NOMINAL DEFINITIONS AND LOGICAL CONSEQUENCE IN THE PEANO SCHOOL[†]

Francisco RODRIGUEZ-CONSUEGRA*

* Departamento de Lógica y Filosofía de la Ciencia, Universidad de Valencia, Apdo. 22.109, 46010 Valencia. E-mail: francisco.rodriguez@uv.es

BIBLID [ISSN 0495-4548 (1997) Vol. 12: No 28; p. 125-137]

ABSTRACT: This paper is devoted to show the development of some of the model-theoretic ideas which are clearly present in the main members of the Peano school (Peano himself, Burali-Forti, Pieri and Padoa) as a result of their conception of nominal definitions. Also, their semantic definition of logical consequence (Pieri, Padoa) is viewed as one of the outcomes of that conception. Some examples of their use of the expression "nominal definition" are presented first. Second, the main advantages of this kind of definition, as they saw them, are briefly explained, mainly in a philosophical context. Finally, already in the kernel of the paper, some of the details of the model-theoretic view itself are shown, first in Peano, then in Pieri and Padoa, including in both cases some study of their semantic definitions of logical consequence.

Keywords: Logical consequence, nominal definition, philosophy of logic.

1. The expression "nominal definition"

The members of the Peano School used not only the expression "nominal definition", but also similar others, like "definition of name", or "symbolic definition". I think they chose the expression to avoid philosophical possible problems, given the traditional distinction between "real" and "nominal", and the danger of regarding mathematical definitions as the ones devoted to clarify the true "essence" of certain "objects" existing by themselves, in the Platonistic style. Thus, Pieri tells us something very explicit about the distinction mentioned. He said that definitions of things, which are opposed to nominal definitions, are definitions of entities in themselves, and are constituted by a system of predicates which are sufficient to qualify a subject. Also, he wrote that the expression "nominal definition" must be preferred to exclude "real" definitions, which are definitions of things (1899, 2). However, they usually insisted that in mathematics most of the definitions are nominal in the sense that they are simply useful conventions.

Regarding examples of "nominal definitions" things are easier. As a nominal definition is the mere convention to abbreviate a group of symbols a with a short name x, so its common form is this:

$$x = a \qquad \text{[Def.]},$$

and so almost every definition in the writings of the members of the Peano School is a nominal definition (see Ch. 3 of my 1991). As for particular examples which they used to give to the reader when explaining their use of "nominal definition", we can for instance mention Peano's efforts to classify nominal mathematical definitions into two classes: those referred to words which cannot be found in ordinary language (e.g. prime number), and those which try to make more precise the sense of words already pertaining to ordinary language (i.e., circle) (Peano OS2, 103). A particular example actually given in Peano (OS2, 166) is this:

$$Np = (N + 1) - [(N + 1) \times (N + 1)] \qquad \text{Def.,}$$

(i.e., Prime number = number which is greater than one and that cannot be decomposed into two numbers greater than one.), and the following is another, this time containing an initial "hypothesis" (OS2, 208):

$$a, b\, \varepsilon\, K . \supset .\!\cdot.\, a \supset b . = : x\, \varepsilon\, a . \supset_x . x\, \varepsilon\, b \qquad \text{Def.}$$

(In this case a "logical" symbol is defined -introduced, so it can be done only "in use". Note that apparently the symbol appears again in the *definiens*, but this is not really so, because in that case it means the conditional, together with universal quantification.)

2. The advantages of nominal definitions

I have already said something on this when I wrote that the members of the School chose nominal definitions to avoid philosophical problems with "real" definitions. But the important point here is the role of nominal definitions as opposed, not to real ones, but to the other two main kinds of definitions they admitted: definitions by abstraction (e.g., two straight lines are parallel iff they have the same direction) and by postulates (e.g., the concept of number as "defined" by the famous five postulates). Here the attitude was not the same for all the members of the School. Peano tended to accept the three kinds, although he always gave nominal definitions when he could, which means that he gave definitions by abstraction and by postulates only when he could not avoid them. But there was an instance of definition by abstraction (that of equality of numbers in terms of the bijective mapping between their corresponding classes) which was transformed by other members of the School (Burali-Forti by following Pieri's ideas) into a nominal one (i.e., the number of a class *a* is the class of all classes similar to *a*), and then Peano did not accept the transformation (see my 1991, p. 112). Also, he at times wrote as if definitions by postulates (e.g. the Peano postulates to characterize the concept of number) could be understood as "defining" their subject somehow.

It was Burali-Forti who best explained the main advantage of nominal definitions: they give us actual *concepts,* while definitions by abstraction and by postulates give us mere *intuitions.* That's why he added that we resort to non-nominal definitions only when we do not know how to use the corresponding nominal definitions. And this is the reason he thought that non-nominal definitions can -and must- be reduced to nominal definitions (see his 1900, 296). Another way to say the same thing is by resorting to Pieri's distinction between explicit and implicit definitions. Nominal definitions are explicit, in the sense that they eliminate, or construct, the *definiendum* in terms of the *definiens*. But implicit definitions do not eliminate the *definiendum*, so they cannot be used to build up chains of definitions which, by starting from a few primitive ideas (the ones from logic), can introduce the rest of ideas of a science. This is the most important sense according to which Russell's logicism can be said to be based on the Peano School, by mainly avoiding the several places where Peano's *Formulaire*. contained "gaps", i.e., places where the chain of definitions was broken by admitting definitions by abstraction or by postulates, which forced him to admit "new" entities, which were then "irreducible" to the primitive ideas of logic (I explained that in several places of my book). This is the sense in which for Burali-Forti and Pieri nominal definitions have to be preferred.

3. Nominal definitions and logical consequence in a model-theoretic context

Some members of the Peano school, remarkably Peano himself, as well as Pieri and Padoa, developed a view of axiom systems according to which they can be regarded as formal systems which can be explained by resorting to a semantic, or model-theoretic approach, where nominal definitions are going to work as schemes which, as a matter of fact, can be seen as being independent of any standard interpretation. It is under this conception that they, rather unexpectedly, arrived at what can be regarded as a full semantic conception of logical consequence. I think they have to be acknowledged as having clearly anticipated Tarski's definition of that concept in his celebrated paper of 1936. To develop this question I will refer separately to them.

3.1. Peano

When Peano posed himself the problem of the status of the definitions by postulates, for instance, the status of his famous five postulates for arithmetic, he sometimes said that those postulates can be understood as that which we can obtain by abstraction from all the possible interpretations which can satisfy the postulates (which, by the way, could also be understood as another way to transform a definition by postulates into a nominal definition; see Pieri below). In this way he was somehow convinced to make precise his well-known expression that the postulates somehow (implicitly, to be sure) "define" the primitive ideas of number, zero and successor, or the conception according to which we can say that <N, 0, +> is "the" system satisfying the five postulates, for he was also convinced that the same set of postulates could be satisfied by an infinity of different systems of entities.

The problem of nominal definitions under this rather model-theoretic viewpoint is that every definition we could build up, that is every definition whose definiens is made out of the primitive ideas, would be a definition which could hardly be depending on the "standard interpretation" of those ideas, given that the same symbolical definition would continue to be valid under a different interpretation which satisfies the postulates as well. So, the constructions actually carried out by the definitions would be mere schemes void of any intuitive content, at least until we point our some particular interpretation. That would be fine under a formal viewpoint. The problem is that for Peano the arithmetical primitives do have intuitive content, which we try to grasp through the postulates, so what every nominal definition actually does is to point out a well-known content in the definiens. Thus for Peano the intuitive or empirical content of the primitive ideas was precisely the ultimate guarantee that when we build up a whole system we are not constructing mere void schemes to be later fulfilled by any system of entities whatsoever, but we actually are handling real entities whose ontological status cannot be dissolved into formal schemes.

To sum up: for Peano the formal aspects of an axiom system seemed to be clear disadvantages in order to try to grasp the ultimate essence of the primitive ideas, which are given to us through a previous process of empirical induction. Thus, we can say that for him the formal character of his axiom systems was somehow opposed to the epistemological character he wanted for their primitive ideas and propositions, which were to be *entities* and *facts* previously known. As we shall see, this problem was later gradually avoided by Pieri and, mainly, by Padoa.

Even so, Peano was perfectly able to use the method of counter-examples, or interpretations (or models) to prove the relative independence of certain axioms. He used the method at least since 1889, but only in 1894 he explicitly gave a theoretic account of the method (OS3, 127):

> Si può provare l'indipendenza di alcuni postulati da altri, mediante esempi. Gli esempi per provare l'indipendenza dei postulati si ottengono attribuendo ai segni non definiti, (...), dei significati affatto qualunque; e se si trova che i segni fondamentali, in questo nuovo significato, soddisfino ad un grupo di proposizioni primitive, e non a tutte, si dedurrà che queste non sono conseguenze necessarie di quelle; ossia che il secondo gruppo di proposizioni esprimono proprietà (...) que ancora non erano espresse da quelle.
>
> ("It is possible to prove that some postulates are independent from others by means of examples. Those examples are obtained by attributing to the undefined symbols, (...), any meanings; and if we find that the fundamental symbols, through the new meaning, do satisfy a set of primitive propositions, but not all of them, it will be inferred that the latter propositions are not necessary consequences of the former; that is to say, that the second set of propositions express properties (...) which were not yet expressed in the first set.")

It seems to me that in this way a first implicit equivalence between independence and the negation of logical consequence appeared, in spite of Peano's lack of interest in following this development.

Also, this may have been a simple consequence of Peano's usual way of understanding logical consequence from the positive side. Thus, already in *Arithmetices Principia* (1889) he wrote (p. viii):

> Si propositiones a, b entia indeterminata continent x, y, ..., scilicet sunt inter ipsa entia conditiones, tunc $a \supset_{x, y} \dots b$ significat: quaecumque sunt x, y, ... a propositione a deducitur b.
>
> ("If propositions a, b contain indeterminate entities x, y, ..., that is, they are conditions between these same entities, then $a \supset_{x, y} \dots b$ means: whatever x, y, ... may be, from proposition a we can deduce proposition b.")

However, Peano never seemed to be interested in clarifying the semantic interpretation which could be involved in this way of describing the meaning of logical implication. Let us see how Pieri and Padoa improved on this still obscure position.

3.2. Pieri

Mario Pieri contributed with three new ideas to our topic: his method to transform definitions by abstraction into nominal ones; his conception of axiom systems as mere hypothetico-deductive systems, which made possible for the first time a true model-theoretic approach; and his method to transform definitions by postulates into nominal definitions, which was a consequence of that formal conception.

Regarding Pieri's method to transform definitions by abstraction into nominal definitions, I have already explained the two instances he offered in his publications elsewhere (see my 1991, 131 ff.), but I can say here that those instances were strictly equivalent to the famous one according to which Peano and Russell later defined the number of a class, not through the equality of two numbers and the bijection between their corresponding classes, but through the class of all classes similar to the original class. With that method Pieri intended to replace the obscure method of abstraction by what he described as "true definitions".

The second, and even more important contribution was Pieri's formal view of axiom systems. According to him axiom systems consist of a set of primitive ideas without meaning and a set of postulates which are neither true nor false, and a set of theorems which are derived from them through a set of logical axioms. Thus, we cannot say anything about the primitives of a system until we provide it with an interpretation of the primitive terms, and so Pieri gave us for the first time a true scheme of the new approach to axiomatics, the one opposed to the old Euclidean view. Therefore, we do not need any epistemological, or ontological justification for the primitive ideas of a system, which are merely implicitly defined by the postulates, which do not represent facts, but mere logical relations between the primitive terms. Thus, the intuitive empirical approaches by Pasch and Peano resulted completely overcome.

From this viewpoint nominal definitions are no longer intuitive constructions which build up derivative concepts out of simple ones, but mere abstract devices, no matter the fact that we may need, to reach the theory, to keep a particular interpretation of the primitive ideas in mind. This led Pieri to a generalization of two notions which had previously appeared in geometry: (i) Gergonne's principle of duality, according to which certain demonstrations are valid for dual forms of the theorems with only changing certain primitive ideas; (ii) Klein's method to classify the different geometries according to the different properties invariant under certain group of transformations. Regarding the principle of duality Pieri thought that it was only a particular form of a more general principle: the principle of plurality, according to which the primitive ideas of a science are indeterminate except by the postulates, so that they may be replaced by many other sets of undefined terms without affecting the proofs of the theorems. Thus, this general principle concerns not only geometry, but any other deductive theory.

Concerning Klein's method of classification, Pieri wrote that it can provide with a method to select the ideas which have to be taken as primitive. In this way, as every science is usually characterized by a certain group of transformations that cannot change the essential properties which are studied by that science, then we could take as primitive concepts those which are invariant under that fundamental group of transformations, but not under a wider group (Pieri, *Opere*, p. 257). Unfortunately Pieri did not work out these ideas nor did he apply them to his particular investigations, but they clearly show the completely modern way in which he saw the more abstract features of axiomatic theories. Needless to say, by resorting to Klein's idea Pieri can be regarded as the first known precedent of the much later use of that idea to describe formal sciences (Tarski and others).

As a whole, that formal view of the primitive ideas and nominal definitions led Pieri even to discuss the sense according to which we could say that some sort of indetermination unavoidably affects the ultimate meaning of the undefined ideas of any deductive science. At that point it seems to me that Pieri was no yet able to draw the ultimate consequences of his own model-theoretic viewpoint as clearly as Padoa did it a little later. For Pieri the plurality affecting those primitive ideas does not lead to a complete indetermination: although it is true that the mind cannot grasp the complete domain of those ideas, however, the primitive propositions always allow us to decide whether or not any given object belongs to the domain of the primitive concepts. Thus apparently Pieri did not see the possibility of non-standard interpretations of his systems, in spite of the fact that he, at least since 1906, was in possession of the concept of categoricity (*Opere,* p. 432), doubtless taken from Huntington and Veblen. Again unfortunately, Pieri did not apply this concept to any of his axiomatizations.

A further consequence of Pieri's abstract conception of definitions is that it made a true model-theoretic description of implicit definitions possible. Thus Pieri said, not only that the postulates somehow "define" the primitive terms of a theory, but also that the whole set of postulates define a global concept. For instance, in speaking of the postulates of his axiomatization of projective geometry of 1898, Pieri wrote that they define the global concept of geometric space, which can be understood as the class of all possible interpretations which are capable of satisfying the postulates (*Opere,* 155-6). Thus, he not only advanced the approach used by Russell in his *Principles of mathematics* (precisely in the sections on Geometry, located in the last part of the work), but showed a mastering of the abstract axiomatic method which has been said to be superior to Hilbert's classic approach of 1899, which is many times associated rather to the implicit definition approach, which was already criticized by Frege, and corrected by Bernays in his famous review about forty years later.

The model-theoretic approach made also possible for Pieri to apply systematically metatheoretic devices at least from 1897 on. Thus, in that year he devoted a whole appendix to his axiomatization of projective geometry in which he constructed models to demonstrate the ordinal independence of the postulates, and offered a whole model to demonstrate their consistency, and all that about two years before Hilbert's *Grundlagen.* It is true that for Pieri the only way to prove consistency was the exhibition of a model, as his later contributions to the problem show, so he explicitly supported Frege's view that the consistency of a concept means essentially the existence of some object falling under it. Curiously, when Hilbert discussed with Frege about this problem in the celebrated correspondence, he defended the view that mere consistency is enough for mathematical existence, in spite of the fact that by those years Hilbert himself had only available the method of the exhibition of a model to prove relative consistency, as his *Grundlagen* shows.

It is in the framework of this full model-theoretic context that Pieri's definition of logical consequence appeared already in 1897 (*Opere* , 60, 45, 9).

There he defined independence in the usual way:

> (...) le proposizioni *P, Q, R,* ... si dirano "*indipendenti* le une dalle altre" se avvien che nessuna sia conseguenza delle rimanenti (dunque se avviene, che per ciascuna si possan trovare degli *x, y, z,* ... che non la verificano, mentre rendon soddisfatte le altre).
>
> ("(...) the propositions *P, Q, R,* ... will be said to be '*independent* from each other' if none of them is a consequence of the rest (therefore, if for any of them we can find certain *x, y, z,* ... which do not verify it, while they do satisfy the rest).")

But he had insisted before that this conception of independence was based on a clear semantic conception of the negation of logical consequence in terms of sets of objects or models, that is in clear semantic terms:

> Date (...) proposizioni condicionali $P(x, y, z, ...)$, $Q(x, y, z, ...)$, $R(x, y, z, ...)$, ecc., sugli enti variabili x, y, z, ... non può cader dubbio sul valore delle asserzioni "dalle *P* e *Q* *non si deduce* la *R*", "la *R non è consequenza* delle *P* e *Q*"; (...) Ambo i modi null'altro esprimono che questa prosizione particolare: "*esistono* degli *x, y, z*, ... per cui son vere la P e la Q, ma non è vera la R".
>
> ("Given (...) conditional propositions $P(x, y, z, ...)$, $Q(x, y, z, ...)$, $R(x, y, z, ...)$, etc., over variable entities x, y, z, ... there could not be any doubt on the meaning of the assertions '*R cannot be deduced* from *P* and *Q*", "*R is not a consequence* of *P* and *Q*"; (...) Both express but this particular proposition: "*there are* certain *x, y, z*, ... from which P and Q are true, but R is not true.")

Again, this semantic conception of the negation of logical consequence was said to be based on a previously introduced positive conception of the same concept:

> (...) il giudizio "da $P(a, b, c, ...)$ si deduce $Q(a, b, c, ...)$" - (...) - è da retenersi il medesimo che: "qualunque siano *a, b, c*, ..., se per essi è vera $P(a, b, c, ...)$ sarà altres" vera $Q(a, b, c, ...)$...".
>
> "(...) the judgement 'from $P(a, b, c, ...)$ is deduced $Q(a, b, c, ...)$' - (...) - has to be kept in mind as being the same as 'for every *a, b, c*, ..., if $P(a, b, c, ...)$ is true for them, $Q(a, b, c, ...)$ will be true as well...")

Or, even in clearer terms:

> Se $P(x, y, z, ...)$, $Q(x, y, z, ...)$ sono propozioni negli enti variabili *x, y, z*, ..., la scrittura "$P(x, y, z, ...) \supset_{x, y} Q(x, y, z, ...)$" significa "qualunque siano *x, y*, purchè soddisfacenti a $P(x, y, z, ...)$, dovranno altres" verificare la $Q(x, y, z, ...)$"; e può leggersi: da *P* si deduce, rispetto ad *x, y, Q*".
>
> ("If $P(x, y, z, ...)$, $Q(x, y, z, ...)$ are propositions over the variable entities *x, y, z*, ..., the writing '$P(x, y, z, ...) \supset_{x, y} Q(x, y, z, ...)$' means 'for every *x, y*, provided they satisfy $P(x, y, z, ...)$, they must verify $Q(x, y, z, ...)$ as well'; and so it can be read: Q is deduced from P, with regard to *x, y, Q*'.")

It is true that here Pieri did not speak explicitly about models or interpretations, and, in particular, he did not speak about "every interpretation" (or "all models"), so making the quantifiers range over every domain. It is also true that it is possible that he was simply thinking of proofs in the usual truth-functional (conditional) way, as Peano himself did before. But I am convinced he was so familiar with the model-theoretic approach that in speaking this way he had in mind something very similar to the now usual way of defining logical consequence. At any rate, only three years later Alessandro Padoa reached the full contemporary definition of logical consequence, as we are going to see to finish.

3.3. Padoa

In his "Logical introduction to any deductive theory", a part of the paper read before the 1900 Paris International Congress of Philosophy, Padoa established his general approach to his formal and model-theoretic conception of axiom systems. Today it is believed that his main contribution in that paper was his method to prove whether the undefined symbols of a system are irreducible with respect to the system of its unproved propositions, which was a further application of Peano's method to prove the independence of the postulates. For Padoa, to do that, it suffices to find, for each undefined symbol, an interpretation of the whole set of primitive terms which verifies the postulates even when the meaning of the symbol considered is suitably changed. But the interesting thing here is pointing out that the introduction of that method was only possible thanks to a general view of axiom systems as mere formal, uninterpreted schemes.

Thus, Padoa needed, before discovering his famous method, to see deductive theories in a new way, according to which the ideas and facts which usually were given as the ones represented by the undefined symbols and the unproved propositions, were actually mere empirical or psychological aids which we use in constructing the theory, but which we must regard as quite independent of the theory itself, regarded as a formal system. Thus Padoa, instead of accepting Pieri's distinction between axioms and postulates, interpreted the postulates of a formal theory as mere conditions imposed upon the undefined symbols and definitely forgot about any criterion of simplicity for them. Under this general view he described for the first time nominal definitions as mere relations between a symbol and the symbols previously introduced which is able to single out a meaning for that symbol as soon as we choose an interpretation of the system as a whole.

However, both aspects of his completely model-theoretic conception of theories, that is, the already quite formal conception and the consequent character of nominal definitions, were exposed in a much clearer way in an unpublished manuscript of the same year entitled "Riassunto delle Conferenze su l'Algebra e la Geometria quali teorie deduttive" in which a series of lectures in the University of Rome were transcribed.[1] In addition, a clearer conception of the relationship between the usual way of proving the independence of the postulates and the notion of logical consequence can also be found in the manuscript, in which we can find even a literally Tarskian definition of logical consequence. Let us see these three points through the corresponding passages.

Regarding the formal view of axiom systems, Padoa writes:

> Noi consideremo le teorie deduttive sotto *l'aspetto formale;* immagineremo cioè che, all'inizio della teoria, I simboli che rappresentano le idee assunte quali primitive, sieno sprovvisti di significato.

> ("We consider deductive theories under the *formal aspect;* that is, we imagine that, at the beginning of the theory, the symbols which represent the ideas assumed as primitive are devoid of meaning".)

To my knowledge this is the first use of the expression "formal" in that context in the Peano school.

Concerning the role of nominal definitions under the model-theoretic conception, we can read:

> Nelle teorie deduttive considerate sotto l'aspetto formale, nemmeno le Df che I logici dicono "*nominali*" individuano il significato delle *idee derivate.* Ogni Df esprime soltanto una *relazione* fra il *significato della idea definita* e *l'interpretazione delle idee primitive*; per modo che, fissata, questa, quello risulta implicitamente individuato.
>
> ("In the deductive theories regarded under the formal aspect the definitions which are called 'nominal' by the logicians do not single out the meaning of the *derivative ideas.* Every definition expresses only a *relationship* between the *meaning of the defined ideas* and the *interpretations of the primitive ideas;* in a way that once we set the later, the first results implicitly singled out.")

Thus, we can say that only Padoa made quite explicit the tendency towards expressing nominal definitions as mere abstract relations, which can be discovered only in an implicit way in Peano and Pieri. And we can add that this was possible thanks to Padoa's explicit effort to distinguish the logical, the semantical and the epistemological elements which were involved in the transition from the old Euclidean axiomatics to the new modern formal one.

Finally, let us see Padoa's model-theoretic definition of logical consequence. Already in his contribution to the Paris philosophy congress Padoa, by following the tradition of Peano and Pieri, wrote that in order to prove the independence of an unproved proposition it suffices with exhibiting an interpretation under which the given proposition could be false and all the others true. However, Padoa also made explicit the ultimate reason for that, which the rest of authors of the school left implicit: when we establish an interpretation of the undefined symbols that verifies the unproved propositions but one, "then this proposition is not a logical consequence of the other propositions; that is, it is not possible to deduce the proposition in question from the other unproved propositions".

In the unpublished manuscript I referred before the corresponding passage contained a full definition of what we call today "logical consequence" on these lines, which precedes the introduction of the method to proving independence:

> Se una Pp, ad es. α, fosse deducibili dalle precedenti, ciò significherebbe che ogni interpretazione delle idee primitive, la quale verifichi le Pp precedenti la Pα, deve pure verificare la Pα.
>
> ("If a primitive proposition, say α, was deducible fro m the precedent propositions, that would mean that every interpretation of the primitive ideas which verify those primitive propositions must also verify the proposition α".)

This is to me strictly equivalent to Tarski's famous definition, according to which a given proposition φ is a logical consequence of a set Γ of propositions iff every model which satisfies Γ also satisfies φ.[2] Therefore, only Padoa was able to make completely explicit the fact that the old method to prove independence was implicitly based on an underlying concept of logical consequence in the modern sense. (No matter the fact that he did not say anything about the difference between derivability and logical consequence, that is, the difference, which was discovered much later, between syntactic and semantic deducibility.)

The fact that only Padoa was able to do that seems to me to be due to his complete mastery of the formal, model-theoretic approach. Before Padoa, every member of the school knew about the usual methods of proving consistency and independence, which were essentially introduced by the discoverers of non-Euclidean geometries, which showed that the postulate of the parallels was independent of the rest of Euclidean postulates by finding the corresponding interpretations, and also gave interpretations of the new geometries which showed that they were consistent if Euclidean geometry was consistent (Beltrami, Klein). But only Padoa, who, in addition, usually saw axiom systems under the model-theoretic viewpoint, and was especially interested in analyzing the abstract structure of deductive theories as a goal which was interesting in itself, was able to draw the corresponding conclusions.

To finish, I can try to make quite explicit Padoa's argument relating independence and logical consequence (for: Γ = set of statements; φ e Γ; M = model; sat = satisfies; ind = is independent of).

The usual definition of independence in model-theoretic terms, assuming the consistency of the whole system, is:

$$\varphi \text{ ind } \Gamma \leftrightarrow \exists M \,(M \text{ sat } \Gamma \wedge M \neg \text{ sat } \varphi);$$

by replacing 'ind' by '$\neg \vDash$', and '$\exists \ldots \neg$' by '$\neg\forall$', we obtain:

$$\Gamma \neg \vDash \varphi \leftrightarrow \forall M \,(M \text{ sat } \Gamma \rightarrow M \text{ sat } \varphi);$$

therefore, logical consequence can be defined this way:

$$\Gamma \vDash \varphi \leftrightarrow \forall M \,(M \text{ sat } \Gamma \rightarrow M \text{ sat } \varphi),$$

which is, precisely, what Padoa, and much later Tarski, actually did.

Notes

† Former versions of parts of this paper were read in the symposium "Logics and the foundations of mathematics (1885-1905)", organized by I. Grattan-Guinness and myself, as a part of the XIXth International Congress of History of Science (Zaragoza, August, 1993), and also in the Department of Mathematics, University of Genova, June, 1995. For the preparation of this paper resources provided by the grant DGICYT PS93-0220 are gratefully acknowledged.

1 The ms was sent to me for study by I. Grattan-Guinness, who found it in the Library of Pavia, Italy. The reader may easily imagine my astonishment in discovering Padoa's definition. We do not know yet whether those lectures were given before of after the Paris Congress of 1900. On the one hand Padoa may have given them before the Congress, whose mentioned contribution must have contained only a set of partial results; on the other, some of the details of the manuscripts might be interpreted as the result of further thinking, as for instances it takes place with his more formal general view.

2 In a recent trip to Italy I have discovered that Padoa (and also Burali-Forti) used similar semantic definitions of logical consequence in some publications, now unfortunately forgotten. I hope to be able to write a further paper on this particular subject in order to develop some of the ideas appearing here with the help of those additional definitions. I have recently dealt with Pieri's and Padoa's ideas in my papers 1996a, b and c.

BIBLIOGRAPHY

Burali-Forti: 1900, 'Sur les différentes méthodes logiques pour la definition du nombre réel', *Congrés Int. de Phil.*, París, 1900, París, Colin, vol. III, 289-307.

Padoa, A.: 1900a, 'Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique a une théorie déductive quelconque', *Congrés Int. de Phil.*, París, 1900, París, Colin, vol. III, 309-65. Trad. ing. de la primera parte en van Heijenoort (1967), 118-23.

--: 1900b, 'Riassunto delle Conferenze su l'Algebra e la Geometria quali teorie deduttive', Manuscrito inédito existente en la Biblioteca de Pavia, descubierto por I. Grattan-Guinness, que me lo remitió para su estudio, en el transcurso del cual me percaté con asombro de la definición que contenía.

Peano, G.: 1957, *Opere scelte* (OS), 3 vols., U. Cassina (ed.), Roma, Cremonese, 1957-9.

--: 1889, *Arithmetices principia, nova methodo exposita.* Rep. and Spanish translation by J. Velarde, Oviedo, Pentalfa, 1979.

--: 1894, 'Sui fondamenti della geometria', OS3, 115-57.

Pieri, M.: 1980, *Opere sui fondamenti della matematica,* Roma, Cremonese.

--: 1898, 'I principii della geometria di posizione composti in sistema logico deduttivo', *Memorie della Reale Accademia delle Scienze di Torino* 48, 1-62. Rep. en *Opere.*

--: 1899, 'Della geometria elementare come sistema ipotetico deduttivo', *Memorie della Reale Accademia delle Scienze di Torino* 49, 173-222. Rep. en *Opere.*

Rodríguez-Consuegra, F.: 1991, *The mathematical philosophy of Bertrand Russell: origins and development,* Basilea, Boston y Berlín, Birkhäuser. 2nd print. in 1993.

--: 1993a (with E.A. Marchisotto), 'The work of Mario Pieri in the philosophy and foundations of mathematics', *Hist. Phil. Logic* 14, 215-20.

--: 1993b, 'Mario Pieri y la naturaleza de la axiomática', in Bustos, E., Echeverría, J., Pérez, E. and Sánchez, M.I. (eds.): *Actas del I Congreso de la Sdad. de Lógica, Metod. y Filos. de la Ciencia en España,* Madrid, UNED, 1993, 502-505.

--: 1996a, La definicion semántica de consecuencia lógica: descubrimiento de la escuela de Peano', *Arbor,* 1996, in print.

--: 1996b, 'Mario Pieri y la filosofía de las ciencias deductivas', in F. Zalamea (ed.): *La trenza y el velo. Lazos entre historia, matemáticas, lógica y filosofía,* Universidad Nacional de Colombia, Bogotá, 1996, in print.

--: 1996c, 'La vía negativa hacia el concepto de consecuencia lógica', in the same as 1996b.

Francisco A. Rodríguez Consuegra es profesor titular de Lógica en el Dpto. de Lógica y Filosofía de la Ciencia de la Universidad de Valencia. Su principal línea de investigación está dirigida a estudiar los nexos entre la filosofía del lenguaje y la filosofía de la lógica y de la matemática en torno a los orígenes de la filosofía analítica clásica. Entre sus publicaciones, aparte de numerosos artículos en revistas especializadas, cabe reseñar *The mathematical philosophy of Bertrand Russell: origins and development*, Basel/Boston/Berlin, Birkhäuser, 1991, y su edición de Kurt Gödel, *Unpublished philosophical essays*, Basel/Boston/Berlin, Birkhäuser, 1995.