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Viscosity Approximation Methods for ∗−Nonexpansive Multi-Valued Mappings in Convex Metric Spaces

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Received: 12 December 2019; Accepted: 12 January 2020; Published: 17 January 2020

Abstract: In this paper, we prove convergence theorems for viscosity approximation processes involving ∗−nonexpansive multi-valued mappings in complete convex metric spaces. We also consider finite and infinite families of such mappings and prove convergence of the proposed iteration schemes to common fixed points of them. Our results improve and extend some corresponding results.

Keywords: ∗−nonexpansive multi-valued mapping; viscosity approximation methods; fixed point; convex metric space

MSC: 47H10; 26A51

1. Introduction

Many of the real world known problems that scientists are looking to solve are nonlinear. Therefore, translating linear version of such problems into their equivalent nonlinear version has a great importance. Mathematicians have tried to transfer the structure of covexity to spaces that are not linear spaces. Takahashi [1], Kirk [2,3], and Penot [4], for example, presented this notion in metric spaces. Takahashi [1] introduced the following notion of convexity in metric spaces:

Definition 1. ([1]) *Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $x, y, u \in X$ and all $t \in I$,*

$$d(u, W(x, y, t)) \leq td(u, x) + (1 - t)d(u, y).$$

A metric space (X, d) together with a convex structure W is called a convex metric space and is denoted by (X, W, d).

A subset C of X is called convex if $W(x, y, t) \in C$, for all $x, y \in C$ and all $t \in I$.

Example 1. *Let $X = M_2(\mathbb{R})$. For any $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ and $t \in I = [0, 1]$, we define the mapping $W : X \times X \times I \to X$ by*

$$W(A, B, t) = \begin{bmatrix} ta_1 + (1 - t)b_1 & ta_2 + (1 - t)b_2 \\ ta_3 + (1 - t)b_3 & ta_4 + (1 - t)b_4 \end{bmatrix}$$

and the metric $d : X \times X \to [0, +\infty)$ *by*

$$d(A, B) = \Sigma_{i=1}^{4} |a_i - b_i|.$$

Then (X, W, d) *is a convex metric space.*

Example 2. *Let* $X = \mathbb{R}^2$ *with the metric*

$$d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

for any $(x_1, x_2), (y_1, y_2) \in X$ *and define the mapping* $W : X \times X \times [0, 1] \to X$ *by*

$$W((x_1, x_2), (y_1, y_2), t) = (tx_1 + (1 - t)y_1, tx_2 + (1 - t)y_2),$$

for each $(x_1, x_2), (y_1, y_2) \in X$ *and* $t \in [0, 1]$*. Then* (X, W, d)*is a convex metric space.*

Example 3. *Let* $X = C([0, 1])$ *be the metric space with the metric* $d(f, g) = \int_0^1 |f(x) - g(x)|dx$ *and define* $W : X \times X \times [0, 1] \to X$ *by* $W(f, g, t) = tf + (1 - t)g$*, for all* $f, g \in X$ *and* $t \in [0, 1]$*. Then* (X, W, d) *is a convex metric space.*

This notion of convex structure is a generalization of convexity in normed spaces and allows us to obtain results that seem to be possible only in linear spaces. One of its useful applications is the iterative approximation of fixed points in metric spaces. All of the sequences that are used in fixed point problems require linearity or convexity of the space. So, this concept of convexity helps us to define various iteration schemes and to solve fixed point problems in metric spaces. In recent years, many authors have established several results on the covergence of some iterative schemes using different contractive conditions in convex metric spaces. For more details, refer to [5–14].

Now, let us recall some definitions and concepts that will be needed to state our results:

Definition 2. ([15]) *Let* (X, d) *be a metric. A subset* D *is called proximinal if for each* $x \in X$ *there exists an element* $y \in D$ *such that* $d(x, y) = d(x, D)$*, where* $d(x, D) = \inf\{d(x, z) : z \in D\}$*.*

We denote the family nonempty proximinal and bounded subsets of D by $P(D)$ and the family of all nonempty closed and bounded subsets of X by $CB(X)$.

For two bounded subsets A and B of a metric space (X, d), the Pompeiu–Hausdorff metric between A and B is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\}.$$

Definition 3. ([16]) *Let* (X, d) *be a metric space. A multi-valued mapping* $T : X \to CB(X)$ *is said to be nonexpansive if* $H(Tx, Ty) \leq d(x, y)$*, for all* $x, y \in X$*.*
An element $p \in X$ *is called a fixed point of* T *if* $p \in T(p)$*. The set of all fixed points of* T *are denoted by* $F(T)$*.*

Definition 4. ([17]) *Let* (X, d) *be a metric space and* D *be a nonempty subset of* X*. A multi-valued mapping* $T : D \to CB(D)$ *is called* $*-$*nonexpansive if for all* $x, y \in D$ *and* $u_x \in T(x)$ *with* $d(x, u_x) = \inf\{d(x, z) : z \in T(x)\}$*, there exists* $u_y \in T(y)$ *with* $d(y, u_y) = \inf\{d(y, w) : w \in T(y)\}$ *such that*

$$d(u_x, u_y) \leq d(x, y).$$

It is clear that if T *is a* $*-$*nonexpansive map, then* P_T *is a nonexpansive map, where* P_T *for* $T : D \to P(D)$ *is defined by*

$$P_T(x) = \{y \in T(x) : d(x, y) = d(x, T(x))\},$$

for all $x \in D$.

Definition 5. ([16]) *Let* (X, d) *be a metric space. A multi-valued mapping* $T : X \to CB(X)$ *is said to satisfy condition (I) if there is a nondecreasing function* $f : [0, \infty) \to [0, \infty)$ *with* $f(0) = 0$, $f(r) > 0$ *for* $r \in (0, \infty)$ *such that* $d(x, T(x)) \geq f(d(x, F(T)))$, *for all* $x \in X$.

First of all, Moudafi [18] introduced the viscosity approximation method for approximating the fixed point of nonexpansive mappings in Hilbert spaces. Since then, many authors have been extending and generalizing this result by using different contractive conditions on several spaces. For some new works in these fields, we can refer to [19–27]. Inspired and motivated by the research work going on in these fields, in this paper we investigate the convergence of some viscosity approximation processes for $*-$nonexpansive multi-valued mappings in a complete convex metric spaces. The convergence theorems for finite and infinite family of such mappings are also presented. Our results can improve and extend the corresponding main theorems in the literature.

2. Main Results

At first, we present two lemmas that are used to prove our main result. Since the idea is similar to the one given in Lemmas 2.1 and 2.2 in [28], we only state the results without the proof:

Lemma 1. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be sequences in a convex metric space* (X, W, d) *and* $\{a_n\}$ *be a sequence in* $[0, 1]$ *such that* $\limsup_n a_n < 1$. *Set*

$$d = \limsup_{n\to\infty} d(u_n, v_n) \;\; or \;\; d = \liminf_{n\to\infty} d(u_n, v_n).$$

Let $u_{n+1} = W(v_n, u_n, a_n)$ *for all* $n \in \mathbb{N}$. *Suppose that*

$$\limsup_{n\to\infty} (d(v_{n+1}, v_n) - d(u_{n+1}, u_n)) \leq 0,$$

and $d < \infty$.*Then*

$$\liminf_{n\to\infty} |d(v_{n+k}, u_n) - (1 + a_n + a_{n+1} + \ldots + a_{n+k-1})d| = 0,$$

for all $k \in \mathbb{N}$.

Lemma 2. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be bounded sequences in a convex metric space* (X, W, d) *and* $\{a_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_n a_n \leq \limsup_n a_n < 1$. *Suppose that* $u_{n+1} = W(v_n, u_n, a_n)$ *and*

$$\limsup_{n\to\infty} (d(v_{n+1}, v_n) - d(u_{n+1}, u_n)) \leq 0.$$

Then $\lim_{n\to\infty} d(v_n, u_n) = 0$

Now, we state and prove the main theorem of this paper:

Theorem 1. *Let* D *be a nonempty, closed and convex subset of a complete convex metric space* (X, W, d) *and* $T : D \to P(D)$ *be a* $*-$*nonexpansive multi-valued mapping with* $F(T) \neq \emptyset$, *such that* T *satisfies condition (I). Suppose that* $a_n \in [0, 1]$ *such that* $0 < \liminf_n a_n \leq \limsup_n a_n < 1$ *and* $c_n \in (0, +\infty)$ *such that* $\lim_{n\to\infty} c_n = 0$. *Let* $\{x_n\}$ *be the Mann type iterative scheme defined by*

$$x_{n+1} = W(z_n, x_n, a_n), \tag{1}$$

where $d(z_{n+1}, z_n) \leq H(P_T(x_{n+1}), P_T(x_n)) + c_n$ *for* $z_n \in P_T(x_n)$. *Then* $\{x_n\}$ *converges to a fixed point of* T.

Proof. Take $p \in F(T)$. Then $p \in P_T(p) = \{p\}$ and we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(z_n, x_n, a_n), p) \\ &\leq a_n d(z_n, p) + (1 - a_n) d(x_n, p) \\ &\leq a_n H(P_T(x_n), P_T(p)) + (1 - a_n) d(x_n, p) \\ &\leq a_n d(x_n, p) + (1 - a_n) d(x_n, p) = d(x_n, p). \end{aligned}$$

Hence, $\{d(x_n, p)\}$ is a decreasing and bounded below sequence and thus $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$. Therefore $\{x_n\}$ is bounded and so $\{z_n\}$ is bounded. On the other hand,

$$d(z_{n+1}, z_n) \leq H(P_T(x_{n+1}), P_T(x_n)) + c_n \leq d(x_{n+1}, x_n) + c_n.$$

Thus

$$\limsup_{n\to\infty} (d(z_{n+1}, z_n) - d(x_{n+1}, x_n)) \leq 0.$$

Applying Lemma 2, we get

$$\lim_{n\to\infty} d(z_n, x_n) = 0.$$

Hence, we have $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Since T satisfies condition (I), we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, thus for $\varepsilon_1 > 0$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$d(x_n, F(T)) \leq \frac{\varepsilon_1}{3}.$$

Thus, there exists $p_1 \in F(T)$ such that for all $n \geq n_1$,

$$d(x_n, p_1) \leq \frac{\varepsilon_1}{2}.$$

It follows that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(p_1, x_n) \leq d(x_n, p_1) + d(p_1, x_n) \\ &\leq \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1, \end{aligned}$$

for all $m, n \geq n_1$. Therefore $\{x_n\}$ is a Cauchy sequence and hence it is convergent. Let $\lim_{n\to\infty} x_n = p^*$. We will show that p^* is a fixed point of T.
Since $\lim_{n\to\infty} x_n = p^*$, thus for given $\varepsilon_2 > 0$, there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$,

$$d(x_n, p^*) \leq \frac{\varepsilon_2}{4}.$$

Moreover, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ implies that there exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$,

$$d(x_n, F(T)) \leq \frac{\varepsilon_2}{12},$$

and thus there exists $p_2 \in F(T)$ such that for all $n \geq n_3$,

$$d(x_n, p_2) \leq \frac{\varepsilon_2}{8}.$$

Therefore

$$
\begin{aligned}
d(T(p^*), p^*) &\leq d(T(p^*), p_2) + d(p_2, T(x_{n_3})) + d(T(x_{n_3}), p_2) + d(p_2, x_{n_3}) + d(x_{n_3}, p^*) \\
&\leq H(P_T(p^*), P_T(p_2)) + 2H(P_T(p_2), P_T(x_{n_3})) + d(p_2, x_{n_3}) + d(x_{n_3}, p^*) \\
&\leq d(p^*, p_2) + 2d(p_2, x_{n_3}) + d(p_2, x_{n_3}) + d(x_{n_3}, p^*) \\
&\leq d(p^*, x_{n_3}) + d(x_{n_3}, p_2) + 2d(p_2, x_{n_3}) + d(p_2, x_{n_3}) + d(x_{n_3}, p^*) \\
&= 2d(x_{n_3}, p^*) + 4d(x_{n_3}, p_2) \leq \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2} = \varepsilon_2.
\end{aligned}
$$

Thus, $p^* \in T(p^*)$ and therefore p^* is a fixed point of T. □

As a result of Theorem 1, Corollaries 1 and 2 are obtained:

Corollary 1. *Let D be a nonempty, closed and convex subset of a complete convex metric space (X, W, d), $T : D \to P(D)$ be $*-$nonexpansive multi-valued mapping with $F(T) \neq \emptyset$ such that T satisfies condition* (I) *and $f : D \to D$ be a contractive mapping with a contractive constant $k \in (0, 1)$. Then the iterative sequence $\{x_n\}$ defined by*

$$x_{n+1} = W(z_n, f(x_n), a_n)$$

where $z_n \in P_T(x_n)$ and $0 < \liminf_n a_n \leq \limsup_n a_n < 1$, converges to a fixed point of T.

Corollary 2. *Let D be a nonempty, closed, and convex subset of a complete convex metric space (X, W, d) and $T : D \to P(D)$ be $*-$nonexpansive multi-valued mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa type iterative scheme defined by*

$$
\begin{aligned}
x_{n+1} &= W(z_n^{'}, x_n, a_n) \\
y_n &= W(z_n, x_n, b_n)
\end{aligned}
$$

where $z_n^{'} \in P_T(y_n)$, $z_n \in P_T(x_n)$, and $\{a_n\}, \{b_n\} \in [0, 1]$. Then $\{x_n\}$ converges to a fixed point of T if and only if $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

The above result can be generalized to the finite and infinite family of $*-$nonexpansive multi-valued mappings:

Theorem 2. *Let D be a nonempty, closed, and convex subset of a complete convex metric space (X, W, d) and $\{T_i : D \to P(D) : \ i = 1, \ldots, k\}$ be a finite family of $*-$nonexpansive multi-valued mappings such that $F := \cap_{i=1}^k F(T_i) \neq \emptyset$. Consider the iterative process defined by*

$$
\begin{aligned}
y_{1n} &= W(z_{1n}, x_n, a_{1n}), \\
y_{2n} &= W(z_{2n}, x_n, a_{2n}), \\
&\cdots \\
y_{(k-1)n} &= W(z_{(k-1)n}, x_n, a_{(k-1)n}), \\
x_{n+1} &= W(z_{kn}, x_n, a_{kn}),
\end{aligned}
$$

where $a_{in} \in [0, 1]$ and $z_{in} \in P_{T_i}(y_{(i-1)n})$ $(y_{0n} = x_n)$, for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, k$. Then $\{x_n\}$ converges to a point in F if and only if $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity of conditions is obvious and we will only prove the sufficiency. Let $p \in F$. we have

$$\begin{aligned}
d(y_{1n}, p) &= d(W(z_{1n}, x_n, a_{1n}), p) \\
&\le a_{1n} d(z_{1n}, p) + (1 - a_{1n}) d(x_n, p) \\
&\le a_{1n} H(P_{T_1}(x_n), P_{T_1}(p)) + (1 - a_{1n}) d(x_n, p) \\
&\le a_{1n} d(x_n, p) + (1 - a_{1n}) d(x_n, p) = d(x_n, p), \\
d(y_{2n}, p) &= d(W(z_{2n}, x_n, a_{2n}), p) \\
&\le a_{2n} d(z_{2n}, p) + (1 - a_{2n}) d(x_n, p) \\
&\le a_{2n} H(P_{T_2}(y_{1n}), P_{T_2}(p)) + (1 - a_{2n}) d(x_n, p) \\
&\le a_{2n} d(y_{1n}, p) + (1 - a_{2n}) d(x_n, p) \\
&\le a_{2n} d(x_n, p) + (1 - a_{2n}) d(x_n, p) = d(x_n, p), \\
&\vdots \\
d(y_{(k-1)n}, p) &= d(W(z_{(k-1)n}, x_n, a_{(k-1)n}), p) \\
&\le a_{(k-1)n} d(z_{(k-1)n}, p) + (1 - a_{(k-1)n}) d(x_n, p) \\
&\le a_{(k-1)n} H(P_{T_{k-1}}(y_{(k-2)n}), P_{T_{k-1}}(p)) + (1 - a_{(k-1)n}) d(x_n, p) \\
&\le a_{(k-1)n} d(y_{(k-2)n}, p) + (1 - a_{(k-1)n}) d(x_n, p) \\
&\le a_{(k-1)n} d(x_n, p) + (1 - a_{(k-1)n}) d(x_n, p) = d(x_n, p).
\end{aligned}$$

Thus

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(z_{kn}, x_n, a_{kn}), p) \\
&\le a_{kn} d(z_{kn}, p) + (1 - a_{kn}) d(x_n, p) \\
&\le a_{kn} H(P_{T_k}(y_{(k-1)n}), P_{T_k}(p)) + (1 - a_{kn}) d(x_n, p) \\
&\le a_{kn} d(y_{(k-1)n}, p) + (1 - a_{kn}) d(x_n, p) \\
&\le a_{kn} d(x_n, p) + (1 - a_{kn}) d(x_n, p) = d(x_n, p).
\end{aligned}$$

Therefore, $\{d(x_n, p)\}$ is a decreasing sequence and so $d(x_{n+m}, p) \le d(x_n, p)$, for all $n, m \in \mathbb{N}$. As in the proof of Theorem 1, $\{x_n\}$ is a Cauchy sequence and thus $\lim_{n\to\infty} x_n$ exists and equals to some $p^* \in D$. Again, with a similar process as in the proof of Theorem 1, we conclude that $p^* \in P_{T_i}(q)$ for all $i = 1, \ldots, k$. Hence $p^* \in F$ and this completes the proof of theorem. □

Theorem 3. *Let D be a nonempty, closed, and convex subset of a complete convex metric space (X, W, d) and $\{T_i : D \to P(D) : i = 1, \ldots\}$ be an infinite family of $*-$nonexpansive multi-valued mappings such that $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Consider the iterative process defined by*

$$\begin{aligned}
x_{n+1} &= W(z_n^{'}, x_n, a_n) \\
y_n &= W(z_n, x_n, b_n)
\end{aligned}$$

where $z_n^{'} \in P_{T_n}(y_n)$, $z_n \in P_{T_n}(x_n)$ and $\{a_n\}, \{b_n\} \in [0, 1]$. Then $\{x_n\}$ converges to a point in F if and only if $\lim_{n\to\infty} d(x_n, F) = 0$.

Author Contributions: Data curation, A.G.; Formal analysis, A.G.; Software, A.G.; Writing—original draft, A.G.; Conceptualization, H.P.M.; Project administration, H.P.M.; Supervision, M.D.L.S.; Funding acquisition, M.D.L.S.; Writing—review and editing, M.D.L.S. and M.R.; Validation, M.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Basque Government through grant IT1207-19.

Acknowledgments: The authors are grateful to the referees for valuable suggestions and to the Basque Government for Grant IT1207-19.

Conflicts of Interest: The authors declare no conflict of interest.

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