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New Fixed Point Theorems in Orthogonal $\mathscr{F}$-Metric Spaces with Application to Fractional Differential Equation

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Received: 27 March 2020; Accepted: 6 May 2020; Published: 19 May 2020

Abstract: We present the notion of orthogonal $\mathscr{F}$-metric spaces and prove some fixed and periodic point theorems for orthogonal $\perp_{\Omega}$-contraction. We give a nontrivial example to prove the validity of our result. Finally, as application, we prove the existence and uniqueness of the solution of a nonlinear fractional differential equation.

Keywords: orthogonal set; $\mathscr{F}$-metric space; Banach fixed point theorem

1. Introduction and Preliminaries

Fixed point theory is one of the important branches of nonlinear analysis. After the celebrated Banach contraction principle [1], a number of authors have been working in this area of research. Fixed point theorems are very significant instruments for proving the existence and uniqueness of the solutions to nonlinear integral and differential equations, variational inequalities, and optimization problems. Metric fixed point theory grew up after the well-known Banach contraction theorem. From that point forward, there have been numerous results related to mappings satisfying various contractive conditions and underlying distance spaces; we refer to [2–15] and the references contained therein.

Recently, Jleli and Samet [16] presented the idea of $\mathscr{F}$-metric space and proved an analogue of Banach contraction principle [1].

They introduced a collection $\mathscr{F}$ defined below and presented the idea of generalized metric space called $\mathscr{F}$-metric space:

Definition 1 ([16]). *Let $\mathscr{F}$ be the set of functions $\zeta : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:*

($\mathscr{F}_1$) ζ is nondecreasing, i.e., $0 < p < q$ iff $\zeta(p) \le \zeta(q)$;
($\mathscr{F}_2$) For every sequence $\{p_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to+\infty} p_n = 0 \Leftrightarrow \lim_{n\to+\infty} \zeta(p_n) = -\infty.$$

Definition 2 ([16]). *Let $\aleph$ be a nonempty set and $\mathscr{D} : \aleph \times \aleph \to [0, \infty)$. Suppose that there exist $(\zeta, a) \in \mathscr{F} \times [0, +\infty)$ such that for all $(p, q) \in \aleph \times \aleph$*

($\mathscr{D}$1) $\mathscr{D}(p,q) = 0 \Leftrightarrow p = q$;
($\mathscr{D}$2) $\mathscr{D}(p,q) = \mathscr{D}(q,p)$;
($\mathscr{D}$3) *For each* $m \in \mathbb{N},\ m \geq 2$, *and for every* $(p_i)_{i=1}^{n} \subset \aleph$ *with* $(p_1, p_m) = (p,q)$, *we have*

$$\mathscr{D}(p,q) > 0 \Rightarrow \zeta(\mathscr{D}(p,q)) \leq \zeta\left(\sum_{i=1}^{m-1} \mathscr{D}(p_i, p_{i+1})\right) + a.$$

Then, $\mathscr{D}$ *is called an* $\mathscr{F}$*-metric on* $\aleph$ *and* $(\aleph, \mathscr{D})$ *is called an* $\mathscr{F}$*-metric space.*

Example 1 ([16]). *A metric* $\mathscr{D} : \mathbb{N} \times \mathbb{N} \to [0, \infty)$, *defined by*

$$\mathscr{D}(p,q) = \begin{cases} (p-q)^2 & if\ \ (p,q) \in [0,3] \times [0,3] \\ |p-q| & if\ \ (p,q) \notin [0,3] \times [0,3], \end{cases}$$

is an $\mathscr{F}$*-metric with* $\zeta(\mu) = \ln(\mu)$ *and* $a = \ln(3)$, *so the pair* $(\mathbb{N}, \mathscr{D})$ *is called an* $\mathscr{F}$*-metric space.*

Definition 3 ([16]). *Let* $(\aleph, \mathscr{D})$ *be an* $\mathscr{F}$*-metric space.*

(i) *A sequence* $\{p_n\}$ *in* $\aleph$ *is* $\mathscr{F}$*-convergent to* $p \in \aleph$ *if* $\{p_n\}$ *is convergent to* p *with respect to the* $\mathscr{F}$*-metric* $\mathscr{D}$;
(ii) *A sequence* $\{p_n\}$ *is* $\mathscr{F}$*-Cauchy if*

$$\lim_{n,\omega \to +\infty} \mathscr{D}(p_n, p_\omega) = 0.$$

(iii) *The space* $(\aleph, \mathscr{D})$ *is* $\mathscr{F}$*-complete if every* $\mathscr{F}$*-Cauchy sequence in* $\aleph$ *is* $\mathscr{F}$*-convergent to a an element of* $\aleph$.

Definition 4 ([17]). *A nonempty set* $\aleph$ *is said to be an orthogonal set (briefly O-set) if the binary relation* $\perp \subset \aleph \times \aleph$ *satisfies the following assertion:*

$$\exists\, p_0 : \ (\forall\, q,\ q \perp p_0) \ \ or \ \ (\forall\, q,\ p_0 \perp q).$$

The O-set is denoted by $(\aleph, \perp)$.

It is to be noted that the element p_0 in the above Definition is an orthogonal element; additionally, if p_0 is to be unique, then we call that p_0 is the unique orthogonal element and $(\aleph, \perp)$ is the uniquely orthogonal set.

Example 2. *Suppose that* $GL_n(\mathbb{R})$ *is a set of all* $n \times n$ *invertible matrices. Define relation* $\perp$ *on* $GL_n(\mathbb{R})$ *by*

$$P \perp Q \Leftrightarrow \exists\, I \in GL_n(\mathbb{R}) : PQ = QP.$$

It is easy to see that $GL_n(\mathbb{R})$ *is an O-set.*

Definition 5 ([17]). *Let* $(\aleph, \perp)$ *be an O-set. A sequence* $\{p_n\}_{n \in \mathbb{N}}$ *is called an orthogonal sequence (briefly, O-sequence) if*

$$(\forall n, p_n \perp p_{n+1}) \ \ or \ \ (\forall n, p_{n+1} \perp p_n).$$

Definition 6 ([17]). *Let* $(\aleph, \perp)$ *be an O-set. A mapping* $\zeta : \aleph \to \aleph$ *is called* $\perp$*-preserving if* $p \perp q$ *implies* $\zeta(p) \perp \zeta(q)$.

Consistent with Jleli and Samet [18], we denote by Δ_Ω the set of all functions $\Omega : (0, \infty) \to [1, \infty)$ satisfying the following conditions:

(Ω_1) Ω is strictly increasing;

(Ω_2) For all sequences $\{\sigma_n \subseteq (0,\infty)\}$,

$$\lim_{n\to\infty} \sigma_n = 0 \Leftrightarrow \lim_{n\to\infty} \Omega(\sigma_n) = 1;$$

(Ω_3) There exist $0 < r < 1$ and $l \in (0,\infty]$ such that

$$\lim_{\mu\to 0^+} \frac{\Omega(\mu) - 1}{\mu^r} = l.$$

2. Fixed Point Theorem

In this section, we first define the notion of orthogonal $\mathscr{F}$-metric space (briefly $\perp$-$\mathscr{F}$-metric space) and then prove a fixed point result for $\perp_\Omega$-contraction in such a generalized structure.

Definition 7. *Let $(\aleph,\perp)$ be an O-set and $\mathscr{D}$ be an $\mathscr{F}$-metric on $\aleph$. The triplet $(\aleph,\perp,\mathscr{D})$ is called an orthogonal $\mathscr{F}$-metric space.*

Example 3. *Let $\aleph = [0,1]$ be a $\mathscr{F}$-metric space with $\mathscr{F}$-metric*

$$\mathscr{D}(p,q) = \begin{cases} \exp(|p-q|) & p \neq q \\ 0 & p = q, \end{cases}$$

for all $p,q \in \aleph$, $\zeta(\mu) = -\frac{1}{\mu}$, $\mu > 0$ and $a = 1$. Define $p \perp q$ if $pq \leq p$ or $pq \leq q$. Then, for all $p \in \aleph$, $0 \perp p$, so $(\aleph,\perp)$ is an O-set. Then, $(\aleph,\perp,\mathscr{D})$ is an orthogonal $\mathscr{F}$-metric space.

From now on, $(\aleph,\perp)$ is an O-set and $(\aleph,\mathscr{D})$ is an $\mathscr{F}$-metric space.

Definition 8. *Let $(\aleph,\perp,\mathscr{D})$ be an orthogonal $\mathscr{F}$-metric space. Then, $\zeta : \aleph \to \aleph$ is called orthogonally continuous (or $\perp$-continuous) at $a \in \aleph$ if, for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\aleph$ with $a_n \to a$, we have $\zeta(a_n) \to \zeta(a)$. Furthermore, ζ is said to be $\perp$-continuous on $\aleph$ if ζ is $\perp$-continuous at each $a \in \aleph$.*

Example 4. *Let $\aleph = [0,1)$ and $\mathscr{F}$-metric on $\aleph$ be $\mathscr{D}(p,q) = |p-q|$ for all $p,q \in \aleph$. Define $p \perp q$ if $pq \leq \frac{p}{3}$. Define a mapping $\zeta : \aleph \to \aleph$ such that*

$$\zeta(p) = \begin{cases} \frac{p}{3} & p \in (0,\frac{1}{3}] \\ 0 & p \notin (0,\frac{1}{3}]. \end{cases}$$

Since $p \perp q$ and $pq \leq \frac{p}{3}$, $p = 0$ or $q \leq \frac{1}{3}$. So, we have following four cases:

Case-I: *$p = 0$ and $q \leq \frac{1}{3}$. Then, $\zeta(p) = 0$ and $\zeta(q) = \frac{q}{3}$;*
Case-II: *$p = 0$ and $q > \frac{1}{3}$. Then, $\zeta(p) = 0$ and $\zeta(q) = 0$;*
Case-III: *$q \leq \frac{1}{3}$ and $p \leq \frac{1}{3}$. Then, $\zeta(q) = \frac{q}{3}$ and $\zeta(p) = \frac{p}{3}$;*
Case-IV: *$q \leq \frac{1}{3}$ and $p > \frac{1}{3}$. Then, $\zeta(q) = \frac{q}{3}$ and $\zeta(p) = 0$.*

Therefore, from all cases, we have $\zeta(p)\zeta(q) \leq \frac{\zeta(p)}{3}$. Clearly, ζ is not continuous, but it is easy to see that ζ is $\perp$-continuous.

Definition 9. *Let $(\aleph,\perp,\mathscr{D})$ be an orthogonal $\mathscr{F}$-metric space. Then, $\aleph$ is said to be orthogonally $\mathscr{F}$-complete (briefly, O-$\mathscr{F}$-complete) if every Cauchy O-sequence is $\mathscr{F}$-convergent in $\aleph$.*

Example 5. *Let $\aleph = [0,1)$ and $\mathscr{F}$-metric on $\aleph$ be $\mathscr{D}(p,q) = |p-q|$ for all $p,q \in \aleph$. Define $p \perp q$ if $pq \leq \max\{\frac{p}{3},\frac{q}{3}\}$. Clearly, $\aleph$ is not complete, but it is O-$\mathscr{F}$-complete.*

Definition 10. *Let $(\aleph, \perp, \mathscr{D})$ be an orthogonal $\mathscr{F}$-metric space and $\mathscr{T} : \aleph \to \aleph$ be a given mapping. Suppose that $\Omega \in \Delta_\Omega$ and $\kappa \in (0,1)$ such that*

$$\forall\, p, q \in \aleph,\ p \perp q, \quad \mathscr{D}(\mathscr{T}p, \mathscr{T}q) \neq 0 \ \Rightarrow\ \Omega(\mathscr{D}(\mathscr{T}p, \mathscr{T}q)) \leq [\Omega(\mathscr{D}(p,q))]^\kappa, \tag{1}$$

is called $\perp_\Omega$-contraction.

Example 6. *Let $\aleph = [1, \infty)$ with $\mathscr{F}$-metric $\mathscr{D}(p,q) = |p - q|$ for all $p, q \in \aleph$, $\zeta(\mu) = \ln(\mu), \mu > 0$ and $a = 0$. Let the set orthogonal relation "$\perp$" be defined as $p \perp q \Leftrightarrow pq \in \max\{p, q\}$. Define $\mathscr{T} : \aleph \to \aleph$ by*

$$\mathscr{T}p = \begin{cases} 1 & p \leq 4 \\ \frac{p}{4} & p > 4. \end{cases}$$

It can easily be seen that $\mathscr{T}$ is $\perp_\Omega$-contraction with $\Omega(\mu) = e^{\sqrt{\mu}}$.

Theorem 1. *Let $(\aleph, \perp, \mathscr{D})$ be an O-complete $\mathscr{F}$-metric space and $\Omega \in \Delta_\Omega$. Let $\mathscr{T} : \aleph \to \aleph$ be $\perp$-continuous, $\perp_\Omega$-contraction, and $\perp$-preserving. Then, $\mathscr{T}$ has a unique fixed point.*

Proof. Let $\epsilon > 0$ be fixed and $(\zeta, a) \in \mathscr{F} \times [0, +\infty)$ be such that $(\mathscr{D}3)$ is satisfied. By $(\mathscr{F}_2)$, there exists $\delta > 0$ such that

$$0 < \mu < \delta \ \text{ implies } \ \zeta(\mu) < \zeta(\epsilon) - a. \tag{2}$$

Since $\aleph$ is an O-set, there is an orthogonal element $p_0 \in \aleph$ such that

$$\forall\, y \in \aleph,\ p_0 \perp y \ \text{ or } \ \forall\, y \in \aleph,\ y \perp p_0.$$

Therefore, $p_0 \perp \mathscr{T}p_0$ or $\mathscr{T}p_0 \perp p_0$. Let

$$p_1 := \mathscr{T}p_0,\ \ p_2 := \mathscr{T}p_1 = \mathscr{T}^2 p_0, ..., p_{n+1} := \mathscr{T}p_n = \mathscr{T}^{n+1} p_0$$

$\forall\, n \in \mathbb{N}$. Since $\mathscr{T}$ is $\perp$-preserving, $\{p_n\}_{n \in \mathbb{N}}$ is an O-sequence. If there exists $n_0 \in \mathbb{N}$, such that $p_{n_0+1} = p_{n_0}$, then p_{n_0} is a fixed point of $\mathscr{T}$. Therefore, we suppose $\mathscr{D}(p_n, p_{n+1}) > 0$ for all $n \in \mathbb{N}$. Now, since $\mathscr{T}$ is $\perp_\Omega$-contraction, then for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \Omega(\mathscr{D}(p_n, p_{n+1})) &= \Omega(\mathscr{D}(\mathscr{T}p_{n-1}, \mathscr{T}p_n)) \\ &\leq [\Omega(\mathscr{D}(p_{n-1}, p_n))]^\kappa \leq [\Omega(\mathscr{D}(p_{n-2}, p_{n-1}))]^{\kappa^2} \\ &\quad \cdots \\ &\leq [\Omega(\mathscr{D}(p_0, p_1))]^{\kappa^n}. \end{aligned}$$

Thus, we have

$$1 \leq \Omega(\mathscr{D}(p_n, p_{n+1})) \leq [\Omega(\mathscr{D}(p_0, p_1))]^{\kappa^n}, \ \text{ for all } n \in \mathbb{N}. \tag{3}$$

Letting $n \to \infty$ in Inequality (3), we get

$$\Omega(\mathscr{D}(p_n, p_{n+1})) \to 1,$$

which implies from (Ω_2) that

$$\lim_{n \to \infty} \mathscr{D}(p_n, p_{n+1}) = 0. \tag{4}$$

From condition (Ω_3), there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} = l.$$

Suppose that $l < \infty$. In this case, let $v = l/2 > 0$. From the Definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} - l \right| \le v, \text{ for all } n \ge n_0.$$

This implies that

$$\frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} \ge l - v = v, \text{ for all } n \ge n_0.$$

Then,

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0,$$

where $\sigma = 1/v$.

The case for $l = \infty$. In this case, let $v > 0$ be arbitrary. By definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} \ge v, \text{ for all } n \ge n_0.$$

Then,

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0,$$

where $\sigma = 1/v$.

Thus, in all cases, there exist $\sigma > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0.$$

Using Inequality (3), we obtain

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_0, p_1)) - 1]^{\kappa^n}, \text{ for all } n \ge n_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n[\mathscr{D}(p_n, p_{n+1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\mathscr{D}(p_n, p_{n+1}) \le \frac{1}{n^{1/r}}, \text{ for all } n \ge n_1,$$

which yields

$$\sum_{i=n}^{\omega-1} \mathscr{D}(p_i, p_{i+1}) \le \sum_{i=n}^{\omega-1} \frac{1}{i^{1/r}}, \ \omega > n.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ is a convergent series, then there exists $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{\omega-1} \frac{1}{i^{1/r}} < \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} < \delta, \;\; n \geq N. \tag{5}$$

Hence, by Inequality (5) and $(\mathscr{F}_1)$, we have

$$\zeta\left(\sum_{i=n}^{\omega-1} \mathscr{D}(p_i, p_{i+1})\right) \leq \zeta\left(\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}\right) < \zeta(\epsilon) - a, \;\; \omega > n \geq N. \tag{6}$$

Using $(\mathscr{D}3)$ and Inequality (6), we obtain

$$\mathscr{D}(p_n, p_\omega) > 0, \;\; \omega > n \geq N \;\; \Rightarrow \zeta(\mathscr{D}(p_n, p_\omega)) \leq \zeta\left(\sum_{i=n}^{\omega-1} \mathscr{D}(p_i, p_{i+1})\right) + a < \zeta(\epsilon),$$

which, from $(\mathscr{F}_1)$, gives that

$$\mathscr{D}(p_n, p_\omega) < \epsilon, \quad \omega > n \geq N.$$

This shows that $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy O-sequence.
Since $\aleph$ is O-complete, there exists $p \in \aleph$ such that

$$\lim_{n\to\infty} p_n = p.$$

On the other hand, $\perp$-continuity of $\mathscr{T}$ gives $\mathscr{T} p_n \to \mathscr{T} p$ as $n \to \infty$. Thus,

$$\mathscr{T} p = \lim_{n\to\infty} \mathscr{T} p_n = \lim_{n\to\infty} p_{n+1} = p.$$

To prove the uniqueness of fixed point, let $v \in \aleph$ be another fixed point of $\mathscr{T}$. Then, we have $\mathscr{T}^n v = v$ for all $n \in \mathbb{N}$. By our choice of p_0 in the first part of proof, we have

$$[p_0 \perp v] \;\text{ or }\; [v \perp p_0].$$

Since $\mathscr{T}$ is $\perp$-preserving, we have

$$[\mathscr{T}^n p_0 \perp \mathscr{T}^n v] \;\; or \;\; [\mathscr{T}^n v \perp \mathscr{T}^n p_0]$$

for all $n \in \mathbb{N}$, since $\mathscr{T}$ is an $\perp_\Omega$-contraction. Then, we have for all $n \in \mathbb{N}$,

$$1 < \Omega(\mathscr{D}(\mathscr{T}^n p_0, v)) \leq [\Omega(\mathscr{D}(\mathscr{T}^{n-1} p_0, v))]^\kappa \leq ... \leq [\Omega(\mathscr{D}(p_0, v))]^{\kappa^n}.$$

Letting $n \to \infty$ in the above inequality and using condition (Ω_2), we get $\lim_{n\to\infty} p_n = v$. Uniqueness of limit gives $p = v$. $\square$

Now, we give an example which shows that Theorem 1 is a real generalization of Theorem 5.1 of [16].

Example 7. *Constructing a sequence $\{\rho_n\}$, $n \in \mathbb{N}$, in the following way:*

$$\begin{aligned}
\xi_1 &= \ln(1) \\
\xi_2 &= \ln(1+2) \\
\xi_3 &= \ln(1+2+3) \\
&\dots \\
\xi_n &= \ln(1+2+3+\cdots+n) = \ln\left(\frac{n(n+1)}{2}\right), \quad n \in \mathbb{N}.
\end{aligned}$$

Let $\aleph = \{\xi_n : n \in \mathbb{N}\}$ endowed with $\mathscr{F}$-metric $\mathscr{D}$ given by

$$\mathscr{D}(p,q) = \begin{cases} e^{|p-q|}, & if \;\; p \neq q \\ 0, & if \;\; p = q, \end{cases}$$

with $\zeta(\mu) = \frac{-1}{\mu}$ and $a = 1$. For all $\xi_n, \xi_\omega \in \aleph$, define $\xi_n \perp \xi_\omega$ iff $(\omega \geq 2 \wedge n = 1)$. Hence, $(\aleph, \perp, \mathscr{D})$ is an O-complete $\mathscr{F}$-metric space. Map $\mathscr{T} : \aleph \to \aleph$ is defined by

$$\mathscr{T}(\xi_n) = \begin{cases} \xi_1 & n = 1, \\ \xi_{n-1} & n > 1. \end{cases}$$

Since

$$\lim_{n\to\infty} \frac{\mathscr{D}(\mathscr{T}(\xi_n), \mathscr{T}(\xi_1))}{\mathscr{D}(\xi_n, \xi_1)} = 1,$$

then $\mathscr{T}$ is not a contraction in the sense of [16].

Let $\Omega : (0,\infty) \to (1,\infty)$, defined by $\Omega(t) = e^{\sqrt{te^t}}$. It is easy to show that $\Omega \in \Delta_\Omega$. Now, to prove $\mathscr{T}$ is an $\perp_\Omega$-contraction, that is

$$\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega) \neq 0 \Rightarrow e^{\sqrt{\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega)e^{\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega)}}} \leq e^{\kappa\sqrt{\mathscr{D}(\xi_n,\xi_\omega)e^{\mathscr{D}(\xi_n,\xi_\omega)}}},$$

for some $\kappa \in (0,1)$. The above condition is equivalent to

$$\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega) \neq 0 \Rightarrow \mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega)e^{\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega)} \leq \kappa^2 \mathscr{D}(\xi_n, \xi_\omega)e^{\mathscr{D}(\xi_n,\xi_\omega)}.$$

So, we have to check that

$$\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega) \neq 0 \Rightarrow \frac{\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega)}{\mathscr{D}(\xi_n, \xi_\omega)} e^{\mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_\omega) - \mathscr{D}(\xi_n,\xi_\omega)} \leq \kappa^2, \tag{7}$$

for some $\kappa \in (0,1)$.

For every $\omega \in \mathbb{N}, \omega \geq 2$, we have

$$\begin{aligned}
\frac{\mathscr{D}(\mathscr{T}\xi_\omega, \mathscr{T}\xi_1)}{\mathscr{D}(\xi_\omega, \xi_1)} e^{\mathscr{D}(\mathscr{T}\xi_\omega, \mathscr{T}\xi_1) - \mathscr{D}(\xi_\omega,\xi_1)} &= \frac{e^{\xi_{\omega-1}-\xi_1}}{e^{\xi_\omega - \xi_1}} e^{e^{\xi_{\omega-1}} - e^{\xi_\omega}} \\
&= \frac{(\omega-1)}{(\omega+1)} e^{-\omega} < e^{-1}.
\end{aligned}$$

Thus, the Inequality (7) is satisfied with $\kappa = e^{-1/2}$. Hence, $\mathscr{T}$ is an $\perp_\Omega$-contraction. So, from Theorem 1 we imply that $\mathscr{T}$ has a unique fixed point $\xi = \ln(1)$.

Example 8. *Consider the sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *as follows:*

$$\begin{aligned}
\rho_1 &= \log_2 1\\
\rho_2 &= \log_2 1+\log_2 2\\
\rho_3 &= \log_2 1+\log_2 2+\log_2 3\\
&\cdots\\
\rho_n &= \log_2 1+\log_2 2+\cdots+\log_2 n=\log_2 n!,\ \ n\in\mathbb{N}.
\end{aligned}$$

Let $\aleph=\{\rho_n : n\in\mathbb{N}\}$ *endowed with* $\mathscr{F}$*-metric* $\mathscr{D}$ *given by*

$$\mathscr{D}(p,q)=\begin{cases} 2^{|p-q|}, & if\ \ p\neq q\\ 0, & if\ \ p=q,\end{cases}$$

with $\zeta(\mu)=\frac{-1}{\mu}$ *and* $a=1$*. For all* $\rho_n,\rho_\omega\in\aleph$*, define* $\rho_n\perp\rho_\omega$ *iff* $(\omega>2\wedge n=1)\vee(\omega>n>1)$*. Hence,* $(\aleph,\perp,\mathscr{D})$ *is an O-complete* $\mathscr{F}$*-metric space. Define the mapping* $\mathscr{T}:\aleph\to\aleph$ *by*

$$\mathscr{T}(\rho_n)=\begin{cases}\rho_1 & n=1,\\ \rho_{n-1} & n>1.\end{cases}$$

Let $\Omega:(0,\infty)\to(1,\infty)$*, defined by* $\Omega(t)=e^{\sqrt{te^t}}$*. It is easy to show that* $\Omega\in\Delta_\Omega$*. Now, to prove* $\mathscr{T}$ *is an* $\perp_\Omega$*-contraction, that is*

$$\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)\neq 0\Rightarrow e^{\sqrt{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)e^{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)}}}\leq e^{\kappa\sqrt{\mathscr{D}(\rho_n,\rho_\omega)e^{\mathscr{D}(\rho_n,\rho_\omega)}}},$$

for some $\kappa\in(0,1)$*. The above condition is equivalent to*

$$\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)\neq 0\Rightarrow \mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)e^{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)}\leq\kappa^2\mathscr{D}(\rho_n,\rho_\omega)e^{\mathscr{D}(\rho_n,\rho_\omega)}.$$

So, we have to check that

$$\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)\neq 0\Rightarrow \frac{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)}{\mathscr{D}(\rho_n,\rho_\omega)}e^{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)-\mathscr{D}(\rho_n,\rho_\omega)}\leq\kappa^2, \tag{8}$$

for some $\kappa\in(0,1)$*.*

For every $\omega\in\mathbb{N},\omega>2$*, we have*

$$\begin{aligned}
\frac{\mathscr{D}(\mathscr{T}\rho_\omega,\mathscr{T}\rho_1)}{\mathscr{D}(\rho_\omega,\rho_1)}e^{\mathscr{D}(\mathscr{T}\rho_\omega,\mathscr{T}\rho_1)-\mathscr{D}(\rho_\omega,\rho_1)} &= \frac{2^{\rho_{\omega-1}-\rho_1}}{2^{\rho_\omega-\rho_1}}e^{2^{\rho_{\omega-1}}-2^{\rho_\omega}}\\
&= \frac{(\omega-1)!}{\omega!}e^{(\omega-1)!-\omega!}\\
&= \frac{1}{\omega}e^{-(\omega-1)(\omega-1)!}<e^{-1}.
\end{aligned}$$

For every $\omega,n\in\mathbb{N},\omega>n>1$*, the following holds:*

$$\begin{aligned}
\frac{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)}{\mathscr{D}(\rho_n,\rho_\omega)}e^{\mathscr{D}(\mathscr{T}\rho_n,\mathscr{T}\rho_\omega)-\mathscr{D}(\rho_n,\rho_\omega)} &= \frac{2^{\rho_{\omega-1}-\rho_{n-1}}}{2^{\rho_\omega-\rho_n}}e^{2^{\rho_{\omega-1}-\rho_{n-1}}-2^{\rho_\omega-\rho_n}}\\
&= \frac{(\omega-1)!}{(n-1)!}\times\frac{(n)!}{(\omega)!}e^{\frac{(\omega-1)!}{(n-1)!}-\frac{(\omega)!}{(n)!}}\\
&= \frac{n}{\omega}e^{\frac{(\omega-1)!}{(n-1)!}\frac{n-\omega}{n}}<e^{-1}.
\end{aligned}$$

Thus, the Inequality (8) is satisfied with $\kappa = e^{-1/2}$. Hence, $\mathscr{T}$ is an $\perp_\Omega$-contraction. So, from Theorem 1, we imply that $\mathscr{T}$ has a unique fixed point $\rho = \log_2 1$.

3. Periodic Point Theorem

Let $\mathscr{T} : \aleph \to \aleph$ be a mapping such that $\mathscr{T}(p) = p$, then for every $n \in \mathbb{N}$, $\mathscr{T}^n(p) = p$. However, the converse of this fact is not true in general. The mapping satisfying $Fix(\mathscr{T}) = Fix(\mathscr{T}^n)$ for each $n \in \mathbb{N}$ is said to have property P.

Definition 11. *Let $(\aleph, \perp, \mathscr{D})$ be an orthogonal $\mathscr{F}$-metric space and $\mathscr{T} : \aleph \to \aleph$ be a self-mapping. The set $O(p) = \{p, \mathscr{T}p, ..., \mathscr{T}^n p, ...\}$ is called the orbit of $\aleph$. A mapping $\mathscr{T}$ is called orbitally O-continuous at p if for each O-sequence $\{\mathscr{T}^n p\}$ in $\aleph$, $\lim_{n\to\infty} \mathscr{T}^n p = x$ implies that $\lim_{n\to\infty} \mathscr{T}^{n+1} p = \mathscr{T}x$. A mapping $\mathscr{T}$ is orbitally continuous on X if $\mathscr{T}$ is orbitally O-continuous at all $p \in \aleph$.*

Theorem 2. *Let $(\aleph, \perp, \mathscr{D})$ be an O-complete $\mathscr{F}$-metric space and $\Omega \in \Delta_\Omega$. Let $\mathscr{T} : \aleph \to \aleph$ be $\perp$-preserving and satisfy*

$$\forall\, p \in \aleph, \quad \mathscr{T}p \perp \mathscr{T}^2 p, \quad \mathscr{D}(\mathscr{T}p, \mathscr{T}^2 p) > 0 \;\Rightarrow\; \Omega(\mathscr{D}(\mathscr{T}p, \mathscr{T}^2 p)) \leq [\Omega(\mathscr{D}(p, \mathscr{T}p))]^\kappa, \tag{9}$$

where $\kappa \in (0, 1)$. Then, $\mathscr{T}$ has the property P provided that $\mathscr{T}$ is orbitally continuous on $\aleph$.

Proof. Let $\epsilon > 0$ be fixed and $(\zeta, a) \in \mathscr{F} \times [0, +\infty)$ be such that $(\mathscr{D}3)$ is satisfied. By $(\mathscr{F}_2)$, there exists $\delta > 0$ such that

$$0 < \mu < \delta \text{ implies } \zeta(\mu) < \zeta(\epsilon) - a. \tag{10}$$

We show that $Fix(\mathscr{T}) \neq \phi$. Define an O-sequence $\{p_n\}_{n\in\mathbb{N}}$ in $\aleph$ such that $p_{n+1} = \mathscr{T}^n p$. If there exists $n_0 \in \mathbb{N}$, such that $p_{n_0+1} = p_{n_0}$, then p_{n_0} is a fixed point of $\mathscr{T}$. Therefore, we suppose $\mathscr{D}(p_n, p_{n+1}) > 0$ for all $n \in \mathbb{N}$. Using Inequality (9), we obtain

$$\begin{aligned}
\Omega(\mathscr{D}(p_n, p_{n+1})) &= \Omega(\mathscr{D}(\mathscr{T}p_{n-1}, \mathscr{T}^2 p_{n-1})) \\
&\leq [\Omega(\mathscr{D}(p_{n-1}, \mathscr{T}p_{n-1}))]^\kappa \\
&= [\Omega(\mathscr{D}(\mathscr{T}p_{n-2}, \mathscr{T}^2 p_{n-2}))]^\kappa \\
&\leq [\Omega(\mathscr{D}(p_{n-2}, \mathscr{T}p_{n-2}))]^{\kappa^2} \\
&\quad \cdots \\
&\leq [\Omega(\mathscr{D}(p_0, p_1))]^{\kappa^n}.
\end{aligned}$$

Thus, we have

$$1 \leq \Omega(\mathscr{D}(p_n, p_{n+1})) \leq [\Omega(\mathscr{D}(p_0, p_1))]^{\kappa^n}, \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

Letting $n \to \infty$ in Inequality (11), we get

$$\Omega(\mathscr{D}(p_n, p_{n+1})) \to 1,$$

which implies from (Ω_2) that

$$\lim_{n\to\infty} \mathscr{D}(p_n, p_{n+1}) = 0. \tag{12}$$

From condition (Ω_3), there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} = l.$$

Suppose that $l < \infty$. In this case, let $v = l/2 > 0$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} - l\right| \le v, \text{ for all } n \ge n_0.$$

This implies that

$$\frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} \ge l - v = v, \text{ for all } n \ge n_0.$$

$$\Omega(\mathscr{D}(p_n, p_{n+1})) - 1 \ge v[\mathscr{D}(p_n, p_{n+1})]^r, \text{ for all } n \ge n_0.$$

Then,

$$[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0,$$

where $\sigma = 1/v$. Multiplying by n on both sides of inequality, we get

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0.$$

Suppose that $l = \infty$. In this case, let $v > 0$ be arbitrary. By Definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\Omega(\mathscr{D}(p_n, p_{n+1})) - 1}{[\mathscr{D}(p_n, p_{n+1})]^r} \ge v, \text{ for all } n \ge n_0.$$

Then,

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0,$$

where $\sigma = 1/v$.

Thus, in all cases, there exist $\sigma > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_n, p_{n+1})) - 1], \text{ for all } n \ge n_0.$$

Using Inequality (11), we obtain

$$n[\mathscr{D}(p_n, p_{n+1})]^r \le \sigma n[\Omega(\mathscr{D}(p_0, p_1)) - 1]^{\kappa^n}, \text{ for all } n \ge n_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n[\mathscr{D}(p_n, p_{n+1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\mathscr{D}(p_n, p_{n+1}) \le \frac{1}{n^{1/r}}, \text{ for all } n \ge n_1,$$

which yields

$$\sum_{i=n}^{\omega-1} \mathscr{D}(p_i, p_{i+1}) \le \sum_{i=n}^{\omega-1} \frac{1}{i^{1/r}}, \ \omega > n.$$

Since $\sum\limits_{i=1}^{\infty}\frac{1}{i^{1/r}}$ is a convergent series, then there exists $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{\omega-1}\frac{1}{i^{1/r}} < \sum_{i=n}^{\infty}\frac{1}{i^{1/r}} < \delta, \;\; n \geq N. \tag{13}$$

Hence, by Inequality (13) and $(\mathscr{F}_1)$, we have

$$\zeta\left(\sum_{i=n}^{\omega-1}\mathscr{D}(p_i, p_{i+1})\right) \leq \zeta\left(\sum_{i=n}^{\infty}\frac{1}{i^{1/r}}\right) < \zeta(\epsilon) - a, \;\; \omega > n \geq N. \tag{14}$$

Using $(\mathscr{D}3)$ and Inequality (14), we obtain

$$\mathscr{D}(p_n, p_\omega) > 0, \;\; \omega > n \geq N \;\; \Rightarrow \zeta(\mathscr{D}(p_n, p_\omega)) \leq \zeta\left(\sum_{i=n}^{\omega-1}\mathscr{D}(p_i, p_{i+1})\right) + a < \zeta(\epsilon),$$

which, from $(\mathscr{F}_1)$, gives that

$$\mathscr{D}(p_n, p_\omega) < \epsilon, \quad \omega > n \geq N.$$

This shows that $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy O-sequence.

Since $\{\mathscr{T}^n p_0 : n \in \mathbb{N}\} \subseteq O(p_0) \subseteq \aleph$ and $\aleph$ is O-complete, there exists $p \in \aleph$ such that

$$\lim_{n\to\infty}\mathscr{T}^n p_0 = p.$$

On the other hand, orbital $\perp$-continuity of $\mathscr{T}$ gives $p = \lim\limits_{n\to\infty}\mathscr{T}^{n-1}p_0 = \mathscr{T}p$. Hence, $\mathscr{T}$ has a fixed point and $Fix(\mathscr{T}^n) = Fix(\mathscr{T})$ is true for $n = 1$. Now, let $n > 1$. Suppose on the contrary that $p \in Fix(\mathscr{T}^n)$ but $p \notin Fix(\mathscr{T})$, then $\mathscr{D}(p, \mathscr{T}p) = a > 0$. Now,

$$\begin{aligned}
\Omega(\mathscr{D}(p, \mathscr{T}p)) &= \Omega(\mathscr{D}(\mathscr{T}(\mathscr{T}^{n-1}p), \mathscr{T}^2(\mathscr{T}^{n-1}p))) \\
&\leq [\Omega(\mathscr{D}(\mathscr{T}^{n-1}p, \mathscr{T}^n p))]^{\kappa} \\
&\leq [\Omega(\mathscr{D}(\mathscr{T}^{n-2}p, \mathscr{T}^{n-1}p))]^{\kappa^2} \\
&\quad \ldots \\
&\leq [\Omega(\mathscr{D}(p, \mathscr{T}p))]^{\kappa^n}.
\end{aligned}$$

Thus, we have

$$1 \leq \Omega(\mathscr{D}(p, \mathscr{T}p)) \leq [\Omega(\mathscr{D}(p, \mathscr{T}p))]^{\kappa^n}, \;\; \text{for all } n \in \mathbb{N}. \tag{15}$$

Letting $n \to \infty$ in Inequality (15), we get

$$\Omega(\mathscr{D}(p, \mathscr{T}p)) \to 1,$$

which, from (Ω_2), implies that

$$\mathscr{D}(p, \mathscr{T}p) = 0 \tag{16}$$

is a contradiction. So, $p \in \mathscr{T}p$. $\square$

4. Application

This section is devoted to show the existence of the solution of the following nonlinear differential equation of fractional order (see [19]) given by

$$^{C}D^{\wp}p(\mu) = \zeta(\mu, p(\mu)) \quad (0 < \mu < 1, 1 < \wp \leq 2), \tag{17}$$

with boundary conditions

$$p(0) = 0,\ p(1) = \int_0^{\pi} p(\nu)d\nu \ \ (0 < \pi < 1), \tag{18}$$

where $^{C}D^{\wp}$ stands for Caputo fractional derivative with order $\wp$, which is defined by

$$^{C}D^{\wp}\zeta(\mu) = \frac{1}{\Gamma(m-\wp)}\int_0^{\mu}(\mu-\nu)^{m-\wp-1}f^{m}(\nu)d\nu,$$

where $m-1 < \wp < m, m = [\wp]+1$ and $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}^{+}$ is a continuous function. We consider $\aleph = \{p | p \in C([0,1], \mathbb{R})\}$ with supremum norm $\|p\|_{\infty} = \sup\limits_{p\in[0,1]} |p(\mu)|$. So, $(\aleph, \|p\|_{\infty})$ is a Banach space. Recall, the Riemann–Liouville fractional integral of order $\wp$ is given by

$$I^{\wp}\zeta(\mu) = \frac{1}{\Gamma(\wp)}\int_0^{\mu}(\mu-\nu)^{\wp-1}\zeta(\nu)d\nu \ \ (\wp > 0).$$

Lemma 1. *The Banach space $(\aleph, \|\cdot\|_{\infty})$ endowed with the metric $\mathscr{D}$ defined by*

$$\mathscr{D}(p, p^*) = \|p - p^*\|_{\infty} = \sup_{\mu\in[0,1]} |p(\mu) - p^*(\mu)|$$

and orthogonal relation $p \perp p^* \Leftrightarrow pp^* \geq 0$, where $p, p^* \in \aleph$, is an orthogonal $\mathscr{F}$-metric space.

Proof. It is clear by definition of $\mathscr{D}$ that it satisfies conditions $(\mathscr{D}1)$ and $(\mathscr{D}2)$. To verify $(\mathscr{D}3)$, for every $(p, p^*) \in \aleph \times \aleph$ where $p \perp p^*$, for every $M \in \mathbb{N}, M \geq 2$, and for every $(p_i)_{i=1}^{M} \subset \aleph$ with $(p_1, p_M) = (p, p^*)$, we have

$$\mathscr{D}(p, p^*) \leq \sum_{i=1}^{M-1} \mathscr{D}(p_i, p_{i+1}),$$

which gives

$$\mathscr{D}(p, p^*) > 0 \Rightarrow \ln(\mathscr{D}(p, p^*)) \leq \ln\left(\sum_{i=1}^{M-1} \mathscr{D}(p_i, p_{i+1})\right).$$

Then, $\mathscr{D}$ verifies $(\mathscr{D}3)$ with $\zeta(p) = \ln(p), p > 0$ and $a = 0$. Hence, $(\aleph, \perp, \mathscr{D})$ is an orthogonal $\mathscr{F}$-metric space. □

Theorem 3. *Suppose that $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}^{+}$ is a continuous function, satisfying the following condition*

$$|\zeta(\mu, p) - \zeta(\mu, q)| \leq K|p - q|,$$

for all $\mu \in [0,1]$ and for all $p, q \in \aleph$ such that $p(\mu)q(\mu) \geq 0$ and a constant K with $K\lambda < 1$, where

$$\lambda = \frac{\mu^{\wp}(2-\pi^2)(\wp+1) + 2\mu(\wp+\pi+1)}{(2-\pi^2)\wp(\wp+1)\Gamma(\wp)},$$

where $0 < \pi < 1$. Then, the differential Equation (17) with boundary conditions Equation (18) has a unique solution.

Proof. For all $\mu \in [0,1]$, assume the orthogonality relation on $\aleph$, by

$$p \perp q \text{ if } p(\mu)q(\mu) \geq 0.$$

Under this relation, the set $\aleph$ is orthogonal because for every $p \in \aleph$ $\exists$ $q(\mu) = 0$ $\forall$ $\mu \in [0,1]$ such that $p(\mu)q(\mu) = 0$. We consider $\mathscr{D}(p,q) = \sup_{\mu \in [0,1]} |p(\mu) - q(\mu)|$ for all $p, q \in \aleph$. So, the triplet $(\aleph, \perp, \mathscr{D})$ is a complete O-$\mathscr{F}$-metric space.

Define a mapping $\mathscr{T} : \aleph \to \aleph$ by

$$\begin{aligned}\mathscr{T}p(\mu) = \quad & \frac{1}{\Gamma(\wp)} \int_0^\mu (\mu - \nu)^{\wp - 1} \zeta(\nu, p(\nu)) d\nu - \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^1 (1 - \nu)^{\wp - 1} \zeta(\nu, p(\nu)) d\nu \\ & + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^\pi \int_0^\nu (\nu - \omega)^{\wp - 1} \zeta(\omega, p(\omega)) d\omega d\nu,\end{aligned}$$

for $\mu \in [0,1]$. Then, $\mathscr{T}$ is $\perp$-continuous.

A function $p \in \aleph$ is a solution of Equation (17) if and only if $p = \mathscr{T}p$. In order to prove the existence of fixed point of $\mathscr{T}$, we prove that $\mathscr{T}$ is $\perp$-preserving and $\perp_\Omega$-contraction.

To show $\mathscr{T}$ is $\perp$-preserving, let $p(\mu) \perp q(\mu)$, for all $\mu \in [0,1]$. Now, we have

$$\begin{aligned}\mathscr{T}p(\mu) = \quad & \frac{1}{\Gamma(\wp)} \int_0^\mu (\mu - \nu)^{\wp - 1} \zeta(\nu, p(\nu)) d\nu - \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^1 (1 - \nu)^{\wp - 1} \zeta(\nu, p(\nu)) d\nu \\ & + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^\pi \int_0^\nu (\nu - \omega)^{\wp - 1} \zeta(\omega, p(\omega)) d\omega d\nu > 0,\end{aligned}$$

which implies that $\mathscr{T}p \perp \mathscr{T}q$, i.e. $\mathscr{T}$ is $\perp$-preserving.

Next, we show that $\mathscr{T}$ is an $\perp_\Omega$-contraction. For all $\mu \in [0,1]$, $p(\mu) \perp q(\mu)$, we have

$$\begin{aligned}|\mathscr{T}p - \mathscr{T}q| \quad = \quad & \Bigg| \frac{1}{\Gamma(\wp)} \int_0^\mu (\mu - \nu)^{\wp - 1} \zeta(\nu, p(\nu)) d\nu - \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^1 (1 - \nu)^{\wp - 1} \zeta(\nu, p(\nu)) d\nu \\ & + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^\pi \left(\int_0^\nu (\nu - \omega)^{\wp - 1} \zeta(\omega, p(\omega)) d\omega \right) d\nu \\ & - \frac{1}{\Gamma(\wp)} \int_0^\mu (\mu - \nu)^{\wp - 1} \zeta(\nu, q(\nu)) d\nu + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^1 (1 - \nu)^{\wp - 1} \zeta(\nu, q(\nu)) d\nu \\ & - \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^\pi \left(\int_0^\nu (\nu - \omega)^{\wp - 1} \zeta(\omega, q(\omega)) d\omega \right) d\nu \Bigg| \\ \leq \quad & \frac{1}{\Gamma(\wp)} \int_0^\mu (\mu - \nu)^{\wp - 1} |\zeta(\nu, p(\nu)) - \zeta(\nu, q(\nu))| d\nu \\ & + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^1 (1 - \nu)^{\wp - 1} |\zeta(\nu, p(\nu)) - \zeta(\nu, q(\nu))| d\nu \\ & + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^\pi \int_0^\nu (\nu - \omega)^{\wp - 1} |\zeta(\omega, p(\omega)) - \zeta(\omega, q(\omega))| d\omega d\nu \\ \leq \quad & \Bigg(\frac{1}{\Gamma(\wp)} \int_0^\mu (\mu - \nu)^{\wp - 1} d\nu + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^1 (1 - \nu)^{\wp - 1} d\nu \\ & + \frac{2\mu}{(2 - \pi^2)\Gamma(\wp)} \int_0^\pi \int_0^\nu (\nu - \omega)^{\wp - 1} d\omega d\nu \Bigg) K \|p - q\|_\infty \\ = \quad & \left(\frac{\mu^\wp (2 - \pi^2)(\wp + 1) + 2\mu(\wp + \pi + 1)}{(2 - \pi^2)\wp(\wp + 1)\Gamma(\wp)} \right) K \|p - q\|_\infty \\ = \quad & K\lambda \|p - q\|_\infty,\end{aligned}$$

which implies that $\|\mathscr{T}p - \mathscr{T}q\|_\infty \leq K\lambda\|p-q\|_\infty$. Thus, for each $p, q \in \aleph$, we have

$$\mathscr{D}(\mathscr{T}p, \mathscr{T}q) \leq K\lambda\mathscr{D}(p,q).$$

Let $\Omega(\nu) = e^{\sqrt{\nu}} \in \Omega$, $\nu > 0$, we have

$$e^{\sqrt{\mathscr{D}(\mathscr{T}p,\mathscr{T}q)}} \leq e^{\sqrt{K\lambda\mathscr{D}(p,q)}} = [e^{\sqrt{\mathscr{D}(p,q)}}]^\kappa, \ \forall\, p, q \in \aleph,$$

where $\kappa = \sqrt{K\lambda}$. Since $K\lambda < 1$, $\kappa \in (0,1)$. Therefore, $\mathscr{T}$ is an $\perp_\Omega$-contraction.

Now, let (p_n) be a Cauchy O-sequence converging in $\aleph$. Therefore, for $n \in \mathbb{N}$, we have $p_n(\mu)p_{n+1}(\mu) \geq 0$ for all $\mu \in [0,1]$. We have two cases: either $p_n(\mu) \geq 0$ or $p_n(\mu) \leq 0$. If $p_n(\mu) \geq 0$ for each $n \in \mathbb{N}$ and $\mu \in [0,1]$. Then, for every $\mu \in [0,1]$, there is a sequence of non-negative real numbers which converges to $p(\mu)$. Hence, we must get $p(\mu) \geq 0$ for each $\mu \in [0,1]$, i.e., $p_n(\mu) \perp p(\mu)$ for all $n \in \mathbb{N}$ and $\mu \in [0,1]$. The second case, $p_n(\mu) \leq 0$ for all $n \in \mathbb{N}$, has to be discarded. So, by Theorem 1, $\mathscr{T}$ has a unique fixed point and hence Equation (17) possesses a unique solution. □

Author Contributions: Conceptualization, T.K. and A.H.; methodology, A.H.; validation, A.H, H.B., and M.d.l.S.; formal analysis, H.B.; writing—original draft preparation, T.K., A.H.; writing—review and editing, A.H., H.B.; supervision, M.S.; funding acquisition, M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Basque Government under the Grant IT 1207-19.

Acknowledgments: The authors are very grateful to the Basque Government for the support.

Conflicts of Interest: Authors have no conflict of interest.

References

1. Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Salazar, L.A.; Reich, S. A remark on weakly contractive mappings. *J. Nonlinear Convex Anal.* **2015**, *16*, 767–773.
3. Alber, Y.I.; Guerre-Delabrere, S. Principles of weakly contractive maps in Hilbert spaces. In *New Results in Operator Theory and Its Applications*; Gohberg, I., Lyubich, Y., Eds.; Birkhauser Verlag: Basel, Switzerland, 1997; Volume 98, pp. 7–22.
4. Berinde, V. Approximating fixed points of weak φ-contractions. *Fixed Point Theory* **2003**, *4*, 131–142.
5. Boyd, D.W.; Wong, J.S. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
6. Ćirić, L.B. A generalization of Banach's contraction principle. *Proc. Am. Math. Soc.* **1974**, *45*, 267–273. [CrossRef]
7. Eshaghi Gordji, M.; Baghani, H.; Kim, G.H. Common fixed point theorems for (ψ, φ)-weak nonlinear contraction in partially ordered sets. *Fixed Point Theory Appl.* **2012**, *2012*, 62. [CrossRef]
8. Nieto, J.J.; Pouso, R.L.; Rodríguez-López, R. Fixed point theorems in ordered abstract sets. *Proc. Am. Math. Soc.* **2007**, *135*, 2505–2517. [CrossRef]
9. Nieto, J.J.; Rodríguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [CrossRef]
10. Nieto, J.J.; Rodríguez-López, R. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Math. Sin.* **2007**, *23*, 2205–2212. [CrossRef]
11. Ran, A.C.M.; Reurings, M.C.B. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [CrossRef]
12. Reich, S. Some remarks concerning contraction mappings. *Canad. Math. Bull.* **1971**, *14*, 121–124. [CrossRef]
13. Rhoades, B.E. Some theorems on weakly contractive maps. *Nonlinear Anal.* **2001**, *47*, 2683–2693. [CrossRef]
14. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
15. Zhang, Q.; Song, Y. Fixed point theory for generalized φ-weak contractions. *Appl. Math. Lett.* **2009**, *22*, 75–78. [CrossRef]

16. Jleli, M.; Samet, B. On a new generalization of metric spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 128. [CrossRef]
17. Eshaghi Gordji, M.; Ramezani, M.; Senb, M.D.L.; Cho, Y.J. On orthogonal sets and Banach fixed Point theorem. *Fixed Point Theory*. **2017**, in press.
18. Jleli, M.; Samet, B. A new generalizations of the Banach contraction principle. *J. Inequal. Appl.* **2014**, *2014*, 38. [CrossRef]
19. Baleanu, D.; Rezapour, S.; Mohammadi, H. Some existance results on nonlinear fractional differential equations. *Philos. Trans. A* **1990**, *371*, 1–7.