MENGER'S TRACE IN FUZZY LOGIC

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ABSTRACT: This paper deals with the relation with Fuzzy Logic of some of the ideas of Karl Menger published between 1942 and 1966 and concerning what he called "Hazy Sets", Probabilistic Relations and Statistical Metric Spaces. The author maintains the opinion that if Lofti A. Zadeh is actually the father of Fuzzy Logic, Menger not only was a forerunner of this field but that his ideas were and still are influential on it.

Keywords: Fuzzy logic, Karl Menger's hazy sets.

1. Introduction

I personally met Karl Menger once in my life; it was in May, 1977 at his Chicago place in a day for me unforgettable. I met Menger several times by letter from 1966 to 1976; his letters were always extremely kind and full of interesting geometrical ideas.

I am very grateful to Karl Menger not only by his suggestions but, specially, as he introduced me to Berthold Schweizer and, as a consequence, Claudi Alsina, then a young graduate student of mine, could go to Amherst to learn with Schweizer more on Probabilistic Metric Spaces and had the opportunity to introduce himself into the exciting kingdom of Functional Equations.

In a time on which our old country was moving to the exit of a long and dark period for freedom, culture and science, Menger, Schweizer and Sklar, among others, allowed us to start for a creative work jointly with people of other countries. Today we are proud of such a help.

But I met and I continue to meet Karl Menger in part of my own time for study and reflexion: I am passionate by many ideas underlying his numerous writings and, following what Menger said for Mach, I think that the best we can do for a scientist passed away is to show that his ideas are still alive, influential. This is what with this short paper I will try to do in memory of Menger's death more than ten years ago.

With the conviction that now, at the very end of this century, many branches of Science and Philosophy need to be refreshed by new and true concepts I got the idea of joining people coming from the fields of Mathematics, Physics, Philosophy, Economy, Didactics, Artificial Intelligence, etc., around a new "Menger Symposium" to be held each two years and matching the huge field of topics on which Menger wrote papers or books.

After discussing this idea with Claudi Alsina, and he with Bert Schweizer and Abe Sklar, I proposed it to Professor Terricabras, currently director of the "Ferrater Mora" chair for Philosophy and Contemporary Thinking at the University of Girona, and he kindly agreed to organize and to eventually host these Menger Symposia. The first will take place in 1997 in Girona, the second in 1999 and the third will be delayed until 2002 to be coincidental with the first hundred years of Menger's birth.

But now let me say something on how Fuzzy Logic, my research field after 1975, benefits from Menger's ideas.

In fact, Fuzzy Logic is in debt with Karl Menger from its very origin as he was one of the forerunners of the same concept of fuzzy set, on which Fuzzy Logic is grounded. However, Fuzzy Logic is not only in debt with Menger on this point but also in the point of expressing the logical connectives *and*, *or*, "*If.., then*" and, very specially, on the problem of individuating fuzzy sets.

2. On Menger's Legacy to Fuzzy Logic

2.1.

The use of a predicate P on a set of objects E could be or not a matter of degree, depending on both the players P and E. If any proposition "x is P" (for short xP) has one of the two classical truth values "true" or "false" (for short, l or 0 respectively), for all x in E for which xP has meaning, it is said that P is a crisp or ungraded predicate on E. By the contrary, if for some x in E the proposition xP has meaning but it is to agree that it has a truth-value different from 0 and 1 (perhaps something like "more true than false"), it is said that P is a vague or graded predicate on E. Most of the predicates we use in everyday reasoning to communicate among us are vague.

Maybe it could be better to say that "the *use* of P on E is crisp or vague to make clear that frequently the crispness or vagueness of a predicate depends on the context in which it is uttered or written.

Max Black (1) tried to associate with each P, in a given E, what he called a profile function, μ_P, empirically defined, and essentially showing the use a social community did of P when affirming "xP" or being in agreement that "it is the case that x is P". In fact, Black accepted that when xP is context-true then $\mu_P(x)=1$, that when xP is context-false then $\mu_P(x)=0$ and that when xP is less context-true than yP it is $\mu_P(x) \leq \mu_P(y)$. This was made by playing with some numerical characteristics of P and taking μ_P as a function of the real line on [0.1]; for example, if P is "tall" and E is some huge population, the numerical characteristic is "height" measured in centimeters. Like it is usual in Statistics.

In several papers (2), (3) and (4), written between 1941 and 1951, Menger said, with other words, that when the use of P on E is vague the only positive thing we can do is to study the frequency on which P fits with x, in a sense to be decided at each case. In fact, he come to assert that $\mu_P(x)$ is some probability of founding xP provide of meaning in the context of E. With this he concluded that the *set* of elements of E for which xP has some meaning should be something like a cloud that he called a *hazy-set* (ensemble flou in his french writings).

He really tried to give some "extensionality" to those *experimental predicates* that being non-crisp on E have nevertheless propositions xP with some meaning. Menger was motivated by understanding which kind of things are those that belong to the subatomic world.

Some years later, in 1965, Lotfi Zadeh (5) motivated by the troubles he got in early attempts to represent knowledge difficult of grasping with systems of differential equations, as it was usual in the world of Control Theory, come to a neighbor idea by thinking that if a crisp predicate P has a classical-set extension **P**, characterized by its indicator function

$$\varphi_{\mathbf{P}}(x) = \begin{cases} 0, \text{ if } x \in \mathbf{P} \textit{ is false} \\ 1, \text{ if } x \in \mathbf{P} \textit{ is true} \end{cases} = \begin{cases} 0, \text{ if } x \textit{ is } \mathrm{P} \textit{ is false} \\ 1, \text{ if } x \textit{ is } \mathrm{P} \textit{ is true} \end{cases} = \mu_P(x) \quad (1)$$

then, by considering the compatibility-function of a vague predicate with E:

$$\mu_P(x) =: \textit{truth value of } xP \in [0,1],$$

when those values exits in [0,1], it is possible to consider a virtual extension of P on E, defined by $\varphi_{\mathbf{P}} = \mu_P$. He called **P** the fuzzy set labeled P.

It is also not to be forgot that, in 1966, Stephan Körner (6) introduced the ideas of Inexact Predicate and Inexact Class just to give logical extensionality to the predicates appearing in the Empirical Sciences. It is important to point out that in considering *quantities* Körner did follow exactly the ideas before published by Karl Menger (7).

Perhaps we can say that this was the way of passing from "being" to "being in", or from "being" to "belong". Today we look at a fuzzy set as the mathematical representation, or extension, of a predicate P in a given context when $\mu_P(x)$ can be operationally defined by a known use of the propositions xP that allows to identify "x is P up to the degree r" with "the degree of truth of xP is r". Only then we talk on the fuzzy-set **P** through $\varphi_{\mathbf{P}}(x) = r$ *or* $x \in \mathbf{P}$

All this is very important for Artificial Intelligence, that needs to represent common sense reasoning as it is addressed by humans: *through an extensive use of short and very informative sentences with many vague predicates that are context-dependent*, like this last phrase.

As it seems that the human brain has an enormous capacity to communicate by that kind of expressions, Artificial Intelligence needs to grasp such capacity by some kind of representation of common-knowledge, and Fuzzy Logic has shown to be very successful in many cases, not only at a theoretical level but also allowing to produce efficient technological devices that markets accept under the label "Fuzzy Logic Product".

And, if everybody recognizes Zadeh as the father of Fuzzy Logic, nobody doubts that Menger and Black are among the grandfathers of the fuzzy-set idea.

2.2.

One of the things that mostly shows the context-dependency of Fuzzy Set Theory lies on the different connectives used to represent sentences composed by several predicates. The most elementary cases are xP *and* yQ, xP *or* yQ, *not* xP, and *If* xP, *then* yQ.

Maybe the question that more clearly shows the pervasive nature of vagueness is the distinguishability of two fuzzy sets.

In both cases, Menger's ideas were influential.

2.3.

In his celebrated paper introducing the Probabilistic Metric Spaces (8), Menger also introduced the concept of triangular-norm to represent the new triangle's inequality for those spaces. Later, Bert Schweizer and Abe Sklar (9) restricted the old, and too wider, definition to be just an operation transforming the linearly ordered unit interval on a commutative ordered semigroup with unit 1 and zero 0. This is what now we call a t-norm.

Bellman and Giertz (10) shown that to keep the properties of a DeMorgan Algebra (i.e., those of a Boolean Algebra except both the laws of Excluded-Middle and Non-Contradiction, that most people considers as only typical of crisp predicates) for the structure of composed vague and crisp predicates with

$$\mu_{PandQ}(x) = F(\mu_P(x), \mu_Q(x))$$
$$\mu_{PorQ}(x) = G(\mu_P(x), \mu_Q(x))$$

it should be F= Min and G= Max, under continuity conditions.

As it is not difficult to show under few, weak and usual logical conditions, it is

$$\begin{aligned} \mu_{PandQ}(x) \quad = \quad & F(\mu_P(x), \mu_Q(x)) \leq Min(\mu_P(x), \mu_Q(x)) \\ & \leq Max(\mu_P(x), \mu_Q(x)) \\ & \leq G(\mu_P(x), \mu_Q(x)) = \mu_{PorQ}(x), \end{aligned} \tag{2}$$

and Alsina, myself and Valverde started to use F as a t-norm and G as a t-conorm in a well known 1983 paper (11), without pressuposing the law of distributivity that only holds for the couple F=Min, G=Max. Usually F and G are N-duals, with $N: [0,1] \rightarrow [0,1]$ what is called, after a 1979 paper of mine (12), a strong-negation function allowing to define $\mu_{notP}(x) = N(\mu_P(x))$. Frequently, F is a 2-copula and belongs to the Frank's family. And families of predicates are now viewed as quantities.

This use of F, G and N opened a lot of possibilities for the study of several kinds of Fuzzy Logics, and most of the papers devoted to such studies were written by spaniards that reached a good consideration among the community of people doing research in the field of Fuzzy Logic. I think that a broader set of researchers in the world were aware on the existence of t-norms and copulas through Fuzzy Logic: we are proud of this and, probably, if Menger was alive he will be also proud of it.

2.4.

The theory of t-norms has shown its usefulness in a second and important aspect that allowed Fuzzy Logic to reach some maturity, to a central aspect in any Logic after Alfred Tarski's work: the way of representing conditional statements and its use either as general or as non-monotonic rules to make inferences.

In Fuzzy Logic there were different reputable expressions to represent "If $x \in \mathbf{P}$, then $y \in \mathbf{Q}$", ranging from $I(\mu_Q(y)/\mu_P(x)) = G(N(\mu_P(x)), \mu_Q(y))$, with the function $I(y/x) = G(N(x), y)$ defined in $[0,1] \times [0,1]$, to much more sophisticated expressions. But the most successful for studying good logical properties of "implications" were obtained by residuation, that is by using $I_\mu^E(y/x) = Sup\{z \in [0,1]; F(z, \mu(x) \leq \mu(y)\} = \hat{F}(\mu(x), \mu(y))$. These "implication-functions" are reflexive and F-transitive; they are F-Preorders, and, for example, the Prod-Preorder $\mathrm{Min}(1, \mu(y)/\mu(x))$ is known as the Menger-Goguen implication-function (13).

More again, to produce "inferences" it is necessary from the logical point of view to have the Meta-Rule of Modus Ponens. And this rule was established in Fuzzy Logic (following what was usual in the Logic of Quantum Mechanics) to be the so-called Rule of Compositional Inference or MP-Inequality:

$$T(\mu(x), I(\mu(y)/\mu(x))) \leq \mu(y), \text{ for any } x, y \text{ in } E,$$

playing a central role in order to obtain admissible approximate consequences from inexact premises. Also in that point the Spanish researchers were the introducers of those mathematical formulations within Fuzzy Logic. The theory of t-norms has played an important role in the development of the logico-mathematical ground of Fuzzy Logic, but this is not the end of Menger's trace.

2.5.

It's clear that there is no a unique Fuzzy Logic and that an important part of its utility in Artificial Intelligence lies in its flexibility on a broad variety of contexts of which common sense reasoning is rich.

At the beginning of Fuzzy Set Theory, by means of which vage predicates are represented in Fuzzy Logic, it was accepted without discussion that **P=Q** was the case when, and only when, $\mu_P = \mu_Q$ pointwise. This was taken as equivalent to both inequalities $\mathbf{P} \subset \mathbf{Q}$ ($\mu_P \leq \mu_Q$ pointwise) and $\mathbf{Q} \subset \mathbf{P}$ (ibidem), also equivalent to accept that $xP \rightarrow yQ$ is represented by means of the pointwise inequality of the respective μ_P and μ_Q.

That was strange because by one side it remembered the strong boolean equivalence $a \leq b$ *iff* $a' + b = a \rightarrow b = 1$, and by the other, if it was in that time usual to define the truth-value of $xP \rightarrow yQ$ through $\mathrm{Min}(1, 1 - \mu_P(x) + \mu_Q(y))$, the Łukasiewicz implication that when $xP \rightarrow yQ$ has the truth-value 1 shows the linear order, it was also very frequent to use $\mathrm{Max}(1 - \mu_P(x), \mu_Q(y))$ that does not allow to reach such order.

When the use of Fuzzy Logic was growing, and more examples were available, the suspicion that the pointwise identification of fuzzy sets is not always neither correct nor useful was also growing. The question is not still completely answered but some steps were doing in the last ten years; an one of these steps is not to consider pointwise equality but indistinguishability (4).

An F-ε-Indistinguishability function is a F-transitive, ε-reflexive and symmetrical fuzzy relation e on [0,1];

- $e(x,x) \geq \varepsilon$, for any $x \in [0,1]$. (3)
- $e(x,y) = e(y,x)$, for any couple $x,y \in [0,1]$. (4)
- $F(e(x,y),e(y,z)) \leq e(x,z)$, for any triad $x,y,z \in [0,1]$, (5)

with $0 \leq \varepsilon \leq 1$. When F=Min, they are called Zadeh's Similarity Relations; when F=W are called Ruspini's Likeness Relations, and when F=Prod are called Menger's Probabilistic Relations (14).

As given any F-Preorder I on [0,1], it is

$$e_I(x,y) = F(I(y/x),I(x/y)) \text{ a } F\text{-1-}Indistinguishability,$$

if the truth of xP$\rightarrow$yQ is given by $I(\mu_Q(y)/\mu_P(x))$ it has sense to assert:

xP is T-1-indistinguishable of yQ, up to the degree $e_I(\mu_P(x),\mu_Q(y))$, and this allows, for example, to define:

$$\mathbf{P{=}Q} \ \textit{iff} \ e_I(\mu_P(x)\cdot\mu_Q(y)) \geq r, \text{ any } x \in E,$$

that contains as a particular case the pointwise, and seems more related to to the phenomena of vagueness.

By extending T-1-Indistinguishabilities E as giving not numbers but fuzzy sets as values and defining, *à la* Tarski, the *falsum* of a fuzzy set as $f(\mu_P)=E(\mu_P,\emptyset)$ (that, in some cases, coincides with the complement of μ_P), the fuzzy set $d(\mu_P)=E(\mu_P,f(\mu_P))$ is said to be the *difussum* of μ_P, and satisfies the properties of a measure of fuzziness or fuzzy-entropy (14).

These definitions open the door towards a possible study of fuzzy objects or vague things. I am sure that those acquainted with the developments of Probabilistic Metric Spaces will see behind this the mixed shadows of Karl Menger, Bert Schweizer and Abe Sklar.

3. To finish, by now

Let me say finally that in the last five or six years there is a growing number of papers, mainly written by mathematicians but also by physicists, on the Logic of Quantum Mechanics (15) and using some ideas coming from Fuzzy Logic rather than from orthomodular classical lattices. For example, it is typical of these papers to consider "fuzzy observables" better that "observables", and those acquainted with (2) and (16) will recognize that Menger was thinking in this direction more than forty years ago.

Menger is still serving us fruitful ideas.

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