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The Riemann Zeta Function and Zeta Regularization in Casimir Effect.

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Preface

Having achieved his objective of solving the so-called Basel problem, Leonhard Euler (1707-1783) turned to the arithmetic properties of a more general series, that consisting of the sum of integer powers of all natural numbers. While he was investigating topics concerning prime numbers, he discovered a remarkable identity expressing this series as a product over all prime numbers. More than a century later, Bernhard Riemann (1826-1866) published a short but ground-breaking paper entitled *On the Number of Primes Less Than a Given Magnitude* [17], in which he used Euler's identity as the starting point. In this memoir, Riemann went much further and realized that the key to a deeper study of the distribution of prime numbers lies in considering this series as a function of a complex variable s. In particular, he introduced its analytic continuation along with its functional equation, and he outlined the eventual proof of the prime number theorem, which constitutes one of the crowning achievements of analytic number theory. The resulting function, denoted by $\zeta(s)$, became worldwide known simply as the *Riemann zeta function.*

Nowadays, the Riemann zeta function is just the prototype of a whole family of zeta functions of various kinds. However, we have limited our work to the analytic structure and the special values of $\zeta(s)$, whereas its connection with number theory and its generalizations are only mentioned briefly. The main purpose of this work is to bring together two fields which are apparently unrelated. Namely, we show how $\zeta(s)$ can be used to derivate the Casimir effect, which describes a non-classical attraction arising between two parallel plates located in vacuum.

This work is divided into three chapters. The aim of Chapter 1 is to introduce some basic tools that will be used throughout the rest of the dissertation, in order to make it as self-contained as possible. In Chapter 2, an elementary overview of the main features of the Riemann zeta function is presented. In Chapter 3, we introduce the more general concept of zeta function associated with a differential operator, showing that it can be used as a summation method for divergent series. Finally, we introduce the Casimir effect and we compute the value of the Casimir force in the simplest scenario.

Chapter 1

Preliminaries

The purpose of this chapter is to set all the material we need to proceed with the theory of the Riemann zeta function, in order to make the work almost self-contained. In the following section, we will limit ourselves to review, without proofs, some well-known results from complex analysis, in which much of the work relies on. We follow [19] as the main reference.

1.1 Complex analysis

Complex analysis is devoted to the study of holomorphic functions. Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on Ω. Then, f is said to be *holomorphic at* $z_0 \in \Omega$ if it is complex differentiable on some open disk around z_0. Moreover, f is said to be *holomorphic on* Ω if it is holomorphic at every point of Ω, and *entire* if it is holomorphic in all of $\mathbb{C}$.

It turns out that this property has much stronger consequences than its real counterpart. For instance, any holomorphic function is indefinitely differentiable, and can be locally expanded into a convergent power series, being therefore also analytic. This means that both notions are actually equivalent. We call a *domain* to a non-empty connected open set in $\mathbb{C}$.

Theorem 1.1.1. (Identity theorem) *Let f be holomorphic on a domain Ω, vanishing on a sequence of distinct points with a limit point in Ω. Then, f vanishes identically in Ω.*

As a consequence, zeros of non-trivial holomorphic functions are isolated, which showcases their strong rigidity. Moreover, as a simple corollary of the theorem, we deduce that any holomorphic function is uniquely determined by its restriction to any arbitrarily small curve segment of its domain. Thus, given a pair of holomorphic functions f and F in domains Ω and Ω' respectively, where $\Omega \subset \Omega'$, if they both agree on the smaller set Ω, we say that F is the unique *analytic continuation* of f to Ω'.

Example. Consider the geometric series $f(z) = \sum_{n=0}^{\infty} z^n$ and $g(z) = \frac{1}{1-z}$. f only converges and equals g when $|z| < 1$, defining there an holomorphic function, whereas g is holomorphic everywhere except for a simple pole at $z = 1$. Thus, g is said to be the analytic continuation of f to $\mathbb{C} \setminus \{1\}$.

Throughout this work we will have to deal with infinite series, products, integrals, or even some combinations of these. Therefore, instead of working on them separately, we will make use of the following results so that we can verify more easily that they are holomorphic.

Theorem 1.1.2. (Weierstrass) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on an open set Ω, converging uniformly to f on every compact subset of Ω. Then, f is holomorphic on Ω and the sequence of derivatives $\{f'_n\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset of Ω.*

We now turn to holomorphic functions defined in terms of integrals depending on a real parameter t, quite usual among special functions.

Theorem 1.1.3. *Let I be a finite interval of real numbers and Ω an open set in $\mathbb{C}$, such that $F(t, z)$ is continuous on $I \times \Omega$ and holomorphic on Ω for every fixed $t \in I$. Then, the function defined by*

$$f(z) = \int_I F(t, z)\, dt$$

is holomorphic on Ω.

On the other hand, infinite products are useful to represent holomorphic functions while showing their zeros explicitly. The Weierstrass factorization theorem states that every entire function can be factorized as a product involving its zeros, as exemplified by Euler's sine product formula

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

which is used to prove certain values of the Riemann zeta function.

Theorem 1.1.4. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on the open set Ω, with constants $M_n > 0$ such that*

$$\sum_{n=1}^{\infty} M_n < \infty \quad \text{and} \quad |f_n(z)| \leq M_n \quad \text{for all } z \in \Omega.$$

Then, $P(z) = \prod_{n=1}^{\infty}(1 + f_n(z))$ is holomorphic on Ω and $P(z_0) = 0$ for some $z_0 \in \Omega$ if and only if one of its factors vanishes in z_0. In addition, if $P(z)$ never vanishes, we can take its logarithmic derivative as

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty} \frac{f'_n(z)}{1 + f_n(z)}.$$

Finally, recall that complex analysis is especially interested in singular behaviour. A function f, holomorphic on a domain Ω except for some poles, is said to be *meromorphic* in Ω. On the other hand, the *residue* of f at a pole z_0 is the coefficient a_{-1} in its Laurent series at that point. Therefore, if a pole is simple at z_0, its residue is just given by

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Keeping the standard notation, from now on s will denote a complex number with real part $\text{Re}(s) = \sigma$. Also, in order to avoid confusion, the set of all integers greater than or equal to $a \in \mathbb{Z}$ will be represented as $\mathbb{Z}_{\geq a}$.

Now we will study some integral transforms, which are simply linear operators mapping a function into another, sometimes making it easier to work with or giving us otherwise hard to figure out information. It turns out that some of them are useful when dealing with zeta functions.

1.2 The Mellin transform

The Mellin transform, which is closely related to the Fourier and Laplace transforms, constitutes a basic tool for analyzing the behavior of many special functions. In the next chapter we will see that it allows to transform symmetries of theta functions into those of zeta functions. Our main reference are [8] and the appendix from Zagier in [23].

Definition 1.2.1. (Mellin transform) Let $f(t)$ be a locally Lebesgue integrable function over $(0, +\infty)$. Then, the *Mellin transform* of $f(t)$ is defined by

$$\mathcal{M}[f(t); s] = f^*(s) = \int_0^\infty f(t) t^{s-1}\, dt.$$

Its domain of definition is the largest open strip $\alpha < \text{Re}(s) < \beta$, also denoted as $\langle \alpha, \beta \rangle$, in which the integral converges. This strip is usually called the *fundamental strip*, and it is essentially determined by the behaviour of $f(t)$ near zero and infinity, as we show in the following proposition.

Proposition 1.2.1. *Let $f(t)$ be continuous and $a < b$ reals such that*

$$f(t) \underset{t \to 0^+}{=} O(t^{-a}), \quad f(t) \underset{t \to +\infty}{=} O(t^{-b}). \tag{1.1}$$

Then, $f^*(s)$ is holomorphic in $\langle a, b \rangle$.

Proof. It is straightforward to check that

$$\left| \int_0^\infty f(t) t^{s-1}\, dt \right| \leq c_1 \int_0^1 t^{\sigma - a - 1}\, dt + c_2 \int_1^\infty t^{\sigma - b - 1}\, dt,$$

where c_1 and c_2 are constants, so the first integral converges for $\mathrm{Re}(s) > a$, whereas the second converges for $\mathrm{Re}(s) < b$. The function $F(t,s) = f(t)t^{s-1}$ satisfies the hypotheses of Theorem 1.1.3 for all $(t,s) \in [1/n, n] \times \langle a, b\rangle$ with $n > 1$. Thus, each $f_n(s) = \int_{1/n}^{n} F(t,s)\,dt$ is holomorphic in $\langle a, b\rangle$. Furthermore, they converge uniformly to $f^*(s)$ in every strip in $\langle a, b\rangle$, and therefore, also in every compact subset contained in $\langle a, b\rangle$. Hence, it follows from Weierstrass' theorem that $f^*(s)$ is holomorphic in $\langle a, b\rangle$. □

Simple changes of variables in the definition of the Mellin transform yield many interesting transformation rules. For instance, provided that λ is a positive real, the substitution $t \mapsto \lambda t$ gives us the *scaling property*

$$\mathcal{M}[f(\lambda t); s] = \lambda^{-s} f^*(s)$$

for all $s \in \langle \alpha, \beta\rangle$. From here, by the linearity of the transform, it also follows that whenever $\mathcal{K}$ is a finite index set and $\lambda_k > 0$ for all $k \in \mathcal{K}$, we have

$$\mathcal{M}\left[\sum_{k\in\mathcal{K}} a_k f(\lambda_k t); s\right] = \left(\sum_{k\in\mathcal{K}} \frac{a_k}{\lambda_k^s}\right) f^*(s). \tag{1.2}$$

The following proposition generalizes this result to infinite sums, which in the future will enable us to work with exponential series instead of Dirichlet series (see Section 2.1.1), making it easier to study the latter.

Proposition 1.2.2. *The property (1.2) holds in the intersection of the fundamental strip of $f^*(s)$ and the domain of absolute convergence of the generalized Dirichlet series $\sum_{k\in\mathcal{K}} a_k/\lambda_k^s$ for any $\mathcal{K}$.*

Proof. If the intersection is not empty, the interchange of summation and integration is justified by Lebesgue's dominated convergence theorem. □

Remark. Although we will not follow this approach, the Hankel integral representation can also be used to transform holomorphic functions. If f is holomorphic in some open set containing $[0, +\infty)$ and satisfies suitable growth conditions, then *Hankel's formula*

$$\mathcal{M}[f(t); s] = \frac{i}{2\sin \pi s} \int_{\mathcal{H}} f(w)(-w)^{s-1} dw$$

holds for all $s \in \langle 0, \beta\rangle$, where $\mathcal{H}$ is the Hankel contour, which starts in the upper half-plane at $+\infty$, circles the origin once counter-clockwise and returns to $+\infty$ in the lower half-plane.

As we have seen, provided that $f(t)$ is well-behaved, i.e., continuous and with ideal growth and decay conditions, $f^*(s)$ actually defines an entire function. Otherwise, it may happen being analytically continuable to a larger region of the plane than the original strip in which it was defined, making use of more precise estimates than the ones from (1.1). In fact, there is a fundamental correspondence between terms in the asymptotic expansion of $f(t)$ (either at zero or infinity) and the set of singularities of the extension of $f^*(s)$ (in a left or right half-plane respectively). Here we will only consider the following case of our interest.

Proposition 1.2.3. *Let $f(t)$ have a Mellin transform $f^*(s)$ with non-empty fundamental strip $\langle \alpha, \beta \rangle$ and admit as $t \to 0^+$ a finite asymptotic expansion of the form*

$$f(t) = \sum_{n=0}^{N-1} a_n t^{i_n} + O(t^{i_N}),$$

where the real exponents satisfy $-\alpha = i_0 < \cdots < i_{N-1} < i_N$ and the coefficients a_n are non-vanishing. Then, $f^*(s)$ can be analytically continued to a meromorphic function in the strip $\langle -i_N, \beta \rangle$, with only simple poles, of residue a_n, at each $s = -i_n$.

Proof. Let us split the integral for $f^*(s)$ defined on $\langle \alpha, \beta \rangle$ as follows,

$$f^*(s) = \int_0^1 \left(f(t) - \sum_{n=0}^{N-1} a_n t^{i_n} \right) t^{s-1}\,dt + \sum_{n=0}^{N-1} \frac{a_n}{s+i_n} + \int_1^\infty f(t) t^{s-1}\,dt.$$

Hence, as in Proposition 1.2.1, we deduce that the first integral is holomorphic for $\text{Re}(s) > -i_N$ while the last is holomorphic for $\text{Re}(s) < \beta$. Note that its singularities and residues are exhibited by the finite sum. □

Remark. If $f(t)$ is analytic at zero, since the theorem holds for every N, $f^*(s)$ becomes meromorphic in a complete left half-plane (or even the whole plane given that $f(t)$ is of rapid decay at infinity). In fact, it extends likewise towards the right by the symmetry relation $-f^*(-s) = \mathcal{M}[f(1/t); s]$.

1.3 The Poisson summation formula

While offering a powerful symmetry between a function and its Fourier transform, the *Poisson summation formula* has many noteworthy consequences. It is used to prove many transformation properties of theta functions, and it will be indeed a key tool in our proof of Theorem 1.4.3, which then leads to the functional equation of the Riemann zeta function.

For the formula to hold, we need the involved function to be well-behaved. For the sake of simplicity, we will restrict our attention to a special class of integrable functions, even though their conditions can be weakened.

Definition 1.3.1. A *Schwartz function* is an infinitely differentiable function which, along with all its derivatives, decays at infinity faster than any negative power of $|x|$, i.e.,

$$\sup_{x\in\mathbb{R}} |x^k||f^{(l)}(x)| < \infty \quad \text{for every } k, l \geq 0.$$

We call the *Schwartz space* $\mathcal{S}(\mathbb{R})$ to the vector space of all Schwartz functions on $\mathbb{R}$, which can be proved to be closed under differentiation, multiplication by polynomials, and linear change of variable. In fact, the *Fourier transform* of $f \in \mathcal{S}(\mathbb{R})$, given by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}\, dx, \quad y \in \mathbb{R},$$

also belongs to $\mathcal{S}(\mathbb{R})$. We are now ready to prove the following theorem.

Theorem 1.3.1. (Poisson summation formula) *Let $\hat{f}$ be the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R})$. Then, the following identity holds:*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m). \tag{1.3}$$

Proof. Let us introduce the auxiliary function $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$, which is clearly periodic. Since f is a Schwartz function, F converges absolutely and uniformly on every compact subset of $\mathbb{R}$, and thus, it is continuous everywhere. Applying the same argument to the derivatives of f, we conclude that F is also infinitely differentiable, so that it admits a Fourier expansion

$$F(x) = \sum_{m\in\mathbb{Z}} c_m e^{2\pi imx},$$

which converges uniformly to F, and whose coefficients are given by

$$\begin{aligned} c_m &= \int_0^1 F(x)e^{-2\pi imx}\, dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi imx}\, dx \\ &= \int_{\mathbb{R}} f(y)e^{-2\pi imy}\, dy = \hat{f}(m). \end{aligned}$$

Note that the interchange of the sum and the integral is valid since the convergence is uniform. Hence, we conclude that

$$\sum_{n\in\mathbb{Z}} f(x+n) = F(x) = \sum_{m\in\mathbb{Z}} c_m e^{2\pi imx} = \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{2\pi imx}.$$

In particular, the desired formula is obtained by setting $x = 0$. □

1.4 Some special functions

Some particular functions emerge in many unrelated contexts in mathematics. This is the case of gamma and theta, both playing a crucial role in the theory of the Riemann zeta function. Let us recall some of their properties.

1.4.1 The gamma function

The gamma function originally arose when it turned out convenient to find a factorial expression for non-integer real values, and it was subsequently extended to the complex numbers. From the two natural approaches to the function, we will introduce it by means of a Mellin transform, while the other is based on the Weierstrass product, which will not be needed here.

Definition 1.4.1. (GAMMA FUNCTION) The *gamma function*, denoted $\Gamma(s)$, is defined for all s with $\mathrm{Re}(s) > 0$ by the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt.$$

An easy computation shows that $\Gamma(1) = 1$, along with the fundamental recurrence relation

$$\Gamma(s+1) = s\Gamma(s), \tag{1.4}$$

which can be proved integrating by parts. Thus, by induction, it follows that $\Gamma(n) = (n-1)!$ for every positive integer n.

Note that $\Gamma(s)$ corresponds simply to the Mellin transform of e^{-t}. Using previous results, this observation enables us to deduce some of its analytic properties and to extend its domain of definition further.

Proposition 1.4.1. $\Gamma(s)$ *is holomorphic for all* s *with* $Re(s) > 0$.

Proof. Since $f(t) = e^{-t}$ is continuous and satisfies the conditions

$$e^{-t} \underset{t\to 0^+}{\sim} 1, \quad e^{-t} \underset{t\to +\infty}{=} O(t^{-N}),\ \forall N > 0,$$

by Proposition 1.2.1, $f^*(s) = \Gamma(s)$ is holomorphic for $\mathrm{Re}(s) > 0$. □

Proposition 1.4.2. $\Gamma(s)$ *can be analytically continued to a meromorphic function in* $\mathbb{C}$, *with only simple poles, of residue* $(-1)^n/n!$, *at each non-positive integer* $s = -n$, $n \in \mathbb{Z}_{\geq 0}$.

Proof. By Proposition 1.2.3, it follows easily from the power series expansion of $f(t) = e^{-t}$ at zero.

□

However, these well-known properties can also be obtained from the functional equation (1.4), m applications of which gives the analytic continuation

$$\Gamma(s) = \frac{\Gamma(s+m)}{s(s+1)\cdots(s+m-1)}$$

to $\mathrm{Re}(s) > -m$. Therefore, since m is arbitrary, we can extend $\Gamma(s)$ to a meromorphic function in the whole complex plane from merely its values in the strip $0 < \mathrm{Re}(s) \leq 1$. Note that the above expression also gives us all the information about its poles and residues. From now on, the function $\Gamma(s)$ will denote this unique continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Furthermore, many other remarkable identities of $\Gamma(s)$ can be proven (see for example [20, pp. 45-48]), such as *Euler's reflection formula*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{1.5}$$

which relates gamma directly with the sine function and shows its symmetry about the line $\mathrm{Re}(s) = 1/2$, or *Legendre's duplication formula*

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right), \tag{1.6}$$

which allows us to express values of gamma at half-integers in terms of its values at integers.

Note that the reflection formula proves that $\Gamma(s)$ never vanishes. If we assume $\Gamma(s_0) = 0$, since the right-hand side of (1.5) is never zero, then $\Gamma(1-s)$ should have a pole at $s_0 \in \mathbb{Z}_{\geq 1}$, which is clearly a contradiction. As a consequence, the reciprocal gamma function, $\Gamma^{-1}(s)$, is entire with simple zeros at non-positive integers. Furthermore, setting $s = 1/2$ yields $\Gamma(1/2) = \sqrt{\pi}$ (since $\Gamma(x) > 0$ for $x > 0$), which determines all values at half-integers by the functional equation.

1.4.2 The theta function

The arc-length of an ellipse and of many other curves cannot be expressed in terms of elementary functions, which leads to a wide class of integrals, whose inverses are called elliptic functions. The theta functions appeared then as auxiliary tools in such calculations. Jacobi introduced four of them of complex variables z and τ, and he derived their properties purely algebraically. Following Riemann, one emphasizes the theta function

$$\vartheta_3(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z},$$

which is holomorphic for all $z \in \mathbb{C}$ and $\mathrm{Im}(\tau) > 0$. Jacobi also discovered the fundamental transformation formula

$$\vartheta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\pi i z^2/\tau} \vartheta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right),$$

where $\sqrt{-i\tau}$ denotes the branch of the square root defined on the upper half-plane. This formula is actually a generalization of our next theorem.

When viewed as a function of z, they can be seen as quasi doubly-periodic elliptic analogues of the basic trigonometric functions, while if considered as a function of τ, they reveal their modular nature. In some contexts like number theory, "the theta function" means $\vartheta_3(z,\tau)$ evaluated at $z = 0$. In our case, we are especially interested in setting $\tau = it$ for real $t > 0$ to define

$$\vartheta_3(0, it) = \theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \quad t \in \mathbb{R}_{>0},$$

which we will simply call the *theta function.*

As an application of the Poisson summation formula, the following theorem shows the modularity of the theta function and gives us a fundamental tool to derive the functional equation of the Riemann zeta function.

Theorem 1.4.3. *$\theta(t)$ satisfies for $t > 0$ the functional equation*

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right).$$

Proof. It is straightforward to check that $f(x) = e^{-\pi x^2 t}$ is a Schwartz function. Taking its Fourier transform and completing the square we reach to

$$\hat{f}(m) = \int_{\mathbb{R}} f(x) e^{-2\pi i x m}\, dx = e^{-\pi m^2/t} \int_{\mathbb{R}} e^{-\pi t (x + im/t)^2}\, dx.$$

Now, changing variables, it is easy to justify the movement in the line of integration by using limits and Cauchy's theorem, so we get

$$\int_{\mathbb{R}} e^{-\pi t(x+im/t)^2}\, dx = \int_{u=im/t+\mathbb{R}} e^{-\pi t u^2}\, du = \int_{\mathbb{R}} e^{-\pi t u^2}\, du,$$

which is no more than the well-known Gauss integral

$$\int_{-\infty}^{\infty} e^{-\pi t u^2} du = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} e^{-v} v^{-1/2}\, dv = \frac{\Gamma(1/2)}{\sqrt{\pi t}} = \frac{1}{\sqrt{t}}.$$

Thus, we have proved the Fourier transform of $f(x)$ to be

$$\hat{f}(m) = \frac{1}{\sqrt{t}} e^{-\pi m^2/t}.$$

Finally, it suffices to apply the Poisson summation formula (1.3) to obtain

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m) = \frac{1}{\sqrt{t}} \sum_{m\in\mathbb{Z}} e^{-\pi m^2/t} = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).$$

□

Remark. This can also be proved by showing that the Gaussian function $g(x) = e^{-\pi x^2}$ is its own Fourier transform and using the scaling property.

Note that we can deduce the asymptotic behaviour near zero of the theta function from its functional equation. Let $N > 0$ be arbitrary. Thus, since we have $\theta(t) = 1 + O(t^{-N})$ as $t \to +\infty$, we deduce that $\theta(t) = t^{-1/2} + O(t^N)$ as $t \to 0^+$. At first sight, it seems unsuitable to take its Mellin transform. However, we can correct this issue by expressing the function as

$$\theta(t) = 1 + 2\psi(t) \quad \text{where} \quad \psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

In this case, we actually have $\psi(t) = O(e^{-\pi t})$ as $t \to +\infty$, since

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} \le \sum_{n=1}^{\infty} e^{-\pi n t} = \frac{e^{-\pi t}}{1 - e^{-\pi t}} < 2e^{-\pi t}$$

for all $t \ge 1$. This replacement will enable us to obtain the functional equation of the Riemann zeta function by means of its Mellin transform. For doing so, we also need its transformation formula, which is immediately inherited from the ordinary theta function.

Corollary 1.4.4. *$\psi(t)$ satisfies for $t > 0$ the functional equation*

$$\psi(t) = \frac{1}{\sqrt{t}}\psi\left(\frac{1}{t}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{t}} - 1\right).$$

We are almost done with the preliminaries. We only need to recall some basic notions of the so-called Bernoulli numbers, which play an important role in the remainder of this work.

1.5 The Bernoulli numbers

The Bernoulli numbers were discovered while trying to give a closed expression for the sum of equal powers of the first n integers. Their most common definition was posed by Euler as follows.

Definition 1.5.1. (Bernoulli numbers) The *Bernoulli numbers*, denoted B_n, are the coefficients of the exponential generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{1.7}$$

Note that expanding the left-hand side as a Maclaurin series and matching coefficients on both sides allows us to compute the first terms of the sequence. However, we can rewrite the expression in the alternative form

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\left(\frac{e^z + 1}{e^z - 1}\right) = \frac{z}{2}\coth\frac{z}{2},$$

which is actually an even function. Therefore, we already know $B_1 = -1/2$ and $B_{2n+1} = 0$ for all $n \geq 1$. In passing, we have also derived the expansion

$$\frac{z}{2}\coth\frac{z}{2} = \sum_{n=0}^{\infty} \frac{B_{2n} z^{2n}}{(2n)!}, \quad |z| < 2\pi,$$

and making the substitution $z \mapsto 2iz$ we obtain

$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{B_{2n} 2^{2n} z^{2n}}{(2n)!}, \quad |z| < \pi, \tag{1.8}$$

which will be used at the end of the next chapter.

From (1.7) we can also derive a simple recurrence relation, useful to generate all B_n efficiently. Replacing $e^z - 1$ with its power series, working out the Cauchy product between both series and equating coefficients yields

$$B_0 = 1, \qquad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad \text{for all } n \geq 1,$$

which proves that they form a rational sequence and leads to an intuitive understanding of their structure. The very first non-zero Bernoulli numbers are $B_2 = 1/6$, $B_4 = -1/30$ and $B_6 = 1/42$, which give the misleading impression that they converge to zero. However, the sequence B_{2n} actually grows unbounded very rapidly in absolute value.

The Bernoulli numbers hold a wide variety of connections with many different topics, such as the Euler-Maclaurin summation formula or even Kummer's regular primes in Fermat's last theorem. In fact, we will prove them to hold a deep relationship with the Riemann zeta function, as they are particularly useful to evaluate some of its values at integer arguments.

Chapter 2

The Riemann zeta function

We are finally in a position to study the main properties of the Riemann zeta function. To begin with, we will present its initial definition along with its analytic properties. Then, two essential representations will be given. There exists a vast literature on this function, our main references are [21][4][5].

2.1 Definition and basic properties

Definition 2.1.1. (Riemann zeta function) The *Riemann zeta function*, denoted $\zeta(s)$, is defined for all s with $\mathrm{Re}(s) > 1$ by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.1}$$

Proposition 2.1.1. *$\zeta(s)$ is holomorphic for all s with $\mathrm{Re}(s) > 1$. Furthermore, its derivative in this region is given by*

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s}.$$

Proof. Note that for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \sigma$ we have

$$|n^{-s}| = |e^{-s \ln n}| = e^{-\sigma \ln n} = n^{-\sigma}.$$

Then, $\zeta(s)$ converges absolutely for s if and only if $\mathrm{Re}(s) > 1$, since the series $\zeta(\sigma)$ converges for any real $\sigma > 1$ (the integral test may be used since all its terms are monotonically decreasing) and diverges otherwise (by comparison with the harmonic series, which does not converge). It follows from the Weierstrass M-test that partial sums of $\zeta(s)$, which are entire, converge uniformly in the half-plane $\mathrm{Re}(s) \geq 1 + \delta$ for every $\delta > 0$, and therefore, also in every compact subset contained in $\mathrm{Re}(s) > 1$. Hence, by Weierstrass' Theorem, $\zeta(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 1$ and its derivatives can be obtained by termwise differentiation of the series. $\square$

Remark. In number theory, the analogue to the Riemann zeta function for an unspecified algebraic number field is called the Dedekind zeta function, where the sum is taken likewise over the non-zero ideals of its ring of integers. The interested reader is referred to [12, p. 132] or [16, p. 457].

Actually, this behaviour is not exclusive to this series, but there is a whole family (which includes the Riemann zeta function) acting very similarly in this sense. These are the so-called Dirichlet series, for which we will give now a brief description. Proofs can be found at [11, pp. 116-119].

2.1.1 Dirichlet series

Definition 2.1.2. A *Dirichlet series* is a series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s \in \mathbb{C}, \tag{2.2}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence of complex numbers.

For each Dirichlet series, there exists a unique number $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$, called the *abscissa of absolute convergence*, such that (2.2) converges absolutely in the half-plane $\mathrm{Re}(s) > \sigma_a$ but does not in the half-plane $\mathrm{Re}(s) < \sigma_a$. Furthermore, a similar argument to that of the Riemann zeta function shows that the convergence is uniform on compact subsets in its half-plane of absolute convergence, so that the series is holomorphic there. Also, even though it is trickier to prove, the *abscissa of convergence* σ_c, defined analogously, satisfies the same analytic properties. However, both numbers do not have to be necessarily equal, as seen in the following example.

Example. Consider the *Dirichlet eta function* (or alternating zeta series)

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \tag{2.3}$$

Its abscissa of absolute convergence is of course $\sigma_a = 1$. However, convergence of $\eta(\sigma)$ at any real $\sigma > 0$ is easily proved by the alternating series test, while it clearly diverges for $\sigma \leq 0$. Hence, its abscissa of convergence is in fact $\sigma_c = 0$, so that $\eta(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

Therefore, we have just shown that σ_a can be strictly larger than σ_c. But this strip of conditional convergence is never wider, since it can be proved that the inequality $\sigma_c \leq \sigma_a \leq \sigma_c + 1$ is always satisfied.

Remark. If the sequence $\{a_n\}$ from (2.2) is non-negative, as in the case of the Riemann zeta function, we necessarily have $\sigma_c = \sigma_a$. This implies that the expression (2.1) actually diverges for any s with $\mathrm{Re}(s) < 1$.

2.1.2 The Euler product

Euler observed that the sum over all natural numbers defining $\zeta(s)$ can be developed as an infinite product over all prime numbers. However, while he studied this duality for real arguments, it was Riemann who considered it for the first time for all complex values of s. The Euler product plays a key role in the analytic theory of primes and it is usually stated as the analytic equivalent to the law of unique prime factorization of integers.

Theorem 2.1.2. (Euler product) *For all s with $Re(s) > 1$, the following identity holds:*

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}, \tag{2.4}$$

where $\mathbb{P}$ denotes the set of all primes. Moreover, $\zeta(s)$ has no zeros there.

Proof. For all s with $\mathrm{Re}(s) > 1$ we have

$$\sum_{p\in\mathbb{P}} |p^{-s}| = \sum_{p\in\mathbb{P}} p^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-\sigma} < \infty,$$

which means, by Theorem 1.1.4, that the right-hand side of (2.4) is holomorphic. Moreover, since none of its factors vanishes, we can also deduce that the equality would imply $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$. Let us prove now that it actually holds. Since $p \geq 2$, each factor in the product can be written as an absolutely convergent geometric series (indeed for $\mathrm{Re}(s) > 0$), i.e.

$$\frac{1}{1-p^{-s}} = \sum_{k=0}^{\infty} \frac{1}{p^{ks}},$$

which allows termwise multiplication in the product. Suppose now that $\mathcal{P} \subset \mathbb{P}$ is any finite set of primes. Thus, by the Fundamental Theorem of Arithmetic, taking the product of these series over $\mathcal{P}$ yields

$$\prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} = \prod_{p\in\mathcal{P}} \left(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \right) = \sum_{n\in\mathbb{N}(\mathcal{P})} \frac{1}{n^s},$$

where $\mathbb{N}(\mathcal{P})$ is the set of all positive integers whose prime decomposition contains only primes from $\mathcal{P}$. Letting $\mathcal{P} = \mathcal{P}_m$ be the set of all primes up to m and taking the limit as $m \to \infty$, we obtain the desired result. □

Remark. The infinitude of primes can be proved as a corollary. If the set $\mathbb{P}$ were finite, the Euler product would have a finite limit as $s \to 1^+$, whereas the limit on $\zeta(s)$ actually diverges. The stronger assertion that the sum of the reciprocals of all primes diverges can also be proved from here.

Exercise. Let $n \in \mathbb{N}$. Prove that the probability P_n of k randomly chosen positive integers up to n being relatively prime as $n \to \infty$ is $1/\zeta(k)$.

Solution. Consider each of the possible n^k choices as a k-tuple $(a_1, a_2, \ldots, a_k)$ with n fixed and $1 \leq a_i \leq n$ for all $1 \leq i \leq k$, and let $p_1, \ldots, p_r$ be all the prime numbers less than or equal to to n. Therefore, by the inclusion-exclusion principle expressed in its complementary form, the number of tuples whose elements share no common prime divisor is given by

$$n^k - \sum_{1 \leq i \leq r} \left\lfloor \frac{n}{p_i} \right\rfloor^k + \sum_{1 \leq i < j \leq r} \left\lfloor \frac{n}{p_i p_j} \right\rfloor^k - \cdots + (-1)^r \left\lfloor \frac{n}{p_i \cdots p_r} \right\rfloor^k.$$

Thus, dividing by n^k to compute P_n and taking the limit $n \to \infty$ (so that $\lfloor n/a \rfloor / n \to 1/a$ for any a), the expression obtained is seen to be

$$\lim_{n\to\infty} P_n = \prod_{p \in \mathbb{P}} \left(1 - p^{-k}\right) = \frac{1}{\zeta(k)}.$$

□

2.1.3 An integral representation

So far we have expressed the Riemann zeta function as an infinite sum and as an infinite product. Surprisingly, it turns out that $\zeta(s)$ can also be written as an infinite integral. Let us consider for $t > 0$ the function

$$f(t) = \sum_{n=1}^{\infty} e^{-nt} = \frac{e^{-t}}{1 - e^{-t}} = \frac{1}{e^t - 1}, \tag{2.5}$$

which can indeed be expanded as a geometric series since $e^{-t} < 1$. Hence, by Proposition 1.2.2, the Mellin transform of $f(t)$ is given by

$$f^*(s) = \left(\sum_{n=1}^{\infty} n^{-s}\right) \mathcal{M}[e^{-t}; s] = \Gamma(s)\zeta(s) \tag{2.6}$$

whenever $\mathrm{Re}(s) > 1$, a condition that simultaneously ensures absolute convergence of the Dirichlet series and of the Mellin transform. Therefore, we have incidentally obtained the following integral representation of $\zeta(s)$.

Proposition 2.1.3. *For all s with $Re(s) > 1$, we have*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt. \tag{2.7}$$

Many of the properties of $\zeta(s)$ are determined by this integral representation. In fact, as we shall see, it turns out to be a very appropriate starting point to extend its domain to a larger region of the plane.

2.2 Analytic continuation of the zeta function

All we have done so far applies only for values in the half-plane $\mathrm{Re}(s) > 1$. However, many of the deeper properties of the Riemann zeta function take place for other values of s, i.e., they concern its analytic continuation. There are several techniques that permit to extend its domain of definition, but as we already know, they all give rise to the same function, due to the uniqueness of analytic continuation.

Now, we will take advantage of the results from last section to characterize its analytic continuation to the whole complex plane. Note that the function $f(t)$ from (2.5), which decays exponentially, also admits at $t = 0$ the complete asymptotic expansion

$$f(t) = \frac{1}{e^t - 1} = \sum_{n=-1}^{\infty} \frac{B_{n+1}}{(n+1)!} t^n, \quad t < 2\pi,$$

where the coefficients B_n are the Bernoulli numbers. Thus, it follows from Proposition 1.2.1 that $f^*(s) = \Gamma(s)\zeta(s)$ is holomorphic for $\mathrm{Re}(s) > 1$, which we already knew. However, by Proposition 1.2.3, this also means that it can be analytically continued to a meromorphic function in $\mathbb{C}$. What is more, its only singularities are simple poles at each integer less than or equal to 1, except for negative even integers, where the corresponding Bernoulli number is zero and thus, they are regular points. In any case, we have

$$\lim_{s \to -n} (s+n)(\Gamma(s)\zeta(s)) = \frac{B_{n+1}}{(n+1)!}, \quad n \in \mathbb{Z}_{\geq -1}.$$

On the other hand, recall from Proposition 1.3.2 that $\Gamma(s)$ is also meromorphic in $\mathbb{C}$, and that its only singularities are simple poles of residue $(-1)^n/n!$ at each non-positive integer $s = -n$, that is,

$$\lim_{s \to -n} (s+n)\Gamma(s) = \frac{(-1)^n}{n!}, \quad n \in \mathbb{Z}_{\geq 0}.$$

Hence, since $\Gamma(s)$ never vanishes and $\Gamma(1) = 1$, we deduce the following.

Proposition 2.2.1. *$\zeta(s)$ can be analytically continued to a meromorphic function in $\mathbb{C}$ with a unique simple pole of residue 1 at $s = 1$. Furthermore, the values of $\zeta(s)$ at non-positive integers are simply given by*

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}, \quad n \in \mathbb{Z}_{\geq 0}.$$

In particular, this proposition proves values at non-positive integers to be rational, and gives some remarkable values as $\zeta(0) = -1/2$, or even the famous $\zeta(-1) = -1/12$, which is often misunderstood as the sum of all positive integers, corresponding to the original series (for which is obviously not defined). In addition, for all $n \in \mathbb{Z}_{\geq 1}$, $B_{2n+1} = 0$ implies $\zeta(-2n) = 0$. Anyway, each of these results will be proved again later by other means.

We have just shown that the definition of $\zeta(s)$ can be extended beyond the half-plane of convergence of the original series, where we have also deduced some of its main properties. Nevertheless, we have not given yet any expression of this analytic continuation. In the following two subsections, we will try to make sense of $\zeta(s)$ for larger regions of the plane.

2.2.1 Extension to the critical strip

Before extending $\zeta(s)$ to the whole complex plane, we shall start first by doing so, by elementary means, to the right half-plane $\mathrm{Re}(s) > 0$. This, together with the functional equation, will give us a full insight of the function. Let us consider the Dirichlet eta function $\eta(s)$ from (2.3), which is holomorphic for $\mathrm{Re}(s) > 0$. Note that for $\mathrm{Re}(s) > 1$ we have

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = \left(1 - \frac{2}{2^s}\right) \zeta(s),$$

or equivalently,

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}. \tag{2.8}$$

Hence, we have just shown $\zeta(s)$ to be extended to a meromorphic function in $\mathrm{Re}(s) > 0$, with perhaps some poles where $1-s$ is an integer multiple of $2\pi i/\ln 2$. However, $\eta(s)$ can be proved to vanish at each of these points except $s = 1$, but it is not evident at first sight. Note that the classical result $\eta(1) = \ln 2$ shows that $\zeta(s)$ has indeed a simple pole at $s = 1$.

On the other hand, we can immediately deduce from the expression (2.8) that $\zeta(\sigma) < 0$ on the real segment $0 < \sigma < 1$, since in that section, the terms from $\eta(\sigma)$ can be grouped pairwise so that each of them is positive, and the denominator is clearly negative. In particular, $\zeta(s)$ has no zeros there.

Another elementary approach to this partial extension consists in comparing the sum $\sum_{n=1}^{\infty} n^{-s}$ with its corresponding integral $\int_1^{\infty} x^{-s}\, dx$. It is straightforward to check that for $\mathrm{Re}(s) > 1$ we have

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} n^{-s} - \int_1^{\infty} x^{-s}\, dx = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\, dx, \tag{2.9}$$

where each integral on the right-hand side is holomorphic on $\mathrm{Re}(s) > 0$ by Theorem 1.1.3. Using the basic relations

$$|f(b) - f(a)| = \left| \int_a^b f'(t)\, dt \right| \leq \max_{a \leq t \leq b} |f'(t)||b - a|,$$

we see that for fixed s with $\mathrm{Re}(s) > 0$ and $x \in [n, n+1]$ where $n \geq 1$, each integrand satisfies the estimate

$$|n^{-s} - x^{-s}| = \left| \int_n^x st^{-s-1}\, dt \right| \leq |s| \int_n^x \left| t^{-s-1} \right| dt \leq \frac{|s|}{n^{\sigma+1}},$$

which follows easily from (2.9) to

$$\left| \zeta(s) - \frac{1}{s-1} \right| \leq |s|\zeta(\sigma + 1).$$

Therefore, $\zeta(s) - 1/(s-1)$ is a sum of holomorphic functions converging uniformly for every $\mathrm{Re}(s) \geq \delta$, and thus, holomorphic in $\mathrm{Re}(s) > 0$. Note that this incidentally exhibits the pole of $\zeta(s)$ at $s = 1$ with residue 1. Next, we could develop this idea to extend $\zeta(s)$ to the half-plane $\mathrm{Re}(s) > -n$ for any positive integer n. However, once we have defined it on this right half-plane, the entire analytic continuation is obtained in one step much more easily from Riemann's functional equation, which we present below.

Remark. As an immediate consequence of (2.9), we note that the zeta function has a certain symmetry about the real axis, namely $\zeta(\overline{s}) = \overline{\zeta(s)}$.

2.2.2 Further extension by means of its functional equation

Riemann realized that the further study of primes was related with the analytic continuation of $\zeta(s)$ to the rest of the plane. However, he did not only develop this analytic continuation, but he also derived a *functional equation* relating the values of $\zeta(s)$ to those of $\zeta(1-s)$. This symmetry about the line $\mathrm{Re}(s) = 1/2$ allows us to easily study the behaviour of $\zeta(s)$ on the left half-plane, where the function is not naturally defined.

The functional equation can be proved in several different ways (see for example [21, pp. 13-27], where seven methods are presented), but here we will follow one of Riemann's original proofs (he also proved it from (2.7) by contour integration and applying the residue theorem), which is still one of the most elegant and maintains a great significance in number theory. However, the proof is slightly modified, since we are using the theory of Mellin transforms to save explanations on convergence.

For this proof we will approach $\zeta(s)$ in an alternative way. Recall that the theta function is defined for real $t > 0$ by

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \quad \text{where} \quad \psi(t) = \frac{\theta(t)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

Now, we introduce after Riemann the following meromorphic function.

Definition 2.2.1. The *completed zeta function* is given for $\mathrm{Re}(s) > 1$ by

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

First, we will express $\xi(s)$ by means of the Mellin transform of $\psi(t)$, which is of exponential decay, as opposed to $\theta(t)$. Finally, using the identity of $\psi(t)$ derived in Chapter 1, we will show that $\xi(s)$ can be analytically continued to all of $\mathbb{C}$ and that it is invariant under the substitution of s by $1-s$. The following provides the integral representation we need for $\xi(s)$.

Lemma 2.2.2. *For all s with Re$(s) > 1$, we have*

$$\xi(s) = \int_0^{\infty} \psi(t) t^{\frac{s}{2}-1}\, dt. \tag{2.10}$$

Proof. Growth and decay conditions of $\psi(t)$ assure its Mellin transform to be holomorphic for $\mathrm{Re}(s) > 1/2$. Then, by Proposition 1.2.2 we obtain

$$\psi^*(s) = \left(\sum_{n=1}^{\infty} \left(\pi n^2\right)^{-s}\right)\mathcal{M}[e^{-t};s] = \pi^{-s}\left(\sum_{n=1}^{\infty} n^{-2s}\right)\Gamma(s) = \pi^{-s}\Gamma(s)\zeta(2s),$$

where the sum and the integral converge absolutely. Consequently, we have

$$\xi(s) = \psi^*\left(\frac{s}{2}\right) = \int_0^{\infty} \psi(t) t^{\frac{s}{2}-1}\, dt$$

for $\mathrm{Re}(s) > 1$, as it was needed. □

Considering the asymptotic behaviour of $\psi(t)$, its transform offers weaker information about the analytic continuation of $\zeta(s)$ than the one from (2.6), as it does not give any expression for non-positive arguments. However, the advantage of this second approach relies on the fact that the transformation formula of $\psi(t)$ leads to the symmetric form of the functional equation.

Theorem 2.2.3. *The function $\xi(s)$ can be analytically continued to a meromorphic function in $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. Furthermore, it satisfies the functional equation*

$$\xi(s) = \xi(1-s) \tag{2.11}$$

for all $s \in \mathbb{C}$.

Proof. Recall from Corollary 1.4.4 that the functional equation for $\psi(t)$ is given by the expression

$$\psi(t) = t^{-\frac{1}{2}}\psi\left(\frac{1}{t}\right) + \frac{1}{2}\left(t^{-\frac{1}{2}} - 1\right), \quad t > 0.$$

Substituting this expression in the integral (2.10) of $\xi(s)$ from 0 to 1 yields

$$\int_0^1 \psi(t)t^{\frac{s}{2}-1}\,dt = \int_0^1 \psi\left(\frac{1}{t}\right)t^{\frac{s-1}{2}-1}\,dt + \frac{1}{2}\int_0^1 \left(t^{\frac{s-1}{2}-1} - t^{\frac{s}{2}-1}\right)dt,$$

valid for $\mathrm{Re}(s) > 1$, where the last integral can be evaluated explicitly, i.e.

$$\frac{1}{2}\int_0^1 \left(t^{\frac{s-1}{2}-1} - t^{\frac{s}{2}-1}\right)dt = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)}.$$

On the other hand, we can make the change of variables $t \leftrightarrow 1/t$ to get

$$\int_0^1 \psi\left(\frac{1}{t}\right)t^{\frac{s-1}{2}-1}\,dt = \int_1^\infty \psi(t)t^{\frac{1-s}{2}-1}\,dt.$$

Finally, bringing back all the pieces together gives

$$\xi(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(t)\left(t^{\frac{s}{2}-1} + t^{\frac{1-s}{2}-1}\right)dt. \tag{2.12}$$

This has been proved under the assumption that $\mathrm{Re}(s) > 1$, but the exponential decay of $\psi(t)$ at infinity shows as usual that the above integral defines an entire function. Therefore, we conclude that $\xi(s)$ has an analytic continuation to a meromorphic function in the whole complex plane, with only simple poles at $s = 0$ and $s = 1$. Moreover, it remains unchanged when replacing s by $1-s$. Thus, it satisfies $\xi(s) = \xi(1-s)$ as we wanted. □

Remark. $\xi(s)$ is sometimes considered including the factor $s(s-1)$ or even $s(s-1)/2$, which do not affect the functional equation and make the function entire (of order 1), leading to its product representation.

As a by-product, the expression (2.12) provides an explicit form of the analytic continuation of $\zeta(s)$ to a meromorphic function in $\mathbb{C}$, given by

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left[\frac{1}{s(s-1)} + \int_1^\infty \psi(t)\left(t^{\frac{1-s}{2}-1} + t^{\frac{s}{2}-1}\right)dt\right].$$

Then, since $\Gamma^{-1}(s/2)$ is entire with simple zeros at non-positive even integers, we know that $\zeta(s)$ has a unique simple pole at $s = 1$ of residue 1, and that it vanishes at each negative even integer. Let us see this more clearly by the following asymmetrical formulation of the functional equation.

Corollary 2.2.4. (Functional equation) *For all $s \in \mathbb{C}$ it holds that*

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s). \tag{2.13}$$

Proof. Starting from the identity (2.11) we have

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now, isolating $\zeta(s)$ and with the aid of Euler's reflection formula (1.5) and Legendre's duplication formula (1.6), we obtain the expression

$$\zeta(s) = 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{2.14}$$

Equivalently, substituting s by $1-s$ proves the corollary. □

Remark. Note that the functional equation is useless to be evaluated at $s=0$. However, since $\Gamma(2-s) = (1-s)\Gamma(1-s)$, equation (2.14) implies

$$-1 = \lim_{s\to 1}(1-s)\zeta(s) = \lim_{s\to 1} 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(2-s)\zeta(1-s) = 2\zeta(0),$$

that is, $\zeta(0) = -1/2$, agreeing with Proposition 2.2.1.

Note that since the Euler product tells us $\zeta(s)$ never vanishes in the half-plane $\mathrm{Re}(s) > 1$, we immediately deduce that the only (simple) zeros in $\mathrm{Re}(s) < 0$ come from the cosine function. These are located at the negative even integers, and they are usually called the *trivial zeros* of the zeta function. Therefore, in conjunction with the fact that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$ either (see [19, pp. 185-187]), which is a key fact in the proof of the prime number theorem (in fact, both results are equivalent), it follows that all non-trivial zeros lie in the *critical strip* $0 < \mathrm{Re}(s) < 1$. This is the only strip in which $\zeta(s)$ is allowed to behave erratically.

On the other hand, by (2.8) we have already proved that $\zeta(\sigma)$ has no zeros in the real segment $0 < \sigma < 1$. Furthermore, since that expression also shows $\zeta(\sigma)$ to be real whenever $\sigma > 0$, the functional equation (2.13) gives that $\zeta(s)$ is real on the real axis, which is equivalent by the Schwarz reflection principle (see [19, pp. 57-60]) to $\zeta(\overline{s}) = \overline{\zeta(s)}$. Hence, non-trivial zeros are all complex and they are distributed symmetrically with respect to both the real axis and the *critical line* $\mathrm{Re}(s) = 1/2$. Moreover, Riemann proved that these zeros get denser as we go up the critical strip.

What is more, the famous *Riemann hypothesis* strengths these results, claiming that all such zeros actually lie on the critical line. The conjecture seems true, since Hardy proved in 1914 that there are infinitely many, while it has been numerically checked that the first 10^{13} zeros belong there. However, it still remains as one of the most important open problems in all mathematics, due primarily to the substantial improvement its assumption implies over the estimate of the error term in the prime number theorem. In fact, the Riemann hypothesis is equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x),$$

where $\pi(x)$ is the prime-counting function and $\mathrm{Li}(x)$ is the offset logarithmic integral, so that its failure would create havoc in the distribution of primes.

2.3 Special values of the zeta function

In the previous section, we gave a rough picture of how $\zeta(s)$ looks like as a whole in the complex plane. Instead, we will now seek exact values of the zeta function in the positive real axis, in particular, for those integer arguments lying in the original domain of convergence.

2.3.1 Values at positive integers

We have already shown that values for non-positive integer arguments are expressible in terms of the Bernoulli numbers, and that they are given by

$$\zeta(1-n) = (-1)^{n-1}\frac{B_n}{n}, \quad n \in \mathbb{Z}_{\geq 1}. \tag{2.15}$$

Furthermore, evaluating the functional equation at $2n$ and using the above yields the following expression for even positive integer arguments,

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \quad n \in \mathbb{Z}_{\geq 1}.$$

However, this formula can be obtained independently of the functional equation and of other values of $\zeta(s)$. For this reason, we will try to verify both results following an alternative approach.

Remark. Euler computed all the values from (2.15) by formal manipulation of power series, using what we now know as Abel summation. Actually, he compared them with those at the positive even integers and essentially conjectured the functional equation for real values a century before Riemann proved it. See [1] for a detailed explanation.

The Basel problem and positive even arguments

Euler's early work on the zeta function was motivated by the so-called *Basel problem*, which consists on finding an exact solution to the sum of the reciprocals of all perfect squares. After long effort, he managed to solve it by equating two different expressions for the sine function, which is quite remarkable, taking into account the tools he had available at the time. On the one hand, applying the infinite product for the sine termwise we have

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = 1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) z^2 + O(z^4).$$

On the other, the Maclaurin series for the sine gives

$$\frac{\sin \pi z}{\pi z} = 1 - \frac{\pi^2}{3!} z^2 + O(z^4).$$

Hence, comparing coefficients of the quadratic term yields

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is, arguably, one of the most beautiful results in all mathematics, since it tells us about deep connections between very distant concepts. Other values like $\zeta(4) = \pi^4/90$ can also be computed similarly. In fact, Euler noticed that values of $\zeta(2n)$ are always a rational multiple of π^{2n}. As we have already anticipated, it turns out that they can also be expressed in terms of the Bernoulli numbers as follows.

Theorem 2.3.1. *The values of $\zeta(s)$ at positive even integers are given by*

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \quad k \in \mathbb{Z}_{\geq 0}. \tag{2.16}$$

Proof. The sketch of the proof is very similar to the one above, but this time we are using the cotangent as a bridge. We compare the expression (1.8) derived in Chapter 1 in terms of Bernoulli numbers with another expansion in terms of the zeta function, obtained from the sine product formula

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right). \tag{2.17}$$

For all $z \in \mathbb{C}$ with $|z| < \pi$, all the hypotheses from Theorem 1.1.4 are satisfied. Therefore, the product is holomorphic in $\mathbb{C}$ and since none of its factors vanishes there, we can take its logarithmic derivative as

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - (n\pi)^2}.$$

As long as $|z| < \pi$, we can use the formula for the geometric series to get

$$\begin{aligned} z\cot z &= 1 - 2\sum_{n=1}^{\infty}\left(\frac{z}{n\pi}\right)^2 \frac{1}{1-\left(\frac{z}{n\pi}\right)^2} \\ &= 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{z}{n\pi}\right)^{2k} \\ &= 1 - 2\sum_{k=1}^{\infty}\frac{z^{2k}}{\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\ &= 1 - \sum_{k=1}^{\infty}\frac{2\zeta(2k)}{\pi^{2k}}z^{2k}, \end{aligned}$$

where the interchange of summations is justified since both series converge absolutely in that region. Finally, manipulating the expression from (1.8) it follows that

$$z\cot z = 1 - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}B_{2k}}{(2k)!}z^{2k}.$$

Hence, by uniqueness of the Maclaurin expansion, equating coefficients from both expressions and rearranging terms, we obtain the desired result.

□

Remark. This theorem additionally proves some of our preliminary observations about the Bernoulli numbers of even index. Also, it proves that they alternate in sign and enables us to estimate their asymptotic growth.

Finally, applying the functional equation to (2.13) yields

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}, \quad k \in \mathbb{Z}_{\geq 0},$$

which combined with trivial zeros and the value at zero, verifies our result from Proposition 2.2.1 and offers an almost complete scope about the values of the zeta function at integer arguments. However, the reader may have noticed that there is an exception, which we discuss below.

Apéry's constant and positive odd arguments

Euler had the goal to present a closed-form formula of $\zeta(k)$ for every $k \in \mathbb{Z}\backslash 1$. Unfortunately, he failed in doing so for odd positive integers. In fact, their arithmetical nature is still a mystery, since no similar expression to (2.16) has been discovered yet (note that the functional equation is useless in this case). Trying to give an answer to this fact has become a tremendous open problem in number theory. Actually, it took until 1979 to shed some light on this issue, when Apéry proved the irrationality of $\zeta(3)$, also known as *Apéry's constant*. In his proof, he used the rapidly converging series

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.$$

In 2000, Rivoal proved that there exist infinitely many odd arguments for which the value of zeta is irrational, while Zudilin proved that at least one among $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational. Besides, any results concerning if they are transcendental or not seems far from being proved today.

2.3.2 Some values arising in physics

Values of the Riemann zeta function are not only relevant in number theory, but they are also found during many physical calculations. In fact, the aforementioned Apéry's constant $\zeta(3)$ is used in quantum electrodynamics to compute the second and third order corrections of the electron magnetic moment, which is one of the best measured and calculated numbers in all of physics. On the other hand, they arise naturally quite often together with the gamma function, when solving certain integrals of the form of (2.6). For instance, $\zeta(4)$ appears in the derivation of the Stefan-Boltzmann law from Planck's law, and $\zeta(3/2)$ is used to calculate the critical temperature for the transition to Bose-Einstein condensation.

Furthermore, the value of $\zeta'(0)$ is sometimes needed in the context of zeta function regularization, as we present in the next chapter. A way to compute this value is to apply the Euler transformation of series to the Dirichlet eta function $\eta(s)$ (see [18]), obtaining for $\mathrm{Re}(s) > -1$ the expression

$$(1-2^{1-s})\zeta(s) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\left[n^{-s} - (n+1)^{-s}\right].$$

Taking the derivative on both sides and evaluating at $s = 0$ yields

$$-\zeta'(0) - \ln 2 = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \ln\frac{n}{n+1} = \frac{1}{2}\ln\frac{\pi}{2},$$

where the last equality follows easily from the basic identities of the logarithm and the famous Wallis product, which is an immediate consequence of applying $z = \pi/2$ in (2.17). Therefore, we have

$$\zeta'(0) = -\frac{1}{2}\ln 2\pi. \tag{2.18}$$

Chapter 3

Zeta function regularization

The Riemann zeta function is known to be of extreme importance in analytic number theory. Actually, it is regarded mostly as falling completely within the realm of pure mathematics, since for example, it is not a solution to any physically motivated differential equation. However, zeta functions are often used in physics, especially quantum field theory, where calculations are plagued with formally divergent expressions from which one needs to extract physically meaningful information.

Among the different procedures for giving meaning to these ill-defined expressions, the so-called *zeta function regularization* is one of the most powerful and elegant methods, since it provides finite results via analytic continuation at once, with no need to remove or subtract divergent quantities. In practice, we need to extrapolate the initial definition of the Riemann zeta function to the relevant differential operator in each case (where the natural numbers are replaced by the eigenvalues of the operator). The zeta function encoding the eigenvalues of the Laplacian on a compact Riemannian manifold was first constructed by Carleman (for the case of a compact region of the plane). Later, Minakshisundaram and Pleijel showed its convergence and its analytic continuation to a meromorphic function in the whole complex plane. Here we will consider a slightly more general setting and study the structure of the associated zeta function.

The use of divergent series has been quite controversial within the mathematical community for many centuries, since giving sense of them is not justified in any logical way (only by the pragmatic discovery that they give useful results). The classical monograph [10] is recommended to the reader. Nevertheless, it has been proved experimentally to give accurate predictions in physics, as in the case of the Casimir effect, for which we will provide an alternative and more modern approach.

3.1 Introduction

Let us see how the Riemann zeta function can be used as a summation method by giving two simple examples. We first consider the sum of all positive integers $S = \sum_{n=1}^{\infty} n$, which is obviously divergent. This series arises naturally in string theory, in particular, in the discussion of transverse Virasoro operators. Roughly speaking, a string is a 1-dimensional object that replaces the notion of a point particle, and whose time-evolution sweeps out a 2-dimensional surface called a world-sheet, as it moves in a d-dimensional spacetime $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$, usually known as target. In particular, for bosonic string theory (where the strings correspond only to bosons and there are no fermions) certain Virasoro constraints force the spacetime dimension d to assume a specific value. Specifically, a condition of the form

$$a = -\left(\frac{d-2}{2}\right) S$$

arises, where $a = 1$ must be imposed to preserve Lorentz invariance. This gives a motivation for reinterpreting S as the analytic continuation of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ at $s = -1$. Thus, if we formally assign to S the unique and finite value $\zeta(-1) = -1/12$, the critical dimension of the bosonic string is forced to be $d = 26$. More can be found at [22].

Similarly, this can also be done with infinite products. We now consider the product of all positive integers $P = \prod_{n=1}^{\infty} n$. Recall from Proposition 2.1.1 that the derivative of $\zeta(s)$ is given for $\mathrm{Re}(s) > 1$ by

$$\zeta'(s) = -\sum_{n=1}^{\infty} (\ln n) n^{-s}.$$

Then, if we formally evaluate this expression at $s = 0$, we find

$$-\zeta'(0) = \sum_{n=1}^{\infty} \ln n = \ln P.$$

Therefore, since we already know the value $\zeta'(0) = -\frac{1}{2}\ln 2\pi$ from (2.18), it seems natural to formally assign to P the finite value

$$P = e^{-\zeta'(0)} = \sqrt{2\pi}. \tag{3.1}$$

Remark. If $a > 0$, the Hurwitz zeta function $\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}$ can be used in the same way to generalize (3.1) to the classical Lerch's formula

$$\prod_{n=0}^{\infty} (n + a) = \frac{\sqrt{2\pi}}{\Gamma(a)},$$

which actually exhibits the structure of relations involving the gamma function in a much clearer way. In fact, it respects the location of its simple poles and helps us understand its functional equation intuitively.

Giving a meaning in this way to otherwise divergent series or products by interpreting them as special values of suitable zeta functions is what we usually call zeta regularization. In fact, these ideas can be generalized to any suitable sequence $\{\lambda_n\}$ of non-vanishing complex numbers. In particular, we will now consider the discrete spectrum of a certain (partial) differential operator, and encode it by means of its associated zeta function.

3.2 Spectral zeta functions

In this section, we will study boundary value problems and their associated zeta functions. However, we will not give any proof, since many of them are similar to the ones concerning the Riemann zeta function and we are not really going to make use of them. Our main reference is [13], the interested reader may consult [14] from the same author or any of the books [3][6][7].

Fundamental properties of physical systems are often encoded in the spectrum of certain differential operators, which leads to the analysis of the so-called *spectral functions*. Many of the following results can be generalized to any elliptic pseudo-differential operator on a compact manifold. But in our case, we are only interested in the most classical setting, that is, the class of Laplace-type operators on a d-dimensional Riemannian manifold M, possibly with a boundary ∂M. By a Laplace-type operator P we mean that it can be written in the unified form

$$P = -g^{ij}\nabla_i^V \nabla_j^V - E,$$

where g^{ij} is the metric of M, ∇^V is the connection on M acting on a smooth vector bundle V over M and E is an endomorphism of V.

The advantage of this particular class of operators lies on the fact that, imposing suitable boundary conditions, the eigenvalue spectrum is real, positive, discrete and explicitly known. Suppose that we have a positive increasing sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ given by

$$P\phi_n(x) = \lambda_n \phi_n(x).$$

Then, we can define the *spectral zeta function* $\zeta_P(s)$ associated to P as

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

for sufficiently large $\mathrm{Re}(s)$.

From now on we consider M to be a d-dimensional smooth compact Riemannian manifold with a smooth boundary ∂M, and we suppose that Dirichlet or Neumann boundary conditions have been imposed. Under these conditions, it follows from the classical Weyl's law, which states that the eigenvalues λ_n behave asymptotically for $n \to \infty$ as

$$\lambda_n^{d/2} \sim \frac{2^{d-1}\pi^{d/2}\Gamma(d/2)d}{\mathrm{vol}(M)} n,$$

that $\zeta_P(s)$ converges in the half-plane $\mathrm{Re}(s) > d/2$.

Example. Consider the Laplacian operator $P = -\partial^2/\partial x^2$ on the interval $M = [0, L]$ and the corresponding Dirichlet boundary value problem

$$P\phi_n(x) = \lambda_n \phi_n(x), \quad \phi_n(0) = \phi_n(L) = 0,$$

which is well-known to have eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}.$$

In this case, the zeta function $\zeta_P(s)$ associated with the boundary value problem turns out to be a multiple of the Riemann zeta function, i.e.,

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^{-2s} = \left(\frac{L}{\pi}\right)^{2s} \zeta(2s) \tag{3.2}$$

for $\mathrm{Re}(s) > 1/2$. Note that its domain of definition agrees with the half-plane of convergence mentioned above, since in this case we have $d = 1$.

Remark. The Riemann zeta function and many of its generalizations, such as the ones from Hurwitz, Barnes or Epstein, can all be thought of as being generated by eigenvalues of specific boundary value problems (see [13]). The first ones are associated with a linear spectrum, while Epstein-type zeta functions are associated with spectra of a quadratic form.

As it happens with the Riemann zeta function, most of the relevant properties of $\zeta_P(s)$ lie to the left of its half-plane of convergence. Indeed, most of the ideas from the previous chapter remain applicable here, and its analytic continuation is performed by a similar method to the one applied to the Riemann zeta function. Its meromorphic structure is revealed by considering the corresponding *heat kernel* $\theta_P(t)$, given by

$$\theta_P(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t},$$

which is of exponential decay and clearly diverges as t tends to zero. As usual, by Proposition 1.2.2 we can express the spectral zeta function $\zeta_P(s)$ in terms of the heat kernel $\theta_P(t)$ as the Mellin transform

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^\infty \theta_P(t) t^{s-1} \, dt,$$

still valid for $\mathrm{Re}(s) > d/2$. Therefore, we know that possible residues and special values of the analytic continuation of $\zeta_P(s)$ arise exclusively from the asymptotic behaviour of $\theta_P(t)$ as $t \to 0$. Although it is certainly not easy to read off, the asymptotic expansion of the heat kernel $\theta_P(t)$ as $t \to 0$ is well known to be (see for example [9]) of the form

$$\theta_P(t) \sim \sum_{k=0}^{\infty} a_k t^{\frac{k-d}{2}}, \tag{3.3}$$

where the a_k are the so-called heat kernel coefficients. These coefficients depend explicitly on the operator P, the geometry of the manifold M under consideration and the chosen boundary conditions. In fact, if the manifold has no boundary, the coefficients a_k with odd index vanish.

Hence, we deduce as always from Proposition 1.2.3 that $\zeta_P(s)$ admits an analytic continuation to a meromorphic function in the whole complex plane, possessing only simple poles in one to one correspondence with the half-integer powers of t in (3.3), and with its residues and special values being determined by the heat kernel coefficients a_k. In particular, we have

$$\mathrm{Res}(\zeta_P(s)\Gamma(s))|_{s=(d-k)/2} = a_k,$$

or equivalently, the residues of $\zeta_P(s)$ are given by

$$\mathrm{Res}(\zeta_P(s)) = \frac{a_{d-2s}}{\Gamma(s)}, \quad s = \frac{d}{2}, \frac{d-1}{2}, \dots, \frac{1}{2}, -\frac{2n+1}{2}, \quad \forall n \in \mathbb{Z}_{\geq 0},$$

and its values at non-positive integers are given by

$$\zeta_P(-m) = (-1)^m m! a_{d+2m}, \quad \forall m \in \mathbb{Z}_{\geq 0}.$$

This means that for manifolds without boundaries, where the coefficients a_k with odd index vanish, the behaviour of $\zeta_P(s)$ depends crucially on whether the dimension d is even or odd. For d even, its poles are simple, finite in number, and can only be located at points $s = 1/2, 1, \dots, d/2$, whereas for d odd, it will generally have in addition infinitely many simple poles located at points $s = -(2n+1)/2$ for $n \in \mathbb{Z}_{\geq 0}$, and it vanishes at each non-positive integer. If we recall the example (3.2), we see that these additional points may not exist due to the vanishing of the corresponding residues.

Remark. The explicit calculation of heat kernel coefficients for different boundary conditions has been an important issue during the last decades. In mathematics, this interest was originated in the connections between the heat equation and the Atiyah-Singer index theorem.

In any case, $\zeta_P(s)$ can be analytically continued to a neighbourhood of the origin $s = 0$ and it is holomorphic there (actually vanishing in manifolds without boundary and with odd dimension). Therefore, similarly to the example we gave in the beginning, by formal differentiation we have

$$-\frac{d}{ds}\zeta_P(s)\bigg|_{s=0} = \sum_{n=1}^{\infty} \ln \lambda_n = \ln\left(\prod_{n=1}^{\infty} \lambda_n\right),$$

which suggests the definition of the *zeta regularized determinant* of P in terms of its associated zeta function, given by

$$\det\ P = \prod_{n=1}^{\infty} \lambda_n = e^{-\zeta_P'(0)}.$$

Remark. This definition was introduced by the mathematicians Ray and Singer in the context of analytic torsion, and later used by Hawking in physics due to ambiguities of dimensional regularization when applied to quantum field theory in curved spacetime.

Furthermore, since $\zeta_P(s)$ is also holomorphic at $s = -1$, we can define the *zeta regularized trace* of P to be

$$\mathrm{tr}\ P = \sum_{n=1}^{\infty} \lambda_n = \zeta_P(-1).$$

As we shall see in the next section, it turns out to be physically useful to assign such finite values to certain divergent series.

It is worth mentioning that explicit knowledge of the spectrum is in general only guaranteed for highly symmetric manifolds, such as the torus, the sphere or regions bounded by parallel planes. However, in the specific example of physical application chosen later we will make sure that the spectrum of the operator is indeed known. In fact, the associated zeta function will be closely related to the Riemann zeta function, for which we have a good knowledge at our disposal. As an example of the many applications of zeta regularization within quantum field theory under the influence of external conditions, we next consider the so-called *Casimir effect*.

3.3 Physical application: the Casimir effect

This phenomenon takes its name after the Dutch prominent physicist H.B.G. Casimir, who in 1948 predicted theoretically an attractive force between two uncharged, perfectly conducting parallel plates in vacuum. The literature on this subject is very extensive, see for example [15].

3.3.1 Background

More than a century ago, Planck described how energy of emitted radiation is quantized, i.e., how it can only assume integral multiples of the basic energy value $h\nu$, where ν is the frequency of the radiation and h is what now is called the Planck constant. This theory led him to his second radiation law, in which oscillators possessed a non-vanishing residual energy of average value $h\nu/2$. This marked the birth of the concept of zero-point energy, as the lowest possible energy that a quantum system may have.

According to quantum field theory, a field in the vacuum does not really vanish, but rather fluctuates, allowed by the fundamental Heisenberg uncertainty principle. Casimir showed that van der Waals interactions could be successfully explained in terms of the change in the zero-point energy of the electromagnetic field, caused by the presence of external constraints. In the original setting considered by Casimir, the presence of the plates determines a boundary condition, so that the frequencies of the radiation between the plates are restricted to a discrete set of values. Therefore, the difference between the energies (with and without plates) is finite, which gives rise to a mechanical pressure on the plates of value

$$P(a) = -\frac{\hbar c \pi^2}{240 a^4}, \tag{3.4}$$

where a is the distance between the plates, $\hbar$ is the reduced Planck constant and c is the speed of light in a vacuum. Note that the negative sign indicates that the force is attractive, and that its quantum mechanical character is revealed by the fact that it vanishes in the classical limit $\hbar \to 0$.

As we can see, the strength of the force is tiny and falls off very rapidly with distance, which means that it is only measurable when the distance between the objects is extremely small, where it becomes dominant. The importance of this topic lies on the continuing miniaturization of technological devices towards the nanometer range. In fact, Casimir forces are of direct practical relevance in nanotechnology, where, for instance, sticking of mobile components in micromachines might be caused by them. Instead of fighting the occurrence of the effect, the tendency now is to try taking technological advantage of it.

Nowadays, the Casimir effect is not only considered in its original form, but also assuming any other kind of external constraint or boundary condition. In fact, the pressure exerted on the surfaces delimiting the boundaries may be either positive or negative, depending on the exact form of the constraints and on the nature of the fields. Actually, two decades after Casimir predicted the effect, Boyer found a repulsive pressure of magnitude $F(R) \sim 1/R^4$ for a perfectly conducting spherical shell of radius R. In the absence of general answers on the dependence of the Casimir energy on the underlying geometry, one approach consists in accumulating further knowledge by studying specific configurations.

Taking into account the small distances required and the corrections imposed by the experimental setup, the early attempts to measure the Casimir force did not verify sufficiently well the effect. But, in spite of the technical difficulties in detecting such an weak force, its verification proved to be feasible in the experiment carried out by Lamoreaux in 1997, which is usually considered to be the first actual reliable measurement of the Casimir force. Since then, this field has undergone an impressive progress, where experimental data and theoretical predictions have been proved to be in excellent agreement. The best tested configurations are those of parallel plates, a plate and a sphere or even a plate and a cylinder. This interplay between theory and experiments and its possible technological applications are the main reasons for the revival of this issue in recent years.

On the other hand, the nature of these forces, as well as their distinction from van der Waals forces, has been far from being clear for many years. Recently, Jaffe stated that experimental verification of the Casimir effect does not establish by itself the reality of zero-point fluctuations. In fact, whether direct evidence of these fluctuations exists or not is still a very controversial topic. Since Einstein's theory of general relativity has much wider consensus, a search at the cosmological level has been proposed, as the accelerated expansion of the universe might indicate its existence. However, quantum theoretical calculations of the contribution of zero-point energy to Einstein's cosmological constant lead to a value which is off by roughly 120 orders of magnitude as compared with observational tests. This large discrepancy is known as the *cosmological constant problem* and it is one of the greatest unsolved mysteries in theoretical physics.

3.3.2 The Casimir energy

Before we restrict ourselves to any specific setting, we will first introduce briefly the zeta function regularization of the Casimir energy, for the case of a massless scalar field. The extension to, for instance, an electromagnetic field, should be straightforward from here. Since we are only concerned on the influence of boundary conditions, we will only have to consider the negative of the Laplacian $P = -\Delta$, for a smooth compact d-dimensional Riemannian manifold with a smooth boundary.

The Casimir energy of a quantum scalar field $\Phi(t, \mathbf{x})$ in a $(d+1)$-dimensional ultrastatic spacetime, satisfying the field equation

$$(\partial_t^2 + P)\Phi(t, \mathbf{x}) = 0, \tag{3.5}$$

and fulfilling suitable boundary conditions, is formally given by

$$E_{\text{Cas}} = \frac{1}{2}\sum_n \omega_n,$$

where the one-particle energies $\omega_n = \sqrt{\lambda_n}$ are obtained from the eigenvalues λ_n of the Laplacian P. Note that the time-dependent part of (3.5) can be separated out from the spatial one and that $\Phi(t, \mathbf{x}) = e^{-i\omega t}\phi(\mathbf{x})$ has been set. Therefore, taking the above expression for granted, we have reached to

$$E_{\text{Cas}} = \frac{1}{2}\sum_n \lambda_n^{1/2}$$

for the Casimir energy. But note that this expression is purely formal, since, in general, the sequence λ_n grows unbounded for $n \to \infty$ and the series diverges. Naively, we could try to regularize it by considering the term $\lambda_n^{1/2}$ as λ_n^{-s} for $s = -1/2$ and formally assigning to it the value

$$E_{\text{Cas}} = \frac{1}{2}\zeta_P\left(-\frac{1}{2}\right), \tag{3.6}$$

where $\zeta_P(s)$ is the analytic continuation of the spectral zeta function associated to P, as seen in the previous section.

Note that the above expression is only valid when $\zeta_P(s)$ is regular at $s = -1/2$. In fact, it depends crucially on the boundary conditions imposed, and on whether the dimension d is even or odd, since as we have already seen, $\zeta_P(s)$ may possess a simple pole at that point whenever d is odd. Thus, the kernel coefficient $a_{(d+1)}$ is usually an obstacle for giving a finite definition for the Casimir energy. In such cases, we would require a more general expression applying a principal part prescription [2, 13], but giving rise to a finite ambiguity, and which reduces to (3.6) when the latter is also well-defined. Nevertheless, for many interesting geometries, we are guaranteed to have $a_{(d+1)} = 0$, so that the residue vanishes and we do not encounter any pole. Thus, we obtain a unique physical result for the Casimir energy. Let us see that this actually happens for the original configuration of parallel plates considered by Casimir.

3.3.3 Derivation for parallel plates

We now take $d = 3$ space dimensions and derive the Casimir effect for the idealized configuration considered by Casimir, where two identical parallel plates are held a distance a apart. For the sake of simplicity, we consider a massless scalar field instead of an electromagnetic field.

Without loss of generality, we may assume the plates to be of infinite extension and perpendicular to the x-axis, and we fix the left plate at $x = 0$. Therefore, the boundary value problem to be solved is simply

$$-\Delta u_k(x, y, z) = \lambda_k u_k(x, y, z),$$

where k is a multi-index and Dirichlet boundary conditions are imposed on the fixed plates $x = 0$ and $x = a$, that is,

$$u_k(0, y, z) = u_k(a, y, z) = 0.$$

Now, we compactify for the moment the (y, z)-directions to a torus with perimeter length R, and impose periodic boundary conditions in both directions, which means that the eigenfunctions have to satisfy

$$\begin{aligned} u_k(x, 0, z) &= u_k(x, R, z), \qquad & \frac{\partial}{\partial y} u_k(x, 0, z) &= \frac{\partial}{\partial y} u_k(x, R, z), \\ u_k(x, y, 0) &= u_k(x, y, R), \qquad & \frac{\partial}{\partial z} u_k(x, y, 0) &= \frac{\partial}{\partial z} u_k(x, y, R). \end{aligned}$$

Later we will consider the limit $R \to \infty$ so that the initial configuration is recovered. By the usual process of separation of variables, we can obtain the normalized eigenfunctions in the form

$$u_{n,m,l}(x, y, z) = \sqrt{\frac{2}{aR^2}} \sin\left(\frac{\pi l}{a} x\right) e^{i\frac{2\pi n}{R} y} e^{i\frac{2\pi m}{R} z},$$

with their corresponding eigenvalues being

$$\lambda_{n,m,l} = \left(\frac{2\pi n}{R}\right)^2 + \left(\frac{2\pi m}{R}\right)^2 + \left(\frac{\pi l}{a}\right)^2, \quad (n, m) \in \mathbb{Z}^2, \quad l \in \mathbb{N}.$$

Therefore, taking $P = -\Delta$ we have to study the spectral zeta function

$$\zeta_P(s) = \sum_{(n,m)\in\mathbb{Z}^2} \sum_{l=1}^{\infty} \left[\left(\frac{2\pi n}{R}\right)^2 + \left(\frac{2\pi m}{R}\right)^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s}$$

and its behaviour at a neighbourhood of $s = -1/2$, in which we are especially interested. As we already know, this expression is only valid for the half-plane $\mathrm{Re}(s) > 3/2$, which means that we need its analytic continuation to the left of the plane, where singularities may exist. Fortunately, it turns out as we will see that in this case it can be expressed in terms of the Riemann zeta function and things will not get too complicated.

The sums can be obviously interchanged, and since we are assuming the limit $R \to \infty$, the distance a between the plates is much smaller than R, so that we may replace the Riemann sum by a double integral. Therefore, by a simple change of variables we reach to the expression

$$\zeta_P(s) = \left(\frac{R}{2\pi}\right)^2 \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[k_1^2 + k_2^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s} dk_1\, dk_2.$$

Now, converting the integral to polar coordinates yields

$$\zeta_P(s) = \left(\frac{R}{2\pi}\right)^2 \sum_{l=1}^{\infty} 2\pi \int_0^{\infty} k \left[k^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s} dk,$$

which follows easily by direct integration to

$$\zeta_P(s) = \frac{R^2}{2\pi} \frac{1}{2(1-s)} \sum_{l=1}^{\infty} \left[k^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s+1} \Bigg|_0^{\infty},$$

since s is such that $\mathrm{Re}(s) > 3/2$. For this reason, the integral converges to

$$\zeta_P(s) = -\frac{R^2}{4\pi(1-s)} \sum_{l=1}^{\infty} \left(\frac{\pi l}{a}\right)^{2(-s+1)}.$$

As a consequence, the above expression can be reduced to obtain

$$\zeta_P(s) = -\frac{R^2}{4\pi(1-s)} \left(\frac{\pi}{a}\right)^{2-2s} \zeta_R(2s-2),$$

where $\zeta_R(s)$ is just the ordinary Riemann zeta function. Recall that according to the results from previous section, since we are in odd dimensions ($d = 3$), the function $\zeta_P(s)$ may diverge at $s = -1/2$ due to the presence of a simple pole. However, we have been lucky enough and there is actually no singularity, which means that the Casimir energy is simply given by

$$E_{\mathrm{Cas}} = \frac{1}{2}\zeta_P\left(-\frac{1}{2}\right) = -\frac{1}{2}\frac{R^2}{4\pi}\frac{2}{3}\left(\frac{\pi}{a}\right)^3 \zeta_R(-3) = -\frac{R^2\pi^2}{12a^3}\zeta_R(-3).$$

At this point, we just have to recall from (2.15) the value

$$\zeta_R(-3) = -\frac{B_4}{4} = \frac{1}{120}$$

to obtain that the corresponding Casimir energy is precisely

$$E_{\text{Cas}}(a) = -\frac{R^2\pi^2}{1440a^3}. \tag{3.7}$$

The Casimir force per unit surface (that is, the pressure) between two parallel plates at a distance a from each other is therefore given by

$$F_{\text{Cas}}(a) = -\frac{\partial}{\partial a}\frac{E_{\text{Cas}}(a)}{R^2} = -\frac{\pi^2}{480a^4}. \tag{3.8}$$

The negative sign on the above equation indicates that the right plate $x = a$ is attracted to the left, and we notice that the force decreases much faster than gravity due to the factor a^4 in the denominator. This is a truly remarkable result, which shows that the force is independent of the nature of the plates and depends exclusively on the distance between them.

Remark. The pressure on an electromagnetic field given by (3.4) differs to ours by the fundamental constants $\hbar$ and c, along with a factor of 2, as a consequence of the physical fact that the two possible polarizations of the electromagnetic field double the number of modes. Note that we have obtained the correct result without any explicit subtractions.

Note that in this computation only those vacuum fluctuations from between the plates have been considered. In case we wanted to find the force acting on, for instance, the plate at $x = a$, we would also have to take into account the contribution from the right to this plate. This can be done by placing a third plate at the position $x = L$, and taking the limit $L \to \infty$ at the end. Following the above calculation, we only need to replace a by $L-a$ on (3.7) to prove that the contribution to the force on the plate at $x = a$ is

$$F_{\text{Cas}}(a) = \frac{\pi^2}{480(L-a)^4}, \tag{3.9}$$

which shows that the plate at $x = a$ is always attracted to the closer plate. As $L \to \infty$ we see that (3.9) vanishes and that (3.8) also describes the total force on the plate at $x = a$ for the parallel plate configuration.

Conclusions

In pure mathematics, many topics are of great intrinsic interest. We have proved the Riemann zeta function to be a clear example. This object appears in a wide variety of beautiful results and open problems in number theory, such as the Euler product, the distribution of prime numbers or the well-known Riemann hypothesis. What is more, the role of other special functions in its functional equation is of great significance, and the relation of some of its particular values with Bernoulli numbers is captivating.

During the last part of the work, we have realized that aside from all these outstanding results, it also encounters applications in other disciplines. At first glance, the Riemann zeta function and physics are clearly disjoint fields of study. But we have shown that if we extend the definition of the zeta function to the eigenvalues of a suitable differential operator, we can use its analytic continuation as a summation method to give physical meaning in a formal way to many divergent expressions arising in quantum field theory. In particular, we have applied these ideas to derivate the famous Casimir effect for the simplest setting, where the relevant zeta function is essentially reduced to the one of Riemann.

Mathematics are undeniably influenced by their applications in other fields. However, they often find many uses which were not initially expected. Sometimes, as in our case, these applications are by no means obvious, so that they would be unknown if it were not for the early development of the theory. This is the reason why, in my humble opinion, mathematics should be studied not only for the sake of usefulness but also for their inner beauty. Otherwise, we would not let the curious minds of tomorrow unveil the deepest secrets of mathematics and enjoy the thrills these have to offer.

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