A proof of the inconsistency of Quine's system

«Mathematical Logic (1951)»

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Especial para THEORIA

On the following pages we shall give a proof of the inconsistency of the logical system proposed by W. V. O. QUINE in his «Mathematical Logic (1951», the proof its consistency being still extant.

Setting out to prove the inconsistency we shall first (A) quote those definitions and theorems from Quine's system «Mathematical Logic (1951)», on which our proof is based. Then (B) after introducing an abstract which may be symbolised by « ιx» (*) and contextually defined by

D $\quad$ '$y \varepsilon \iota x$' for '$x \varepsilon y$'

we shall deduce within QUINE's system the theorem

T_1 $\quad (y)(x) \quad x \varepsilon y \cdot \supset \cdot y \varepsilon V$

from that follow as corollaries the theorems

T_2 $\quad x \varepsilon \hat{x}(x \tilde{\varepsilon} x) \cdot \supset \cdot \hat{x}(x \tilde{\varepsilon} x) \varepsilon V$

and

T_3 $\quad x \varepsilon \hat{x}(x \varepsilon x) \cdot \supset \cdot \hat{x}(x \varepsilon x) \varepsilon V$

With their help we deduce the contradiction proving the inconsistency

C $\quad V \varepsilon V \cdot \sim (V \varepsilon V)$

A.

Number (**)			Page (**)
D 10		'$(\zeta = \eta)$' for '$(\alpha)(\alpha \varepsilon \zeta \cdot \equiv \cdot \alpha \varepsilon \eta)$'	136
D 13		'$(\zeta \tilde{\varepsilon} \eta)$' for '$\sim (\zeta \varepsilon \eta)$'	140
D 15		'V' for '$\hat{x}(x = x)$'	144

** 103 If Φ' is like Φ except for containing free ocurences of α' wherever Φ contains free ocurrences of α $\quad \vdash$ '$(\alpha) \Phi \supset \Phi$' $\quad$ 88

+ 183	$(y)(x)$	$x = y \cdot \equiv \cdot y = x$	137
+ 190	$(y)(x)$	$x \varepsilon y \cdot \supset \cdot x \varepsilon V$	144
+ 191	$(y)(x)$	$x \varepsilon y \cdot \equiv \cdot x \varepsilon V \cdot x \varepsilon y$	144
+ 210		$V \varepsilon V$	163
** 230		'$\alpha \varepsilon \hat{\alpha} \Phi \cdot \equiv \cdot \alpha \varepsilon V \cdot \Phi$'	171
+ 260		$\hat{x}(x \tilde{\varepsilon} x) \tilde{\varepsilon} V$	179
+ 261		$\hat{x}(x \varepsilon x) \tilde{\varepsilon} V$	179

(*) The sign 'ιx' may not be confounded with the symbols for description '(ιx)' and the unit class 'ιx' that are not used in the following.

(**) The numbers of definitions, theorems and pages are referring to QUINE's «Mathematical Logic (1951)».

B.

T_1 $\quad (y)(x) \quad x \varepsilon y \cdot \supset \cdot y \varepsilon V$

Proof:

D		'$y \varepsilon \iota x$' for '$x \varepsilon y$'	(1)
+ 191		$y \varepsilon \iota x \cdot \equiv \cdot y \varepsilon V \cdot y \varepsilon \iota x$	(2)
D		$\equiv \cdot y \varepsilon V \cdot x \varepsilon y$	(3)
** 230		$\equiv \cdot y \varepsilon \hat{y}(x \varepsilon y)$	(4)
D 10	(x)	$\iota x = \hat{y}(x \varepsilon y)$ (***)	(5)
D	$(y)(x)$	$x \varepsilon y \cdot \equiv \cdot y \varepsilon \iota x$	(6)
		$\equiv \cdot y \varepsilon \hat{y}(x \varepsilon y)$	(7)
** 230		$\equiv \cdot y \varepsilon V \cdot x \varepsilon y$	(8)
		$\supset \cdot y \varepsilon V$	(9)

T_2 $\quad x \varepsilon \hat{x}(x \tilde{\varepsilon} x) \cdot \supset \cdot \hat{x}(x \tilde{\varepsilon} x) \varepsilon V$

Proof. ** 103 $\quad [T_1 \supset] T_2$

T_3 $\quad x \varepsilon \hat{x}(x \varepsilon x) \cdot \supset \cdot \hat{x}(x \varepsilon x) \varepsilon V$

Proof. ** 103 $\quad [T_1 \supset] T_3$

C $\quad V \varepsilon V \cdot \sim (V \varepsilon V)$

Proof:

+260, T_2 $\quad (x) \quad [\hat{x}(x \tilde{\varepsilon} x) \tilde{\varepsilon} V \cdot]$

$x \varepsilon \hat{x}(x \tilde{\varepsilon} x) \cdot \supset \cdot \hat{x}(x \tilde{\varepsilon} x) \tilde{\varepsilon} V \cdot \hat{x}(x \tilde{\varepsilon} x) \varepsilon V$ $\quad$ (1)

D 13 $\quad [(1) \cdot \equiv \cdot] \, x \varepsilon \hat{x}(x \tilde{\varepsilon} x) \cdot \supset \cdot$

$\cdot \sim (\hat{x}(x \tilde{\varepsilon} x) \varepsilon V) \cdot \hat{x}(x \tilde{\varepsilon} x) \varepsilon V$ $\quad$ (2)

$[(2) \cdot \equiv \sim R(2) \supset] \sim (x \varepsilon \hat{x}(x \tilde{\varepsilon} x))$ $\quad$ (3)

** 230 $\quad [(3) \equiv] \sim (x \varepsilon V \cdot x \tilde{\varepsilon} x)$ $\quad$ (4)

$[(4) \equiv] \; x \varepsilon V \cdot \supset \cdot \sim (x \tilde{\varepsilon} x)$ $\quad$ (5)

(***) Since '$x \varepsilon y$' is stratified, the formula holds:

'$x \varepsilon V \cdot \supset \cdot \hat{y}(x \varepsilon y) \varepsilon V$'

D 13	$[(5) \equiv]$	$x \varepsilon V \cdot \supset \cdot x \varepsilon x$	(6)
+260, T_3	$[\hat{x}(x \varepsilon x) \tilde{\varepsilon} V \cdot]$		
		$x \varepsilon \hat{x}(x \varepsilon x) \cdot \supset \cdot \hat{x}(x \varepsilon x) \tilde{\varepsilon} V \cdot \hat{x}(x \varepsilon x) \varepsilon V$	(7)
D 13	$[7) \cdot \equiv \cdot]$		
		$x \varepsilon \hat{x}(x \varepsilon x) \cdot \supset \cdot \sim (\hat{c}(x \varepsilon x) \varepsilon V) \cdot \hat{x}(x \varepsilon x) \varepsilon V$	(8)
	$[8) \cdot \equiv \cdot \sim R(8 \supset]$	$\sim (x \varepsilon \hat{x}(x \varepsilon x))$	(9)
** 230	$[(9) \equiv]$	$\sim (x \varepsilon V \cdot x \varepsilon x)$	(10)
	$[(10) \equiv]$	$x \varepsilon V \cdot \supset \sim (x \varepsilon x)$	(11)
(6), (11)		$x \varepsilon V \cdot \supset x \varepsilon x \cdot \sim (x \varepsilon x)$	(12)
+210	$[V \varepsilon V \cdot \supset]$	$V \varepsilon V \cdot \sim (V \varepsilon V)$	

It, when proving theorem T_1 we have introduced the abstract « $\imath x$» by the contextual definition D rather than by abstract definition

$$'\imath x' \quad \text{for} \quad '\hat{y}(x \varepsilon y)'$$

this method is absolutely correct: it is true, theoren T_1 and, consequently, theorems T_2, T_3 and C could not be deduced if the abstract «$\imath x$» is introduced by an abstract definition, and the inconsistency of the system would remain undiscovered. There is no precept, however, in QUINE.s system of «Mathematical Logic (1951)» that an abstract may be introduced by an abstract definition only, and not by a contextual one; indeed, such a precept could be neither justified nor pronounced a general rule as the abstract, which is expressed by the abstractive symbol «$\hat{\alpha}\Phi$», can be introduced solely by a contextual definition.

If we have also introduced the abstract « $\imath x$» by the contextual definition D and not by the definition

$$'y \varepsilon \imath x' \quad \text{for} \quad 'y \varepsilon V \cdot x \varepsilon y'$$

containing the elementhood clause in the definiens, this method must likewise appear above reproach: it is true, the introduction of the abstract « $\imath x$» by the definition as quoted in the last instance would not lead to theorem T_1, nor, thus, to theorems T_2, T_3 and C, and the inconsistency of the system would remain undiscovered. There is, however, no precept in QUINE's system limiting the freedom of definition and demanding that the contextual definition of the abstract « $\imath x$» contain the elementhood clause in the definiens. Such a precept would be arbitrary in any case, and its imposition would have to be regarded as a means of obscuring the inconsistency of the system. Nor could such a precept be deduced from QUINE's axiom of membership

$$(7) \qquad '(\exists \beta)(\alpha)(\alpha \varepsilon \beta \cdot \equiv \cdot \alpha \varepsilon V \cdot \Phi)' \qquad p\ 162$$

For (7) constitutes merely the general warrant that, so long as «$\alpha \varepsilon V \cdot \Phi$» is valid, an abstract will exist, so that

$$'(\alpha)(\alpha \varepsilon \beta \cdot \equiv \cdot \alpha \varepsilon V \cdot \Phi)'$$

is fulfilled; however (7), does not disallow the existence of an abstract β for a formula Φ in a specific case, so that

$$'(\alpha)(\alpha \varepsilon \beta \cdot \equiv \Phi)'$$

is fufilled. The assumption of the existence of an abstract « $\imath x$» for «$x \varepsilon y$» in our definition D, so that

$$'(y)(y \varepsilon \imath x \cdot \equiv \cdot x \varepsilon y)'$$

is fulfilled, is not in contradiction to (7). An additional postulation would be required that the existence of an abstract β shall never be assumed for a specific formula Φ, so that

$$(\alpha)(\alpha \varepsilon \beta \cdot \equiv \Phi)'$$

would be fulfilled; contextual difinitions of the form

$$'\alpha \varepsilon \beta' \quad \text{for} \quad '\Phi'$$

would thus be barred and contextual definitions of the form

$$'\alpha \varepsilon \beta' \quad \text{for} \quad '\alpha \varepsilon V \cdot \Phi'$$

would be permissible only.