Generating cluster submodels from two-stage stochastic mixed integer optimization models

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Abstract

Stochastic optimization problems of practical applications lead, in general, to some large models. The size of those models is linked to the number of scenarios that defines the scenario tree. This number of scenarios can be so large that decomposition strategies are required for problem solving in reasonable computing time. Methodologies such as Branch-and-Fix Coordination and Lagrangean Relaxation make use of these decomposition approaches, where independent scenario clusters are given. In this work, we present a technique to generate cluster submodel structures from the decomposition of a general two-stage stochastic mixed integer optimization model. Scenario cluster submodels are generated from the original stochastic problem by combining the compact and splitting variable representations in some of the variables related to the nodes that belong to the first stage. We consider a two-stage stochastic capacity expansion problem as illustrative example where several decompositions are provided.

Keywords: Stochastic Optimization, Scenario Cluster Partitioning, C++ code , MPS format.

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1 Introduction

Stochastic optimization problems of practical applications lead, in general, to some large models. The size of those models is linked to the number of scenarios that define the scenario tree. This number of scenarios can be so large that decomposition strategies are required for problem solving in reasonable time. For theory, methodologies and algorithms in relation with stochastic optimization see [4, 6, 25, 29]and [30], among many others.

Most of the optimization problems require integer variables, mainly 0-1 variables besides continuous ones. The advantage of a mixed 0-1 approach is that it provides a general framework for modeling a large variety of problems.

The traditional aim in this type of problems is to solve the Deterministic Equivalent Model (for short, DEM), which usually is a mixed 0-1 problem with a special structure. Exact decomposition algorithms for solving the DEM have been studied for different types of problems, see [21, 32]. Some of them combining with a Branch-and-Bound method to deal with the integer variables, see our work in [8, 10, 12] as well as [26], among others.

Some approaches for multistage problems where appear most promising to use the information about the separability of the problem are presented in [24, 27, 28], just for naming a few works. Specifically, see in [8, 9, 10, 11, 12, 15, 31, 32, 35], among others, some decomposition approaches that consider scenario clustering or grouping for solving large-scale multistage stochastic mixed integer problems. The so-named Branch-and-Fix Coordination (BFC) methodology is used in the decomposition approaches presented in [8, 10, 12, 15] to generate independent scenario clusters such that multiplicity of any scenario is not allowed among the clusters. Moreover, Lagrangean Decomposition (for short, LD) [13, 14, 16, 17, 18] can be used as part of a solution methodology. In a first step, consists of relaxing the so-called nonanticipativity constraints (for short, NAC) for some of the stages. Then, scenario cluster submodels are generated from the original stochastic problem by dualizing the NAC related to the nodes that belong to the stages up to the break one. Subgradient Method (SM)[19, 20], Volume Algorithm (VA) [3], Lagrangean Progressive Hedging Algorithm (LPHA) [30, 13] and Dynamic Constrained Cutting Plane (DCCP) algorithm [22] can then be used in the Lagrangean multipliers updating scheme, while solving multi-stage mixed 0-1 submodels based on scenario cluster decomposition, see [14]. The proximal bundle method presented in [23] can be also used as a Lagrangean multipliers updating scheme. In all these cases it is proved that the tightness of the bounds can be increased “by grouping together scenarios”.

In general, for any multi-stage stochastic problem with T stages and $|\Omega|$ scenarios, the information about until what stage the scenario submodels have common information, is saved in the scenario tree. See in [2], for the details of the general procedure of scenario cluster partitioning, based in the break stage concept in case of general multi-stage problems.

In the case of two-stage problems the simpler structure of the scenario tree allows to use an easy procedure to decompose the full model. About scenario cluster partitioning in two-stage models, see [13]. In this paper, a scenario clustering is presented where the number of clusters can be chosen from the set of divisors of the number of scenarios, $|\Omega|$. Now, we propose the generalization of this procedure such that be possible to choose the number of clusters as any value from the set $\mathscr{C} = \{1, 2, \cdots, |\Omega|\}$.

The rest of the paper is organized as follows. Section 2 deals with the main concepts and related notation of a two-stage optimization model. Section 3 presents the scenario clustering scheme. Section 4 presents an example as illustrative case. Section 5 introduces the files required for the decomposition. Section 6 reports the results of the decomposition procedure over the illustrative case; and, Section 7 concludes.

Some Appendices present the detail of some MPS files, a part of the C++ code, and some other information.

2 Two-stage model

Let us consider the following two-stage stochastic mixed 0-1 model in *compact* representation:

$$\begin{array}{lll} (MIP): z_{MIP} & = \min & a_1 x_1 + b_1 y_1 + E_\psi[a_2^\omega x_2^\omega + b_2^\omega y_2^\omega] \\ s.t. & & \\ & & A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \leq h_1 \\ & & T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + W^\omega \begin{pmatrix} x_2^\omega \\ y_2^\omega \end{pmatrix} \leq h_2^\omega, \quad \forall \omega \in \Omega \\ & & y_1, y_2^\omega \geq 0, \quad \forall \omega \in \Omega \\ & & x_1, x_2^\omega \in \{0,1\}, \quad \forall \omega \in \Omega \end{array} \tag{1}$$

where a_1 and b_1 are known vectors of the objective function coefficients for the 0-1 and continuous variables in the first stage x_1 and y_1, respectively, h_1 is the right hand side vector for the first stage constraints, and A is the known matrix of coefficients for the first stage constraints. For each scenario ω, h_2^ω is the right hand side vector for the second stage constraints; and a_2^ω and b_2^ω are the objective function coefficients for the second stage variables 0-1 and continuous, x_2^ω and y_2^ω respectively. Furthermore, T and W^ω are the technology matrices, this second under scenario ω, for $\omega \in \Omega$, where Ω is the set of scenarios to consider. Piecing together the stochastic components of the problem, we have a vector $\psi^\omega = (a_2^\omega, b_2^\omega, h_2^\omega, W^\omega)$, with $\omega \in \Omega$. Finally, E_ψ represents the mathematical expectation with respect to ψ over the set of scenarios Ω.

All this information can be represented as a tree in which each path from the root to the leaf corresponds to a specific scenario ω. In addition, each node of the tree can be associated with a group of scenarios g where $\mathscr{G}$ denotes the set of scenario groups (i.e. nodes in the underlying scenario tree), and $\mathscr{G}_t$, the subset of scenario groups that belong to stage $t \in \mathscr{T}$, such that $\mathscr{G} = \cup_{t\in\mathscr{T}} \mathscr{G}_t$ and Ω_g is the set of the scenarios related to group g. Finally, $\mathscr{T}$ denotes the set of stages and in our particular case, $|\mathscr{T}| = 2$.

Sometimes it can be useful to work with scenarios instead of with nodes or groups, for example if you want to split the set of scenarios into different subsets. Ω_g denotes the set of scenarios that belong to group g, for $g \in \mathscr{G}$.

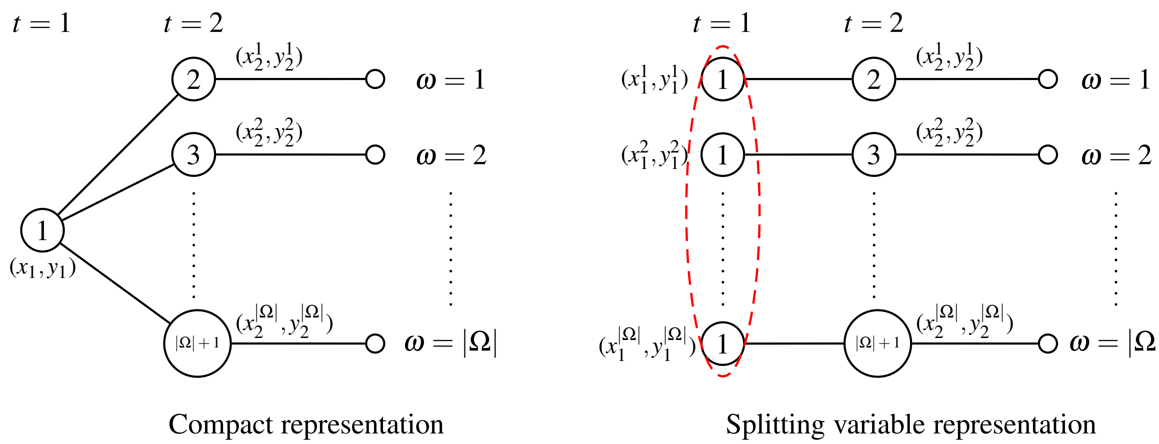

Figure 1: Scenarios tree

The left part of Figure 1 corresponds to the compact representation of the problem in which there is only one copy of each first stage variable. In this part, the concept of scenario group is used to represent the nonanticipativity constrains implicitly. It corresponds to the representation given in model (1).

The right part of Figure 1 corresponds to the splitting variable representation of the problem, given in model (4), in which there is a replica of each first stage variable for each scenario. In this way, in the first stage we must impose the nonanticipativity constrains in the variables x_1^ω and y_1^ω for the scenarios ω that belong the same group Ω_g, $g \in \mathcal{G}_1 = \{1\}$, i.e., for all scenarios, $\omega \in \Omega$.

Following the nonanticipativity principle stated in [34] and restated in [30] see also [4], among many others, both scenarios should have the same value for the related variables with the same index up to the given stage. The conditions called Non-Anticipativity Constrains (NAC) establish in particular in the two-stage models that the decisions of the first stage must be independent of the scenario in which they occur and, therefore, be the same under the different scenarios. So that

$$x^\omega = x^{\omega'}, \quad \forall \omega, \omega' \in \Omega, \omega \neq \omega' \tag{2}$$

$$y^\omega = y^{\omega'}, \quad \forall \omega, \omega' \in \Omega, \omega \neq \omega' \tag{3}$$

Similar the NAC must be satisfied by all the coefficients vectors and matrices of the first stage $(a_1^\omega, b_1^\omega, h_1^\omega, A^\omega, T^\omega)$.

Then, the splitting variable representation of a two-stage model is given by:

$$
\begin{array}{rcl}
(MIP): \; z_{MIP} & = & \min \sum_{\omega \in \Omega} \sum_{t=1}^{2} w^{\omega} (a_t^{\omega} \, x_t^{\omega} + b_t^{\omega} y_t^{\omega}) \\
s.t. & & \\
& & A^{\omega} \begin{pmatrix} x_1^{\omega} \\ y_1^{\omega} \end{pmatrix} \leq h_1^{\omega}, \quad \forall \omega \in \Omega \\
& & T^{\omega} \begin{pmatrix} x_1^{\omega} \\ y_1^{\omega} \end{pmatrix} + W^{\omega} \begin{pmatrix} x_2^{\omega} \\ y_2^{\omega} \end{pmatrix} \leq h_2^{\omega}, \quad \forall \omega \in \Omega \\
& & x^{\omega} = x^{\omega'}, \quad \forall \omega, \omega' \in \Omega, \omega \neq \omega' \\
& & y^{\omega} = y^{\omega'}, \quad \forall \omega, \omega' \in \Omega, \omega \neq \omega' \\
& & y^{\omega} \geq 0, \quad \forall \omega \in \Omega \\
& & x^{\omega} \in \{0,1\}, \quad \forall \omega \in \Omega
\end{array} \tag{4}
$$

where w^{ω} is the likelihood attributed to each scenario ω.

Throughout the following sections we will describe a procedure to generate the cluster submodel structures from the descomposition of a general full model.

3 Scenario Cluster Partitioning

In order to combine the compact and the splitting variable representation, we propose a cluster scenario partition of the full model, being a scenario cluster a set of scenarios where the NAC constrains are implicitly considered.

In general, for any multi-stage stochastic problem with T stages and $|\Omega|$ scenarios, the information about until what stage the scenario submodels have common information, and when the NAC must be explicit, is saved in the subsets $\mathscr{G}_t$ and Ω_g, $g \in \mathscr{G}_t$, $t \in \mathscr{T}$, i.e., in the scenario tree. Then, a general procedure of scenario cluster partitioning based in the break stage concept is developped. See [10], [12] and [2], for the definition of this concept and related topics.

In the case of two-stage problems the simple structure of the scenario tree allows to use an easy procedure to decompose the whole model. See [13] a scenario cluster partitioning of two-stage models,where the number of clusters can be chosen from the set of divisors of the number of scenarios, $|\Omega|$. In this work , we propose the generalization of this procedure such that is possible to choose the number of clusters as any value from the set $\mathscr{C} = \{1, 2, \cdots, |\Omega|\}$.

Definition 1 *The* scenario tree matrix, *$ST \in \mathscr{M}_{|\Omega|x|\mathscr{G}|}$, is a matrix where the corresponding value for the pair (ω, g) gives the related stage t,such that,*

$$
ST(\omega, g) \;=\; \begin{cases} t, & if \;\; \omega \in \Omega_g, \; g \in \mathscr{G}_t \\ 0, & otherwise \end{cases} \tag{5}
$$

Notice that the scenario tree matrix reproduces the structure given by the scenario tree. This matrix can be built using the sets Ω_g and $\mathscr{G}_t$, i.e., the scenario tree, but these sets can be also generated from the matrix. For each stage $t \in \mathscr{T}$, the sets of scenario groups in such stage, $\mathscr{G}_t$, can be identified as the column of the position (ω, g) for which the corresponding element in the scenario tree matrix is equal to t; then $\mathscr{G}_t = \{g \in \mathscr{G} : \exists \omega \in \Omega : ST(\omega, g) = t\}$. See also that the set of scenarios related to group g is $\Omega_g = \{\omega \in \Omega : ST(\omega, g) \neq 0\}$.

Then, given the choice of the number of clusters, $C = |\mathscr{C}|$, we will decompose the scenario tree into a subset of scenario clusters subtrees, each one for each scenario cluster in the set denoted as $\mathscr{C} = \{1, ..., C\}$.

Let Ω^c denote the set of scenarios that belongs to cluster c, where $c \in \mathscr{C}$ and $\sum_{c=1}^{C} |\Omega^c| = |\Omega|$. Notice that $\Omega^c \bigcap \Omega^{c'} = \emptyset$, $c, c' = 1, \ldots, C : c \neq c'$ and $\Omega = \cup_{c=1}^{C} \Omega^c$.

Let also $\mathscr{G}^c \subset \mathscr{G}$ denote the set of scenario groups for cluster c, such that $\Omega_g \cap \Omega^c \neq \emptyset$ means that $g \in \mathscr{G}^c$, and let $\mathscr{G}_t^c = \mathscr{G}_t \cap \mathscr{G}^c$ denote the set of scenario groups for cluster $c \in \mathscr{C}$ in stage $t \in \mathscr{T}$.

Definition 2 *The* scenario cluster *models are those that result from the relaxation of some of the NAC in model (1).*

In the same way that for the full scenario tree, for a cluster subtree, we can reproduce its structure by using a matrix.

Definition 3 *The* cluster tree matrix *associated with a partition into C clusters, $CT \in \mathscr{M}_{Cx|\mathscr{G}|}$, is a matrix where the corresponding value for the pair (c, g) gives the related stage t,such that,*

$$CT(c,g) = \begin{cases} t, & if \;\; g \in \mathscr{G}_t^c \\ 0, & otherwise \end{cases} \tag{6}$$

Once decided the number of clusters C, the corresponding cluster partition is obtained, and its structure is defined by the related cluster tree matrix. In two-stage models, and from the definition of cluster submodels, a way to create each partition is to consider the integer division of the number of scenarios over the desired number of clusters, $\frac{|\Omega|}{C}$, and if this quotient is not equal to zero, distribute the remaining scenarios among the different clusters, starting by the first cluster and continuing until the remaining scenarios are finished. The implementation in C++ of this procedure is shown in Appendix B. See also the example of scenario cluster partitioning in Section 6.

Notice that $\mathscr{G}_t^c = \mathscr{G}_t \cap \mathscr{G}^c$, is the set of scenario groups for cluster $c \in \mathscr{C}$, in stage $t \in \mathscr{T}$. The subsets $\mathscr{G}^c$ and $\mathscr{G}_t$ and, consequently $\mathscr{G}_t^c$ can be obtained from the cluster tree matrix defined above. For each cluster $c \in \mathscr{C}$ (i.e., $c-$row in the CT matrix), the set of scenario groups $\mathscr{G}^c$ can be obtained as the set of columns in the cluster tree matrix with a nonzero element, i.e., $\mathscr{G}^c = \{g \in \mathscr{G} : \exists c \in \mathscr{C} : CT(c, g) = t\}$.

Then, we can decompose the whole model into splitting variable representation between the cluster models and add explicitly the nonanticipativity constraints into the diferent clusters. Notice also that we consider compact representation into each cluster model.

The submodel to consider for each scenario cluster $c \in \mathscr{C}$, can be expressed as:

$$
\begin{array}{lll}
(MIP^c): \; z^{\mathbf{c}} & = \min & \sum_{\omega \in \Omega^{\mathbf{c}}} w^{\omega} [a_1^T x_1^{\mathbf{c}} + b_1^T y_1^{\mathbf{c}} + a_2^{\omega T} x_2^{\omega} + b_2^{\omega T} y_2^{\omega}] \\
\qquad\quad s.t. & & \\
& & A \begin{pmatrix} x_1^{\mathbf{c}} \\ y_1^{\mathbf{c}} \end{pmatrix} \leq h_1 \\
& & T \begin{pmatrix} x_1^{\mathbf{c}} \\ y_1^{\mathbf{c}} \end{pmatrix} + W^{\omega} \begin{pmatrix} x_2^{\omega} \\ y_2^{\omega} \end{pmatrix} \leq h_2^{\omega}, \quad \forall \omega \in \mathbf{\Omega c} \\
& & x_1^{\mathbf{c}}, x_2^{\omega} \in \{0,1\}, \quad \forall \omega \in \Omega^{\mathbf{c}} \\
& & y_1^{\mathbf{c}}, y_2^{\omega} \geq 0, \quad \forall \omega \in \Omega^{\mathbf{c}}
\end{array}
\tag{7}
$$

The following figure illustrates the cluster generation proposed depending on whether the number of scenarios of the problem is even or odd.

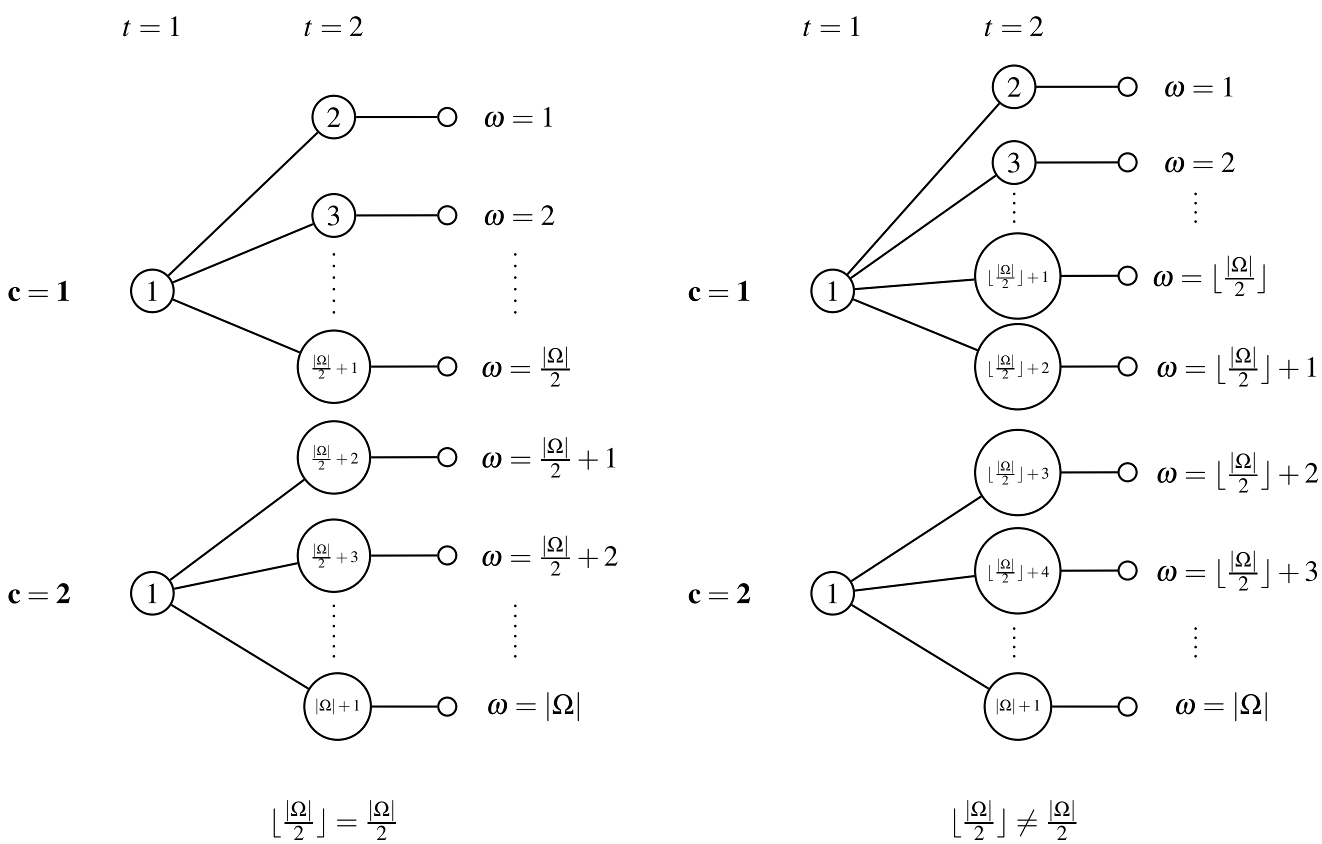

Figure 2: Scenario cluster partitioning

4 Illustrative example

Consider the following two-stage stochastic capacity expansion problem based on the multistage one presented in [1].

The mixed 0-1 model has the following form:

$$
\begin{aligned}
\min & \sum_{i\in\mathscr{I}} (a_{1i}\, x_{1i} + e_{1i}\, y_{1i}) + \sum_{\omega\in\Omega} w^{\omega} \sum_{i\in\mathscr{I}} (a_{2i}^{\omega}\, x_{2i}^{\omega} + e_{2i}^{\omega}\, y_{2i}^{\omega}) & & \\
\text{s.t. } & y_{1i} \leq M_{1i}\, x_{1i} & & \forall i \in \mathscr{I} \\
& y_{2i}^{\omega} \leq M_{2i}^{\omega}\, x_{2i}^{\omega} & & \forall i \in \mathscr{I}, \quad \forall \omega \in \Omega \\
& \sum_{i\in\mathscr{I}} y_{1i} \geq d_1 & & \\
& \sum_{i\in\mathscr{I}} (y_{2i}^{\omega} + y_{1i}) \geq d_2^{\omega} & & \forall \omega \in \Omega \\
& x_{1i},\ x_{2i}^{\omega} \in \{0,1\},\ \ y_{1i},\ y_{2i}^{\omega} \in \mathbb{R}^{+} & & \forall i \in \mathscr{I}, \quad \forall \omega \in \Omega
\end{aligned}
\tag{8}
$$

where $\mathscr{I}$ denotes the set of resources or technology types.The goal is to determine a schedule of timing and level of capacity acquisitions of set $\mathscr{I}$ to satisfy the demands d_1 and d_2^{ω} , while minimizing the expected discounted cost over the set of scenarios along the time horizon. The decision continuous variables y_{1i} and y_{2i}^{ω} denote the capacity expansion of resource type $i \in \mathscr{I}$ and x_{1i} and x_{2i}^{ω} denote the 0-1 decision variables for the corresponding capacity expansion decision for $i \in \mathscr{I}$. Without loss of generality, we assume zero initial capacities. Moreover, a_{1i} and a_{2i}^{ω} are the discounted fixedand for resource $i \in \mathscr{I}$; e_{1i} and e_{2i}^{ω} are the variable investment cost for resource $i \in \mathscr{I}$ and M_{1i} and M_{2i}^{ω} are the variable upper bound on the capacity additions for $i \in \mathscr{I}$.

Table 1 gives the problem data. It is assumed that the uncertain parameters given in Table 1 refer to the scenario tree structure depicted in Figure 3, being the solution value $z = 78.841185$.
Notice that the number of scenarios in the example is seven and there are three 0-1 variables and three continuous variables in each scenario group and each stage, all of them with the same probability. Moreover, the order of these variables in the input data is exactly the given in Figure 3, as it can be seen in the MPS format of Appendix A.

Table 1: Illustrative example parameters

		a_{11}	a_{12}	a_{13}	e_{11}	e_{12}	e_{13}	M_{11}	M_{12}	M_{13}	d_1	w_1
$t=1$		10	15	5	2	1	2	4.5	2.8	2.7	5	1
	ω	a_{21}^{ω}	a_{22}^{ω}	a_{23}^{ω}	e_{21}^{ω}	e_{22}^{ω}	e_{23}^{ω}	M_{21}^{ω}	M_{22}^{ω}	M_{23}^{ω}	d_2^{ω}	w_2^{ω}
	1	10	5	3	1	3	2	3.8	3.5	4.7	15	0.14285
	2	10	30	20	2	1	2	4	3.7	3.8	20	0.14285
	3	11	5	10	1	1	1	4.8	4.9	4.3	20	0.14285
$t=2$	4	5	10	3	3	1	2	4.4	5.3	5.6	20	0.14285
	5	10	3	5	1	2	1	4.5	3.5	3.6	15	0.14285
	6	3	10	5	2	1	1	4.5	4.7	4.5	20	0.14285
	7	4	10	5	2	1	3	4	4	4	15	0.14285

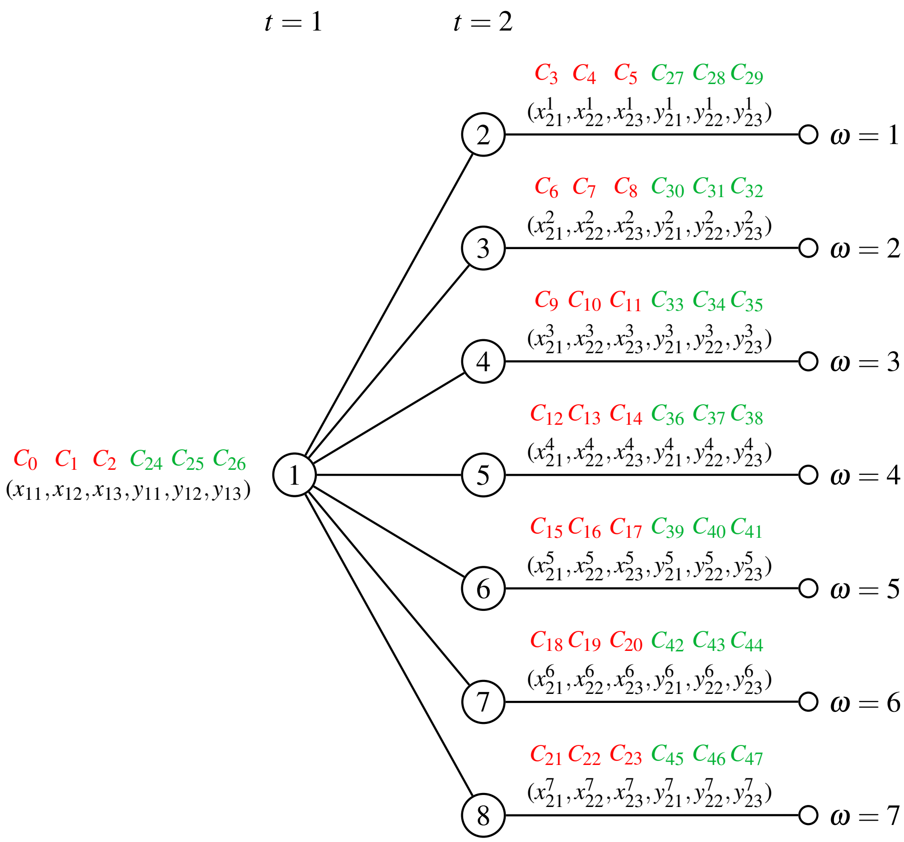

Figure 3: Scenarios tree

The problem can be formulated as follows:

$$
\begin{aligned}
\min \quad & 10x_{11} + 15x_{12} + 5x_{13} + 1.4285x_{21}^{1} + 0.71425x_{22}^{1} + 0.42855x_{23}^{1} \\
+ \quad & 1.4285x_{21}^{2} + 4.2855x_{22}^{2} + 2.857x_{23}^{2} + 1.57135x_{21}^{3} + 0.71425x_{22}^{3} + 1.4285x_{23}^{3} \\
+ \quad & 0.71425x_{21}^{4} + 1.4285x_{22}^{4} + 0.42855x_{23}^{4} + 1.4285x_{21}^{5} + 0.42855x_{22}^{5} + 0.71425x_{23}^{5} \\
+ \quad & 0.42855x_{21}^{6} + 1.4285x_{22}^{6} + 0.71425x_{23}^{6} + 0.5714x_{21}^{7} + 1.4285x_{22}^{7} + 0.71425x_{23}^{7} \\
+ \quad & 2y_{11} + y_{12} + 2y_{13} + 0.14285y_{21}^{1} + 0.42855y_{22}^{1} + 0.2857y_{23}^{1} \\
+ \quad & 0.2857y_{21}^{2} + 0.14285y_{22}^{2} + 0.2857y_{23}^{2} + 0.14285y_{21}^{3} + 0.14285y_{22}^{3} + 0.14285y_{23}^{3} \\
+ \quad & 0.42855y_{21}^{4} + 0.14285y_{22}^{4} + 0.2857y_{23}^{4} + 0.14285y_{21}^{5} + 0.2857y_{22}^{5} + 0.14285y_{23}^{5} \\
+ \quad & 0.2857y_{21}^{6} + 0.14285y_{22}^{6} + 0.14285y_{23}^{6} + 0.2857y_{21}^{7} + 0.14285y_{22}^{7} + 0.42855y_{23}^{7} \\
\text{s.t.} \quad & \\
& -4.5x_{11} + y_{11} \le 0 \\
& -2.8x_{12} + y_{12} \le 0 \\
& -2.7x_{13} + y_{13} \le 0 \\
& -3.8x_{21}^{1} + y_{21}^{1} \le 0 \\
& -3.5x_{22}^{1} + y_{22}^{1} \le 0 \\
& -4.7x_{23}^{1} + y_{23}^{1} \le 0 \\
& -4x_{21}^{2} + y_{21}^{2} \le 0 \\
& -3.7x_{22}^{2} + y_{22}^{2} \le 0 \\
& -3.8x_{23}^{2} + y_{23}^{2} \le 0 \\
& -4.8x_{21}^{3} + y_{21}^{3} \le 0 \\
& -4.9x_{22}^{3} + y_{22}^{3} \le 0 \\
& -4.3x_{23}^{3} + y_{23}^{3} \le 0 \\
& -4.4x_{21}^{4} + y_{21}^{4} \le 0 \\
& -5.3x_{22}^{4} + y_{22}^{4} \le 0 \\
& -5.6x_{23}^{4} + y_{23}^{4} \le 0 \\
& -4.5x_{21}^{5} + y_{21}^{5} \le 0 \\
& -3.5x_{22}^{5} + y_{22}^{5} \le 0 \\
& -3.6x_{23}^{5} + y_{23}^{5} \le 0 \\
& -4.5x_{21}^{6} + y_{21}^{6} \le 0 \\
& -4.7x_{22}^{6} + y_{22}^{6} \le 0 \\
& -4.5x_{23}^{6} + y_{23}^{6} \le 0 \\
& -4x_{21}^{7} + y_{21}^{7} \le 0 \\
& -4x_{22}^{7} + y_{22}^{7} \le 0 \\
& -4x_{23}^{7} + y_{23}^{7} \le 0 \\
& y_{11} + y_{12} + y_{13} \ge 5 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{1} + y_{22}^{1} + y_{23}^{1} \ge 15 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{2} + y_{22}^{2} + y_{23}^{2} \ge 20 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{3} + y_{22}^{3} + y_{23}^{3} \ge 20 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{4} + y_{22}^{4} + y_{23}^{4} \ge 20 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{5} + y_{22}^{5} + y_{23}^{5} \ge 15 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{6} + y_{22}^{6} + y_{23}^{6} \ge 20 \\
& y_{11} + y_{12} + y_{13} + y_{21}^{7} + y_{22}^{7} + y_{23}^{7} \ge 15 \\
& y_{1i}, y_{2i}^{\omega} \ge 0, \quad \forall i \in \mathscr{I}, \quad \forall \omega \in \Omega \\
& x_{1i}, x_{2i}^{\omega} \in \{0,1\}, \quad \forall i \in \mathscr{I}, \quad \forall \omega \in \Omega
\end{aligned} \tag{9}
$$

5 Basic requeriments

Given a two-stage stochastic mixed integer optimization model, and for the aim of obtaining the scenario cluster partition, some additional information is required. This information has been organized in two files:

1. A file (in this case called *total.mps*, see Appendix A) with the two-stage model in compact representation (4) in MPS format.

2. An input file so-called *inputData.dat* with the following information:

   - C, number of clusters in which the model is going to be descomposed
   - Number of scenario groups in each stage
   - nx_t, $t = 1,2$, number of 0-1 variables by stage (number of 0-1 variables in any scenario group)
   - ny_t, $t = 1,2$, number of continuous variables by stage (number of continuous variables in any scenario group)
   - w^{ω}, $\omega \in \Omega$, vector of likelihood for scenarios. If all scenarios have the same probability of occurrence, 0 value appears in the corresponding line.
   - γ, parameter that takes value 1 if the order of variables is the expected one, and 0 in other case. The order of variables is the expected one when first all the 0-1 variables, and then, all the continuous one are stored. Moreover, in each variable type, they are ordered by stage and in each stage, ordered by scenario.
   - ox_i, $i = 1, \cdots, nx$; oy_i, $i = 1, \cdots, ny$, position of the variables such they are included in the model, being nx and ny the number of 0-1 and continuous variables respectively. If the original order of variables in the MPS file is the expected one, 0 value appears in this line.

```
4
1 2 4 8
2 2 2 2 2 2 2
0 0 0 0
2 2 2 2
16 23 17 24 18 25 19 26 20 27 21 28 22 29 0 8 1 9 2 10 3 11 4 12 5 13 6 14 7 15
```

inputData.dat

The first data in file inputData.dat corresponds to the number of clusters, C, that in this example is 2. There are $|G_1| = 1$, $|G_2| = |\Omega| = 7$ scenario groups. Then, in both stages, there are (nx_1, ny_1)=(nx_2, ny_2) = $(3,3)$ variables in each scenario group. So, in the first stage there are 3 binary variables and 3 continuos ones, while in the second stage there are 21 binary and 21 continuous, respectively.

The likelihood for each scenario is the same, equal to $\frac{1}{7}$, and then, the corresponding data in the file is a 0.

Finally, the 0-1 variables in the model (total.mps file) appear in the following order:

0 1 2 6 7 8,...

While the order for the continuos ones, is:

3 4 5 9 10 11,...

This means, that these variables (0-1 and continuos) appear mixed at each stage and scenario group:

$$x_{11}, x_{12}, x_{13}, y_{11}, y_{12}, y_{13}, x_{21}^1, x_{22}^1, x_{23}^1, y_{21}^1, y_{22}^1, y_{23}^1, \ldots, x_{21}^7, x_{22}^7, x_{23}^7, y_{21}^7, y_{22}^7, y_{23}^7$$

while the expected or required order in the model, needed to start the decomposition is, first all the binaries and after, all the continuos:

$x_{11}, x_{12}, x_{13}, x_{21}^1, x_{22}^1, x_{23}^1, \ldots, x_{21}^7, x_{22}^7, x_{23}^7$

$y_{11}, y_{12}, y_{13}, y_{21}^1, y_{22}^1, y_{23}^1, \ldots, y_{21}^7, y_{22}^7, y_{23}^7$

In this way, they are saved in the model after the first steps of the procedure, see below.

6 Example. Scenario Cluster Partitioning

The scenario tree matrix $ST(\omega, g)$ corresponding to the illustrative example is given in (10).

$$ST(\omega, g) = \left(\begin{array}{c|ccccccc} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{array} \right) \tag{10}$$

In the ilustrative example depicted in Figure 3, firstly, we are going to consider in a first partition that $C = 2$. As $\frac{|\Omega|}{C} = \frac{7}{2} = 3.5$, the integer division is equal to 3, and, then, the rest is equal to 1. As output of the procedure, we obtain that the first cluster is given by the scenarios among $minwc[0] = 1$ and $maxwc[0] = 4$ and the second one, by the scenarios among $minwc[1] = 5$ and $maxwc[1] = 7$. See the left part of Figure 4.
Then, the cluster tree matrix associated to this partition is given in (11).

$$CT(c, g) = \left(\begin{array}{c|ccccccc} 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \end{array} \right) \tag{11}$$

In this case, the set of scenarios are $\Omega^1 = \{1,2,3,4\}$ for cluster $c = 1$ and $\Omega^2 = \{5,6,7\}$ for cluster $c = 2$. The related cluster models are linked by the NAC for $g = 1$ that can be expressed

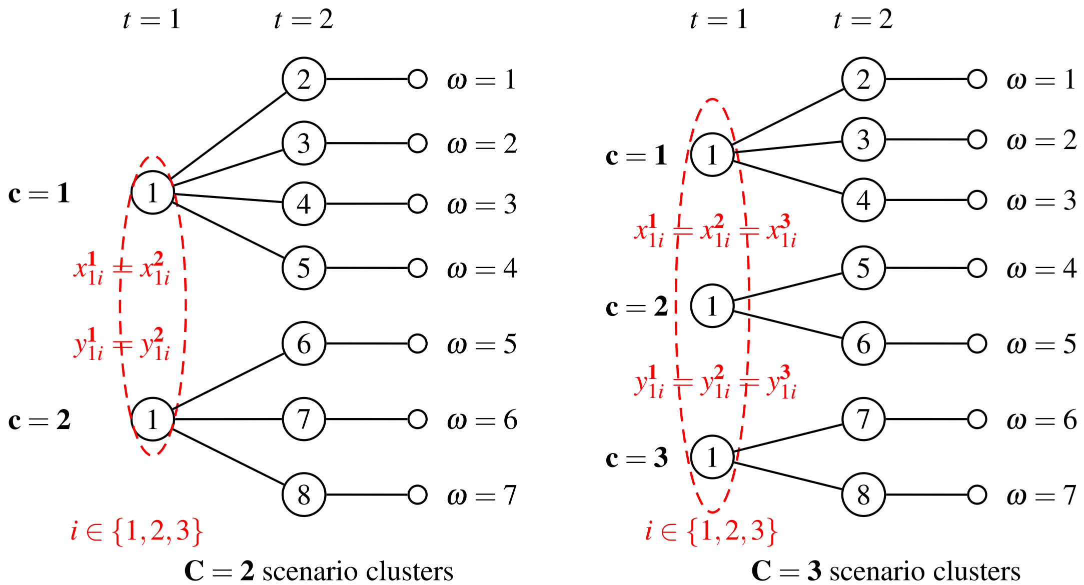

Figure 4: Scenario cluster partitioning

$$x^{\mathbf{1}}_{1i}=x^{\mathbf{2}}_{1i}, \qquad \forall i\in\mathscr{I} \tag{12}$$
$$y^{\mathbf{1}}_{1i}=y^{\mathbf{2}}_{1i}, \qquad \forall i\in\mathscr{I} \tag{13}$$

The corresponding model to cluster $c=1$ is the following:

$$
\begin{array}{lll}
\min & & 10x^{\mathbf{1}}_{11} + 15x^{\mathbf{1}}_{12} + 5x^{\mathbf{1}}_{13} + 1.4285x^{1}_{21} + 0.71425x^{1}_{22} + 0.42855x^{1}_{23} \\
& + & 1.4285x^{2}_{21} + 4.2855x^{2}_{22} + 2.857x^{2}_{23} + 1.57135x^{3}_{21} + 0.71425x^{3}_{22} + 1.4285x^{3}_{23} \\
& + & 0.71425x^{4}_{21} + 1.4285x^{4}_{22} + 0.42855x^{4}_{23} + 2y^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{12} + 2y^{\mathbf{1}}_{13} \\
& + & 0.14285y^{1}_{21} + 0.42855y^{1}_{22} + 0.2857y^{1}_{23} + 0.2857y^{2}_{21} + 0.14285y^{2}_{22} + 0.2857y^{2}_{23} \\
& + & 0.14285y^{3}_{21} + 0.14285y^{3}_{22} + 0.14285y^{3}_{23} + 0.42855y^{4}_{21} + 0.14285y^{4}_{22} + 0.2857y^{4}_{23} \\
\text{s.t.} & & \\
& & -4.5x^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{11} \leq 0 \\
& & -2.8x^{\mathbf{1}}_{12} + y^{\mathbf{1}}_{12} \leq 0 \\
& & -2.7x^{\mathbf{1}}_{13} + y^{\mathbf{1}}_{13} \leq 0 \\
& & -3.8x^{1}_{21} + y^{1}_{21} \leq 0 \\
& & -3.5x^{1}_{22} + y^{1}_{22} \leq 0 \\
& & -4.7x^{1}_{23} + y^{1}_{23} \leq 0 \\
& & -4x^{2}_{21} + y^{2}_{21} \leq 0 \\
& & -3.7x^{2}_{22} + y^{2}_{22} \leq 0 \\
& & -3.8x^{2}_{23} + y^{2}_{23} \leq 0 \\
& & -4.8x^{3}_{21} + y^{3}_{21} \leq 0 \\
& & -4.9x^{3}_{22} + y^{3}_{22} \leq 0 \\
& & -4.3x^{3}_{23} + y^{3}_{23} \leq 0 \\
& & -4.4x^{4}_{21} + y^{4}_{21} \leq 0 \\
& & -5.3x^{4}_{22} + y^{4}_{22} \leq 0 \\
& & -5.6x^{4}_{23} + y^{4}_{23} \leq 0 \\
& & y^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{12} + y^{\mathbf{1}}_{13} \geq 5 \\
& & y^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{12} + y^{\mathbf{1}}_{13} + y^{1}_{21} + y^{1}_{22} + y^{1}_{23} \geq 15 \\
& & y^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{12} + y^{\mathbf{1}}_{13} + y^{2}_{21} + y^{2}_{22} + y^{2}_{23} \geq 20 \\
& & y^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{12} + y^{\mathbf{1}}_{13} + y^{3}_{21} + y^{3}_{22} + y^{3}_{23} \geq 20 \\
& & y^{\mathbf{1}}_{11} + y^{\mathbf{1}}_{12} + y^{\mathbf{1}}_{13} + y^{4}_{21} + y^{4}_{22} + y^{4}_{23} \geq 20 \\
& & y^{\mathbf{1}}_{1i}, y^{\omega}_{2i} \geq 0, \quad \forall i \in \mathscr{I}, \omega \in \Omega^{1} \\
& & x^{\mathbf{1}}_{1i}, x^{\omega}_{2i} \in \{0,1\}, \quad \forall i \in \mathscr{I}, \omega \in \Omega^{1}
\end{array}
\tag{14}
$$

Notice that this cluster has 15 integer variables and 15 continuous variables, 20 constraints, 57 nonzero elements and the value of the objective function is $z_{MIP} = 49.5845$.

The corresponding model to cluster $c = 2$ is the following:

$$
\begin{array}{rl}
\min & 10x^{\mathbf{2}}_{11} + 15x^{\mathbf{2}}_{12} + 5x^{\mathbf{2}}_{13} + 1.4285x^{5}_{21} + 0.42855x^{5}_{22} + 0.71425x^{5}_{23} \\
+ & 0.42855x^{6}_{21} + 1.4285x^{6}_{22} + 0.71425x^{6}_{23} + 0.5714x^{7}_{21} + 1.4285x^{7}_{22} + 0.71425x^{7}_{23} \\
+ & 2y^{\mathbf{2}}_{11} + y^{\mathbf{2}}_{12} + 2y^{\mathbf{2}}_{13} + 0.14285y^{5}_{21} + 0.2857y^{5}_{22} + 0.14285y^{5}_{23} \\
+ & 0.2857y^{6}_{21} + 0.14285y^{6}_{22} + 0.14285y^{6}_{23} + 0.2857y^{7}_{21} + 0.14285y^{7}_{22} + 0.42855y^{7}_{23} \\
\text{s.t.} & \\
& -4.5x^{\mathbf{2}}_{11} + y^{\mathbf{2}}_{11} \leq 0 \\
& -2.8x^{\mathbf{2}}_{12} + y^{\mathbf{2}}_{12} \leq 0 \\
& -2.7x^{\mathbf{2}}_{13} + y^{\mathbf{2}}_{13} \leq 0 \\
& -4.5x^{5}_{21} + y^{5}_{21} \leq 0 \\
& -3.5x^{5}_{22} + y^{5}_{22} \leq 0 \\
& -3.6x^{5}_{23} + y^{5}_{23} \leq 0 \\
& -4.5x^{6}_{21} + y^{6}_{21} \leq 0 \\
& -4.7x^{6}_{22} + y^{6}_{22} \leq 0 \\
& -4.5x^{6}_{23} + y^{6}_{23} \leq 0 \\
& -4x^{7}_{21} + y^{7}_{21} \leq 0 \\
& -4x^{7}_{22} + y^{7}_{22} \leq 0 \\
& -4x^{7}_{23} + y^{7}_{23} \leq 0 \\
& y^{\mathbf{2}}_{11} + y^{\mathbf{2}}_{12} + y^{\mathbf{2}}_{13} \geq 5 \\
& y^{\mathbf{2}}_{11} + y^{\mathbf{2}}_{12} + y^{\mathbf{2}}_{13} + y^{5}_{21} + y^{5}_{22} + y^{5}_{23} \geq 15 \\
& y^{\mathbf{2}}_{11} + y^{\mathbf{2}}_{12} + y^{\mathbf{2}}_{13} + y^{6}_{21} + y^{6}_{22} + y^{6}_{23} \geq 20 \\
& y^{\mathbf{2}}_{11} + y^{\mathbf{2}}_{12} + y^{\mathbf{2}}_{13} + y^{7}_{21} + y^{7}_{22} + y^{7}_{23} \geq 15 \\
& y^{\mathbf{2}}_{1i}, y^{\omega}_{2i} \geq 0, \quad \forall i \in \mathscr{I}, \omega \in \Omega^2 \\
& x^{\mathbf{2}}_{1i}, x^{\omega}_{2i} \in \{0,1\}, \quad \forall i \in \mathscr{I}, \omega \in \Omega^2
\end{array}
\tag{15}
$$

The cluster $c = 2$ has 12 integer variables and 12 continuous variables, 16 constraints, 45 nonzero elements and the value of the objective function for this cluster is $z_{MIP} = 24.3994$.

If we choose, $C = 3$, i.e., a partition into three clusters, folowing the same procedure ($\frac{|\Omega|}{C} = \frac{7}{3} = 2.33$, and rest equal to 1) we obtain that the first cluster is given by the scenarios among $minwc[0] = 1$ and $maxwc[0] = 3$, the second one by the scenarios among $minwc[1] = 4$ and $maxwc[1] = 5$, and the third one by the scenarios among $minwc[2] = 6$ and $maxwc[2] = 7$. Then, the corresponded cluster tree matrix will be,

$$
CT(c,g) = \left(\begin{array}{c|ccccccc} 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \end{array} \right) \tag{16}
$$

In this partition, the set of scenarios are $\Omega^1 = \{1,2,3\}$ for cluster $c = 1$, $\Omega^2 = \{4,5\}$ for cluster $c = 2$ and $\Omega^3 = \{6,7\}$ for cluster $c = 3$. The related cluster models are linked by the NAC for $g = 1$ that can be expressed

$$
x^{\mathbf{1}}_{1i} = x^{\mathbf{2}}_{1i} = x^{\mathbf{3}}_{1i}, \qquad \forall i \in \mathscr{I} \tag{17}
$$

$$
y^{\mathbf{1}}_{1i} = y^{\mathbf{2}}_{1i} = y^{\mathbf{3}}_{1i}, \qquad \forall i \in \mathscr{I} \tag{18}
$$

The cluster $c = 1$ has 12 integer variables and 12 continuous variables, 16 constraints, 45 nonzero elements and the value of the objective function is $z_{MIP} = 38.799$. The cluster $c = 2$ has 9 integer variables and 9 continuous variables, 12 constraints, 33 nonzero elements and the value of the objective function is $z_{MIP} = 17.3995$ and the cluster $c = 3$ has 9 integer variables and 9 continuous variables, 12 constraints, 33 nonzero elements and the value of the objective function is $z_{MIP} = 16.971$.

7 Conclusions

In this brief note we have presented a general scenario cluster partitioning procedure for two-stage models. To generate the new structure and representation of the two-stage model some information is required, like the whole model in compact representation and *mps* format, the number of clusters in which the model is going to be decomposed, the number of scenarios, the number and the order of the variables (binary and continuous) at each stage and scenario group and the likelihood of each scenario.

The use of this procedure embedded in a cluster based Lagrangian decomposition scheme, and for a choice of a small number of clusters, will allow to obtain strong (lower) bounds to the solution value of the original two-stage stochastic problem.

Appendix A Illustrative stochastic example in MPS format

The model corresponding to the example presented in (9) can be represented with MPS format as follows:

```
NAME                BLANK
ROWS
 N  OBJROW
 L   R0000000
 L   R0000001
 L   R0000002
 L   R0000003
 L   R0000004
 L   R0000005
 L   R0000006
 L   R0000007
 L   R0000008
 L   R0000009
 L   R0000010
 L   R0000011
 L   R0000012
 L   R0000013
 L   R0000014
 L   R0000015
 L   R0000016
 L   R0000017
 L   R0000018
 L   R0000019
 L   R0000020
```

```
 L   R0000021
 L   R0000022
 L   R0000023
 G   R0000024
 G   R0000025
 G   R0000026
 G   R0000027
 G   R0000028
 G   R0000029
 G   R0000030
 G   R0000031
COLUMNS
     C0000000  OBJROW    10.       R0000000  -4.5
     C0000001  OBJROW    15.       R0000001  -2.8
     C0000002  OBJROW    5.        R0000002  -2.7
     C0000003  OBJROW    2.        R0000000  1.
     C0000003  R0000024  1.        R0000025  1.
     C0000003  R0000026  1.        R0000027  1.
     C0000003  R0000028  1.        R0000029  1.
     C0000003  R0000030  1.        R0000031  1.
     C0000004  OBJROW    1.        R0000001  1.
     C0000004  R0000024  1.        R0000025  1.
     C0000004  R0000026  1.        R0000027  1.
     C0000004  R0000028  1.        R0000029  1.
     C0000004  R0000030  1.        R0000031  1.
     C0000005  OBJROW    2.        R0000002  1.
     C0000005  R0000024  1.        R0000025  1.
     C0000005  R0000026  1.        R0000027  1.
     C0000005  R0000028  1.        R0000029  1.
     C0000005  R0000030  1.        R0000031  1.
     C0000006  OBJROW    1.4285    R0000003  -3.8
     C0000007  OBJROW    0.71425   R0000004  -3.5
     C0000008  OBJROW    0.42855   R0000005  -4.7
     C0000009  OBJROW    0.14285   R0000003  1.
     C0000009  R0000025  1.
     C0000010  OBJROW    0.42855   R0000004  1.
     C0000010  R0000025  1.
     C0000011  OBJROW    0.2857    R0000005  1.
     C0000011  R0000025  1.
     C0000012  OBJROW    1.4285    R0000006  -4.
     C0000013  OBJROW    4.2855    R0000007  -3.7
     C0000014  OBJROW    2.857     R0000008  -3.8
     C0000015  OBJROW    0.2857    R0000006  1.
     C0000015  R0000026  1.
     C0000016  OBJROW    0.14285   R0000007  1.
     C0000016  R0000026  1.
     C0000017  OBJROW    0.2857    R0000008  1.
     C0000017  R0000026  1.
     C0000018  OBJROW    1.57135   R0000009  -4.8
     C0000019  OBJROW    0.71425   R0000010  -4.9
     C0000020  OBJROW    1.4285    R0000011  -4.3
     C0000021  OBJROW    0.14285   R0000009  1.
     C0000021  R0000027  1.
     C0000022  OBJROW    0.14285   R0000010  1.
```

```
    C0000022  R0000027  1.
    C0000023  OBJROW    0.14285   R0000011  1.
    C0000023  R0000027  1.
    C0000024  OBJROW    0.71425   R0000012  -4.4
    C0000025  OBJROW    1.4285    R0000013  -5.3
    C0000026  OBJROW    0.42855   R0000014  -5.6
    C0000027  OBJROW    0.42855   R0000012  1.
    C0000027  R0000028  1.
    C0000028  OBJROW    0.14285   R0000013  1.
    C0000028  R0000028  1.
    C0000029  OBJROW    0.2857    R0000014  1.
    C0000029  R0000028  1.
    C0000030  OBJROW    1.4285    R0000015  -4.5
    C0000031  OBJROW    0.42855   R0000016  -3.5
    C0000032  OBJROW    0.71425   R0000017  -3.6
    C0000033  OBJROW    0.14285   R0000015  1.
    C0000033  R0000029  1.
    C0000034  OBJROW    0.2857    R0000016  1.
    C0000034  R0000029  1.
    C0000035  OBJROW    0.14285   R0000017  1.
    C0000035  R0000029  1.
    C0000036  OBJROW    0.42855   R0000018  -4.5
    C0000037  OBJROW    1.4285    R0000019  -4.7
    C0000038  OBJROW    0.71425   R0000020  -4.5
    C0000039  OBJROW    0.2857    R0000018  1.
    C0000039  R0000030  1.
    C0000040  OBJROW    0.14285   R0000019  1.
    C0000040  R0000030  1.
    C0000041  OBJROW    0.14285   R0000020  1.
    C0000041  R0000030  1.
    C0000042  OBJROW    0.5714    R0000021  -4.
    C0000043  OBJROW    1.4285    R0000022  -4.
    C0000044  OBJROW    0.71425   R0000023  -4.
    C0000045  OBJROW    0.2857    R0000021  1.
    C0000045  R0000031  1.
    C0000046  OBJROW    0.14285   R0000022  1.
    C0000046  R0000031  1.
    C0000047  OBJROW    0.42855   R0000023  1.
    C0000047  R0000031  1.
RHS
    RHS       R0000024  5.        R0000025  15.
    RHS       R0000026  20.       R0000027  20.
    RHS       R0000028  20.       R0000029  15.
    RHS       R0000030  20.       R0000031  15.
BOUNDS
 BV BOUND     C0000000  1.
 BV BOUND     C0000001  1.
 BV BOUND     C0000002  1.
 BV BOUND     C0000006  1.
 BV BOUND     C0000007  1.
 BV BOUND     C0000008  1.
 BV BOUND     C0000012  1.
 BV BOUND     C0000013  1.
 BV BOUND     C0000014  1.
```

```
  BV BOUND      C0000018   1.
  BV BOUND      C0000019   1.
  BV BOUND      C0000020   1.
  BV BOUND      C0000024   1.
  BV BOUND      C0000025   1.
  BV BOUND      C0000026   1.
  BV BOUND      C0000030   1.
  BV BOUND      C0000031   1.
  BV BOUND      C0000032   1.
  BV BOUND      C0000036   1.
  BV BOUND      C0000037   1.
  BV BOUND      C0000038   1.
  BV BOUND      C0000042   1.
  BV BOUND      C0000043   1.
  BV BOUND      C0000044   1.
ENDATA
```

total.mps

Appendix B C++ implementation for the clusters building

```
int *minwc; minwc=new int[nmodel];
int *maxwc; maxwc=new int[nmodel];

nmodel=numcluster;
nomega=ng-1;
int nc=nomega/numcluster;
int rest,ic;

for(ic=0;ic<nmodel;ic++){
   minwc[ic]=ic*nc;
   maxwp[ic]=(ic+1)*nc-1;
}

if(nomega%numcluster!=0){
  rest=nomega%numcluster;

  for(ic=0;ic<nmodel;ic++){
    if(ic<nomega%numcluster){
       minwc[ic]=ic*nc+ic;
       maxwc[ic]=(ic+1)*nc+ic;
    }
    if(ic>=nomega%numcluster){
       minwc[ic]=ic*nc+rest;
       maxwc[ic]=(ic+1)*nc+rest-1;
    }
```

```
    }
  }

  outputData<<"\n\n Cluster number c: ";
  outputData<<"\n Min-max w: ";
  for(ic=0;ic<nmodel;ic++) outputData<<" "<<minwc[ic]<<"-"<<maxwc[ic];
```

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