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Some New Results on Coincidence Points for Multivalued Suzuki-Type Mappings in Fairly Complete Spaces

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Abstract: In this paper, we introduce Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized and modified proximal contractive mappings. We establish some coincidence and best proximity point results in fairly complete spaces. Also, we provide coincidence and best proximity point results in partially ordered complete metric spaces for Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized and modified proximal contractive mappings. Furthermore, some examples are presented in each section to elaborate and explain the usability of the obtained results. As an application, we obtain fixed-point results in metric spaces and in partially ordered metric spaces. The results obtained in this article further extend, modify and generalize the various results in the literature.

Keywords: Coincidence best proximity point; Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction; Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contraction; fairly complete space; fixed point; partially ordered metric space

1. Introduction and Preliminaries

S. Banach [1] stated and proved the *Banach contraction principle*. This principle has wide applications due to its simple and constructive nature of proof. The constructive proof leads to developing algorithms and can be easily applied in computer and data sciences (see in [2]) as well. The application of this principle is not limited to these areas, it is extensively used in dynamically programming ([3]) and biosciences as well. Due to wide range of applications, researchers around the globe are attracted towards this principle to generalize, modify and extend this pioneer result (for detail, see [4–12]). These modifications are consisting upon three pillars (1) generalizing the contractive conditions, (2) generalizing the underlying space and (3) modifying the single valued mapping with multivalued mapping. In all the three modifications, the Banach contraction principle gets modification with three different aspects.

The *"fixed point"* $\mathfrak{q}$ of a self-mapping M is actually a solution of an operator equation $M\mathfrak{q} = \mathfrak{q}$ (i.e., $\mathsf{d}(\mathfrak{q}, M\mathfrak{q}) = 0$). Among these three aspects of generalization of *"Banach contraction principle"*, it would be quite interesting to discuss, if the operator equation $M\mathfrak{q} = \mathfrak{q}$ has no solution. In this case, when $\mathsf{d}(\mathfrak{q}, M\mathfrak{q}) \neq 0$ then it is evident to minimize the distance between $\mathfrak{q}$ and $M\mathfrak{q}$ which leads to the following optimization problem:

$$\min_{\mathfrak{q} \in Y} \mathsf{d}(\mathfrak{q}, M\mathfrak{q}).$$

Now, if M is non-self-mapping, so we cannot find the *"fixed point"* of M, in this case we can optimize the distance between $\mathfrak{q}$ and $M\mathfrak{q}$, but in the case of non-self-mapping such that if $M : \mathcal{Q} \to \mathcal{R}$ then we cannot reduce the $\mathsf{d}(\mathfrak{q}, M\mathfrak{q})$ to zero but it can minimize up to $\mathsf{d}(\mathcal{Q}, \mathcal{R})$ (the distance between set $\mathcal{Q}$ and set $\mathcal{R}$). Any point $\mathfrak{q} \in \mathcal{Q}$ is called a *"best proximity point"* of mapping M, if it satisfies $\mathsf{d}(\mathfrak{q}, M\mathfrak{q}) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$. Please note that if $\mathcal{Q} \cap \mathcal{R}$ is nonempty then any *"best proximity point"* of the mapping M becomes a *"fixed point"* of the mapping M.

An element $\mathfrak{q} \in \mathcal{Q}$ is said to be a *"coincidence best proximity point"* of the pair of mappings $(\mathfrak{g}, M)$, if $\mathfrak{q}$ satisfy $\mathsf{d}(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$, where $M : \mathcal{Q} \to \mathcal{R}$ and $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$. If $\mathfrak{g} = I_{\mathcal{Q}}$ then *"coincidence best proximity point"* becomes a *"best proximity point"* of mapping M.

One of the interesting generalizations of the *"Banach contraction principle"* was given by V. Berinde ([13]) and proved the following result.

Theorem 1. *([13]) Let (Y, d) be a complete metric space and mapping $M : Y \to Y$ satisfies*

$$\mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) \leq \alpha \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathsf{d}(M\mathfrak{q}, \mathfrak{r}),$$

for all $\mathfrak{q}, \mathfrak{r} \in Y$ where $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$ then the mapping M has a "fixed point".

T. Suzuki ([14]) introduced *"Suzuki contraction"*, which generalized the *"Banach contraction"* and he proved the following *"fixed-point theorem"*.

Theorem 2. *([14]) Let (Y, d) be a complete metric space and mapping $M : Y \to Y$ satisfies*

$$\frac{1}{2}\mathsf{d}(\mathfrak{q}, M\mathfrak{q}) < \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ implies } \mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) < \mathsf{d}(\mathfrak{q}, \mathfrak{r}),$$

for all $\mathfrak{q}, \mathfrak{r} \in Y$ then mapping M has a unique fixed point in Y.

In 2014, M. Gabeleh ([15]) revised and generalized the contractions presented in Theorem 1 and in Theorem 2 to prove the single valued and multivalued *"best proximity point results"* . Recently S. Basha ([16]) introduced the concept of *"fairly and proximally complete spaces"* and proved *"best proximity point results"* in these spaces.

In this paper, we will modify Suzuki-type *"best proximity point results"* of M. Gabeleh ([15]) and prove Suzuki-type *"coincidence best proximity point results"* in the setting of *"fairly complete space"* and *"partially ordered fairly complete space"*.

We will use the following notations in the entire article and assume that $\mathcal{Q}$ and $\mathcal{R}$ are nonempty subsets of a metric space (Y, d), further

$$\begin{aligned} \mathsf{d}(\mathcal{Q}, \mathcal{R}) &= \inf\{\mathsf{d}(\mathfrak{q}, \mathfrak{r}) : \mathfrak{q} \in \mathcal{Q} \text{ and } \mathfrak{r} \in \mathcal{R}\} \text{ (distance between two sets } \mathcal{Q} \text{ and } \mathcal{R}), \\ \mathcal{Q}_0 &= \{\mathfrak{q} \in \mathcal{Q} \text{ such that } \mathsf{d}(\mathfrak{q}, \mathfrak{r}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}), \text{ for some } \mathfrak{r} \in \mathcal{R}\}, \\ \mathcal{R}_0 &= \{\mathfrak{r} \in \mathcal{R} \text{ such that } \mathsf{d}(\mathfrak{q}, \mathfrak{r}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}), \text{ for some } \mathfrak{q} \in \mathcal{Q}\}, \\ \text{also } \mathsf{d}^*(\mathfrak{q}, \mathfrak{r}) &= \mathsf{d}(\mathfrak{q}, \mathfrak{r}) - \mathsf{d}(\mathcal{Q}, \mathcal{R}), \text{ for some } \mathfrak{q} \in \mathcal{Q} \text{ and } \mathfrak{r} \in \mathcal{R}. \end{aligned}$$

Raj introduced the $\mathcal{P}-$property in ([17]), which is defined as:

Definition 1. *A pair $(\mathcal{Q}, \mathcal{R})$ is said to satisfy the $\mathcal{P}-$property if and only if*

$$\left.\begin{array}{l} \mathsf{d}(\mathfrak{q}_1, \mathfrak{r}_1) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathsf{d}(\mathfrak{q}_2, \mathfrak{r}_2) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \end{array}\right\} \textit{ implies } \mathsf{d}(\mathfrak{q}_1, \mathfrak{q}_2) = \mathsf{d}(\mathfrak{r}_1, \mathfrak{r}_2),$$

for all $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{Q}$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathcal{R}$.

We recall the following notions of cyclically Cauchy sequence and fairly Cauchy sequence.

Definition 2. *([16]) Consider two sequences $\{\mathfrak{q}_n\}$ in $\mathcal{Q}$ and $\{\mathfrak{r}_n\}$ in $\mathcal{R}$. The sequence $\{(\mathfrak{q}_n, \mathfrak{r}_n)\}$ in $(\mathcal{Q}, \mathcal{R})$ is said to be:*

- *A cyclically Cauchy sequence if there exists a natural number N such that*

$$\mathsf{d}(\mathfrak{q}_m, \mathfrak{r}_n) < \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \epsilon,$$

  for every $\epsilon > 0$ and for all $\mathfrak{m}, \mathfrak{n} \geq N \in \mathbb{N}$
- *A fairly Cauchy sequence if the following conditions are satisfied*

  (1) $\{(\mathfrak{q}_n, \mathfrak{r}_n)\}$ is a cyclically Cauchy sequence,
  (2) $\{\mathfrak{q}_n\}$ and $\{\mathfrak{r}_n\}$ are Cauchy sequences,

  for all $\mathfrak{n} \geq N \in \mathbb{N}$.

Next, we recall a special type of completeness for a pair of nonempty subsets $(\mathcal{Q}, \mathcal{R})$ of (Y, d).

Definition 3. *([16]) A pair $(\mathcal{Q}, \mathcal{R})$ is a fairly complete space if and only if for every fairly Cauchy sequence $\{(\mathfrak{q}_n, \mathfrak{r}_n)\}$ converges in $(\mathcal{Q}, \mathcal{R})$, also the sequences $\{\mathfrak{q}_n\}$ and $\{\mathfrak{r}_n\}$ are converging in $\mathcal{Q}$ and $\mathcal{R}$, respectively.*

The notion of uniform $M-$approximation of a set is described in the following definition.

Definition 4. *([16]) Let $M : \mathcal{Q} \to \mathcal{R}$ be a mapping. The set $\mathcal{R}$ is said to have uniform $M-$approximation in set $\mathcal{Q}$ if and only if there exist $\delta > 0$ and $\epsilon > 0$, such that*

$$\left.\begin{array}{l} \mathsf{d}(\mathfrak{q}_1, M\mathfrak{r}_1) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathsf{d}(\mathfrak{q}_2, M\mathfrak{r}_2) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathsf{d}(M\mathfrak{r}_1, M\mathfrak{r}_2) < \delta \end{array}\right\} \textit{ implies } \mathsf{d}(\mathfrak{q}_1, \mathfrak{q}_2) < \epsilon,$$

for all $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{r}_1$ and $\mathfrak{r}_2$ in $\mathcal{Q}$.

In 2012, Samet et al. ([18]) introduced $\alpha - \psi-$contractive and $\alpha-$admissible mappings and established some fixed-point theorems for such mappings in complete metric spaces. Samet et al. ([18]) defined the notion of $\alpha-$admissible mapping as follows.

Definition 5. *A mapping $M : Y \to Y$ is said to be $\alpha-$admissible if there exists $\alpha : Y \times Y \to [0, \infty)$ such that*

$$\alpha(\mathfrak{q}, \mathfrak{r}) \geq 1 \textit{ imply } \alpha(M\mathfrak{q}, M\mathfrak{r}) \geq 1, \tag{1}$$

for all $\mathfrak{q}, \mathfrak{r} \in Y$.

The concept of $\alpha-$admissible mapping was generalized and extended in many directions. Jleli et al. ([19]) introduced $\alpha-$proximal admissible mapping as follows.

Definition 6. *Let $\mathcal{Q}$ and $\mathcal{R}$ be the nonempty subsets of metric space (Y, d). A mapping $M : \mathcal{Q} \to \mathcal{R}$ is said to be $\alpha-$proximal admissible if there exists $\alpha : Y \times Y \to [0, \infty)$ such that*

$$\left.\begin{array}{l} \alpha(\mathfrak{q}_1, \mathfrak{q}_2) \geq 1 \\ \mathsf{d}(\mathfrak{r}_1, M\mathfrak{q}_1) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathsf{d}(\mathfrak{r}_2, M\mathfrak{q}_2) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \end{array}\right\} \textit{ imply } \alpha(\mathfrak{r}_1, \mathfrak{r}_2) \geq 1,$$

for all $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{r}_1, \mathfrak{r}_2 \in \mathcal{Q}$.

Please note that if $\mathcal{Q} = \mathcal{R} = Y$ then every $\alpha-$proximal admissible mapping is an $\alpha-$admissible mapping.

Definition 7. *([20]) A mapping $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ satisfies the α_R-property if there exists a function $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ such that*

$$\alpha(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \geq 1 \textit{ imply } \alpha(\mathfrak{q}, \mathfrak{r}) \geq 1,$$

for all $\mathfrak{q}, \mathfrak{r} \in \mathcal{Q}$.

Let $\mathcal{C}_{\mathcal{B}}(Y)$ be a closed and bounded subset of the metric space (Y, d). Then the Pompeiu–Hausdroff metric ([21]) on $\mathcal{C}_{\mathcal{B}}(Y)$, is defined as

$$\mathcal{H}(\mathcal{Q}, \mathcal{R}) = \max\{\sup_{\mathfrak{q} \in \mathcal{Q}} \mathcal{D}(\mathfrak{q}, \mathcal{R}), \sup_{\mathfrak{r} \in \mathcal{R}} \mathcal{D}(\mathfrak{r}, \mathcal{Q})\},$$

for $\mathcal{Q}, \mathcal{R} \in \mathcal{C}_{\mathcal{B}}(Y)$ where

$$\begin{aligned} \mathcal{D}(\mathfrak{q}, \mathcal{R}) &= \inf\{\mathsf{d}(\mathfrak{q}, \mathfrak{r}) : \mathfrak{r} \in \mathcal{R}\} \text{ (distance of a point } \mathfrak{q} \text{ to a set } \mathcal{R}), \\ &\quad \text{and} \\ \mathcal{D}^*(\mathfrak{q}, \mathfrak{r}) &= \mathcal{D}(\mathfrak{q}, \mathfrak{r}) - \mathsf{d}(\mathcal{Q}, \mathcal{R}), \text{ for all } \mathfrak{q} \in \mathcal{Q} \text{ and } \mathfrak{r} \in \mathcal{R}. \end{aligned}$$

2. Main Results

To obtain the main results, we need to define the Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal and Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contractions as follows:

Definition 8. *1. A pair $(\mathfrak{g}, M)$ where $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ and $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contractive condition if $\gamma \in (0, 1]$ such that $0 \leq \beta < \gamma$ and*

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \textit{ imply } \alpha(\mathfrak{q}, \mathfrak{r})\, \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}).$$

2. A mapping $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contractive condition if $\gamma \in (0, 1]$ such that $0 \leq \beta < \gamma$ and

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ imply } \alpha(\mathfrak{q}, \mathfrak{r})\, \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{r}, M\mathfrak{q}),$$

where $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ and $\alpha(\mathfrak{q}, \mathfrak{r}) \geq 1$, for all $\mathfrak{q}, \mathfrak{r} \in \mathcal{Q}$.

The constants γ and β satisfies the condition C, if $\gamma \in (0, 1]$ *and* $0 \leq \beta < \gamma$ *such that* $0 < \beta + \gamma \leq 1$.

In the first result we will prove that the pair $(\mathfrak{g}, M)$ which satisfies the Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction has a coincidence best proximity point in the frame work of fairly complete spaces.

Theorem 3. *Let $\mathcal{Q}$ and $\mathcal{R}$ be nonempty closed subsets of a complete metric space (Y, d) such that the pair $(\mathcal{Q}, \mathcal{R})$ is a fairly complete space and satisfy the $\mathcal{P}-$property. Consider a pair $(\mathfrak{g}, M)$ satisfying Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contractive condition with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$ where mapping $\mathfrak{g}$ satisfy the α_R-property and mapping M is an $\alpha-$proximal admissible. Furthermore, suppose that there exist some $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that*

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \textit{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1.$$

Then the pair $(\mathfrak{g}, M)$ possesses a coincidence best proximity point.

Proof. Let $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that $\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$ and $\alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1$. As $M\mathfrak{q}_1 \in M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, there exists an element $\mathfrak{g}\mathfrak{q}_2 = \mathfrak{q}_2^{'} \in \mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$ such that $\mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_1) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$. As M is an $\alpha-$proximal admissible, it follows that $\alpha(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) \geq 1$. Since $\mathfrak{g}$ satisfy the α_R-property therefore $\alpha(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) \geq 1$ imply $\alpha(\mathfrak{q}_1, \mathfrak{q}_2) \geq 1$. Since $M\mathfrak{q}_2 \in M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, there exists an element $\mathfrak{g}\mathfrak{q}_3 = \mathfrak{q}_3^{'} \in \mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$ such that

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_3, M\mathfrak{q}_2) = \mathsf{d}(\mathcal{Q}, \mathcal{R}),$$

as M is an $\alpha-$proximal admissible, it follows that $\alpha(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) \geq 1$. Also $\mathfrak{g}$ possesses the α_R-property therefore $\alpha(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) \geq 1$ implies $\alpha(\mathfrak{q}_2, \mathfrak{q}_3) \geq 1$. Continuing the same reasoning we get a sequence $\{\mathfrak{g}\mathfrak{q}_\mathfrak{n}\}$ in $\mathcal{Q}_0$ such that

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, M\mathfrak{q}_\mathfrak{n}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}), \tag{2}$$

with $\alpha(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}) \geq 1$. Since mapping $\mathfrak{g}$ satisfies the α_R-property we have $\alpha(\mathfrak{q}_\mathfrak{n}, \mathfrak{q}_{\mathfrak{n}+1}) \geq 1$. We know that if $\gamma \in (0, 1]$ and $0 \leq \beta < \gamma$ then $1 + \beta + \gamma \geq 1$ and $\frac{1}{1+\beta+\gamma} \leq 1$. Now consider the case when $\mathfrak{q}_{\mathfrak{n}+1} = \mathfrak{q}_\mathfrak{n}$ then from Equation (2), we have $\mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$, which proves the theorem. Now if $\mathfrak{q}_{\mathfrak{n}+1} \neq \mathfrak{q}_\mathfrak{n}$ for all $\mathfrak{n} \in \mathbb{N}$ then we have

$$\begin{aligned}
\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) \leq \mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) &= \mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\
&\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) + \mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_1) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\
&= \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2),
\end{aligned}$$

and so, the above inequality can be written as

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2).$$

Since $\alpha(\mathfrak{q}_1, \mathfrak{q}_2) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfies Suzuki-type $(\alpha, \beta, \gamma_\mathfrak{g})-$generalized proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \alpha(\mathfrak{q}_1, \mathfrak{q}_2)\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_1),$$

above inequality becomes

$$\begin{aligned}
\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) &\leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) + \beta[\mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_1) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\
&= \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2).
\end{aligned}$$

Since the pair $(\mathcal{Q}, \mathcal{R})$ satisfy the $\mathcal{P}-$property, using the $\mathcal{P}-$property the above inequality can be written as

$$\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) = \gamma\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1), \tag{3}$$

which shows that $\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \gamma\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1)$. Continuing on the same lines for $\mathfrak{q}_2$ we can verify the following

$$\begin{aligned}
\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) \leq \mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) &= \mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\
&\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) + \mathcal{D}(\mathfrak{g}\mathfrak{q}_3, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\
&= \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3),
\end{aligned}$$

and so the above inequality can be written as

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3).$$

Since $\alpha(\mathfrak{q}_2, \mathfrak{q}_3) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfies Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) \leq \alpha(\mathfrak{q}_2, \mathfrak{q}_3)\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_3, M\mathfrak{q}_2),$$

which can be written as

$$\begin{aligned} \mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) &\leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) + \beta[\mathcal{D}(\mathfrak{g}\mathfrak{q}_3, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\ &= \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3). \end{aligned}$$

The pair $(\mathcal{Q}, \mathcal{R})$ satisfy the $\mathcal{P}-$property, using inequality (3) and the $\mathcal{P}-$property in above inequality, we obtain

$$\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) = \gamma\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \gamma^2\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1),$$

thus, for a sequence $\{M\mathfrak{q}_\mathfrak{n}\}$ in $\mathcal{R}_0$ we have

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}+1}) \leq \gamma^\mathfrak{n}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1). \tag{4}$$

Therefore

$$\sum_{\mathfrak{n}=1}^{\infty} \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+2}) = \sum_{\mathfrak{n}=1}^{\infty} \mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}+1}) \leq \sum_{\mathfrak{n}=1}^{\infty} \gamma^\mathfrak{n}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1),$$

which leads $\{M\mathfrak{q}_\mathfrak{n}\}$ to be a Cauchy sequence in $\mathcal{R}$ and $(\mathcal{Q}, \mathcal{R})$ is a pair of nonempty closed subsets of a complete metric space (Y, d) and so $\{M\mathfrak{q}_\mathfrak{n}\}$ converges to some point $\mathfrak{q} \in \mathcal{R}_0$. In the same way, the sequence $\{\mathfrak{g}\mathfrak{q}_\mathfrak{n}\}$ is convergent to some point $\mathfrak{g}p \in \mathcal{Q}_0$. So, we have

$$\mathsf{d}(\mathfrak{g}p, \mathfrak{q}) = \lim_{\mathfrak{n}\to\infty} \mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}-1}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}).$$

Using triangular inequality, we can write

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_\mathfrak{n}) \leq \mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_{\mathfrak{m}-1}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{m}-1}, M\mathfrak{q}_\mathfrak{n}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{m}-1}, M\mathfrak{q}_\mathfrak{n}). \tag{5}$$

If $\mathfrak{m} - 1 < \mathfrak{n}$ then we have

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_\mathfrak{n}) \leq \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{m}-1}, M\mathfrak{q}_\mathfrak{m}) + \mathcal{H}(M\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_{\mathfrak{m}+1}) + \cdots + \mathcal{H}(M\mathfrak{q}_{\mathfrak{n}-1}, M\mathfrak{q}_\mathfrak{n}). \tag{6}$$

By Equations (4) and (6) we have

$$\begin{aligned} \mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_\mathfrak{n}) &\leq \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \gamma^{\mathfrak{m}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \gamma^{\mathfrak{m}}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \cdots + \gamma^{\mathfrak{n}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) \\ &= \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \gamma^{\mathfrak{m}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1)[1 + \gamma + \cdots + \gamma^{\mathfrak{n}-\mathfrak{m}}] \\ &= \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \gamma^{\mathfrak{m}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1)[\frac{(1 - \gamma^{\mathfrak{n}-\mathfrak{m}+1})}{(1-\gamma)}] \\ &\leq \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \frac{\gamma^{\mathfrak{m}-1}}{1-\gamma}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1). \end{aligned}$$

If $\mathfrak{n} < \mathfrak{m} - 1$ then the inequality (5) implies

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_\mathfrak{n}) \leq \mathsf{d}(\mathcal{Q}, \mathcal{R}) + \mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}+1}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{n}+1}, M\mathfrak{q}_{\mathfrak{n}+2}) + \cdots + \mathcal{H}(M\mathfrak{q}_{\mathfrak{m}-2}, M\mathfrak{q}_{\mathfrak{m}-1}), \tag{7}$$

using inequality (4), the inequality (7) becomes

$$\begin{aligned}\mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{m}}, M\mathfrak{q}_{\mathfrak{n}}) &\leq \mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma^{\mathfrak{n}}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \gamma^{\mathfrak{n}+1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \cdots + \gamma^{\mathfrak{m}-2}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1)\\ &= \mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma^{\mathfrak{n}}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1)[1+\gamma+\cdots+\gamma^{\mathfrak{m}-\mathfrak{n}-2}]\\ &= \mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma^{\mathfrak{n}}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1)[\frac{(1-\gamma^{\mathfrak{m}-\mathfrak{n}-1})}{(1-\gamma)}]\\ &\leq \mathsf{d}(\mathcal{Q},\mathcal{R}) + \frac{\gamma^{\mathfrak{n}}}{1-\gamma}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1).\end{aligned}$$

Thus, $\{(\mathfrak{g}\mathfrak{q}_{\mathfrak{m}}, M\mathfrak{q}_{\mathfrak{n}})\}$ is a cyclically Cauchy sequence. Since $\mathfrak{g}\mathfrak{q}_{\mathfrak{n}} \to \mathfrak{g}p$ there exists $N_1 \in \mathbb{N}$ such that

$$\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}p) \leq \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p), \tag{8}$$

for all $\mathfrak{n} \geq N_1 \in \mathbb{N}$. Since $1+\beta+\gamma > 1$ and $\frac{1}{1+\beta+\gamma} < 1$ we can write

$$\begin{aligned}\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}_{\mathfrak{n}}) &\leq \mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}_{\mathfrak{n}}) - \mathsf{d}(\mathcal{Q},\mathcal{R})\\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}p) + \mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}) + \mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, M\mathfrak{q}_{\mathfrak{n}}) - \mathsf{d}(\mathcal{Q},\mathcal{R})\\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}p) + \mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1})\\ &\leq \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) + \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p)\\ &= \frac{2}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p)\\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) - \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p)\\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) - \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}p) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}\mathfrak{q}).\end{aligned}$$

Since $\alpha(\mathfrak{q}_{\mathfrak{n}}, \mathfrak{q}) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfies Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}) \leq \alpha(\mathfrak{q}_{\mathfrak{n}}, \mathfrak{q})\mathcal{H}(M\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}\mathfrak{q}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}_{\mathfrak{n}}),$$

above inequality can be written as

$$\mathcal{H}(M\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}\mathfrak{q}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}_{\mathfrak{n}}). \tag{9}$$

We can write

$$\mathcal{D}(\mathfrak{g}p, M\mathfrak{q}) = \lim_{\mathfrak{n}\to\infty}\mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}) \leq \lim_{\mathfrak{n}\to\infty}[\mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}_{\mathfrak{n}-1}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{n}-1}, M\mathfrak{q}_{\mathfrak{n}}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q})], \tag{10}$$

using inequalities (4) and (9), inequality (10) becomes

$$\begin{aligned}\mathcal{D}(\mathfrak{g}p, M\mathfrak{q}) &\leq \lim_{\mathfrak{n}\to\infty}[\mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma^{\mathfrak{n}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}\mathfrak{q}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}_{\mathfrak{n}})]\\ &= \mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}) + \beta\mathsf{d}^*(q, \mathfrak{g}\mathfrak{q}),\end{aligned}$$

after simplification above inequality can be written as

$$\mathcal{D}^*(\mathfrak{g}p, M\mathfrak{q}) \leq \gamma\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}) + \beta\mathsf{d}^*(q, \mathfrak{g}\mathfrak{q}). \tag{11}$$

From triangular inequality we have

$$\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, M\mathfrak{q}_{\mathfrak{n}}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}}, \mathfrak{g}p) + \mathcal{D}^*(\mathfrak{g}p, M\mathfrak{q}_{\mathfrak{n}}). \tag{12}$$

Using inequality (11), inequality (12) becomes

$$\begin{aligned}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \gamma\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_\mathfrak{n}) + \beta\mathsf{d}^*(q, \mathfrak{g}\mathfrak{q}_\mathfrak{n}) \\ &\leq (1+\gamma)\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_\mathfrak{n}) + \beta[\mathcal{D}(q, M\mathfrak{q}_{\mathfrak{n}-1}) + \mathcal{D}(M\mathfrak{q}_{\mathfrak{n}-1}, \mathfrak{g}\mathfrak{q}_\mathfrak{n}) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\ &= (1+\gamma)\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_\mathfrak{n}) + \beta\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_\mathfrak{n}) \\ &= (1+\beta+\gamma)\mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_\mathfrak{n}),\end{aligned}$$

after simplification we have

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) \leq \mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_\mathfrak{n}).$$

Since $\alpha(\mathfrak{q}_\mathfrak{n}, p) \geq 1$ and furthermore the pair $(\mathfrak{g}, M)$ satisfy Suzuki-type $(\alpha, \beta, \gamma_\mathfrak{g})-$generalized proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) \leq \alpha(\mathfrak{q}_\mathfrak{n}, p)\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \beta\mathcal{D}^*(M\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p),$$

further we have

$$\begin{aligned}\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) &\leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \beta\mathcal{D}^*(M\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) \\ &\leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \beta[\mathcal{D}(M\mathfrak{q}_\mathfrak{n}, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}) + \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, \mathfrak{g}p) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\ &= \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \beta\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, \mathfrak{g}p),\end{aligned}$$

since $\mathfrak{g}\mathfrak{q}_\mathfrak{n} \to \mathfrak{g}p$. In above relation if $\mathfrak{n} \to \infty$ then we conclude that $M\mathfrak{q}_\mathfrak{n} \to Mp$, that is, $q = Mp$ and we have

$$\mathcal{D}(\mathfrak{g}p, Mp) = \mathsf{d}(\mathcal{Q}, \mathcal{R}).$$

Therefore, p is a coincidence best proximity point of the pair $(\mathfrak{g}, M)$. □

The subsequent example corroborates the result proved in Theorem 3.

Example 1. *Let* $Y = \mathbb{R}^2$ *be a metric space with Euclidean metric* d. *Suppose* $\mathcal{Q} = \{(-1,1), (-1,0), (-1,-1)\}$ *and* $\mathcal{R} = \{(-4,1), (-4,0), (-4,-1)\}$ *are nonempty subsets of* Y. *After simple calculation, we obtain* $\mathsf{d}(\mathcal{Q}, \mathcal{R}) = 3$ *and the pair* $(\mathcal{Q}, \mathcal{R})$ *satisfy the* $\mathcal{P}-$*property, also* $\mathcal{Q}_0 = \mathcal{Q}$ *and* $\mathcal{R}_0 = \mathcal{R}$. *Now define a mapping* $M : \mathcal{Q} \to \mathcal{C}_\mathcal{B}(\mathcal{R})$ *as:*

$$M(\mathfrak{q}) = \begin{cases} \{(-4,0)\}, \textit{ if } \mathfrak{q} \in \{(-1,1)\} \\ \{(-4,1)\}, \textit{ if } \mathfrak{q} \in \{(-1,-1)\} \\ \{(-4,0), (-4,1)\}, \textit{ if } \mathfrak{q} \in \{(-1,0)\}, \end{cases}$$

clearly $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ *and mapping* $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ *as:*

$$\mathfrak{g}(\mathfrak{q}) = \begin{cases} (-1,1), \textit{ if } \mathfrak{q} \in \{(-1,-1)\} \\ (-1,-1), \textit{ if } \mathfrak{q} \in \{(-1,0)\} \\ (-1,0), \textit{ if } \mathfrak{q} \in \{(-1,1)\}, \end{cases}$$

which satisfies $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$. *Now we must show that the pair* $(\mathfrak{g}, M)$ *satisfies Suzuki-type* $(\alpha, \beta, \gamma_\mathfrak{g})-$*generalized proximal contraction.*

Case 1. If $\mathfrak{q}, \mathfrak{r} \in \{(-1,1),(-1,-1)\} \subseteq \mathcal{Q}$ *then following condition of Suzuki-type* $(\alpha, \beta, \gamma_{\mathfrak{g}})-$*generalized proximal contractive condition holds for all* $\mathfrak{q}, \mathfrak{r} \in \{(-1,1),(-1,-1)\}$ *for* $\beta = 0.5$ *and* $\gamma = 0.7$.

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}). \tag{13}$$

Now, we must show that the second condition of Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$*generalized proximal contraction holds for all* $\mathfrak{q}, \mathfrak{r} \in \{(-1,1),(-1,-1)\}$.

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}). \tag{14}$$

Define $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ *as* $\alpha(\mathfrak{q}, \mathfrak{r}) = e^{\frac{1}{\mathsf{d}(\mathfrak{q},\mathfrak{r})+5}}$. *After calculation we have*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.15,$$

and

$$\gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}) = 1.2,$$

inequality (14) holds.

Case 2. The inequality (13) holds for all $\mathfrak{q}, \mathfrak{r} \in \{(-1,0),(-1,-1)\}$ *and*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.18,$$

for $\mathfrak{q} = (-1,0)$ *and* $\mathfrak{r} = (-1,-1)$ *we have*

$$\gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}) = 1.4,$$

when $\mathfrak{q} = (-1,-1)$ *and* $\mathfrak{r} = (-1,0)$ *we have*

$$\gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}) = 2.4,$$

inequality (14) holds.

Case 3. If $\mathfrak{q} = (-1,1)$ *and* $\mathfrak{r} = (-1,0)$ *then inequality (13) holds and after simple calculation we have*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.18,$$

and

$$\gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}) = 1.2,$$

inequality (14) holds. If we choose $\mathfrak{q} = (-1,0)$ *and* $\mathfrak{r} = (-1,1)$ *then the inequality (13) does not holds. This shows that the pair* $(\mathfrak{g}, M)$ *satisfy Suzuki-type* $(\alpha, \beta, \gamma_{\mathfrak{g}})-$*generalized proximal contractive condition; further remaining conditions of Theorem 3 holds, therefore the pair* $(\mathfrak{g}, M)$ *has two coincidence best proximity points* $(-1,1)$ *and* $(-1,-1)$. *Please note that in this example the contractive condition of Theorem 3.1 of M. Gabeleh ([15]) does not hold. Indeed,* $\mathfrak{q} = (-1,1)$ *and* $\mathfrak{r} = (-1,0)$ *we have*

$$1 = \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma\mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta\mathcal{D}^*(M\mathfrak{q}, \mathfrak{r}) = 0.7.$$

M. Gabeleh in ([15]) proved the best proximity point results but did not discussed the uniqueness of the best proximity point results. In this paper, we will need an additional condition C (2) to prove the uniqueness of coincidence best proximity point results for Suzuki-type generalized proximal contractions.

Theorem 4. *Let $\mathcal{Q}$ and $\mathcal{R}$ be nonempty closed subsets of a complete metric space (Y, d) such that the pair $(\mathcal{Q}, \mathcal{R})$ is a fairly complete space and satisfies the $\mathcal{P}-$property. Consider a pair $(\mathfrak{g}, M)$ satisfying Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contractive condition with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$ where $\mathfrak{g}$ is a one-to-one mapping and satisfies the α_R-property. Mapping M is an $\alpha-$proximal admissible further suppose that there exist some $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that*

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1.$$

If the constants β and γ satisfy the condition C (2) then the pair $(\mathfrak{g}, M)$ possesses a unique coincidence best proximity point.

Proof. Following arguments similar to those in the proof of Theorem 3, we get the existence of the coincidence best proximity point of the pair of mappings $(\mathfrak{g}, M)$. Now, we must prove the uniqueness of coincidence best proximity point of the pair of mappings $(\mathfrak{g}, M)$. On contrary suppose that $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{Q}$ are two coincidence best proximity points of the pair of mappings $(\mathfrak{g}, M)$ with $\mathfrak{q}_1 \neq \mathfrak{q}_2$ that is

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) = \mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) = \mathsf{d}(\mathcal{Q}, \mathcal{R}),$$

the pair $(\mathcal{Q}, \mathcal{R})$ possesses the $\mathcal{P}-$property and $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ is a one-to-one mapping, we can write

$$\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) = \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) \neq 0. \tag{15}$$

Since $1 + \beta + \gamma \geq 1$ and $\frac{1}{1+\beta+\gamma} \leq 1$ we have

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) \leq \mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) = \mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) = 0 < \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_1).$$

As $\alpha(\mathfrak{q}_2, \mathfrak{q}_1) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfies Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_1) \leq \alpha(\mathfrak{q}_2, \mathfrak{q}_1)\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_1) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_1) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_2),$$

from above inequality it can be written as

$$\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_1) \leq \gamma\mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_1) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_2),$$

and by using Equation (15) the above inequality becomes

$$\begin{aligned}\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_1) &\leq \gamma\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) + \beta[\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) + \mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\ &\leq \gamma\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) + \beta\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2).\end{aligned}$$

After simple calculation we have $1 \leq \gamma + \beta$ which is a contradiction. Hence, $\mathfrak{q}_1 = \mathfrak{q}_2$ and the pair $(\mathfrak{g}, M)$ possesses a unique coincidence best proximity point. □

Let us visualize Theorem 4 with the example which follows.

Example 2. *Let $Y = \mathbb{R}^2$ be a metric space with Euclidean metric d. Suppose that $\mathcal{Q} = \mathcal{Q}_0 = \{(-1,1), (-1,0), (-1,-1)\}$ and $\mathcal{R} = \mathcal{R}_0 = \{(-4,1), (-4,0), (-4,-1)\}$ are nonempty subsets of Y.*

After calculation we can see that $\mathsf{d}(\mathcal{Q}, \mathcal{R}) = 3$ *and the pair* $(\mathcal{Q}, \mathcal{R})$ *satisfies the* $\mathcal{P}-$*property. Define mappings* $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$, $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ *as:*

$$\mathfrak{g}(\mathfrak{q}) = \begin{cases} (-1,1), \text{ if } \mathfrak{q} \in \{(-1,-1)\} \\ (-1,0), \text{ if } \mathfrak{q} \in \{(-1,0)\} \\ (-1,-1), \text{ if } \mathfrak{q} \in \{(-1,1)\}, \end{cases}$$

and

$$M(\mathfrak{q}) = \begin{cases} \{(-4,0),(-4,1)\}, \text{ if } \mathfrak{q} \in \{(-1,-1)\} \\ \{(-4,1)\}, \text{ else}, \end{cases}$$

clearly $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ *and* $\mathfrak{g}$ *is an one-to-one mapping. Pair* $(\mathfrak{g}, M)$ *satisfy the Suzuki-type* $(\alpha, \beta, \gamma_{\mathfrak{g}})-$*generalized proximal contraction for* $\beta = 0$, $\gamma = 1$ *such that* $0 < \beta + \gamma \leq 1$ *and for function* $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ *defined as* $\alpha(\mathfrak{q}, \mathfrak{r}) = 1$. *Hence all the conditions of Theorem 4 hold and* $\mathfrak{q} = (-1,-1)$ *is a unique coincidence best proximity point of the pair of mappings* $(\mathfrak{g}, M)$.

The coincidence best proximity point results discussed below can be obtained directly from Theorem 3.

Corollary 1. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Y, d) *such that the pair* $(\mathcal{Q}, \mathcal{R})$ *is a fairly complete space and satisfies the* $\mathcal{P}-$*property. Consider* $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ *and* $M : \mathcal{Q} \to \mathcal{R}$ *satisfy the following, if*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \text{ imply } \alpha(\mathfrak{q}, \mathfrak{r})\, \mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta \mathsf{d}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}), \tag{16}$$

with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$ *where mapping* $\mathfrak{g}$ *satisfies the* α_R-*property and* M *is an* $\alpha-$*proximal admissible mapping. Furthermore, suppose that there exists some* $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ *such that*

$$\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1,$$

where $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$. *Then the pair* $(\mathfrak{g}, M)$ *has a coincidence best proximity point.*

Corollary 2. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Y, d) *such that the pair* $(\mathcal{Q}, \mathcal{R})$ *is a fairly complete space and satisfy the* $\mathcal{P}-$*property. Consider* $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ *be a one-to-one mapping and* $M : \mathcal{Q} \to \mathcal{R}$ *satisfy*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \text{ imply } \alpha(\mathfrak{q}, \mathfrak{r})\, \mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta \mathsf{d}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}), \tag{17}$$

with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$ *where mapping* $\mathfrak{g}$ *satisfies the* α_R-*property and mapping* M *is an* $\alpha-$*proximal admissible. Furthermore, suppose that there exist some* $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ *such that*

$$\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1,$$

where $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$. *Then the pair* $(\mathfrak{g}, M)$ *has unique coincidence best proximity point if the constants* β, γ *satisfies the condition C (2).*

The subsequent result is a best proximity point theorem for the Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contraction in the framework of fairly complete space.

Theorem 5. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Y, d) *such that the pair* $(\mathcal{Q}, \mathcal{R})$ *is a fairly complete space and satisfy the* $\mathcal{P}-$*property. Consider the mapping* M *satisfy the Suzuki-type*

$(\alpha, \beta, \gamma)-$generalized proximal contractive condition with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ and M is an $\alpha-$proximal admissible mapping. Furthermore, suppose that there exist some $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that

$$\mathcal{D}(\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1.$$

Then the mapping M has a best proximity point.

Proof. If we take $\mathfrak{g} = I_{\mathcal{Q}}$ in Theorem 3 then Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal mapping becomes Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal mapping, remaining aspects of Theorem 5 are same as in the proof of Theorem 3. Hence we have a best proximity point of mapping M. □

Corollary 3. *Let $\mathcal{Q}$ and $\mathcal{R}$ be nonempty closed subsets of a complete metric space (Y, d) such that the pair $(\mathcal{Q}, \mathcal{R})$ is a fairly complete space and satisfies the $\mathcal{P}-$property. Consider the mapping M satisfies the Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contractive condition with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ and M is an $\alpha-$proximal admissible mapping. Suppose that there exist some $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that*

$$\mathcal{D}(\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1.$$

Furthermore, if the constants β, γ satisfy the condition C (2) then unique best proximity point of mapping M exists.

The following example will illustrate the result presented in Corollary 3.

Example 3. *Let $Y = \mathbb{R}^2$ be a complete metric space with metric d defined as in Example 1. Suppose that $\mathcal{Q} = \mathcal{Q}_0 = \{(-2, 2), (-2, 1), (-2, -2)\}$ and $\mathcal{R} = \mathcal{R}_0 = \{(-5, 2), (-5, 1), (-5, -2)\}$ are nonempty subsets of Y. After simple calculation, we have $\mathsf{d}(\mathcal{Q}, \mathcal{R}) = 3$ and the pair $(\mathcal{Q}, \mathcal{R})$ satisfy the $\mathcal{P}-$property. Define $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ as*

$$M(\mathfrak{q}) = \begin{cases} \{(-5, 1), (-5, 2)\}, & \text{if } \mathfrak{q} \in \{(-2, -2)\} \\ \{(-5, 2)\} & \text{else,} \end{cases}$$

clearly $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$. Now we must show that the mapping M satisfy Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contractive condition. The subsequent condition of Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contractive condition holds for all $\mathfrak{q}, \mathfrak{r} \in \mathcal{Q}_0$ and for $\beta = 0.4$, $\gamma = 0.6$.

$$\frac{1}{1 + \beta + \gamma} \mathcal{D}^*(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}).$$

Now we must show that the subsequent condition of Suzuki-type $(\alpha, \beta, \gamma)-$generalized proximal contraction holds for all $\mathfrak{q}, \mathfrak{r} \in \mathcal{Q}_0$.

$$\alpha(\mathfrak{q}, \mathfrak{r}) \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{r}, M\mathfrak{q}). \tag{18}$$

Define $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ as

$$\alpha(\mathfrak{q}, \mathfrak{r}) = e^{\frac{1}{\mathsf{d}(\mathfrak{q}, \mathfrak{r})}}.$$

Case 1. If we take $\mathfrak{q}, \mathfrak{r} \in \{(-2, 2), (-2, 1)\}$ then after simple calculation we have

$$\alpha(\mathfrak{q}, \mathfrak{r}) \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 0,$$

inequality (18) holds trivially.
Case 2. If we take $\mathfrak{q}, \mathfrak{r} \in \{(-2, 2), (-2, -2)\}$ then we have

$$\alpha(\mathfrak{q}, \mathfrak{r}) \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.28.$$

For $\mathfrak{q} = (-2, 2)$, $\mathfrak{r} = (-2, -2)$ *we get*

$$\gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{r}, M\mathfrak{q}) = 4,$$

and for $\mathfrak{q} = (-2, -2)$, $\mathfrak{r} = (-2, 2)$ *we get*

$$\gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{r}, M\mathfrak{q}) = 2.4,$$

for any choice of $\mathfrak{q}$ *and* $\mathfrak{r}$, *the inequality (18) holds.*
Case 3. For $\mathfrak{q}, \mathfrak{r} \in \{(-2, 1), (-2, -2)\}$ *we get*

$$\alpha(\mathfrak{q}, \mathfrak{r}) \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.4.$$

For $\mathfrak{q} = (-2, 1)$, $\mathfrak{r} = (-2, -2)$ *we have*

$$\gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{r}, M\mathfrak{q}) = 3.4,$$

and for $\mathfrak{q} = (-2, -2)$, $\mathfrak{r} = (-2, 1)$ *we have*

$$\gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{r}, M\mathfrak{q}) = 1.8,$$

inequality (18) holds. Hence, M satisfy Suzuki-type $(\alpha, \beta, \gamma)-$*generalized proximal contraction, remaining aspects of Theorem 3 are fulfilled. Therefore, the mapping M has unique best proximity point* $(-2, 2)$.

The following results are the nice consequences of Theorem 5.

Corollary 4. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Υ, d) *such that the pair* $(\mathcal{Q}, \mathcal{R})$ *is a fairly complete space and satisfy the* $\mathcal{P}-$*property. Consider an* $\alpha-$*proximal admissible mapping* $M : \mathcal{Q} \to \mathcal{R}$ *satisfy the following contractive condition*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}^*(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \text{ imply } \alpha(\mathfrak{q}, \mathfrak{r}) \mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathsf{d}^*(\mathfrak{r}, M\mathfrak{q}), \tag{19}$$

with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$. *Furthermore, suppose that there exist some* $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ *such that*

$$\mathsf{d}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1,$$

where $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$. *Then the mapping M has a best proximity point.*

Corollary 5. *If we add the condition (2) to the statement of Corollary 4 we obtain that the mapping M possesses a unique best proximity point.*

3. Suzuki-Type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$Modified Proximal Contractive Mapping

We begin this section with the subsequent definitions.

Definition 9. *1. A pair of mappings* $(\mathfrak{g}, M)$ *where* $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ *and* $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ *is said to be Suzuki-type* $(\alpha, \beta, \gamma_{\mathfrak{g}})-$*modified proximal contraction if* $\gamma \in (0, 1]$ *such that* $0 \leq \beta < \gamma$ *and*

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \text{ imply } \alpha(\mathfrak{q}, \mathfrak{r}) \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta \mathcal{D}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}).$$

2. A mapping $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is said to be Suzuki-type $(\alpha, \beta, \gamma)-$modified proximal contraction if $\gamma \in (0, 1]$ such that $0 \leq \beta < \gamma$ and

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ imply } \alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma\mathcal{D}^*(\mathfrak{q}, M\mathfrak{q}) + \beta\mathcal{D}^*(M\mathfrak{q}, \mathfrak{r}),$$

where $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ and we have $\alpha(\mathfrak{q}, \mathfrak{r}) \geq 1$ for all $\mathfrak{q}, \mathfrak{r} \in \mathcal{Q}$.

The following result is a coincidence best proximity point theorem for Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contraction in the setting of a fairly complete space.

Theorem 6. *Let $\mathcal{Q}$ and $\mathcal{R}$ be nonempty closed subsets of a complete metric space (Y, d) such that the pair $(\mathcal{Q}, \mathcal{R})$ is fairly complete space. Consider the pair of mappings $(\mathfrak{g}, M)$ satisfy Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contraction with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$. Set $\mathcal{R}$ has the property of uniform $M-$approximation in set $\mathcal{Q}$ and mapping $\mathfrak{g}$ satisfy the α_R-property. Furthermore, assume the existence of some $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that $\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$ and $\alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1$. Then the pair $(\mathfrak{g}, M)$ possesses a coincidence best proximity point.*

Proof. If we follow the steps of Theorem 3 then we obtain a sequence $\{\mathfrak{q}_\mathfrak{n}\}$ in $\mathcal{Q}_0$ such that $\mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, M\mathfrak{q}_\mathfrak{n}) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$ and $\alpha(\mathfrak{q}_\mathfrak{n}, \mathfrak{q}_{\mathfrak{n}+1}) \geq 1$ with $\mathfrak{q}_\mathfrak{n} \neq \mathfrak{q}_{\mathfrak{n}+1}$. Since $1 + \beta + \gamma \geq 1$ and $\frac{1}{1+\beta+\gamma} \leq 1$, so for $\mathfrak{q}_1$ we have

$$\begin{aligned} \frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) \leq \mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) &= \mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) + \mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_1) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2), \end{aligned}$$

above inequality can be written as

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2).$$

Since $\alpha(\mathfrak{q}_1, \mathfrak{q}_2) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfy Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \alpha(\mathfrak{q}_1, \mathfrak{q}_2)\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_1) + \beta\mathcal{D}^*(M\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2),$$

after simplification we have the following

$$\begin{aligned} \mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) &\leq \gamma[\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) + \mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] + \beta[\mathcal{D}(M\mathfrak{q}_1, \mathfrak{g}\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\ &= \gamma\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1). \end{aligned} \tag{20}$$

On the same lines we can verify for $\mathfrak{q}_2$

$$\begin{aligned} \frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) \leq \mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) &= \mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) + \mathcal{D}(\mathfrak{g}\mathfrak{q}_3, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3), \end{aligned}$$

above inequality becomes

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3).$$

Since $\alpha(\mathfrak{q}_2, \mathfrak{q}_3) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfy Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) \leq \alpha(\mathfrak{q}_2, \mathfrak{q}_3)\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_2) + \beta\mathcal{D}^*(M\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3),$$

from above inequality we have

$$\begin{aligned} \mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) &\leq \gamma[\mathcal{D}(\mathfrak{g}\mathfrak{q}_2, M\mathfrak{q}_1) + \mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] + \beta[\mathcal{D}(M\mathfrak{q}_2, \mathfrak{g}\mathfrak{q}_3) - \mathsf{d}(\mathcal{Q}, \mathcal{R})] \\ &= \gamma\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2), \end{aligned}$$

using inequality (20), above inequality becomes

$$\mathcal{H}(M\mathfrak{q}_2, M\mathfrak{q}_3) \leq \gamma\mathcal{H}(M\mathfrak{q}_1, M\mathfrak{q}_2) \leq \gamma^2\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1).$$

Thus, for a sequence $\{M\mathfrak{q}_\mathfrak{n}\}$ in $\mathcal{R}_0$, we have

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}+1}) \leq \gamma^\mathfrak{n}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1). \tag{21}$$

Therefore

$$\sum_{\mathfrak{n}=1}^{\infty} \mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}+1}) \leq \sum_{\mathfrak{n}=1}^{\infty} \gamma^\mathfrak{n}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1),$$

which implies that $\{M\mathfrak{q}_\mathfrak{n}\}$ is a Cauchy sequence and $(\mathcal{Q}, \mathcal{R})$ is a pair of nonempty closed subsets of a complete metric space (Y, d), $\{M\mathfrak{q}_\mathfrak{n}\}$ converges to some point $\mathfrak{q} \in \mathcal{R}$. Therefore, we have for any $\delta > 0$

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}+1}) < \delta.$$

Since the set $\mathcal{R}$ has the property of uniform $M-$approximation in set $\mathcal{Q}$ which implies that $\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+2}) < \epsilon$, hence $\{\mathfrak{g}\mathfrak{q}_\mathfrak{n}\}$ is a Cauchy sequence and converges to $\mathfrak{g}p \in \mathcal{Q}$ and we have

$$\mathsf{d}(\mathfrak{g}p, \mathfrak{q}) = \lim_{\mathfrak{n}\to\infty} \mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}-1}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}).$$

Like Theorem 3 we can prove that $\{(\mathfrak{g}\mathfrak{q}_\mathfrak{m}, M\mathfrak{q}_\mathfrak{n})\}$ is a cyclically Cauchy sequence. Since $\mathfrak{g}\mathfrak{q}_\mathfrak{n} \to \mathfrak{g}p$ there exists $N_1 \in \mathbb{N}$ such that

$$\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) \leq \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p),$$

for all $\mathfrak{n} \geq N_1 \in \mathbb{N}$. Now we can write for $\mathfrak{q}_\mathfrak{n}$

$$\begin{aligned} \frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) &\leq \mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}) + \mathcal{D}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, M\mathfrak{q}_\mathfrak{n}) - \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \mathsf{d}(\mathfrak{g}p, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}) \\ &\leq \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) + \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) \\ &= \frac{2}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) \\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) - \frac{1}{3}\mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) \\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}p) - \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}\mathfrak{q}). \end{aligned}$$

Since $\alpha(\mathfrak{q}_\mathfrak{n}, \mathfrak{q}) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfies Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contraction which implies that

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}) \leq \alpha(\mathfrak{q}_\mathfrak{n}, \mathfrak{q})\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}_\mathfrak{n}),$$

after simplification, we have

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}_\mathfrak{n}). \tag{22}$$

We can write

$$\mathcal{D}(\mathfrak{g}p, M\mathfrak{q}) = \lim_{\mathfrak{n}\to\infty} \mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}) \leq \lim_{\mathfrak{n}\to\infty}[\mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}-1}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{n}-1}, M\mathfrak{q}_\mathfrak{n}) + \mathcal{H}(M\mathfrak{q}_\mathfrak{n}, M\mathfrak{q})],$$

using inequalities (21) and (22), above inequality becomes

$$\begin{aligned}\mathcal{D}(\mathfrak{g}p, M\mathfrak{q}) &\leq \lim_{\mathfrak{n}\to\infty}[\mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma^{\mathfrak{n}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}_\mathfrak{n})] \\ &= \mathsf{d}(\mathcal{Q},\mathcal{R}) + \gamma\mathsf{d}^*(\mathfrak{g}p, q) + \beta\mathsf{d}^*(\mathfrak{g}\mathfrak{q}, q).\end{aligned}$$

After simplification, above inequality can be written as following

$$\mathcal{D}^*(\mathfrak{g}p, M\mathfrak{q}) \leq \gamma\mathsf{d}^*(\mathfrak{g}p, q) + \beta\mathsf{d}^*(\mathfrak{g}\mathfrak{q}, q). \tag{23}$$

Using triangular inequality, we have

$$\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \mathcal{D}^*(\mathfrak{g}p, M\mathfrak{q}_\mathfrak{n}),$$

using inequality (23), above inequality becomes

$$\begin{aligned}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \gamma\mathsf{d}^*(\mathfrak{g}p, q) + \beta\mathsf{d}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, q) \\ &\leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \gamma[\mathsf{d}(\mathfrak{g}p, q) - \mathsf{d}(\mathcal{Q},\mathcal{R})] + \beta[\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \mathsf{d}(\mathfrak{g}p, q) - \mathsf{d}(\mathcal{Q},\mathcal{R})] \\ &= \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) + \beta\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p) \\ &\leq (1+\beta+\gamma)\mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p),\end{aligned}$$

after further simplification, we can write the above inequality as

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p).$$

Since $\alpha(\mathfrak{q}_\mathfrak{n}, p) \geq 1$ and the pair $(\mathfrak{g}, M)$ satisfies Suzuki-type $(\alpha, \beta, \gamma_\mathfrak{g})-$modified proximal contraction, which implies that

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) \leq \alpha(\mathfrak{q}_\mathfrak{n}, p)\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_\mathfrak{n}) + \beta\mathcal{D}^*(M\mathfrak{q}_\mathfrak{n}, \mathfrak{g}p),$$

above inequality becomes

$$\begin{aligned}\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) \leq\; &\gamma[\mathcal{D}(\mathfrak{g}\mathfrak{q}_\mathfrak{n}, M\mathfrak{q}_{\mathfrak{n}-1}) + \mathcal{H}(M\mathfrak{q}_{\mathfrak{n}-1}, M\mathfrak{q}_\mathfrak{n}) - \mathsf{d}(\mathcal{Q},\mathcal{R})] \\ &+\beta[\mathcal{D}(M\mathfrak{q}_\mathfrak{n}, \mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}) + \mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, \mathfrak{g}p) - \mathsf{d}(\mathcal{Q},\mathcal{R})],\end{aligned}$$

using inequality (21), we can write it as

$$\mathcal{H}(M\mathfrak{q}_\mathfrak{n}, Mp) \leq \gamma^{\mathfrak{n}-1}\mathcal{H}(M\mathfrak{q}_0, M\mathfrak{q}_1) + \beta\mathsf{d}(\mathfrak{g}\mathfrak{q}_{\mathfrak{n}+1}, \mathfrak{g}p),$$

since $\mathfrak{g}\mathfrak{q}_\mathfrak{n} \to \mathfrak{g}p$. In above relation if $\mathfrak{n} \to \infty$ then we conclude that $M\mathfrak{q}_\mathfrak{n} \to Mp$, that is, $q = Mp$ and we have

$$\mathcal{D}(\mathfrak{g}p, Mp) = \mathsf{d}(\mathcal{Q},\mathcal{R}).$$

Hence, p is a coincidence best proximity point of the pair $(\mathfrak{g}, M)$. □

The following example is given to support the usability of Theorem 6.

Example 4. *Let $Y = \mathbb{R}$ with metric d be defined as $\mathsf{d}(\mathfrak{q}, \mathfrak{r}) = |\mathfrak{q} - \mathfrak{r}|$. Also suppose that $\mathcal{Q} = \{2, 4, 6, 9, 12\}$ and $\mathcal{R} = \{3, 5, 7, 8, 14\}$ are the nonempty subsets of Y. We have $\mathsf{d}(\mathcal{Q}, \mathcal{R}) = 1$, $\mathcal{Q}_0 = \{2, 4, 6, 9\}$ and $\mathcal{R}_0 = \{3, 5, 7, 8\}$, further the pair $(\mathcal{Q}, \mathcal{R})$ does not satisfy the $\mathcal{P}-$property. Now consider mappings $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$, $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ be defined as:*

$$M(\mathfrak{q}) = \begin{cases} \{7,8\}, \text{ if } \mathfrak{q} \in \{4\} \\ \{8\}, \text{ else} \end{cases}$$

and

$$\mathfrak{g}(\mathfrak{q}) = \begin{cases} \frac{\mathfrak{q}}{2}, \text{ if } \mathfrak{q} \in \{4\} \\ 2\mathfrak{q}, \text{ if } \mathfrak{q} \in \{2\} \\ \mathfrak{q}+3, \text{ if } \mathfrak{q} \in \{6\} \\ \mathfrak{q}-3, \text{ if } \mathfrak{q} \in \{9\}, \end{cases}$$

clearly $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ and $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$.
If we choose $\mathfrak{q} = 4$ and $\mathfrak{r} = 2$ then after simple calculation we can show that the following inequality

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}),$$

does not hold and for all the remaining cases above contraction holds for $\beta = 0.3$, $\gamma = 0.4$. Now it must be shown that the subsequent condition of Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$modified proximal contractive condition holds.

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta\mathcal{D}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}). \tag{24}$$

Define $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ by

$$\alpha(\mathfrak{q}, \mathfrak{r}) = e^{\frac{1}{\mathfrak{q}\mathfrak{r}}}.$$

Case 1. If we take $\mathfrak{q} = 2$, $\mathfrak{r} = 4$ then after calculation we have

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.13,$$

and

$$\gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta\mathcal{D}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 2.7,$$

inequality (24) holds.
Case 2. If $\mathfrak{q}, \mathfrak{r} \in \{2, 6, 9\}$ then after simple calculation we have

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 0,$$

inequality (24) holds trivially.
Case 3. Now consider $\mathfrak{q}, \mathfrak{r} \in \{4, 6\}$ we have

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) = 1.04,$$

for $\mathfrak{q} = 4$, $\mathfrak{r} = 6$, we have

$$\gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta\mathcal{D}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 1.6,$$

and for $\mathfrak{q} = 6$, $\mathfrak{r} = 4$, after calculation we have

$$\gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta\mathcal{D}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 1.5,$$

inequality (24) holds.
Case 4. If $\mathfrak{q},\mathfrak{r}\in\{4,9\}$ *then we have*

$$\alpha\left(\mathfrak{q},\mathfrak{r}\right)\mathcal{H}(M\mathfrak{q},M\mathfrak{r})=1.03,$$

for $\mathfrak{q}=4$, $\mathfrak{r}=9$, *we have*

$$\gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q},M\mathfrak{q})+\beta\mathcal{D}^*(M\mathfrak{q},\mathfrak{g}\mathfrak{r})=1.6,$$

for $\mathfrak{q}=9$, $\mathfrak{r}=4$, *we have*

$$\gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q},M\mathfrak{q})+\beta\mathcal{D}^*(M\mathfrak{q},\mathfrak{g}\mathfrak{r})=1.9,$$

inequality (24) holds. Hence all the conditions of Theorem 6 hold and $p=6$ *is a coincidence best proximity point of the pair* $(\mathfrak{g},M)$*. Please note that in above example, contractive condition of M. Gabeleh ([15]) is not satisfied and so this is not applicable here. Indeed* $\mathfrak{q}=4$ *and* $\mathfrak{r}=6$ *we get*

$$\mathcal{H}(M\mathfrak{q},M\mathfrak{r})=1>0.98=\gamma\mathsf{d}(\mathfrak{q},\mathfrak{r})+\beta\mathcal{D}^*(M\mathfrak{q},\mathfrak{r}).$$

The next coincidence best proximity point result follows from Theorem 6 directly.

Corollary 6. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Y,d) *such that the pair* $(\mathcal{Q},\mathcal{R})$ *is a fairly complete space. Consider mappings* $\mathfrak{g}:\mathcal{Q}\to\mathcal{Q}$ *and* $M:\mathcal{Q}\to\mathcal{R}$ *satisfy the following contractive condition*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}^*(\mathfrak{g}\mathfrak{q},M\mathfrak{q})\leq\mathsf{d}(\mathfrak{g}\mathfrak{q},\mathfrak{g}\mathfrak{r})\text{ imply }\alpha\left(\mathfrak{q},\mathfrak{r}\right)\mathsf{d}(M\mathfrak{q},M\mathfrak{r})\leq\gamma\mathsf{d}^*(\mathfrak{g}\mathfrak{q},M\mathfrak{q})+\beta\mathsf{d}^*(M\mathfrak{q},\mathfrak{g}\mathfrak{r}),\tag{25}$$

with $M(\mathcal{Q}_0)\subseteq\mathcal{R}_0$, $\mathcal{Q}_0\subseteq\mathfrak{g}(\mathcal{Q}_0)$*. Set* $\mathcal{R}$ *has the property of uniform* $M-$*approximation in set* $\mathcal{Q}$ *and mapping* $\mathfrak{g}$ *satisfies the* α_R-*property. Furthermore, suppose that there exist some* $\mathfrak{q}_0,\mathfrak{q}_1\in\mathcal{Q}_0$ *such that* $\mathsf{d}(\mathfrak{g}\mathfrak{q}_1,M\mathfrak{q}_0)=\mathsf{d}(\mathcal{Q},\mathcal{R})$ *and* $\alpha(\mathfrak{q}_0,\mathfrak{q}_1)\geq 1$ *where* $\alpha:\mathcal{Q}\times\mathcal{Q}\to[0,\infty)$*. Then the pair* $(\mathfrak{g},M)$ *has a coincidence best proximity point.*

Next example is given to corroborates the usability of Corollary 6.

Example 5. *Consider* $\mathcal{Q}=\{-5,0,5,7,8\}$ *and* $\mathcal{R}=\{-3,2,3,11,12\}$ *are subsets of* $Y=\{-5,-4,\ldots,11,12\}$ *with metric defined as in Example 4. After calculation we have* $\mathsf{d}(\mathcal{Q},\mathcal{R})=2$, $\mathcal{Q}_0=\{-5,0,5\}$ *and* $\mathcal{R}_0=\{-3,2,3\}$*. Consider mappings* $M:\mathcal{Q}\to\mathcal{R}$, $\mathfrak{g}:\mathcal{Q}\to\mathcal{Q}$ *defined as follows:*

$$\begin{aligned}M(\mathfrak{q}) &= \begin{cases}2, & \text{if } \mathfrak{q}\in\{5\}\\ 3, & \text{else}\end{cases}\\ &\text{and}\\ \mathfrak{g}(\mathfrak{q}) &= -\mathfrak{q},\end{aligned}$$

clearly $M(\mathcal{Q}_0)\subseteq\mathcal{R}_0$, $\mathcal{Q}_0\subseteq\mathfrak{g}(\mathcal{Q}_0)$ *and the pair* $(\mathcal{Q},\mathcal{R})$ *do not satisfy the* $\mathcal{P}-$*property. Now we must show that the pair* $(\mathfrak{g},M)$ *satisfy the inequality (25). The following part of inequality (25) holds for all* $\mathfrak{q},\mathfrak{r}\in\mathcal{Q}_0$ *for* $\beta=0.46$ *and* $\gamma=0.47$*.*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}^*(\mathfrak{g}\mathfrak{q},M\mathfrak{q})\leq\mathsf{d}(\mathfrak{g}\mathfrak{q},\mathfrak{g}\mathfrak{r}).$$

Now it must be shown that the subsequent condition of a Suzuki-type $(\alpha,\beta,\gamma_{\mathfrak{g}})-$*modified proximal contractive condition holds for all* $\mathfrak{q},\mathfrak{r}\in\mathcal{Q}_0$*.*

$$\alpha\left(\mathfrak{q},\mathfrak{r}\right)\mathsf{d}(M\mathfrak{q},M\mathfrak{r})\leq\gamma\mathsf{d}^*(\mathfrak{g}\mathfrak{q},M\mathfrak{q})+\beta\mathsf{d}^*(M\mathfrak{q},\mathfrak{g}\mathfrak{r}).\tag{26}$$

Define $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ *by*

$$\alpha(\mathfrak{q}, \mathfrak{r}) = \begin{cases} e^{|\frac{\mathfrak{q}}{\mathfrak{r}}|}, \text{ if } \mathfrak{q}, \mathfrak{r} \in \{-5, 0, 5\} \text{ and } \mathfrak{r} \neq 0 \\ 2, \text{ if } \mathfrak{r} = 0. \end{cases}$$

Case 1. For $\mathfrak{q}, \mathfrak{r} \in \{-5, 0\}$ *we have*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) = 0,$$

inequality (26) holds trivially.
Case 2. If $\mathfrak{q}, \mathfrak{r} \in \{-5, 5\}$ *then we have*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) = 2.71,$$

and

$$\gamma \mathsf{d}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta \mathsf{d}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 2.8,$$

inequality (26) holds.
Case 3. If $\mathfrak{q} = 0$ *and* $\mathfrak{r} = 5$ *we get*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) = 1,$$

and

$$\gamma \mathsf{d}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta \mathsf{d}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 3.23,$$

if $\mathfrak{q} = 5$ *and* $\mathfrak{r} = 0$ *then*

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) = 2,$$

and

$$\gamma \mathsf{d}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta \mathsf{d}^*(M\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 2.35,$$

inequality (26) holds. This shows that the pair $(\mathfrak{g}, M)$ *satisfies the inequality (25), further remaining conditions of Corollary 6 hold true. Hence the pair* $(\mathfrak{g}, M)$ *has a coincidence best proximity point* $\mathfrak{q} = -5$.

The subsequent result is a best proximity point theorem for the Suzuki-type $(\alpha, \beta, \gamma)-$modified proximal contraction in the framework of fairly complete space.

Theorem 7. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Y, d) *such that the pair is a fairly complete space. Consider a mapping* M *is an* $\alpha-$*proximal admissible and satisfy Suzuki-type* $(\alpha, \beta, \gamma)-$*modified proximal contraction with* $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$. *Further set* $\mathcal{R}$ *has the property of uniform* $M-$*approximation in set* $\mathcal{Q}$ *and suppose that there exist some* $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ *such that* $\mathcal{D}(\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$ *with* $\alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1$. *Then mapping* M *possesses a best proximity point.*

Proof. If we take $\mathfrak{g} = I_{\mathcal{Q}}$ in Theorem 6, the remaining aspects follow from the same lines. □

The next best proximity point result directly follows from Theorem 7.

Corollary 7. *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be nonempty closed subsets of a complete metric space* (Y, d) *such that the pair* $(\mathcal{Q}, \mathcal{R})$ *is a fairly complete space. Consider a mapping* $M : \mathcal{Q} \to \mathcal{R}$ *satisfies the following contractive condition*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}^*(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \text{ imply } \alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathsf{d}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}^*(\mathfrak{q}, M\mathfrak{q}) + \beta \mathsf{d}^*(M\mathfrak{q}, \mathfrak{r}), \tag{27}$$

with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ *and set* $\mathcal{R}$ *has the property of uniform* $M-$*approximation in set* $\mathcal{Q}$. *Furthermore, suppose that there exist some* $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ *such that* $\mathsf{d}(\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R})$ *and* $\alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1$ *where* $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$. *Then mapping* M *has a best proximity point.*

4. Some Results Related to Partially Ordered Metric Space

Here, we are concerned with coincidence best proximity point results for generalized and modified Suzuki-type contractions in partially ordered metric space.

From now and onward Δ defines:

$$\Delta = \{\mathfrak{q}, \mathfrak{r} \in \mathcal{Q} \text{ such that } \mathfrak{q} \preceq \mathfrak{r} \text{ or } \mathfrak{r} \preceq \mathfrak{q}\}.$$

Definition 10. *([22]) Suppose Y be a nonempty set, a triplet $(Y, \mathsf{d}, \preceq)$ is called a partially ordered metric space if it satisfies the following conditions:*

1. d *is metric on* Y.
2. $\preceq$ *is partial order on* Y.

Definition 11. *[22] A mapping $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is proximally order-preserving, if*

$$\left.\begin{array}{l} \mathfrak{r}_1 \preceq \mathfrak{r}_2 \\ \mathcal{D}(\mathfrak{q}_1, M\mathfrak{r}_1) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathcal{D}(\mathfrak{q}_2, M\mathfrak{r}_2) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \end{array}\right\} \text{imply } \mathfrak{q}_1 \preceq \mathfrak{q}_2,$$

for all $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{r}_1$ and $\mathfrak{r}_2 \in \mathcal{Q}$.

Definition 12. *A pair $(\mathfrak{g}, M)$ where $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ and $M : \mathcal{Q} \to \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is an:*

1. *Ordered Suzuki-type $(\beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction if*

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \text{ imply } \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}).$$

2. *Ordered Suzuki-type $(\beta, \gamma_{\mathfrak{g}})-$modified proximal contraction if*

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \text{ imply } \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) + \beta\mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}),$$

for all $(\mathfrak{q}, \mathfrak{r}) \in \Delta$ and $\gamma \in (0, 1]$ such that $0 \leq \beta < \gamma$.

Theorem 8. *Let $\mathcal{Q}$ and $\mathcal{R}$ are nonempty closed subsets of complete partially ordered metric space $(Y, \mathsf{d}, \preceq)$. Suppose that the pair $(\mathfrak{g}, M)$ is an ordered Suzuki-type $(\beta, \gamma_{\mathfrak{g}})-$generalized proximal contractive condition with $M(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$. Mapping $\mathfrak{g}$ satisfies the α_R-property and M is proximally order-preserving. Also, the pair $(\mathcal{Q}, \mathcal{R})$ possesses the $\mathcal{P}-$property. Further let the existence of some $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathcal{Q}_0$ such that*

$$\mathcal{D}(\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } (\mathfrak{q}_0, \mathfrak{q}_1) \in \Delta.$$

Then the pair $(\mathfrak{g}, M)$ possesses a unique coincidence best proximity point.

Proof. Define $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ as

$$\alpha(\mathfrak{q}, \mathfrak{r}) = \begin{cases} 1, \text{ if } (\mathfrak{q}, \mathfrak{r}) \in \Delta, \\ 0, \text{ otherwise.} \end{cases}$$

Also, the mapping M is an $\alpha-$proximal admissible

$$\begin{cases} \alpha(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \geq 1 \\ \mathcal{D}(\mathfrak{g}u, M\mathfrak{q}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathcal{D}(\mathfrak{g}v, M\mathfrak{r}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}), \end{cases}$$

equivalently we have

$$\begin{cases} (\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \in \Delta \\ \mathcal{D}(\mathfrak{g}u, M\mathfrak{q}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \\ \mathcal{D}(\mathfrak{g}v, M\mathfrak{r}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}). \end{cases}$$

As M is proximally order-preserving $(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \in \Delta$ that is, $\alpha(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \geq 1$ we have

$$\mathcal{D}(\mathfrak{g}\mathfrak{q}_1, M\mathfrak{q}_0) = \mathsf{d}(\mathcal{Q}, \mathcal{R}) \text{ and } \alpha(\mathfrak{q}_0, \mathfrak{q}_1) \geq 1.$$

If $(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \in \Delta$ then $\alpha(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 1$ otherwise $\alpha(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) = 0$. As the mapping M satisfies ordered Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized proximal contraction we have

$$\frac{1}{1+\beta+\gamma}\mathcal{D}^*(\mathfrak{g}\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) \text{ and } \alpha(\mathfrak{q}, \mathfrak{r}) \geq 1,$$

which implies

$$\alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{r}) + \beta \mathcal{D}^*(\mathfrak{g}\mathfrak{r}, M\mathfrak{q}).$$

Let us consider $\{\mathfrak{q}_\mathfrak{n}\}$ as a sequence then $\alpha(\mathfrak{q}_\mathfrak{n}, \mathfrak{q}_{\mathfrak{n}+1}) \geq 1$ with $\mathfrak{q}_\mathfrak{n} \to \mathfrak{q}$ as $\mathfrak{n} \to \infty$ then it follows that $(\mathfrak{q}_\mathfrak{n}, \mathfrak{q}_{\mathfrak{n}+1}) \in \Delta$ with $\mathfrak{q}_\mathfrak{n} \to \mathfrak{q}$ as $\mathfrak{n} \to \infty$. Hence remaining conditions of Theorem 3 fulfilled so that pair $(\mathfrak{g}, M)$ possesses a coincidence best proximity point. □

Theorem 9. *Let $\mathcal{Q}$ and $\mathcal{R}$ are the same sets as in Theorem 8. Suppose that the pair $(\mathfrak{g}, M)$ where $\mathfrak{g} : \mathcal{Q} \to \mathcal{Q}$ and $M : \mathcal{Q} \to \mathcal{C}_\mathcal{B}(\mathcal{R})$ satisfies an ordered Suzuki-type $(\beta, \gamma_\mathfrak{g})-$modified proximal contractive condition with all assumptions of Theorem 8. Then the pair $(\mathfrak{g}, M)$ possesses a unique coincidence best proximity point.*

5. Application to Fixed-Point Theory

Here, we will discuss some results about the fixed-point theory for generalized and modified Suzuki-type contraction.

If $\mathcal{Q} = \mathcal{R} = Y$ then the following contractive conditions can be define.

Definition 13. *A mapping $M : Y \to \mathcal{C}_\mathcal{B}(Y)$ is called a:*

1. *Suzuki-type $(\alpha, \beta, \gamma)-$generalized contraction if*

$$\frac{1}{1+\beta+\gamma}\mathcal{D}(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ imply } \alpha\,(\mathfrak{q}, \mathfrak{r})\,\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathcal{D}(\mathfrak{r}, M\mathfrak{q}).$$

2. *Suzuki-type $(\alpha, \beta, \gamma)-$modified contraction if*

$$\frac{1}{1+\beta+\gamma}\mathcal{D}(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ imply } \alpha(\mathfrak{q}, \mathfrak{r})\mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathcal{D}(\mathfrak{q}, M\mathfrak{q}) + \beta \mathcal{D}(\mathfrak{r}, M\mathfrak{q}),$$

where $\alpha : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ and we have $\alpha(\mathfrak{q}, \mathfrak{r}) \geq 1$ for all $\mathfrak{q}, \mathfrak{r} \in Y$ and $\gamma \in (0, 1]$ such that $0 \leq \beta < \gamma$.

From Theorems 5 and 7 we can find following new fixed-point results.

Theorem 10. *Suppose that if there exists $\mathfrak{q}_0$ with $\alpha(\mathfrak{q}_0, M\mathfrak{q}_0) \geq 1$ then the mapping $M : Y \to \mathcal{C}_\mathcal{B}(Y)$ which satisfy Suzuki-type $(\alpha, \beta, \gamma)-$generalized contractive condition on a complete metric space (Y, d) has a unique fixed point.*

Proof. If we take $\mathcal{Q} = \mathcal{R} = Y$ in Theorem 5 then proximal Suzuki-type $(\alpha, \beta, \gamma)-$generalized contraction implies Suzuki-type $(\alpha, \beta, \gamma)-$generalized contraction. According to Theorem 5 we can find point $\mathfrak{q}$ which satisfies

$$\mathcal{D}(\mathfrak{q}, M\mathfrak{q}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}).$$

However, here we have $\mathcal{Q} = \mathcal{R} = Y$, so we have $\mathcal{D}(\mathfrak{q}, M\mathfrak{q}) = 0$ and there exists a fixed point $\mathfrak{q}$ of Suzuki-type $(\alpha, \beta, \gamma)-$generalized contraction of mapping M. □

Theorem 11. *Suppose that if there exists* $\mathfrak{q}_0$ *with* $\alpha(\mathfrak{q}_0, M\mathfrak{q}_0) \geq 1$ *then the mapping* $M : Y \to \mathcal{C}_{\mathcal{B}}(Y)$ *which satisfy Suzuki-type* $(\alpha, \beta, \gamma)-$*modified contractive condition on a complete metric space* (Y, d) *has unique fixed point.*

Proof. If we take $\mathcal{Q} = \mathcal{R} = Y$ in Theorem 7 then proximal Suzuki-type $(\alpha, \beta, \gamma)-$modified contractive condition implies Suzuki-type $(\alpha, \beta, \gamma)-$modified contractive condition. According to Theorem 7 we can find a point $\mathfrak{q}$ satisfying

$$\mathcal{D}(\mathfrak{q}, M\mathfrak{q}) = \mathsf{d}(\mathcal{Q}, \mathcal{R}),$$

but for self-mapping $\mathcal{Q} = \mathcal{R} = Y$. So, we have $\mathcal{D}(\mathfrak{q}, M\mathfrak{q}) = 0$ and there exists a fixed point $\mathfrak{q}$ of Suzuki-type $(\alpha, \beta, \gamma)-$modified contraction of mapping M. □

Definition 14. *A mapping* $M : Y \to \mathcal{C}_{\mathcal{B}}(Y)$ *is called an:*

1. *Ordered Suzuki-type* $(\beta, \gamma)-$*generalized contraction if*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ imply } \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{q}, \mathfrak{r}) + \beta \mathsf{d}(\mathfrak{r}, M\mathfrak{q}).$$

2. *Ordered Suzuki-type* $(\beta, \gamma)-$*modified contraction if*

$$\frac{1}{1+\beta+\gamma}\mathsf{d}(\mathfrak{q}, M\mathfrak{q}) \leq \mathsf{d}(\mathfrak{q}, \mathfrak{r}) \textit{ imply } \mathcal{H}(M\mathfrak{q}, M\mathfrak{r}) \leq \gamma \mathsf{d}(\mathfrak{q}, M\mathfrak{q}) + \beta \mathsf{d}(\mathfrak{r}, M\mathfrak{q}),$$

for all $(\mathfrak{q}, \mathfrak{r}) \in \Delta$ *where* $\gamma \in (0, 1]$ *such that* $0 \leq \beta < \gamma$.

Theorem 12. *If a mapping* $M : Y \to \mathcal{C}_{\mathcal{B}}(Y)$ *satisfy an ordered Suzuki-type* $(\beta, \gamma)-$*generalized contractive condition with* $\mathfrak{q}_0 \in Y$ *such that* $(\mathfrak{q}_0, M\mathfrak{q}_0) \in \Delta$ *on complete partially ordered metric space* $(Y, \mathsf{d}, \preceq)$ *then mapping* M *has unique fixed point.*

Proof. By following the prove of Theorem 8, we can say that for self mapping every ordered Suzuki-type $(\alpha, \beta, \gamma)-$generalized contractive condition implies ordered Suzuki-type $(\beta, \gamma)-$generalized contractive condition. The remaining aspects of Theorem 8 fulfilled on the same lines and mapping M possesses a unique fixed point. □

Theorem 13. *If a mapping* $M : Y \to \mathcal{C}_{\mathcal{B}}(Y)$ *satisfies an ordered Suzuki-type* $(\beta, \gamma)-$*modified contractive condition with* $\mathfrak{q}_0 \in Y$ *such that* $(\mathfrak{q}_0, M\mathfrak{q}_0) \in \Delta$ *on complete partially ordered metric space* $(Y, \mathsf{d}, \preceq)$ *then mapping* M *possesses a unique fixed point.*

6. Conclusions

In this article, we defined Suzuki-type $(\alpha, \beta, \gamma_{\mathfrak{g}})-$generalized and modified proximal contractive mappings. Furthermore, some coincidence and best proximity point results are obtained in fairly complete spaces, which generalized the result discussed by M. Gabeleh in ([15]). As an application, we obtained some fixed point and coincidence point results in partially ordered metric spaces for modified and generalized Suzuki-type contractions. Some illustrative examples are also provided to visualize and support to the results obtained herein.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund Math.* **1922**, *3*, 133–181. [CrossRef]
2. Saleem, N.; Abbas, M.; Raza, Z. Fixed Fuzzy Point Results of Generalized Suzuki Type $F-$contraction Mappings in Ordered Metric Spaces. Available online: https://www.researchgate.net/publication/321411875_Fixed_fuzzy_point_results_of_generalized_Suzuki_type_F-contraction_mappings_in_ordered_metric_spaces (accessed on 16 March 2020).
3. Saleem, N.; Abbas, M.; Ali, B.; Raza, Z. Fixed points of Suzuki type generalized multivalued mappings in fuzzy metric spaces with applications. *Fixed Point Theory Appl.* **2015**, *1*, 1–18. [CrossRef]
4. Ciric, L. Some Recent Results in Metrical Fixed Point Theory. Available online: https://carma.newcastle.edu.au/resources/jon/Preprints/Papers/CAT(0)/Papers/kirk07.pdf (accessed on 16 March 2020).
5. Todorcevic, V. Harmonic Quasiconformal Mappings and Hyper-bolic Type Metrics. Available online: https://www.springer.com/gp/book/9783030225902 (accessed on 16 March 2020).
6. Sen, M.; Abbas, M.; Saleem, N. On optimal fuzzy best proximity coincidence points of proximal contractions involving cyclic mappings in non-Archimedean fuzzy metric spaces. *Mathematics* **2017**, *5*, 22. [CrossRef]
7. Abbas, M.; Saleem, N.; De la Sen, M. Optimal coincidence point results in partially ordered non-Archimedean fuzzy metric spaces. *Fixed Point Theory Appl.* **2016**, *1*, 44. [CrossRef]
8. Abbas, M.; Saleem, N.; Sohail, K. Optimal coincidence best approximation solution in b-fuzzy metric spaces. *Commun. Nonlinear Anal.* **2019**, *6*, 1–12.
9. Raza, Z.; Saleem, N.; Abbas, M. Optimal coincidence points of proximal quasi-contraction mappings in non-Archimedean fuzzy metric spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3787–3801. [CrossRef]
10. Shatanawi, W.; Mitrovic, Z.; Hussain, N.; Radenovic, S. On Generalized Hardy-Rogers type $\alpha-$admissible mapping in cone b-metric spaces over Banach algebras. *Symmetry* **2020**, *21*, 81. [CrossRef]
11. Abodayeh, K.; Shatanawi, W. Common fixed point for mappings under contractive condition based on almost perfect functions and $\alpha-$admissibility. *Nonlinear Funct. Anal. Appl.* **2018**, *23*, 247–257.
12. Saleem, N.; Abbas, M.; Raza, Z. Optimal coincidence best approximation solution in non-Archimedean fuzzy metric spaces. *Iran. J. Fuzzy Syst.* **2016**, *13*, 113–124.
13. Berinde , V. Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal.* **2004**, *9*, 43–53.
14. Suzuki, T. A new type of fixed point theorem in metric spaces. *Nonlinear Anal.* **2004**, *7*, 5313–5317. [CrossRef]
15. Gabeleh, M. Best proximity point theorems for single and set-valued non-self mappings. *Acta Math. Sci.* **2014**, *34*, 1661–1669. [CrossRef]
16. Basha, S. Best proximity point theorems in the frameworks of fairly and proximally complete spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 1939–1951. [CrossRef]
17. Raj, V.S. A best proximity theorem for weakly contractive non-self mappings. *Nonlinear Anal.* **2011**, *74*, 4804–4808.
18. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $\alpha-\psi-$contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
19. Jleli, M.; Karapinar, E.; Samet, B. Best Proximity Points for Generalized $\alpha-\psi-$proximal Contractive Type Mappings. Available online: https://www.hindawi.com/journals/jam/2013/534127/ (accessed on 16 March 2020).
20. Saleem, N.; Abbas, M.; Mohsin, B.B.; Radenovic, S. Pata type best proximity point results in metric spaces. *Mathematics* **2019**, *7*, 1017. [CrossRef]
21. Rockafellar, T.R.; Wets, R.J.V. *Variational Analysis*; Springer: Berlin, Germany, 2005.
22. Basha, S. Best proximity point theorems on partially ordered sets. *Optim. Lett.* **2013,** *7*, 1035–1043. [CrossRef]