Some contributions to the theory of singularities and their characteristic classes

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A Rosa i Emilio,
els meus estimats pares.

Diuen que cada nou matí ens porta mil roses; sí, però,
on són els pètals de les roses d'ahir?

Omar Khayyam

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Resumen

En esta tesis doctoral se aportan contribuciones a la teoría de singularidades, así como a la teoría de clases características de variedades singulares. La primera parte de esta tesis se centra en el estudio de singularidades que aparecen en la imagen de aplicaciones. Los principales objetos de estudio en la teoría de singularidades de aplicaciones son los gérmenes de aplicación, y uno de los objetivos de esta teoría es la clasificación de dichos gérmenes. En esta dirección, D. Mond formuló una conjetura muy relevante en el área. Aunque algunos casos de esta conjetura han sido resueltos, el caso general aún permanece abierto a día de hoy. La conjetura relaciona dos importantes invariantes analíticos de gérmenes de aplicación de $(\mathbb{C}^n, 0)$ a $(\mathbb{C}^{n+1}, 0)$ de distinta naturaleza, estos son la $\mathcal{A}_e$-codimensión y el número de Milnor en la imagen μ_I. Dado $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ un germen de aplicación con singularidad inestable en 0 y con $(n, n+1)$ en el rango de buenas dimensiones de Mather, la conjetura establece que $\mathcal{A}_e$-codim(f) es menor o igual que $\mu_I(f)$, y con igualdad cuando f es un germen de aplicación casi-homogéneo. La conjetura de Mond se conoce para los siguientes casos: para $n \leq 2$, para gérmenes de aplicación de tipo fold, y para gérmenes de corrango 1 con $\mathcal{A}_e$-codimensión 1. Una de las dificultades de esta conjetura es determinar μ_I, ya que por su naturaleza topológica es difícil de calcular en general. El capítulo 3 y el capítulo 4 de esta tesis doctoral están dedicados a la obtención fórmulas que permiten calcular de forma efectiva μ_I para ciertos gérmenes de aplicación. En el capítulo 3, probamos una fórmula para el número de Milnor en la imagen μ_I de gérmenes de aplicación de corrango 1, mientras que en el capítulo 4 obtenemos dos fórmulas para el número de Milnor en la imagen μ_I en el caso de gérmenes de aplicación casi-homogéneos de $(\mathbb{C}^n, 0)$ a $(\mathbb{C}^{n+1}, 0)$, con $n = 4$ y 5. Estas últimas fórmulas se basan en un resultado introducido por T. Ohmoto que involucra clases características de espacios singulares, dando lugar a fórmulas que permiten calcular fácilmente el número de Milnor en la imagen. Estas fórmulas son obtenidas a través la interacción entre la teoría de singularidades de aplicaciones y la teoría de clases características, y muestran un claro ejemplo de la utilidad de estas clases para el estudio de espacios singulares.

Las clases características fueron introducidas en la década de 1930 por E. Stiefel como parte de la teoría de la obstrucción en el estudio de fibrados vectoriales de variedades lisas. Estas clases son clases de cohomología que miden la trivialidad del fibrado vectorial. En los años siguientes, se definieron diferentes clases características para fibrados vectoriales, y estas se generalizaron

para variedades singulares. Las clases características de variedades singulares son clases de homología que recuperan, para el caso no singular, la clase característica de cohomología correspondiente tomando el producto cap con la clase fundamental. Uno de los principales intereses en la teoría de clases características de espacios singulares es la comparación de distintas clases de cierto espacio singular. Por un lado, con el objetivo de unificarlas. Por otro lado, para estudiar qué información captura la diferencia entre dos clases distintas sobre la variedad singular. Una de las técnicas para definir estas clases es mediante el uso de transformaciones naturales. Estas trasformaciones parten de cierto funtor que depende de la clase característica y llegan al funtor de homología de Borel-Moore. Además, esta transformación natural tiene un elemento distinguido en el funtor de partida de modo que la transformación aplicada a este elemento, recupera la clase característica de cohomología correspondiente para el caso no singular.

J. P. Brasselet, J. Schürmann y S. Yokura, respondiendo a una pregunta formulada por R. MacPherson sobre la teoría de unificación de clases características de espacios singulares, definen la clase de homología de Hirzebruch $T_{y,*}$ como una transformación natural a través de teoría de Hodge. Esta transformación parte del funtor relativo de Grothendieck de variedades algebraicas y llega al funtor de homología de Borel-Moore, y recupera, para el caso no singular, la importante clase característica de cohomología de Hirzebruch T_y^*.

La clase de Hirzebruch T_y^* para variedades no singulares, nace del teorema de Hirzebruch-Riemman-Roch generalizado (g-HRR) probado por F. Hirzebruch. Este teorema, en términos de la clase de Chern y la clase de Hirzebruch T_y^*, recupera para distintos valores de y los siguientes invariantes: Para $y = -1$, la característica de Euler, para $y = 0$, el género aritmético y, para $y = 1$, la signatura de la variedad. La clase cohomológica de Hirzebruch $T_y^*(Y)$ es una generalización a clases características de estos tres invariantes, es decir, se especializa en la clase total de Chern (para $y = -1$), en la clase total de Todd (para $y = 0$) y en la L-clase de Thom-Hirzebruch (para $y = 1$). En los años 1980, M. Goresky y R. MacPherson introducen la homología de intersección dando lugar a la noción de signatura para una variedad singular. Además, M. Goresky y R. MacPherson generalizan la L-clase característica de Thom-Hirzebruch para espacios singulares, conocida como la L-clase de Goresky-MacPherson.

J. P. Brasselet, J. Schürmann y S. Yokura formularon la siguiente conjetura: La clase de homología de Hirzebruch $T_{y,*}$ coincide, para $y = 1$, con la L-clase de Goresky-MacPherson para variedades algebraicas complejas y compactas que son de homología racional. Esta conjetura es la generalización a clases características del importante Teorema del Índice de Hodge. Este teorema calcula la signatura de una variedad algebraica compacta lisa a través de los números de Hodge. Así pues, la conjetura de Brasselet-Schürmann-Yokura establece una generalización del Teorema del Índice de Hodge incluso para variedades de homología racional, dando así una comprensión de la L-clase de Goresky-MacPherson a través de la teoría de Hodge. En el capítulo 5, probamos la conjetura de Brasselet-Schürmann-Yokura para variedades proyectivas, y este trabajo compone el resultado principal de esta tesis doctoral.

Esta tesis se divide en cinco capítulos. El capítulo 1 se corresponde a una introducción de esta tesis doctoral. En el capítulo 2 se presentan los preliminares necesarios para los resultados principales que se exponen en los restantes tres capítulos. La sección 2.1 del capítulo 2, está dedicada a los preliminares sobre teoría de singularidades de aplicaciones necesarios para el desarrollo de los capítulos 3 y 4. En la sección 2.5 se introduce la teoría de clases características y se presentan las clases que involucran los capítulos 4 y 5. El resto de secciones del capítulo de preliminares están dedicadas a las nociones y resultados principales utilizados en el capítulo 5. En estas secciones se incluyen teoría de Hodge clásica, la teoría de t-estructuras, teoría de haces perversos, el Teorema de Descomposición y la teoría de hiperresoluciones cúbicas.

En el capítulo 3, probamos la primera fórmula obtenida para el número de Milnor en la imagen. Esta fórmula es una versión de la clásica *fórmula de Lê-Greuel* para el número de Milnor de la imagen dando lugar a un método recursivo para calcularlo.

Para un germen de aplicación $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ con $n>1$, $\mathcal{A}$-finito (con singularidad inestable en 0) de corrango 1, y $g\colon (\mathbb{C}^{n-1},0)\to(\mathbb{C}^n,0)$ otro germen de aplicación que es el corte transversal de f con respecto a una forma lineal genérica $p\colon \mathbb{C}^{n+1}\to\mathbb{C}$. Entonces, la suma de los números de Milnor en la imagen $\mu_I(f)$ y $\mu_I(g)$ de f y g, respectivamente, es igual a

$$\mu_I(f)+\mu_I(g)=\#\Sigma(p|_{X_s}),$$

donde $\#\Sigma(p|_{X_s})$ denota el número de puntos críticos de la restricción $p|_{X_s}$ de p a la imagen X_s de una perturbación estable f_s de f.

Para el caso $n=1$, la fórmula correspondiente es la siguiente

$$\mu_I(f)+m_0(f)-1=\#\Sigma(p|_{X_s}),$$

donde $m_0(f)$ es la multiplicidad de la curva parametrizada por f. Este trabajo es un trabajo conjunto con el Prof. Juan José Nuño Ballesteros.

En el capítulo 4, se expone el segundo trabajo de esta tesis doctoral que combina la teoría de singularidades de aplicaciones con la teoría de clases características. En este capítulo damos dos fórmulas que calculan el número de Milnor en la imagen para gérmenes de aplicación de $(\mathbb{C}^n,0)$ a $(\mathbb{C}^{n+1},0)$ casi-homogéneos y $\mathcal{A}$-finitos, para $n=4$ y 5, en términos de los pesos y grados asociados a la aplicación. Este es un trabajo en colaboración con el Prof. Guillermo Peñafort.

Las fórmulas se basan en un enfoque topológico que se remonta a R. Thom que conecta la geometría de aplicaciones singulares con ciertas clases de características. T. Ohmoto adaptó estas técnicas probando la existencia de fórmulas que calculan el número de Milnor en la imagen para gérmenes de aplicación casi-homogéneos en términos de sus pesos y grados, para $n \leq 5$. Estas fórmulas predichas por T. Ohmoto tienen una forma específica; son funciones racionales con denominador conocido, y cuyo numerador se obtiene del truncamiento del n-ésimo grado de la serie llamada *polinomio de Segre-MacPherson-Thom.* Esta serie tiene coeficientes $b_\alpha \in \mathbb{Q}$, y nuestro objetivo fue encontrar estos coeficientes con la siguiente técnica: Para un germen de aplicación fijado f, hay una forma de calcular $\mu_I(f)$ con del software SINGULAR, a través de los resultados obtenidos por J. Fernández de Bobadilla, J. J. Nuño y G. Peñafort sobre la conjectura de Mond. Conociendo el valor de $\mu_I(f)$, los pesos w y los grados d de f, se pueden determinar algunas relaciones entre los coeficientes b_α. Tomando suficientes f con los valores conocidos de $\mu_I(f)$ y (w, d), se puede determinar el b_α deseado. Usamos esta técnica para recuperar las fórmulas para $n = 2$ y 3 obtenidas previamente, por D. Mond y T. Ohmoto usando diferentes técnicas, respectivamente, y para determinar las nuevas fórmulas para $n = 4$ y 5. El desafío para los casos $n = 4$ y 5 fue encontrar los ejemplos para calcular los b_α debido a lo siguiente: Por un lado, los gérmenes de aplicación que son demasiado simples no aportan nueva información sobre los b_α. Por otro lado, los candidatos demasiado degenerados pueden dificultar el cálculo de μ_I, o la verificación de la $\mathcal{A}$-finitud.

En el capítulo 5, desarrollamos el trabajo principal de esta tesis doctoral, donde probamos junto al Prof. Javier Fernández de Bobadilla el caso proyectivo de la conjetura de Brasselet-Schürmann-Yokura para variedades de homología racional.

Como mencionamos anteriormente, la conjetura establece que la clase de Hirzebruch (para $y = 1$) coincide con la L-clase de Goresky-MacPherson para variedades algebraicas complejas y compactas que son de homología racional. La clase de homología de Hirzebruch $T_{y,*}$ es la generalización para variedades singulares de la clase de cohomología de Hirzebruch T_y^*, definida para variedades lisas. La clase de Hirzebruch se generalizó al caso singular definiendo una transformación natural $T_{y,*}$ del funtor de Grothendieck relativo $K_0(var/-)$ de variedades algebraicas complejas al funtor de homología de Borel-Moore $H_*^{BM}(-, \mathbb{Q})$. Además, esta transformación, para diferentes valores de y, unifica las siguientes transformaciones para variedades singulares: Para $y = -1$, esta transformación recupera la transformación de Chern-Schwartz-MacPherson, generalizando la clase Chern. Para $y = 0$, la transformación da lugar a la versión singular de la clase de Todd, la transformación de Baum-Fulton-MacPherson Todd. Para $y = 1$, la clase de Hirzebruch especializa en la L-transformación de Cappell-Shaneson, introducida por S. E. Cappell, J. L. Shaneson y S. Weinberger como una generalización de la L-clase de

Goresky-MacPherson que extiende la L-clase de Thom-Hirzebruch para el caso singular. Además, para una variedad algebraica compleja Y, la transformación $T_{y,*}$ satisface que esta aplicada a la clase identidad $[Y \to Y] \in K_0(var/Y)$, especializa para $y = -1$, en $T_{-1,*}(Y) = c_*^{SM}(Y)$ la clase (racionalizada) de Chern-Schwartz-MacPherson de Y, para $y = 0$, en $T_{0,*}(Y) = td_*^{BFM}(Y)$ la clase de Baum-Fulton-MacPherson Todd de Y, si Y tiene singularidades tipo du Bois, y para $y = 1$, tenemos la conjetura de Brasselet-Schürmann-Yokura: Si Y es una variedad algebraica compleja compacta y de homología racional, entonces

$$T_{1,*}(Y) = L_*(Y),$$

donde $L_*(Y)$ es la L-clase de Goresky-MacPherson de Y.

Además, J. P. Brasselet, J. Schürmann y S. Yokura definen una transformación natural sd definida del funtor $K_0(var/-)$ al funtor de cobordismo $\Omega_{\mathbb{K}}(-)$ ($\mathbb{K}$ un subcuerpo de $\mathbb{R}$) de $\mathbb{K}$-complejos de haces acotados, cohomológicamente construibles y auto-duales, cumpliendo la siguiente igualdad de transformaciones naturales: $L_* \circ sd = T_{1,*}$, esto es, que el siguiente diagrama sea commutativo:

$$\begin{array}{ccc} K_0(var/-) & \xrightarrow{\quad sd \quad} & \Omega_{\mathbb{K}}(-) \\ & {\scriptstyle T_{1,*}} \searrow \qquad \swarrow {\scriptstyle L_*} & \\ & H_{2*}(-;\mathbb{Q}) & \end{array}$$

En este capítulo, probamos para variedades proyectivas, el siguiente resultado también conjeturado por J. P. Brasselet, J. Schürmann y S. Yokura. Además, este resultado implica la conjetura de Brasselet-Schürmann-Yokura después de aplicar la L-transformación de Cappell-Shaneson L_*:

Si Y es una variedad algebraica compleja, compacta y de homología racional, entonces tenemos la siguiente igualdad en $\Omega_{\mathbb{R}}(Y)$

$$sd_{\mathbb{R}}([Y \to Y]) = [IC_Y]$$

donde IC_Y es el complejo de haces de cohomología de intersección en Y, $\Omega_{\mathbb{R}}(Y)$ es el grupo de cobordismo de $\mathbb{R}$-complejos de haces acotados cohomológicamente construibles y auto-duales y $sd_{\mathbb{R}}$ denota la transformación sd en $\Omega_{\mathbb{R}}(Y)$.

En la demostración de la conjetura de Brasselet-Schürmann-Yokura para variedades proyectivas nos basamos en la combinación del profundo Teorema de Descomposición y la teoría clásica de Hodge, así como la teoría de hiperresoluciones cúbicas. La prueba se organiza de la siguiente forma: En la sección 5.1, obtenemos una identidad en $K_0(var/Y)$ expresando la clase identidad

$[Y \to Y]$ como una suma alternada de clases de variedades lisas procedente de una variedad semi-simplicial aumentada a Y. Esta suma alternada permite calcular $sd_{\mathbb{R}}([Y \to Y])$, donde la clase del complejo de haces de cohomología de intersección $[IC_Y]$ aparece en la expresión obtenida junto con otros términos. El objetivo es probar que la suma de términos adicionales, aparte de $[IC_Y]$, que aparecen en la expresión obtenida de $sd_{\mathbb{R}}([Y \to Y])$ se anulan. Para demostrar esto usamos ciertas sucesiones exactas de haces perversos que se obtienen en la sección 5.2. Para probar que estas sucesiones de haces perversos son exactas utilizaremos una sucesión espectral de haces perversos asociada a la variedad semi-simplicial que degenera en la segunda página de la sucesión espectral. Es en este paso donde se necesita la hipótesis de proyectividad, debido al uso de secciones hiperplanas.

Contents

Chapter 1

Introduction

In this Ph.D. thesis, we give some contributions to the singularity theory, as well as to the theory of characteristic classes of singular varieties. The first part of this work is focused on the study of singularities appearing in the image of mappings. The main objects of study in the theory of singularities of mappings are map-germs, and one of the goals in this theory is their classification. In this direction, D. Mond in [64] formulated a relevant conjecture relating two important analytical invariants of map-germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ of different nature, they are the $\mathcal{A}_e$-codimension and the image Milnor number μ_I . The conjecture states that the $\mathcal{A}_e$-codimension is less than or equal to μ_I, and with equality for weighted-homogeneous map-germs. The conjecture is proved for some particular cases: for $n \leq 2$, fold map-germs, and corank 1 map-germs with $\mathcal{A}_e$-codimension 1. One of the difficulties of this conjecture is to determine μ_I which is hard to compute in general by its topological nature. Chapter 3 and Chapter 4 are devoted to obtain formulas computing μ_I. In Chapter 3, we give a formula for μ_I for corank 1 map-germs, while in Chapter 4 we obtain two formulas for μ_I for weighted-homogeneous map-germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ with $n = 4$ and 5. The latter formulas connect the theory of singularities of mappings with the theory of characteristic classes. They are based on a result formulated by T. Ohmoto [69] which involves characteristic classes of singular spaces, giving rise to very simple computable formulas for μ_I. These formulas are an example of the usefulness of the characteristic classes to the study of singular spaces.

The classical characteristic classes were introduced in the 1930s by E. Stiefel as a part of the obstruction theory in the study of vector bundles of smooth manifolds. These classes are cohomology classes that measure the triviality of the vector bundle. Several characteristic classes were defined and were generalized to singular varieties. The characteristic classes of singular varieties are usually homology classes that recover in the non-singular case the corresponding cohomological characteristic class by capping with the fundamental class. One of the main goals in the theory of characteristic classes of singular spaces is to compare different ones in order to unify them, as well as studying what information captures the difference between two classes about the singular variety. One of the techniques to define these classes for singular varieties is by using natural transformations from certain functor depending on the characteristic class to the homology functor. Moreover, this natural transformation has a distinguished element in the source functor for which recovers the corresponding cohomology characteristic class for the non-singular case. In [9], the authors answered a question formulated by R. MacPherson about the unification of characteristic classes. They defined the Hirzebruch homology class $T_{y,*}$ as a natural transformation from the relative Grothendieck group of algebraic varieties to the Borel-Moore homology functor, which recovers for the non-singular case the important Hirzebruch cohomology characteristic class. Moreover, this class unifies three relevant characteristic classes defined as natural transformations, for different values of y. For a distinguished element in the Grothendieck group, the same authors formulated the following conjecture: The Hirzebruch homology class for $y = 1$ applied to its distinguished element coincides with the Goresky-MacPherson L-class for compact complex algebraic varieties that are rational homology manifolds. In Chapter 5, we prove the conjecture for projective varieties which composes the main work of this Ph.D. thesis.

This Ph.D. thesis is divided in the following chapters. In Chapter 2, we introduce the preliminaries needed for the main results.

In Chapter 3, we prove the first formula given for the image Milnor number. This formula is a version of the *Lê-Greuel formula* [34, 47] for the image Milnor number which provides a recursive method to compute it.

For an $\mathcal{A}$-finite corank 1 map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ with $n > 1$, and $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ another map-germ which is the transverse slice of f with respect to a generic linear form $p\colon \mathbb{C}^{n+1} \to \mathbb{C}$, that is, g has image $(X \cap H, 0)$, where $(X, 0)$ is the image of f and $H = p^{-1}(0)$, the sum of their image Milnor numbers $\mu_I(f)$ and $\mu_I(g)$ is equal to

$$\mu_I(f) + \mu_I(g) = \#\Sigma(p|_{X_s}),$$

where $\#\Sigma(p|_{X_s})$ is the number of critical points of the restriction $p|_{X_s} : X_s \to \mathbb{C}$ to the image X_s of a stable perturbation f_s of f. In the case of $n = 1$, the formula is

$$\mu_I(f) + m_0(f) - 1 = \#\Sigma(p|_{X_s}),$$

where $m_0(f)$ is the multiplicity of the curve parametrized by f. This work is a joint work with Prof. Juan José Nuño Ballesteros.

Chapter 4 is devoted to the second work which combines the theory of singularities of mappings with the theory of characteristic classes. We give two formulas which compute the image Milnor number for weighted-homogeneous map-germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$, for $n = 4$ and 5, in terms of the weights and degrees associated to the mapping. This is a work in collaboration with Prof. Guillermo Peñafort.

The formulas are based on a topological approach that goes back to R. Thom [80] connecting the geometry of singular maps to certain characteristic classes. T. Ohmoto in [69], adapted these techniques to show the existence of formulas computing the image Milnor number for weighted-homogeneous map-germs in terms of weights and degrees (for $n \leq 5$). These formulas have a specific form; they are rational functions with known denominator, whose numerator is obtained from the n-th degree truncation of the *Segre-MacPherson Thom polynomial* series. This series has coefficients $b_\alpha \in \mathbb{Q}$, and our goal was to find these coefficients by using the following technique: For fixed map-germ f, there is a way to compute $\mu_I(f)$ with the software SINGULAR, based on results in [25]. Having the value of $\mu_I(f)$ and the grading (w, d) of f at hand, one can determine some relations between the coefficients b_α. And, sampling enough f, one can determine the desired b_α. The challenge was to find these examples: On one hand, map-germs that are too simple do not yield new information about the b_α. On the other hand, degenerate candidates can be too complicated to compute their μ_I, or to check $\mathcal{A}$-finiteness. We use this approach to recover the formulas for $n = 2$ and 3 given, respectively, by D. Mond and T. Ohmoto using different techniques, and to derive new ones for $n = 4$ and 5.

In Chapter 5, we develop the main work of this Ph.D. thesis. We prove together with Prof. Javier Fernández de Bobadilla the projective case of the Brasselet-Schürmann-Yokura conjecture for rational homology manifolds formulated in [9]. As we mentioned previously, the conjecture states that the Hirzebruch homology class (for $y = 1$) coincides with the Goresky-MacPherson L-class for compact complex algebraic varieties that are rational homology manifolds. The Hirzebruch homology class $T_{y,*}$ is the generalization for singular varieties the Hirzebruch cohomology class T_y^*, defined for smooth manifolds. This homology class starts from the generalized Hirzebruch-Riemman-Roch Theorem (g-HRR) proved by F. Hirzebruch for Y non-singular, which computes the χ_y-characteristic of Y in terms of the Chern classes and the Hirzebruch cohomology class $T_y^*(Y)$. The (g-HRR) computes the Euler characteristic (for $y = -1$), the arithmetic genus (for $y = 0$), and the signature (for $y = 1$). Moreover, the Hirzebruch class $T_y^*(Y)$ specializes in the total Chern class (for $y = -1$), the total Todd class (for $y = 0$), and the Thom-Hirzebruch L-class (that is the L-polynomial in the Pontrjagin classes) (for $y = 1$). In [9], the Hirzebruch class was generalized to the singular case by defining a natural transformation $T_{y,*}$ from the relative Grothendieck functor $K_0(var/-)$ of complex algebraic varieties to the Borel-Moore homology functor $H_*^{BM}(-, \mathbb{Q})$. It satisfies that, for different values of y, unifies the following transformations: For $y = -1$, it recovers the Chern-Schwartz-MacPherson transformation, which gives a generalization of the Chern class given by [52]. For $y = 0$, it recovers the singular version of the Todd class, the Baum-Fulton-MacPherson Todd transformation [5] For $y = 1$, it recovers the Cappell-Shaneson L-transformation, introduced by S. E. Cappell, J. L. Shaneson and S. Weinberger in [15] as a generalization of the Goresky-MacPherson L-class

[30] which extends the Thom-Hirzebruch L-class for the singular case. Furthermore, for a complex algebraic variety Y, the transformation $T_{y,*}$ satisfies that, it applied to the identity class $[Y \to Y] \in K_0(var/Y)$, specialices for $y = -1$, $T_{-1,*}(Y) = c_*^{SM}(Y)$ is the (rationalized) Chern-Schwartz-MacPherson class of Y, for $y = 0$, $T_{0,*}(Y) = td_*^{BFM}(Y)$ is the Baum-Fulton-MacPherson Todd class of Y, if Y has du Bois singularities, and for $y = 1$, we have the Brasselet-Schürmann-Yokura conjecture: If Y is a compact complex algebraic variety that is a rational homology manifold, then

$$T_{1,*}(Y) = L_*(Y),$$

where $L_*(Y)$ is the Goresky-MacPherson L-class of Y.

Furthermore, there is a natural transformation sd defined in [9] from $K_0(var/-)$ to the cobordism functor $\Omega_{\mathbb{K}}(-)$ of cohomologically constructible bounded self-dual $\mathbb{K}$-complexes ($\mathbb{K}$ a subfield of $\mathbb{R}$) of sheaves, satisfying $L_* \circ sd = T_{1,*}$.

In this chapter, we prove, for projective varieties, the following result also conjectured by J. P. Brasselet, J. Schürmann and S. Yokura in [9] implying the BSY-conjecture after applying the Cappell-Shaneson L-transformation L_*:

If Y is a compact complex algebraic variety that is a rational homology manifold, then we have the equality

$$sd_{\mathbb{R}}([Y \to Y]) = [IC_Y] \in \Omega_{\mathbb{R}}(Y),$$

where IC_Y is the intersection cohomology sheaf complex on Y, and $\Omega_{\mathbb{R}}(Y)$ cobordism group for $\mathbb{R}$-complexes and $sd_{\mathbb{R}}$ denotes the transformation sd in $\Omega_{\mathbb{R}}(Y)$.

The proof is organized as follows: In Section 5.1, we obtain an identity in $K_0(var/Y)$ expressing the class $[Y \to Y]$ as an alternate sum of classes of smooth varieties coming from a semi-simplicial variety over Y. This alternate sum allows to compute $sd_{\mathbb{R}}([Y \to Y])$ where $[IC_Y]$ appears in the obtained expression together with other terms. In order to show that the sum of extra terms vanishes, we use certain exact sequences of perverse sheaves obtained in Section 5.2. To obtain the exact sequences of perverse sheaves we use the degeneration at the second page of a spectral sequence of perverse sheaves associated with the semi-simplicial variety. It is at this step where the projectivity assumption is needed, due to the use of hyperplane sections.

In [27], we prove in collaboration with Prof. M. Saito the general case of the Brasselet-Schürmann-Yokura conjecture for rational homology manifolds. However, this work will not be included as a part of this Ph.D. thesis.

Chapter 2

Preliminaries

In this chapter, we give the main results and definitions used in Chapter 3, Chapter 4, and Chapter 5. Section 2.1.1 is based on the theory of singularities of mappings under which are Chapter 3 and Chapter 4. In Section 2.5, we expose the theory of characteristic classes of singular varieties which will be needed in Chapter 4 and Chapter 5. The rest of the sections in this chapter are devoted to the basics used in Chapter 5.

2.1. Singularities of mappings

Here we introduce the main definitions and results used in Chapter 3 and Chapter 4. This section is based primarily on the general reference the theory of singularities of mappings [66]. The rest of the references used here will be indicated in the corresponding section.

2.1.1. Map-germs

Here, we expose the basics about the main objects in the theory of singularities of mappings, they are the map-germs. For more details about the general theory of map-germs see [66, 2.1].

Let X and Y be two topological spaces, and let $S \subset X$.

Definition 2.1.1. Two subsets X_1 and X_2 of X *have the same germ at* S if there is a neighborhood U of S in X, such that $X_1 \cap U = X_2 \cap U$. A *set-germ* at X is an equivalence class of subset under this relation.

Definition 2.1.2. Let $f\colon U \to Y$ and $g\colon V \to Y$ be two maps, where U and V are open neighborhoods of S in X. We say that f and g *have the same germ at* S, if there is a neighborhood $W \subset U \cap V$ of S in X, such that f and g coindice on W. A *map-germ at* S is an equivalence class under this relation.

We denote the set-germ of X_1 at S by (X_1, S), and X_1 is called the *representative* of the set-germ. The map-germ is denoted by $f\colon (X, S) \to Y$ or $f\colon (X, S) \to (Y, T)$ if $f(S) \subset T \subset Y$. For a map-germ $f\colon (X, S) \to Y$, each member $f\colon U \to Y$ of the class is called a *representative*.

A germ at one point set is called *mono-germ*, and a germ at a finite set with more than one point is called *multi-germ*.

Definition 2.1.3. A map-germ $f\colon (X,S) \to Y$ is *continuous* if there exists a continuous representative $f\colon U \to Y$.

Let $f\colon (X,S) \to (Y,T)$ be a continuous map-germ, and let $g\colon (Y,T) \to Z$ be a map-germ. The *composition* $g \circ f\colon (X,S) \to Z$ is the germ of $g \circ f\colon U \to Z$ at S. A map-germ $\phi\colon (X,S) \to (Y,T)$ is called a *homeomorphism* if there exists a representative $\phi\colon U \to V$ which is a homeomorphism, for U and V open neighborhoods of S and T in X and Y, respectively. Equivalently, ϕ is invertible as a continuous map-germ.

Let $\mathbb{F}^n$ be the affine space, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. By convention, a *smooth mapping* will mean any mapping $f\colon A \to \mathbb{F}^p$, where $A \subset \mathbb{F}^n$ is an open subset, which is differentiable of class C^∞ in the case $\mathbb{F} = \mathbb{R}$ or holomorphic (complex analytic) in the case $\mathbb{F} = \mathbb{C}$. However, we only consider the complex case.

A continuous mapping $f\colon X \to Y$ between manifolds is *smooth* if for every $x \in X$ there exists charts $\phi\colon U \to A$ in X and $\psi\colon V \to B$ in Y such that $x \in U \subset f^{-1}(V)$ and the mapping $\psi \circ f \circ \phi^{-1}\colon A \to B$ is smooth. Hence, a map-germ $f\colon (X,S) \to Y$ is *smooth* if there is a smooth representative $f\colon U \to Y$.

A smooth map-germ $\phi : (X,S) \to (Y,T)$ is a *diffeomorphism* if there exists a representative $\phi : U \to V$, where U, V are open neighborhoods of S and T in X and Y, respectively, which is a diffeomorphism. The *rank* of a smooth mapping at a point is the rank of the differential at that point. The *rank* of a map-germ $f\colon (X,S) \to (Y,T)$ at $x \in S$ is the rank of the differential of a representative of f at x. If the dimension of the domain is less than or equal to the dimension of the target, the *corank* of a map-germ is the dimension of the kernel of the differential at x of a representative; if greater than or equal, the corank is the dimension of the cokernel of the differential at x.

2.1.2. The $\mathcal{A}$-equivalence of map-germs

We define an important equivalence relation in the study of map-germs, this is the $\mathcal{A}$-equivalence of map-germs.

Definition 2.1.4. Let $f\colon (X,S) \to (Y,y)$ and $g\colon (X',S') \to (Y',y')$ be smooth map-germs. They are *left-right-equivalent* if there exist map-germs of diffeomorphism $\phi\colon (X,S) \to (X',S')$ and $\psi\colon (Y,y) \to (Y',y')$ such that $g = \psi \circ f \circ \phi^{-1}$. That is, the following diagram is commutative:

$$\begin{array}{ccc} (X,S) & \xrightarrow{f} & (Y,y) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ (X',S') & \xrightarrow{g} & (Y',y') \end{array}$$

By taking charts in X and Y, any map-gem from X to Y is right-left-equivalent to a map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$. Hence, since the source and target are fixed the equivalence can be seen as a group action. Let $\mathcal{A} = \mathrm{Diff}(\mathbb{C}^n, S) \times \mathrm{Diff}(\mathbb{C}^p, 0)$ be the group of pairs (φ, ψ) such that $\varphi\colon (\mathbb{C}^n, S) \to (\mathbb{C}^n, S)$ and $\psi\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ are map-germs of diffeomorphisms. Then, $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ and $g\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ are $\mathcal{A}$-equivalent if they are in the same $\mathcal{A}$-orbit. We denote by $\mathcal{O}_n$ the ring of function germs from $(\mathbb{C}^n, S)$ to $\mathbb{C}$.

For a map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ where $S = \{x_1, \dots, x_r\}$ is a finite set of points, and for each $k \geq 0$. The *k-jet* of f at S is

$$j^k f := (j^k f(x_1), \dots, j^k f(x_r)),$$

where $j^k f(x_i)$ is the k-jet of f at x_i, that is, the degree k Taylor polynomial of f at x_i without its constant term. The Taylor polynomial of f is determined by partial derivatives of order $\leq k$ of the component functions of f at x_i, so the k-jet can be thought of as simply recording these partial derivatives.

Definition 2.1.5. Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ a map-germ. We say f is *k-determined for $\mathcal{A}$-equivalence* if whenever the k-jet at S of another map-germ g coincides with that of f, we have that f and g are $\mathcal{A}$-equivalent, and *finitely determined* if it is k-determined for some $k \in \mathbb{N}$.

2.1.3. The $\mathcal{A}_e$-codimension of a map-germ

We define an important analytical invariant, the $\mathcal{A}_e$-codimension. It measures the obstruction of an unfolding of a map-germ to be versal. We introduce the concepts of unfolding, stability and $\mathcal{A}_e$-codimension of a map-germ.

Definition 2.1.6. Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map-germ. A *d-parameter unfolding* of f is a map-germ

$$F\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

of the form

$$F(x, u) = (\tilde{f}(x, u), u)$$

such that $\tilde{f}(x, 0) = f(x)$. If we denote the map $x \mapsto \tilde{f}(x, u)$ by f_u, then the above condition becomes simply $f_0 = f$. We call f_u a *d-parameter deformation* of f.

Two d-parameter unfoldings F, $G\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ of $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ are *equivalent* if there exist map-germs of diffeomorphisms

$$\varphi\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\})$$

and

$$\psi\colon (\mathbb{C}^p \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

which are themselves unfoldings of the identity in $\mathbb{C}^n$ and $\mathbb{C}^p$, respectively, such that $G = \psi \circ F \circ \varphi^{-1}$.

Definition 2.1.7. An unfolding $F\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ of f is *trivial* if it is equivalent to $f \times id$, i.e. the constant unfolding $(x, u) \mapsto (f(x), u)$.

Definition 2.1.8. A map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is *stable* if every unfolding of f is trivial.

Let $F\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be a d-parameter unfolding of f. Let

$$h : (\mathbb{C}^l, 0) \to (\mathbb{C}^d, 0)$$

be a map-germ, such that $v \mapsto h(v) = u$. We define $G := h^*F$ an l-parameter unfolding of f as

$$G : (\mathbb{C}^n \times \mathbb{C}^l, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^l, 0)$$

given by $(x, v) \mapsto (f(x, h(v)), v)$. The unfolding G is called the *unfolding induced from* F by h.

Definition 2.1.9. An unfolding $F\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ of $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is *versal* if every unfolding of f is $\mathcal{A}$-equivalent to h^*F for some mapping h.

By Definition 2.1.8, a map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable if every unfolding of f is trivial. Hence, the map-germ f is stable if every deformation is trivial. Then, if f_t is a 1-parameter deformation of f, there should exist deformations ϕ_t and ψ_t, such that

$$f_t = \psi_t \circ f \circ \phi_t^{-1}.$$

We define

$$ID(f) := \{\frac{df_t}{dt}|_{t=0} : F(x,t) = (f_t(x), t) \text{ any 1-parameter unfolding of } f\}$$

the space of all *infinitesimal deformations* of f.

Definition 2.1.10. Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map-germ, we define

$$T_{\mathcal{A}_e} f = \{\frac{d}{dt}(\psi_t \circ f_t \circ \varphi_t^{-1})|_{t=0} : \varphi_0 = id \text{ and } \psi_0 = id\},$$

and the quotient

$$T^1_{\mathcal{A}_e} f = \frac{ID(f)}{T_{\mathcal{A}_e} f}.$$

Definition 2.1.11. The $\mathcal{A}_e$*-codimension* of f is

$$\mathcal{A}_e - \operatorname{codim}(f) := \dim_{\mathbb{C}} T^1_{\mathcal{A}_e} f$$

Definition 2.1.12. A map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is $\mathcal{A}$*-finite* if

$$\dim_{\mathbb{C}} T^1_{\mathcal{A}_e} f < \infty.$$

The following important theorem known as Mather's infinitesimal criterion for finite determinacy was proved by J. N. Mather in [57]. A proof of this theorem can be found in [66, Theorem 6.2]:

Theorem 2.1.13. *A map-germ f is finitely determined if, and only if, it is $\mathcal{A}$-finite.*

Proposition 2.1.14. *A map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable if, and only if, $T^1_{\mathcal{A}_e} f = 0$.*

The converse of this proposition was proved by J. N. Mather in [58]. A proof of this proposition can be found in [66, Proposition 3.5, Theorem 3.2.].

We give below an alternative definition for $\mathcal{A}_e$-codimension in terms of vector fields.

Let $f\colon X \to Y$ be a smooth mapping, hence the *differential* of f is the mapping

$$df\colon TX \to TY,$$

such that $df(v) = d_x f(v)$, for each $v \in T_x X$. A *vector field* on X is a section $\xi\colon X \to TX$ of the tangent bundle. The set of vector fields on X is denoted by θ_X, and it has structure of $\mathcal{O}_X$-module, where $\mathcal{O}_X$ is the set of smooth functions from X to $\mathbb{C}$.

Definition 2.1.15. Let $f\colon X \to Y$ be a smooth mapping. A *vector field along* f is a smooth mapping $\xi\colon X \to TY$, such that $\pi \circ \xi = f$, where $\pi\colon TY \to Y$ is the canonical projection.

The set of vector fields along f is denoted by $\theta(f)$, and it has structure of $\mathcal{O}_X$-module. Notice that if $\xi \in \theta_X$ and $\eta \in \theta_Y$, then $df \circ \xi$ and $\eta \circ f$ are vector fields along f.

Lemma 2.1.16. *If φ_t and ψ_t are parameterised families of diffeomorphisms, then*

$$\frac{d}{dt}(\psi_t \circ f_t \circ \varphi_t^{-1})|_{t=0} = df \circ (\frac{d\varphi_t{}^{-1}}{dt}|_{t=0}) + (\frac{d\psi_t}{dt}|_{t=0}) \circ f$$

See a proof of this lemma in [66, Lemma 3.2].

The derivatives $\frac{d\varphi_t}{dt}|_{t=0}$ and $\frac{d\psi_t}{dt}|_{t=0}$ determine germs of vector fields on $(\mathbb{C}^n, S)$ and $(\mathbb{C}^p, 0)$, respectively. The set of all germs of vector fields on

$(\mathbb{C}^n, S)$ is denoted by $\theta_n := \theta_{\mathbb{C}^n,S}$. Moreover, there is a canonical identification between $\theta(f)$ and $ID(f)$, see Section 3.2 in [66].

We consider the following mappings:

$$tf\colon \theta_n \to \theta(f)$$

defined by $\xi \mapsto df \circ \xi$, and

$$\omega f\colon \theta_{n+1} \to \theta(f)$$

the map $\eta \mapsto \eta \circ f$.

Corollary 2.1.17. *For any map-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, then*

$$T_{\mathcal{A}_e} f = tf(\theta_n) + \omega f(\theta_p),$$

and

$$T^1_{\mathcal{A}_e} f = \frac{\theta(f)}{T_{\mathcal{A}_e} f}.$$

Theorem 2.1.18. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map-germ. A d-parameter unfolding of f,*

$$F\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0),$$

$F(x, u) = (x, f_u(x))$ is versal if, and only if,

$$T^1_{\mathcal{A}_e} f + \mathbb{C}\{\frac{\partial f_u}{\partial u_1}|_{u=0}, \dots, \frac{\partial f_u}{\partial u_d}|_{u=0}\} = \theta(f).$$

This relevant theorem was proved by J. Martinet in [56]. This means that the $\mathcal{A}_e$-codimension of f is the minimum number of parameters needed to obtain a versal unfolding.

2.1.4. Stable singularity types of mappings

We introduce the notion of stable singularity type appearing in a stable mapping. This section is based on [69] and [66].

Definition 2.1.19. Two map-germs $f\colon (\mathbb{C}^{n+s}, 0) \to (\mathbb{C}^{p+s}, 0)$, $g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are *stably $\mathcal{A}$-equivalent* if f is $\mathcal{A}$-equivalent to the trivial unfolding $g \times id_{(\mathbb{C}^s,0)}$.

Denote by η the equivalence class under this relation, and it will be called an *$\mathcal{A}$-singularity type.*

Let $f\colon X \to Y$ be a smooth map. We define the set

$$\eta(f) := \{x \in X : \text{ the germ of } f \text{ at } x \text{ is stably } \mathcal{A}\text{-equivalent to } \eta\},$$

its closure $\overline{\eta(f)} \subset X$ is called the *singular locus of f of type η.*

Remark 2.1.20. If the map $f\colon X \to Y$ is *locally stable*, that is, if the map-germ $f\colon (X, f^{-1}(y)) \to (Y, y)$ is stable for each $y \in Y$. Then, $\eta(f)$ consists of the stable singularities of type η.

Definition 2.1.21. A *multi-singularity* is an ordered set

$$\underline{\eta} := (\eta_1, \ldots, \eta_r)$$

of mono-singularities η_i of map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. We distinguish the first entry η_1 from others.

For a stable map $f\colon X \to Y$, we set

$$\underline{\eta}(f) := \left\{ \begin{array}{cc} x_1 \in \eta_1(f): & \exists x_2, \ldots, x_r \in f^{-1}(f(x_1)) \text{ s.t. } x_i \neq x_j, \\ & f \text{ at } x_i \text{ is of type } \eta_i,\ 2 \leq i \leq r \end{array} \right\},$$

and we call the closure $\overline{\underline{\eta}(f)} \in X$ the *multi-singularity locus of type $\underline{\eta}$ in the source*. The image is

$$f(\underline{\eta}(f)) := \left\{ \begin{array}{cc} y \in Y: & \exists x_1, \ldots, x_r \in f^{-1}(y), \text{ s.t. } x_i \neq x_j, \\ & f \text{ at } x_i \text{ is of type } \eta_i,\ 1 \leq i \leq r \end{array} \right\},$$

we call the closure $\overline{f(\underline{\eta}(f))} \subset Y$ the *multi-singularity locus of type $\underline{\eta}$ in the target*. The restriction map

$$f\colon \overline{\underline{\eta}(f)} \to \overline{f(\underline{\eta}(f))}$$

is finite-to-one on the critical locus. Let $\deg_1\underline{\eta}$ be the degree of this map, then

$$\deg_1\underline{\eta} = \text{the number of } \eta_1 \text{ appearing in the tuple } \underline{\eta}.$$

Let $f\colon X \to Y$ be a stable map, and let $\underline{\eta}$ be a stable type. We define

$$\underline{\eta}^o(f) := \overline{\underline{\eta}(f)} \setminus (\bigsqcup \underline{\xi}(f)) \subset X$$

where $\bigsqcup \underline{\xi}(f)$ runs all types $\underline{\xi}$ such that $\underline{\xi} \subset \overline{\underline{\eta}}$. The restriction map

$$f\colon \underline{\eta}^o(f) \to f(\underline{\eta}^o(f))$$

over the stratum $\underline{\eta}^o(f)$ is $\deg_1\underline{\eta}$-to-one. Hence, the source X is decomposed in a disjoint union of multi-singularities types $\underline{\eta}^o(f)$, and the image $f(X) \subset Y$ is decomposed in the corresponding strata of their images.

In order to introduce the remark below, we give the following important theorem due to J. N. Mather in [59]:

Theorem 2.1.22 (Mather's nice dimensions)**.** *Proper stable mappings from an n-dimensional manifold N to a p-dimensional manifold P are dense in the set of proper mappings from N to P if, and only if, the pair (n, p) satisfies one of the following conditions:*

$$\begin{aligned} n < \frac{6}{7}p + \frac{8}{7} \quad &\textit{and} \quad p - n \geq 4, \\ n < \frac{6}{7}p + \frac{9}{7} \quad &\textit{and} \quad 3 \geq p - n \geq 0, \\ p < 8 \quad &\textit{and} \quad p - n = -1, \\ p < 6 \quad &\textit{and} \quad p - n = -2, \\ &\textit{or} \\ p < 7 \quad &\textit{and} \quad p - n \geq -3. \end{aligned}$$

If (n, p) satisfies one of these conditions, we say that the pair (n, p) is in the range of *"Mather's nice dimensions"*.

Remark 2.1.23. If f is a stable map, or $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ an $\mathcal{A}$-finite germ, or has corank 1 and $n \leq p$, or (n, p) in the Mather's nice dimensions. Then, the stratification of the image of f by stable types is a Whitney stratification (see Definition B.3. [66]).

This remark follows for instance by [66, Remark 5.3, Corollary 7.5].

2.1.5. The image Milnor number of a map-germ

Here we introduce the main object of study in Chapter 3 and Chapter 4. This is the image Milnor number associated to an $\mathcal{A}$-finite map-germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$.

Definition 2.1.24. A *stabilisation* of a map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a 1-parameter unfolding $F\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ such that, there exists a representative $\tilde{F}\colon U \to V \times T$ of F, such that the mapping $f_s\colon U_s = \tilde{F}^{-1}(V \times \{s\}) \to V$ is locally stable, for all $0 \neq s \in T$.

Proposition 2.1.25. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite map-germ, and assume $(n, n+1)$ in Mather's nice dimensions or f has corank 1. Then, f admits a stabilisation.*

See a proof of the above proposition in [66, Corollary 5.4].

We need the notion of stable perturbation of a map-germ. In order to define this, we introduce the following theorem which is a simple consequence of Thom–Mather first isotopy lemma the proof can be found in [29, Theorem II. 5.2]. See [66, Theorem 5.7] for a more general statement.

Theorem 2.1.26. *Let $F\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ be a stabilisation of an $\mathcal{A}$-finite map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, with $(n, n+1)$ in Mather's nice dimensions, and let $\tilde{F}\colon X \to Y \times T$ be a representative of F:*

1. *There exists $\epsilon > 0$ such that the image $\mathrm{Im}(f_0)$ is stratified transverse to the sphere $\mathbb{S}_{\epsilon'}$, for all ϵ' with $0 < \epsilon' \leq \epsilon$.*

2. *There exists $\delta > 0$, such that for $|s| < \delta$, the image $\mathrm{Im}(f_s)$ is stratified transverse to $\mathbb{S}_\epsilon$.*

3. *The map $\pi\colon\ \mathrm{Im}(\tilde{F}) \cap (B_\epsilon \times (B_\delta \setminus \{0\})) \to B_\delta \setminus \{0\}$ is locally trivial fiber bundle.*

Let $\tilde{F}$ be a representative of F, and ϵ, δ as in Theorem 2.1.26. The following theorem holds:

Theorem 2.1.27. *The diagram*

$$\begin{array}{ccc}
\tilde{F}^{-1}(\mathrm{Im}(\tilde{F}) \cap (B_\epsilon \times (B_\delta \setminus \{0\}))) & \xrightarrow{\tilde{F}} & \mathrm{Im}(\tilde{F}) \cap (B_\epsilon \times (B_\delta \setminus \{0\})) \\
\downarrow & \swarrow & \\
B_\delta \setminus \{0\} & &
\end{array}$$

is locally trivial family of mappings.

For all pairs $s_1,\ s_2 \in B_\delta \setminus \{0\}$, $f_{s_1}\colon f_{s_1}^{-1}(B_\epsilon) \to B_\epsilon$ and $f_{s_2}\colon f_{s_2}^{-1}(B_\epsilon) \to B_\epsilon$ are *left-right equivalent*, that is, if there are homeomorphisms φ and ψ such that $f_{s_1} = \psi^{-1} \circ f_{s_2} \circ \varphi$. A member of this family is called a *stable perturbation* of f. The image X_s of a stable perturbation f_s of f, is called the *disentanglement* of f. Theorem 2.1.27 follows from the Thom–Mather second isotopy lemma [29, Theorem II. 5.8] (see also [66, 5.5]).

In [64], D. Mond proved the following important theorem:

Theorem 2.1.28. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite map-germ with $(n, n+1)$ in Mather's nice dimensions. Then, the disentanglement X_s of f has the homotopy type of a wedge of n-spheres.*

The number of such n-spheres is called the *image Milnor number* of f, and it is denoted by $\mu_I(f)$.

D. Mond also in [64] formulated a relevant conjecture relating the image Milnor number with the $\mathcal{A}_e$-codimension (see Section 2.1.3) of an $\mathcal{A}$-finite map-germ. To introduce the conjecture, we need the notion of weighted-homogeneous map-germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$.

Definition 2.1.29. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a map-germ

$$f(x) = (f_0(x), \dots f_n(x)).$$

We say that f is *weighted-homogeneous* with weights $w = (w_1, \dots, w_n)$ and degrees $d = (d_0, \dots, d_n)$, if each f_i is a weighted-homogeneous polynomial of degree d_i with weights w. This means that

$$f_i(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^{d_i} f_i(x_1, \dots, x_n),$$

for all $\lambda \in \mathbb{C}^*$.

Conjecture 2.1.30 (Mond's conjecture)**.** Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite map-germ with $(n, n+1)$ in Mather's nice dimensions. Then

$$\mu_I(f) \geq \mathcal{A}_e\text{-codim}(f),$$

with equality in the weighted-homogeneous case.

The conjecture is known to be true in some cases. It holds for $n \leq 2$ [65],[64]. For fold map-germs, by work of K. Houston [41]. And for singularities of corank 1 with $\mathcal{A}_e$-codimension 1, by work of T. Cooper, D. Mond and R. W. Atique [17].

2.1.6. Multiple point spaces of corank 1 map-germs

In this section, we give the definitions and main results about multiple point spaces of corank 1 map-germs developed by T. Marar and D. Mond in [54].

Consider $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ $(n < p)$ a corank 1 map-germ. We can choose coordinates in the source and the target, such that f is written in *prenormal form*, that is:

$$f(z, y) = (f_n(z, y), \dots, f_p(z, y), y), \ z \in \mathbb{C}, \ y \in \mathbb{C}^{n-1}.$$

Let $I_k(f)$ be the ideal generated by $(k-1)(p-n+1)$ functions $\Delta_i^{(j)} \in \mathcal{O}_{n+k-1}$, $1 \leq i \leq k-1$, $n \leq j \leq p$. Each $\Delta_i^{(j)}$ is a function only of the variables $z_1, \dots, z_{i+1}, y$ such that:

$$\Delta_1^{(j)}(z_1, z_2, y) = \frac{f_j(z_1, y) - f_j(z_2, y)}{z_1 - z_2},$$

and for $1 \leq i \leq k-2$,

$$\Delta_{i+1}^{(j)}(z_1, \dots, z_{i+2}, y) = \frac{\Delta_i^{(j)}(z_1, \dots, z_i, z_{i+1}, y) - \Delta_i^{(j)}(z_1, \dots, z_i, z_{i+2}, y)}{z_{i+1} - z_{i+2}}.$$

Definition 2.1.31. The *k-th multiple point space* is $D^k(f) = V(I_k(f))$, the zero locus in $(\mathbb{C}^{n+k-1}, 0)$ of the ideal $I_k(f)$.

If f is stable, then, set-theoretically, $D^k(f)$ is the Zariski closure of the set of points $(z_1, \ldots, z_k, y) \in \mathbb{C}^{n+k-1}$ such that:

$$f(z_1, y) = \ldots = f(z_k, y), \quad z_i \neq z_j, \text{ for } i \neq j,$$

(see [54, 68]). But, in general, this may be not true if f is not stable. For instance, consider the cusp $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $f(z) = (z^2, z^3)$. Since f is one-to-one, the closure of the double point set is empty, but

$$D^2(f) = V(z_1 + z_2, z_1^2 + z_1 z_2 + z_2^2).$$

This example also shows that the k-th multiple point space may be non-reduced in general.

The following theorem is main result of Marar-Mond in [54] which states that the k-th multiple point spaces can be used to characterize the stability and the $\mathcal{A}$-finiteness of f (see [54, 2.12]):

Theorem 2.1.32. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ $(n < p)$ be a corank 1 map-germ. Then:*

1. *f is stable if, and only if, $D^k(f)$ is smooth of dimension $p - k(p - n)$, or empty, for $k \geq 2$.*
2. *f is $\mathcal{A}$-finite if, and only if, for each k with $p - k(p - n) \geq 0$, $D^k(f)$ is either an ICIS of dimension $p - k(p - n)$ or empty, and if, for those k such that $p - k(p - n) < 0$, $D^k(f)$ consists at most of the point $\{0\}$.*

The following construction is also due to Marar-Mond in [54] and gives a refinement of the types of multiple points.

Definition 2.1.33. Let $\mathcal{P} = (r_1, \ldots, r_m)$ be a partition of k, that is, $r_1 + \ldots + r_m = k$, with $r_1 \geq \cdots \geq r_m$. Let $I(\mathcal{P})$ be the ideal in $\mathcal{O}_{n-1+k}$ generated by the $k - m$ elements $z_i - z_{i+1}$ for $r_1 + \ldots + r_{j-1} + 1 \leq i \leq r_1 + \ldots + r_j$ for $j = 1, \ldots, m$. Define the ideal $I_k(f, \mathcal{P}) = I_k(f) + I(\mathcal{P})$ and the *k-th multiple point space* of f with respect to the partition $\mathcal{P}$ as $D^k(f, \mathcal{P}) = V(I_k(f, \mathcal{P}))$.

Definition 2.1.34. We define a *generic point* of $D^k(f, \mathcal{P})$ as a point

$$(z_1, \ldots, z_1, \ldots, z_m, \ldots, z_m, y),$$

(z_i iterated r_i times, and $z_i \neq z_j$ if $i \neq j$) such that the local algebra of f at (z_i, y) is isomorphic to $\mathbb{C}[t]/(t^{r_i})$, and such that

$$f(z_1, y) = \ldots = f(z_m, y).$$

If f is stable, then $D^k(f, \mathcal{P})$ is equal to the Zariski closure of its generic points (see [54]).

The following corollary extends Theorem 2.1.32 to the multiple point spaces with respect to the partitions (see [2.15,[54]]):

Corollary 2.1.35. *If f is $\mathcal{A}$-finite (resp. stable), then for each partition $\mathcal{P} = (r_1, \ldots, r_m)$ of k satisfying $p - k(p-n+1) + m \geq 0$, the germ of $D^k(f, \mathcal{P})$ at $\{0\}$ is either an ICIS (resp. smooth) of dimension $p - k(p-n+1) + m$, or empty. Moreover, those $D^k(f, \mathcal{P})$ for $\mathcal{P}$ not satisfying the inequality consist at most of the single point $\{0\}$.*

2.2. Hodge Theory

In this section, we give an overview of classical Hodge theory. The name of this theory is devoted to W. Hodge [40], who introduced the pure Hodge structures. Later, P. Deligne in [21] and [22], provided the notion of mixed Hodge structure. Here, we give introduce these structures which will be relevant in the proof of the main result in Chapter 5. For a comprehensive reference consider [71].

2.2.1. Pure Hodge structures

Let H be a finite dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$), and let $H_{\mathbb{C}} := H \otimes \mathbb{C}$ be its complexification.

Definition 2.2.1. A $\mathbb{K}$-*pure Hodge structure of weight* k on H is a direct sum decomposition, the *Hodge decomposition*

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$, where $\bar{\cdot}$ denotes the complex conjugation.

The numbers

$$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$$

are called the *Hodge numbers* of the Hodge structure H.

Definition 2.2.2. Let H and H' be two $\mathbb{K}$-pure Hodge structures of weight k. A *morphism of pure Hodge structures of weight* k is a linear map $h\colon H \to H'$ of $\mathbb{K}$-vector spaces whose complexification $h_{\mathbb{C}} := h \otimes id_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ maps $H^{p,q}$ to $H'^{p,q}$.

The classical result below follows from Hodge's Decomposition Theorem [40] which carries a pure Hodge structure on the cohomology groups of compact Kälher manifolds.A proof of this theorem can be found in [71, Corollary 1.13]:

Theorem 2.2.3. *Let X be a compact Kälher manifold. The k-th cohomology group $H^k(X;\mathbb{K})$ is a pure Hodge structure of weight k. If $f\colon X \to Y$ is a holomorphic map between compact Kälher manifolds, then*

$$f^*\colon H^k(Y;\mathbb{K}) \to H^k(X;\mathbb{K})$$

is a morphism of pure Hodge structures of weight k.

Definition 2.2.4. Let H be a pure Hodge structure of weight k. The *Hodge filtration* $F^\bullet$ on $H_{\mathbb{C}}$ is a decreasing filtration given by

$$F^p := \bigoplus_{r \geq p} H^{r,k-r}.$$

Conversely, let H be a finite dimensional $\mathbb{K}$-vector space. Consider

$$H_{\mathbb{C}} \supset \cdots \supset F^p \supset F^{p+1} \supset \ldots$$

a decreasing filtration on the complexification $H_{\mathbb{C}}$ of H, with the property that $F^p \cap \overline{F^q} = 0$ if $p+q = k+1$. Then, it defines a pure Hodge structure of weight k on H by setting

$$H^{p,q} := F^p \cap \overline{F^q}.$$

Hence, we obtain an equivalent definition of pure Hodge structure in terms of filtrations:

Definition 2.2.5 (bis)**.** Let H be a finite dimensional $\mathbb{K}$-vector space. A pure Hodge structure of weight k is a decreasing filtration $F^{\bullet}$ on the complexification $H_{\mathbb{C}}$ of H satisfying $F^p \cap \overline{F^q} = 0$, if $p+q = k+1$.

Definition 2.2.6 (bis)**.** If H and H' are pure Hodge structures of the same weight k, a linear map $h\colon H \to H'$ is called a *morphism of pure Hodge structures* if $h_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ satisfies

$$h_{\mathbb{C}}(F^p(H_{\mathbb{C}})) \subset F^p(H'_{\mathbb{C}}),$$

for all p.

Remark 2.2.7. If $h\colon H \to H'$ is a morphism of pure Hodge structures, then the vector spaces $\operatorname{Ker} h$, $\operatorname{Im} h$, and $\operatorname{Coker} h$ have canonically induced pure Hodge structures of the same weight.

Proposition 2.2.8. *Let $h\colon H \to H'$ be a morphism of pure Hodge structures. Then, $h_{\mathbb{C}}$ is strictly compatible with the Hodge filtration $F^{\bullet}$, that is,*

$$h_{\mathbb{C}}(F^p(H_{\mathbb{C}})) = F^p(H'_{\mathbb{C}}) \cap \operatorname{Im} h_{\mathbb{C}},$$

for all p.

Corollary 2.2.9. *The category HS of pure Hodge structures is an abelian category.*

Definition 2.2.10. Let H be a pure Hodge structure. The *Weil operator* $C\colon H_{\mathbb{C}} \to H_{\mathbb{C}}$ is defined by

$$C(u) := i^{p-q}u$$

for every $u \in H^{p,q}$.

Definition 2.2.11. A *polarization* of a pure Hodge structure H of weight k is a bilinear form

$$Q\colon H \otimes H \to \mathbb{R}$$

that is $(-1)^k$-symmetric, and it satisfies:

1. The orthogonal complement of F^m is F^{k-m+1},
2. The hermitian form on $H_{\mathbb{C}}$ given by
$$Q(Cu, \bar{v})$$
is positive-definite.

A pure Hodge structure that admits a polarization is said to be a *polarizable* Hodge structure.

Example 2.2.12 (Hodge-Riemann bilinear relations)**.** Let X be a compact Kälher manifold of dimension n, and let η be an ample line bundle on X. Define the *primitive part* P^{n-r} of $H^{n-r}(X;\mathbb{K})$ by

$$P^{n-r} := \mathrm{Ker}(\eta^{r+1} \colon H^{n-r}(X;\mathbb{K}) \to H^{n+r+2}(X;\mathbb{K})).$$

and it is a polarizable pure Hodge structure of weight $n-r$. The following results hold:
The classical Hard-Lefschetz Theorem: For $r \geq 0$

$$\eta^r \colon H^{n-r}(X;\mathbb{K}) \simeq H^{n+r}(X;\mathbb{K}).$$

The Primitive Lefschetz Decomposition: For $r \geq 0$, there is a direct sum decomposition

$$H^{n-r}(X;\mathbb{K}) = \bigoplus_{l \geq 0} \eta^l P^{n-r-2l},$$

where the summands are mutually orthogonal with respect to the bilinear form

$$\int_X - \wedge - \wedge \eta^r.$$

The Hodge-Riemann bilinear relations: For $k \in \mathbb{Z}$, the Hodge-Riemann bilinear form is a bilinear form

$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \eta^{n-k}, \quad [\alpha], [\beta] \in H^k(X;\mathbb{C})$$

that is $(-1)^k$-symmetric. The two Hodge-Riemann relations are:

1. $Q(H^{p,q}, H^{r,s}) = 0$ if $(r,s) \neq (q,p)$,
2. For $u \in P^k \cap H^{p,q}(X;\mathbb{C})$, $i^{p-q}Q(u,\bar{u}) = Q(Cu,\bar{u}) = (u,u)$ and hence > 0 if $u \neq 0$.

2.2.2. Mixed Hodge structures

Definition 2.2.13. A *mixed* $\mathbb{K}$*-Hodge structure* ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$) is a finite dimensional $\mathbb{K}$-vector space H endowed with an increasing *weight filtration* $W_\bullet$, and a decreasing *Hodge filtration* $F^\bullet$ on $H_\mathbb{C}$ which has the additional property that it induces a pure $\mathbb{K}$-Hodge structure of weight k on each graded piece

$$Gr_k^W(H) = W_k/W_{k-1}.$$

To a mixed Hodge structure $(H, W_\bullet, F^\bullet)$ one associates *(mixed) Hodge numbers* defined by

$$h^{p,q}(H) = \dim_\mathbb{C} Gr_F^p Gr_{p+q}^W(H_\mathbb{C}).$$

We say that the mixed Hodge structure is *graded-polarizable* if the $Gr_k^W(H)$ are polarizable pure Hodge structures.

Definition 2.2.14. A linear map $h\colon H \to H'$ between two mixed Hodge structures is a *morphism of mixed Hodge structures* if h is compatible with the two filtrations $F^\bullet$ and $W_\bullet$, that is,

$$h(W_k(H)) \subset W_k(H') \text{ for all } k,$$

and

$$h_\mathbb{C}(F^p(H_\mathbb{C})) \subset F^p(H'_\mathbb{C}) \text{ for all } p.$$

As a consequence of this definition, we have the following corollary:

Proposition 2.2.15. *A morphism $h\colon H \to H'$ with H and H' pure Hodge structures of weights k and k', respectively. If $k \neq k'$, the morphism h is the zero morphism.*

Proposition 2.2.16. *Let $h\colon H \to H'$ be a morphism of mixed Hodge structures. Then, h is strictly compatible with the filtrations $W_\bullet$ and $F^\bullet$, that is,*

$$h(W_k(H)) = W_k(H') \cap \operatorname{Im} h$$

for all k, and

$$h_\mathbb{C}(F^p(H_\mathbb{C})) = F^p(H'_\mathbb{C}) \cap \operatorname{Im} h_\mathbb{C},$$

for all p.

Corollary 2.2.17. *The category MHS of mixed Hodge structures is abelian.*

In [21], P. Deligne proved the following fundamental result which shows that the cohomology groups of complex algebraic varieties have canonical mixed Hodge structures. For more details see [71, Part III, Theorem 5.33]:

Theorem 2.2.18. *Let X be a complex algebraic variety of complex dimension n. There is a canonical mixed Hodge structure on the cohomology groups $H^k(X;\mathbb{K})$, for all k. Furthermore, if $f\colon X \to Y$ is a morphism of complex algebraic varieties, the induced homomorphism on cohomology is a morphism of mixed Hodge structures.*

The cohomology groups $H^k(X;\mathbb{K})$ have the following weights depending on if the variety X is non-singular, compact or general:

	non-singular	compact	general
$k \leq n$	$[k, 2k]$	$[0, k]$	$[0, 2k]$
$k \geq n$	$[k, 2n]$	$[2k-2n, k]$	$[2k-2n, 2n]$

Table 2.1: Weights on $H^k(X;\mathbb{K})$

2.3. Perverse sheaves and the Decomposition Theorem

This section is mainly devoted to introduce the notion of perverse sheaf as well as the deep and important Decomposition Theorem which composes one of the central tools in Chapter 5. We give the standard properties about the six functor formalism for bounded derived categories of sheaves, that is, Grothendieck's six operations Rf_*, $Rf_!$,f^*, $f^!$, $R\mathcal{H}om$, and $\overset{L}{\otimes}$. For comprehensive references see [7], [31] or [3]. Then, we give the basics about the theory of perverse sheaves developed by A.A. Beilinson, J. N. Bernstein, P. Deligne and O. Gabber in [6]. For generalities about t-structures see [3] or [61]. Finally, we give the main results about the Decomposition Packadge developed in [6], [21], [22] (see also [18]). For generalities about additive and abelian categories see [28]. For triangulated categories and derived categories see [7, V.5], [3] or [61].

2.3.1. Functors in the derived category

Here we give the standard properties about Grothendieck's six functors in the derived category. Let X be a complex algebraic variety and let $\mathbb{K}$ be a subfield of $\mathbb{R}$.

Definition 2.3.1. A $\mathbb{K}$-complex of sheaves $\mathcal{F}$ on X is called *cohomologically locally constant* if the associated local cohomology sheaves are locally constant. Let $\mathcal{X}$ be a Whitney stratification of X. The complex $\mathcal{F}$ is called *cohomologically constructible* with respect to $\mathcal{X}$ if, for each stratum X_j, $\mathcal{F}|_{X_j}$, is cohomologically locally constant and has finitely generated stalk cohomology.

We denote by $C^b_c(X)$ the full subcategory, of the category of bounded $\mathbb{K}$-complexes of sheaves $C^b(X)$, consisting of the cohomologically constructible bounded $\mathbb{K}$-complexes of sheaves on X. Similarly, we denote by $D^b_c(X)$ the full subcategory of the bounded derived category $D^b(X)$, consisting of all cohomologically constructible bounded $\mathbb{K}$-complexes of sheaves (see [81], [82], [31]). In this section, we restrict ourselves to the categories $C^b_c(X)$ and $D^b_c(X)$.

Let $f\colon X \to Y$ be a continuous map, and let $\mathcal{F} \in \mathrm{Ob}(C^b_c(X))$. In the category $C^b_c(X)$, we have defined the following functors: the direct image f_*, the direct image with proper supports $f_!$, the inverse image f^*, $\mathcal{H}om(\mathcal{F}, -)$, $\mathcal{F} \otimes -$,.

See for instance [3] for the definitions of the functors $\mathcal{H}om(\mathcal{F}, -)$, $\mathcal{F} \otimes -$. For the definitions of the functors f_*, $f_!$ and f^* see [7, VI] or [3].

In the cohomologically constructible bounded derived category $D^b_c(X)$, the corresponding right derived functors of the functors $\mathcal{H}om(\mathcal{F}, -)$, f_* and f^* are the following: $R\mathcal{H}om(\mathcal{F}, -)$, Rf_* and f^*, respectively (see [3, 2.4.3]). In the case of the functor $f_!$, we have the derived functor $Rf_!$ (see [7, V.7,VI] or [3, 3.1]). In [81], J. L. Verdier introduced a functor in $D_c^b(X)$, this is the inverse image functor $f^!$ (see for instance [7, VI] or [3, 3.2]). For the functor $\mathcal{F} \otimes -$, the corresponding functor in $D_c^b(X)$ is a left derived functor $\mathcal{F} \overset{L}{\otimes} -$ (see [3, 2.4.3]), but in our case it coincides with $\mathcal{F} \otimes -$, since $\mathbb{K}$ is a field (see [31]).

We give below the standard properties relating the six functors: Rf_*, $Rf_!$, f^*, $f^!$, $R\mathcal{H}om(\mathcal{F}, -)$ and $\mathcal{F} \otimes -$. For comprehensive references see[7, V,VI], [31] or [3].

Theorem 2.3.2. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two continuous maps. Then,*

1. $(g \circ f)^* = f^* \circ g^*$, $R(g \circ f)_* = Rg_* \circ Rf_*$.
2. $(g \circ f)^! = f^! \circ g^!$, $R(g \circ f)_! = Rg_! \circ Rf_!$.

A proof of this theorem can be found in [7, V, Theorem 10.6].

Remark 2.3.3. Let $\mathcal{F} \in \mathrm{Ob}(D_c^b(X))$ and $\mathcal{G} \in \mathrm{Ob}(D_c^b(Y))$. If $i\colon X \hookrightarrow Y$ is the inclusion map, then we have the following two properties: If X is open in Y, then

$$f^!\mathcal{G} \simeq f^*\mathcal{G}, \tag{2.1}$$

and if X is closed in Y, then

$$Rf_!\mathcal{F} \simeq Rf_*\mathcal{F}. \tag{2.2}$$

Proposition 2.3.4. *Let*

$$\begin{array}{ccc} T & \xrightarrow{\bar{f}} & Z \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian diagram of spaces. For any $\mathcal{F} \in Ob(D^b(Z))$, then

$$R\bar{\pi}_*\bar{f}^!\mathcal{F} \simeq f^!R\pi_*\mathcal{F}. \tag{2.3}$$

For a proof of this proposition see for instance [7, V, Propostion 10.7].

The following properties come from the adjointness between the above functors: The functors Rf_* and f^*, $Rf_!$ and $f^!$, and $R\mathcal{H}om(\mathcal{F}, -)$ and $\mathcal{F} \otimes -$ are adjoints, respectively (see [7, V], [3]). For generalities about adjoint functors see [43].

Theorem 2.3.5 (Verdier duality)**.** *Let $f\colon X \to Y$ be a continuous map, and let $\mathcal{F} \in \mathrm{Ob}(D_c^b(X))$, $\mathcal{G} \in \mathrm{Ob}(D_c^b(Y))$. In $D_c^b(Y)$, we have the canonical isomorphism*

$$R\mathcal{H}om(Rf_!\mathcal{F}, \mathcal{G}) \simeq Rf_*R\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}). \tag{2.4}$$

The above theorem was obtained by J. L. Verdier in [81]. See also [7, V, Theorem 7.17] for a proof.

Corollary 2.3.6. *Let f, $\mathcal{F}$, and $\mathcal{G}$ be as in Theorem 2.3.5. Then, there is a canonical isomorphism in $D^b_c(X)$*

$$\mathrm{Hom}(Rf_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}). \tag{2.5}$$

Proposition 2.3.7. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Ob}(D^b_c(X))$, then we have a canonical isomorphism*

$$R\mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq R\mathcal{H}om(\mathcal{F}, R\mathcal{H}om(\mathcal{G}, \mathcal{H})). \tag{2.6}$$

A proof of this result can be found in [7, V, Proposition 10.2].

Corollary 2.3.8. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Ob}(D^b_c(X))$, we have a canonical isomorphism in $D^b_c(X)$*

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F}, R\mathcal{H}om(\mathcal{G}, \mathcal{H})) \tag{2.7}$$

Definition 2.3.9. Let $f\colon X \to \{pt\}$ be the map to a point. The *dualizing complex* is an object $\mathbb{D}_X$ in $D^b(X)$, defined by

$$\mathbb{D}_X := f^!\mathbb{R}_{pt}.$$

Definition 2.3.10. Let $\mathcal{F} \in \mathrm{Ob}(D^b_c(X))$. The *Borel-Moore-Verdier dualizing functor* is

$$\mathcal{D}(\mathcal{F}) := R\mathcal{H}om(\mathcal{F}, \mathbb{D}_X).$$

Theorem 2.3.11. *Let $\mathcal{F} \in Ob(D^b_c(X))$. Then, there is a canonical isomorphism*

$$can\colon \mathcal{F} \to \mathcal{D}(\mathcal{D}(\mathcal{F})) \tag{2.8}$$

For more details about the canonical isomorphism *can* see [7, V, 8.9].

The following proposition is an important consequence of Verdier duality. See for instance [3, Proposition 3.4.5] for a proof.

Proposition 2.3.12. *Let $f\colon X \to Y$ be a continuous map and let $\mathcal{F} \in \mathrm{Ob}(D^b_c(X))$, $\mathcal{G} \in \mathrm{Ob}(D^b_c(Y))$. Then, there are canonical isomorphisms*

$$\mathcal{D}(Rf_!\mathcal{F}) \simeq Rf_*(\mathcal{D}(\mathcal{F})), \quad \mathcal{D}(f^*\mathcal{G}) \simeq f^!(\mathcal{D}(\mathcal{G})).$$

Then, the functors Rf_* and $Rf_!$ are dual to each other, as well as the functors f^* and $f^!$.

Definition 2.3.13. A *self-dual* complex is a pair $(\mathcal{F}, \alpha)$ where $\mathcal{F} \in \mathrm{Ob}(D^b_c(X))$ and $\alpha\colon \mathcal{F} \to \mathcal{D}(\mathcal{F})$ is an isomorphism.

Remark 2.3.14. Let $(\mathcal{F}, \alpha)$ be a self-dual complex, and let $f\colon X \to Y$ be a proper map. Then, we have $Rf_* \simeq Rf_!$ and, by Proposition 2.3.12, Rf_* commutes with the Borel-Moore-Verdier duality functor, that is,

$$\mathcal{D}(Rf_*\mathcal{F}) \simeq Rf_*(\mathcal{D}(\mathcal{F})).$$

Hence, $(Rf_*\mathcal{F}, Rf_*(\alpha))$ is a self-dual complex on Y.

2.3.2. *t*-structures

We give the notion of t-structure on a triangulated category and overview the main properties. An example of t-structure is the perverse structure which will be exposed in this section. The original construction of perverse sheaves in [6] is given by the machinery of triangulated categories introduced by J. L. Verdier in [82]. For the proofs of the results below see for instance [3, 7].

Definition 2.3.15. A *t-structure* on a triangulated category D is a pair $(D^{\leq 0}, D^{\geq 0})$ of strictly full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ of D, such that, by setting $D^{\leq n} := D^{\leq 0}[-n]$ and $D^{\geq n} := D^{\geq 0}[-n]$:

1. $\mathrm{Hom}_D(A, B) = 0$ if $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$,
2. $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$, and
3. for any object A in D, there exists a triangle
$$A' \longrightarrow A \longrightarrow A'' \xrightarrow{[1]}$$
with $A' \in D^{\leq 0}$ and $A'' \in D^{\geq 1}$.

A triangulated category D together with a t-structure will be called a *t-category.*

Remark 2.3.16. If $(D^{\leq 0}, D^{\geq 0})$ is a t-structure, then so is $(D^{\leq n}, D^{\geq n})$, that is, the shifted t-structure.

Definition 2.3.17. The full subcategory
$$\mathcal{C} := D^{\leq 0} \cap D^{\geq 0}$$
of D is called the *heart* of the t-structure.

Proposition 2.3.18. *Let D be a t-category.*

1. *The inclusion $D^{\leq n} \hookrightarrow D$ has a right adjoint functor $\tau_{\leq n} \colon D \to D^{\leq n}$, that is, there exists canonical morphisms $\tau_{\leq n} A \to A$ such that the induced map*
$$\mathrm{Hom}_{D^{\leq n}}(A, \tau_{\leq n}(B)) \longrightarrow \mathrm{Hom}_D(A, B)$$
is an isomorphism for all $A \in D^{\leq n}$ and B.
2. *The inclusion $D^{\geq n} \hookrightarrow D$ has a left adjoint functor $\tau_{\geq n} \colon D \to D^{\geq n}$, that is, there exists canonical morphisms $A \to \tau_{\geq n} A$ such that the induced map*
$$\mathrm{Hom}_{D^{\geq n}}(\tau_{\geq n} A, B) \longrightarrow \mathrm{Hom}_D(A, B)$$
is an isomorphism for all A and $B \in D^{\geq n}$.

3. *For any object A in D, there is a distinguished triangle:*

$$\tau_{\leq n}A \longrightarrow A \longrightarrow \tau_{\geq n+1}A \xrightarrow{[1]}$$

Remark 2.3.19. Let D be a t-category.

- Let A be an object in D. The following properties are equivalent:
    1. $A \in D^{\leq n}$ (respectively, $A \in D^{\geq n}$),
    2. the canonical morphism $\tau_{\leq n}\colon D \to D^{\leq n}$ (respectively, $\tau_{\geq n}\colon D \to D^{\geq n}$) is an isomorphism,
    3. $\tau_{\geq n+1}(A) = 0$ (respectively, $\tau_{\leq n}(A) = 0$).
- Let $A' \longrightarrow A \longrightarrow A'' \xrightarrow{[1]}$ be a distinguished triangle in D. If $A', A'' \in D^{\leq 0}$ (respectively, $A', A'' \in D^{\geq 0}$), then $A \in D^{\leq 0}$ (respectively, $A \in D^{\geq 0}$).

Notice that, for any A in t-category D and integers n, m:

$$\tau_{\leq n}(A[m]) \simeq \tau_{\leq n+m}(A)[m],$$
$$\tau_{\geq n}(A[m]) \simeq \tau_{\geq n+m}(A)[m].$$

Proposition 2.3.20. *For any integers $n \leq m$, there is a unique isomorphism*

$$\tau_{\geq m}\tau_{\leq n}(A) \simeq \tau_{\leq n}\tau_{\geq m}(A).$$

Proposition 2.3.21. *The heart $\mathcal{C}$ of a t-structure is an abelian category.*

The notion of cohomology groups can be extended to any t-category by using the truncation functors $\tau_{\leq}$ and $\tau_{\geq}$ as follows:

Definition 2.3.22. The functor

$${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0}\colon D \to \mathcal{C}$$

is called the *cohomology functor* of the t-structure. Moreover, we set

$${}^tH^i := {}^tH^0 \circ [i],$$

that is, ${}^tH^i(A) = {}^tH^0(A[i]) = (\tau_{\geq i}\tau_{\leq i}(A))[i]$ for any object A in D.

Proposition 2.3.23. *Let $A' \longrightarrow A \longrightarrow A'' \xrightarrow{[1]}$ be a distinguished triangle in a t-category D, then the cohomology functor ${}^tH^0$ induces a long exact sequence*

$$\cdots \longrightarrow {}^tH^{-1}(A'') \longrightarrow {}^tH^0(A') \longrightarrow {}^tH^0(A) \longrightarrow {}^tH^0(A'') \longrightarrow {}^tH^1(A') \longrightarrow \ldots.$$

Remark 2.3.24. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$, then there exists a unique morphism $C \to A[1]$ such that

$$A \longrightarrow B \longrightarrow C \xrightarrow{[1]}$$

is a distinguished triangle in D.

Example 2.3.25 (The standard t-structure)**.** Let $C(X)$ be the abelian category of $\mathbb{K}$-complexes ($\mathbb{K}$ subfield of $\mathbb{R}$) of sheaves on X, and let $D(X)$ be the corresponding derived category. Then

$$D^{\leq 0} := \{\mathcal{F} \in \mathrm{Ob}(D(X)) : \mathcal{H}^i(\mathcal{F}) = 0, \quad i > 0\}$$

and

$$D^{\geq 0} := \{\mathcal{F} \in \mathrm{Ob}(D(X)) : \mathcal{H}^i(\mathcal{F}) = 0, \quad i < 0\}$$

yields a t-structure on $D(X)$, see Example 7.1.3 in [3]. It is called the *standard t-structure.* The truncation functors are defined by

$$\tau_{\leq 0}(\mathcal{F}) = \{\cdots \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \ker(d^0) \longrightarrow 0 \dots\}$$

and

$$\tau_{\geq 0}(\mathcal{F}) = \{\cdots \longrightarrow 0 \longrightarrow \mathrm{coker}(d^{-1}) \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \dots\}.$$

The heart of the standard t-structure is

$$D^{\leq 0} \cap D^{\geq 0} = \{\cdots \longrightarrow 0 \longrightarrow \mathcal{H}^0(\mathcal{F}) \longrightarrow 0 \longrightarrow \dots\},$$

and it is equivalent to the category $C(X)$.

The perverse t-structure

Definition 2.3.26. The *perverse t-structure* is defined by

$${}^pD^{\leq 0}(X) := \{\mathcal{F} \in \mathrm{Ob}(D^b_c(X)) : \dim_{\mathbb{C}} \ \mathrm{supp}^{-j}(\mathcal{F}) \leq j, \forall j \in \mathbb{Z}\},$$

and

$${}^pD^{\geq 0}(X) := \{\mathcal{F} \in \mathrm{Ob}(D^b_c(X)) : \dim_{\mathbb{C}} \ \mathrm{cosupp}^{j}(\mathcal{F}) \leq j, \forall j \in \mathbb{Z}\},$$

where, for the inclusion $i \colon \{x\} \hookrightarrow X$, the *support* and *cosupport* are defined by

$$\mathrm{supp}^j(\mathcal{F}) := \overline{\{x \in X : \mathcal{H}^j(i^*(\mathcal{F})) \neq 0\}},$$

and

$$\mathrm{cosupp}^j(\mathcal{F}) := \overline{\{x \in X : \mathcal{H}^j(i^!(\mathcal{F})) \neq 0\}},$$

respectively.

Definition 2.3.27. A complex $\mathcal{F} \in \mathrm{Ob}(D_c^b(X))$ is called a *perverse sheaf* if

$$\mathcal{F} \in Perv(X) := {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X),$$

that is, if $\mathcal{F}$ is in the heart of the perverse t-structure.

The generalities about t-structures introduced in Section 2.3.2, can be translated into the following properties:

There exist perverse truncations ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$ that are adjoints to the inclusions ${}^pD^{\leq 0}(X) \hookrightarrow D_c^b(X)$ and ${}^pD^{\geq 0}(X) \hookrightarrow D_c^b(X)$, respectively, that is, for every integer n,

$$\mathrm{Hom}_{{}^pD^{\leq n}(X)}(\mathcal{F}, {}^p\tau_{\leq n}(\mathcal{G})) = \mathrm{Hom}_{D_c^b(X)}(\mathcal{F}, \mathcal{G}), \tag{2.9}$$

if $\mathcal{F} \in {}^pD^{\leq n}(X)$, and

$$\mathrm{Hom}_{{}^pD^{\geq n}(X)}({}^p\tau_{\geq n}(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}_{D_c^b(X)}(\mathcal{F}, \mathcal{G}), \tag{2.10}$$

if $\mathcal{G} \in {}^pD^{\geq n}(X)$.

Definition 2.3.28. The *i-th perverse cohomology* of $\mathcal{F}$ is defined as

$${}^p\mathcal{H}^i(\mathcal{F}) := {}^p\tau_{\leq 0}{}^p\tau_{\geq 0}(\mathcal{F}[i]).$$

Proposition 2.3.29. *The following properties hold:*

1. $\mathcal{F} \in Perv(X)$ *if, and only if,* ${}^p\mathcal{H}^0(\mathcal{F}) = \mathcal{F}$, *and* ${}^p\mathcal{H}^i(\mathcal{F}) = 0$, *for* $i \neq 0$.
2. *For every distinguished triangle*

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \xrightarrow{[1]}$$

in $D_c^b(X)$, *there is an associted long exact sequence in* $Perv(X)$:

$$\cdots \to {}^p\mathcal{H}^{i-1}(\mathcal{C}) \to {}^p\mathcal{H}^i(\mathcal{A}) \to {}^p\mathcal{H}^i(\mathcal{B}) \to {}^p\mathcal{H}^i(\mathcal{C}) \to {}^p\mathcal{H}^{i+1}(\mathcal{A}) \to \ldots.$$

3. *If* $\mathcal{A}$, $\mathcal{C} \in Perv(X)$, *and* $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \xrightarrow{[1]}$ *is a distinguished triangle in* $D_c^b(X)$, *then* $\mathcal{B} \in Perv(X)$.

Remark 2.3.30. Let $\mathcal{F}$, $\mathcal{G} \in \mathrm{Ob}(D_c^b(X))$.

- $\mathcal{F} \cong 0$ if, and only if, ${}^p\mathcal{H}^i(\mathcal{F}) = 0$.
- Let $u\colon \mathcal{F} \to \mathcal{G}$ be a morphism in $D_c^b(X)$. Then, u is a quasi-isomorphism if, and only if
$${}^p\mathcal{H}^i(u)\colon {}^p\mathcal{H}^i(\mathcal{F}) \to {}^p\mathcal{H}^i(\mathcal{G})$$
is an isomorphism of perverse sheaves for every $i \in \mathbb{Z}$.

Since $\mathbb{K}$ is a field, by the Universal Coefficient Theorem (see for instance [61, 5.4]), then

$$\mathrm{cosupp}^j(\mathcal{F}) = \mathrm{supp}^{-j}(\mathcal{F}).$$

Remark 2.3.31. A complex $\mathcal{F}$ is in ${}^pD^{\leq 0}(X)$ if, and only if, $\mathcal{D}(\mathcal{F}) \in {}^pD^{\geq 0}(X)$, and the Borel-Moore-Verdier duality functor $\mathcal{D}(-)$ (see Definition 2.3.10) preserves perverse sheaves. Moreover, there are canonical isomorphisms of functors:

$${}^p\tau_{\leq 0} \circ \mathcal{D} \simeq \mathcal{D} \circ {}^p\tau_{\geq 0}, \quad {}^p\tau_{\geq 0} \circ \mathcal{D} \simeq \mathcal{D} \circ {}^p\tau_{\leq 0},$$

and

$$\mathcal{D} \circ {}^p\mathcal{H}^i \simeq {}^p\mathcal{H}^{-i} \circ \mathcal{D}.$$

2.3.3. The intersection cohomology complex

In [30], M. Goresky and R. MacPherson introduced the intersection homology, and defined the intersection cohomology complex $IC^\bullet_Y$ of a topological pseudomanifold Y. In [31], the same authors showed a second construction of $IC^\bullet_Y$ considering it as an object in the derived category of sheaves, conjectured by P. Deligne (I thank J. P. Brasselet for this historical remark). This complex is called the intersection cohomology sheaf complex. This construction works in the context of stratifications and sheaf theory, and it produces intersection homology groups for pseudomanifolds and algebraic varieties. Furthermore, the intersection cohomology sheaf complex is characterized under some axioms. For a detailed account about this see [30], [31], [7] or [3].

We focus here on the complex algebraic context. Let Y be a complex algebraic variety of complex dimension n. The variety Y admits a Whitney stratification making Y into an oriented topological pseudomanifold of real dimension $2n$ with all strata of even real dimension. Consider Y_j the strata of a stratification $\mathcal{Y}$ of Y where $\mathrm{codim}_{\mathbb{C}}(Y_j) = j$. We use Deligne's indexing convention, that is, we consider the intersection cohomology sheaf complex shifted by $[-n]$, we denote this by IC_Y. In the following, we call the complex IC_Y the intersection cohomology complex of Y.

The intersection cohomology complex IC_Y in the derived category $D^b_c(Y)$ is uniquely characterized up to canonical isomorphism by the following axioms:

(AX1) $(IC_Y)_{|Y \setminus \Sigma} \simeq \mathbb{R}_{Y \setminus \Sigma}[n]$ where Σ is the singular locus of Y.

(AX2) $\mathcal{H}^k(IC_Y) = 0$ for all $k < -n$.

(AX3) For all $y \in Y_j$, $\mathcal{H}^k(IC_Y)_y = 0$ for all $k \geq -n + j$ and $j \geq 1$.

(AX4) For all $y \in Y_j$, with $i_y \colon \{y\} \hookrightarrow Y$ the inclusion, $\mathcal{H}^k(i^!_y IC_Y)_y = 0$ for all $k \leq n - j$ and $j \geq 1$.

The intersection cohomology complex IC_Y is an object in $Perv(Y)$, and it is a self-dual complex, that is, $IC_Y \simeq \mathcal{D}(IC_Y)$ (see Definition 2.3.13). Furthermore, this complex was generalized for local systems: Let $\mathcal{L}$ be a local system on $Y \setminus \Sigma$. The intersection cohomology complex $IC_Y(\mathcal{L})$ associated with $\mathcal{L}$ is such that satisfies the corresponding axioms (AX1) - (AX4) taking $\mathcal{L}[n]$ instead of $\mathbb{R}_{Y\setminus\Sigma}[n]$.

Remark 2.3.32. An object $P \in Perv(Y)$ is *simple* if it has no non-trivial sub-objects. P is *semi-simple* if it is isomorphic to a direct sum of simple objects. Moreover, $P \in Perv(Y)$ is simple if, and only if $P \simeq IC_{Y'}(\mathcal{L}')$ for some closed subvariety Y' of Y, and some simple local system $\mathcal{L}'$ defined on an open subvariety of the regular part of Y'. Hence, a semi-simple perverse sheaf P is a finite direct sum of such simple objects. Notice that, if Y' and Y'' are distinct closed subvarieties of Y,

$$\mathrm{Hom}_{D(Y)}(IC'_Y(\mathcal{L}'), IC''_Y(\mathcal{L}'')) = 0.$$

By Axiom (AX1), the intersection cohomology complex $IC_Y \simeq \mathbb{R}_Y[n]$, if Y is non-singular. There are another type of varietes satisfying that the intersection cohomology complex is quasi-isomorphic to the constant sheaf, they are the rational homology manifolds:

Definition 2.3.33. A topological space Y is a *rational homology manifold* of real dimension n if $H_i(Y, Y \setminus \{y\}; \mathbb{Q})$ is equal to $\mathbb{Q}$ if $i = n$, and 0 if $i \neq n$.

Proposition 2.3.34. *A complex algebraic variety Y is a rational homology manifold if, and only if, the link of every point in Y is a rational homology sphere.*

Theorem 2.3.35. *If Y is complex algebraic variety which is rational homology manifold, then*

$$IC_Y \simeq \mathbb{Q}_Y[\dim_{\mathbb{C}} Y].$$

The proofs of the above two results can be found in [61, Proposition 6.6.2] and [61, Theorem 6.6.3], respectively.

As a consequence of the theory of mixed Hodge modules developed by M. Saito [74], [75], [76] (see also [18]) one can obtain that the intersection cohomology of a complex algebraic variety carries a mixed Hodge structure. This result was originally proved by J. Steenbrink and S. Zucker in [78] for the curve case and by F. El Zein in [23], [24] for the general situation. For a developed and detailed references about this consider for instance [71] or [61]. As a particular case, we have the following theorem:

Theorem 2.3.36. *If Y is a compact complex algebraic variety that is a rational homology manifold. Then, the k-th cohomology group $H^k(Y;\mathbb{Q})$ of Y carries a pure (polarizable) Hodge structure of weight k, for every $k \in \mathbb{Z}$.*

For the general theory of pure and mixed Hodge structures see Section 2.2.

2.3.4. Decomposition Packadge

The Decomposition Theorem was proved in [6] for $\mathbb{C}$-coefficients. For $\mathbb{R}$- and $\mathbb{Q}$-coefficients follows from [20]. In this section, we give the results of the Decomposition Packadge for $\mathbb{R}$-coefficients which will be used in Chapter 5. The Decomposition Theorem was re-proved later by M. Saito, using the theory of Hodge modules in [75], [74], [76], and also by M. A. de Cataldo and L. Migliorini in [18].

Let $\varepsilon\colon Z \to Y$ be a projective morphism of complex algebraic varieties, with Z non-singular of dimension d. Let $\eta \in H^2(Z;\mathbb{R})$ be the first Chern class of an η-ample line bundle on X. The class η corresponds to a map of complexes

$$\eta\colon \mathbb{R}_Z \to \mathbb{R}_Z[2],$$

and it induces a map $\eta\colon R\varepsilon_*\mathbb{R}_Z \to R\varepsilon_*\mathbb{R}_Z[2]$. After taking perverse cohomology, we obtain a map of perverse sheaves on Y:

$$\eta\colon {}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d]) \to {}^p\mathcal{H}^{i+2}(R\varepsilon_*\mathbb{R}_Z[d]).$$

Iterating, we have maps of perverse sheaves

$$\eta^i\colon {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \to {}^p\mathcal{H}^{i}(R\varepsilon_*\mathbb{R}_Z[d]), \tag{2.11}$$

for every $i \geq 0$.

The Decomposition Packadge is composed by the following three theorems: the *relative Hard Lefschetz Theorem*, the *Decomposition Theorem*, and the *Semi-simplicity Theorem*. The corresponding theorems state the following (see [18]):

Theorem 2.3.37 (Relative Hard Lefschetz Theorem)**.** *Let ε and η as above. For every $i \geq 0$, the induced map by η of perverse cohomologies (2.11), i.e.,*

$$\eta^i\colon {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \xrightarrow{\cong} {}^p\mathcal{H}^{i}(R\varepsilon_*\mathbb{R}_Z[d])$$

is an isomorphism. In particular, by setting

$$\mathcal{P}_\eta^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) := \operatorname{Ker}(\eta^{i+1}\colon {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \to {}^p\mathcal{H}^{i+2}(R\varepsilon_*\mathbb{R}_Z[d])),$$

we have equalities:

$$ {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) = \bigoplus_{l\geq 0} \eta^l \mathcal{P}_\eta^{-i-2l}(R\varepsilon_*\mathbb{R}_Z[d]) \tag{2.12}$$

$$ {}^p\mathcal{H}^{i}(R\varepsilon_*\mathbb{R}_Z[d]) = \bigoplus_{l\geq 0} \eta^{i+l} \mathcal{P}_\eta^{-i-2l}(R\varepsilon_*\mathbb{R}_Z[d]) \tag{2.13}$$

for $i \geq 0$.

The term
$$\mathcal{P}_\eta^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \subset {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])$$
is called the *primitive part* of the perverse cohomology ${}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])$. The decompositions (2.12) and (2.13) are called the *primitive decomposition* of ${}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])$ and ${}^p\mathcal{H}^{i}(R\varepsilon_*\mathbb{R}_Z[d])$, respectively.

Theorem 2.3.38 (The Decomposition Theorem)**.** *Let ε be as above, there is a non-canonical isomorphism in $D^b_c(Y)$:*

$$R\varepsilon_*\mathbb{R}_Z[d] \simeq \bigoplus_i {}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])[-i], \tag{2.14}$$

The sum appearing in the decomposition is finite, the perverse cohomologies run all the degrees for $M \leq i \leq M$, where M is the defect of semi-smallness of ε. However, since the value of M is not relevant in the tratement of the Decomposition Theorem in Chapter 5, then we only consider M as a certain integer. See [18] for more details about semi-smallness and the Decomposition Theorem.

Theorem 2.3.39 (The Semi-simplicity Theorem)**.** *Each ${}^p\mathcal{H}^i(R\varepsilon * \mathbb{R}_Z[d])$ is a semi-simple object in $Perv(Y)$, i.e., if $\mathcal{Y}$ is the set of connected components of strata of Y in a stratification of ε, there is a canonical isomorphism in $Perv(Y)$:*
$${}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d]) \simeq \bigoplus_{S\in\mathcal{Y}} IC_{\overline{S}}(\mathcal{L}_{i,S}), \tag{2.15}$$
where the local systems $\mathcal{L}_{i,S}$ on S are semi-simple (see Remark 2.3.32).

The collection of subvarieties $\overline{S}$ appearing in (2.15) are called the *set of supports* of ε with some associated non-zero local system $\mathcal{L}_{i,S}$.

Remark 2.3.40. If $\varepsilon\colon Z \to Y$ is a resolution of singularities of Y, by Theorem 2.3.38 and Theorem 2.3.39, there is a stratification of Y such that the intersection cohomology complex IC_Y is a direct summand of the decompositions (2.14) and (2.15). That is,

$$R\varepsilon_*\mathbb{R}_Z[d] \simeq IC_Y \oplus (\text{contribution from singularities of } Y). \tag{2.16}$$

2.4. Theory of Cubical Hyperresolutions

In this section, we give an overview of the theory of cubical hyperresolutions due to F. Guillén, V. Navarro Aznar, P. Pascual-Gainza, and F. Puerta in [35]. The cubical hyperresolutions will be one of the central tools appearing throughout Chapter 5. The results and definitions appearing in this section are based on [71, Part III, Chapter 5].

2.4.1. Semi-simplicial varieties and cubical varieties

Definition 2.4.1. The *semi-simplicial category* $\triangle$ is the category with objects the ordered sets $\{0,\ldots,n\}$, for $n \in \mathbb{N}$, and with morphisms strictly increasing maps.

Notice that the morphisms of the semi-simplicial category $\triangle$ are obtained as composition of maps $\delta^j \colon \{0,\ldots,k\} \to \{0,\ldots,k+1\}$ defined by $\delta^j(p) = p$, for $p < j$, and $\delta^j(p) = p+1$, for $p \geq j$. Then, the semi-simplicial category can be seen as the category of ordered sets $[k]$ and with morphisms generated by the face maps. This information can be captured in a diagram:

$$\{0\} \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} \{0,1\} \;\overset{\delta^0,\,\delta^1,\,\delta^2}{\rightarrow}\; \{0,1,2\} \;\overset{\delta^0,\,\delta^1,\,\delta^2,\,\delta^3}{\rightarrow}\; \{0,1,2,3\} \qquad \ldots$$

Definition 2.4.2. The *cubical category* $\square$ is the category with objects the finite subsets of $\mathbb{N}$, and for which $\mathrm{Hom}(I,J)$ consists of a single element if $I \subset J$, and empty if else.

Set $[k] := \{0,\ldots,k\}$. The *n-truncated semi-simplicial category* $\triangle_n$ is the full sub-category of $\triangle$ whose objects are the $[k]$ with $k \in [n-1]$. The *cubical category* $\square_n$ is the full sub-category of $\square$ whose objects are the subsets of $[n-1]$,

Definition 2.4.3. A *semi-simplicial object* (*co-semi-simplicial*) in a category $\mathcal{C}$ is a contravariant functor $K_\bullet \colon \triangle \to \mathcal{C}$ (co-variant functor $C^\bullet \colon \triangle \to \mathcal{C}$). A morphism between such objects is to be understood as a morphism of corresponding functors.

We define *cubical objects*, *co-cubical objects* in a similar way. We get an *n-(co)semi-simplicial object* by replacing $\triangle$ by $\triangle_n$, and similarly for *n-(co)cubical-object*.

In particular, for the face maps δ^j, we define objects

$$K_n := K_\bullet[n], \quad C^n := C^\bullet[n]$$

in $\mathcal{C}$, and morphisms

$$d_j := K(\delta^j), \quad d^j = C(\delta^j).$$

If the category is additive, we can consider

$$\delta_n := \sum_{j=0}^{n} (-1)^j d_j \colon K_n \to K_{n-1}, \quad \delta^n := \sum_{j=0}^{n} (-1)^j d^j \colon C^n \to C^{n+1}$$

and define complexes in $\mathcal{C}$:

$$CK_\bullet := \{\dots \xrightarrow{\delta_1} K_1 \xrightarrow{\delta_0} K_0\}, \qquad CC^\bullet := \{C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \dots\}.$$

For a cubical object X and $I \subset \mathbb{N}$ finite, we define

$$X_I := X(I), \quad d_{IJ} := X(I \hookrightarrow J) \colon X_J \to X_I, \quad I \subset J.$$

Let K be an object in $\mathcal{C}$, a *constant semi-simplicial object* K is obtained by setting $K_n = K$ for all n, and taking the identity for the corresponding maps.

Definition 2.4.4. An *augmentation* of a semi-simplicial object $K_\bullet$ to constant semi-simplicial object Y is a morphism

$$\varepsilon \colon K_\bullet \to Y$$

of semi-simplicial objects.

If the category $\mathcal{C}$ is the category of topological spaces, we speak of a *semi-simplicial space*, if $\mathcal{C}$ is the category of complex algebraic varieties, we speak of a *semi-simplicial complex algebraic variety*, etc.

Definition 2.4.5. Let $K_\bullet$ be a semi-simplicial space. Using the convention that every strictly increasing map $f \colon [q] \to [p]$ has geometric realizations $|f| \colon \Delta_q \to \Delta_p$, the *geometric realization* of $K_\bullet$ is

$$|K_\bullet| := \coprod_{p=0}^{\infty} \Delta_p \times K_p / R,$$

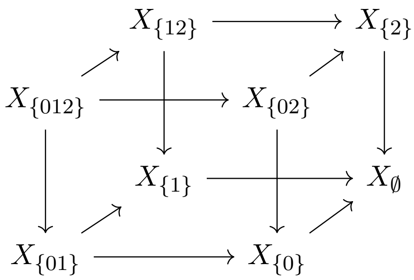
where the equivalence relation R is generated by identifying $(s, x) \in \Delta \times K_q$ with $(|f|(s), y) \in \Delta_p \times K_p$, if $x = K(f)y$ for all strictly increasing maps $f \colon [q] \to [p]$.

The topology on $|K_\bullet|$ is the quotient topology under R obtained from the direct product topology. There is a natural augmentation

$$K_\bullet \to |K_\bullet|$$

defined by sending $x \in K_n$ to the equivalence class of (x, z_n), where z_n is the barycenter of Δ_n.

Example 2.4.6. Let $Y = \bigcup_{i=0}^{n} Y_i$ be a variety with irreducible components $Y_0, \ldots, Y_n$. Let $X_\emptyset := Y$, and $X_I := \bigcap_{i \in I} Y_i$ for $I \subset [n]$ non-empty. The maps $d_{IJ} \colon X_J \to X_I$ are given by the inclusions $I \subset J$. This defines an $(n+1)$-cubical variety. For $n = 2$, the 3-cubical variety is



(2.17)

Remark 2.4.7. Any $(k+1)$-cubical variety X can be considered as a morphism of k-cubical varieties $Y \to Z$ where $Z_I = X_I$ and $Y_I = X_{I \cup \{k\}}$ for $I \subset [k-1]$. In particular, a 1-cubical variety is the same as a morphism of varieties.

Remark 2.4.8. Every $(n+1)$-cubical variety X gives rise to an augmented n-semi-simplicial variety $X_\bullet \to Y$ captured in the following diagram:

$$\cdots \qquad X_3 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \; X_2 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; X_1 \rightrightarrows X_0 \longrightarrow Y$$

The objects are

$$X_k := \coprod_{|I|=k+1} X_I, \quad k = 0, \ldots, n.$$

The morphisms are the following: For each inclusion $\beta \colon [s] \to [r]$ and $I \subseteq [n]$ with cardinality $|I| = r+1$, writting $I = \{i_0, \ldots, i_r\}$, $i_0 < \cdots < i_r$, then the morphisms are

$$X(\beta)|_{X_I} := d_{JI}$$

where $J = \{i_{\beta(0)}, \ldots, i_{\beta(s)}\} \subset I$.

For all $I \subseteq [n]$, we have a well-defined map $d_{\emptyset I} \colon X_I \to X_\emptyset = Y$. So, we have an augmentation of the above semi-simplicial variety.

Definition 2.4.9. Let X be a cubical variety, and let $\varepsilon \colon X_\bullet \to X_\emptyset$ be its asociated augmented semi-simplicial variety, the continuous map

$$|\varepsilon| \colon |X_\bullet| \to X_\emptyset,$$

is the *geometric realization of the cubical variety* X.

2.4.2. Sheaves on semi-simplicial spaces and their cohomology

Consider the category with objects the pairs $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf on X, and morphisms the pairs $(f, f^{\#})\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ where $f\colon X \to Y$ is a continuous map and $f^{\#}\colon \mathcal{G} \to f_*\mathcal{F}$ is a sheaf homorphism on Y.

Definition 2.4.10. A *sheaf on a semi-simplicial space* $X_\bullet$ is a semi-simplicial object $(X_\bullet, \mathcal{F}^\bullet)$ in the above category which consists of a family of pairs $(X_k, \mathcal{F}^k)$, where $\mathcal{F}_k$ is a sheaf on X_k, and, for increasing morphisms $\beta\colon [n] \to [m]$, the morphisms are pairs $(X(\beta), X(\beta)^{\#})$ where $X(\beta)\colon X_m \to X_n$ and $X(\beta)^{\#}\colon \mathcal{F}^n \to X(\beta)_*\mathcal{F}^m$ satisfying $(X(\beta) \circ X(\gamma))^{\#} = X(\beta)^{\#} \circ X(\gamma)^{\#}$, for any increasing map $\gamma\colon [m] \to [l]$. That is, it is covariant functor in the second factor.

Example 2.4.11. Let $X_\bullet$ be a semi-simplicial space, define the constant sheaf $\mathbb{R}_{X_\bullet}$ on the semi-simplicial space $X_\bullet$. Taking the family of constant sheaves $\mathbb{R}_{X_k}$ over X_k and morphisms $X(\beta)^{\#}\colon \mathbb{R}_{X_n} \to X(\beta)_*\mathbb{R}_{X_m}$ for ingreasing maps $\beta\colon [n] \to [m]$.

Let Y be a topological space, and let $\mathcal{F}$ be a sheaf on Y. For each $y \in Y$, let $\mathcal{F}_y$ be the stalk of $\mathcal{F}$ at y. The *Godement sheaf* $\mathcal{C}_{Gdm}(\mathcal{F})$ of $\mathcal{F}$ is defined by taking for open subset $U \subseteq Y$, sections $\mathcal{C}_{Gdm}(\mathcal{F})(U) := \prod_{y\in U} \mathcal{F}_y$, and for open subsets $U \subseteq V$, restriction maps $\mathcal{C}_{Gdm}(\mathcal{F})(V) \to \mathcal{C}_{Gdm}(\mathcal{F})(U)$. There is a resolution of the sheaf $\mathcal{F}$,

$$0 \longrightarrow \mathcal{F} \xrightarrow{d^0} \mathcal{C}^0_{Gdm}(\mathcal{F}) \xrightarrow{d^1} \mathcal{C}^1_{Gdm}(\mathcal{F}) \xrightarrow{d^2} \dots$$

by taking $\mathcal{C}^0_{Gdm}(\mathcal{F}) := \mathcal{C}_{Gdm}(\mathcal{F})$, and $\mathcal{C}^i_{Gdm}(\mathcal{F}) := \mathcal{C}_{Gdm}(\operatorname{Coker} d^{i-1})$ for $i > 0$. It is called the *Godement resolution* of $\mathcal{F}$.

Consider $\varepsilon\colon X_\bullet \to Y$ a semi-simplicial space augmented to Y and $\mathcal{F}^\bullet$ a sheaf on $X_\bullet$. Notice that the sheaves $\varepsilon_*\mathcal{C}^q_{Gdm}(\mathcal{F}^p)$ form a double complex of sheaves on Y; and its associated simple complex defines

$$R\varepsilon_*\mathcal{F}^\bullet := s[\varepsilon_*\mathcal{C}^\bullet_{Gdm}(\mathcal{F}^\bullet)].$$

Definition 2.4.12. A semi-simplicial space $\varepsilon\colon X_\bullet \to Y$ augmented to Y is said to be *of cohomological descent* if the natural map

$$\varepsilon^{\#}\colon \mathbb{R}_Y \to R\varepsilon_*\mathbb{R}_{X_\bullet}$$

is a quasi-isomorphism.

2.4.3. Cubical hyperresolutions

Definition 2.4.13. Let Y be a variety and let D be a closed subvariety of Y. A *semi-simplicial resolution* of the pair (Y, D) is a semi-simplicial variety $\varepsilon\colon X_\bullet \to Y$ augmented to Y satisfying the following properties:

1. All maps $X_k \to Y$ are proper, and X_k is smooth for all k.

2. ε is of cohomological descent.

3. The inverse image of D on each irreducible component X_k^i is: either all of X_k^i, or empty, or a divisor with simple normal crossings on X_k^i.

Definition 2.4.14. A cubical variety is a *cubical hyperresolution* of Y if its associated semi-simplicial variety $\varepsilon\colon X_\bullet \to Y$ augmented to Y is a semi-simplicial resolution.

Remark 2.4.15. Let X be a n-cubical variety, and let $\varepsilon\colon X_\bullet \to X_\emptyset$ be its associated augmented semi-simplicial variety. Let $C^\bullet(X)$ be the cone of the morphism $\mathbb{R}_{X_\emptyset} \to R\varepsilon_*\mathbb{R}_{X_\bullet}$. Then, the n-cubical variety X is of cohomological descent if and only if the cone $C^\bullet(X)$ is acyclic.

Construction of Cubical Hyperresolutions

Definition 2.4.16. 1. A *proper modification* of a variety Y is a proper morphism $f\colon \tilde{Y} \to Y$ such that there exists an open dense set U in Y for which the morphism f induces an isomorphism $f^{-1}(U) \to U$.

2. A *resolution* of Y is a proper modification $f\colon \tilde{Y} \to Y$ such that $\tilde{Y}$ is smooth.

3. The *discriminant* of a proper morphism $f\colon \tilde{Y} \to Y$ is the minimal closed subset $\Delta(f)$ of Y such that the morphism f induces an isomorphism $\tilde{Y} \setminus f^{-1}(\Delta(f)) \to Y \setminus \Delta(f)$.

The following theorem was proved independently by D. Abramovich and J. de Jong in [1], and F. A. Bogomolov and T. G. Pantev in [8], respectively:

Theorem 2.4.17. *Let Y be an (irreducible) algebraic variety and let D be a closed subset of Y. Then, there exists a resolution $f\colon \tilde{Y} \to Y$ which is a projective morphism and such that the inverse image $f^{-1}(D)$ of D in $\tilde{Y}$ is a simple normal crossings divisor.*

Lemma-Definition 2.4.18. *Let $f\colon \tilde{Y} \to Y$ be a proper modification with discriminant D. The* discriminant square *of f is the commutative diagram*

$$\begin{array}{ccc} f^{-1}(D) & \xrightarrow{\ j\ } & \tilde{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{\ i\ } & Y \end{array} \tag{2.18}$$

and let X be the corresponding 2-cubical variety. Then, the associated semi-simplicial variety of X is of cohomological descent.

Definition 2.4.19. The *discriminant* of a proper morphism $f\colon Y \to Z$ of cubical varieties is the smallest closed cubical variety D of Z such that f induces isomorphisms $Y_I \setminus f^{-1}(D_I) \to Z_I \setminus D_I$ for all I.

Notice that we can construct a discriminant square for a proper morphism between k-cubical varieties as in Lemma-Definition 2.4.18 between ordinary varieties.

Definition 2.4.20. Let $f\colon Y \to Z$ be a proper morphism of cubical varieties with discriminant D and let T be a closed cubical subspace of Z. Then, we call f a *resolution* of (Z,T) if Y_I is smooth, $f_I^{-1}(T_I)$ consists of certain components of Y_I and divisors with simple normal crossings on some other components of Y_I, and $\dim f_I^{-1}(D_I) < \dim Z_I$ for all I.

Theorem 2.4.21. *Let Z be an n-cubical variety and let T be a closed cubical subvariety. Then, there exists a resolution $f\colon Y \to Z$ of (Z,T).*

The proof of this theorem can be found in [71, Theorem 5.25].

Theorem 2.4.22. *For any variety Y of dimension n and any Zariski closed subset T with dense complement, there exists an $(n+1)$-cubical hyperresolution X of (Y,T) such that $\dim X_I \leq n - |I| + 1$.*

This theorem is proved in [71, Theorem 5.26], and it gives an explicit construction of a cubical hyperresolution of (Y,T). Since this construction will be used to prove Lemma 5.1.1 in Chapter 5, we explain below this construction of a cubical hyperresolution and illustrate it with two examples.

The construction of the hyperresolution is given step by step. The first step is to choose a resolution $\pi\colon \tilde{Y} \to Y$ of (Y,T) with discrimiant D

$$\begin{array}{ccc} X^{(1)}_{\{0,1\}} = \pi^{-1}(D) & \longrightarrow & \tilde{Y} = X^{(1)}_{\{0\}} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X^{(1)}_{\{1\}} = D & \longrightarrow & Y = X^{(1)}_{\emptyset} \end{array} \tag{2.19}$$

and consider this as a 2-cubical variety which is of cohomological descent. The hypothesis of induction is the following: After k steps, we obtain a $(k+1)$-cubical variety $X^{(k)}$ which is proper, of cohomological descent, with $X_\emptyset^{(k)} = X$, $X_I^{(k)}$ smooth for all non-empty $I \subseteq [k-1]$ and $\dim X_I^{(k)} \leq n - |I| + 1$ for all $I \subseteq [k]$, and the inverse image of T in $X_I^{(k)}$ is a union of irreducible components of $X_I^{(k)}$ and a simple normal crossings divisor.

We proceed to construct the next step: Consider $X^{(k)}$ as a morphism $f^{(k)}\colon Y^{(k)} \to Z^{(k)}$ of k-cubical varieties. Notice that Z_I is smooth for $I \neq \emptyset$ and $Y^{(k)}$ is possibly singular. Let $T^{(k)}$ be the inverse image of T in $Y^{(k)}$. We choose a resolution $\pi_{Y^{(k)}}\colon \tilde{Y}^{(k)} \to Y^{(k)}$ of $(Y^{(k)}, T^{(k)})$ and construct its discriminant square

$$\begin{array}{ccc} E^{(k)} & \longrightarrow & \tilde{Y}^{(k)} \\ \downarrow & & \downarrow{\scriptstyle \pi_{Y^{(k)}}} \\ D^{(k)} & \longrightarrow & Y^{(k)} \end{array} \tag{2.20}$$

where $D^{(k)}$ is the discriminant of the resolution $\pi_{Y^{(k)}}$, and $E^{(k)}$ is the inverse image of $D^{(k)}$ in $\tilde{Y}^{(k)}$. Then, we can find the following commutative square

$$\begin{array}{ccc} E^{(k)} & \longrightarrow & \tilde{Y}^{(k)} \\ \downarrow & & \downarrow \\ D^{(k)} & \longrightarrow & Z^{(k)} \end{array} \tag{2.21}$$

through the composition of $\pi_Y^{(k)}$ with $Y^{(k)} \to Z^{(k)}$. Square (2.21) can be seen as a $(k+2)$-cubical variety $X^{(k+1)}$. Indeed, for all $I \subseteq [k-1]$, we set

$$X_I^{(k+1)} := Z_I^{(k)}, \quad X_{I\cup\{k\}}^{(k+1)} := \tilde{Y}_I^{(k)}, \quad X_{I\cup\{k+1\}}^{(k+1)} := D_I^{(k)}, \quad X_{I\cup\{k,k+1\}}^{(k+1)} := E_I^{(k)}. \tag{2.22}$$

Furthermore, $\dim X_I^{(k+1)} \leq n - |I| + 1$ for all $I \subseteq [k+1]$ and $X^{(k+1)}$ is of cohomological descent (see the proof of [71, Theorem 5.26]). Notice that in the last step, all terms appearing in the $(n+1)$-cubical variety are smooth for $I \neq \emptyset$. So, this construction gives a hyperresolution of (Y, T).

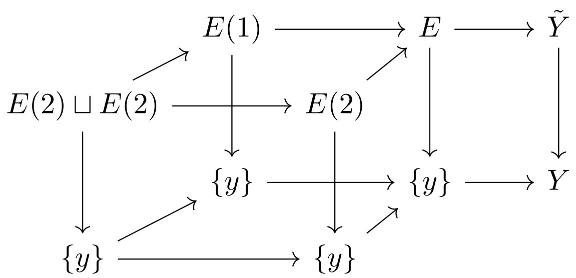
Example 2.4.23. Let Y be a complex surface with only one isolated singularity y. We obtain a hyperresolution of Y following the above construction. First, consider a resolution of singularities $\pi\colon \tilde{Y} \to Y$ of (Y, y), with discriminant $D = \{y\}$ and $E := \pi^{-1}(y)$ the exceptional divisor with simple normal crossings. The discriminant square of π is

$$\begin{array}{ccc} E & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{y\} & \longrightarrow & Y \end{array} \tag{2.23}$$

By Lemma-Definition 2.4.18, the square (2.23) is a 2-cubical variety $X^{(1)}$ which is of cohomological descent.

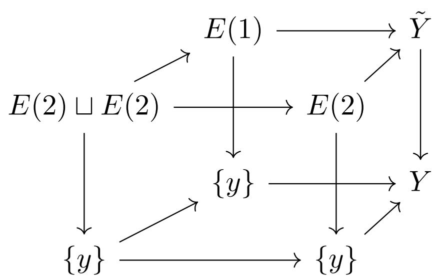
We introduce the following notation: For $E = \bigcup_{i\in I} E_i$, we set $E(k) := \bigsqcup_{|J|=k} E_J$, where $E_J = \bigcap_{j\in J} E_j$, $J \subseteq I$.

By taking $X^{(1)}$ as a morphism $f^{(1)}\colon Y^{(1)} \to Z^{(1)}$ of 1-cubical varieties $Y^{(1)}$ and $Z^{(1)}$, where $Y^{(1)}_{\emptyset} = \{y\}$, $Y^{(1)}_0 = E$, and $Z^{(1)}_{\emptyset} = Y$, and $Z^{(1)}_0 = \tilde{Y}$, respectively. Notice that $Z^{(1)}$ is smooth for $I \neq \emptyset$, and $Y^{(1)}$ is not smooth. We choose a resolution $\tilde{Y}^{(1)} \to Y^{(1)}$ of $Y^{(1)}$, where $\tilde{Y}^{(1)}$ is $\tilde{Y}^{(1)}_{\emptyset} = \{y\}$ the resolution of $(Y^{(1)}_{\emptyset}, \{y\})$ and $\tilde{Y}^{(1)}_{\{0\}} = E(1)$ the resolution of $(Y^{(1)}_{\{0\}}, E(2))$. So, the discriminant square of $\pi_{Y^{(1)}}$ is



(2.24)

then, we obtain the following 3-cubical variety



(2.25)

whose associated semi-simplicial variety augmented to Y is

$$E(2) \sqcup E(2) \Rrightarrow E(1) \sqcup E(2) \sqcup \{y\} \rightrightarrows \tilde{Y} \sqcup \{y\} \sqcup \{y\} \longrightarrow Y.$$

By Theorem 2.4.22, the semi-simplicial variety (2.25) is a cubical hyperresolution of Y.

Example 2.4.24. Let Y be a normal complex 3-fold with Σ_1 the singular locus of Y of dimension 1. Consider Σ_1 with r connected components and with only one special point $\{y\}$. We construct a hyperresolution of Y step by step. First, consider a resolution of singularities $\pi\colon \tilde{Y} \to Y$ of Y with simple normal crossing exceptional divisor $E = \pi^{-1}(\Sigma_1)$. The discriminat of π is Σ_1, then the discriminant square is the following 2-cubical variety

$$\begin{array}{ccc} E & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \Sigma_1 & \longrightarrow & Y \end{array} \tag{2.26}$$

Since the 1-cubical variety $Y^{(1)}$ (that is $E \to \Sigma_1$) is not smooth, we continue with the second step of the construction. We introduce some notation: For $E = \cup_{i\in I} E_i$, we set $E(k) := \bigsqcup_{|J|=k} E_J$, where $E_J = \bigcap_{j\in J} E_j$, $J \subseteq I$, $E^l := \bigcup_{|L|=l} E_L$ in E and $E^l(k) := \bigsqcup_{|J|=k} E^l_J$ with $J \subseteq L$.

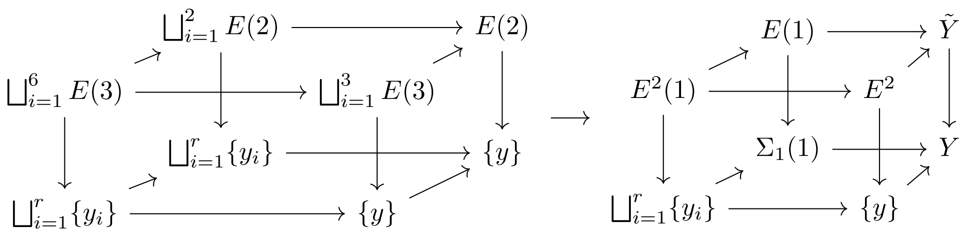
Consider the normalization of $(Y^{(1)}, T^{(1)})$, where $T^{(1)}$ is the 1-cubical variety $E^2 \to \{y\}$, and its discriminant square is

$$\begin{array}{ccccccc} & & E(1) & \longrightarrow & E & \longrightarrow & \tilde{Y} \\ & \nearrow & \downarrow & \nearrow & \downarrow & & \downarrow \\ E^2(1) & \longrightarrow & & E^2 & & & \\ \downarrow & & \Sigma_1(1) & \longrightarrow & \Sigma_1 & \longrightarrow & Y \\ & \nearrow & & \downarrow & \nearrow & & \\ \bigsqcup_{i=1}^r \{y_i\} & \longrightarrow & & \{y\} & & & \end{array} \tag{2.27}$$

Since the 2-cubical variety

$$\begin{array}{ccc} E^2(1) & \longrightarrow & E^2 \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \bigsqcup_{i=1}^r \{y_i\} & \longrightarrow & \{y\} \end{array} \tag{2.28}$$

is not smooth, we continue with a last step of the construction, and we obtain the 4-cubical variety



(2.29)

which is a cubical hyperresolution of Y.

2.5. Characteristic classes of singular varieties

This section is devoted to the theory of characteristic classes which involves Chapter 4 and Chapter 5. The theory of characteristic classes started as a part of the obstruction theory in the study of vector bundles of smooth manifolds. Later, these classes were generalized to singular spaces in several ways. In Section 2.5.1, we introduce the theory of characteristic classes of vector bundles and smooth manifolds, and exhibit some examples of these classes. In Section 2.5.2 characteristic classes of singular spaces will be introduced, as well as the corresponding singular characteristic classes generalizing the ones in the previous section. We focus on the main ones that will appear in Chapter 4 and Chapter 5.

2.5.1. Characteristic classes of vector bundles

The classical characteristic classes were introduced by E. Stiefel [79] in the 1930s as part of the obstruction theory. They are cohomology classes of the base space of a vector bundle that measure the obstruction of the existence of linearly independent sections of the vector bundle, that is, the triviality of the vector bundle.

A characteristic class of a vector bundle over a topological space X is an assignment from the set of isomorphism classes of vector bundles over X to the cohomology group of X. Let $\mathrm{Vect}(-)$ be the contravariant functor of isomorphism classes of vector bundles over a topological space, and let $H^*(-;R)$ be the contravariant cohomology functor with coefficient ring R. The caracteristic classes of vector bundles are defined as follows:

Definition 2.5.1. A *characteristic class of vector bundles* is a natural transformation

$$cl^*\colon \mathrm{Vect}(-) \to H^*(-;R),$$

which assigns to a vector bundle $\xi\colon E \to X$ a cohomology class

$$cl^*(E) := cl^*(\xi) \in H^*(X;R)$$

on the base space X of the vector bundle $\xi\colon E \to X$.

One of the most fundamental characteristic classes of vector bundles are *Chern classes*. They are defined axiomatically as follows:

The *Chern class* c^* is an assignation to a complex vector bundle $\xi\colon E \to X$ a cohomology class

$$c^i(E) \in H^{2i}(X;\mathbb{Z}),$$

satisfying the following axioms:

(A1) $c^0(E) := 1$ and $c^i(E) = 0$ for $i > \text{rank}(E)$.

$$c^*(E) := 1 + c^1(E) + \cdots + c^{\text{rank}(E)}(E) \in H^*(X; \mathbb{Z})$$

is called the *total Chern class* of E.

(A2) (Naturality) Let $f\colon Y \to X$ be a map, then

$$c^*(f^*E) = f^*(c^*(E)).$$

(A3) (Wu product) Let $\xi\colon E \to X$ and $\eta\colon F \to X$ be two complex vector bundles, then

$$c^*(E \oplus F) = c^*(E) \cup c^*(F).$$

(A4) (Normalization) The total Chern class $c^*(\gamma_n^1(\mathbb{C}))$ of the tautological line bundle γ_n^1 over $\mathbb{P}^n(\mathbb{C})$ is

$$c^*(\gamma_n^1(\mathbb{C})) = 1 - g,$$

where $g \in H^2(\mathbb{P}^n(\mathbb{C}); \mathbb{Z})$ is the Poincaré dual of the hyperplane $\mathbb{P}^{n-1}(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$.

In the theory of vector bundles, the Chern classes are useful to reduce computations using line bundles through the following theorem, the *splitting principle*:

Theorem 2.5.2. *Let $\xi\colon E \to X$ be a complex vector bundle of rank n over a paracompact space X. There is an associated space $F(E)$ to E and a map $f\colon F(E) \to X$ such that:*

1. *The vector bundle $f^*(E)$ is a direct sum of complex line bundles, that is*

$$f^*(E) = L_1 \oplus L_2 \oplus \cdots \oplus L_n. \tag{2.30}$$

2. *The map $f^*\colon H^*(X) \to H^*(F(E))$ is injective.*

The first Chern classes of the line bundles L_i appearing in Equation (2.30) are called the *Chern roots* α_i of E.

A proof of this theorem can be found in [37, Proposition 3.3].

There is another characteristic class defined through the Chern classes, they are the *Pontrjagin classes* p^*. This class p^* is an assignation to a real vector bundle $\xi\colon E \to X$ a cohomology class

$$p^i(E) \in H^{4i}(X; \mathbb{Z})$$

defined by $(-1)^i c^{2i}(E \otimes \mathbb{C})$, where c^k is the k-th Chern class.

Characteristic classes of smooth manifolds

Let X be a smooth manifold. A characteristic class $cl^*(TX)$ of the tangent bundle TX of X is called a *characteristic cohomology class* $cl^*(X)$ of the manifold X. We denote by

$$cl_*(X) := cl^*(TX) \cap [X] \in H_*^{BM}(X; R)$$

the corresponding *characteristic homology class* of the manifold X, with $[X]$ the fundamental class in Borel-Moore homology of X.

Furthermore, the *characteristic number* $\#(X)$ of X is defined as

$$\#(X) := deg(cl_*(X)) = \int_X cl^*(TX) \cap [X].$$

For X a compact complex manifold, the *Gauss-Bonet-Chern Theorem* states that the characteristic number associated to the Chern class $c^*(X)$ of X

$$e(X) = \int_X c^*(TX) \cap [X] \tag{2.31}$$

is the Euler characteristic $e(X)$ of X.

In [38], F. Hirzebruch answering affirmatively a question of J. P. Serre, gave the called *Hirzebruch-Riemann-Roch*: For X a non-singular complex projective variety, and E a holomorphic vector bundle over X, then

$$\chi(X, E) = \int_X (ch^*(E)td^*(TX)) \cap [X], \tag{2.32}$$

where $\chi(X, E)$ is the Euler-Poincaré characteristic of E. Here

$$ch^*(E) := \sum_{j=1}^{\text{rank E}} e^{\beta_j}$$

is the *Chern character*, and

$$td^*(TX) := \prod_{i=1}^{\dim X} \frac{\alpha_i}{1 - e^{-\alpha_i}},$$

is the *Todd class*, where β_j and α_i are the Chern roots of E and TX, respectively. Moreover, when E is a trivial line bundle, the Hirzebruch-Riemann-Roch theorem recovers the *arithmetic genus* $\chi(X)$ of X.

Furthermore, in [39], F. Hirzebruch proved the *Hirzebruch's Signature Theorem*. This theorem states that there are unique polynomials in the Pontrjagin classes p^*, called the *Thom-Hirzebruch L-classes* L^*, satisfying that for any closed oriented manifold X of dimension divisible by 4, the evaluation of these classes in the fundamental class of X recovers its signature. That is:

$$\sigma(X) = \int_X L^*(TX) \cap [X], \tag{2.33}$$

where $\sigma(X)$ is the *signature* of X (i.e. the signature of the intersection form associated to X).

The Hirzebruch cohomology class

In this section, we introduce the *Hirzebruch cohomology class* of smooth manifolds. One of the special features of this characteristic class is that it depends on an indeterminate y, such that for different values of y, this class specializes in three different characteristic classes.

In [38], F. Hirzebruch proved the *generalized Hirzebruch Riemann-Roch theorem*, which recovers for different values of y, the Gauss-Bonnet-Chern theorem, the Hirzebruch Riemann-Roch theorem, and Hirzebruch's Signature theorem introduced in the previous section:

Theorem 2.5.3 (g-HRR). *Let X be a non-singular complex projective variety and E a holomorphic vector bundle over X. The χ_y-characteristic of E equals to*

$$\chi_y(X, E) = \int_X T_y^*(TX) ch_{(1+y)}(E) \cap [X] \in \mathbb{Q}[y],$$

with

$$ch_{(1+y)}(E) := \sum_{j=1}^{\mathrm{rank}(E)} e^{\beta_j(1+y)}$$

and

$$T_y^*(TX) := \prod_{i=1}^{\dim(X)} \frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y.$$

Here β_j and α_i are the Chern roots of E and TX, respectively.

The *Hirzebruch cohomology class* $T_y^*(X) := T_y^*(TX)$ specializes in the following three characteristic cohomology classes of TX:

For $y = -1$,

$$T_{-1}^*(X) = c^*(X)$$

the total Chern class of X.
For $y = 0$,

$$T_0^*(X) =: td^*(X)$$

the total Todd class of X.
For $y = 1$,

$$T_1^*(X) = L^*(X)$$

the total Thom-Hirzebruch L-class of X.

The g-HRR theorem for a trivial line bundle E specializes in the following:
For $y = -1$, *Gauss-Bonnet-Chern Theorem:*

$$e(X) = \int_X c^*(X) \cap [X]$$

the Euler characteristic of X.
For $y = 0$, *Hirzebruch Riemann-Roch Theorem:*

$$\chi(X) = \int_X td^*(X) \cap [X]$$

the arithmetic genus of X.
For $y = 1$, *Hirzebruch's Signature Theorem*:

$$\sigma(X) = \int_X L^*(X) \cap [X]$$

the signature of X.

2.5.2. Characteristic classes of singular varieties

Characteristic classes of manifolds were generalized in several ways to the singular case. For a singular complex algebraic variety X, its tangent bundle is not available because of the existence of singularities. Characteristic classes of singular varieties are usually homology classes which, in the non-singular case, recover the corresponding characteristic cohomology class by capping with the fundamental class. Indeed, they can be seen as natural transformations from a functor depending on the characteristic class to the homology functor. The first characteristic class of singular complex varieties formulated as natural transformations was the Chern-Schwartz-MacPherson class transformation [51] (Section 2.5.2). After this, P. Baum, W. Fulton, and R. MacPherson [5] defined a singular version of the Todd class (Section 2.5.2). Later, S. E. Cappell, J. L. Shaneson and S. Weinberger in [15] (see also [14] and [84]) introduced a homology L-class transformation recovering the Thom-Hirzebruch L-class for the non-singular case (Section 2.5.2).

As we defined in the previous section, the characteristic classes of vector bundles are natural transformations from the functor Vect$(-)$ of vector bundles to the cohomology functor $H^*(-; R)$. This naturality motivates the following definition of characteristic classes for singular varieties.

Definition 2.5.4. A *characteristic class* (for singular varieties) is a natural transformation

$$cl_* \colon A(-) \to H_*^{BM}(-; R)$$

from a suitable covariant functor $A(-)$ depending on the choice of cl_* to the Borel-Moore homology functor $H_*^{BM}(-;R)$. This transformation satisfies the following two properties:

1. There is always a *distinguished element* $1_X \in A(X)$, such that the *characteristic class of the singular variety* X is defined by

$$cl_*(X) := cl_*(1_X).$$

2. It satisfies the following *normalization condition*:

$$cl_*(1_X) = cl^*(TX) \cap [X],$$

when X is a smooth manifold.

The characteristic class cl_* should be seen as a homology class version of the *characteristic number* of the singular variety X:

$$\#(X) := cl_*(pt_* 1_X) = pt_*(cl_*(1_X)) \in H_*(\{pt\}; R) = R,$$

where $pt\colon X \to \{pt\}$ is a constant map. Moreover, the normalization condition recovers, for X smooth, the characteristic number of X:

$$\#(X) = \int_X cl^*(TX) \cap [X].$$

The Chern-Schwartz-MacPherson class

The Chern-Schwartz-MacPherson characteristic class comes from the unification of two different generalizations for singular varieties of the Chern class of smooth manifolds. In [77], M. H. Schwartz generalized the Chern class via obstruction theory. In [51], proving a conjecture formulated by P. Deligne and A. Grothendieck, a generalization of the Chern class via natural transformations was obtained by R. MacPherson. Then, J. P. Brasselet and M. H. Schwartz in [10] showed that the distinguished value of this natural transformation coincides with the definition of M. H. Schwartz via Alexander duality.

Let X be a complex algebraic variety. A *constructible function* $\alpha\colon X \to \mathbb{Z}$ is a function on X given by a finite sum $\alpha = \sum_i n_i 1_{W_i}$, where $n_i \in \mathbb{Z}$, W_i a subvariety of X, and 1_{W_i} is the characteristic function of W_i. Denote by $F(X)$ the abelian group of *constructible functions* on X. The *integral* of α is defined by

$$\int_X \alpha := \sum_i \chi(W_i),$$

where χ is the Euler characteristic taking Borel-Moore homology. For a morphism $f\colon X \to Y$, the push-forward $f_*\colon F(X) \to F(Y)$ is defined by

$$f_*(\alpha)(y) := \int_{f^{-1}(y)} \alpha.$$

The group of constructible functions define a covariant functor $F(-)$ from the category of complex algebraic varieties and proper morphisms to the category of abelian groups. In [51], R. MacPherson proved the following theorem:

Theorem 2.5.5. *There is a unique natural transformation*

$$c_*\colon F(-) \to H^{BM}_{2*}(-;\mathbb{Z})$$

from the constructible function functor $F(-)$ to the Borel-Moore homology functor $H^{BM}_{2*}(-;\mathbb{Z})$ in even degrees, satisfying:

1. *The distinguished element is the characteristic function $1_X \in F(X)$ of a complex algebraic variety X, and*
2. *if X is non-singular, then $c_*(1_X)$ is the Poincaré dual of the total Chern cohomology class:*
$$c_*(1_X) = c^*(TX) \cap [X].$$

Definition 2.5.6. The class $c^{SM}_*(X) := c_*(1_X)$ is called the *Chern-Schwartz-MacPherson class* of X.

By considering the map from X to a point, we obtain

$$e(X) = deg(c^{SM}_*(X)) = \int_X c^{SM}_*(X),$$

that is, a singular version of the Gauss-Bonnet-Chern theorem.

Segre-Schwart-MacPherson class

After defining the Chern-Schwartz-MacPherson class, another characteristic class was given defined through the Chern-Schwartz-MacPherson class and the total Chern class.

Let $i\colon X \hookrightarrow M$ be a closed embedding of a complex algebraic variety X in a complex manifold M. The *Segre-Schwartz-MacPherson class* of the embedding i is defined by

$$s^{\mathrm{SM}}(X,M) := c^*(i^*TM)^{-1} \cap c^{SM}_*(X).$$

The Baum-Fulton-MacPherson Todd class

P. Baum, W. Fulton, and R. MacPherson in [5] defined a generalization for singular varieties of the Todd class, by proving the following theorem:

Theorem 2.5.7. *There is a unique natural transformation*

$$td_*\colon G_0(-) \to H_{2*}^{BM}(-;\mathbb{Q})$$

from the Grothendieck functor $G_0(-)$ of coherent sheaves (see Chapter II in [36]) to the Borel-Moore homology functor $H_{2*}^{BM}(-;\mathbb{Q})$. It satisfies

1. *for a complex algebraic variety X, the distinguished element is $1_X := [\mathcal{O}_X] \in G_0(X)$, that is, the class of the structure sheaf.*
2. *If X is non-singular,*

$$td_*([\mathcal{O}_X]) = td^*(TX) \cap [X].$$

Definition 2.5.8. The class $td_*(X) := td_*([\mathcal{O}_X])$ is called the *Baum-Fulton-MacPherson Todd class* of X.

Considering the mapping from X to a point, we obtain the singular version of the Riemann-Roch theorem:

$$\chi(X) = \int_X td_*(X).$$

The Cappell-Shaneson L-class

M. Goresky and R. MacPherson developed the intersection homology theory in [30] and [31]. Using this theory, they defined a homology class $L_*(X)$ for a stratified space X with even codimensional strata, such that if X is non-singular, it recovers the Thom-Hirzebruch L-class of X by capping with its fundamental class:

$$L_*(X) = L^*(TX) \cap [X].$$

S. E. Cappell, J. L. Shaneson and S. Weinberger in [15] (see also [14], [84]) using some topological aspects of perverse sheaves, introduced a homology L-class transformation L_*. It is a natural transformation from the cobordism functor $\Omega_{\mathbb{K}}(-)$ of self-dual cohomologically constructible bounded $\mathbb{K}$-complexes ($\mathbb{K}$ a subfield of $\mathbb{R}$) of sheaves to the rational homology functor $H_{2*}(-;\mathbb{Q})$. Moreover, for a compact complex algebraic variety X, the transformation L_* applied to the intersection cohomology complex IC_X (see 2.3.3) recovers the Goresky-MacPherson definition of L-class.

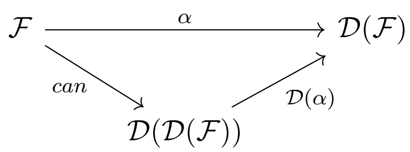
In Chapter 5, we use the Cappell-Shaneson interpretation of the L-class. In order to introduce the Cappell-Shaneson transformation, we define the cobordism group $\Omega_{\mathbb{K}}(X)$ ($\mathbb{K}$ a subfield of $\mathbb{R}$) on a compact complex algebraic variety

X. The group $\Omega_{\mathbb{K}}(X)$ was introduced in [14]. A problem with the ambiguity of mapping cones related with the definition in [14] was improved in [85] (see [9]). First, we define the cobordism group given in [85]. After this, we give some relations between this definition with another definition of cobordism introduced in [27].

Let X be a compact complex algebraic variety. Consider $(\mathcal{F}, \alpha)$ a self-dual complex of sheaves on X, that is $(\mathcal{F}, \alpha)$ is a pair such that $\mathcal{F} \in \mathrm{Ob}(D^b_c(X))$ and the morphism $\alpha\colon \mathcal{F} \to \mathcal{D}(\mathcal{F})$ is an isomorphism. The functor $\mathcal{D}(-)$ is the Borel-Moore-Verdier dualizing functor (see Definition 2.3.10). It satisfies that

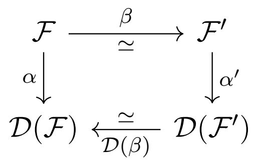
$$\alpha = \pm can \circ \mathcal{D}(\alpha),$$

that is, if the following diagram



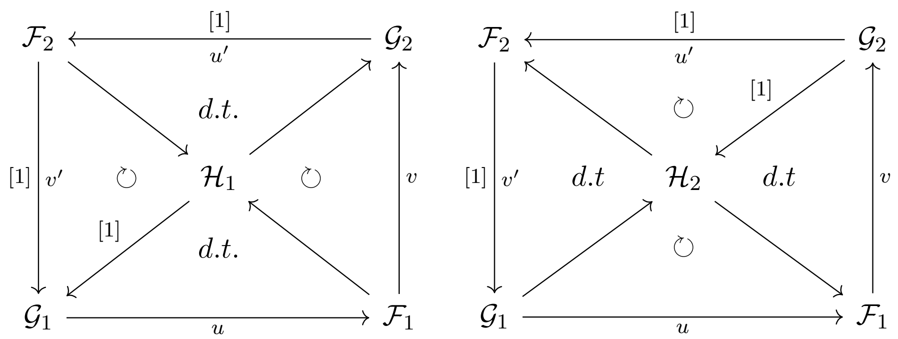
is either commutative or anti-commutative, respectively (for *can* see (2.8)). The self-dual complex $(\mathcal{F}, \alpha)$ is called *symmetric* if $\alpha = can \circ \mathcal{D}(\alpha)$ and *skew-symmetric* if $\alpha = -can \circ \mathcal{D}(\alpha)$.

Definition 2.5.9. Let $(\mathcal{F}, \alpha)$ and $(\mathcal{F}', \alpha')$ be two self-dual complexes on X. Then, they are *isomorphic* if there is an isomorphism $\beta\colon \mathcal{F} \to \mathcal{F}'$ such that the following diagram



commutes.

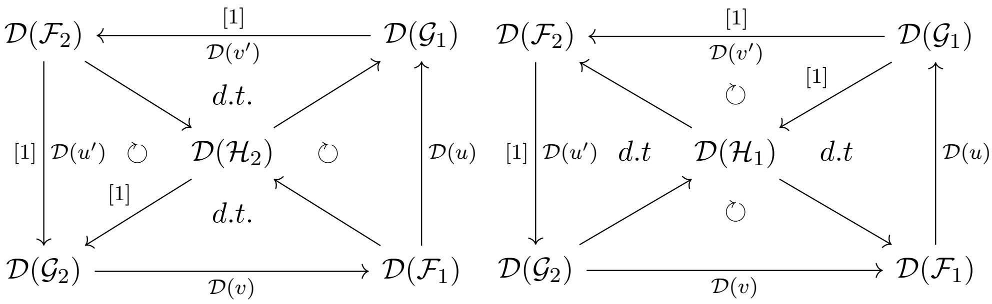
In order to define the cobordism relation, we consider the following octahedral diagram (Oct) in $D^b_c(X)$:



(2.34)

where the morphisms marked [1], are of degree one, the triangles marked with *d.t* are distinguished and the ones marked ↻ are commutative. The two composite morphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$ (via $\mathcal{G}_1$ and $\mathcal{G}_2$) have to be the same. Similarly for the two composite morphisms from $\mathcal{H}_2$ to $\mathcal{H}_1$ (via $\mathcal{F}_1$ and via $\mathcal{F}_2$).

Consider the octahedral diagram (R$\mathcal{D}$(Oct)) given by the application of the Borel-Moore-Verdier duality functor $\mathcal{D}$ and a rotation by 180^o about the axis connecting upper-left and lower-right corner. That is, the following octahedral diagram:



(2.35)

Moreover, by applying again $\mathcal{D}$ with the above rotation to the obtained octahedral diagram (R$\mathcal{D}$(Oct)) we have the octahedral diagram ($\mathcal{D}(\mathcal{D}$(Oct))), that is the one obtained from (Oct) by application of $\mathcal{D}^2$. The octahedral (Oct) is called *symmetric* or *skew-symmetric* if there is an isomorphism $\alpha\colon$ (Oct)→ (R$\mathcal{D}$(Oct)) of octahedral diagrams, such that

$$\mathrm{R}\mathcal{D}(\alpha)\circ can = \alpha \quad \text{or} \quad -\mathrm{R}\mathcal{D}(\alpha)\circ can = \alpha$$

as morphisms of octahedral diagrams, respectively. Here *can* is the canonical isomorphism $can\colon$ (Oct)→ ($\mathcal{D}(\mathcal{D}$(Oct))) of octahedral diagrams.

Notice that the isomorphism $\alpha\colon$ (Oct)→ (R$\mathcal{D}$(Oct)) gives (skew)-symmetric self-dual complexes $(\mathcal{F}_1,\alpha_1)$ and $(\mathcal{F}_2,\alpha_2)$ in the corners corresponding to $\mathcal{F}_1$ and $\mathcal{F}_2$. We say that ((Oct), α) is an *elementary cobordism* between $(\mathcal{F}_1,\alpha_1)$ and $(\mathcal{F}_2,\alpha_2)$. This relation of elementary cobordism is reflexive and symmetric (Remark 6.2, [85]).

Definition 2.5.10. Let $(\mathcal{F},\alpha)$ and $(\mathcal{F}',\alpha')$ be two self-dual complexes on X. We say that $(\mathcal{F},\alpha)$ is *cobordant* to $(\mathcal{F}',\alpha')$ if there is a sequence

$$(\mathcal{F},\alpha)=(\mathcal{F}_0,\alpha_0),(\mathcal{F}_1,\alpha_1),\ldots,(\mathcal{F}_r,\alpha_r)=(\mathcal{F}',\alpha')$$

such that $(\mathcal{F}_i,\alpha_i)$ is elementary cobordant to $(\mathcal{F}_{i+1},\alpha_{i+1})$, for $i=0,\ldots,r-1$.

Thus, by definition of cobordism, the cobordism relation is an equivalence relation.

Definition 2.5.11. The *symmetric cobordism group* $\Omega_{\mathbb{K}+}(X)$ of self-dual $\mathbb{K}$-complexes on X is the quotient of the monoid of isomorphism classes of cohomologically constructible bounded symmetric self-dual complexes by this cobordism relation. Similarly for the *skew-symmetric cobordism group* $\Omega_{\mathbb{K}-}(X)$ replacing in the above definition symmetric by skew-symmetric.

The sum in $\Omega_{\mathbb{K}+}(X)$ and $\Omega_{\mathbb{K}-}(X)$ is given by the direct sum. They are abelian groups, since $(\mathcal{F}, \alpha) \oplus (\mathcal{F}, -\alpha)$ is cobordant to 0 (see [85], [9]). We denote by $\Omega_{\mathbb{K}}(X) := \Omega_{\mathbb{K}+}(X) \oplus \Omega_{\mathbb{K}-}(X)$.

The following proposition was proved by B. Youssin in [85, Example 6.6], which will be relevant in Chapter 5 simplifying computations.

Proposition 2.5.12. *Let $(\mathcal{F}, \alpha)$ be a (skew-) symmetric self-dual complex on X. Then, $(\mathcal{F}, \alpha)$ is elementary cobordant to the (skew-) symmetric self-dual complex $({}^p\mathcal{H}^0(\mathcal{F}), {}^p\mathcal{H}^0(\alpha))$.*

For the definition of the 0-th perverse cohomology ${}^p\mathcal{H}^0(\mathcal{F})$ of $\mathcal{F}$ see Section 2.3.2.

Remark 2.5.13. By Remark 2.3.14, if $(\mathcal{F}, \alpha)$ is a (skew-) symmetric self-dual complex on X, then the corresponding pair after applying the functor Rf_* to $(\mathcal{F}, \alpha)$ is a (skew)-symmetric self-dual complex on Y, for $f\colon X \to Y$ a proper morphism. Moreover, the functor Rf_* induces a group homomorphism $f_*\colon \Omega_{\mathbb{K}\pm}(X) \to \Omega_{\mathbb{K}\pm}(X)$ defined by $[(\mathcal{F}, \alpha)] \mapsto [(Rf_*\mathcal{F}, Rf_*(\alpha))]$.

After expose the definition of cobordism through elementary cobordisms given by B. Youssin in [85], we introduce the notion of cobordism given in [27]. In [27], the self-dual complexes are considered by taking pairings $\mathcal{F} \otimes \mathcal{F} \to \mathbb{D}_X$ instead of isomorphisms $\mathcal{F} \xrightarrow{\simeq} \mathcal{D}(\mathcal{F})$. And, the notion of directly cobordim is introduced. In a proposition below, we will show that two elementary cobordant self-dual complexes are directly cobordant.

Let X be a compact complex algebraic variety, and let $\mathbb{K}$ be a subfield of $\mathbb{R}$. For $\mathcal{F}$, $\mathcal{G} \in \mathrm{Ob}(D^b_c(X))$, a morphism

$$S\colon \mathcal{F} \otimes \mathcal{G} \to \mathbb{D}_X,$$

where $\mathbb{D}_X$ is the dualizing complex on X (see Definition 2.3.9), is a called a *perfect pairing* if the corresponding morphism $\alpha\colon \mathcal{F} \to \mathcal{D}(\mathcal{G})$ given by adjuntion (see Corollary 2.3.8) is an isomorphism in $D^b_c(X)$.

Definition 2.5.14. A pair $(\mathcal{F}, S)$ is called a *self-dual* complex if $\mathcal{F} \in \mathrm{Ob}(D^b_c(X))$ and $S\colon \mathcal{F} \otimes \mathcal{F} \to \mathbb{D}_X$ is a perfect pairing.

Remark 2.5.15. Notice that Definition 2.3.13 of self-dual complex is equivalent to this one, by using Corollary 2.3.8. That is, by the canonical isomorphism

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{F}, \mathbb{D}_X) \simeq \mathrm{Hom}(\mathcal{F}, R\mathcal{H}om(\mathcal{F}, \mathbb{D}_X)).$$

We say that the perfect pairing $S\colon \mathcal{F}\otimes\mathcal{F}\to\mathbb{D}_X$ is *symmetric* if $S\circ\iota = S$ and *skew-symmetric* if $S\circ\iota=-S$, for $\iota\colon \mathcal{F}\otimes\mathcal{F}\to\mathcal{F}\otimes\mathcal{F}$ the involution defined by

$$u\otimes v\mapsto(-1)^{ij}v\otimes u,$$

where $u\in\mathcal{F}^i$, and $v\in\mathcal{F}^j$. A self-dual complex $(\mathcal{F},S)$ is called *symmetric* (or *skew-symmetric*) if the pairing S is symmetric (or skew-symmetric).

Remark 2.5.16. Since we are using Deligne's indexing convention, if Z is smooth of complex dimension d, then $\mathbb{D}_Z=\mathbb{K}_Z[2d]$ and $IC_Z=\mathbb{K}_Z[d]$. Considering $\mathbb{K}=\mathbb{R}$, then the canonical pairing given by usual real number multiplication

$$\sigma_Z:\mathbb{R}_Z[d]\otimes\mathbb{R}_Z[d]\to\mathbb{R}_Z[2d]$$

defines a perfect pairing which is symmetric for d even, and skew-symmetric for d odd.

Definition 2.5.17. Let $(\mathcal{F},S)$, $(\mathcal{F}',S')$ be two self-dual complexes. We say that $(\mathcal{F},S)$ is *directly cobordant to* $(\mathcal{F}',S')$ if there is a commutative diagram in $D^b_c(X)$

$$\begin{array}{ccc}\mathcal{G} & \xrightarrow{\rho'} & \mathcal{F}'\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'}\\ \mathcal{F} & \xrightarrow{\rho} & \mathcal{G}'\end{array}\tag{2.36}$$

together with a perfect pairing $S''\colon\mathcal{G}\otimes\mathcal{G}'\to\mathbb{D}_X$, such that

$$\begin{aligned}S\circ(\pi\otimes id_{\mathcal{F}})&=S''\circ(id_{\mathcal{G}}\otimes\rho)\colon\mathcal{G}\otimes\mathcal{F}\to\mathbb{D}_X,\\ S'\circ(\rho'\otimes id_{\mathcal{F}'})&=S''\circ(id_{\mathcal{G}}\otimes\pi')\colon\mathcal{G}\otimes\mathcal{F}'\to\mathbb{D}_X,\end{aligned}\tag{2.37}$$

and the morphism of mapping cones $\operatorname{cone}(\rho')\to\operatorname{cone}(\rho)$ induced (non-canonically) by (π,π') is an isomorphism in $D^b_c(X)$.

Proposition 2.5.18. *The self-dual complex* $(\mathcal{F},S)\oplus(\mathcal{F},-S)$ *is directly cobordant to* 0.

Proof. Considering in the diagram (2.36) $\mathcal{F}:=0$, $(\mathcal{G},S)=(\mathcal{G}',S'):=(\mathcal{F},S)$, $(\mathcal{F}',S'):=(\mathcal{F},S)\oplus(\mathcal{F},-S)$, together with $\rho':=(id,id)$ and $\pi':=(id,-id)$ the proposition holds. □

Proposition 2.5.19. *The self-dual complex* $(\mathcal{F},S)$ *is directly cobordant to the self-dual complex* $({}^p\mathcal{H}^0(\mathcal{F}),{}^p\mathcal{H}^0(\mathcal{F})(S))$.

Proof. Notice that ${}^p\mathcal{H}^0(\mathcal{F})(S)$ is a perfect pairing induced by S, since ${}^p\mathcal{H}^0$ commutes with the Borel-Moore-Verdier duality functor $\mathcal{D}$. We have the following commutative diagram

$$\begin{array}{ccc}{}^p\tau_{\le0}\mathcal{F} & \longrightarrow & \mathcal{F}\\ \downarrow & & \downarrow\\ {}^p\mathcal{H}^0(\mathcal{F}) & \longrightarrow & {}^p\tau_{\ge0}\mathcal{F}\end{array}$$

and a perfect pairing $S'' \colon {}^p\tau_{\leq 0}\mathcal{F} \otimes {}^p\tau_{\geq 0}\mathcal{F} \to \mathbb{D}_X$ induced by the perfect pairing S, and the proposition is satisfied. □

Remark 2.5.20. Let $(\mathcal{F}, S)$ be a self-dual complex. For a proper morphism $f \colon X \to Y$, the induced pairing by

$$\mathrm{Tr} \circ Rf_*(S) \colon Rf_*\mathcal{F} \otimes Rf_*\mathcal{F} \to \mathbb{D}_Y$$

is also a perfect pairing by Verdier duality (see Theorem 2.3.5), where the morphism $\mathrm{Tr} \colon Rf_! f^! \to id$ is the trace morphism defined by adjuntion.

Definition 2.5.21. Let $(\mathcal{F}, S)$, $(\mathcal{F}', S')$ be two self-dual complexes. We say that $(\mathcal{F}, S)$ is *(directly) cobordant to* $(\mathcal{F}', S')$ if there are $(\mathcal{F}_i, S_i)$, for $i = 1, \ldots, r$, such that $(\mathcal{F}_0, S_0) = (\mathcal{F}, S)$, $(\mathcal{F}_r, S_r) = (\mathcal{F}', S')$, and $(\mathcal{F}_{i-1}, S_{i-1})$ is directly cobordant to $(\mathcal{F}_i, S_i)$ for any $1 \leq i \leq r$.

The condition of directly cobordism is reflexive and symmetric. By Definition 2.5.21, this relation is an equivalence relation.

The *symmetric (directly) cobordism group* $\Omega_{\mathbb{K}+}(X)$ is defined to be the quotient of the monoid of isomorphism classes of symmetric self-dual complexes on X which is divided by this cobordism relation. The *skew-symmetric (directly) cobordism group* $\Omega_{\mathbb{K}-}(X)$ is defined in the same way replacing symmetric by skew-symmetric. The sum in $\Omega_{\mathbb{K}+}(X)$ and $\Omega_{\mathbb{K}-}(X)$ is given by the direct sum. Hence, they are abelian groups, since

$$[(\mathcal{F}, S)] + [(\mathcal{F}, -S)] = 0.$$

We define $\Omega_{\mathbb{K}}(X) := \Omega_{\mathbb{K}+}(X) \oplus \Omega_{\mathbb{K}-}(X)$.

Proposition 2.5.22. *If $(\mathcal{F}_1, \alpha_1)$ is elementary cobordant to $(\mathcal{F}_2, \alpha_2)$, then there are perfect parings S_1 and S_2 associated to α_1 and α_2, respectively, such that $(\mathcal{F}_1, S_1)$ is directly cobordant to $(\mathcal{F}_2, S_2)$.*

Proof. Since $(\mathcal{F}_1, \alpha_1)$ and $(\mathcal{F}_2, \alpha_2)$ are self-dual complexes, we have perfect pairings

$$S_1 \colon \mathcal{F}_1 \otimes \mathcal{F}_1 \to \mathbb{D}_X \quad \text{and} \quad S_2 \colon \mathcal{F}_2 \otimes \mathcal{F}_2 \to \mathbb{D}_X.$$

In order to show that the pairs $(\mathcal{F}_1, S_1)$ and $(\mathcal{F}_2, S_2)$ are directly cobordant, we consider the octahedral diagram (2.34) and the isomorphism of octahedral diagrams $\alpha \colon (\mathrm{Oct}) \to (\mathrm{R}\mathcal{D}(\mathrm{Oct}))$: We have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{\rho'} & \mathcal{F}_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathcal{F}_1 & \xrightarrow{\rho} & \mathcal{H}_1 \end{array},$$

and an isomorphism $\beta_2 \colon \mathcal{H}_2 \to \mathcal{D}(\mathcal{H}_1)$, hence we have a perfect pairing

$$S'' \colon \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathbb{D}_X$$

satisfying

$$S_1 \circ (\pi \otimes id_{\mathcal{F}_1}) = S'' \circ (id_{\mathcal{H}_2} \otimes \rho) \tag{2.38}$$

and

$$S_2 \circ (\rho' \otimes id_{\mathcal{F}_2}) = S'' \circ (id_{\mathcal{H}_2} \otimes \pi'). \tag{2.39}$$

Indeed, by the isomorphism $\alpha\colon$ (Oct)$\to$ (R$\mathcal{D}$(Oct)) we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{\beta_2} & \mathcal{D}(\mathcal{H}_1) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathcal{D}(\rho)} \\ \mathcal{F}_1 & \xrightarrow{\alpha_1} & \mathcal{D}(\mathcal{F}_1) \end{array}$$

then $\alpha_1 \circ \pi = \mathcal{D}(\rho) \circ \beta_2$. Considering the natural isomorphism

$$\phi_1\colon \mathrm{Hom}(\mathcal{H}_2 \otimes \mathcal{F}_1, \mathbb{D}_X) \simeq \mathrm{Hom}(\mathcal{H}_2, \mathcal{D}(\mathcal{F}_1)),$$

coming from adjuntion, we have

$$\phi_1(S_1 \circ (\pi \otimes id_{\mathcal{F}_1})) = \alpha_1 \circ \pi = \mathcal{D}(\rho) \circ \beta_2 = \phi_1(S'' \circ (id_{\mathcal{H}_2} \otimes \rho)).$$

Then, the desired equality (2.38) follows. Similarly for (2.39). Furthermore, we have the morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathcal{G}_1 & \longrightarrow & \mathcal{H}_2 & \xrightarrow{\rho'} & \mathcal{F}_2 & \xrightarrow{[1]} & \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \\ \mathcal{G}_1 & \xrightarrow{u} & \mathcal{F}_1 & \xrightarrow{\rho} & \mathcal{H}_1 & \xrightarrow{[1]} & \end{array}$$

hence the morphism of mapping cones $\mathrm{cone}(\rho') \to \mathrm{cone}(\rho)$ is an isomorphism. □

After giving the definition of cobordism group, we introduce the Cappell-Shaneson L-transformation given in [15] (see also [14], [84]) defined from the cobordism functor $\Omega_{\mathbb{K}}(-)$ to the homology functor $H_{2*}(-;\mathbb{Q})$ for rational coefficients :

Theorem 2.5.23. *There is a unique a natural transformation*

$$L_*\colon \Omega_{\mathbb{K}}(-) \to H_{2*}(-;\mathbb{Q})$$

with distinguished element $[(IC_X, \alpha)] \in \Omega_{\mathbb{K}}(X)$ *the cobordism class of the intersection cohomology complex on* X*. If* X *is non-singular,*

$$L_*([(\mathbb{K}_X[\dim_{\mathbb{C}} X], \alpha)]) = L^*(TX) \cap [X].$$

The Cappell-Shaneson L-class of the intersection cohomology complex IC_X satisfies $L_*([(IC_X, \alpha)]) = L_*(X)$, where $L_*(X)$ is the Goresky-MacPherson homology L-class. See [14], [13], and [15] for more details.

The degree of $L_0([(\mathcal{F}, \alpha)])$ is the signature of the induced pairing

$$H^0(X;\mathcal{F}) \otimes_{\mathbb{K}} \mathbb{R} \times H^0(X;\mathcal{F}) \otimes_{\mathbb{K}} \mathbb{R} \to \mathbb{R},$$

by definition, the signature is 0 if the pairing is skew-symmetric. In particular, for the intersection cohomology complex IC_X, we have

$$\sigma(X) = \int_X L_*([(IC_X, \alpha)]),$$

this is the singular version of Hirzebruch's signature theorem.

The Hirzebruch homology class

The Chern-Schwartz-MacPherson, the Baum-Fulton-MacPherson Todd, and the Cappell-Shaneson transformations generalize for singular varieties the Chern classes, Todd classes and Thom-Hirzebruch L-classes, respectively. As we showed in Section 2.5.1, the Hirzebruch cohomology class unifies these three characteristic classes of vector bundles. Nevertheless, it is natural to ask if there is a characteristic homology class for singular varieties which unifies the Chern-Schwartz-MacPherson, the Baum-Fulton-MacPherson Todd, and the Cappell-Shaneson classes. MacPherson's formulated this question in [52]. The difficulty of this problem is that the source covariant functors of these three natural transformations are all different.

In [9], J. P. Brasselet, J. Schürmann, and S. Yokura answered MacPherson's question. They defined a Hodge-theoretical natural transformation, depending on an indeterminate, from the relative Grothendieck functor of complex algebraic varieties to the Borel-Moore homology functor with rational coefficients.

Let X be a complex algebraic variety. E. Looijenga in [49] and F. Bittner in [7] introduced the *relative Grothendieck group of complex algebraic varieties over* X, denoted by $K_0(var/X)$.

The relative Grothendieck group $K_0(var/X)$ over X is the quotient of the free abelian group of isomorphism classes of morphisms to X, denoted by $[Y \to X]$, modulo the following *additivity relation*:

$$[Y \to X] = [Z \to X] + [Y \setminus Z \to X] \tag{2.40}$$

for $Z \subset Y$ a closed subvariety of Y. For a morphism $f : X' \to X$, the push-forward

$$f_* : K_0(var/X') \to K_0(var/X)$$

is defined by $f_*([Y \to X']) := [Y \to X]$.

J. P. Brasselet, J. Schürmann and S. Yokura in [9] proved the following theorem:

Theorem 2.5.24. *There is a unique natural transformation with respect to proper maps,*

$$T_{y,*}\colon K_0(var/-) \to H_{2*}^{BM}(-;\mathbb{Q})[y] \tag{2.41}$$

such that, for X non-singular complex algebraic variety,

$$T_{y,*}([X \to X]) = T_y^*(TX) \cap [X],$$

that is, $T_{y,*}([X \to X])$ of the class of the identity map of X recovers the Hirzebruch cohomology class of X.

Definition 2.5.25. The class $T_{y,*}(X) := T_{y,*}([X \to X])$ is called the *Hirzebruch homology class* of X.

In [9], the authors showed that the Hirzebruch natural transformation (2.41), unifies the Chern-Schwartz-MacPherson trasformation (for $y = -1$), the Baum-Fulton-MacPherson Todd transformation (for $y = 0$), and the Cappell-Shaneson L-transformation (for $y = 1$):

Theorem 2.5.26. *For $y = -1$, there is a unique natural transformation*

$$\epsilon\colon K_0(var/-) \to F(-) \tag{2.42}$$

such that, for X non-singular, $\epsilon([X \to X]) = 1_X$. And the following diagram is commutative:

$$\begin{array}{ccc} K_0(var/-) & \xrightarrow{\quad\epsilon\quad} & F(-) \\ & {\scriptstyle T_{-1,*}} \searrow \quad \swarrow {\scriptstyle c_*^{SM}\otimes\mathbb{Q}} & \\ & H_{2*}^{BM}(-;\mathbb{Q}) & \end{array} \tag{2.43}$$

For $y = 0$, there is a unique natural transformation

$$\gamma\colon K_0(var/-) \to G_0(-) \tag{2.44}$$

such that, for X non-singular, $\gamma([X \to X]) = [\mathcal{O}_X]$. And the following diagram is commutative:

$$\begin{array}{ccc} K_0(var/-) & \xrightarrow{\quad\gamma\quad} & G_0(-) \\ & {\scriptstyle T_{0,*}} \searrow \quad \swarrow {\scriptstyle td_*} & \\ & H_{2*}^{BM}(-;\mathbb{Q}) & \end{array} \tag{2.45}$$

For $y = 1$, there is a unique natural transformation

$$sd\colon K_0(var/-) \to \Omega_{\mathbb{K}}(-) \tag{2.46}$$

such that, for X non-singular, $sd([X \to X]) = [\mathbb{K}_X[\dim_{\mathbb{C}} X]]$. And the following diagram is commutative:

$$\begin{array}{ccc} K_0(var/-) & \xrightarrow{\quad sd \quad} & \Omega_{\mathbb{K}}(-) \\ & {\scriptstyle T_{1,*}} \searrow \quad \swarrow {\scriptstyle L_*} & \\ & H_{2*}(-;\mathbb{Q}) & \end{array} \tag{2.47}$$

Chapter 3

A Lê-Greuel formula for the image Milnor number

This chapter is devoted to my first work in the theory of singularities of mappings. We obtain a version of the *Lê-Greuel formula* for the image Milnor number of corank 1 map-germs, which provides a recursive method to compute it. Our proof is based on the *Marar's formula* that computes the Euler characteristic of the disentanglement of a corank 1 map-germ, in terms of the Milnor numbers of the multiple point spaces associated with the germ. This is a joint work with Prof. Juan José Nuño Ballesteros, [67].

The Lê-Greuel formula [34, 47] provides a recursive method to compute the Milnor number of an isolated complete intersection singularity (ICIS). Let $(X, 0)$ be a d-dimensional ICIS defined as the zero locus of a map-germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$. The *Milnor fibre* $X_s := g^{-1}(s)$, for s a generic value in $\mathbb{C}^{n-d}$, has the homotopy type of a wedge of d-spheres, and the number of such spheres is called the *Milnor number* $\mu(X, 0)$ of $(X, 0)$. If $d > 0$, we can consider $p : \mathbb{C}^n \to \mathbb{C}$ a generic linear projection, with $H = p^{-1}(0)$, such that $(X \cap H, 0)$ is a $(d-1)$-dimensional ICIS. Then, the *Lê-Greuel formula* states:

$$\mu(X, 0) + \mu(X \cap H, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(g) + J(g, p)}, \tag{3.1}$$

where (g) is the ideal in $\mathcal{O}_n$ generated by the components of g, and $J(g, p)$ is the *Jacobian ideal* of (g, p), that is, the ideal generated by the maximal minors of the Jacobian matrix. Note that X_s is smooth, and if p is generic enough, then the restriction $p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function, and the dimension appearing in the right hand side of (3.1) is equal to the number of critical points of $p|_{X_s}$.

We prove the following version of the Lê-Greuel formula: Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite corank 1 map-germ with $n > 1$. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear form, then f can be seen as a 1-parameter unfolding of another map-germ $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ which is the transverse slice of f with respect to p, that is, g has image $(X \cap H, 0)$, where $(X, 0)$ is the image of f and $H = p^{-1}(0)$. The disentanglement X_s of a stable perturbation f_s of f (see Section 2.1.5) is not smooth, but it has a natural Whitney stratification given by the stable types (see Section 2.1.4). If p is generic enough, the restriction

$p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function on each stratum. The Lê-Greuel type formula for the image Milnor number states:

$$\mu_I(f) + \mu_I(g) = \#\Sigma(p|_{X_s}), \tag{3.2}$$

where the right hand side of equation is the number of critical points of $p|_{X_s}$ on all the strata of X_s. The case $n = 1$, we have

$$\mu_I(f) + m_0(f) - 1 = \#\Sigma(p|_{X_s}), \tag{3.3}$$

where $m_0(f)$ is the multiplicity of the curve parametrized by f. This makes sense, since $\mu(X, 0) = m_0(X, 0) - 1$ for a 0-dimensional ICIS $(X, 0)$.

In Section 2.1.5, we introduced Mond's conjecture (Conjecture 2.1.30) which relates the image Milnor number with the $\mathcal{A}_e$-codimension of an $\mathcal{A}$-finite map-germ. We feel that this Lê-Greuel type formula for the image Milnor number could be useful to prove the conjecture. Indeed, it would be enough to prove that the module which controls the number of critical points of a generic linear function is Cohen-Macaulay and then, use an induction argument on the dimension n, see [25] for details about Mond's conjecture in this direction.

3.1. Marar's formula and multiple point spaces

Here we expose Marar's formula [53] which relates the Euler characteristic of the disentanglement associated to a stable perturbation of a corank 1 map-germ with the corresponding multiple point spaces. We base on Section 2.1.6 for the Marar-Mond construction of the k-th multiple point spaces for corank 1 map-germs.

Definition 3.1.1. Let M be a $\mathbb{Q}$-vector space upon which S_k acts. Then the *alternating part* of M, denoted by $\text{Alt}_k M$, is defined to be

$$\text{Alt}_k M := \{m \in M : \sigma(m) = \text{sign}(\sigma)m, \text{ for all } \sigma \in S_k\}.$$

Given a topological space X on which S_k acts, the *alternating Euler characteristic* is defined by

$$\chi^{alt}(X) := \sum_i (-1)^i \dim_{\mathbb{Q}} \text{Alt}_k(H_i(X, \mathbb{Q})).$$

The following theorem is proved by V. Goryunov and D. Mond in [33, 2.6] and it allows to compute the image Milnor number of f in terms of the k-th multiple point spaces $D^k(f)$ which are invariant under the action of the k-th symmetric group S_k.

Theorem 3.1.2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 map-germ, f_s be a stable perturbation of f, for $s \neq 0$, and let X_s be the disentanglement of f. Then,*

$$H_n(X_s, \mathbb{Q}) \cong \bigoplus_{k=2}^{n+1} \mathrm{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})). \tag{3.4}$$

By Theorem 2.1.28, the disentanglement X_s has the homotopy type of a wedge of n-spheres, hence the image Milnor number $\mu_I(f)$ is the rank of the homology group $H_n(X_s, \mathbb{Q})$, that is,

$$\mu_I(f) = \dim_{\mathbb{Q}} H_n(X_s, \mathbb{Q}). \tag{3.5}$$

By Equation (3.4) and (3.5), then

$$\mu_I(f) = \sum_{k=2}^{n+1} \dim_{\mathbb{Q}} \mathrm{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})). \tag{3.6}$$

By [46, Corollary 2.8], we can compute the alternating Euler characteristic of $D^k(f_s)$ as follows: For each partition $\mathcal{P} = (r_1, \dots, r_s)$, we set

$$\beta(\mathcal{P}) = \frac{\mathrm{sign}(\mathcal{P})}{\prod_i \alpha_i! i^{\alpha_i}},$$

where $\alpha_i := \#\{j : \; r_j = i\}$ and $\mathrm{sign}(\mathcal{P})$ is the number $(-1)^{k - \sum_i \alpha_i}$. Then,

$$\chi^{alt}(D^k(f_s)) = \sum_{|\mathcal{P}|=k} \beta(\mathcal{P}) \chi(D^k(f_s, \mathcal{P})). \tag{3.7}$$

By Theorem 2.1.32 and Corollary 2.1.35, the spaces $D^k(f_s)$ and $D^k(f_s, \mathcal{P})$ are Milnor fibres of the ICIS $D^k(f)$ and $D^k(f, \mathcal{P})$, respectively. Then, they have the homotopy type of a wedge of spheres of real dimension $\dim D^k(f) = n - k + 1$ and $\dim D^k(f, \mathcal{P})$, respectively. Thus,

$$\dim_{\mathbb{Q}} \mathrm{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})) = (-1)^{n-k+1}(\chi^{alt}(D^k(f_s)) - 1), \tag{3.8}$$

and

$$\chi(D^k(f_s, \mathcal{P})) = 1 + (-1)^{\dim D^k(f, \mathcal{P})} \mu(D^k(f, \mathcal{P})). \tag{3.9}$$

Substituting (3.7), (3.8), and (3.9) in Equation (3.6), we obtain the following version of Marar's formula [53]:

$$\mu_I(f) = \sum_{k=2}^{n+1} \sum_{|\mathcal{P}|=k} (-1)^{n-k+1} \beta(\mathcal{P})(1 + (-1)^{\dim D^k(f, \mathcal{P})} \mu(D^k(f, \mathcal{P}))), \tag{3.10}$$

where the coefficients $\beta(\mathcal{P}) = 0$ when the sets $D^k(f, \mathcal{P})$ are empty, for $k = 2, \dots, n+1$.

3.2. Lê-Greuel type formula

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite corank 1 map-germ. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection, such that $H = p^{-1}(0)$ is a generic hyperplane through the origin in $\mathbb{C}^{n+1}$. We can choose linear coordinates in $\mathbb{C}^{n+1}$ such that $p(y_1, \dots, y_{n+1}) = y_1$. Then, we choose the coordinates in $\mathbb{C}^n$ in such a way that f is written in the form

$$f(z, y_1, \dots, y_{n-1}) = (h_1(z, y_1, \dots, y_{n-1}), h_2(z, y_1, \dots, y_{n-1}), y_1, \dots, y_{n-1}),$$

for some holomorphic functions h_1, h_2. Notice that f can be seen as a 1-parameter unfolding of the map-germ $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ given by

$$g(z, y_1, \dots, y_{n-2}) = (h_1(z, y_1, \dots, y_{n-2}, 0), h_2(z, y_1, \dots, y_{n-2}, 0), y_1, \dots, y_{n-2}).$$

We say that g is the *transverse slice* of f with respect to the generic hyperplane H. If f has image $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$, then the image of g in $(\mathbb{C}^n, 0)$ is isomorphic to $(X \cap H, 0)$.

Let f_s be a stable perturbation of f, and X_s its disentanglement. Since f has corank 1, X_s has a natural Whitney stratification given by the stable types associated to f_s (see Remark 2.1.20). Indeed, the strata in X_s can be described as the following submanifolds in terms of the multiple point spaces and the partitions:

$$M^k(f_s, \mathcal{P}) := \epsilon^k(D^k(f_s, \mathcal{P})^0) \setminus \epsilon^{k+1}(D^{k+1}(f_s)),$$

where $D^k(f_s, \mathcal{P})^0$ is the set of generic points of $D^k(f_s, \mathcal{P})$, $\epsilon^k : \mathbb{C}^{n+k-1} \to \mathbb{C}^{n+1}$ is the map $(z_1, \dots, z_k, y) \mapsto f_s(z_1, y)$ and $\mathcal{P}$ runs through all the partitions of k with $k = 2, \dots, n+1$. We can choose the generic linear projection $p : \mathbb{C}^{n+1} \to \mathbb{C}$ in such a way that the restriction to each stratum $M^k(f_s, \mathcal{P})$ is a Morse function. In other words, such that the restriction $p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function on each stratum, this is one of the conditions of be a stratifed Morse function in the sense of [32]. We will denote by $\#\Sigma(p|_{X_s})$ the number of critical points on all the strata of X_s.

Theorem 3.2.1. *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be an injective map-germ. Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection, then*

$$\#\Sigma(p|_{X_s}) = \mu_I(f) + m_0(f) - 1,$$

where $m_0(f)$ is the multiplicity of f.

Proof. After a change of coordinates, we can assume that f has the form

$$f(t) = (t^k, c_m t^m + c_{m+1} t^{m+1} + \dots),$$

where $k = m_0(f)$, $m > k$ and $c_m \neq 0$. The stable perturbation f_s is an immersion with only transverse double points, then its disentaglement X_s has only two strata: $M^2(f_s,(1,1))$ is a 0-dimensional stratum composed by the transverse double points, and $M^1(f_s,(1))$ is a 1-dimensional stratum given by the smooth points of X_s. Notice that the number of double points of f_s is the delta invariant of the plane curve, $\delta(X,0)$, which is equal to $\mu_I(f)$ by [65, Theorem 2.3].

Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection, such that $p|_{X_s}$ is a Morse function on each stratum. Then:

$$\#\Sigma(p|_{X_s}) = \#M^2(f_s,(1,1)) + \#\Sigma(p|_{M^1(f_s,(1))}) = \mu_I(f) + \#\Sigma(p|_{M^1(f_s,(1))}).$$

Since f_s is a local diffeomorphism on the stratum $M^1(f_s,(1))$, the number of critical points of $p|_{M^1(f_s,(1))}$ is equal to the number of critical points of $p \circ f_s$. Notice that the points of $M^2(f_s,(1,1))$ can be excluded by the genericity of p. Assume that $p(x,y) = Ax + By$ with $A \neq 0$. Then $p \circ f_s$ is a morsification of the function

$$p \circ f(t) = At^k + B(c_m t^m + c_{m+1}t^{m+1} + \ldots)$$

The number of critical points of $p \circ f_s$ is equal to $\mu(p \circ f) = k - 1 = m_0(f) - 1$, which proves the formula. □

In order to prove the case for $n > 1$, we consider the following notation introduced in [53] and [50], respectively: Let $\mathcal{P}$ be a partition of k, we denote by $\rho_{\mathcal{P}}$ the mapping given by the composition of the mappings: the inclusion $D^k(f_s,\mathcal{P}) \hookrightarrow D^k(f_s)$, the projection $D^k(f_s) \to U_s$ and the stable perturbation f_s.

Remark 3.2.2. Let $\mathcal{P} = (a_1,\ldots,a_h)$ be a partition of k, with $a_i \geq a_{i+1}$. If y is a generic point of $D^k(f_s,\mathcal{P}')$, where $\mathcal{P}' = (b_1,\ldots,b_q)$, with $b_i \geq b_{i+1}$ and $\mathcal{P} < \mathcal{P}'$ then $\#\rho_{\mathcal{P}}^{-1}(\rho_{\mathcal{P}'}(y))$ is the coefficient of the monomial $x_1^{b_1}x_2^{b_2}\ldots x_q^{b_q}$ in the polynomial $\prod_{i\geq 1}(x_1^{a_i} + x_2^{a_i} + \ldots x_q^{a_i})$.

Lemma 3.2.3. *Let h_k be the k-th complete symmetric function in variables $x_1,\ldots,x_q$, i.e., h_k is the sum of all monomials of degree k in the variables $x_1,\ldots,x_q$. Then*

$$h_k = \sum_{\mathcal{P}} \frac{1}{\prod_{i\geq 1}\alpha_i! i^{\alpha_i}} \prod_{i\geq 1}(x_1^i + \ldots + x_q^i)^{\alpha_i},$$

where $\mathcal{P}$ runs through the set of all ordered partitions of k.

Theorem 3.2.4. *Let $f : (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ be an $\mathcal{A}$-finite corank 1 map-germ with $n > 1$. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection that defines a transverse slice $g : (\mathbb{C}^{n-1},0) \to (\mathbb{C}^n,0)$. Then,*

$$\#\Sigma(p|_{X_s}) = \mu_I(f) + \mu_I(g).$$

Proof. By Marar's formula (3.10):

$$\mu_I(f)+\mu_I(g)=\sum_{k=2}^{n+1}(-1)^{n-k+1}\sum_{|\mathcal{P}|=k}\beta(\mathcal{P})(1+(-1)^{\dim D^k(f,\mathcal{P})}\mu(D^k(f,\mathcal{P})))+$$
$$+\sum_{k=2}^{n}(-1)^{n-k}\sum_{|\mathcal{P}|=k}\beta(\mathcal{P})(1+(-1)^{\dim D^k(g,\mathcal{P})}\mu(D^k(g,\mathcal{P}))).$$

If $\dim D^k(f,\mathcal{P})>0$, then $\dim D^k(f,\mathcal{P})=1+\dim D^k(g,\mathcal{P})$, and if $\dim D^k(f,\mathcal{P})=0$, then $D^k(g,\mathcal{P})=\emptyset$. Then, we can divide the formula into two parts, the first one for partitions with $\dim D^k(f,\mathcal{P})=0$, and the second one for partitions with $\dim D^k(f,\mathcal{P})>0$:

$$\mu_I(f)+\mu_I(g)=\sum_{k=2}^{n+1}(-1)^{n+k-1}\sum_{\substack{|\mathcal{P}|=k\\ \dim D^k(f,\mathcal{P})=0}}\beta(\mathcal{P})(1+\mu(D^k(f,\mathcal{P})))$$
$$+\sum_{k=2}^{n}(-1)^{n+k-1}\sum_{\substack{|\mathcal{P}|=k\\ \dim D^k(f,\mathcal{P})>0}}\beta(\mathcal{P})(-1)^{\dim D^k(f,\mathcal{P})}(\mu(D^k(f,\mathcal{P}))+\mu(D^k(g,\mathcal{P})))$$

If $\dim D^k(f,\mathcal{P})=0$, the Milnor number of $D^k(f,\mathcal{P})$ is

$$\mu(D^k(f,\mathcal{P}))=deg(D^k(f,\mathcal{P}))-1,$$

where deg is the degree of the map-germ that defines the 0-dimensional ICIS $D^k(f,\mathcal{P})$. Notice that $deg(D^k(f,\mathcal{P}))$ can be seen as the number of critical points of $\tilde{p}|_{D^k(f_s,\mathcal{P})}$.

Choosing coordinates such that $p(y_1,\dots,y_{n+1})=y_1$, we denote by $\tilde{p}:\mathbb{C}^{n+k-1}\to\mathbb{C}$ the projection onto the first coordinate. Then:

$$D^k(g,\mathcal{P})=D^k(f,\mathcal{P})\cap\tilde{p}^{-1}(0).$$

By the Lê-Greuel formula for ICIS [34, 47],

$$\mu(D^k(f,\mathcal{P}))+\mu(D^k(g,\mathcal{P}))=\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})}).$$

It is easy to check that $(-1)^{\dim D^k(f)}\text{sign}(\mathcal{P})(-1)^{\dim D^k(f,\mathcal{P})}=1$ for any partition $\mathcal{P}$. Thus, we get:

$$\mu_I(f)+\mu_I(g)=\sum_{k=2}^{n+1}\sum_{|\mathcal{P}|=k}\frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})})}{\gamma(\mathcal{P})},$$

where $\gamma(\mathcal{P})=\prod_i\alpha_i!i^{\alpha_i}$.

Let $\mathcal{P}$ be a partition of k, if $|\mathcal{P}'| = k$ and $\mathcal{P}' \geq \mathcal{P}$ then any critical point of $\tilde{p}|_{D^k(f_s,\mathcal{P}')}$ is a critical point of $\tilde{p}|_{D^k(f_s,\mathcal{P})}$. This implies

$$\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})}) = \sum_{\substack{|\mathcal{P}'|=k \\ \mathcal{P}' \geq \mathcal{P}}} \alpha(\mathcal{P},\mathcal{P}')\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0}),$$

where $\alpha(\mathcal{P},\mathcal{P}')$ is defined by

$$\alpha(\mathcal{P},\mathcal{P}') := \frac{\#\rho_{\mathcal{P}}^{-1}(\rho_{\mathcal{P}'}(y))}{\#\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y))}$$

for a generic point y in $D^k(f_s,\mathcal{P}')$. We can see $\alpha(\mathcal{P},\mathcal{P}')$ as the number of times that a generic point of $D^k(f_s,\mathcal{P}')$ appears repeated in $D^k(f_s,\mathcal{P})$. By Remark 3.2.2 and Lemma 3.2.3, we have

$$\begin{aligned}
\mu_I(f) + \mu_I(g) &= \sum_{k=2}^{n+1} \sum_{|\mathcal{P}|=k} \frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})})}{\gamma(\mathcal{P})} \\
&= \sum_{k=2}^{n+1} \sum_{|\mathcal{P}|=k} \sum_{\substack{|\mathcal{P}'|=k \\ \mathcal{P}' \geq \mathcal{P}}} \frac{\alpha(\mathcal{P},\mathcal{P}')}{\gamma(\mathcal{P})} \#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0}) \\
&= \sum_{k=2}^{n+1} \sum_{|\mathcal{P}'|=k} \left(\sum_{\substack{|\mathcal{P}|=k \\ \mathcal{P} \leq \mathcal{P}'}} \frac{\#\rho_{\mathcal{P}}^{-1}(\rho_{\mathcal{P}'}(y))}{\gamma(\mathcal{P})} \right) \frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0})}{\#\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y))} \\
&= \sum_{k=2}^{n+1} \sum_{|\mathcal{P}'|=k} \frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0})}{\#\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y))} \\
&= \sum_{k=2}^{n+1} \sum_{|\mathcal{P}'|=k} \#\Sigma(p|_{M^k(f_s,\mathcal{P}')}),
\end{aligned}$$

which is nothing but the number of critical points of $p|_{X_s}$. □

3.3. Examples

In this section, we give two examples to illustrate the formulas obtained in Theorem 3.2.1 and Theorem 3.2.4. Moreover, we explain the procedure followed for $n > 1$ for computing examples.

Example 3.3.1. (The singular plane curve E_6) Consider the singular plane curve E_6, given by the parameterization

$$f(z) = (z^3, z^4)$$

and let

$$f_s(z) = (z^3 + sz, z^4 + \frac{5}{4}sz^2)$$

be a stable perturbation of f, for $s \neq 0$.

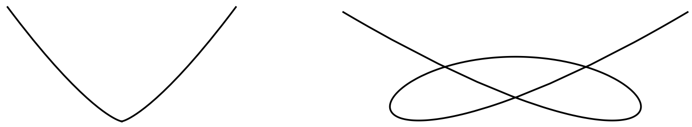


Figure 3.1: The curve E_6 and its stable perturbation for fixed $s < 0$

Let $M^2(f_s, (1,1))$ be the 0-dimensional stratum of $X_s = \mathrm{Im}(f_s)$. It is composed by three points, they correspond to three transverse double points. Let $M^1(f_s, (1))$ be the 1-dimensional stratum. If we compose f_s with $p(z, u) = z$ there are two critical points in a neighborhood of the origin, then

$$\# \sum p_{|X_s} = 5.$$

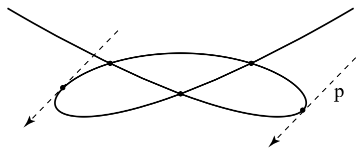


Figure 3.2: Critical points in X_s

Now, since the multiplicity of f is $m_0(f) = 3$, and the image Milnor number of f is $\mu_I(f) = 3$,

$$\mu_I(f) + m_0(f) - 1 = 5$$

as predicted by the formula.

Now, we explain the technique for the case of $n > 1$. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite corank 1 map-germ written in prenormal form

$$f(z, y) = (h_1(z, y), h_2(z, y), y)$$

with $z \in \mathbb{C}$, and $y \in \mathbb{C}^{n-1}$.

Consider f_s a stable perturbation of f, and its image X_s. First, we calculate the number of critical points of the restriction of p to X_s, for the generic linear projection $p(y_1, \ldots, y_{n+1}) = y_1$. We divide the image set X_s in strata of different dimensions given by stable types, which correspond to the sets $M^k(f_s, \mathcal{P})$. The n-dimensional stratum, $M^1(f_s, (1))$, is composed of the regular part of f_s. Then, the restriction $p|_{M^1(f_s)}$ has not critical points.

The $(n-1)$-dimensional stratum is composed of $M^2(f_s,(1,1))$. In order to obtain the critical points, we will work with the inverse image of the map $\epsilon^k : \mathbb{C}^{n+k-1} \to \mathbb{C}^{n+1}$, given by $(z_1,\ldots,z_k,y) \mapsto f_s(z_1,y)$ for $k=2$, that is, $D^2(f_s,(1,1)) = D^2(f_s)$. The double point space $D^2(f_s)$ is a subset of $\mathbb{C}^{n+1}$, but we take a projection of $D^2(f_s)$ in the first n variables. We denote by $D(f_s)$ the projection of the double point space in $\mathbb{C}^n$. The double point space $D(f_s)$ is a hypersurface in $\mathbb{C}^n$ given by the resultant of P_s and Q_s with respect to z_2, where

$$P_s = \frac{h_{1,s}(z_2,y) - h_{1,s}(z_1,y)}{z_2 - z_1}$$

and

$$Q_s = \frac{h_{2,s}(z_2,y) - h_{2,s}(z_1,y)}{z_2 - z_1}.$$

This gives the defining equation of $D(f_s)$, denoted by $\lambda_s(x,z) = 0$.

To compute the critical points of the set $D(f_s)$, we take the linear projection $\tilde{p}(z,y_1,\ldots,y_{n-1}) = y_{n-1}$. Note that the hypersuface $D(f_s)$ also contains the critical points of the other k-dimensional strata, with $k < n-1$. Then, it will be enough to compute critical points here, in order to obtain all the critical points.

We have that $(z,y_1,\ldots,y_{n-1})$ is a critical point of $\tilde{p}_{|_{D(f_s)}}$ if $\lambda_s(z,y) = 0$ and $J(\lambda_s,\tilde{p})(z,y) = 0$, where $J(\lambda_s,\tilde{p})$ is the Jacobian determinant of λ and $\tilde{p}$.

If a critical point of $\tilde{p}_{|_{D(f_s)}}$ corresponds to an m-multiple point, then we will have m critical points in $D(f_s)$ for each one in the image of f_s. Thus, once the critical points of each type are obtained, we have to divide by the multiplicity of the point. In this way, we obtain the number of critical points of p in the image of f_s. Then, we will compute separately the image Milnor numbers of f and g in order to check the formulas.

Example 3.3.2. (The germ F_4 in $\mathbb{C}^3$) Let

$$f(z,y) = (z^2, z^5 + y^3 z, y)$$

be the germ F_4, and let

$$f_s(z,y) = (z^2, z^5 + ysz^3 + (y^3 - 5ys - s)z, y)$$

be a stable perturbation of f, for fixed $s \neq 0$. By [55], f is a 1-parameter unfolding of the plane curve A_4, that is

$$g(z) = (z^2, z^5),$$

indeed g is the transverse slice of f.

Let $M^3(f_s,(1,1,1)) \cup M^2(f_s,(2))$ be the 0-dimensional strata of X_s. In our case, there are not triple points and there are three cross-caps in $M^2(f_s,(2))$.

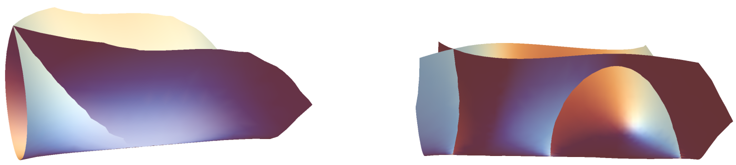


Figure 3.3: The surface F_4 and its stabe perturbation for a fixed $s > 0$

Let $M^2(f_s,(1,1))$ be the 1-dimensional stratum of X_s. Let $D^2(f_s)$ be the double point curve in $\mathbb{C}^3$ and, projecting in the first two coordinates, we have the double point curve in $\mathbb{C}^2$, denoted by $D(f_s)$.

We compute the resultant of P_s and Q_s respect to z_2, where P_s and Q_s are the divided differences. The double point curve of f_s in $\mathbb{C}^2$ is the plane curve

$$\lambda_s(z,y) = -s - 5sy + y^3 + syz^2 + z^4.$$

The critical points of the restriction $p|_D(f_s)$ are given by $\lambda_s(z_0,y_0) = 0$ and $J(\lambda_s,\tilde{p})(z_0,y_0) = 0$, where $\tilde{p}(z,y) = y$.

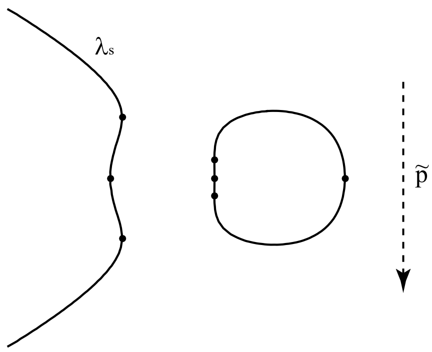


Figure 3.4: Cusps and tacnodes in the double point curve

Nine critical points are obtained. Three of these points are cusps in $g_{y,s}$ which correspond to the three cross-caps of f_s. Then, the other six critical points in $\tilde{p}|_{\lambda_s(z_0,y_0)=0}$ correspond to three tacnodes in $g_{y,s}$ which are represented in the double point curve when a vertical line is tangent at two points of $D(f_s)$. So, each two of these critical points in λ_s correspond to one tacnode of $g_{y,s}$ in $M^2(f_s,(1,1))$. Note that in the Fig. 3.4 there are only two tacnodes, that is because the other is a complex tacnode.

Finally, in the 2-dimensional stratum $M^1(f_s,(1))$ there are not critical points. So, the number of critical points in X_s is $\#\Sigma p|_{X_s} = 6$, three cusps, three tacnodes and zero triple points. Then,

$$\#\Sigma p|_{X_s} = C + J + T$$

where C, J, T are the numbers of cusps, tacnodes and triple points of $g_{y,s}$, respectively. By [55], the image Milnor number of f is

$$\mu_I(f) = C + J + T - \delta(g).$$

Since g is a plane curve, we have that $\mu_I(g) = \delta(g)$ (see [65]). Then,

$$\#\Sigma p|_{X_s} = C + J + T = \mu_I(f) + \mu_I(g).$$

Chapter 4

Image Milnor number for weighted-homogeneous map-germs

This chapter is based on my second work which combines the theory of singularities of mappings with the theory of characteristic classes of singular spaces. We obtain two formulas (for $n = 4$ and 5) for the image Milnor number of weighted-homogeneous map-germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ in terms of the weigths and the degrees associated to the mapping. This work is in collaboration with Prof. Guillermo Peñafort Sanchis [70].

In Section 2.1.1, we show that a map-germ is *stable* if its $\mathcal{A}$-class does not change after a small perturbation, and it is $\mathcal{A}$-*finite* if stability fails at most on an isolated point. Some common invariants associated to $\mathcal{A}$-finite map-germs are the 0-*stable invariants*, the $\mathcal{A}_e$-*codimension* and the *image Milnor number* μ_I. The 0-stable invariants count the number of appearances of particular stable singularity types, the $\mathcal{A}_e$-codimension measures the number of parameters of a versal unfolding and μ_I counts the rank in the middle dimension of the homology of the image of a stable perturbation.

All these invariants are hard to compute directly from the definition, but many of them can be computed as dimensions of suitable vector spaces. This applies for many of the 0-stable invariants and for the $\mathcal{A}_e$-codimension (see Section 4.4), but not so far for μ_I. This work is devoted to establish formulas of a different nature for μ_I in the case of weighted-homogeneous map-germs, extending results of T. Ohmoto and D. Mond. Apart from being interesting on their own, the formulas for μ_I bring us closer to the proof of Mond's conjecture, see Conjecture 2.1.30, which claims that $\mathcal{A}_e$-codim $\leq \mu_I$, with equality in the case of weighted-homogeneous map-germs. Thanks to results from [25], it suffices to check the statement of the conjecture for a family of finitely determined map-germs with unbound multiplicity. Given such a family, one can compute its $\mathcal{A}_e$-codimension via the already known formula, the only part missing is the μ_I computation. It is worth mentioning that the possibility of finding a formula that computes μ_I as the dimension of a vector space prior to proving Mond's conjecture is unlikely. This is because there is already a candidate for such a formula found in [25] but, as explained there, proving that it actually computes μ_I is equivalent to proving Mond's conjecture.

T. Ohmoto in [69] has adapted these techniques to show the existence of formulas computing the 0-stable invariants and the image Milnor number, for weighted-homogenenous map-germs, for $n \leq 5$, in terms of weights and degrees. The formulas are conjectured to exist for arbitrary n (see [44, 45, 69]), while the expressions for 0-stable invariants follow easily from their *Thom polynomials*, the image Milnor number formulas are harder to obtain.

The μ_I formulas predicted by T. Ohmoto in [69] are rational functions with known denominator, whose numerator is obtained from the n-th degree truncation of the *Segre-MacPherson Thom polynomial* $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})$ series (see Section 4.2). Adapting R. Rimányi's *restriction method* [72] which allows to compute Thom polynomials, T. Ohmoto determined $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})$ up to degree three, recovering the μ_I formula for $n = 2$ due to D. Mond [63], and giving the formula for $n = 3$.

The series $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})$ has coefficients $b_\alpha \in \mathbb{Q}$ and variables s_0 and c_i. If $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a weighted-homogeneous mapping with grading (w, d), then the image Milnor number $\mu_I(F)$ depends on the evaluation on certain functions $s_0(w, d), c_i(w, d)$ of the n-th truncation of $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})$. Our goal is to find the b_α up to degree n in the followging way: For fixed F, we compute $\mu_I(F)$ with the software Singular, based on results in [25]. In Section 4.2, we show that using the value of $\mu_I(F)$, and the weights and degrees (w, d) of F, one can find relations between the coefficients b_α. Sampling enough map-germs F, one can determine the desired b_α (see Section 4.2). Notice that this interpolative method does not involve the characteristic classes construction of T. Ohmoto's approach. We recover the formulas for $n = 2,\ 3$, and obtain the rest of the cases, that is, for $n = 4$ and 5.

The first steps of the process are easy, consisting only on sampling singularities found in the literature. Indeed, a surprisingly big portion of the interpolation can be completed just by sampling different gradings of stable map-germs. The challenge starts once the information coming from known singularities has been exhausted. On one hand, too simple mappings do not give new information about the b_α (for example, the case of $n = 4$ requires sampling at least one map-germ with quintuple points, while the case of $n = 5$ requires considering no less than three map-germs of corank two). On the other hand, degenerate candidates can be too complicated to compute their μ_I, or to check $\mathcal{A}$-finiteness. The difficulty of this work strives on navigating between these two extremes. In a series of remarks, we emphasize on the key strategies that have made our interpolative approach successful.

4.1. Formulas for μ_I and $\#\eta$

In this section we list the formulas for the image Milnor number μ_I and 0-stable invariants $\#\eta$ of an $\mathcal{A}$-finite weighted-homogeneous germ $F\colon (\mathbb{C}^n, 0) \to$

$(\mathbb{C}^{n+1}, 0)$, for $n = 4$ and 5 (see Definition 2.1.29). The expressions depend on some coefficients $c_{k,n}$ and $s_{0,n}$ of *Chern* and *Landweber-Novikov* classes associated to F (and not on the classes themselves). For the shake of completeness, these classes are introduced briefly in Section 4.2, the proofs being found in subsequent sections.

Image Milnor number for $n = 4$ and 5

T. Ohmoto has shown that, for weighted-homogeneous map-germs, the image Milnor number μ_I can be expressed in terms of the weights and degrees for $n \leq 5$. The restriction $n \leq 5$ comes from what follows: certain results are only known for Morin singularities, that is, for stable corank 1 map-germs. In dimensions $n \leq 5$, all stable germs have corank 1. This suggests that Ohmoto's work (and also the methods of the present paper) should work for corank 1 singularities of higher dimensions.

The μ_I expression for $n = 2$ was obtained by D. Mond in [63] with a different approach. T. Ohmoto in [69] recovers the formula for $n = 2$ and obtains the one for $n = 3$. The following theorem, whose proof will be given in Section 4.3, includes the two remaining cases.

Theorem 4.1.1. *Let $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a weighted-homogeneous $\mathcal{A}$-finite map-germ with weights $w = (w_1, \dots, w_n)$ and degrees $d = (d_0, \dots, d_n)$. If $n = 4$, then*

$$\begin{aligned}\mu_I(F) = \frac{1}{\sigma_4}\Big(&\frac{1}{2!}(-s_0 + c_1)\sigma_3 + \frac{1}{3!}(s_0^2 - c_1^2 - c_2)\sigma_2\\
&+ \frac{1}{4!}(-s_0^3 - 2s_0^2c_1 + s_0c_1^2 + 16s_0c_2 + 2c_1^3 - 10c_1c_2)\sigma_1\\
&+ \frac{1}{5!}(s_0^4 + 5s_0^3c_1 + 5s_0^2c_1^2 - 50s_0^2c_2 - 5s_0c_1^3 - 20s_0c_1c_2\\
&+ 60s_0c_3 - 6c_1^4 + 34c_1^2c_2 - 64c_1c_3 + 108c_2^2 + 4c_4)\Big).\end{aligned}$$

If $n = 5$, then

$$\begin{aligned}\mu_I(F) = -\frac{1}{\sigma_5}\Big(&\frac{1}{2!}(-s_0 + c_1)\sigma_4 + \frac{1}{3!}(s_0^2 - c_1^2 - c_2)\sigma_3 +\\
&+ \frac{1}{4!}(-s_0^3 - 2s_0^2c_1 + s_0c_1^2 + 16s_0c_2 + 2c_1^3 - 10c_1c_2)\sigma_2\\
&+ \frac{1}{5!}(s_0^4 + 5s_0^3c_1 + 5s_0^2c_1^2 - 50s_0^2c_2 - 5s_0c_1^3 - 20s_0c_1c_2\\
&+ 60s_0c_3 - 6c_1^4 + 34c_1^2c_2 - 64c_1c_3 + 108c_2^2 + 4c_4)\sigma_1\\
&+ \frac{1}{6!}(-s_0^5 - 9s_0^4c_1 - 25s_0^3c_1^2 + 110s_0^3c_2 - 15s_0^2c_1^3 + 270s_0^2c_1c_2\\
&- 240s_0^2c_3 + 26s_0c_1^4 + 16s_0c_1^2c_2 + 24s_0c_1c_3 - 1138s_0c_2^2 + 336s_0c_4\\
&+ 24c_1^5 - 156c_1^3c_2 + 276c_1^2c_3 + 108c_1c_2^2 - 396c_1c_4 + 600c_2c_3)\Big).\end{aligned}$$

The coefficients σ_k, c_k and s_0 are determined by w and d as follows: For fixed n, set

$$\sigma_k = \sigma_{k,n} = \sum_{1\leq j_1<\ldots<j_k\leq n} w_{j_1}\cdot\ldots\cdot w_{j_k},$$

for $k = 1,\ldots,n$. To obtain the $c_k = c_{k,n}$, we set

$$\delta_k = \delta_{k,n} = \sum_{0\leq i_1<\ldots<i_k\leq n} d_{i_1}\cdot\ldots\cdot d_{i_k},$$

for $k = 1,\ldots,n+1$. Then,

$$c_{k,n} = \sum_{0\leq i\leq k} (-1)^{k-i}\delta_i \sum_{|\alpha|=k-i} w^\alpha,$$

with the usual multi-index notation for α. Finally, $s_0 = s_{0,n}$ is the rational function

$$s_0 = \frac{\delta_{n+1}}{\sigma_n}.$$

Zero-stable invariants

For any fixed n, certain stable multi-germs types appear, at most, on isolated points in the target of the stable multi-germs $F\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. Such a stable type η is called a 0-*stable type* for the dimensions $(n, n+1)$. Whenever a map-germ F is stabilised, the target of the stable perturbation exhibits a certain number of multi-germs of type η. If F is $\mathcal{A}$-finite, this number is independent of the chosen stabilisation and it is $\mathcal{A}$-invariant. This number is called the 0-*stable invariant* $\#\eta(F)$ (see Section 2.1.4). We write $\#\eta$ for $\#\eta(F)$ if there is no risk of confusion.

As in the case of the image Milnor number, T. Ohmoto shows the existence of expressions for the 0-stable invariants of weighted-homogeneous map-germs in terms of σ_n, s_0 and c_k, for $n \leq 5$. By [69, Theorem 5.3], the 0-invariants admit the expression

$$\#\eta(F) = \frac{[tp(\eta)]_n}{\deg_1(\eta) w_1 \ldots w_n}.$$

The coefficient $\deg_1(\eta)$ is determined by the repetitions of branches defining η (see Section 2.1.4) and $[\omega]_n$ stands for the coefficient of the n-th degree term of ω. The only non-trivial task is to obtain the Thom polynomial $tp(\eta)$ in the c_k and s_0, which can be acomplished based on works R. Rimányi [72, 73] and M. E. Kazarian [44, 45].

For completeness, we start with the formulas for $n \leq 3$, due to Ohmoto.

We use the following notation: the A_i represent the Morin singularities, that is, A_0 is a regular map, A_1 a cross-cap, and so on. The product of A_i

stands for a stable multi-germ whose branches are the factors in the product. For instance, $A_0^2A_1$ consists of two regular branches and a cross-cap.

The only 0-stable invariant for $n = 1$ is the number of double points:

$$\#A_0^2 = \frac{s_0 - c_1}{2!\sigma_1}.$$

For $n = 2$, the number of triple points and cross-caps are:

$$\#A_0^3 = \frac{s_0^2 - 3s_0c_1 + 2c_1^2 + 2c_2}{3!\sigma_2},$$

$$\#A_1 = \frac{c_2}{\sigma_2}.$$

Finally, for $n = 3$ the invariants are the number of quadruple points and number of transverse incidences of a curve of cross-caps with a regular branch:

$$\#A_0^4 = \frac{1}{4!\sigma_3}(s_0^3 - 6s_0^2c_1 + 11s_0c_1^2 + 8s_0c_2 - 6c_1^3 - 18c_1c_2 - 12c_3),$$

$$\#A_0A_1 = \frac{1}{\sigma_3}(s_0c_2 - 2c_1c_2 - 2c_3).$$

The invariants for $n = 4$, are the number of quintuple points, the incidence of two regular branches and surface of cross-caps, and the number of A_2 singularities:

$$\begin{aligned}\#A_0^5 = \frac{1}{5!\sigma_4}(&s_0^4 - 10s_0^3c_1 + 35s_0^2c_1^2 + 20s_0^2c_2 - 50s_0c_1^3 - 110s_0c_1c_2 \\ &- 60s_0c_3 + 24c_1^4 + 144c_1^2c_2 + 216c_1c_3 + 48c_2^2 + 144c_4)\end{aligned}$$

$$\#A_0^2A_1 = \frac{1}{2!\sigma_4}\big(s_0^2c_2 - 5s_0c_1c_2 - 4s_0c_3 + 6c_1^2c_2 + 14c_1c_3 + 4c_2^2 + 12c_4\big)$$

$$\#A_2 = \frac{1}{\sigma_4}\big(c_1c_3 + c_2^2 + 2c_4\big).$$

The Thom polynomials which lead to the above expressions are obtained by dividing Kazarian's polynomials m_η by a certain correction coefficient, see [44, 2, 5] for details.

For $n = 5$, the invariants are the incidence of three regular branches and a 3-space of cross-caps, the incidence a regular branch with a curve of A_2, and the incidence of two three-spaces of cross-caps:

$$\begin{aligned}\#A_0^6 = \frac{1}{6!\sigma_5}\Big(&s_0^5 - 15s_0^4c_1 + 5s_0^3(17c_1^2 + 8c_2) - 15s_0^2(15c_1^3 + 26c_1c_2 + 12c_3) \\ &+ 2s_0(137c_1^4 + 607c_1^2c_2 + 164c_2^2 + 738c_1c_3 + 432c_4) \\ &- 120(c_1^5 + 10c_1^3c_2 + 10c_1c_2^2 + 25c_1^2c_3 + 12c_2c_3 + 38c_1c_4 + 24c_5)\Big),\end{aligned}$$

$$\#A_0^3A_1 = \frac{1}{3!\sigma_5}\Big((s_0^3c_2 - 3s_0^2(3c_1c_2 + 2c_3) + 2s_0(13c_1^2c_2 + 7c_2^2 + 24c_1c_3 + 18c_4) - 24(c_1^3c_2 + 4c_1^2c_3 + 3c_2c_3 + 2c_1(c_2^2 + 4c_4) + 6c_5)\Big),$$

$$\#A_0A_2 = \frac{1}{\sigma_5}\Big(s_0(c_2^2 + c_1c_3 + 2c_4) - 3(c_1^2c_3 + 2c_2c_3 + c_1(c_2^2 + 4c_4) + 4c_5)\Big),$$

$$\#A_1^2 = \frac{1}{\sigma_5}(s_0c_2^2 - 2c_1^2c_3 - 4c_1c_2^2 - 8c_2c_3 - 10c_1c_4 - 12c_5).$$

In this case, the polynomials m_η cannot be found in Kazarian's paper; they are obtained by putting together Theorem 5.3 and the corresponding ingredients from the lists about residual classes R_η and classes n_η from [44].

Relations between 0-stable invariants in corank 1

To better relate the corank with the weights and degrees, we restrict ourselves to map-germs in *prenormal form.* For a given $\mathcal{A}$-class of rank r, we only consider the representatives $F\colon (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0)$ of the form

$$(z, y) \mapsto (f_y(z), y), \quad y \in \mathbb{C}^r.$$

Consider the germ

$$f_{y_1,\dots,y_k}\colon (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^{n+1+k}, 0),$$

obtained by making the parameters $y_{k+1}, \dots, y_r$ equal to zero. To such an F, and a 0-stable type $\eta\colon \mathbb{C}^{n+k} \to \mathbb{C}^{n+1+k}$, we associate the number

$$\#\eta = \#\eta(f_{y_1,\dots,y_k})$$

if $f_{y_1,\dots,y_k}$ is $\mathcal{A}$-finite, and $\#\eta = \infty$ otherwise.

Observe that only the $\#\eta$ of stable types $\eta\colon \mathbb{C}^{n+r} \to \mathbb{C}^{n+1+r}$ are $\mathcal{A}$-invariants of F. The numbers $\#\eta$ for lower dimensional η are just numbers that come in handy.

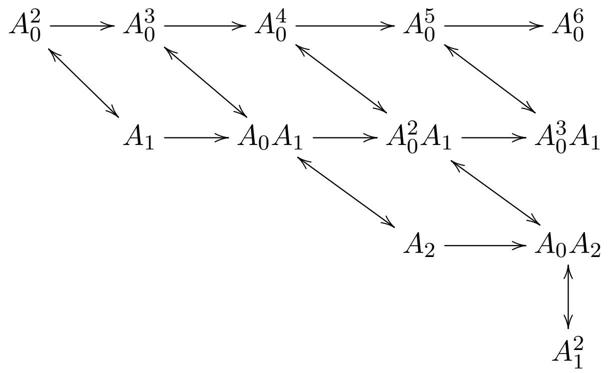
Proposition 4.1.2. *Let $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite weighted-homogeneous map-germ of corank 1 in prenormal form, with $2 \le n \le 5$.*

If $f_{y_1,\dots,y_{n-2}}$ is $\mathcal{A}$-finite, then

$$\begin{aligned}
\#A_0^{n+1}(F) &= \#A_0^n(f_{y_1,\dots,y_{n-2}})\frac{(d_0-nw_1)(d_1-nw_1)}{nw_1w_n}\\
\#A_0^{n-2}A_1(F) &= \#A_0^{n-3}A_1(f_{y_1,\dots,y_{n-2}})\frac{(d_0-(n-1)w_1)(d_1-(n-1)w_1)}{(n-2)w_1w_n}\\
\#A_0^{n-2}A_1(F) &= \#A_0^n(f_{y_1,\dots,y_{n-2}})\frac{n(n-1)w_1}{w_n}\\
\#A_0^{n-4}A_2(F) &= \#A_0^{n-3}A_1(f_{y_1,\dots,y_{n-2}})\frac{(n-3)w_1}{w_n}\\
\#A_0A_2(F) &= 2\#A_1^2(F).
\end{aligned}$$

Proof. The proof is a case by case calculation. We ilustrate the procedure but omit the actual computations as they are simple but too long to be included here. Each formula is obtained as follows: Take the corresponding expressions for $\#\eta(F)$ and $\#\eta'(f_{y_1,\dots,y_{n-2}})$ in terms of $c_{i,n}$, $s_{0,n}$ and $c_{i,n-1}$, $s_{0,n-1}$, respectively. Now compare the result of expanding the previous expressions in terms of weights and degrees taking into account the following two things: since F is a corank 1 mapping, the grading can be taken of the form $(w_1,\dots,w_n)$ and $(d_0,d_1,w_2,\dots,w_n)$. Since f_0 is a slice of F, its grading is $(w_1,\dots,w_{n-1})$ and $(d_0,d_1,w_2,\dots,w_{n-1})$. The formula $\#A_0A_2(F) = 2\#A_1^2(F)$ depends only on the observation about the grading of corank 1 map-germs. □

The following diagram indicates when the vanishing of an invariant implies the vanishing of another.



(4.1)

For instance, $\#A_0^2(f_0) = 0$ implies $\#A_0^3((f_y, y)) = 0$, but not the other way around. As another example, for a map-germ $F\colon (\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ given by $F(z, y_1, \dots, y_4) = (f_{y_1,\dots,y_4}(z), y_1, \dots, y_4)$, the following things are equivalent:

$$\#A_0A_2(F) = 0,$$

$$\#A_0^2A_1(f_{y_1,y_2,y_3}, y_1, y_2, y_3) = 0,$$

$$\#A_0^4(f_{y_1,y_2}, y_1, y_2)) = 0.$$

Remark 4.1.3. These relations do not hold in higher corank, as the singularities $\hat{P}_1$ and $\hat{N}_1$ from Table 4.5 show. One checks that $\#A_1^2(\hat{P}_1) = 2$, $\#A_0A_2(\hat{P}_1) = 6$, $\#A_1^2(\hat{N}_1) = 40$ and $\#A_0A_2(\hat{N}_1) = 84$.

Remark 4.1.4. The relations between the invariants $\#\eta(F)$ and $\#\eta(f_{y_1,\ldots,y_{n-2}})$ are found by brute force and only for certain cases. However, the expressions for $c_{k,n}$ and $s_{0,n}$ can be studied in greater generality.

If F is a weighted-homogeneous 1-parameter unfolding, and hence $d_n = w_n$, from the geometric construction giving rise to the functions σ, s_0 and c_k follows

$$c_{k,n}(w_1,\ldots,w_n,d_0,\ldots,d_{n-1},w_n) = c_{k,n-1}(w_1,\ldots,w_{n-1},d_0,\ldots,d_{n-1}),$$

for all k, and that

$$s_{0,n}(w_1,\ldots,w_n,d_0,\ldots,d_{n-1},w_n) = s_{0,n-1}(w_1,\ldots,w_{n-1},d_0,\ldots,d_{n-1}).$$

In the case general case of mappings with $w_n \neq d_n$, the previous equalities suggest the existence of a function $q(w,d)$ in variables w and d satisfying

$$c_{k,n}(w,d) = c_{k,n-1}(w_1,\ldots,w_{n-1},d_0,\ldots,d_{n-1}) + (d_n - w_n)q(w,d).$$

Indeed, a little combinatorics shows

$$c_{k,n} = c_{k,n-1} + (d_n - w_n)\sum_{i=0}^{k-1}(-1)^i w_n^i c_{k-i-1,n-1}$$

and

$$s_{0,n} = s_{0,n-1}\frac{d_n}{w_n}.$$

4.2. Image Milnor number formulas

Here, we give some theoretical background appearing in T. Ohmoto's work [69] and explain our methods used to prove formulas in Theorem 4.1.1. The method is illustrated in detail for $n = 2$ and briefly for $n = 3$.

Characteristic classes and image Milnor number

Our interpolation method is based entirely on Proposition 4.2.1, which is an immediate consequence of Ohmoto's Theorem 4.2.2. Notice that Proposition 4.2.1 expresses μ_I in terms of weights and degrees, exclusively. This allows for the interpolation method to be applied blindly, independently of its origins in the theory of characteristic classes.

For $\alpha = (\alpha_0,\ldots,\alpha_n)$ we let $\|\alpha\| = \alpha_0 + \sum_{k=1}^n k\alpha_k$ and $c^\alpha = s_0^{\alpha_0}c_1^{\alpha_1}\ldots c_n^{\alpha_n}$.

Proposition 4.2.1. *There are unique $b_\alpha \in \mathbb{Q}$, with $0 \neq \alpha \in \mathbb{N}^6$, such that any $\mathcal{A}$-finite weighted-homogeneous map-germ $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, with $n \leq 5$, satisfies*

$$\mu_I(F) = (-1)^n \frac{\sum_{\|\alpha\| \leq n} b_\alpha c^\alpha \sigma_{n-\|\alpha\|}}{\sigma_n}. \tag{4.2}$$

The remaining of the section is devoted to explaining how Proposition 4.2.1 derives immediately from the following result proved by T. Ohmoto in [69, Theorem 6.20]:

Theorem 4.2.2 (Ohmoto's Theorem)**.** *Let $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite and $n \leq 5$. The Euler characteristic of the image of a stable perturbation F_y of F is*

$$\chi(\mathrm{Im}(F_y)) = \frac{[c(E_0) \cdot tp^{SM}(\alpha_{image})(c(F))]_n}{[c_n(E_0)]_n}. \tag{4.3}$$

Observe that the left hand side of Equation (4.2) is

$$\mu_I(F) = (-1)^n (\chi(\mathrm{Im}(F_y)) - 1).$$

Now we proceed to describe the ingredients in the right hand side of Formula (4.3).

Let ℓ be the dual tautological line bundle over $\mathbb{P}^\infty$. Associated to the grading (w, d), there are the two bundles

$$E_0 := \ell^{\otimes w_1} \oplus \cdots \oplus \ell^{\otimes w_n} \quad \text{and} \quad E_1 := \ell^{\otimes d_0} \oplus \cdots \oplus \ell^{\otimes d_n}.$$

The cohomology of $\mathbb{P}^\infty$ is isomorphic to the polynomial ring $\mathbb{Z}[a]$ and, under this isomorphism, the total Chern class of ℓ is $c(\ell) = 1 + a$. From this we obtain the total Chern classes

$$c(E_0) = \prod_{j=1}^{n} (1 + w_j a) \quad \text{and} \quad c(E_1) = \prod_{i=0}^{n} (1 + d_i a).$$

One can construct a *universal map* $\tilde{F}\colon E_0 \to E_1$ whose restriction to each fiber is $\mathcal{A}$-equivalent to F [69]. By abuse of notation, one writes $c(F)$ for the total chern class $c(\tilde{F}) = c(\tilde{F}^* TE_1 - TE_0)$ of the virtual normal bundle. One checks that

$$c(F) = \frac{\prod_{i=0}^{n} (1 + d_i a)}{\prod_{j=1}^{n} (1 + w_j a)}.$$

The functions $\sigma_j(w, d)$, $\delta_i(w, d)$ and $c_k(w, d)$ from Section 4.1 are precisely the coefficients in the graded decompositions

$$\begin{aligned} c(E_0) &= 1 + \sigma_1 a + \cdots + \sigma_n a^n, \\ c(E_1) &= 1 + \delta_1 a + \cdots + \delta_{n+1} a^{n+1}, \\ c(F) &= 1 + c_1 a + c_2 a^2 + c_3 a^3 + \ldots. \end{aligned}$$

The term $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})$ is the Segre-MacPherson Thom polynomial of the constructible function α_{image}. This is an extension of the classical Thom polynomial in the following sense:

The classical Thom polynomial [80] of a stable mono-singularity type $\eta\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$ is the unique polynomial $tp(\eta) \in \mathbb{Z}[c_1, c_2, \dots]$ such that, for any stable map $f\colon M^m \to N^{m+k}$, the following holds

$$\mathrm{Dual}[\overline{\eta(f)}] = tp(\eta)(c(f)).$$

The left hand side of the equality is the Poincaré dual of the fundamental class of the closure of the singularities of type η exhibited by f. The right hand side is the evaluation of $tp(\eta)$ in the total Chern class $c(f) := c(f^*TN - TM)$ of the virtual normal bundle of f.

For multi-singularity types $\underline{\eta}$, the definition was extended by M. E. Kazarian [44] to Thom polynomials $tp(\underline{\eta})$ depending on further variables s_I. They satisfy the analogous property $\mathrm{Dual}[\overline{\underline{\eta}(f)}] = tp(\underline{\eta})(s_I(f), c(f))$. Here, the class $s_I(f)$ is the *Landweber-Novikov class* of f, it is defined by

$$s_I(f) = f^* f_*(c_1(f)^{i_1} c_2(f)^{i_2} \dots),$$

for the multi-index $I = (i_1 i_2 \dots)$. The *0-th Landweber-Novikov class* of f is

$$s_0(f) = f^* f_*(1) = \frac{c_{top}(f^*TN)}{c_{top}(TM)}.$$

For simplicity, we consider the evaluation of $tp(\overline{\eta})$ in the Chern classes $c_i(f)$ and the class $s_0(f)$. In particular, the universal map above gives

$$s_0(F) = \frac{c_{n+1}(E_1)}{c_n(E_0)} = s_0 a.$$

The Segre-MacPherson Thom polynomial $tp^{\mathrm{SM}}(\underline{\eta})$ is the unique series in s_0, c_i satisfying the following similar property [69]:

$$\mathrm{Dual}\ (i_* s^{\mathrm{SM}}(\overline{\underline{\eta}(f)}, M)) = tp^{\mathrm{SM}}(\underline{\eta})(s_0(f), c(f)),$$

where $s^{\mathrm{SM}}(\overline{\underline{\eta}(f)}, M)$ is the Segre-Schwartz-MacPherson class of the embedding $i\colon \overline{\underline{\eta}(f)} \hookrightarrow M$ (see Section 2.5.2).

The Segre-MacPherson Thom polynomial has the form

$$tp^{\mathrm{SM}}(\eta) = tp(\eta) + \text{ higher degree terms.}$$

The definition of Segre-MacPherson Thom polynomials extends to certain constructible functions α, so that $tp^{\mathrm{SM}}(\mathbb{1}_{\underline{\eta}}) = tp^{\mathrm{SM}}(\underline{\eta})$. For $f\colon M \to N$, there is such a constructible function α_{image}, determined by $\mathbb{1}_{f(M)} = f_*(\alpha_{image})$.

Finally, $[c(E_0) \cdot tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})(s_0(F), c(F))]_n$ stands for the n-th degree part of the series $c(E_0) \cdot tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})(s_0(F), c(F))$ in the variable a (note that, by abuse of notation, the term $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})(s_0(F), c(F))$ appears in [69] as $tp^{\mathrm{SM}}(\alpha_{\mathrm{image}})$).

How to obtain the μ_I formulas

We write the multi-indices $\alpha \in \mathbb{N}^6$ of Proposition 4.2.1 only up to their last non zero entry. For example, we write $(0,1)$ for $\alpha = (0,1,0,0,0,0)$.

Our strategy is based on the following simple interpolation idea: pick any $\mathcal{A}$-finite map-germ F, with known image Milnor number. Every possible weigths w and degrees d of F determine values σ_k, s_0 and c_k, and the formula yields a linear equation in the variables b_α. The data

$$\tau(F) := (w, d, \mu_I(F))$$

will be called a *sample* of F. The b_α are determined after sampling singularities as many times as the number of b_α, provided that each sample gives an equation which is independent from the preceding ones.

The initial cases are somehow trivial, because the literature contains enough singularities to complete the interpolation. The difficulties arise in $n \geq 4$, due to the lack of $\mathcal{A}$-finite map-germs with known μ_I. The key points of our interpolation strategy are contained in Remarks 4.2.3 and 4.2.4, Proposition 4.2.5 and 4.4.4.

Ohmoto's formula for $n = 2$ (see [69, Example 6.21]) can be rewriten as

$$\mu_I(F) = \frac{1}{\sigma_2}\big(\frac{1}{2!}(-s_0 + c_1)\sigma_1 + \frac{1}{3!}(s_0^2 - c_1^2 - c_2)\big).$$

To recover the formula, we need to determine the six b_α with $\|\alpha\| \leq 2$, hence we must find six samples giving rise to independent equations. Again, all maps will be chosen in prenormal form. Since the computations involved often become hard, we want to sample the simplest singularities first.

We start in corank 0, that is, with $d_1 = w_1$ and $d_2 = w_2$. Since corank 0 map-germs are regular, we know $\mu_I(F) = 0$. Replacing d_1 and d_2 by w_1 and w_2, the formula (4.2) reads

$$0 = \frac{d_0((b_{02} + b_{11} + b_2)d_0 + (b_{01} + b_1)(w_1 + w_2))}{w_1 w_2}. \tag{4.4}$$

The regular map

$$R\colon (z, y) \mapsto (0, z, y)$$

admits samples $\tau_1(R) = ((1,1),(1,1,1),0)$ and $\tau_2(R) = ((1,1),(2,1,1),0)$. By substitution, it follows

$$b_{02} + b_{11} + b_2 = b_{01} + b_1 = 0.$$

Remark 4.2.3. *Different samples of a same singularity may give independent equations.* Observe the following:

1. this is exactly what happened with $\tau_1(R)$ and $\tau_2(R)$, and it will continue to happen for all higher degrees.

2. However, let (w, d) and (w', d') be two gradings of a map-germ F. If $(w', d') = \lambda(w, d)$, for $\lambda \in \mathbb{Q}$, then the samples $\tau_1(F) = (w, d, \mu_I(F))$ and $\tau_2(F) = (w', d', \mu_I(F))$ give rise to the same equation on b_α. This is because the coefficients of each b_α in Formula (4.2) are homogeneous rational functions of degree zero in the weights and degrees.

3. Two representatives F and F' of the same $\mathcal{A}$-class may produce different sets of samples. For example, the representative (z, z, y) in the $\mathcal{A}$-class of R admits $\tau_1(R)$ as a sample but not $\tau_2(R)$. The map (z^2, z, y) admits the opposite combination. An strategy to find better representatives is to eliminate monomials in the coordinate functions. For instance, the cross-cap $(z^2, z^3 + yz, y)$ is 2-determined and by eliminating the z^3 term, the resulting representative admits more samples.

We claim that no further independent equations can be found by sampling the regular map R. This is because for any b_α, satisfying the above equations, the right hand side of equation (4.4) vanishes. Having exhausted R, we must move to the case where $d_1 \neq w_1$, and the simpler such singularities are the map-germs of corank 1.

Every singular map-germ has $\#A_1 > 0$, but we still want to start with the simpler ones, having $\#A_0^3 = 0$ and $\#A_1$ as low as possible. Consider the cross-cap, parameterised as

$$A_1\colon (z, y) \mapsto (z^2, yz, y),$$

and the samples

$$\tau_i(A_1) = ((1, i), (2, i + 1, i), 0).$$

The samples $\tau_1(A_1)$ and $\tau_2(A_1)$ give equations

$$0 = 8b_1 + 6b_{01} + 16b_2 + 12b_{11} + 9b_{02} + b_{001}$$

and

$$0 = 9b_1 + 6b_{01} + 18b_2 + 12b_{11} + 8b_{02} + b_{001},$$

respectively. We show now that no more independent equations can be obtained from map-germs having $\#A_1 = 1$ and $\#A_0^3 = 0$. The idea is to look at the expressions of these invariants for corank 1 germs:

$$\#A_1 = \frac{(d_0 - w_1)(d_1 - w_1)}{w_1 w_2}, \quad \#A_0^3 = \frac{(d_0 - 2w_1)(d_1 - 2w1)}{6w_1^2} \#A_1.$$

If $\#A_1$ does not vanish, the condition $\#A_0^3 = 0$ implies $d_0 = 2w_1$ or $d_1 = 2w_1$ and, by a permutation of the coordinate functions of the map-germ, we may choose $d_1 = 2w_1$. Replacing d_1 by $2w_1$ in the expression $\#A_1 = 1$, we obtain $w_2 = d_0 - w_1$. Eliminating four of the b_α by means of the previous equations and impossing the conditions $d_1 = 2w_1$ and $d_2 = w_2 = d_0 - w_1$, we obtain a closed expression for μ_I, independent of the remaining b_α. This means that the last two b_α cannot be found by taking samples satisfying such conditions. This illustrates another key point of the interpolation strategy:

Remark 4.2.4. *The numbers $\#\eta$ may be used to decide whether a singularity has been exhausted.*

Since the cross-cap is the only singular stable mono-germ for dimensions $(2,3)$, from now on we need to take non-stable map-germs into account. A known singularity with $\#A_1 = 2$ and $\#A_0^3 = 0$ is

$$S_1\colon (z,y) \mapsto (z^2, z^3 + y^2 z, y).$$

It is well known that S_1 has $\mathcal{A}_e$-codimension one and, since Mond's conjecture holds for $n \leq 2$ (see [65, 19]), this number is precisely $\mu_I(S_1)$.

After one sampling of S_1, one checks easily that we need a sample with $\#A_0^3 \neq 0$. The interpolation is finished after sampling Mond's map-germ

$$H_2\colon (z,y) \mapsto (z^3, z^5 + yz, y),$$

which has $\mu_I(H_2) = 2$.

Table 4.1 contains numbers associated to the interpolation samples. Horizontal lines separate changes in corank. The number d_0 is only included for $\tau_1(R)$ and $\tau_2(R)$, because it does not carry any clear geometric information about the rest of singularities. The ∞ symbol means that $\#A_0^2$ is not well defined for the corresponding slice. For higher n, there will be too many associated numbers, and we will include only the essential ones, based on Diagram (4.1).

Sample	d_0	$\#A_0^2$	$\#A_1$	$\#A_0^3$
$\tau_i(R)$, $i = 1, 2$	i	0	0	0
$\tau_i(A_1), i = 1, 2$		∞	1	0
$\tau(S_1)$		1	2	0
$\tau(H_2)$		4	2	1

Table 4.1: Numbers associated to the samples for $n = 2$.

The μ_I formula for $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ (see [69, Example 6.22]) reads

$$\mu_I(F) = -\frac{1}{\sigma_3}\Big(\frac{1}{2!}(-s_0 + c_1)\sigma_2 + \frac{1}{3!}(s_0^2 - c_1^2 - c_2)\sigma_1 + \\ + \frac{1}{4!}(-s_0^3 - 2s_0^2 c_1 + s_0 c_1^2 + 16 s_0 c_2 + 2c_1^3 - 10 c_1 c_2)\Big).$$

Observe that, from the b_α with $\|\alpha\| \leq 3$ to be found, we already know the ones with $\|\alpha\| \leq 2$. This leaves us with the seven unknown b_α.

The first equations are obtained automatically from the following result, based on sampling trivial unfoldings of stable singularities of smaller dimensions.

Proposition 4.2.5. *Let* $F\colon (\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ *be a weighted-homogeneous stable map-germ and let* (w,d) *be a grading of* F*. With the notations above, the coefficients* b_α *satisfy*

$$0=\sum_{\|\alpha\|\leq n+r} b_\alpha c^\alpha\Big(\sum_{k=0}^{n+r-\|\alpha\|}\binom{k}{r}\sigma_{n+r-\|\alpha\|-k}\Big),$$

for all $r\geq 0$.

Proof. Observe that any trivial r-parameter unfolding of F is also stable and admits the grading $((w,1,\dots,1),(d,1,\dots,1))$. The result follows putting together Proposition 4.2.1, Remark 4.1.4 and the equality

$$\sigma_{\ell,n+r}(w,1,\dots,1)=\sum_{k=0}^{\ell}\binom{k}{r}\sigma_{\ell-k}(w).$$ □

Applying this property to the (w,d) from $\tau_1(R)$, $\tau_1(A_1)$ and $\tau_2(A_1)$ with $r=1$, we obtain three independent equations.

Notice that, for $r=1$, the equation from $\tau_2(R)$ is not independent of the one from $\tau_1(R)$. The sample $\tau_3(A_1)$ which did not produce an independent equation for $n=2$, does give a new equation for $n=3$, that is, for $r=1$. We have used the singularities from Table 4.2 to finish the interpolation. That is, Houston's and Kirk's singularities P_1 and P_2 [42], and Sharland's singularity $\hat{B}_3$ [2].

Label	Map-germ	μ_I
P_1	$(z(z^3+y),z(z^2+x),y,x)$	1
P_2	$(z(z^4+y),z(z^2+x),y,x)$	2
$\hat{B}_3$	$(y^2+xz,z^2-xy,y(y^2+z^2)+z(y^2-z^2),x)$	33

Table 4.2: Singularities sampled beside R and A_1 for $n=3$.

From the diagram (4.1), it follows that the crucial invariants are the ones in Table 4.3.

Notice that the singularities in Table 4.2 have corank 1, with the exception of Sharland's singularity $\hat{B}_3$ of corank 2. However, the interpolation method can be completed without resort to corank 2 map-germs. We may use the singularity

$$F\colon (z,y,x)\mapsto(z^4-xz,(y+z)^5+xz^2,y,x).$$

Sample	$\#A_1$	$\#A_0A_1$	$\#A_0^4$
$\tau_1(R)$	0	0	0
$\tau_i(A_1)$, $i = 1,2,3$	1	0	0
$\tau(P_1)$	3	2	0
$\tau(P_2)$	2	3	0
$\tau(\hat{B}_3)$	5	16	1

Table 4.3: Numbers associated to the samples for $n = 3$.

The previous map-germ of corank 2 was included in order to avoid justifying that F is $\mathcal{A}$-finite with $\mu_I(F) = 52$. Criteria for $\mathcal{A}$-finiteness and computation of μ_I will be discussed in Section 4.4.

4.3. μ_I formulas for $n = 4$, 5

Here we give the steps to prove Theorem 4.1.1. The same interpolation idea used for $n = 2, 3$ applies just as fine for $n = 4, 5$ but, as far we know, the examples found in the literature do not suffice to complete the associated system of equations.

As it turns out, it is not always easy to produce $\mathcal{A}$-finite singularities giving new independent equations. One has to bear in mind that checking $\mathcal{A}$-finiteness and computing μ_I are often computationally unfeasible tasks. For $\mathcal{A}$-finiteness, we use a geometric criteria based on multiple points. For μ_I, we first compute the $\mathcal{A}_e$-codimension (for which commutative algebra algorithms exist), then we justify that the germ satisfies Mond's conjecture, ensuring the equality of μ_I and the computed $\mathcal{A}_e$-codimension.

The μ_I formula for $n = 4$

From Proposition 4.2.5 applied to R and A_1, for $r = 2$, we find five independent equations from $\tau_1(R)$ and $\tau_i(A_1)$, for $i = 1, \ldots, 4$. One can check that no more samples from singularities $\#A_1 = 1$ can be used.

Our next move is to consider the stable singularity

$$A_2\colon (z, y, x, t) \mapsto (z^3 + tz, yz^2 + xz, y, x, t)$$

with samples

$$\tau_i(A_2) = ((1, i+1, 2, i), (3, i+2, i+1, 2, i), 0).$$

Three new equations arise from $\tau_1(A_2)$, $\tau_2(A_2)$ and $\tau_3(A_2)$. One checks that map-germs with $\#A_2 \leq 1$ and $\#A_0^2A_1 = 0$ do not provide new information.

Also, nothing new comes from map-germs that the authors could find in the literature. We consider the new map-germs

$$L_1\colon (z,y,x,t)\mapsto (z^4-tz,(y+z)^6+xz,y,x,t),$$

$$L_2\colon (z,y,x,t)\mapsto (z^4+xz^2+tz,(y+z)^5+(x^2+ty)z,y,x,t),$$

which have $\#A_0^5=0$ and $\mu_I(L_1)=39$ and $\mu_I(L_2)=87$. At this point, it is not possible to obtain further equations if $\#A_0^5=0$. We take another map-germ

$$L_3\colon (z,y,x,t)\mapsto (z^5-tz,(y+z)^7+xz,y,x,t),$$

whith $\mu_I(L_3)=178$. To avoid disrupting the flow of the explanation, the μ_I values and $\mathcal{A}$-finiteness of L_1, L_2 and L_3 will be justified in Section 4.4. The singularities L_i were not our first candidates for the interpolation. In Remark 4.4.4 we explain what brought us to them.

At this stage, one can check that it is necessary to introduce map-germs of corank 2. For instance, Sharland's

$$\hat{D}_1\colon (z,y,x,t)\mapsto (y^2+xz+(x^2+t)y,yz,z^2+y^3+t^2y,x,t),$$

which is known to have $\mu_I(\hat{D}_1)=27$ [2]. This finishes the proof of Theorem 4.1.1 for $n=4$, except from the claimed $\mathcal{A}$-finiteness and image Milnor numbers of L_1, L_2 and L_3.

Remark 4.3.1. If one does not care about introducing more map-germs of corank 2, L_1 and L_2 can be interchanged by Sharland's $\hat{E}_1$ and $\hat{K}_1$ [2]. It is however unavoidable to study the $\mathcal{A}$-finiteness and the μ_I of at least one new map-germ. This is because the system of equations cannot be closed without resorting to a map-germ with $\#A_0^5\neq 0$, such as L_3.

Here, we include the table with the numbers associated to the interpolation samples for completeness.

The μ_I formula for $n=5$

There are 19 unknown b_α to be determined. We will need six new map-germs and to stablish their $\mathcal{A}$-finiteness and μ_I values.

Again, Proposition 4.2.5 is applied a number of times, in this case to $\tau_1(R)$, $\tau_i(A_1)$, for $i=1,\dots,5$, and $\tau_i(A_2)$, for $i=1,\dots,4$. Next samples need to satisfy $\#A_2>1$ and hence they cannot be stable. By a similar argument, at least three map-germs of corank 2 will be necessary to close the formula. These will be Sharland's map-germs [2], $\hat{M}_{1,1}$, $\hat{P}_1$ and $\hat{N}_1$[1] with image Milnor numbers 13, 24, and 1400, respectively, and coordinate functions as in Table 4.5.

[1]There seems to be a typo in Sharland's parameterisation of $\hat{N}_1$. Our term x^4y replaces her x^2y, inconsistent with the claim that $\hat{N}_1$ unfolds $\hat{E}_1$.

Sample	$\#A_1$	$\#A_2$	$\#A_0^2A_1$	$\#A_0^5$
$\tau_1(R)$	0	0	0	0
$\tau_i(A_1)$, $i = 1, \dots, 4$	1	0	0	0
$\tau_i(A_2)$, $i = 1, 2, 3$	∞	1	0	0
L_1	15	8	12	0
L_2	12	12	12	0
L_3	24	15	60	3
$\hat{D}_1$	∞	9	0	0

Table 4.4: Numbers associated to the samples for $n = 4$.

Label	Map-germ
$\hat{M}_{1,1}$	$(y^2 + xz + (x^2 + s)y, yz + ty, z^2 + y^3 + s^2y, x, t, s)$
$\hat{P}_1$	$(y^2 + (x + s)z, z^2 + xy, y^3 + s^2y + z^3 + yz^2 + tz, x, t, s)$
$\hat{N}_1$	$(y^3 + (x^4 + t)y + xz, (y + s)z, z^2 + y^5 + s^3y^2 + (t^2 + s^4)y, x, t, s)$

Table 4.5: Sharland's singularities of corank 2.

Once $\hat{M}_{1,1}$, $\hat{P}_1$ and $\hat{N}_1$ are included, no other known singularity will contribute an independent equation. We produce the new non-stable map-germs of corank 1 found in Table 4.6, whose $\mathcal{A}$-finiteness and μ_I are determined case by case. Again, the singularities $\tilde{L}_i$ and Q_i were not our first examples for the interpolation. We shall explain the details in the following section.

Label	Map-germ	$\mu_I(F)$
$\tilde{L}_2$	$(z^4 + tz + xz^2 + s^3z, (y + z)^5 + x^2z + tyz + s^2z^3, y, x, t, s)$	321
$\tilde{L}_1$	$(z^4 - tz + s^2z^2, (y + z)^6 + xz + s^3z^3, y, x, t, s)$	149
Q_1	$(z^4 + tz^2 + tyz + s^3z, (y + z)^7 - xz + s^4z^3, y, x, t, s)$	711
Q_2	$(z^5 - xz + tz^2 + s^2z, (y + z)^5 + sz^3 + xz, y, x, t, s)$	144
Q_3	$(z^5 + (x^2 + t)z - sz^2 + xz^3, (y + z)^6 + sxz - tz^2, y, x, t, s)$	654
Q_4	$(z^8 - xz + syz^3, (y + z)^6 + tz - sz^2, y, x, t, s)$	862

Table 4.6: Some new $\mathcal{A}$-finite singularities.

This finishes the interpolation for $n = 5$. Table 4.7 contains the numbers associated to the samples.

Sample	$\#A_1$	$\#A_2$	$\#A_0A_2$	$\#A_0^3A_1$	$\#A_0^6$
$\tau_1(R)$	0	0	0	0	0
$\tau_i(A_1)$, $i=1,\dots,5$	1	0	0	0	0
$\tau_i(A_2)$, $i=1,\dots,4$	∞	1	0	0	0
$\tau(\tilde{L}_2)$	12	12	0	0	0
$\tau(\tilde{L}_1)$	15	8	24	0	0
$\tau(Q_1)$	18	15	60	0	0
$\tau(Q_2)$	16	12	24	4	0
$\tau(Q_3)$	20	30	60	20	0
$\tau(Q_4)$	35	24	90	120	3
$\tau(\hat{M}_{1,1})$	3	6	0	0	0
$\tau(\hat{P}_1)$	5	6	6	0	0
$\tau(\hat{N}_1)$	5	33	84	0	0

Table 4.7: Numbers associated to the samples for $n = 5$.

4.4. $\mathcal{A}$-finiteness, stabilisations and image Milnor number

The remaining map-germs whose $\mathcal{A}$-finiteness we must justify have corank 1 and can be studied in terms of their multiple point spaces, thanks to work by T. Marar and D. Mond [54] (see Section 2.1.6).

The criterion of Theorem 2.1.32 has been used for L_1, L_2, L_3, $\tilde{L}_1$, $\tilde{L}_2$ and $Q_1, \dots, Q_4$, by means of a Singular [83] implementation of the divided differences.

Our methods to compute the image Milnor number require finding stable unfoldings or stabilisations of F. Stable unfoldings are easier to obtain, by means of a well known procedure due to J. N. Mather in [60]. We have used stable unfoldings of L_1, L_2, L_3, $\tilde{L}_1$, $\tilde{L}_2$ and Q_1.

Remark 4.4.1. In certain cases, stable unfoldings are too complicated for the computations we need to perform. For these maps it is worth spending some time in finding a stabilisation (see Definition 2.1.24).

We do not know a method to produce stabilizations other than just trial and error, but a candidate can be checked to be a stabilization in the following way:

Let J_y be the *relative jacobian ideal* of $I^k(\mathcal{F})$, i.e. the ideal generated by the divided differences of an unfolding $\mathcal{F}(z, y) = (F_y(z), y)$. To be precise, J_y is generated by the maximal minors of the matrix of partial derivatives, only with respect to z, of the generators of $I^k(\mathcal{F})$. Inspection of the divided differences gives the equality

$$D^k(\mathcal{F}) \cap \{y = y_0\} = D^k(F_{y_0}).$$

By Theorem 2.1.32, the germ F_{y_0} is stable for all $y_0 \neq 0$ if, and only if,

$$D^k(\mathcal{F}) \cap V(J_y) \subseteq \{y = 0\}.$$

This can be checked with the help of SINGULAR, as follows.

Proposition 4.4.2. *With the previous notations, $\mathcal{F}$ is a stabilization of F_0 if and only if $y \in \sqrt{J_y + I^k(\mathcal{F})}$.*

This method has been used to find stabilizations $(\mathbb{C}^6, 0) \to (\mathbb{C}^7, 0)$ of Q_2, Q_3 and Q_4 mapping (z, y, x, t, s, u), respectively, to

$$\big(z^5 + u^2z^3 + tz^2 + (s^2 - x)z, (y + z)^5 + sz^3 + (u^4 + x)z, y, x, t, s, u\big),$$

$$\big(z^5 + (u^2 + x)z^3 - sz^2 + (x^2 + t)z, (y + z)^6 - tz^2 + (u^5 + sx)z, y, x, t, s, u\big),$$

$$\big(z^8 + syz^3 + u^6z^2 - xz, (y + z)^6 + u^2z^4 + sz^2 + tz, y, x, t, s, u\big).$$

This covers the required techniques to check $\mathcal{A}$-finiteness and find stabilization and stable unfoldings. Because of its topological nature, computing μ_I directly is a hard task; we do it via Mond's conjecture (see Conjecture 2.1.30).

Our strategy to compute μ_I for a weighted-homogeneous germ is based on results from [25] and consists on computing $\mathcal{A}_e$-codim(F) first and then justifying that Mond's conjecture holds for F. The $\mathcal{A}_e$-codimension of weighted-homogeneous $\mathcal{A}$-finite map-germs can be computed with SINGULAR, as follows: let $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite map-germ. Let $g \in \mathcal{O}_{n+1}$ be a function-germ such that $g = 0$ is a reduced equation for the image of F, and let $J(g)$ be the jacobian ideal of g. Then

$$\mathcal{A}_e\text{-codim}(F) = \dim_{\mathbb{C}} \frac{(f^*)^{-1}(J(g) \cdot \mathcal{O}_n)}{J(g)}.$$

To check that Mond's conjecture holds for F, let $\mathcal{F}\colon (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0)$ be either a stable unfolding or a stabilisation of F. Let G be an equation of the image of F and $\mathcal{G}$ be an equation of the image of $\mathcal{F}$ which specialises to G. Let

$$M_z(\mathcal{G}) = \frac{J(\mathcal{G})}{J_z(\mathcal{G})},$$

where $J(\mathcal{G})$ is the jacobian ideal and $J_z(\mathcal{G})$ the relative jacobian ideal of $\mathcal{G}$, for $z \in \mathbb{C}^{n+1}$.

The following result was proved by J. Fernández de Bobdadilla, J. J. Nuño and G. Peñafort in [25, Theorem 6.1]:

Theorem 4.4.3. *Let F and $M_z(\mathcal{G})$ be as above. If $M_z(\mathcal{G})$ is a Cohen-Macaulay module, then F satisfies Mond's conjecture.*

We have used this criterion on stable unfoldings of our new samples, with the exception of Q_2, Q_3 and Q_4 where computations became unfeasible. Mond's conjecture for these three examples was checked by means of the stabilisations above, instead of stable unfoldings.

Remark 4.4.4. As pointed out before, the new singularities used for interpolation in the cases $n = 4$ and 5 were not our first candidates. Observe that not all choices of (w, d) have $\mathcal{A}$-finite map-germs associated to them (for instance, all (w, d) for which the μ_I formula predicts a non-integer value). The first $\mathcal{A}$-finite germs we found had extremely high $\mathcal{A}_e$-codimension, making impracticable to check Mond's conjecture for them. Our strategy was as follows:

1. Assume that Mond's conjecture holds for these maps.
2. Use their conjectured values of μ_I to obtain a candidate μ_I formula.
3. Use a computer to find weights and degrees (w, d) with small (conjectured) μ_I values and such that they determine enough linearly independent equations.
4. Try to find $\mathcal{A}$-finite candidates for these (w, d) and check that they satisfy Mond's conjecture.
5. Reproof the μ_I formula by sampling these new examples.

The μ_I formula for $n = 4$ expressed in weights and degrees.

As a curiosity, we include the expanded formula for the image Milnor number $\mu_I(F)$ of an $\mathcal{A}$-finite weighted-homogeneous map-germ $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, for $n = 4$, given in Theorem 4.1.1 in terms of its weights $w = (w_1, \dots, w_4)$ and degrees $d = (d_0, \dots, d_4)$. The formula is as follows:

$$
\begin{aligned}
\mu_I(F) = {} & \frac{1}{120w_1^5w_2^5w_3^5w_4^5}(d_0^4d_1^4d_2^4d_3^4d_4^4 + 5d_0^3d_1^3d_2^3d_3^3d_4^3w_1w_2w_3(d_0 + d_1 + d_2 + d_3 \\
& + d_4 - w_1 - w_2 - w_3 - w_4)w_4 + 5d_0^2d_1^2d_2^2d_3^2d_4^2w_1^2w_2^2w_3^2(d_0 + d_1 + d_2 + d_3 \\
& + d_4 - w_1 - w_2 - w_3 - w_4)^2w_4^2 - 5d_0d_1d_2d_3d_4w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 \\
& + d_4 - w_1 - w_2 - w_3 - w_4)^3w_4^3 - 6w_1^4w_2^4w_3^4(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
& - w_2 - w_3 - w_4)^4w_4^4 - 120w_1^5w_2^5w_3^5w_4^5 - 50d_0^2d_1^2d_2^2d_3^2d_4^2w_1^2w_2^2w_3^2w_4^2(d_0d_1 \\
& + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 \\
& + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 \\
& + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 \\
& - w_3)w_4 + w_4^2) - 20d_0d_1d_2d_3d_4w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 \\
& - w_3 - w_4)w_4^3(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 \\
& - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 \\
& - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 \\
& - w_1 - w_2 - w_3)w_4 + w_4^2) + 34w_1^4w_2^4w_3^4(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 \\
& - w_3 - w_4)^2w_4^4(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 \\
& - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 \\
& - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 \\
& - w_1 - w_2 - w_3)w_4 + w_4^2) + 108w_1^4w_2^4w_3^4w_4^4(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 \\
& + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 \\
& - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
& - w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2)^2 \\
& + 60d_0d_1d_2d_3d_4w_1^3w_2^3w_3^3w_4^3(d_0d_1d_2 + (d_1d_2 + d_0(d_1 + d_2))d_3 + (d_2d_3 \\
& + d_1(d_2 + d_3) + d_0(d_1 + d_2 + d_3))d_4 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 \\
& + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4)w_1 + (d_0 + d_1 + d_2 + d_3 + d_4)w_1^2 \\
& - w_1^3 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 \\
& - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2)w_2 + (d_0 + d_1 + d_2 + d_3 + d_4 \\
& - w_1)w_2^2 - w_2^3 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 \\
& + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3
\end{aligned}
$$

$$
\begin{aligned}
&+ d_4 - w_1)w_2 + w_2^2)w_3 + (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3^2 - w_3^3 \\
&- (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 \\
&+ w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2)w_4 + (d_0 + d_1 \\
&+ d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4^2 - w_4^3) - 64w_1^4w_2^4w_3^4(d_0 + d_1 + d_2 \\
&+ d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4^4(d_0d_1d_2 + (d_1d_2 + d_0(d_1 + d_2))d_3 \\
&+ (d_2d_3 + d_1(d_2 + d_3) + d_0(d_1 + d_2 + d_3))d_4 - (d_0d_1 + (d_0 + d_1)d_2 \\
&+ (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4)w_1 + (d_0 + d_1 + d_2 + d_3 \\
&+ d_4)w_1^2 - w_1^3 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 \\
&+ d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2)w_2 + (d_0 + d_1 + d_2 \\
&+ d_3 + d_4 - w_1)w_2^2 - w_2^3 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 \\
&+ (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2)w_3 + (d_0 + d_1 + d_2 + d_3 \\
&+ d_4 - w_1 - w_2)w_3^2 - w_3^3 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 \\
&+ (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 \\
&- w_1 - w_2)w_3 + w_3^2)w_4 + (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4^2 \\
&- w_4^3) + 4w_1^4w_2^4w_3^4w_4^4(d_0d_1d_2d_3 + (d_0d_1d_2 + d_0d_1d_3 \\
&+ (d_0 + d_1)d_2d_3)d_4 - (d_0d_1d_2 + d_0d_1d_3 + (d_0 + d_1)d_2d_3 + d_0d_1d_4 \\
&+ (d_0 + d_1)d_2d_4 + (d_0 + d_1 + d_2)d_3d_4)w_1 + (d_0d_1 + (d_0 + d_1)d_2 \\
&+ (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4)w_1^2 - (d_0 + d_1 + d_2 + d_3 \\
&+ d_4)w_1^3 + w_1^4 - (d_0d_1d_2 + (d_1d_2 + d_0(d_1 + d_2))d_3 + (d_2d_3 + d_1(d_2 + d_3) \\
&+ d_0(d_1 + d_2 + d_3))d_4 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 \\
&+ d_2 + d_3)d_4)w_1 + (d_0 + d_1 + d_2 + d_3 + d_4)w_1^2 - w_1^3)w_2 + (d_0d_1 + (d_0 + d_1)d_2 \\
&+ (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + \\
&w_1^2)w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2^3 + w_2^4 - (d_0d_1d_2 + (d_1d_2 + d_0(d_1 \\
&+ d_2))d_3 + (d_2d_3 + d_1(d_2 + d_3) + d_0(d_1 + d_2 + d_3))d_4 - (d_0d_1 + (d_0 + d_1)d_2 \\
&+ (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4)w_1 + (d_0 + d_1 + d_2 + d_3 \\
&+ d_4)w_1^2 - w_1^3 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 \\
&+ d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2)w_2 + (d_0 + d_1 + d_2 \\
&+ d_3 + d_4 - w_1)w_2^2 - w_2^3)w_3 + (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 \\
&+ (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 \\
&+ d_2 + d_3 + d_4 - w_1)w_2 + w_2^2)w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
&- w_2)w_3^3 + w_3^4 - (d_0d_1d_2 + (d_1d_2 + d_0(d_1 + d_2))d_3 + (d_2d_3 + d_1(d_2 + d_3)
\end{aligned}
$$

$$
\begin{aligned}
&+ d_0(d_1 + d_2 + d_3))d_4 - (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0\\
&+ d_1 + d_2 + d_3)d_4)w_1 + (d_0 + d_1 + d_2 + d_3 + d_4)w_1^2 - w_1^3 - (d_0d_1\\
&+ (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2\\
&+ d_3 + d_4)w_1 + w_1^2)w_2 + (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2^2 - w_2^3 - (d_0d_1\\
&+ (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2\\
&+ d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2)w_3 + (d_0 + d_1\\
&+ d_2 + d_3 + d_4 - w_1 - w_2)w_3^2 - w_3^3)w_4 + (d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1\\
&+ d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2\\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1\\
&- w_2)w_3 + w_3^2)w_4^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4^3 + w_4^4)\\
&- 5w_1w_2w_3w_4^2(d_0^3d_1^3d_2^3d_3^3d_4^3 + 2d_0^2d_1^2d_2^2d_3^2d_4^2w_1w_2w_3(d_0 + d_1 + d_2 + d_3 + d_4\\
&- w_1 - w_2 - w_3 - w_4)w_4 - d_0d_1d_2d_3d_4w_1^2w_2^2w_3^2(d_0 + d_1 + d_2 + d_3 + d_4\\
&- w_1 - w_2 - w_3 - w_4)^2w_4^2 - 2w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2\\
&- w_3 - w_4)^3w_4^3 - 16d_0d_1d_2d_3d_4w_1^2w_2^2w_3^2w_4^2(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1\\
&+ d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2\\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1\\
&- w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2)\\
&+ 10w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4^3(d_0d_1\\
&+ (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1\\
&+ d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2\\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4\\
&- w_1 - w_2 - w_3)w_4 + w_4^2)) - 5w_1w_2w_3^2w_4(d_0^3d_1^3d_2^3d_3^3d_4^3\\
&+ 2d_0^2d_1^2d_2^2d_3^2d_4^2w_1w_2w_3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4\\
&- 4d_0^2d_1^2d_2^2d_3^2d_4^2w_1w_2w_3w_4^2 - d_0d_1d_2d_3d_4w_1^2w_2^2w_3^2(d_0 + d_1 + d_2 + d_3 + d_4\\
&- w_1 - w_2 - w_3 - w_4)^2w_4^2 - 2w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2\\
&- w_3 - w_4)^3w_4^3 + 4w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3\\
&- w_4)^2w_4^4 - 16d_0d_1d_2d_3d_4w_1^2w_2^2w_3^2w_4^2(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1\\
&+ d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1\\
&+ d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2\\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2) + 10w_1^3w_2^3w_3^3(d_0 + d_1 + d_2\\
&+ d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4^3(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3\\
&+ (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3\\
&+ d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 - (d_0 + d_1\\
&+ d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2) + 4w_1^3w_2^3w_3^3w_4^4(d_0d_1 + (d_0 + d_1)d_2
\end{aligned}
$$

$$
\begin{aligned}
&+ (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 \\
&+ w_1^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 \\
&- w_1 - w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2)) \\
&+ 20w_1^3w_2^3w_3^2w_4^2(d_0^2d_1^2d_2^2d_3^2d_4^2 - w_1^2w_2^2w_3^2(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 \\
&- w_3 - w_4)^2w_4^2 - 3w_1w_2w_3w_4^2(d_0d_1d_2d_3d_4 - w_1w_2w_3(d_0 + d_1 + d_2 + d_3 \\
&+ d_4 - w_1 - w_2 - w_3 - w_4)w_4) - w_1^2w_2^2w_3^2w_4^2(d_0d_1 + (d_0 + d_1)d_2 + (d_0 \\
&+ d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
&- w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2) \\
&- 3w_1w_2w_3^2w_4(w_1w_2w_3(-d_1 - d_2 - d_3 - d_4 + w_1 + w_2 + w_3 - w_4)w_4 \\
&+ d_0(d_1d_2d_3d_4 - w_1w_2w_3w_4))) - 5w_1w_2(w_1 + w_2)w_3w_4(d_0^3d_1^3d_2^3d_3^3d_4^3 \\
&+ 2d_0^2d_1^2d_2^2d_3^2d_4^2w_1w_2w_3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4 \\
&- 4d_0^2d_1^2d_2^2d_3^2d_4^2w_1w_2w_3w_4^2 - d_0d_1d_2d_3d_4w_1^2w_2^2w_3^2(d_0 + d_1 + d_2 + d_3 + d_4 \\
&- w_1 - w_2 - w_3 - w_4)^2w_4^2 - 2w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 \\
&- w_3 - w_4)^3w_4^3 + 4w_1^3w_2^3w_3^3(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3 \\
&- w_4)^2w_4^4 - 16d_0d_1d_2d_3d_4w_1^2w_2^2w_3^2w_4^2(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 \\
&+ (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 \\
&+ d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2) + 10w_1^3w_2^3w_3^3(d_0 + d_1 \\
&+ d_2 + d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4^3(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 \\
&+ d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
&- w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2) \\
&+ 4w_1^3w_2^3w_3^3w_4^4(d_0d_1 + (d_0 + d_1)d_2 + (d_0 + d_1 + d_2)d_3 + (d_0 + d_1 + d_2 \\
&+ d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 - (d_0 + d_1 + d_2 + d_3 \\
&+ d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2)w_3 + w_3^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2) \\
&- 4w_1w_2w_3^2w_4(d_0^2d_1^2d_2^2d_3^2d_4^2 - w_1^2w_2^2w_3^2(d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
&- w_2 - w_3 - w_4)^2w_4^2 - 3w_1w_2w_3w_4^2(d_0d_1d_2d_3d_4 - w_1w_2w_3(d_0 + d_1 + d_2 \\
&+ d_3 + d_4 - w_1 - w_2 - w_3 - w_4)w_4) - w_1^2w_2^2w_3^2w_4^2(d_0d_1 + (d_0 + d_1)d_2 + (d_0 \\
&+ d_1 + d_2)d_3 + (d_0 + d_1 + d_2 + d_3)d_4 - (d_0 + d_1 + d_2 + d_3 + d_4)w_1 + w_1^2 \\
&- (d_0 + d_1 + d_2 + d_3 + d_4 - w_1)w_2 + w_2^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 \\
&- w_2)w_3 + w_3^2 - (d_0 + d_1 + d_2 + d_3 + d_4 - w_1 - w_2 - w_3)w_4 + w_4^2)))).
\end{aligned}
$$

Chapter 5

The Brasselet-Schürmann-Yokura conjecture on L-classes

In this chapter, we develop the main work of this Ph.D. thesis. Prof. Javier Fernández de Bobadilla and I, proved the Brasselet-Schürmann-Yokura conjecture for projective varieties [26]. This conjecture is a conjecture of characteristic classes of singular varieties which states that the Hirzebruch homology class $T_{y,*}$ (for $y = 1$) coincides with the Goresky-MacPherson L-class for compact complex algebraic varieties that are rational homology manifolds.

In Section 2.5.2, we introduced the Hirzebruch homology characteristic class $T_{y,*}$. This class is defined as a natural transformation from the relative Grothendieck functor $K_0(var/-)$ of complex algebraic varieties to the Borel-Moore homology functor $H_{2*}^{BM}(-, \mathbb{Q})$ with rational coefficients (see Theorem 2.5.24).

The Hirzebruch class $T_{y,*}$ unifies, for different values of y, the following characteristic classes (see Theorem 2.5.26): for $y = -1$, the Chern-Schwartz-MacPherson transformation (see Theorem 2.5.5), for $y = 0$ the Baum-Fulton-MacPherson Todd transformation (see Theorem 2.5.7), and for $y = 1$, the Cappell-Shaneson L-transformation (see Theorem 2.5.23).

Furthermore, the transformation $T_{y,*}$ applied to the distinguished element $[Y \to Y] \in K_0(var/Y)$, that is $T_{y,*}(Y) := T_{y,*}([Y \to Y])$, specializes: for $y = -1$, in the (rationalized) Chern-Schwartz-MacPherson class of Y, for $y = 0$, in the Baum-Fulton-MacPherson Todd class of Y, if Y has du Bois singularities, and, for $y = 1$, J. P. Brasselet, J. Schürmann and S. Yokura conjectured the following equality of characteristic classes:

Theorem 5.0.1 (The BSY-conjecture)**.** *If Y is a compact complex algebraic variety that is a rational homology manifold, then*

$$T_{1,*}(Y) = L_*(Y),$$

where $L_*(Y)$ is the Goresky-MacPherson L-class of Y.

In the 1980s, M. Goresky and R. MacPherson introduced the intersection homology, and a notion of signature for singular varieties was given (see Section 2.3.3 and Section 2.5.2). The Brasselet-Schürmann-Yokura conjecture is the characteristic class version of the important Hodge Index Theorem, which computes the signature of a compact complex algebraic manifold in terms of Hodge numbers. Hence, this conjecture establishes a generalization of Hodge's Index Theorem for higher-degree homology groups giving rise a Hodge-theoretical realization of the Goresky-MacPherson L-class even if the variety is a rational homology manifold.

The conjecture was previously solved for the following special cases: In [11], S. E. Cappell, L. G. Maxim, J. Schürmann and J. L. Shaneson solved it for varieties with isolated singularities that are hypersurfaces in an smooth algebraic variety. In [12], the same authors proved the case for $X = Y/G$, where Y is a projective G-manifold and G is a finite group of algebraic automorphisms. L. G. Maxim and J. Schürmann in [62] gave a proof for simplicial projective toric varieties. In the projective case, the degree 0 case holds by a direct consequence of Saito's intersection cohomology Hodge Index Theorem (details can be found in M. Banagl's paper [4]). M. Banagl in [4], showed the case for normal projective complex 3-folds at worst canonical singularities, trivial canonical divisor, and $H^1(X; \mathcal{O}_X) \neq 0$.

Let $\Omega_{\mathbb{K}}(-)$ be the cobordism functor of cohomologically constructible bounded self-dual $\mathbb{K}$-complexes ($\mathbb{K}$ a subfield of $\mathbb{R}$) of sheaves (see Section 2.5.2). By Theorem 2.5.26, there is a natural transformation

$$sd\colon K_0(var/-) \to \Omega_{\mathbb{K}}(-) \tag{5.1}$$

such that, for Y non-singular, $sd([Y \to Y]) = [\mathbb{K}_Y[\dim_{\mathbb{C}} Y]]$, and

$$\begin{array}{ccc} K_0(var/-) & \xrightarrow{\quad sd \quad} & \Omega_{\mathbb{K}}(-) \\ & {\scriptstyle T_{1,*}} \searrow \quad \swarrow {\scriptstyle L_*} & \\ & H_{2*}(-;\mathbb{Q}) & \end{array} \tag{5.2}$$

In this chapter, we prove the following result also conjectured in [9] implying the BSY-conjecture:

Theorem 5.0.2. *If Y is a projective complex variety that is a rational homology manifold, then we have the equality*

$$sd_{\mathbb{R}}([Y \to Y]) = [IC_Y] \in \Omega_{\mathbb{R}}(Y),$$

where $\Omega_{\mathbb{R}}(Y)$ is the cobordism group of cohomologically constructible bounded self-dual $\mathbb{R}$-complexes, $sd_{\mathbb{R}}$ denotes the natural transformation sd in $\Omega_{\mathbb{R}}(Y)$ and IC_Y is the intersection cohomology complex on Y (see Section 2.3.3).

Since the Cappell-Shaneson L-transformation applied to the intersection cohomology complex IC_Y recovers the Goresky-MacPherson L-class $L_*(Y)$

(see Section 2.5.2), the previous theorem, implies the Brasselet-Schürmann-Yokura conjecture for projective varieties by using the commutative diagram (5.2).

The proof of Theorem 5.0.2 is organized in the following steps: First, to compute the left-hand side of Theorem 5.0.2, we obtain an identity in the Grothendieck group $K_0(var/Y)$ expressing the class $[Y \to Y]$ as an alternate sum of classes of smooth varieties coming from a semi-simplicial resolution of Y (Section 5.1). The expression obtained for $sd_{\mathbb{R}}([Y \to Y])$ includes the class $[IC_Y]$ together with other terms. The goal is to show that the extra terms vanish. The second step is to obtain exact sequences coming from a spectral sequence of perverse sheaves associated with the semi-simplicial resolution (Section 5.2) which give an identity in $\Omega_{\mathbb{R}}(Y)$ giving rise to the desired vanishing (Section 5.3). To obtain the exact sequences, we prove the degeneration at the second page of this spectral sequence where the projectivity assumption is needed. Our proof of the degeneration uses classical Hodge theory, and the ideas resemble the way that M. A. A. de Cataldo and L. Migliorini used classical Hodge theory for their proof of the Decomposition Theorem in [18] (see also Section 2.3.4).

An important intermediate step that is needed is the fact that the classes of certain cohomologically constructible complexes together with perfect pairings in the cobordism group are independent of the perfect pairing as long as the cohomologically constructible complex and the pairing satisfy certain Hodge theoretic compatibility properties. This is Lemma 5.3.6 and the proof is based on a representation theory argument that may be of some utility elsewhere.

Let us make precise here what we do mean by "proof based on classical Hodge theory": Our proof uses the Decomposition Theorem for $\mathbb{R}$-coefficients in the form as in Theorem 2.3.38, that is: Let $\varepsilon : Z \to Y$ be a projective morphism from a smooth complex projective variety Z of dimension d. Then,

$$R\varepsilon_*\mathbb{R}_Z[d] \cong \bigoplus_{i=-M}^{M} {}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])[-i], \tag{5.3}$$

where M is a positive integer, and ${}^p\mathcal{H}^i(-)$ denotes the i-th cohomology functor for the perverse t-structure introduced in Section 2.3.2. Moreover, the perverse sheaves ${}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])$ are semi-simple (see Theorem 2.3.39).

As we mentioned in Chapter 2, the Decomposition Theorem was proved originally in [6] for $\mathbb{C}$-coefficients. For $\mathbb{R}$-coefficients, the equation (5.3), and even its analogue for $\mathbb{Q}$-coefficients follows from [20]; then the semi-simplicity of the perverse sheaves ${}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_X)$ can be obtained by an argument involving the exactness of the scalar extension functor from the abelian category of $\mathbb{R}$-perverse sheaves to the abelian category of $\mathbb{C}$-perverse sheaves. The Decomposition Theorem in the form we need was re-proved later by M. Saito, using Hodge modules, and also by M. A. de Cataldo and L. Migliorini. M. A.

de Cataldo and L. Migliorini's proof is geometric and rests in classical Hodge theory in the sense that it only uses the formalism of perverse sheaves and Hodge theory as developed in [21], [22].

We also use the theory of cubical hyperresolutions using the treatment in Section 2.4). Given the Decomposition Theorem as stated above, the theory of cubical hyperresolutions and generalities in perverse sheaves, our proof only needs fairly elementary Hodge theory computations (see Section 2.2).

In [27], we prove in collaboration with M. Saito Theorem 5.0.2 in the general case of compact complex algebraic varietes that are rational homology manifolds, using the theory of mixed Hodge modules. However, this proof will not be covered in this chapter.

5.1. An identity in the relative Grothendieck group of algebraic varieties

Let Y be a compact complex algebraic variety of dimension n. In this section, we prove that the class $[Y \to Y]$ in the relative Grothendieck group $K_0(var/Y)$ of complex algebraic varieties over Y (see Section 2.5.1), can be expressed as a sum of classes $[X_k \to Y] \in K_0(var/Y)$ for X_k non-singular varieties using cubical hyperresolutions (see Definition 2.4.14). We will denote by $[X]$ the class $[X \to Y] \in K_0(var/Y)$ for simplicity.

Lemma 5.1.1. *There exists an $(n+1)$-semi-simplicial resolution*

$$\varepsilon \colon X_\bullet \to Y, \tag{5.4}$$

such that in the relative Grothendieck group $K_0(var/Y)$ over Y, the following identity holds:

$$[Y] = [\tilde{Y}] + [\bigsqcup_{i=1}^{n} X_{0,i}] + \sum_{k=1}^{n} (-1)^k [X_k], \tag{5.5}$$

where $X_0 = \tilde{Y} \bigsqcup (\bigsqcup_{i=1}^{n} X_{0,i})$, and $\tilde{Y} \to Y$ is a resolution of singularities of Y which restricts to an isomorphism over the regular locus of Y.

Proof. We prove the equality (5.5) following the procedure to construct a cubical hyperresolution given in Section 2.4.3 combined with the application of the additive relation (2.40) in $K_0(var/Y)$ in each step of the construction.

In the first step, we choose a resolution of singularities $\pi \colon \tilde{Y} \to Y$ of Y which restricts to an isomorphism over the regular locus of Y. For the discriminant $D := \Sigma$ of π, we consider its discriminant square (see Lemma-Definition 2.4.18 and also (2.19)). Since the discriminant square is cartesian and the map $D \to Y$ is a closed inclusion, then we have the inclusion $E :=$

$\pi^{-1}(D) \to \tilde{X}$. By applying the additive relation (2.40) to this diagram, and by taking into account that $\tilde{Y} \setminus E \simeq Y \setminus D$, the following identity holds

$$[Y] = [\tilde{Y}] + [D] - [E]. \tag{5.6}$$

Suppose now, we are in the $(k+1)$-th step of the construction. Consider the $(k+1)$-cubical variety obtained in the k-th step as a morphism $f^{(k)} \colon Y^{(k)} \to Z^{(k)}$ between k-cubical varieties. We choose a resolution $\tilde{Y}^{(k)}$ of $Y^{(k)}$ and consider its discriminant square (2.20). Notice that the k-cubical varieties $Y^{(k)}$ and $Z^{(k)}$ are the k-cubical variety $E^{(k-1)} \to D^{(k-1)}$ and the k-cubical variety given by the composition of k-cubical varieties $\tilde{Y}^{(k-1)} \to Y^{(k-1)} \to Z^{(k-1)}$, respectively. Since the discriminant square of $\tilde{Y}^{(k)} \to Y^{(k)}$ is composed by the discriminant squares of resolutions $\tilde{Y}_I^{(k)} \to Y_I^{(k)}$ for all I, then each discriminant square satisfies the identity (5.6) proved in step 1, that is,

$$[Y_I^{(k)}] = [\tilde{Y}_I^{(k)}] + [D_I^{(k)}] - [E_I^{(k)}] \tag{5.7}$$

where $D_I^{(k)} = Y_I^{(k+1)}$ and $E_I^{(k)} = Y_{I\cup\{k\}}^{(k+1)}$. Then, after replacing the identities (5.7) in all the steps of the construction, and writing them as an associated augmented semi-simplicial variety as in Remark 2.4.8, we find the identity

$$[Y] = \sum_{k=0}^{n} (-1)^k [X_k]. \tag{5.8}$$

Writting $X_0 = \tilde{Y} \bigsqcup_{i=1}^{n} X_{0,i}$ where $X_{0,i}$ is the term X_i in the $(n+1)$-cubical variety, we obtain the desired identity. □

The semi-simplicial resolution satisfies that each X_k is a disjoint union of smooth varieties $X_{k,i}$, such that $\dim X_{k,i} \leq n-k$. Moreover, the only component of X_0 of dimension n is the smooth variety $\tilde{Y}$ to which ε restricts to a resolution of singularities $\varepsilon|_{\tilde{Y}} : \tilde{Y} \to Y$. Observe also that all morphisms involved in the cubical hyperresolution are projective.

Remark 5.1.2. Let $\{Z_I\}_{I\in\{0,1\}^n}$ be a cubical hyperresolution of $Y \subset \mathbb{P}^n_{\mathbb{C}}$, constructed following the procedure of Section 2.4.3. Suppose $Y \subset \mathbb{P}^N_{\mathbb{C}}$. Let $H \in (\mathbb{P}^N_{\mathbb{C}})^*$ be a generic hyperplane. Then the cubical variety formed taking the fibre product of $\{Z_I\}_{I\in\{0,1\}^n}$ by $Y \cap H$ is a cubical hyperresolution of $Y \cap H$. Moreover, a variation H_t of the generic hyperplane where $H_t \in U \subset (\mathbb{P}^N_{\mathbb{C}})^*$ is a small neighborhood of the point H_0 in the dual projective space $(\mathbb{P}^N_{\mathbb{C}})^*$, yields a topologically trivial family of cubical hyperresolutions.

Iterating we obtain the same statement for generic linear sections of arbitary codimension.

Proof. The proof of the first assertion is an inspection on the construction of Section 2.4.3, combined with the fact that if $|L| \subset (\mathbb{P}^M_{\mathbb{C}})^*$ is a linear system without base points in a smooth projective manifold $Z \subset \mathbb{P}^M_{\mathbb{C}}$, then a generic hyperplane section $Z \cap H$ in $|L|$ is smooth. □

5.2. Some exact sequences of perverse sheaves

In this section, we construct a spectral sequence of perverse sheaves associated with the semi-simplicial resolution of Y given in Lemma 5.1.1. We prove that the rows on the first page of this spectral sequence are exact. This exactness will allow giving the needed cancellations to prove Theorem 5.0.2.

For any k, the variety X_k is a disjoint union of smooth varieties of different dimensions, and $\varepsilon|_{X_k}$ is a projective morphism. So, by Theorem 2.3.38, we have a decomposition

$$R\varepsilon_*\mathbb{R}_{X_k} \cong \bigoplus_{q\geq 0} {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})[-q].$$

Furthermore, Theorem 2.3.39 predicts that the perverse sheaves ${}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})$ decomposes as a direct sum of simple intersection cohomology complexes (see Remark 2.3.32). Let $\{Y = \Sigma_0, \Sigma_1, ..., \Sigma_N\}$ be the collection of subvarieties in Y which are the support of simple direct summands of the perverse sheaves ${}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})$. We have the further decomposition

$${}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k}) \cong \bigoplus_{j\in J} {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})_{\Sigma_j}, \tag{5.9}$$

where ${}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})_{\Sigma_j}$ denotes the direct sum of the simple summands in ${}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})$ whose support is Σ_j.

Remark 5.2.1. Notice that ${}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})_Y = 0$ unless $k = 0$ and $q = n = \dim(Y)$, and that ${}^p\mathcal{H}^n(R\varepsilon_*\mathbb{R}_{X_0})_Y = IC_Y[-n]$, since $\varepsilon|_{\tilde{Y}}$ is a resolution of singularities of Y (see Remark 2.3.40).

Lemma 5.2.2. *If Y is a projective complex variety that is a rational homology manifold, for each support Σ_j strictly contained in Y, and for each $q \geq 0$, we have an exact sequence of perverse sheaves*

$$0 \to {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_0})_{\Sigma_j} \to {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_1})_{\Sigma_j} \to ... \to {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_n})_{\Sigma_j} \to 0. \tag{5.10}$$

We reduce the proof to the degeneration at the second page of a spectral sequence of perverse sheaves. The proof only uses classical Hodge theory, and is more in the spirit of [18], but only works when Y is projective because we use hyperplane sections.

Proof. The proof has 2 parts. In the first part, we identify a spectral sequence converging to the perverse cohomology of an acyclic complex whose E_1 page splits as the direct sum of the complexes (5.10), and in the second part we use Hodge theory to prove its degeneration at the second page.

Part 1: Denote by $\mathcal{C}^\bullet_{X_k}$ the canonical Godement resolution of the constant sheaf $\mathbb{R}_{X_k}$ in X_k (see Section 2.4.2). By cohomological descent (see Definition 2.4.12), of the semi-simplicial resolution (5.4), we obtain a double complex of sheaves $I^{\bullet,\bullet}$ in Y such that each column $I^{k,\bullet}$ is equal to $\varepsilon_*\mathcal{C}^\bullet_{X_k}$ (and hence computes $R\varepsilon_*\mathbb{R}_{X_k}$), and such that there is a quasi-isomorphism $\mathbb{R}_Y \to s(I^{\bullet,\bullet})$, where $s(I^{\bullet,\bullet})$ denotes the simple complex associated to $I^{\bullet,\bullet}$. It is important to notice that the horizontal differentials in the double complex $I^{\bullet,\bullet}$ give rise to morphisms of complexes $\varepsilon_*\mathcal{C}^\bullet_{X_k} \to \varepsilon_*\mathcal{C}^\bullet_{X_{k+1}}$ which are induced from the alternating sum of the pullbacks by the $(k+1)$-morphisms $X_{k+1} \to X_k$ appearing in the semi-simplicial variety $X_\bullet$.

Consider the sequence of morphisms of double complexes

$$I^{\bullet,\bullet} \xrightarrow{\beta_1} I^{0,\bullet} \xrightarrow{\beta_2} R(\varepsilon|_{\tilde{Y}})_*\mathbb{R}_{\tilde{Y}} \xrightarrow{\beta_3} IC_Y[-n],$$

where the simple complex $I^{0,\bullet}$ is seen as a double complex that has non-zero objects only at the 0-th column, β_1 is the natural morphism of double complexes, β_2 is the composition of the natural projection from $I^{0,\bullet}$ to $R(\varepsilon|_{\tilde{Y}})_*\mathbb{R}_{\tilde{Y}}$ given by the decomposition of X_0 in connected components, and β_3 is described as follows: by Theorem 2.3.38, there is a non-canonical direct sum decomposition

$$\Phi : R(\varepsilon|_{\tilde{Y}})_*\mathbb{R}_{\tilde{Y}} \to IC_Y[-n] \oplus L,$$

where L is a direct sum of shifted simple perverse sheaves with support strictly contained in Y. We define $\beta_3 := \rho \circ \Phi$, where ρ is the canonical projection $IC_Y[-n] \oplus L \to IC_Y[-n]$.

Even if β_3 is not unique, by the uniqueness of Proposition in [31, 5.1], restricting to the non-singular stratum of Y we obtain that the simple complex morphism

$$\eta : s(I^{\bullet,\bullet}) \cong \mathbb{R}_Y \to IC_Y[-n]$$

associated with the composition $\beta_3 \circ \beta_2 \circ \beta_1$ is, up to multiplication with a non-zero real number, the canonical morphism connecting cohomology with intersection cohomology complexes. By uniqueness of the intersection cohomology complex, if Y is a rational homology manifold then η is a quasi-isomorphism, and hence $\mathrm{cone}(\eta)[-1]$ is an acyclic complex.

Notice that we can form a double complex $K^{\bullet,\bullet}$, whose columns are

$$K^{0,\bullet} = \mathrm{cone}(\beta_3 \circ \beta_2)[-1], \quad \text{and} \quad K^{p,\bullet} = I^{p,\bullet} \text{ for } p > 0,$$

here $\beta_3 \circ \beta_2$ denotes the simple complex morphism induced at the 0-th column.

Since there is a quasi-isomorphism

$$s(K^{\bullet,\bullet}) \cong \mathrm{cone}(\eta)[-1]$$

because Y is a rational homology manifold, we have shown that the simple complex $s(K^{\bullet,\bullet})$ associated to the double complex $K^{\bullet,\bullet}$ is acyclic.

We notice that the double complex $K^{\bullet,\bullet}$ depends on the choice of β_3. However, this non-uniqueness does not affect our proof.

The single complex $s(I^{\bullet,\bullet})$ is decreasingly filtered by the subcomplexes $F^p s(I^{\bullet,\bullet})$, where $F^p s(I^{\bullet,\bullet})$ is the simple complex of the double sub-complex of $I^{\bullet,\bullet}$ formed by the direct sum of $I^{a,b}$ for $a \geq p$. A similar filtration is defined on $s(K^{\bullet,\bullet})$

We want to construct a spectral sequence of perverse sheaves associated with the double complexes $I^{\bullet,\bullet}$ and $K^{\bullet,\bullet}$. In order to do this, we use the general technique of construction of spectral sequences given by H. Cartan and S. Eilenberg in [16, XV.7] (see Example 1 in *loc.cit.*). Define the Cartan-Eilenberg systems of perverse sheaves as follows:

$$H[I^{\bullet,\bullet}](p,q) := \sum_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(F^p s(I^{\bullet,\bullet})/F^q s(I^{\bullet,\bullet})),$$

$$H[K^{\bullet,\bullet}](p,q) := \sum_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(F^p s(K^{\bullet,\bullet})/F^q s(K^{\bullet,\bullet})),$$

for $p \geq q$. The morphisms $H[I^{\bullet,\bullet}](p,q) \to H[I^{\bullet,\bullet}](p',q')$ are induced from the natural morphism of complexes, and similarly for $K^{\bullet,\bullet}$. The connecting morphisms coincide with the connecting morphism for the exact sequence of complexes

$$0 \to F^q s(I^{\bullet,\bullet})/F^r s(I^{\bullet,\bullet}) \to F^p s(I^{\bullet,\bullet})/F^r s(I^{\bullet,\bullet}) \to F^p s(I^{\bullet,\bullet})/F^q s(I^{\bullet,\bullet}) \to 0,$$

and similarly for $K^{\bullet,\bullet}$. The defining sums above are finite, and it is straightforward to check, using that $X_k = \emptyset$ for $k > n$, that conditions $(SP.1)-(SP.5)$ of [16, XV.7] are satisfied. The perverse sheaves $H[I^{\bullet,\bullet}](p,q)$ are graded by the defining sum, and the same for $K^{\bullet,\bullet}$. There is an obvious morphism of Cartan-Eilenberg systems $H[I^{\bullet,\bullet}](p,q) \to H[K^{\bullet,\bullet}](p,q)$.

Therefore, we obtain two spectral sequences of graded objects in $Perv(Y)$, and a morphism between them. We obtain the following terms in the page one of the spectral sequences:

$$E(I)_1^{p,q} \cong {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_p}). \tag{5.11}$$

Since $K^{p,q} \to I^{p,q}$ is an isomorphism for $p > 0$, we have an isomorphism

$$E(K)_1^{p,q} \cong E(I)_1^{p,q} \cong {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_p}). \tag{5.12}$$

Considering the decomposition (5.9), and by definition of $K^{\bullet,\bullet}$, for $p = 0$ we have

$$E(K)_1^{0,q} := \bigoplus_{j\neq 0} {}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_0})_{\Sigma_j}; \tag{5.13}$$

that is, all the summands except $IC_Y[-n]$ if $q = n$.

Theorem 2.3.39 implies that the complexes appearing in the E_1 page of the spectral sequence associated with the acyclic complex $s(K^{\bullet,\bullet})$ splits in a direct sum of complexes of perverse sheaves with strict support Σ_j, for $j > 0$. These complexes coincide with the complexes (5.10). So, proving degeneration at the second page is enough to finish the proof.

PART 2: The proof of degeneration at E_2 is by double induction on $\dim(Y)$ and $\operatorname{codim}(\Sigma_j)$. Suppose that the lemma holds for $\dim(Y) < n$, and for $\operatorname{codim}(\Sigma_j) < d$ when $\dim(Y) = n$. Assuming $\dim(Y) = n$, we prove the exactness simultaneously for all supports Σ_j of codimension d in Y.

CASE $d < n$: For any Σ_j, there exists a dense open subset U_j over which all the perverse sheaves $(E_1(K)^{p,q})_{\Sigma_j}$ are local systems. In order to prove the exactness of (5.10) it is enough to prove the exactness of the stalk

$$0 \to ((E(K)_1^{0,q})_{\Sigma_j})_z \to ((E(K)_1^{1,q})_{\Sigma_j})_z \to ... \to ((E(K)_1^{n,q})_{\Sigma_j})_z \to 0, \quad (5.14)$$

of the complex (5.10) at a point z in each connected component of each of the open subsets U_j. Let H be a generic linear section of dimension d such that the intersection $\Sigma_j \cap H$ is a finite set of points contained in U_j for every d-codimensional component Σ_j. Then, $Y \cap H$ is a projective rational homology manifold of dimension $n - \dim(\Sigma_j)$ (see Lemma 5.2.3), and by Remark 5.1.2 the pullback to $Y \cap H$ of the semisimplicial resolution of $X_\bullet \to Y$ gives a semi-simplicial resolution of $(X|_H)_\bullet \to Y \cap H$. Construct the perverse spectral sequence (5.11) for the semi-simplicial resolution $(X|_H)_\bullet \to Y \cap H$, and split it as a direct sum of spectral sequences of perverse sheaves with common support as above.

For any $z \in \Sigma_j \cap H$, the point z is a support for the E_1 page of the perverse spectral sequence associated with the hyperresolution $(X|_H)_\bullet \to Y \cap H$ and the complex (5.14) is the analog of the complex (5.10) for the support z. This follows because $(X|_H)_p$ is the fibre product $X_p \times_Y (Y \cap H)$, and then, by the topological triviality statement of Remark 5.1.2, we have $R\varepsilon_*\mathbb{R}_{(X|_H)_p} = \iota^*_{Y\cap H} R\varepsilon_*\mathbb{R}_{X_p}$, where $\iota_{Y\cap H}$ denotes the inclusion of $Y \cap H$ into Y.

Since $\dim(Y \cap H) = n - \dim(\Sigma_j) < n$, by induction hypothesis the lemma is true for $Y \cap H$ and the semisimplicial resolution $(X|_H)_\bullet \to Y \cap H$, we have the exatness of the sequence (5.14).

CASE $d = n$:

Applying the functor $H^*(Y, -)$ to the quotients $F^p s(I^{\bullet,\bullet})/F^q s(I^{\bullet,\bullet})$ and $F^p s(K^{\bullet,\bullet})/F^q s(K^{\bullet,\bullet})$ we obtain two Cartan-Eilenberg systems and a morphism between them, in a similar way as above for the construction of spectral sequences of perverse sheaves. They induce spectral sequences of real vector spaces, whose terms are denoted by $'E(K)_r^{p,b}$ and $'E(I)_r^{p,b}$. The morphism between the Cartan-Eilenberg systems induces homomorphisms $'E(K)_r^{p,b} \to {}'E(I)_r^{p,b}$ which are compatible with the differentials. The E_1 terms are the following:

$$'E(I)_1^{p,b} \cong H^b(Y, R\varepsilon_*\mathbb{R}_{X_p}). \quad (5.15)$$

For $p > 0$, we have an isomorphism

$$'E(K)_1^{p,b} \cong {'E(I)}_1^{p,b} \cong H^b(Y, R\varepsilon_*\mathbb{R}_{X_p}), \tag{5.16}$$

and, for $p = 0$, we have

$$'E(K)_1^{0,b} := \mathrm{Ker}(H^b(Y, R\varepsilon_*\mathbb{R}_{X_0}) \overset{\beta_3\circ\beta_2}{\longrightarrow} H^b(Y, IC_Y[-n])). \tag{5.17}$$

The spectral sequence $'E(I)$ coincides with the spectral sequence induced by the filtration by columns of the double complex $\Gamma(Y, I^{\bullet,\bullet})$. By [71, Theorem 3.18, Theorem 5.33], the spectral sequence $'E(I)$ lifts to a spectral sequence of real mixed Hodge structures, degenerates at E_2, and converges to the mixed Hodge structure $H^*(Y;\mathbb{R})$. Since $H^k(Y;\mathbb{R}) \cong \bigoplus_{p+b=k} {'E_2}^{p,b}(I)$, each term $'E_2^{p,b}(I)$ has weight b, and Y is compact (see Table 2.1), we have that $W_{k-r}H^k(Y;\mathbb{R}) \cong \bigoplus_{p\geq r} {'E(I)}_2^{p,b}$, for any k. Since Y is a rational homology manifold, by Theorem 2.3.36, then $H^k(Y;\mathbb{R})$ is a pure Hodge structure of weight k, and so $'E(I)_2^{p,b} = 0$ for $p \geq 1$.

By the isomorphism (5.16), we deduce that $'E(K)_2^{p,b} \cong {'E(I)}_2^{p,b} = 0$ for $p \geq 2$. Therefore $'E(K)_r^{p,b} = 0$ for any r and $p \geq 2$, and the spectral sequence $'E(K)$ degenerates at the E_2 page.

By the degeneration at the second page, since the complex $s(K^{\bullet,\bullet})$ is acyclic, we have the vanishing $'E(K)_2^{p,b} = 0$ for every p, b, and therefore for every b we have the exact sequence

$$0 \to {'E(K)}_1^{0,b} \to {'E(K)}_1^{1,b} \to \ldots \to {'E(K)}_1^{n,b} \to 0. \tag{5.18}$$

We denote by d_1' the differential of this complex.

By Theorem 2.3.38, and the E_1 term description (5.16) and (5.17), we have the splitting

$$'E(K)_1^{p,b} = \bigoplus_{1\leq j\leq N} \bigoplus_{q\geq 0} H^b(Y, (E(K)_1^{p,q})_{\Sigma_j}[-q]) \tag{5.19}$$

in the category of real vector spaces.

Denote by

$$d_1'(p, b, j_1, j_2, q_1, q_2)\colon H^b(Y, (E(K)_1^{p,q_1})_{\Sigma_{j_1}}[-q_1]) \to H^b(Y, (E(K)_1^{p+1,q_2})_{\Sigma_{j_2}}[-q_2])$$

the composition of d_1' with the inclusion of the source in $'E(K)_1^{p,b}$ and the projection from $'E(K)_1^{p+1,b}$ to the target.

For $j_1 = j_2$, $q_1 = q_2$, the morphism $d_1'(p, b, j_1, j_1, q_1, q_1)$ is obtained applying the functor $H^b(Y, -)$ to the differential $d_1 : (E(K)_1^{p,q_1})_{\Sigma_{j_1}} \to (E(K)_1^{p+1,q_1})_{\Sigma_{j_1}}$ appearing in the complex (5.10) for $q = q_1$.

For $j_1 \neq j_2$, $q_1 = q_2$, the morphism $d_1'(p, b, j_1, j_2, q_1, q_1)$ vanishes since the terms $(E(K)_1^{p,q_1})_{\Sigma_{j_1}}$ and $(E(K)_1^{p,q_1})_{\Sigma_{j_2}}$ are semi-simple perverse sheaves of disjoint support.

By induction, if $\Sigma_j \neq Y$ is not of dimension 0, then the sequence of semi-simple perverse sheaves (5.10) is exact. Semi-simplicity and exactness implies that the sequence (5.10) is isomorphic to the direct sum of exact sequences of perverse sheaves of the form

$$0 \to P[-l] \to P[-(l+1)] \to 0, \tag{5.20}$$

where P is simple and perverse, the map is the identity and $0 \leq l \leq n-1$.

Pick some $\Sigma_j \neq Y$, and some simple perverse sheaf such that (5.20) is a direct summand of (5.10). Pick any b such that $H^b(Y, P[-q]) \neq 0$. Then the sequence (5.18) is isomorphic to one of the form

$$... \to A^{l-1} \to A^l \oplus H^b(Y, P[-q]) \xrightarrow{d_1'} A^{l+1} \oplus H^b(Y, P[-q]) \to A^{l+2} \to ... \tag{5.21}$$

where the differential d_1' restricted and projected to $H^b(Y, P[-q])$ is the identity. Then the sequence

$$... \to A^{l-1} \to A^l \to (A^{l+1} \oplus H^b(Y, P[-q]))/d_1'(H^b(Y, P[-q])) \to A^{l+2} \to ...,$$

is also exact, and identifying

$$A^{l+1} \cong (A^{l+1} \oplus H^b(Y, P[-q]))/d_1(H^b(Y, P[-q]))$$

is isomorphic to

$$... \to A^{l-1} \to A^l \to A^{l+1} \to A^{l+2} \to ...$$

with differential induced from d_1'.

Define $d_j := \dim(\Sigma_j)$. Proceed in the same way with all the direct summands (5.20) appearing in all the exact sequences (5.10) for any $\Sigma_j \neq Y$ not of dimension 0, and any q. Taking into account the splitting (5.19), from the exact sequence (5.18), we obtain an exact sequence of the form

$$... \to \bigoplus_{d_j=0} \bigoplus_{q \geq 0} H^b(Y, (E(K)_1^{p,q})_{\Sigma_j}[-q]) \to \bigoplus_{d_j=0} \bigoplus_{q \geq 0} H^b(Y, (E(K)_1^{p+1,q})_{\Sigma_j}[-q]) \to ...$$

The group $H^b(Y, (E(K)_1^{p,q})_{\Sigma_j})[-q]$ vanishes unless $q = b$, because $(E(K)_1^{p,q})_{\Sigma_j}$ has 0-dimensional support, so the sequence becomes

$$... \to \bigoplus_{d_j=0} H^q(Y, (E(K)_1^{p,q})_{\Sigma_j}[-q]) \to \bigoplus_{d_j=0} H^q(Y, (E(K)_1^{p+1,q})_{\Sigma_j}[-q]) \to ...$$

Using that the morphism $d_1'(p, b, j_1, j_2, q_1, q_1)$ vanishes if $j_1 \neq j_2$, we obtain that this sequence is the direct sum over the set of 0-dimensional supports (for fixed q) of the sequences (5.10). $\square$

Lemma 5.2.3. *Any generic hyperplane section of a rational homology manifold Y is a rational homology manifold.*

Proof. Choose a Whitney stratification of Y. A generic hyperplane H does not meet the 0 dimensional stratum, and so for any point $y \in Y \cap H$ there exists a neighborhood U of y in Y such that $U = (U \cap H) \times D$, where D is a disk. The proof follows now from easy homological considerations. □

5.3. A computation in the cobordism group of self-dual complexes

In this section, we give the proof of Theorem 5.0.2. We will show in Section 5.3.1 that

$$sd_{\mathbb{R}}([Y]) - [IC_Y] = 0 \tag{5.22}$$

in $\Omega_{\mathbb{R}}(Y)$ when Y is a projective rational homology manifold.

To obtain this vanishing, we will combine Lemma 5.3.9 with Lemma 5.2.2 proved in the previous section. In order to prove Lemma 5.3.9, we show two preliminary results involving polarizations of Hodge structures (see Definition 2.2.11).

In Section 2.5.2, we introduced two notions of cobordism. The first one was given by B. Youssin in [85] through the notion of elementary cobordism (see Definition 2.5.10). The second one was given in [27] considering the directly cobordism relation (see Definition 2.5.17). In this chapter, we consider this second definition of cobordism group and its properties given in Section 2.5.2. However, our result holds as well for both definitions of cobordism, since the key identity in $\Omega_{\mathbb{R}}(X)$, which is provided in Lemma 5.3.6 holds for it (see Remark 5.3.7).

Let Y be a compact complex algebraic variety. Consider a self-dual $\mathbb{R}$-complex $(\mathcal{F}, S)$, that is a pair $(\mathcal{F}, S)$ with $\mathcal{F} \in \mathrm{Ob}(D^b_c(Y))$ and $S : \mathcal{F} \otimes \mathcal{F} \to \mathbb{D}_Y$ is a perfect pairing (see Definition 2.5.14), where $\mathbb{D}_Y$ is the dualizing complex on Y (see Definition 2.3.9). Notice that we can consider equivalently the definition of self-dual complex given by B. Youssin (see Definition 2.3.13 and Remark 2.5.15) instead of perfect pairings.

Let $\varepsilon : Z \to Y$ be a projective morphism of complex algebraic varieties, with Z smooth of dimension d. By Remark 2.5.16, we have the pair $(\mathbb{R}_Z[d], \sigma_Z)$ is a self-dual complex, where $\sigma_Z : \mathbb{R}_Z[d] \otimes \mathbb{R}_Z[d] \to \mathbb{D}_Z$ is the perfect pairing given by usual real numbers multiplication. By Remark 2.5.20, we have that $R\varepsilon_*\mathbb{R}_Z[d]$ inherits a perfect pairing

$$S : R\varepsilon_*\mathbb{R}_Z[d] \otimes R\varepsilon_*\mathbb{R}_Z[d] \to \mathbb{D}_Y. \tag{5.23}$$

In this section, the complexes C appearing will be direct sums of intersection cohomology complexes associated with local systems. Given such a complex C and a subvariety $Y_j \subset Y$, we denote by C_{Y_j} the direct sum of those direct summands of C whose support is exactly Y_j.

Theorem 2.3.38 gives the direct sum decomposition

$$R\varepsilon_*\mathbb{R}_Z[d] \cong \bigoplus_{i=-M}^{M} {}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])[-i]. \tag{5.24}$$

The pairing (5.23) induces a perfect pairing

$${}^p\mathcal{H}^0(S) : {}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d]) \otimes {}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d]) \to \mathbb{D}_Y, \tag{5.25}$$

and, by Proposition 2.5.19 (see also Proposition 2.5.12), we have the equality of classes

$$[(R\varepsilon_*\mathbb{R}_Z[d], S)] = [({}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d]), {}^p\mathcal{H}^0(S))] \tag{5.26}$$

in $\Omega_{\mathbb{R}}(Y)$.

Let η be the first Chern class of a relative ample bundle for ε. Relative Hard-Lefschetz Theorem (see Theorem 2.3.37) is satisfied, that is, η induces isomorphisms

$$\eta^i : {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \to {}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d]), \tag{5.27}$$

and we have the direct sum decomposition

$${}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \cong \bigoplus_{l\geq 0} \mathcal{P}^{-i-2l}(R\varepsilon_*\mathbb{R}_Z[d]) \tag{5.28}$$

for every non-negative i, where $\mathcal{P}^{-i-2l}(R\varepsilon_*\mathbb{R}_Z[d])$ denotes the primitive part of ${}^p\mathcal{H}^{-i-2l}(R\varepsilon_*\mathbb{R}_Z[d])$. We remind that $\mathcal{P}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])$ is defined to be the kernel of $\eta^{i+1} : {}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \to {}^p\mathcal{H}^{i+2}(R\varepsilon_*\mathbb{R}_Z[d])$ in the abelian category of perverse sheaves (see Section 2.3.4). For $i = 0$, this decomposition is orthogonal for the self-duality (5.25).

The Decomposition Theorem also implies that each $\mathcal{P}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])$ is a direct sum of simple intersection cohomology complexes (see Section 2.3.4 and Section 2.3.3). Then, we have the decomposition

$$\mathcal{P}^{-i}(R\varepsilon_*\mathbb{R}_Z[d]) \cong \bigoplus_{j\in J} \mathcal{P}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \tag{5.29}$$

where $\{Y_j\}_{j\in J}$ is the collection of possible supports. Since any morphism between two simple perverse sheaves with different strict support vanishes (see Remark 2.3.32), this decomposition is also orthogonal for the pairing (5.25). As a consequence we get the following equality in $\Omega_{\mathbb{R}}(Y)$:

$$[({}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d]), {}^p\mathcal{H}^0(S))] = \sum_{j\in J}\sum_{l\geq 0}[(\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, S^l_{Y_j})], \tag{5.30}$$

where $S^l_{Y_j}$ denotes the restriction of the perfect pairing ${}^p\mathcal{H}^0(S)$ to the orthogonal direct summand $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$.

The complex $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$ is isomorphic to $IC_{Y_j}(\mathcal{L})$, for a certain local system $\mathcal{L}$ in a Zariski open subset U_j of Y_j. For each $y \in U_j$, we consider the inclusion $i_y : \{y\} \to Y_j$. We have the identification $i^!_y IC_{Y_j}(\mathcal{L})[\dim(Y_j)] \cong \mathcal{L}_y$ and applying the functor $i^!_y(-)[2\dim(Y_j)]$, we obtain a perfect pairing

$$Q_{\alpha_{Y_j},y} := i^!_y S^l_{Y_j}[2\dim(Y_j)] : \mathcal{L}_y \otimes \mathcal{L}_y \to \mathbb{R}.$$

In order to fix our convention, we follow the definition of polarization as Definition 2.2.11.

Lemma 5.3.1. *Define $d(j,i) := \dim(Z) - \dim(Y_j) - i$. For any point $y \in U_j$ the stalk $\mathcal{L}_y$ has a pure $\mathbb{R}$-Hodge structure of weight $d(j,2l)$ and*

$$(-1)^{(1/2)d(j,2l)(d(j,2l)+1)+(d-\dim(Y_j))\dim(Y_j)} Q_{\alpha_{Y_j},y}$$

is a polarization.

Remark 5.3.2. A proof of this lemma is also possible using M. Saito theory of Hodge modules as follows: by [74, Theorem 5.3.1] $IC_{Y_j}(\mathcal{L})$ underlies a polarized pure Hodge module, whose polarization is the perfect pairing $S^l_{Y_i}$ up to a sign which is precisely determined. Such a pure Hodge module corresponds to a polarized variation of Hodge structures whose local system is $\mathcal{L}$. A dictionary comparing the signs of polarizations of pure Hodge modules and polarizations of their corresponding variation of pure Hodge structures is provided in [74, 5.2.12]. Here, we have to notice that since in our convention for polarization we insert Weil's operator on the left, and in Saito's convention it is inserted in the right, one need to multiply by the extra sign $(-1)^w$, where w is the weight of the variation of pure Hodge structures. Then, the sign dictionary is the following: if a perfect pairing S induces a polarization of a variation of pure Hodge structures of weight w and support of pure dimension d, then $(-1)^{(1/2)d(d-1)+w}S$ induces a polarization of the corresponding pure Hodge module.

However, for our proof, no Hodge modules or variations of Hodge structures are really needed. For us it is enough to understand a sigle stalk of $\mathcal{L}$. So, below we prove the lemma using computations based on classical Hodge theory.

Proof of Lemma 5.3.1. First we prove the lemma for the case of the structure morphism $\varepsilon : Z \to Y = \{pt\}$. Due to the shift, the intersection form

$$Q : H^d(Z,\mathbb{R}) \otimes H^d(Z,\mathbb{R}) \to H^{2d}(X,\mathbb{R}) \cong \mathbb{R}$$

that this pairing induces equals $(-1)^d\langle -,-\rangle$, where

$$\langle -,-\rangle : H^d(Z,\mathbb{R}) \otimes H^d(Z,\mathbb{R}) \to H^{2d}(X,\mathbb{R}) \cong \mathbb{R}$$

is the usual intersection form, induced by the pairing $\mathbb{R}_Z \otimes \mathbb{R}_Z \to \mathbb{R}_Z$. Indeed, let $A_Z^\bullet$ be the de Rham complex of Z. We have a chain of isomorphisms

$$A_Z^\bullet[d] \otimes A_Z^\bullet[d] \cong A_Z^\bullet \otimes \mathbb{R}[d] \otimes A_Z^\bullet \otimes \mathbb{R}[d] \cong A_Z^\bullet \otimes A_Z^\bullet \otimes \mathbb{R}[d] \otimes \mathbb{R}[d] \cong A_Z^\bullet \otimes A_Z^\bullet \otimes \mathbb{R}[2d],$$

of which the second maps

$$\beta \otimes \lambda[d] \otimes \gamma \otimes \mu[d] \to (-1)^{dl} \beta \otimes \gamma \otimes \lambda[d] \otimes \mu[d]$$

for $\beta \otimes \lambda[d] \otimes \gamma \otimes \mu[d] \in A_Z^k \otimes \mathbb{R}[d] \otimes A_Z^l \otimes \mathbb{R}[d]$. This induces a $(-1)^{d^2} = (-1)^d$ sign comparing the pairings Q and $\langle -, - \rangle$.

In this case, the stalk $\mathcal{L}_y$ is identified with $\mathcal{P}^{d-2l}(Z)$, where $\mathcal{P}^{d-2l}(Z)$ denotes the η-primitive part of the cohomology $H^{d-2l}(Z, \mathbb{R})$. By the classical Hodge-Riemann bilinear relations (see Example 2.2.12), we have that $(-1)^{(1/2)(d-2l)(d-2l-1)} \langle -, - \rangle$ is a polarization of $\mathcal{P}^{d-2l}(Z)$, with the pure Hodge structure inherited from $H^{d-2l}(Z, \mathbb{R})$. The lemma holds in this case because $(-1)^d = (-1)^{d-2l}$ and

$$(1/2)(d-2l)(d-2l-1) + d - 2l = (1/2)(d-2l)(d-2l+1).$$

Now we consider the general case. Let $\iota \colon H_j \hookrightarrow Y$ be the inclusion map, where H_j is the intersection of $\dim(Y_j)$ generic hyperplanes in Y. By genericity, we have that

1. $Z_{H_j} := \varepsilon^{-1}(H_j)$ is smooth, and $\varepsilon|_{Z_{H_j}} : Z_{H_j} \to H_j$ is a resolution of singularities.

2. Set $c := \dim(Y_j)$. The intersection $Y_{j'} \cap H_j$ is of dimension $\dim(Y_{j'}) - c$, and empty if $c > \dim(Y_{j'})$. If the intersection is not empty, then $U_{j'} \cap H_j$ is dense in $Y_{j'} \cap H_j$.

3. There is a tubular neighborhood $T(H_j)$ in Y and a continuous retraction map $\pi : T(H_j) \to H_j$, such that π is topologically equivalent to a real vector bundle over H_j of rank $2c$, and such that for every $j' \in J$ such that $\dim(Y_{j'}) \geq c$, we have $Y_{j'} \cap T(H_j) = \pi^{-1}(Y_{j'} \cap H_j)$ and $U_{j'} \cap T(H_j) = \pi^{-1}(U_{j'} \cap H_j)$.

By the third property above and [31, 5.4.1, 5.4.3], we obtain that for every j' such that $\dim(Y_{j'}) \geq c$, the complex $\iota^! \mathcal{P}^{-2l}(R\varepsilon_* \mathbb{R}_Z[d])_{Y_{j'}}[\dim(Y_j)]$ is the intersection cohomology complex associated with the restriction to $U_{j'} \cap H_j$ of the local system corresponding to $\mathcal{P}^{-2l}(R\varepsilon_* \mathbb{R}_Z[d])_{Y_{j'}}[\dim(Y_j)]$.

By [31, 5.4.1], we have $\iota^! \mathbb{R}_Z[d][\dim(Y_j)] = \mathbb{R}_{Z_{H_j}}[d - \dim(Y_j)]$ and by applying (2.3), we obtain that $R(\varepsilon|_{Z_{H_j}})_* \mathbb{R}_{Z_{H_j}}[d - \dim(Y_j)] = \iota^! R\varepsilon_* \mathbb{R}_Z[d][\dim(Y_j)]$. Then, applying ${}^p\mathcal{H}^0(\iota^!(-)[2\dim(Y_j)])$ to the decompositions (5.24), (5.28) and (5.29), and noticing that $\iota^!(-)[2\dim(Y_j)]$ transforms the shifted perverse sheaves appearing in the decomposition into shifted perverse sheaves, we obtain the following decomposition

$$^p\mathcal{H}^0(R(\varepsilon|_{Z_{H_j}})_*\mathbb{R}_{Z_{H_j}}[d-\dim(Y_j)]) \cong \bigoplus_{l\geq 0}\bigoplus_{j'\in J}\iota^!\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y'_j}[\dim(Y_j)],$$

which is the η-primitive decomposition of $^p\mathcal{H}^0(R(\varepsilon|_{Z_{H_j}})_*\mathbb{R}_{Z_{H_j}}[d-\dim(Y_j)])$. We denote by $S^l_{Y_{j'}\cap H_j}$ the restriction of the perfect pairing $R(\varepsilon|_{Z_{H_j}})_*\sigma_{Z_{H_j}}$ to the orthogonal summand $\iota^!\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_{j'}}[\dim(Y_j)]$.

We have the equality of perfect pairings

$$S^l_{Y_{j'}\cap H_j} = (-1)^{(d-\dim(Y_j))\dim(Y_j)}\iota^! S^l_{Y_{j'}}[2\dim(Y_j)]. \tag{5.31}$$

This sign comes by a reason analogous to the sign comparing Q and $\langle -,-\rangle$ above, since the dimension of Z_{H_j} is $d-\dim(Y_j)$ and $\iota^!\mathbb{R}_Z[d][\dim(Y_j)] = \iota^*\mathbb{R}_Z[d][-\dim(Y_j)]$ by [31, 5.4.1]. Furthermore, the bilinear form $Q_{\alpha_{Y_{j'}},y}$ coincides with the bilinear form $Q_{\alpha_{Y_{j'}\cap H_j},y}$ associated with $e^!_y S^l_{Y_{j'}\cap H_j}$, up to the sign $(-1)^{(d-\dim(Y_j))\dim(Y_j)}$. This reduces the proof to the case in which Y_j equals a point.

Assume that $Y_j = \{y\}$. The vector space $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$ inherits a perfect pairing $S^l_{Y_j}$, which coincides with the bilinear form $Q_{\alpha_{Y_j},y}$ since $Y_j = \{y\}$. By [18, Corollary 2.1.7, Theorem 2.1.8], we have $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$ is a $\mathbb{R}$-Hodge structure of weight $d-2l$ and that $Q_{\alpha_{Y_j},y}$ is a polarization up to a sign. Below we reduce a self contained argument proving this and determining the sign.

Consider the structure morphisms $f: Z \to \{pt\}$ and $g: Y \to \{pt\}$. Notice that $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_y$ is a direct summand both $R\varepsilon_*\mathbb{R}_Z[d]$ supported at the point y and of the primitive part $\mathcal{P}^{d-2l}(Z)$ of $H^{d-2l}(Z,\mathbb{R}) = R^{-2l}f_*\mathbb{R}_Z[d]$. Since we have proven the lemma for the case of the structure morphims, the perfect pairing $Rf_*\sigma_Z$ induces a bilinear form Q'_y on $\mathcal{P}^{d-2l}(Z)$ such that $(-1)^{(1/2)(d-2l)(d-2l+1)}Q'_y$ is a polarization. Then the proof of the lemma is finished because the restriction of Q'_y to $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_y$ coincides with $Q_{\alpha_{Y_j},y}$. The last claim holds because we have the equality of perfect pairings

$$Rf_*\sigma_Z = Rg_*R\varepsilon_*\sigma_Z,$$

(see Theorem 2.3.2) and since $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_y$ is supported at a point y the functor Rg_* restricted to it takes global sections and identifies the perfect pairing $Rf_*\sigma_Z$ with the restriction of $R\varepsilon_*\sigma_Z$ to $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_y$.

□

Remark 5.3.3. The use of hyperplane sections in Lemma 5.3.1 does not force projectivity assumptions. Indeed, the statement is local in Y, and Y can be covered by affine patches that can be completed to projective varieties for which the proof works (the completion is needed because the compactness of the resolution Z is used in the last part of the proof).

Let $\mathcal{V}$ be a semi-simple $\mathbb{R}$-perverse sheaf with strict support in an irreducible variety Y (that is, the support of any of its simple components is Y). Let U be a Zariski open subset such that $\mathcal{V}|_U = \mathcal{L}$ where $\mathcal{L}$ is a $\mathbb{R}$-local system. Assume that for a given $y \in U$ the fibre $\mathcal{L}_y$ is endowed with a pure Hodge structure.

Definition 5.3.4. Let $\alpha : \mathcal{V} \otimes \mathcal{V} \to \mathbb{D}_Y$ be a perfect paring. Then, α is called a *polarizing self-duality for the Hodge structure at* y if α induces a polarization $Q_{\alpha,y} : \mathcal{L}_y \times \mathcal{L}_y \to \mathbb{R}$ of the Hodge structure. If the negative of the self-duality is polarizing, then we say that the self-duality is (-1)*-polarizing for the Hodge structure at* y.

Remark 5.3.5. Lemma 5.3.1 states that the pairing $S^l_{Y_j}$ on $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$ induces a $(-1)^{(1/2)d(j,2l)(d(j,2l)+1)+(d-\dim(Y_j))\dim(Y_j)}$-polarizing self-duality for the Hodge structure induced at $\mathcal{L}_y$ for any point $y \in U_j$. In other words, the fact that $S^l_{Y_j}$ is polarizing or (-1)-polarizing does not depend on the point $y \in U_j$.

Lemma 5.3.6. *Let α and α' be polarizing self-dualities of $\mathcal{V}$ for the same Hodge structure at y. Then $(\mathcal{V}, \alpha)$ and $(\mathcal{V}, \alpha')$ represent the same element in $\Omega_{\mathbb{R}}(Y)$.*

Proof. For any $s \in [0, 1]$ the morphism $s\alpha + (1-s)\alpha'$ is a polarizing self-duality. Indeed, since both α and α' are polarizing there exits a common open subset U of Y such that $Q_{\alpha,y}$ and $Q_{\alpha',y}$ are polarizations for any $y \in U$. Therefore an straightforward check of the conditions of Definition 2.2.11 imply that for any $y \in U$, the bilinear form

$$Q_{s\alpha+(1-s)\alpha',y} = sQ_{\alpha,y} + (1-s)Q_{\alpha',y}$$

is a polarization of the Hodge structure $\mathcal{L}_y$. This implies that $Q_{s\alpha+(1-s)\alpha',y}$ is non-degenerate for any s. Then, in order to complete the proof it is enough to use Proposition 5.3.8 below. □

Remark 5.3.7. The equality proved in the previous Lemma holds for all the possible definitions of $\Omega_{\mathbb{R}}(Y)$ mentioned in Section 2.5.2.

Proposition 5.3.8. *Let $\mathcal{L}$ be a $\mathbb{R}$-local system on U and Q_s a smooth* 1*-parameter family of non-degenerate symmetric or anti-simmetric pairings of $\mathcal{L}$ parametrized by an interval I. Then I admits an open cover $I = \cup_{j \in J} I_j$ such that for any j there exists a smooth family of automorphisms $T(s)$ of $\mathcal{L}$*

, $s \in I_j$, so that $Q_s(T(s)(-), T(s)(-))$ is independent of s. Consequently, for any $s, s' \in I$ there is an automorphism $T_{s,s'}$ of $\mathcal{L}$ such that

$$Q_{s'}(T_{s,s'}(-), T_{s,s'}(-)) = Q_s(-,-).$$

Proof. The local system $\mathcal{L}$ is a representation $\rho : \pi_1(U, y) \to GL(\mathcal{L}_y)$ which is orthogonal for $Q_{s,y}$ for any s.

By Gram-Schmidt process in the symmetric case, and by the proof of uniqueness of non-degenerate anti-symmetric real bilinear forms, for any $s_0 \in I$ there exists a neighborhood I_{s_0} of s_0 in I and a smooth family of automorphisms $N(s)$, $s \in I_{s_0}$ (not necessarily compatible with the monodromy) such that $N(s_0) = Id$ and

$$Q_{s,y}(N(s)(-), N(s)(-)) = Q_{s_0,y}(-,-)$$

for all s. Considering $M_{\gamma(s)} := N(s)^{-1}\rho(\gamma)N(s)$ we obtain a smooth family of real orthogonal representations for $Q_{s_0,y}$.

If $Q_{s_0,y}$ is symmetric and (n,m) is its signature of $Q_{s_0,y}$ we define W to be the diagonal matrix $I_{n,m}$ be the diagonal matrix of size $n+m$ such that the (i,i) component equals 1 if $i \leq n$ and -1 if $i > n$. If $Q_{s_0,y}$ is anti-symmetric the rank of the local system is even and we define W to be the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{5.32}$$

where I denotes the identity matrix. Denote $O(W)$ the orthogonal group for the bilinear form W. Then, the proposition is reduced to the following claim:

Claim: Let $M_\gamma(s) : \pi_1(U_j, y) \to O(W)$ be a smooth family of orthogonal representations for the quadratic form W. If there exists a family $N(s)$ of invertible matrices such that $N(0) = Id$ and we have the conjugation $M_\gamma(s) = (N(s))^{-1}M_\gamma(0)N(s)$ for any $\gamma \in \pi_1(U_j, y)$, then there exists a family $P(s)$ of orthogonal matrices such that $M_\gamma(s) = (P(s))^{-1}M_\gamma(0)P(s)$ for any $\gamma \in \pi_1(U_j, y)$.

In order to make the proof of the claim, we recall some facts about Lie groups (it can be considered only the case of matrix Lie groups, this is the case that will be used). See [48] for generalities on smooth manifolds, and Lie groups; flows associated to time dependent vector fields are discussed in Exercise 12-7 of loc. cit.

Let G be a matrix Lie group. Its Lie algebra $\mathfrak{g}$ is identified with the spaces of left-invariant vector fields. Denote by $L_g : G \to G$ the left multiplication by g. We denote the set of smooth paths $g : [0,1] \to G$ by $C^\infty([0,1], G)$. A path $v(t) \in C^\infty([0,1], \mathfrak{g})$ is viewed as a left-invariant time dependent vector field, that is, a vector field in $G \times [0,1]$ such that its $[0,1]$-component is the unit vector field $\frac{\partial}{\partial t}$ in positive direction, and such that its G-component is invariant by the action of G by left multiplication. We say that $v(t)$ is *integrable* if the integral flow associated with it is defined in the domain $G \times [0,1]$. In order to check integrability, by left invariance, it is enough to check the existence of an integral curve whose domain is $[0,1]$. We denote by $C^\infty([0,1],\mathfrak{g})^{int}$ the set of maps giving rise to integrable left invariant time dependent vector fields.

We define the *left bijection*

$$\mathcal{L} : C^\infty([0,1],G) \to G \times C^\infty([0,1],\mathfrak{g})^{int}$$

as follows. Given a smooth path $g : [0,1] \to G$ we define $g' : [0,1] \to \mathfrak{g}$ by the formula

$$g'(s) := DL_{g(s)^{-1}}(g(s))(\frac{dg(u)}{du}|_{u=s}).$$

Given $g(s) \in C^\infty([0,1],G)$ we define $\mathcal{L}(g(s))$ to be the pair $(g(0),g'(s))$. Conversely, given a pair $(g_0,v(t)) \in G \times C^\infty([0,1],\mathfrak{g})^{int}$, we view $v(t)$ as a left-invariant time dependent vector field and define $\mathcal{L}^{-1}(g_0,v(t))$ to be the unique integral curve of the time dependent vector field $v(t)$ with initial point $g(0) = g_0$.

The set $C^\infty([0,1],G)$ has a group structure. Given $h \in G$ and $g(s) \in C^\infty([0,1],G)$, define $h(s) := g(s)^{-1}hg(s)$, a straightforward Lie group computation shows the formula

$$h'(s) = g'(s) - h(s)^{-1}g'(s)h(s). \tag{5.33}$$

Indeed, since left multiplcation by a matrix is a linear transformation at the space of matrices we may write

$$DL_{g(s)^{-1}}(g(s))(\frac{dg(u)}{du}|_{u=s}) = g(s)^{-1}(\frac{dg(u)}{du}|_{u=s}).$$

Using this, Leibnitz rule for derivation and the formula for the derivation of the inverse we obtain

$$\begin{aligned}
&h'(s) = h(s)^{-1}(\frac{dh(u)}{du}|_{u=s}) = \\
&= h(s)^{-1}(-g(s)^{-1}(\frac{dg(u)}{du}|_{u=s})g(s)^{-1}hg(s) + g(s)^{-1}h(\frac{dg(u)}{du}|_{u=s})) = \\
&= -h(s)^{-1}g(s)^{-1}(\frac{dg(u)}{du}|_{u=s})g(s)^{-1}hg(s) + g(s)^{-1}h^{-1}g(s)g(s)^{-1}h(\frac{dg(u)}{du}|_{u=s}) = \\
&= -h(s)^{-1}g'(s)h(s) + g'(s).
\end{aligned}$$

Let W be $(-1)^\beta$-symmetric. Then we have $W^2 = (-1)^\beta Id$ and the Lie algebra $\mathfrak{o}(W)$ of the Lie group is $O(W)$ the subspace of matrices N satisfying

$$(-1)^{\beta+1}WN^tW = N.$$

So $\mathfrak{o}(W)$ is the eigenspace for eigenvalue 1 of the involution $N \mapsto (-1)^{\beta+1}WN^tW$ in the real vector space of square matrices. The vector space of square matrices splits as the direct sum of the eigenspaces with eigenvalues $+1$ and -1

respectively for the involution. Given any square matrix N we decompose it accordingly as $N = N_+ + N_-$.

Formula (5.33) applied to $M_\gamma(s) = N(s)^{-1}M_\gamma(0)N(s)$ yields

$$M'_\gamma(s) = N'(s) - M_\gamma(s)^{-1}N'(s)M_\gamma(s).$$

We have

$$M'_\gamma(s) = M'_\gamma(s)_+ = N'(s)_+ - M_\gamma(s)^{-1}N'(s)_+M_\gamma(s). \tag{5.34}$$

The first equality is because $M_\gamma(s)$ belongs to $O(W)$ for all s. For the second equality write $N'(s) = N'(s)_+ + N'(s)_-$. It is enough to show the equalities

$$(M_\gamma(s)^{-1}N'(s)M_\gamma(s))_+ = M_\gamma(s)^{-1}N'(s)_+M_\gamma(s),$$
$$(M_\gamma(s)^{-1}N'(s)M_\gamma(s))_- = M_\gamma(s)^{-1}N'(s)_-M_\gamma(s),$$

but they follow from an elementary matrix computation using that since $M_\gamma(s) \in O(W)$ we have

$$M_\gamma(s)^{-1} = (-1)^\beta W M_\gamma(s)^t W.$$

Define $P(s) := \mathcal{L}^{-1}(Id, N'(s)_+)$. Since $N'(s)_+ \in \mathfrak{o}(W)$ we have $P(s) \in O(W)$. The equality $M_\gamma(s) = (P(s))^{-1}M_\gamma(0)P(s)$ is obtained by applying $\mathcal{L}^{-1}(Id, -)$ to Equation (5.34). □

In the following Lemma we will use the following notation: certain semi-simple perverse sheaves with strict support $\mathcal{V}$ are endowed with polarizable Hodge structures at their stalks at generic points by Lemma 5.3.1. We denote by $[\mathcal{V}, \oplus]$ the class in $\Omega_{\mathbb{R}}(Y)$ represented by $\mathcal{V}$ together with a polarizing self-duality (this definition makes sense by Remark 5.3.5 and Lemma 5.3.6). We denote by $[\mathcal{V}, \ominus]$ the class with a (-1)-polarizing self-duality. If a semi-simple perverse $\mathcal{W}$ together with a self duality sheaves is a direct sum of polarizing semi-simple perverse sheaves with strict support, then we denote by $[\mathcal{W}, \oplus]$ its class in $\Omega_{\mathbb{R}}(Y)$; we denote by $[\mathcal{W}, \ominus]$ the class of the opposite self-duality.

Lemma 5.3.9. *Define $\beta_{d,j} := (d - \dim(Y_j))\dim(Y_j)$. We have the following equality in $\Omega_{\mathbb{R}}(Y)$:*

$$\begin{aligned}
&[({}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d]), {}^p\mathcal{H}^0(S))] = \\
&= \sum_{j\in J}\sum_{i=-M}^{M}(-1)^{(1/2)d(j,-i)(d(j,-i)+1)+\beta_{d,j}}[{}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus] = \\
&= \sum_{j\in J}\Big(\sum_{i=even}(-1)^{(1/2)d(j,-i)(d(j,-i)+1)+\beta_{d,j}}[{}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus]+ \\
&\quad + \sum_{i=odd}(-1)^{(1/2)d(j,-i)(d(j,-i)+1)+\beta_{d,j}}[{}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \ominus]\Big).
\end{aligned} \tag{5.35}$$

Proof. For each $j \in J$, the Equation (5.28) gives the decomposition

$$ {}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j} \simeq \bigoplus_{l\geq 0} \mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j} $$

where the decomposition is orthogonal for the perfect pairing of ${}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$, and the perfect pairing of $\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$ induced from the perfect pairing of ${}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}$ is $(-1)^{(1/2)d(j,2l)(d(j,2l)+1)+\beta_{d,j}}$-polarizing by Lemma 5.3.1. Then we have the equality

$$ [{}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}] = \sum_{l\geq 0}(-1)^{(1/2)d(j,2l)(d(j,2l)+1)+\beta_{d,j}}[\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus]. \tag{5.36} $$

in $\Omega_{\mathbb{R}}(Y)$.

By Equations (5.27) and (5.28), we have the equality

$$ [{}^p\mathcal{H}^{-i}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus] = [{}^p\mathcal{H}^{i}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus] = \sum_{l\geq 0}[\mathcal{P}^{-i-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus] $$

in $\Omega_{\mathbb{R}}(Y)$ for any $i \geq 0$.

Plugging this equalities into the middle term of (5.35), making the needed *cancellations* and comparing with the left hand side of (5.35) expressed as in (5.36), the first equality of (5.35) follows.

We show here part of the cancellation process. We write

$$ {}^p\mathcal{H}^i := [{}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus] \quad \text{and} \quad \mathcal{P}^{-2l} := [\mathcal{P}^{-2l}(R\varepsilon_*\mathbb{R}_Z[d])_{Y_j}, \oplus]. $$

Then, using Hard-Lefschetz Theorem, we have

$$ \begin{aligned} &(-1)^{(1/2)d(j,0)(d(j,0)+1)+\beta_{d,j}} \sum_{i=-M}^{M} (-1)^{(1/2)d(j,-2i)(d(j,-2i)+1)+\beta_{d,j}}\,{}^p\mathcal{H}^{2i} = \\ &= {}^p\mathcal{H}^0 - 2\,{}^p\mathcal{H}^{-2} + 2\,{}^p\mathcal{H}^{-4} - 2\,{}^p\mathcal{H}^{-6} + ... = \\ &= (\mathcal{P}^0 + \mathcal{P}^{-2} + \mathcal{P}^{-4} + \mathcal{P}^{-6} + ...) - 2(\mathcal{P}^{-2} + \mathcal{P}^{-4} + \mathcal{P}^{-6} + ...) + \\ &+ 2(\mathcal{P}^{-4} + \mathcal{P}^{-6} + ...) - 2(\mathcal{P}^{-6} + ...) + ... = \\ &= \mathcal{P}^0 - \big((\mathcal{P}^{-2} + \mathcal{P}^{-4} + \mathcal{P}^{-6} + ...) - 2(\mathcal{P}^{-4} + \mathcal{P}^{-6} + ...) + \\ &+ 2(\mathcal{P}^{-6} + ...) - ...\big) = \mathcal{P}^0 - \mathcal{P}^{-2} + \mathcal{P}^{-4} - \mathcal{P}^{-6} + ... = \\ &= (-1)^{(1/2)d(j,0)(d(j,0)+1)+\beta_{d,j}} \sum_{l\geq 0}(-1)^{(1/2)d(j,2l)(d(j,2l)+1)+\beta_{d,j}}\mathcal{P}^{-2l}. \end{aligned} $$

This shows the cancellations in the middle term of formula (5.35) when i is even. The i odd part of the middle term of formula (5.35) cancels completely by a similar process, and by this complete cancellation the equality with the right hand side follows. □

5.3.1. Proof of Theorem 5.0.2

Here, we conclude the proof of Theorem 5.0.2. We will show that the difference of cobordism classes $sd_{\mathbb{R}}([Y]) - [IC_Y]$ vanishes after applying the obtained results in the previous sections.

For any k, the variety X_k is a disjoint union of smooth varieties of different dimensions. By d_k we denote the function that assigns to each connected component of X_k its dimension, and given a complex of sheaves C on X_k we denote by $C[d_k]$ the same complex, shifted at the dimension in each connected component.

We have to prove

$$sd_{\mathbb{R}}([Y]) - [IC_Y] = 0$$

in $\Omega_{\mathbb{R}}(Y)$. Indeed, Equation (5.5) in Lemma 5.1.1 implies

$$sd_{\mathbb{R}}([Y]) = sd_{\mathbb{R}}([\tilde{Y}]) + \sum_{i=1}^{n} sd_{\mathbb{R}}([X_{0,i}]) + \sum_{k=1}^{n}(-1)^k sd_{\mathbb{R}}([X_k]). \tag{5.37}$$

Since at the cobordism group $\Omega_{\mathbb{R}}(Y)$ only the 0-th perverse cohomology matters, by applying Equation (5.26), we also have

$$sd_{\mathbb{R}}([\tilde{Y}]) + \sum_{i=1}^{n} sd_{\mathbb{R}}([X_{0,i}]) = [IC_Y] + \sum_{j=1}^{N}[R\varepsilon_*\mathbb{R}_{X_0}[d_0]_{\Sigma_j}] =$$

$$= [IC_Y] + \sum_{j=1}^{N}[{}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_{X_0}[d_0])_{\Sigma_j}]$$

and

$$sd_{\mathbb{R}}([X_k]) = \sum_{j=1}^{N}[R\varepsilon_*\mathbb{R}_{X_k}[d_k]_{\Sigma_j}] = \sum_{j=1}^{N}[{}^p\mathcal{H}^0(R\varepsilon_*\mathbb{R}_{X_k}[d_k])_{\Sigma_j}]$$

for every $k > 0$.

Substituting the above expressions in Equation (5.37) and by applying Lemma 5.3.9, we obtain

$$sd_{\mathbb{R}}([Y]) - [IC_Y] =$$

$$= \sum_{j=1}^{N} \sum_{k=0}^{n} \sum_{i=-M_k}^{M_k} (-1)^{k+(1/2)(d_k-d_j+i)(d_k-d_j+i+1)+\beta_{d_k,j}} [{}^p\mathcal{H}^i(R\varepsilon_*\mathbb{R}_{X_k}[d_k])_{\Sigma_j}, \oplus] =$$

$$= \sum_{j=1}^{N} \sum_{k=0}^{n} \sum_{q=0}^{2d_k} (-1)^{k+(1/2)(q-d_j)(q-d_j+1)+\beta_{d_k,j}} [{}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})_{\Sigma_j}, \oplus] =$$

$$= \sum_{j=1}^{N} \sum_{k=0}^{n} \sum_{q-d_k=\text{even}} (-1)^{k+(1/2)(q-d_j)(q-d_j+1)+\beta_{d_k,j}} [{}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})_{\Sigma_j}, \oplus] +$$

$$+ \sum_{j=1}^{N} \sum_{k=0}^{n} \sum_{q-d_k=\text{odd}} (-1)^{k+(1/2)(q-d_j)(q-d_j+1)+(d_k-d_j+1)d_j} [{}^p\mathcal{H}^q(R\varepsilon_*\mathbb{R}_{X_k})_{\Sigma_j}, \oplus].$$

The last equality uses the last equality of Equation (5.35) when d_j is odd.

The proof concludes noticing that for any q, k, k' such that $d_k - q$ is even and $d_{k'} - q$ is odd the signs $(-1)^{\beta_{d_k,j}}$ and $(-1)^{(d_k-d_j+1)d_j}$ coincide. Then, since the sign $(-1)^{(1/2)(q-d_j)(q-d_j+1)}$ is constant for q and Σ_j fixed, one can use the exact sequences of Lemma 5.2.2 for each support Σ_j and any q and conclude using Lemma 5.3.6.

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