-1} d(T^n x, T^n y)\right) \vee d(y, T^n y)$$
$$\le \min\left(d(xy), \beta^{-1} d(T^n x, T^n y)\right); \quad \forall x, y \in X. \tag{3.40}$$

Theorem 3.15. *The following properties hold:*(i)$T : X \to X$ *fulfils simultaneously* (3.38) *and the* (L, m)*-property for some* $L \in \mathbf{R}_{0+}$ *if*

$$\frac{1-m}{L+m} d(Tx, q) \le d(x, Tx) \le \frac{1}{\beta} d\left(T^{n-1}x, T^n x\right); \quad \forall x \in X,\ \forall q \in F(T) \tag{3.41}$$

A necessary condition for (3.41) *to hold is* $d(Tx, q) \le ((L+m)/\beta(1-m)) d(T^{n-1}x, T^{\ n}x)$; $\forall x \in X$, $\forall q \in F(T)$.

Another necessary condition for (3.41) *to hold is*

$$d(Tx, q) \le \frac{\beta(1-m)}{\beta(1-m) - L - m}\left(d\left(T^{n-1}x, Tx\right) + d(T^n x, q)\right); \quad \forall x \in X,\ \forall q \in F(T) \tag{3.42}$$

provided that $L \in [0, \beta]$ *and* $m \in [0, \min(1, (\beta - L)/(1+\beta)))$

(ii) $T : X \to X$ *fulfils simultaneously* (3.38) *and the* (L, m)*-property for some* $L \in [0,1) \cap \mathbf{R}_{0+}$, $\forall q \in F(T)$ *if*

$$d(Tx,q) \le \min\left(\frac{L+m}{1-m} d(x,Tx), \frac{L+m}{1-L} d(x,q)\right) \\ \le \min\left(\frac{L+m}{\beta\,(1-m)} d\left(T^{n-1}x, T^n x\right), \frac{L+m}{1-L} d(x,q)\right); \quad \forall x \in X. \tag{3.43}$$

Proof. The sufficiency parts of Properties (i) and (ii) follow directly from Remark 3.14 and (3.38) and (3.39), respectively. The first necessary condition of Property (i) is a direct need for the lower-bound of $d(x,Tx)$ in (3.41) do not exceed its upper-bound, $\forall x \in X$. The second necessary condition is proven as follows. From the first necessary condition and the triangle inequality for distances, one gets:

$$d(Tx,q) \le \frac{L+m}{\beta(1-m)} d\left(T^{n-1}x, T^n x\right) \\ \le \frac{L+m}{\beta(1-m)} \left(d\left(T^{n-1}x, Tx\right) + d(Tx,q) + d(q, T^n x)\right); \quad \forall x \in X \Longrightarrow d(Tx,q) \\ \le \frac{\beta(1-m)}{\beta(1-m) - L - m}\left(d\left(T^{n-1}x, Tx\right) + d(T^n x, q)\right); \quad \forall x \in X \tag{3.44}$$

if $(L+m)/\beta(1-m) < 1 \Leftrightarrow L \in [0,\beta] \wedge m \in [0, \min(1, (\beta - L)/(1+\beta)))$. □

Theorem 3.16. *The following properties hold:*

(i) *A necessary condition for* (3.39) *to hold with* $T : X \to X$ *being an* $n(\mathbf{Z}_+ \ni n \ge 2)$*-times reasonable expansive self-mapping is*

$$d\ (x,Tx) \le \min\left(d\left(Tx, T^{n+1}x\right), \beta^{-1} d\left(T^n x, T^{n+1}x\right), \gamma^{-1} d(x, T^n x)\right); \quad \forall x \in X \tag{3.45}$$

(ii) *A necessary condition for* $T : X \to X$ *to possess, in addition, the* (L,m)*-property for some* $L \in \mathbf{R}_{0+}$ *and* $m \in [0,1) \cap \mathbf{R}_{0+}$ *is*

$$d(x,Tx) \\ \le \min\left(\frac{1}{1-L}\left[m d(x,q) + d\left(T^{n+1}x, q\right)\right], \frac{1}{1-L}\left[m(d(x,q) + d(T^n x, q)) + d\left(T^{n+1}x, T^n x\right)\right], \right. \\ \left. \beta^{-1} d\left(T^n x, T^{n+1}x\right), \frac{1}{\gamma - L}\left[(m+1) d(x,q) + d(Tx, T^n x)\right]\right); \quad \forall x \in X,\ \forall q \in F(T). \tag{3.46}$$

Proof. (i) Take $x \in X$, $y = Tx \in X$ so that

$$d\left(T^n x, T^{n+1}x\right) \ge \beta \min\left(d(x,Tx), d\left(Tx, T^{n+1}x\right)\right) \tag{3.47}$$

There are two potential possibilities for each $x \in X$, since (3.39) holds, namely, either:

(a)

$$\begin{aligned}
&\left[d(x,Tx) \leq d\left(Tx,T^{n+1}x\right) \wedge d\left(T^{n}x,T^{n+1}x\right) \geq \beta d(x,Tx)\right] \\
&\qquad \Longrightarrow d(x,Tx) \leq \min\left(d\left(Tx,T^{n+1}x\right),\beta^{-1}d\left(T^{n}x,T^{n+1}x\right)\right) \\
&\qquad \Longrightarrow d(x,Tx) \leq \min\left(d\left(Tx,T^{n+1}x\right),\beta^{-1}d\left(T^{n}x,T^{n+1}x\right),\gamma^{-1}d(x,T^{n}x)\right)
\end{aligned} \tag{3.48}$$

for some real constants $\beta > 1, \gamma > 1$ since, in addition, $T : X \to X$ is n ($\mathbf{Z}_{+} \ni n \geq 2$)-times reasonable expansive, so that Property (i) holds directly, or

(b)

$$\begin{aligned}
&\left[d(x,Tx) \geq d\ \left(Tx,T^{n+1}x\right) \wedge d\left(T^{n}x,T^{n+1}x\right) \geq \beta d\left(Tx,T^{n+1}x\right) \wedge d(x,T^{n}x) \geq \gamma \mathrm{d}(x,Tx)\right] \\
&\qquad \Longrightarrow d\left(Tx,T^{n+1}x\right) > \beta^{-1}d\left(Tx,T^{n+1}x\right) \geq d\left(Tx,T^{n+1}x\right) \leq d(x,Tx) \\
&\qquad \leq \gamma^{-1}d(x,T^{n}x) < d(x,T^{n}x); \quad \forall x \in X
\end{aligned} \tag{3.49}$$

what leads to the contradiction $d(Tx,T^{n+1}x) > d(Tx,T^{n+1}x)$. Thus, the above result of logic implications cannot hold if $(x,Tx) \in X \times X$, as a result, if (3.39) holds then (3.48) is a necessary condition for $T : X \to X$ to be an n ($\mathbf{Z}_{+} \ni n \geq 2$)-times reasonable expansive self-mapping. Property (i) has been proven.

(ii) Property (i) is equivalent to

$$d(x,Tx) \leq d\left(Tx,T^{n+1}x\right) \wedge d(x,Tx) \leq \beta^{-1}d\left(T^{n}x,T^{n+1}x\right) \wedge d(x,Tx) \leq \gamma^{-1}d(x,T^{n}x); \quad \forall x \in X \tag{3.50}$$

so that if $T : X \to X$ satisfies, in addition, the (L,m)-property for some $L \in \mathbf{R}_{0+}$ and $m \in [0,1) \cap \mathbf{R}_{0+}$ then:

(a)

$$d(x,Tx) \leq \beta^{-1}d\left(T^{n}x,T^{n+1}x\right); \quad \forall x \in X \tag{3.51}$$

(b)

$$\begin{aligned}
d(x,Tx) &\leq \gamma^{-1}d(x,T^{n}x) \leq \gamma^{-1}(d(x,Tx) + d(Tx,T^{n}x)) \\
&\leq \gamma^{-1}(d(Tx,q) + d(Tx,T^{n}x) + d(q,x)); \quad \forall x \in X,\ \forall q \in F(T)
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
&\Longrightarrow d(x,Tx) \le \gamma^{-1}\left(d(Tx,q) + d(Tx,T^n x) + d(q,x)\right) \\
&\qquad \le \gamma^{-1}\left(Ld(x,Tx) + d(Tx,T^n x) + (m+1)d(q,x)\right); \quad \forall x \in X,\ \forall q \in F(T) \\
&\qquad \Longrightarrow d(x,Tx) \le \frac{1}{\gamma - L}\left(d(Tx,T^n x) + (m+1)d(q,x)\right); \quad \forall x \in X,\ \forall q \in F(T)
\end{aligned} \tag{3.53}$$

provided that $L \in [0,\gamma) \cap \mathbf{R}_{0+}$

$$\begin{aligned}
d(x,Tx) &\le d\left(Tx,T^{n+1}x\right) \le d(Tx,q) + d\left(q,T^{n+1}x\right) \\
&\le Ld(x,Tx) + md(q,x) + d\left(q,T^{n+1}x\right) \\
&\le Ld(x,Tx) + md(q,x) + Ld\left(T^n x,T^{n+1}x\right) + md(q,T^n x) \Longrightarrow d(x,Tx) \\
&\le \left(\frac{1}{1-L}\right)\min\left(md(q,x) + d\left(q,T^{n+1}x\right), Ld\left(T^n x,T^{n+1}x\right) + m\left(d(q,x) + d(q,T^n x)\right)\right)
\end{aligned} \tag{3.54}$$

$\forall x \in X, \forall q \in F(T)$ provided that $L \in [\,0,1) \cap \mathbf{R}_{0+}$. The combination of (3.52) to (3.54) proves the result. □

4. Examples

Example 4.1. *Consider the one -dimensional linear unforced discrete dynamic system*

$$x_{i+1} = tx_i; \quad \forall i \in \mathbf{Z}_+ \tag{4.1}$$

under initial conditions $-\infty > -R \ge x_0\ (\in \mathbf{R}) \le R < \infty$. *The distance function is taken as the usual Euclidean norm, namely,* $d(x,y) = |x-y|; \forall x,y \in \mathbf{R}$. *It turns out that if* $|t| < 1$ *then* $x_i = t^i x_0 \to 0$ *as* $i \to \infty$ *irrespective of* x_0 *so that* $0 \in \mathbf{R}$ *is the only stable attractor, which is the only equilibrium point, and the system is globally asymptotically stable.* $0 \in \mathbf{R}$ *is also the only fixed point of the self-mapping* T *on* $\mathbf{R}$ *in the complete metric space* $(\mathbf{R}, d)$ *defined by* $Tx = tx; \forall x \in \mathbf{R}$ *which is* k*-contractive for any real* $k \in [0,1)$ *provided that* $t \in [-k,k]$. *It is now tested when* $T : \mathbf{R} \to \mathbf{R}$ *is* α*-Kannan. Note that*

$$\begin{aligned}
d(x_{i+1},y_{i+1}) &= d(Tx_i,Ty_i) = |x_{i+1} - y_{i+1}| = |t||x_i - y_i| \le |t|(|x_i| + |y_i|), \\
d(x_i,x_{k+1}) &= d(x_i,Tx_i) = |x_{i+1} - x_i| = |t-1||x_i|, \\
d(y_i,y_{i+1}) &= d(y_i,Ty_i) = |y_{i+1} - y_i| = |t-1||y_i|,
\end{aligned} \tag{4.2}$$

for any sequences $x_{i+1} = tx_i$; $y_{i+1} = ty_i$; $\forall i \in \mathbf{Z}_+$ *for initial conditions* $-\infty > -R \ge x_0,\ y_0\ (\in \mathbf{R}) \le R < \infty$ *so that by combining the above three relations:*

$$d(Tx_i,Ty_i) \le |t|(|x_i| + |y_i|) = \left|\frac{t}{1-t}\right|\left(d(x_i,Tx_i) + d(y_i,Ty_i)\right) \tag{4.3}$$

and $T : \mathbf{R} \to \mathbf{R}$ is α-Kannan if $0 \le \alpha := |t/(1-t)| < 1/2$ which is guaranteed for $|t| < 1$ if $|t|/(1-|t|) < 1/2 \Rightarrow |t| \le k < 1/3$ which is the condition of Theorem 2.1(i) guaranteeing that if $T : \mathbf{R} \to \mathbf{R}$ is k-contractive, it is also α-Kannan.

Example 4.2. *Now consider the n-th dimensional linear unforced discrete dynamic system*

$$x_{i+1} = Ax_i; \quad \forall i \in \mathbf{Z}_+ \tag{4.4}$$

under initial conditions $\|x_0\|_2 (\in \mathbf{R}^n) \le R < \infty$ where $\|\cdot\|_2$ is the ℓ_2(or spectral)-norm which coincides with the Euclidean (or Froebenius) norm for vectors. For the matrix $A \in \mathbf{R}^{n\times n}$, we define the vector-induced ℓ_2-norm by

$$\|A\|_2 = \max_{\|x\|_2 \le 1} = \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} = \|Ax\|_2 = \sqrt{\lambda_{\max}(A^T A)} \tag{4.5}$$

where $\lambda_{\max}(A^T A)$ is the maximum (real) eigenvalue of $A^T A$. The distance function is taken as the usual Euclidean norm in $\mathbf{R}^n$, namely, $d(x, y) = \|x - y\|_2$; $\forall x, y \in \mathbf{R}^n$. Assume that $\|A\|_2 \le k < 1$. Define the self-mapping T on $\mathbf{R}^n$ as $Tx = Ax$; $\forall x \in \mathbf{R}^n$. It follows that $0 \in \mathbf{R}^n$ is the only equilibrium point, which is stable, and $F(T) = \{0 \in \mathbf{R}^n\}$. The relations obtained for the scalar case still hold with the replacements $|\ \ | \to \|\ \ \|_2$, $t \to A$, $|t| \to \|A\|_2$, $0 \le \alpha := \|A\|_2/(1 - \|A\|_2) < 1/2$ and the k-contractive self-mapping T on $\mathbf{R}^n$ is also α-Kannan if $\|A\|_2 \le k < 1/3$ which is still the sufficient condition of Theorem 2.1.

Example 4.3. *Now consider the n-th dimensional, perhaps nonlinear, unforced time-varying discrete dynamic system subject to perturbations:*

$$x_{i+1} := Tx_i \equiv A_i x_i + F_i(x_i)x_i; \quad \forall i \in \mathbf{Z}_+ \tag{4.6}$$

under initial conditions $\|x_0\|_2 (\in \mathbf{R}^n) \le R < \infty$ and $F : \mathbf{Z}_{0+} \times \mathbf{R}^n \to \mathbf{R}^n$ is a uniformly bounded sequence of real n-vectors $\{v_i\}_0^\infty$ for any bounded x_0 whose elements satisfy $v_{i+1} = F_i(x_i)x_i$. Now consider two solution sequences $\{x_i\}_{i\in\mathbf{Z}_{0+}}$, $\{y_i\}_{i\in\mathbf{Z}_{0+}}$ under initial conditions $\|x_0\|_2, \|y_0\|_2 (\in \mathbf{R}^n) \le R < \infty$. Let $\beta = \beta_1 + \beta_2$ be defined from real finite constants $\beta_1 := \sup_{i\in\mathbf{Z}_0^+} \|A_i\|_2$, $\beta_2 := \sup_{\|z\| \le \sup R_i;\, i\in\mathbf{Z}_0^+} \|F_i(z)\|_2 \le \gamma := \sup_{z\in\mathbf{R}} \|F(z)\|_2 < \infty$ where $R_i := \beta_1^i R + (1 - \beta_1^i)/(1 - \beta_1)\gamma$ provided that $\beta_1 < 1$, that is, all the matrices A_i; $i \in \mathbf{Z}_{0+}$ are stability matrices. Consider the distance being the Euclidean norm. If $\beta < 1$ then,

$$\|x_i\|_2 \le \beta_1^i \|x_0\|_2 + \sum_{j=0}^{i-1} \beta_1^{i-j-1} \left\|F_j(x_0)\right\|_2 \left\|x_j\right\|_2 \le \beta_1^i \|x_0\|_2 + \frac{1 - \beta_1^i}{1 - \beta_1}\gamma \le \mathbf{R}_i < \infty, \tag{4.7}$$

So that the solution sequence is bounded for any bounded initial conditions. Furthermore,

$$d(x_{i+1}, y_{i+1}) \equiv d(Tx_i, Ty_i) \le \left|\frac{\beta}{1-\beta}\right| \left(\|x_i\|_2 + \left\|y_i\right\|_2\right) = \frac{\beta}{1-\beta}\left(d(x_i, Tx_i) + d(y_i, Ty_i)\right). \tag{4.8}$$

Thus, the self-mapping $T : \mathbf{R}^n \times \mathbf{Z}_{0+} \to \mathbf{R}^n \times \mathbf{Z}_{0+}$ is $\beta/(1-\beta)$-Kannan if $\beta/(1-\beta) < 1/2$, that is if $\beta = \beta_1 + \beta_2 < 1/3$, irrespective of its contractiveness or not. The above condition is guaranteed with $\gamma < 1/3 - \beta_1$ and $\beta_1 \in [0, 1/3)$.

Now, assume that the discrete dynamic system is defined by:

$$x_{i+1} := Tx_i \equiv A_i x_i + F_i(x_i); \quad \forall i \in \mathbf{Z}_+ \tag{4.9}$$

$\|F(z)\|_2 \leq f\|z\|_2$ *for some* $f \in \mathbf{R}_{0+}$. *Then,*

$$\begin{aligned} \|x_i\|_2 &\leq \|x_0\|_2 + \frac{f}{1-\beta_1} \sup_{0\leq j\leq i-1} \left\|x_j\right\|_2 \leq \|x_0\|_2 + \frac{f}{1-\beta_1} \sup_{0\leq j\leq i} \left\|x_j\right\|_2 \Longrightarrow \sup_{0\leq j\leq i} \left\|x_j\right\|_2 \\ &\leq \|x_0\|_2 + \frac{f}{1-\beta_1} \sup_{0\leq j\leq i} \left\|x_j\right\|_2 \Longrightarrow \sup_{j\in \mathbf{Z}_+} \left\|x_j\right\|_2 \leq \frac{(1-\beta_1)R}{1-\beta_1-f} \end{aligned} \tag{4.10}$$

since $\|x_0\|_2 < R$ *provided that* $\beta_1 + f < 1$. *In this case, one also has:*

$$\begin{gathered} \infty > \frac{(1-\beta_1)R}{1-\beta_1-f} \geq \|x_i\|_2 \leq (\beta_1 + f)\|x_{i-1}\|_2 \leq (\beta_1+f)^i \|x_0\|_2 \longrightarrow 0 \text{ exponentially as } i \longrightarrow \infty, \\ \left\|x_i - y_i\right\|_2 \leq k\left\|x_{i-1} - y_{i-1}\right\|_2 \quad \text{with } \beta = \beta_1 + f < 1,\ \beta_2 = f. \end{gathered} \tag{4.11}$$

Then the following hold.

(1) *First,* $T : \mathbf{R}^n \times \mathbf{Z}_{0+} \to \mathbf{R}^n \times \mathbf{Z}_{0+}$ *is* β*-contractive with* $\beta = \beta_1 + f$ *with* $q = 0 \in \mathbf{R}^n$ *being its unique stable equilibrium point and its unique fixed point provided that* $0 \leq f < 1 - \beta_1$ *and* $0 \leq \beta_1 < 1$. *The time-varying system is globally asymptotically stable.*

(2) *If* $\beta_1 + \beta_2 < \beta_1 + f < 1/3$, *that is* $f \in [0, 1/3 - \beta_1)$ *and* $\beta_1 \in [0, 1/3)$ *then the* β*-contractive self-mapping* $T : \mathbf{R}^n \times \mathbf{Z}_{0+} \to \mathbf{R}^n \times \mathbf{Z}_{0+}$ *is furthermore* $\beta/(1-\beta)$*-Kannan. Those results still agree with* Theorem 2.1. *On the other hand, the* (L, m)*-property of contractive Kannan self-mappings can be tested for this example according to the formula*

$$\|x_{i+1}\|_2 = d(Tx_i, 0) \leq Ld(x_{i+1}, x_i) + md(x_{i,}, 0) = \frac{1-\beta}{\beta - m}\|x_{i+1} - x_i\|_2 + m\|x_i\|_2, \quad \forall i \in \mathbf{Z}_+ \tag{4.12}$$

from Theorem 2.9 *with* $\alpha = \beta/(1-\beta) = (L+m)/(1-m) \Leftrightarrow L = (1-\beta)/(\beta - m)$ *with* $\beta \in [0, 1/3)$, $m \in [0, \beta)$ *since* $T : \mathbf{R}^n \times \mathbf{Z}_{0+} \to \mathbf{R}^n \times \mathbf{Z}_{0+}$ *is* $\beta/(1-\beta)$*-Kannan and* β*-contractive. Note that*

$$\frac{1-\beta}{\beta - m}\|x_{i+1} - x_i\|_2 + m\|x_i\|_2 \geq 2\|x_{i+1} - x_i\|_2; \quad \beta \in \left[0, \frac{1}{3}\right),\ m \in [0, \beta) \tag{4.13}$$

with the above lower-bound being reached for $m = 0$, $\beta = 1/3$. Note also that $\|x_{i+1}\|_2 \le 2\|x_{i+1} - x_i\|_2$. since otherwise, one would have

$$\begin{aligned} \|x_{i+1}\|_2 > 2\|x_{i+1} - x_i\|_2 \ge 2(\|x_i\|_2 - \| \, x_{i+1}\|_2) \Longrightarrow \frac{1}{3}\|x_i\|_2 \\ > \beta\|x_i\|_2 \ge \|x_{i+1}\|_2 > \frac{2}{3}\|x_i\|_2 \end{aligned} \tag{4.14}$$

what is a contradiction.

Example 4.4. *A forced version of the equation of Example 4.1 is*

$$x_i \equiv Tx_{i-1} = tx_{i-1} + r = t^i x_0 + \frac{1-t^i}{1-t} r; \quad \forall i \in \mathbf{Z}_+ \tag{4.15}$$

with $r \in \mathbf{R}$. If $|t| < 1$ then

$$|x_i| \le |x_0| + \frac{|r|}{1-t}; \quad \forall i \in \mathbf{Z}_{0+}, \qquad x_i \longrightarrow \frac{r}{1-t} \quad \text{as } i \to \infty \tag{4.16}$$

independent of the initial condition for any bounded initial condition. Also, it is direct by complete induction the property

$$|x_0| \le \frac{|r|}{1-|t|} \Longrightarrow |x_i| \le \frac{|r|}{1-|t|}; \quad \forall i \in \mathbf{Z}_{0+}. \tag{4.17}$$

On the other hand, if $|x_0| \le |r|/(1-|t|)$ and $|t| < 1$ then

$$\begin{aligned} |x_{i+2} - x_i| = \left|(1+t)r - \left(1-t^2\right)x_i\right| \ge (1+t)(|r| - (1-t)|x_i|) \\ \ge |x_{i+1} - x_i| \ge (|r| - (1-t)|x_i|); \quad \forall i \in \mathbf{Z}_{0+}. \end{aligned} \tag{4.18}$$

If, in addition, the system is positive and stable, that is $t \in [0,1)$, with positive initial conditions $\mathbf{R}_+ \ni x_0 \le r/(1-t)$ and forcing term $r \in \mathbf{R}_+$ then $T : \mathbf{R}_+ \to \mathbf{R}_+$ is not contractive since $x_{i+1} > x_i$ for any finite $i \in \mathbf{Z}_{0+}$, and

$$|x_{i+2} - x_i| = (1+t)(r - (1-t)x_i) \ge \beta|x_{i+1} - x_i| = \beta(r - (1-t)x_i); \quad \forall i \in \mathbf{Z}_{0+} \tag{4.19}$$

for any real $\beta \in [0, 1+t)$ since it holds that

$$(1+t-\beta)r \ge (1-t)(1+t-\beta)\frac{r}{1-t}. \tag{4.20}$$

Thus, for $\mathbf{R}_+ \ni x_0 \le r/(1-t), r \in \mathbf{R}_+$, $\mathbf{R}_+ \ni t \in [0,1)\mathbf{R}_+ \ni \varepsilon \in [0,t)$, one has

$$| \, x_{i+n} - x_i| \ge (1+\varepsilon)|x_{i+1} - x_i|; \quad \forall i \in \mathbf{Z}_{0+}, \ \forall i (\ge 2) \in \mathbf{Z}_{0+}, \ F(T) = \left\{\frac{r}{1-t}\right\} \tag{4.21}$$

so that the self-mapping $T : \mathbf{R}_+ \to \mathbf{R}_+$ *has a fixed point while it is reasonable expansive (see Definition 3.1 and Theorem 3.2). Extensions to the non positive first-order system and the* n*-th order discrete dynamic system can be addressed in the same way. If the system is time-varying with the sequence of parameters* $\{t_i\}_{i\in\mathbf{Z}_{0+}}$ *fulfilling* $0 \le t_i \le t < 1$ *then* $x_i = \bar{t}^i x_0 + ((1-\bar{t}^i)/(1-\bar{t}))r \to r/(1-\bar{t})$ *as* $i \to \infty$ *where* $\bar{t}$ *is the geometric mean of the elements of* $\{t_i\}_{i\in\mathbf{Z}_{0+}}$*. Thus, there is still a unique fixed point* $r/(1-\bar{t})$*. Also, if there is a finite subset* $V \subset \mathbf{Z}_{0+}$ *such that* $t_i \ge 1$ *if and only if* $i \in V$ *then there is a unique fixed point* $F(T) = \{(1/(1-\bar{t}) + \sum_{i\in V}(t_i - \bar{t}))r\}$ *since* $\lim_{i\to\infty} x_i = (1/(1-\bar{t}) + \sum_{i\in V}(t_i - \bar{t}))r$ *despite the fact that* $T : \mathbf{R}_+ \to \mathbf{R}_+$ *is not contractive.*

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