A GENERAL REPRESENTATION FOR INTERNAL PROPORTIONAL COMBINATORIAL MEASUREMENT SYSTEMS WHEN THE OPERATION IS NOT NECESSARILY CLOSED†

José A. DIEZ*

Manuscript received: 1997.5.24.
Final version: 1998.1.19.

* Departament d'Antropologia Social i Filosofia, Universitat Rovira i Virgili, Plaça Imperial Tàrraco 1, 43005 Tarragona, Spain. E-mail: jadc@fll.urv.es

BIBLID [0495-4548 (1999) 14: 34; p. 157-178]

ABSTRACT: The aim of this paper is to give one kind of internal proportional systems with general representation and without closure and finiteness assumptions. First, we introduce the notions of internal proportional system and of general representation. Second, we briefly review the existing results which motivate our generalization. Third, we present the new systems, characterized by the fact that the linear order induced by the comparison weak order $\succsim$ at the level of equivalence classes is also a well order. We prove the corresponding representation theorem and make some comments on strong limitations of uniqueness; we present in an informal way a positive result, restricted uniqueness for what we call *connected objects*. We conclude with some final remarks on the property that characterizes these systems and on three possible empirical applications.

Keywords: measurement, intensive magnitudes, internal combinatorial systems.

CONTENTS

1. Internal measurement systems and proportional representation
2. Existing results
3. A generalization: IPWE systems
4. Representation
5. Restricted uniqueness
6. Concluding remarks and empirical applications
Bibliography

Measurement systems with a combination operation $\circ$ have a representation f of the form $f(a\circ b)=F(f(a),f(b))$, for some F. *Internal*, or intensive, combinatorial systems are characterized by the fact that the combination $a\circ b$ lies always, relative to the comparison order, between a and b. In these systems, then, the value $f(a\circ b)$ is always inside the interval $[f(a),f(b)]$, and internal *proportional* systems are characterized by the fact that this value maintains always the same proportion for any two combinable objects. In measurement literature we find conditions for a general representation $f(a\circ b)=rf(a)+(1-r)f(b)$ ($r\in(0,1)$) for internal proportional systems, but these conditions include the closure of $\circ$, i.e. that any two objects are combinable. It is a standard goal of Measurement Theory the search of conditions for not necessarily closed systems. In the internal proportional case, the only result without closure assumption at hand in the literature is restricted to finite systems with a specific mean representation, i.e. with $r=1/2$. The aim of this paper is to give one kind of internal proportional systems with general representation and without closure and finiteness assumptions.

1. Internal measurement systems and proportional representation

Measurement is the representation of empirical properties by numbers. Such properties are qualitatively expressed by a comparison relation (i.e. a weak order[1]) $\succsim$ on a domain A of objects. For some properties we also have a *combination* operation $\circ$ (a function from a, may be proper, subset of AxA into A) which help us to find the representation. We call these $\langle A, \succsim, \circ\rangle$ systems, *comparison combinatorial systems*, from here on, briefly, combinatorial systems.[2] Despite some formal analogies, combinatorial systems must be carefully distinguished from interval systems and conjoint systems. For the purpose of this paper this caution is specially important in the second case. In conjoint systems the pairs (a,b) could be interpreted as some kind of "combination" of magnitudes, but two features distinguishes these systems from combinatorial ones: first, in conjoint systems usually the two members of the pairs represent *different* properties; secondly, in conjoint systems the primitive qualitative comparison relation compares *pairs* of objects.[3] We shall come back to this issue briefly in the last section.

A representation of a combinatorial system is a function $f:A\rightarrow Re$ such that (1) $a \succsim b$ iff $f(a)\geq f(b)$ and (2) $f(a\circ b)=F(f(a),f(b))$, for a certain concrete F which is the quantitative representation of the qualitative operation $\circ$. Combinatorial systems have (more or less strong) quantitative representations, with specific F, depending on certain conditions the structure satisfies. One kind of combinatorial systems, the *internal* ones, are those in which $\circ$ is *intern*, i.e. $a\circ b$ lies always between the components:

(3) Internality: for every combinable a, b, a $\succsim$ a∘b $\succsim$ b or b $\succsim$ a∘b $\succsim$ a.

This[4] internality implies *idempotency*: if a, b are combinable and a~b then a~a∘b~b; here ~ is the coincidence or indifference relation induced by $\succsim$, i.e. $\sim\ = \succsim \cap \precsim$ (because $\succsim$ is a weak order, ~ is of equivalence). (3) also implies that, if the internal system has any representation f, f assigns to the compound a∘b a number f(a∘b) between the numbers f(a) and f(b) assigned to the components, i.e. f(a∘b) equals the assignment to the smaller plus a piece of their difference. Then, it trivially follows from internality that, if an internal combinatorial system has any representation f, then

(4) for every combinable a, b, there is $r \in (0,1)$ such that
$f(a \circ b) = rf(a) + (1-r)f(b)$.

What does not follow from internality is that r is the same for every pair of objects, i.e. that a∘b lies always, for every combinable a, b, at the same distance of the extremes. We call *proportional* these internal systems in order to connote that in them the combination maintains always *the same proportion*: their representation not only satisfies (4) but also

(5) there is $r \in (0,1)$ such that for every combinable a, b:
$f(a \circ b) = rf(a) + (1-r)f(b)$.

Although measurement literature has not emphasized it explicitly, the internal combinatorial systems in which the theory of measurement has focused are the proportional ones; this kind of "constant" representation is what is searched when the theory investigates measurement conditions for internal combinatorial systems. Proportional systems, because their constance, are surely the most interesting internal systems but, since proportionality does not follow from internality, it must be emphasized that there can be, at least in principle, internal systems with non-proportional representation. Of course, the possible non-proportional systems of interest are not the ones with mere internal representation (4); the interesting possibility is the existence of non-proportional internal systems with representational properties different to (5) but also stronger than (4), i.e. *special* internal systems different to the main, proportional, ones. How really interesting these other systems are is matter of further theoretical, and empirical, research; here we wanted only to point out their possibility.

2. Existing results

Measurement literature has dealt with general representation for internal systems in the context of the study of bisymmetric structures. Up to my knowledge, the first author in doing so was Pfanzagl,[5] but using specially strong conditions. New results, with weaker conditions, are summarized in volume 1 of *Foundations of Measurement.* The authors define a bisymmetric structures in the following way.[6]

D1 $<A, \succsim, \circ>$ is a *bisymmetric system* iff

(1) A is nonempty, $\succsim$ is a weak order on A and $\circ$ is an operation from AxA into A.

(2) $\forall a,b,c\ (a \succsim b \leftrightarrow a\circ c \succsim b\circ c \leftrightarrow c\circ a \succsim c\circ b)$

(3) $\forall a,b,c,d\ (a\circ b)\circ(c\circ d) \sim (a\circ c)\circ(b\circ d)$

(4) $\forall a,b,c,d\ ((c\circ b \succsim a \succsim d\circ b \rightarrow \exists e\ e\circ b \sim a) \wedge (b\circ c \succsim a \succsim b\circ d \rightarrow \exists e\ b\circ e \sim a))$

(5) Every strictly bounded standard sequence is finite, where $S=\{a_i / a_i \in A, i \in N\}$ is a standard sequence iff exist $p,q \in A$ such that $p \succ q$ and, for every $a_i \in S$ $a_i \circ p \sim a_{i+1} \circ q$, or for every $a_i \in S$ $p \circ a_i \sim q \circ a_{i+1}$.

(2) is monotonicity, (3) bisymmetry, (4) restricted solvability and (5) archimedianity. What is proved (p. 295) is that bisymmetric systems have representations of the following form:

(6) there are $p,q \in Re^+$ and $k \in Re$ such that for every a, b
$f(a\circ b) = pf(a) + qf(b) + k$.

This is the context where we find our internal proportional systems since, as the authors show, it is an immediate result that if we add idempotency we have $p+q=1$ and $k=0$,[7] i.e. exactly our representation (5).

This general proportional representation for internal (bisymmetric) systems is as general as we were looking for, except for one thing. In D1 the operation $\circ$ has AxA as its domain, i.e. A is closed under $\circ$, for any two objects a, b in A, there always exists their combination $a\circ b$. There are many empirical reasons that militate against this requirement and, from the very beginning of the theory, it has been a goal of Measurement Theory (exemplified e. g. by large in *Foundations*) to look for representational conditions which do not require the closure of $\circ$. It seems natural then to apply this goal to our internal proportional systems and to look for conditions for general representation dropping the closure assumption. And all that we found in *Foundations* is a brief presentation of a very special not necessarily closed internal systems. These systems do not satisfy our goal for two reasons: first, they are not general, their representation is not as (5) but a specific case of proportional representation, namely, the *mean* representation; and second, they are finite. Despite this, we briefly review them because they are all we have about internal proportional systems without the closure assumption.[8] Hereafter 'D∘' denotes the domain of ∘, and '$<a,b> \in D_\circ$' means then that a, b are combinable; i.e. D∘ is the nonempty subset (may be proper, may be not) of AxA which defines combinability.

D2 $<A, \succsim, \circ>$ is a *finite halfing system* iff

(1) A is finite, $\succsim$ is a weak order on A and ∘ is a function from a nonempty subset of AxA into A.

(2) $\forall a,b,c\ (<a,c> \in D_\circ \wedge <b,c> \in D_\circ \rightarrow (a \succsim b \leftrightarrow a \circ c \succsim b \circ c)$.

(3) $\forall a\ (<a,a> \in D_\circ \wedge a \circ a \sim a)$.

(4) $\forall a,b\ (<a,b> \in D_\circ \rightarrow <b,a> \in D_\circ \wedge a \circ b \sim b \circ a)$.

(5) $\forall a,b,c,d\ (<b,c> \in D_\circ \wedge aJb \wedge cJd \rightarrow <a,d> \in D_\circ \wedge a \circ d \sim b \circ c)$.

(Here 'aJb' means "$a \succ b$ and there is no c such that $a \succ c \succ b$".)

Condition (2) is monotonicity, (3) is a version of idempotency and (4) is commutativity. The critical condition is (5). The idea is the following. In the case of a finite domain, the linear order generated by the weak order at the level of equivalence classes is also a well order. This means that, for every object in the domain, we can talk of his "immediate followers", those who are exactly "one step" away from him (one step in the sense that they are "the next"). And that is what 'aJb' means, that a is away from b just one step. Then, if the system is one of halfing we should find this situation:

```
              b∘c
-a----b----x----x----x----c----d--
              a∘d
```

That is, since the combination is "the mean", if you take two pairs of consecutive objects, the combination of the external ones is equivalent to the combination of the internal ones (if such combinations exist). In another more general way: the combination of any two objects agrees with the combination of any two other which are equidistant (by different sides) from the first two. This has to happen if the system is one of halfing, and this is what (5) says. In this way we express the "mean combination": the combination of a, b (if it exists) lies in the position where we meet at each other when for each step from a to b we make just another one from b to a. Now, if systems behave qualitatively in this way, it can be easily proved that they have a mean representation (unique up to positive linear transformation), i.e. a representation $f:A \rightarrow Re$ such that (1) and

(7) for every a, b: if $\langle a,b\rangle \in D_\circ$ then $f(a \circ b) = [f(a)+f(b)]/2$.

So far the known internal proportional systems. The former have a general representation, but they include the closure assumption; the later drop these assumption, but include the finiteness one and do not have a general representation. We are going to see now *one* kind of systems which satisfy the desired requirements: general representation with no closure, neither finiteness, assumption.

3. A generalization: IPWE systems

First, it is worth insisting in that, roughly speaking, we are not looking for a *concrete* representation. We want conditions that, together with internality, guarantee the existence of a general internal proportional representation for $\langle A, \succsim, \circ\rangle$, i.e. the existence of a function $f:A \rightarrow Re$ such that (1) and (5). Because in (5) r, though not free, is indeterminate, the representation theorem has in this case an additional degree of indetermination.[9] Roughly, then, what we shall prove is that systems with suitable conditions have a representation *of a certain kind.* This general representation can be specified into a "unique" one (up to admissible transformations, of course) imposing additional conditions; for instance, adding commutativity we can prove the existence of a concrete mean representation.

We obtain the conditions for this general representation in the not necessarily closed case generalizing in a certain way those given for the special mean representation. How to do it? A brief reflection shows that, first of all, we must drop (4), for commutativity, with the other conditions, forces halfing, i.e. mean representation. We must maintain (2), since monotonicity is necessary for proportionality. (3) will be not necessary if, as we do, we introduce, for the above-mentioned reasons, internality (remember that idempotency follows from internality). What to do with (1) and (5)?

First (1). The requirement at issue is finiteness, we of course must maintain that $\succsim$ is a weak order on A and that $\circ$ is a not necessarily close binary operation on A. What about finiteness? It is easy to see that this condition is not necessary for the kind of representation here involved. $\succsim$ is a weak order on A, then $\sim$ ($\sim = \succcurlyeq \cap \preccurlyeq$) is an equivalence relation, $[A]/\sim$ is the quotient set under $\sim$ and $\succcurlyeq$ induces an order relation $[\succcurlyeq]$ at the level of equivalence classes which is a linear order on $[A]/\sim$. What is necessary for the kind of representation involved is that the induced linear order $[\succcurlyeq]$ is also a well order, i.e. that we can go through equivalence classes step by step. Since every finite linear order is a well order, finiteness of A will suffice, but it is not necessary. Then we remove finiteness and we simply impose this weaker condition on the induced relation $[\succcurlyeq]$.[10]

Now (5). The idea for its generalization is the following. In halfing systems we have r=1/2 because, there, the combination of a, b (if it exists) is in the position where we meet at each other when for each step we take from a to b we take just another one from b to a. But this is only one possibility. There are many others. For instance, we could do, for each step from a to b, two steps from b to a. In this case r would equal 2/3. Or we could do two steps from a to b and three from b to a, being then r=3/5. And so on. Different possibilities give rise to different values for r. And each possibility corresponds to a case in which we take n steps from a to b for each m steps from b to a. Then, if we want to generalize D1 in order to obtain a general representation, we must generalize D1(5) leaving these n and m indeterminate. In order to express such generalization we define first a general *distanc*[11] relation J^n (for any integer n):

D3 Let $\succcurlyeq$ be a relation on A, where a, b belong.
(1) aJb iff $a \succ b$ and $\neg\exists c\ a \succ c \succ b$
(2) $\forall n \in N$: aJ^nb iff (n=0 and $a \sim b$) or ($n \geq 1$ and $\exists c(aJc$ and $cJ^{n-1}b$)).
(3) $\forall n \in Z$: aJ^nb iff ($n \in N$ and aJ^nb) or ($n<0$ and $bJ^{-n}a$).

The next lemma, which we present without proof and without additional comments, establishes some properties (which we shall use later) of this relation J^n when the lineal order relation $[\succcurlyeq]$ induced by a weak order $\succcurlyeq$ is also a well order.

L1 Let $\succcurlyeq$ be a weak order on A, $\sim = \succcurlyeq \cap \preccurlyeq$, $[A]/\sim$ the quotient set under $\sim$ and $[\succcurlyeq]$ the relation on $[A]/\sim$ such that $[x][\succcurlyeq][y]$ iff $x \succcurlyeq y$. If $[\succcurlyeq]$ well orders $[A]/\sim$ then, for every a, b in A:
(1) $a \succcurlyeq b \leftrightarrow \exists n \in Z\ (n \geq 0 \wedge aJ^nb)$
(2) $\exists 1\ n \in Z\ aJ^nb$
(3) $\forall n \in Z\ (aJ^nb \leftrightarrow bJ^{-n}a)$
(4) $\forall n,m \in Z\ (aJ^{n+m}b \leftrightarrow \exists c \in A\ aJ^ncJ^mb)$
(5) $\forall c \in A\ \forall n \in Z\ (aJ^nb \wedge b \sim c \rightarrow aJ^nc)$.

We can see now how to generalize D1(5) in order to express the idea that, in a general internal proportional system, the combination of two objects (if it exists) is in the point where we meet at each other when we take n steps from one to the other for each m steps from the second to the first (leaving n, m indeterminate). If this is so, then the combination of two objects b, c is equivalent to the combination of another two a, d when a is n steps away from b and d is m steps away from c, or a multiple of such numbers:

(*) be $k \in Z$, if $aJ^{kn}b$, $cJ^{km}d$ and $<a,d> \in D_{\circ}$, then $<b,c> \in D_{\circ}$ and $a \circ d \sim b \circ c$.

Actually it is possible to obtain the desired result with a slightly weaker condition from which, together with a very weak definibility condition for $\circ$ (already included in (*)), we obtain (*).

In order to obtain the desired representation another condition, which ensures that the combinability agrees with expected results, is still necessary:

($** \rightarrow$) if two objects are combinable, then their distance is k(n+m), $k \in Z$.

Without this assumption we cannot expect to find the results we are searching. Supose that two combinable objects a, b are not separated k(n+m) steps, for instance, in the most dramatic case, supose that they are consecutive, aJb, then "there is no place" for their combination $a \circ b$ to exist; the same problem would happen in less apparent, but equally fatal, cases: if combinable objects are not k(n+m) far away from each other, we cannot expect, as we want, to find their combination doing m steps from b to a for each n steps we do from a to b. It is worth noting that this condition ensures that the combination of two objects, *when it exists*, works as we expect, but it is not a definibility condition for $\circ$ ensuring that certain combinations exist. This would be the case if we take the biconditional instead of the conditional form, i.e. ading:

($** \leftarrow$) every pair of objects separated k(n+m) steps, $k \in Z$, are combinable.

Note that this condition is really strong. It ammounts to the "completeness" of the combination $\circ$ in the following sense: k(n+m)-

distance is a reasonable necessary condition for combinability, if we require it to be also a sufficient condition then the combination would be, though not necessarily closed, "complete", as rich as possible; in short, every possible combination would always exist. Because this condition is not necessary for the representation, and because (though not forcing closure of $\circ$) it goes against the spirit of our generalization, we do not include it in the definition of the systems.

A last comment before the definition. We have two options: we can give a scheme definition relativizing it to the free variables n, m ("$<A, \succsim, \circ>$ is a finite n-m internal general system iff ..."), or a proper definition without relativization bounding n, m with existential quantifier. Although it is a bit more complicated notationly, we follow the second option for its formal congruency with another results in the literature. I call these systems WE-systems in order to connote that they are such that the induced relation $[\succsim]$ well orders equivalence classes ((2) and (3) are, respectively, non-closed monotonicity and internality, and (4) is a reasonable combinability condition).

D4 $<A, \succsim, \circ>$ is an *internal proportional WE-system* (IPWE) iff
- (1) (1.1) A is nonempty, $\succsim$ is a weak order on A and $\circ$ is a function from a nonempty subset of AxA into A.
  - (1.2) The relation $[\succsim]$ on $[A]/\sim$ such that $[x][\succsim][y]$ iff $x \succsim y$, well orders $[A]/\sim$.
- (2) $\forall a,b,c\ (<a,c> \in D_\circ \wedge <b,c> \in D_\circ \rightarrow (a \succsim b \leftrightarrow a \circ c \succsim b \circ c))$.
- (3) $\forall a,b\ (<a,b> \in D_\circ \rightarrow a \succsim a \circ b \succsim b \vee b \succsim a \circ b \succsim a)$.
- (4) $\forall a,b,c,d\ (<a,b> \in D_\circ \wedge a \sim c \wedge b \sim d \rightarrow <c,d> \in D_\circ)$.
- (5) There are non zero natural numbers n, m such that:
  - (5.1) $\forall a,b,c,d\ (aJ^{n}b \wedge cJ^{m}d \wedge (<a,d> \in D_\circ \vee <b,c> \in D_\circ) \rightarrow <a,d> \in D_\circ \wedge <b,c> \in D_\circ \wedge a \circ d \sim b \circ c)$.
  - (5.2) $\forall a,b\ (<a,b> \in D_\circ \rightarrow \exists k \in Z\ aJ^{k(n+m)}b)$.

4. Representation

Before proving the representation theorem we need some previous results. The first is that D4 implies (*):

T1 If $<A, \succsim, \circ>$ is IPWE, there are non zero natural numbers n, m such that $\forall a,b,c,d\ \forall k \in Z\ (aJ^{kn}b \wedge cJ^{km}d \wedge (<a,d> \in D_\circ \vee <b,c> \in D_\circ) \rightarrow <a,d> \in D_\circ \wedge <b,c> \in D_\circ \wedge a \circ d \sim b \circ c)$.

Proof. Be n, m non zero naturals of the kind guaranteed by D3(5). We prove the theorem first for k=0, then by induction for k≥1, and finally for negative integers. (1) Be k=0. aJ^0b and cJ^0d means (see D2) $a\sim b$ and $c\sim d$. If $\langle a,d\rangle\in D_\circ$, by D4(4) we have $\langle b,c\rangle\in D_\circ$, $\langle a,c\rangle\in D_\circ$ and $\langle b,d\rangle\in D_\circ$. By monotonicity, $a\sim b$, and $c\sim d$ we get $a\circ d\sim b\circ d\sim b\circ c$. Analogously if $\langle b,c\rangle\in D_\circ$. (2) Induction on positive naturals. For k=1 is direct from D4(5.1). Let be true for k, we see it is true for k+1. Suppose $aJ^{(k+1)n}b$, $cJ^{(k+1)m}d$ and $\langle a,d\rangle\in D_\circ$. By L1(4) there are e, h such that $aJ^{kn}eJ^nb$ and $cJ^mhJ^{km}d$. From $aJ^{kn}e$, $hJ^{km}d$, $\langle a,d\rangle\in D_\circ$ and inductive step k=k we get $\langle e,h\rangle\in D_\circ$ and $a\circ d\sim e\circ h$. Now, from eJ^nb, cJ^mh, $\langle e,h\rangle\in D_\circ$ and inductive step k=1 we get $\langle b,c\rangle\in D_\circ$ and $e\circ h\sim b\circ c$. Hence $\langle b,c\rangle\in D_\circ$ and $a\circ d\sim b\circ c$. Analogous if $\langle b,c\rangle\in D_\circ$. (3) Be k negative integer. Suppose $aJ^{kn}b$, $cJ^{km}d$ and $\langle a,d\rangle\in D_\circ$. By L1(3), $bJ^{-kn}a$, $dJ^{-km}c$, with -k positive natural. By step (2) of this proof, $\langle b,c\rangle\in D_\circ$ and $b\circ c\sim a\circ d$. Analogously if $\langle c,b\rangle\in D_\circ$.

The next theorem establishes, with a previous intermediate result, the fundamental fact for the representation: $a\circ b$, if it exists, is kn steps away from a and km steps away from b (for some integer k).

T2 If $\langle A,\succsim,\circ\rangle$ is IPWE, there are non zero natural numbers n, m such that

(1) $\forall a,b,c\ \forall k\in Z\ (aJ^{kn}cJ^{km}b \wedge \langle a,b\rangle\in D_\circ \rightarrow a\circ b\sim c)$.

(2) $\forall a,b\ (\langle a,b\rangle\in D_\circ \rightarrow \exists k\in Z\ aJ^{kn}a\circ bJ^{km}b)$.

Proof. Be n, m non zero naturals of the kind guaranteed by T1. (1) Suppose $aJ^{kn}cJ^{km}b$ and $\langle a,b\rangle\in D_\circ$. By T1 $\langle c,c\rangle\in D_\circ$ and $a\circ b\sim c\circ c$; from internality we get $c\circ c\sim c$; hence $a\circ b\sim c$. (2) If $\langle a,b\rangle\in D_\circ$, by D4(5.2) there is $k\in Z$ such that $aJ^{k(n+m)}b$ and, by L1(4), there is c such that $aJ^{kn}cJ^{km}b$. We have just seen that then $a\circ b\sim c$, and by L1(5), we obtain $aJ^{kn}a\circ bJ^{km}b$.

We can see now the representation theorem. The suitable $r\in(0,1)$ whose existence ensures the general representation will be, of course, the quotient m/(n+m) for the n, m whose existence ensures D4 via T2(2). The strong version of the representation theorem ensures the existence of a not only real-valued but *integer*-valued function; this is the version we are going to prove, but it has strong limitations on uniqueness results that we shall comment later.

T3 If $\langle A,\succsim,\circ\rangle$ is IPWE, there are non zero natural numbers n, m such that

There is a function f:A→Z such that for every a, b of A:
(1) a ≿ b ↔ f(a)≥f(b)
(2) <a,b>∈D∘ → f(a∘b) = [m/(n+m)]f(a) + [n/(n+m)]f(b).

Proof. Be n, m non zero naturals of the kind guaranteed by T2. Take any object e of A and assign to it any integer u∈Z. By L1(2), for every member of A there is one and only one δ∈Z such that $aJ^{\delta}e$; for every a, we write 'δ(a,e)' for "its" δ (the distance between a and e). Now we define a function f from A into Z as follows: for every a∈A f(a) $=_{def.}$ u + δ(a,e). From this definition we obtain that for every a, $aJ^{f(a)-u}e$, wich, together with L1(3-4), implies (#) for every a, b, $aJ^{f(a)-f(b)}b$ (for $aJ^{f(a)-u}e$ and $eJ^{-(f(b)-u)}b$).
We shall see that f satisfies (1.1) and (1.2).
(1) (→) Be a ≿ b. By L1(1-2), there is one and only one δ∈Z such that δ≥0 and $aJ^{\delta}b$. Then, by (#) we get δ=f(a)-f(b), hence f(a)-f(b)≥0 and f(a)≥f(b).
(1) (←) Be f(a)≥f(b). Then f(a)-f(b)≥0 and, since $aJ^{f(a)-f(b)}b$, there is integer δ≥0 such that $aJ^{\delta}b$; hence, by L1(1), a ≿ b.
(2) Be <a,b>∈D∘. First, by T2(2), there is k∈Z such that $aJ^{kn}a{\circ}bJ^{km}b$. Second, by (#), $aJ^{f(a)-f(a\circ b)}a{\circ}b$ and $a{\circ}bJ^{f(a\circ b)-f(b)}b$. Then, with L1(2), kn=f(a)-f(a∘b) and km=f(a∘b)-f(b), hence mf(a)-mf(a∘b)=nf(a∘b)-nf(b), i.e. f(a∘b) = [m/(n+m)]f(a) + [n/(n+m)]f(b).

Note that this representation is not "proportional" in the sense that f(a∘b) is f of the smaller plus an always-the-same-portion of |f(a)-f(b)|. This would be commutative, and ours not necessarily. With a simple numerical example: be n=2 and m=3, A=Z and f(x)=x; then 17∘2 = 11 = 2 + (3/5)|17-2|, 23∘13 = 19 = 13 + (3/5)|23-13|, but 2∘17 = 8 = 2 + (2/5)|2-17| (though, of course, 17∘2 = (3/5)17 + (2/5)2 and 2∘17 = (3/5)2 + (2/5)17). This representation is proportional in the above-mentioned sense: there exist r∈(0,1) (r=m/(n+m)) such that for every combinable a, b, f(a∘b) = rf(a) + (1-r)f(b), i.e. f(a∘b) = f(b) + r(f(a)-f(b)), the assignment to the second plus an always-the-same-portion of the difference between the first and the second. We have then the desired general proportional representation (5) for one kind of not necessarily closed systems. As we pointed out, mean representation for halfing systems appears as a specialization of our IPWE systems when we add commutativity to them:

T4 Be <A, ≿ ,∘> a IPWE which satisfies ∀a,b (<a,b>∈D∘ → <b,a>∈D∘ ∧ a∘b~b∘a).
Then there is a function f:A→Z such that for every a, b,
(1) a ≿ b ↔ f(a)≥f(b) and (2) <a,b>∈D∘ → f(a∘b) = [f(a)+f(b)]/2.

Proof. It suffices to prove (2). Case 1: for every combinable a, b, a~b. Then $f(a) = f(a \circ b) = f(b) = [f(b)+f(b)]/2 = [f(a)+f(b)]/2$. Case 2 (the interesting one): there are a, b combinable and no equivalent. We'll see that then n=m. Be a, b two such objects, by T3(1.1) $f(a) \neq f(b)$. From commutativity and T3(1.2) we have $mf(a)+nf(b) = nf(a)+mf(b)$, i.e. $(m-n)f(a) = (m-n)f(b)$. Since $f(a) \neq f(b)$, the only possibility is n=m.

5. *Restricted uniqueness*[12]

The integer-valued version of the representation theorem has fatal consequences for uniqueness. At a first sight, any positive linear transformation $\alpha x + \beta$ ($\alpha \in Re^+$, $\beta \in Re$) is an admissible transformation, i.e. preserves T3 (1) and (2). (1) is immediate, and (2) is also easy: suppose for every $a \in A$ $g(a) = \alpha f(a) + \beta$, then $g(a \circ b) = [m/(n+m)]g(a) + [n/(n+m)]g(b) = [m/(n+m)][\alpha f(a)+\beta] + [n/(n+m)][\alpha f(b)+\beta] = \alpha f(a)m/(n+m) + \alpha f(b)m/(n+m) + \beta m/(n+m) + \beta n/(n+m) = \alpha f(a \circ b) + \beta$. But if f, g must be integer- valued, positive linear transformations do not work for every $\alpha \in Re^+$, $\beta \in Re$. Neither they work with the restriction $\alpha \in N$, $\beta \in Z$, since the inverse of any admissible transformations must be also admissible, which does not happen with this restriction. So there is no alternative for uniqueness if representation must be integer-valued. If we weaken the representation, the stronger possibility which solves this problem is to substitute integer-valued functions by rational-valued ones: if in T3 the function is rational-valued, then every positive linear transformation with rational coefficients (hereafter, Q-transformation) is admissible, gives raise to another representation of the same kind.[13] But this is only half of the story, for uniqueness must say, not only which transformations of a given representation are also representations, but also which is the relation between any two representations. And at this point our IPWE systems have a strong limitation, since positive linear Q-transformations are all admissible but they are not the only ones. When n+m>2 there can be admissible transformations which are not Q-transformations, therefore the representation may have more, *additional*, degrees of indetermination or freedom than usual. The following examples illustrate the situation.

Be a IPWE system with five objects in its domain, the four single objects a, b, c, d, and the compound $a \circ d$ (*without* the converse compound $d \circ a$). Let the intended case be such that n=1 and m=2, and be the order as follows (because all the objects of the domain appear we would describe the situation using '$\succ$' instead of 'J'):

E1 (n=1, m=2) aJb~a∘dJcJd.

Be f a representation such that, e.g., f(a)=2, f(b)=f(a∘d)=(3), f(c)=4 and f(d)=5. The function g such that g(a)=2, f(b)=f(a∘d)=(4), f(c)=6 and f(d)=8 is admissible and it is a positive linear Q-transformation of f, but a function h which coincides with g in the values assigned to the objects different to c and which assigns to c any rational number different to 6 in the interval (4,8), this function is also admissible, a proper representation, but it is not a positive linear Q-transformation of f. Then in this example representations have three degrees of freedom, one more than usual. It is easy to see that this does not happen when n=m=1. This situation may suggest that the representations of IPWE systems have at least n+m degrees of freedom. Because uniqueness up to positive linear transformations implies two degrees of freedom, what is at issue here is, as we announced, that IPWE systems seem to have at least n+m-2 *additional* degrees of freedom or indetermination. But this is not always the case, as the next example shows.

E2 (n=1, m=2) aJb~a∘dJc~d∘aJd.

Given aJbJcJd and that there are no more single objects, it follows from D4(5.2) that only a and d can be combinable. In E1 the system contains a∘d but not d∘a, now we build E2 adding d∘a~c to the former system. In this system representations have only two degrees of freedom, they are unique up to positive linear Q-transformations, then they have *no additional* degrees of freedom. Note that this system satisfies not only D4(5.2) but also (**←),[14] it is ∘-complete in the above-mentioned sense.

Additional degrees of freedom are typically[15] caused by *disconnected objects*, where a disconnected object is an object which is (i) not equivalent to a compound object (this implies that it is a single object) and (ii) not equivalent to an object which is a component of a compound object. In E1 we have one disconnected object, c, in E2 we have none. This does not mean that additional degrees of freedom depend on the *number* of disconnected objects. If in E1 we add a new object e such that e~c, this new disconnected object does not increase the degrees of freedom; but if the new object e is, e.g., such that dJe, now the indeterminacy do increase. Therefore, additional degrees of freedom does not depend on the number of disconnected objects but on the number of *disconnected equivalence classes*, where a disconnected equivalence class is an equivalence class (under the equivalence relation ~) made up of disconnected objects. Because ∘ is non empty, there are at least n+m+1, say K, different equivalence classes and at least three of them are not disconnected. Let us denote by 'D' the number of disconnected equivalence classes; D is smaller than K-3 (K≥n+m+1) and obviously depends on the combination ∘: the richer the combination is, the lower the number of disconnected equivalence classes is. Therefore the satisfaction of (**←), or ∘-completeness, reduces additional degrees of freedom to the minimum. In E2 this minimum is 0,[16] but this is not always the case, ∘-completeness does not eliminate in general additional indetermination, as E3 shows:

E3 (n=2, m=3) aJbJc~a∘fJd~f∘aJeJf, D=2.

E2 may suggest that when n=1 or m=1 then ∘-completeness implies D=0, but it doesn't:

E4 (n=1, m=3) aJb~a∘eJcJd~e∘aJe, D=1.
E5 (n=1, m=4) aJb~a∘fJcJdJe~f∘aJf, D=2.

E2 and E3 also show that (even under ∘-completeness) D does not depend on the difference n-m. E3 and E5 may suggest that D depends on the sum n+m, but E4 and E6 show it does not:

E6 (n=2, m=2) aJbJc~a∘e~e∘aJdJe, D=2.

What is worst, even under ∘-completeness n and m do not determine D, which depends not only on n, m but also on the number of equivalence classes, as E4 and E7 show:

E7 (n=1, m=3) aJb~a∘eJc~b∘fJd~e∘aJe~f∘b~f, D=0.

This is the situation concerning disconnected equivalence classes generating additional degrees of freedom or indetermination. This is a negative result impliying a considerable additional indeterminacy of uniqueness. One possibility of a positive result is to confine ourselves in the uniqueness part to non-disconnected, say connected, objects: the representation seems to be unique up to positive linear Q-transformations as far as connected objects are concerned. But in order to do so we need to include some additional condition in D4, for as it stands we can not rule out the possibility of "strange" systems which do not have the desired uniqueness even restricted to connected objects. These systems look like the mere juxtaposition (or superposition) of two isolated subsystems. Take the following (non-∘-complete), case:

E8 (n=1, m=2) aJb~a∘dJcJdJeJf~e∘hJgJh.

Here the connected objects are a, b, d, e, f and h, but it is easy to see that the representation has not the expected uniqueness even restricted to these objects; the reason is that the first three are connected, so to say, "from one side", and the last three "from other side"; we have two *dislinked*[17] subgroups of connected objects. It is worth noting that these dislinked subgroups that block restricted uniqueness are not necessarily "consecutive", as in E8. Connected objects of dislinked subgroups may appear intercalated in the sequence of equivalence classes, as happens with the subgroups {b,d,g} and {a,c,f} in E9 (e is disconnected):

E9 (n=2, m=3) aJbJc~a∘fJd~b∘gJeJfJg.

In order to avoid this situation an additional property should be included in D4. It is easy to see that this property cannot be ∘-completeness: it is not sufficient, neither it is necessary, for the non existence of dislinked groups of connected objects, as respectively E10 and E11 show.

E10 (n=2, m=2) aJbJc~a∘e~e∘aJd~b∘f~f∘bJeJf,
E11 (n=1, m=2) aJb~a∘dJcJdJe~d∘gJf~e∘hJgJh.

In E11 there are no dislinked subgroups of connected objects (c is disconnected) and ∘-completeness is not satisfied. In E10 ∘-completeness is satisfied but there are dislinked subgroups, {a,c,e} and {b,d,f}, of connected objects.

E10 also illustrates an important fact which motivated a qualification above. We said that additional degrees of freedom are *typically* caused by the existence of disconnected objects and now we know the reason of such qualification: E10 contains no disconnected objects and its representation, *even restricted to connected objects* (in this case the very universe of the system), is not unique up to positive linear Q-transformations because the existence of dislinked subgroups of connected objects. The same happens in the other cases with n=m>1, and not only in these commutative cases, this fact happens in general when n and m have a common factor. Let us call *simplifiable systems* the systems in which n and m have a common factor; let j be the greatest common factor of a simplifiable system; and let us say that two objects are *congruent* when they are separated a multiple of j, say kj ($k \in Z$), steps; from this follows that there are at most j non-congruent compound objects (and that ∘-completeness is sufficient, but not necessary, for the number of non-congruent compound objects be j). The general fact is this: if a simplifiable system has q ($1 \leq q \leq j$) non-congruent compound objects then it has at least q dislinked subgroups of connected objects.[18]

The situation concerning uniqueness is then as follows. Additional degrees of freedom are typically caused by the presence of disconnected objects. Additional degrees of freedom caused in this way are not eliminable, the only thing we can do is to restrict uniqueness to connected objects, what does not seem too drastic. Call *restricted 2-uniqueness* the fact that the representation is unique up to positive linear Q-transformations when restricted to connected objects. Restricted 2-uniqueness is not guarantied because the possible existence of dislinked subgroups of connected objects. Then, the property which is necessary for this restricted uniqueness is the *non-existence of dislinked subgroups of connected objects.* We could include this condition in D4, which is not necessary for the mere existence of the representation, in order to obtain a minimum uniqueness result, but we have seen that simplifiable systems with at least two non-congruent compound objects do not satisfy this condition. Then, the inclusion of this condition would imply the exclusion of such systems from the family of IPWE systems. We must choose among restricted uniqueness and simplifiable systems with two or more non-congruent compound objects. As far as the form of the representation is concerned, the exclusion of such systems has no consequences: if S is a simplifiable system with n, m, and j is the greatest common factor, then a system S' with n'(=n/j) and m'(=m/j) is not simplifiable and has the same intern representation than S. Therefore, drooping simplifiable systems (with at least two non-congruent compound objects) we do not lose any specific concrete form of intern representation. But we do lose something. In a nutshell, we lose some systems.

E12 (n=1, m=2) aJc~a∘gJeJg,
E13 (n=2, m=4) aJbJc~a∘gJd~b∘hJeJfJgJh.

It is true that the number $r \in (0,1)$ which characterizes the representation of E13 is the same than in E12, but they simply are different systems and E13 is not reducible to E12. Then if we exclude, via the inclusion of the new critical property in D4, systems like E13 we do lose something. One possibility is to reduce simplifiable systems to a *proper* combination or *superposition* of several non-simplifiable systems, one for each dislinked subgroup of connected objects, e.g. to see E13 as a proper superposition of E12 and E14:

E14 (n=1, m=2) bJd~b∘hJfJh.

The superposition must be proper in the sense that it must preserve the order of E13. This way of regarding simplifiable systems makes justice to the fact that they really are like superpositions of isolated non-simplified systems and this characteristic explains why these systems simply do not have minimum uniqueness. If we don't want to miss them, we may keep them as IPWE systems but then we must, so to say, restrict restricted uniqueness even more; this is simply the way the things are, in the very nature of simplifiable systems there is the impossibility of being represented with minimum uniqueness.

To summarize. Even if, because the possible existence of disconnected objects, a general uniqueness result is not possible for IPWE systems, it is possible under certain conditions a weaker uniqueness result restricted to connected objects. I do not think that this restriction implies a fatal handicap for IPWE systems since disconnected objects are somehow "quantitatively irrelevant". In order to obtain restricted uniqueness, we should include in D4 (together with the substitution of integer-valued functions by rational-valued ones), an additional property necessary, not for representation but for this minimum uniqueness. This new condition, the non-existence of dislinked groups of connected objects, implies the exclusion of simplifiable systems from the IPWE family. If this move is judged as unacceptably arbitrary, then we can maintain these anomalous systems in the family at the cost of a new limitation of uniqueness results: uniqueness up to positive linear Q-transformations is restricted to connected objects *in* non-simplifiable systems satisfying the critical condition. But the difference between these two options is probably only nominal, they are two ways of describing the very same facts.

6. *Concluding remarks and empirical applications*

I said that IPWE are *one* kind of internal systems that satisfy the desired requirements: general proportional representation and no closure, neither finiteness, assumptions. I want to emphasize it because IPWE are not maximally general. They drop closure and finiteness assumptions but they introduce another, quite strong, assumption, namely, that $\succsim$ induces a well order on equivalence classes (I shall call this property 'WE-ordination'). Then IPWE represent *some* progress, but not all progress we wanted, they satisfy our goal only partially, because WE-ordination assumption and because the limits to the uniqueness of their representation. It is true that internal bisymmetric systems (D1 plus idempotency) have also, besides closure of $\circ$, strong assumptions, specially restricted solvability and Archimedianity. But in the context of Measurement Theory they are natural structural conditions, in any case weaker than our WE-ordination.

I do not think that the above-discussed limitations of uniqueness imply a fatal objection to IPWE systems. If IPWE systems were of interest, then when they do not contain dislinked subgroups of connected objects they do have sufficient minimum restricted uniqueness, and it is a matter of fact which IPWE systems satisfy the critical condition. On the other hand, it is for me an open question the extend to which WE-ordination assumption undermines the interest of IPWE systems. From a purely mathematical point of view, of course they are not an impressive result. And from an empirical perspective, their possible empirical applications should be investigated. In *Foundations* 1 (pp. 294-295) the authors mention the psychophysical method of bisection as a possible example of an empirical bisymetric operation[19] and point out the fact that empirical data make clear that this empirical operation is not commutative in general, then r does not equal 1/2 and we have a nice candidate for a non-mean representation. At a first sight, any empirical application of bisymmetric (idempotent) systems of D1 is a candidate for IPWE, then psychophysical bisection could also be an empirical application of IPWE systems. The crucial point is what happens with D4(1.2), i.e. whether there are empirical bisection systems in which the empirical comparison relation is a WE-order. I think there is no *a priori* answer to this question and, therefore, it must be answered only after empirical investigation. On the other hand, it should also be investigated if empirical bisection is in fact proportional, i.e. if bisection maintains the proportion for any two bisectionable pairs constant. If the answer were negative, then these empirical systems would not be a case of IPWE, *neither* of bisymmetric systems of D1 for they are also proportional. In this case, bisection would be a good motivation for the study of non-proportional internal systems whose possibility we pointed out in the first section.

I conclude with a brief comment on two other possible empirical applications, utility and gambles.

Utility representation in expected utility analysis is of the form

(8) $U(a \cup b) = rU(a) + (1-r)U(b), r \in (0,1).$

It is not clear whether this result can come from an application of internal *combinatorial* system. This utility representation is a special case of a more general one including subjective probabilities as well as expected utilities. Be A, B two (non null) events and a, b two of their respective consequences. Then, under some conditions, we have $U(a_A \cup b_B) = P(A)U(a) + P(B)U(b)$, where P is the subjective probability and U the expected utility. In some special cases B equals -A and then we have (8). But if (8) is obtained in such context, it is hard to consider it as an application of combinatorial system. What we have here is a case of *conjoint* representation of expected utility *and* subjective probability, and actually this is the way in which these cases are treated in Measurement Theory.[20] Because in conjoint measurement the primitive comparison relation compares the conjoint effect of *both* magnitudes, if we arrive at (8) in this manner I think that it can not be regarded as a representation of internal *combinatorial* system. In order to see it in this way we should obtain (8) as the representation of an empirical system where the primitive comparison relation compares utilities *only*. I don't know any result which establishes this fact. It is possible to represent utilities comparing utilities only, but by the procedure comparing utility intervals,[21] and this procedure does not give rise a representation of the form (8). Then, in the absence of news results I think that representation (8) can not be regarded as an empirical application of internal proportional combinatorial systems.

In *Foundations* 3, ch. 19 (pp. 29 ff.), the authors comment another intern empirical operation, namely, the combination of gambles. Under one approach to gambles, this operation is related to a situation similar to the previous utility case (cf. pp. 29-30 and ch. 20.4.5). Under another approach (pp. 33-31) the combination is relativized to a probability p, and the intended interpretation of $a \circ_p b$ is that it is the gamble "in which gamble a is played with probability p and otherwise b is played" (p. 30). At least in this presentation, they explicitly say that this combination of gambles is a closed operation (p. 30), assumption which undermines the interest of IPWE systems for such empirical operation.[22] It should be studied if there are empirical situations in which it is reasonable to think that combination of gambles is not (necessarily) closed. But again, like in the first, psychophysical bisection, case the crucial point is whether or not this empirical systems are WE-ordered. We conclude leaving these issues open.

Notes

† This work has been made with the support of the DGICYT research projects PB95-0125-CO6-05 and PB96-1091-C03-03, Spanish Ministry of Education. I owe to two anonymous referees a number of corrections and useful sugestions on a previous version of this paper.

1 In some measurement systems the comparison relation is not even a weak order. The reason is the possible failure of transitivity of the coincidence or indifference relation ~ induced by the comparison relation. In these cases an usual alternative is the notion of semiorder (this notion is introduced for the first time in the context of measurement theory in Luce 1956, and later simplified in Scott-Suppes 1958). Here I am not going to deal with this "non-classical" approach.

2 With this label I want to refer to the most general kind of measurement structures with operation. For similar purposes, other authors use 'concatenation structures', for instance, *Foundations* vol. 3 p. 26 and (Narens 1985, p. 73). Besides the fact that those approaches are not totally general, I prefer 'combinatorial' for in many cases the physical operation is some kind of combination which can not be described as "concatenation" (for instance, the combination of densities by mixing the liquids); but of course this is only a terminological point, for the substantive one, an appropriate characterization of general combinatorial systems, as well as for a general typology of them, see (Moulines-Díez 1994).

3 This primitive relation, when certain additional facts happen, *induces* two other comparison relations, one for each component.

4 (3) is the non-strict version of *Foundations* 3 p. 26; see also (Pfanzagl 1968, p. 80).

5 See (Pfanzagl 1968, ch. 5 p. 86 and ch. 7, p. 122).

6 P. 294; I have made small notational changes and, for coherence with the rest of the text, I use 'system' instead of 'structure'.

7 P. 295. The authors say that idempotency is true of *intensive quantities*; remember that idempotency follows from our internality, which, in my opinion, not only is true but *characterize* such quantities, i.e. characterize internal systems, systems that qualitatively express intensive properties. On the other hand, the idempotency they use is "for all $a \in A$ $a \circ a \sim a$", slightly weaker than ours above: "for every a, b combinable, if $a \sim b$ then $a \sim a \circ b \sim b$". The former is weaker only if we left aside the closure of the operation; if, as in D1, A is closed under $\circ$ then the former (plus monotonicity) implies the second.

8 Cf. ibd. p. 297 for the definition (the name 'halfing' is mine); I have made again small notational changes.

9 The *usual* degree of indetermination comes from alternative representations under the kind of admissible transformation that the uniqueness theorem establishes.

10 This condition interacts with the not necessary closure of $\circ$ in the following way: it does not preclude the closure of $\circ$, i.e. it does not make $\circ$ necessarily non close, but under this new condition closed systems will not be very interesting since now closed systems are such that $f(a)=f(b)$ for every a, b.

11 In geometry the concept of *distance* (or *metric*) has an established different meanning. In this paper the word is used with this defined new meanning; I use the term 'distance'

because it connotes very well the intended idea of "number of steps that separate two objects (in a well order)".

12 Here I shall present some results only in an informal way.

13 Other, weaker, possibility is of course the representation being real-valued; this is the alternative choice made in *Foundations* for finite halfing systems here presented in D2 (cf. vol. 1 p. 297 theor. 12).

14 Hereafter, when referring to (**←), we omit in the examples the cases for k=0, i.e. the combination of equivalent objects, including every object with itself; this omission simplifies the exposition and does not affect the issue here discussed.

15 We qualify 'typically' for in some atypical cases there can be additional degrees of freedom even in the absence of disconnected objects (see below the discussion concerning "simplifiable" systems).

16 Then the minimum is not always n+m-2, as an anonymous referee suggests (see the previous footnote).

17 To avoid confusion, I shall use 'connecteness' for a property of objects and of their equivalence classes, and 'linkage' for a relation between groups of objects.

18 At least q, and not exactly q, since there may be more than q, actually more than j, dislinked groups of connected objects because the dislinkage of congruent compound objects; $\circ$-completeness is sufficient, but not necessary, for the number of such groups be q (which in turns, as we have said, coincides with j under completeness). This comments also show that the converse conditional is not true, e.g. it may be two dislinked groups of connected objects even if all compound objects are congruent; the reason is again the possible dislinkage of congruent compound objects.

19 The authors do not discuss here how reasonable or unreasonable is the closure assumption in this empirical example. If there were cases in which the subject can not identify the bisection of two stimulus then we had a good empirical motivation for looking for non-closed internal systems.

20 See, for instance, *Foundations* 1 ch. 8, where these cases are treated as a very special type of conjoint measurement; see also vol. 3, ch. 20, p. 152.

21 In the context of Measurement Theory this procedure was proposed for the first time in (Suppes-Winet 1955).

22 Although IPWE systems do not preclude closure of $\circ$, closed IPWE systems are uninteresting, since D4(5.2) implies that closed IPWE systems have only one equivalence class.

BIBLIOGRAPHY

Krantz, D., Luce, D., Suppes, P., Tversky, A.: 1971, *Foundations of Measurement I*, New York, Academic P.

Luce, D.: 1956, 'Semiorders and the theory of utility discrimination', *Econometrica* 24, 178-191.

Luce, D., Krantz, D., Suppes, P., Tversky, A.: 1990, *Foundations of Measurement 3*, New York, Academic P.

Moulines, C.U., Díez, J.A.: 1994, 'Theories as Nets: Combinatorial Measurement Theory', in P. Humphreys (ed.): *Patrick Suppes, Mathematical Philosopher*, Kluwer, pp. 275-297.
Narens, L.: 1985, *Abstract Measurement Theory*, Cambridge, M.I.T. Press.
Pfanzagl, J.: 1968, *Theory of Measurement*, New York, Wiley.
Scott, D., Suppes, P.: 1958, 'Foundational aspects of theories of measurement', *Journal of Symbolic Logic* 23, 113-128.
Suppes, P., Krantz, D., Luce, D., Tversky, A.: 1989, *Foundations of Measurement II*, New York, Academic P.
Suppes, P., Winet, M.: 1955, 'An axiomatization of utility based based on the notion of utility differences', *Management Sci.* 1, 259-270.

José A. Díez is PhD in Philosophy, University of Barcelona, with a dissertation on representational measurement theories. He works on philosophy of science within the semantic approach, he has many papers in international reviews and anthologies and is co-author, with C.U. Moulines, of *Fundamentos de Filosofía de la Ciencia* (Barcelona, 1997). Now J.A. Díez is professor of Logic and Philosophy of Science in the Universitat Rovira i Virgili (Tarragona, Spain).

RECENSIONES
BOOK REVIEWS