UNIVERSITÀ DI TRENTO
Dottorato in Matematica

UNIVERSIDAD DEL PAÍS VASCO - EUSKAL HERRIKO UNIBERTSITATEA
Doctorado en Matemáticas y Estadística

Ph.D. Thesis

Some questions regarding groups of automorphisms of primary trees

by

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A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the
University of Trento and University of the Basque Country

Trento, 2022

Declaration of Authorship

I, Elena Di Domenico, declare that this thesis titled, 'Some questions regarding groups of automorphisms of primary trees' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

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Groups acting on p-adic trees have been well studied over the past decades since they represent a source of examples with interesting properties in group theory. Groups with intermediate growth or counterexamples to the General Burnside Problem can be found inside this class of groups. In this thesis we analyze some properties concerning the structures of two families of groups acting over primary regular rooted trees, i.e. regular rooted trees such that every vertex has a number of descendants equal to a power of a prime. These two families are the GGS-groups acting over the p^n-adic tree and the p-Basilica groups, a generalization of the Basilica group over the p-adic tree for a prime p. A GGS-group over the p^n-adic tree is defined by a vector $\mathbf{e}$ whose components are elements in $\mathbb{Z}/p^n\mathbb{Z}$. Depending on the defining vector of the GGS-group, we determine which of them are branch. We reduce our study to the fractal GGS-groups, since the non-fractal ones cannot be branch. We prove that all of them, except the ones acting over the 2^n-adic tree whose defining vectors have only one invertible component in position 2^{n-1}, are weakly regular branch. The GGS-groups with constant defining vector are weakly regular branch but not branch. For the other GGS-groups, we prove that they are all regular branch with some small exceptions for which the question is still open.

The p-Basilica groups are weakly branch but not branch for any prime p. These provide the first examples of groups with these properties that are super strongly fractal. For this class of groups we study also other problems. We show that they have the p-congruence subgroup property but not the congruence subgroup property nor the weak congruence subgroup property, providing the first examples of weakly branch groups with such properties. We compute the orders of the congruence quotients of these groups, which enables us to determine the Hausdorff dimensions of the p-Basilica groups. Lastly, we show that the p-Basilica groups do not possess maximal subgroups of infinite index and that they have infinitely many non-normal maximal subgroups.

Sommario. I gruppi che agiscono su alberi p-adici sono stati ampiamente studiati negli ultimi decenni poiché rappresentano una fonte di esempi con interessanti proprietà nella teoria dei gruppi. All'interno di questa classe di gruppi si possono trovare esempi di gruppi con crescita intermedia o controesempi al Problema Generale di Burnside. In questa tesi analizzeremo alcune proprietà riguardanti le strutture di due famiglie di gruppi che agiscono su alberi regolari di grado pari ad una potenza di un primo. Queste due famiglie sono i gruppi GGS che agiscono sull'albero p^n-adico e i gruppi p-Basilica, una generalizzazione del gruppo Basilica agli alberi p-adici dove p è un primo. Un gruppo GGS sull'albero p^n-adico è definito da un vettore $\mathbf{e}$ con $p^n - 1$ componenti in $\mathbb{Z}/p^n\,\mathbb{Z}$. A seconda del vettore che definisce il gruppo GGS, determiniamo quali di essi sono branch. Abbiamo ridotto il nostro studio ai gruppi GGS frattali, poiché quelli non frattali non possono essere branch per definizione. Abbiamo dimostrato che tutti, eccetto i GGS che agiscono sull'albero 2^n-adico la cui unica componente invertibile nel vettore di definizione è in posizione 2^{n-1}, sono debolmente branch regolari. I gruppi GGS con vettore costante sono debolmente branch regolari ma non branch. Per gli altri gruppi GGS, abbiamo dimostrato che sono tutti branch regolari con qualche piccola eccezione per le quali il problema è ancora aperto. I gruppi p-Basilica sono debolmente branch regolari ma non branch per ogni primo p, e rappresentano il primo esempio di gruppi super fortemente frattali con questa proprietà. Questa classe di gruppi è anche il primo esempio di gruppi debolmente branch con la p-proprietà del sottogruppo di congruenza ma senza la proprietà del sottogruppo di congruenza e senza la proprietà debole del sottogruppo di congruenza. Inoltre il calcolo degli ordini dei quozienti di congruenza di questi gruppi ci permettono di determinare la dimensione di Hausdorff dei gruppi p-Basilica. Infine, mostriamo che i gruppi p-Basilica non possiedono sottogruppi massimali di indice infinito e che hanno infiniti sottogruppi massimali non normali.

Resumen. Los grupos que actúan sobre árboles p-ádicos han sido extensamente estudiados durante las últimas décadas dado que representan una fuente de ejemplos con interesantes propiedades en teoría de grupos. Dentro de esta clase de grupos se pueden encontrar ejemplos de grupos con crecimiento intermedio o contraejemplos al Problema General de Burnside. En esta tesis analizamos algunas propiedades con respecto a las estructuras de dos familias de grupos que actúan sobre árboles regulares primarios con raíz, es decir, árboles con raíces tales que cada vértice tiene un número de descendientes igual a una potencia de un primo. Estas dos familias son los grupos GGS que actúan sobre el árbol p^n-ádico y los grupos p-Basílica, una generalización del grupo Basílica sobre el árbol p-ádico para un primo p. Un grupo GGS sobre el árbol p^n-ádico está definido por un vector $\mathbf{e}$ cuyas componentes son elementos en $\mathbb{Z}/p^n\mathbb{Z}$. Dependiendo del vector de definición del grupo GGS, determinamos cuáles de ellos son ramificados. Reducimos nuestro estudio a los grupos GGS fractales, ya que los no fractales no pueden ser ramificados. Probamos que todos ellos, excepto los GGS que actúan sobre el árbol 2^n-ádico cuyos vectores de definición tienen solo una componente invertible en la posición 2^{n-1}, son débilmente ramificados regulares. Los grupos GGS con un vector de definición constante son débilmente ramificados regulares pero no son ramificados regulares. Para los otros grupos GGS, probamos que todos son ramificados regulares con algunas pequeñas excepciones para las cuales el problema aún está abierto. Los grupos p-Basílica son débilmente ramificados pero no son ramificados para ningún primo p. Estos son los primeros ejemplos de grupos con estas propiedades que son súper fuertemente fractales. Para esta clase de grupos estudiamos también otros problemas. Mostramos que tienen la p-propiedad de subgrupos de congruencia pero no la propiedad de subgrupos de congruencia ni la propiedad débil de subgrupos de congruencia, proporcionando los primeros ejemplos de grupos débilmente ramificados con tales propiedades. Calculamos los órdenes de los cocientes de congruencia de estos grupos, lo que nos permite determinar las dimensiones de Hausdorff de los grupos p-Basílica. Por último, mostramos que los grupos p-Basílica no poseen subgrupos maximales de índice infinito y que tienen infinitos subgrupos maximales no normales.

Acknowledgements

I would like to thank all the people who shared this special moment of my life with me.
First of all, I thank my advisors Andrea and Gustavo. Thank you Andrea for the continuous support to my research, and for encouraging me during my study. Thank you Gustavo for the work done, for your constant support, for all suggestions and for helping me improve all works. Thanks to both of you I had the opportunity to discover new fields of research and enjoy this work. Thanks for everything.
I would like to thank my referees and thesis committee. Thanks to Agnieszka, Leire, Carlo and Antonio for correcting my thesis and for giving me valuable feedback and comments.
This journey would not have been the same without the friends I met.
Thank you Alberto, Mattia, Marco, Francesco and Agnese, and to the whole department of Trento. Thank you for the company, for the evenings together and for the discussions that allowed us to make important decisions for our future.
A special thanks to the whole group of the department in Bilbao. Marialaura, Bruno, Natalia, Matteo, Xuban, Iker, Andoni... my path would not have been the same without having met you. Every moment was special for some reason, from the lunches together, to the conferences spent abroad, to the evenings together for relaxing and having dinners. Your sympathy and your enthusiasm have made this journey unforgettable.
I wish to thank Oihana, Federico, Albert, Jone, Şükran, Matteo, for the suggestions, for the help, for organizing the seminars and for encouraging me to give talks.
I would like to thank Marithania, Jordi, Sasha, Lander, Igor, Javier, Urban, Sheila and also the friends met at BCAM: Javi C., Javi M., Daniel, David. Thank you for the fun moments we had together.
Thank you Marialaura for your enthusiasm, your happiness and for encouraging me to do more. I realized I could count on you as a friend from the first day I met you, already smiling and open. Thanks for the travels we enjoyed together, for all the advice and for always thinking of me. And thanks to your family for welcoming me to Salerno and for being available to help me at any time.
Thank you Carmine for the research and for the fun days we spent during the conferences.

Thank you Anitha, my stay in Lincoln was an opportunity for me to learn a lot and proceed with research. All this combined with the beautiful days spent walking around Lincoln with you and Harald. Thanks to both of you.
Norberto, thank you for encouraging me, for always finding the time to work with me and for involving me in other activities related to research and mathematics.
Thanks to the Department of L'Aquila where my journey started.
Thank you Montse and Ilya, your activity has made the department really enjoyable, thanks to you there was always time to give us the opportunity to enrich our knowledge and meet researchers.
Leire, thank you for always listening to me and for your feedback after the talks.
Thank you Jon for your help and for spending every minute in Madrid trying to find a solution.
Thank you Naiara for the nice evenings spent talking at home.
Silvana and Francis, thank you for the days spent chatting together while enjoying a tea, and for the wonderful trips around Trento.
Antonella, thank you for sharing this beautiful moment of growth with me.
Giulia and Viviana, I could not imagine my life without having met you. Thank you for the wonderful moments spent together, for the organized trips, for listening to me and for all the advice you have given me. And thanks also to your families who welcomed me and loved me.
Last but not least, I would like to thank my family. Whatever I tried to write wouldn't be enough. Thanks for the esteem, for the support in all my choices, for having always believed in me. For all the opportunities you have given me and for all that you have been able to achieve.

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To my family

Introduction

Groups of automorphisms of regular rooted trees are a source of groups with interesting properties in group theory. The first Grigorchuk group, defined by Rostislav Grigorchuk in 1980 [28], is the first example in this family of an infinite finitely generated periodic group, providing a counterexample to the General Burnside Problem. It is also the first example of a group with intermediate growth [29], solving the Milnor Problem [14]. Later on many examples of groups of tree automorphisms, together with many generalizations of the Grigorchuk group, were defined and studied. Important examples are the family of Gupta-Sidki groups [36] and the second Grigorchuk group [28]. The Grigorchuk-Gupta-Sidki groups, also called GGS-groups, provide a further generalization. A necessary and sufficient condition for a GGS-group to be a counterexample to the General Burnside Problem, in the case when m is a prime power, has been given by Vovkivsky [56].

Another well known example of group of automorphisms of a binary tree is the Basilica group introduced by Grigorchuk and Żuk in [34]. This group is torsion-free, of exponential growth and it was the first example of an amenable but not subexponentially amenable group.

An automorphism f of a regular rooted tree $\mathcal{T}$ of degree m can be identified with its isomorphic images under ψ defined as follows

$$\begin{array}{rccc} \psi: & \operatorname{Aut}\mathcal{T} & \to & \operatorname{Aut}\mathcal{T} \wr \operatorname{Sym}(m) \\ & f & \to & \psi(f) = (f_1, \dots, f_m)\tau. \end{array}$$

where τ is a permutation in $\operatorname{Sym}(m)$ representing the action of f on the first level of $\mathcal{T}$ and $f_1, \dots, f_m$ are automorphisms in $\operatorname{Aut}\mathcal{T}$ that represent the action of f on the subtrees $\mathcal{T}_i$ hanging from the vertices of the first level. When f stabilizes the vertices of the first level, i.e when τ is trivial, the automorphism f can be identified with the vector $(f_1, \dots, f_m) \in \operatorname{Aut}\mathcal{T} \times \overset{m}{\cdots} \times \operatorname{Aut}\mathcal{T}$, thus the group $\operatorname{Aut}\mathcal{T}$

contains an isomorphic copy of the direct product of itself m times. One of the most studied problems in the literature regarding groups of automorphisms of a tree, is to determine which of them are branch, where branchness is a measure of how close the structure of the group resembles the structure of the full automorphism group $\operatorname{Aut}\mathcal{T}$ of the tree.

A subgroup G of $\operatorname{Aut}\mathcal{T}$ is called *weakly regular branch* over a subgroup K if K contains the direct product of itself as an isomorphic image. The group G is called *regular branch* over K if it is weakly regular branch over K and $|G : K| < \infty$. Regular branch groups are also branch groups. The definition of branch groups was first given by Grigorchuk in his talk at the Groups St. Andrews Conference in Bath in 1997. A subgroup G of $\operatorname{Aut}\mathcal{T}$ is called *weakly branch* if for every $n \in \mathbb{N}$ the n-th rigid stabilizers are non-trivial, where the rigid stabilizer is defined as the biggest subgroup of G that maps under ψ onto a direct product of m^n copies of some group. If for all $n \in \mathbb{N}$ the n-th rigid stabilizer has finite index in the group G, the group G is called *branch*.

This thesis is devoted to the study of two families of groups generalizing the GGS-groups and the Basilica group.

Chapter 1 is devoted to definitions and known results about groups acting on regular rooted trees. After giving the definition of groups of automorphisms of regular rooted trees and after seeing several ways to describe them, we see properties related to the structures of these groups. In particular, we define self-similar and fractal groups, we see the definitions of the four types of branch structures mentioned above, and we identify isomorphisms that preserve these branch structures. In this preliminary chapter are also included definitions and basic properties regarding growth and amenability. We will also give the definitions of some generalizations of the congruence subgroup property. The congruence subgroup property asks whether every finite index subgroup contains some level stabilizer, and consequently the completion with respect to the topology defined by the basis $\{\mathrm{st}_G(m) \mid m \in \mathbb{N}\}$, called the congruence completion, coincides with the profinite completion.

This property holds for some well-known examples of groups acting on rooted trees, such as the Grigorchuk group [6] and GGS-groups over the p-adic tree with non-constant defining vectors [20].

Most of the well known subgroups of $\operatorname{Aut}\mathcal{T}$ are groups acting over the p-adic tree for a prime p, and they are often subgroups of the pro-p subgroup Γ of $\operatorname{Aut}\mathcal{T}$

that coincides with the following iterated wreath product

$$\Gamma \cong \varprojlim_{n\in\mathbb{N}} C_p \wr \overset{n}{\cdots} \wr C_p.$$

Thus all the stabilizers have index in the group equal to a power of p, while a normal subgroup need not have a p-power index. Thus when G does not have the congruence subgroup property it is natural to ask if G has the *p-congruence subgroup property*, i.e. if every normal subgroup of G with index a power of p contains some level stabilizer. This weaker version of the congruence subgroup property was introduced by Garrido and Uria-Albizuri in [27]. They provide examples of groups without the congruence subgroup property and with the p-congruence subgroup property, like the GGS-groups with constant defining vector over the p-adic tree and the Basilica group.

In [44] Pervova proved that some of the well known finitely generated branch groups like the Grigorchuk groups and the periodic GGS-groups acting over the p-adic tree do not contain maximal subgroups of infinite index. The first example of a finitely generated branch group that does have maximal subgroups of infinite index was given by Bondarenko in [12].

In this preliminary chapter we collect results related to this topic and we see that for finitely generated groups the existence of maximal subgroups of infinite index is related to the existence of proper *prodense* subgroups. We use this result to prove that the p-Basilica groups do not possess maximal subgroups of infinite index. This result in the case of the Basilica group was proved in [24].

Chapter 2 is devoted to the study of the branch structures of the GGS-groups acting over the p^n-adic tree where p is a prime. The first section of this chapter deals with some properties about the structures of these groups and some results that let us reduce our study to some specific GGS-groups.

The GGS-groups are generated by a *rooted* automorphism a and a *directed* automorphism b defined by:

$$\psi(a) = (1, \overset{m}{\ldots}, 1)\sigma$$

$$\psi(b) = (a^{e_1}, a^{e_2}, \ldots, a^{e_{m-1}}, b).$$

where $\sigma = (12\ldots m) \in \mathrm{Sym}(m)$ and $\mathbf{e} = (e_1, \ldots, e_{m-1}) \in (\mathbb{Z}/m\,\mathbb{Z})^{m-1}$.

Vovkivsky proved in [56] some criteria for GGS-groups over the p^n-adic tree to be infinite, periodic and regular branch. He proved that an infinite periodic GGS-group G acting over the p^n-adic tree corresponding to the defining vector

$$\mathbf{e} = (e_1, \ldots, e_{p^n-1}) \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}$$

is a regular branch group over G'' if and only if there exists at least an integer $k \in \{1, \ldots, p^n - 1\}$ such that e_k is not congruent to 0 modulo p. He also proved that G is periodic if and only if for each $k = 0, \ldots, n-1$ the following conditions hold

$$S_k := e_{p^k} + e_{2p^k} + \cdots + e_{p^n-p^k} \equiv 0 \pmod{p^{k+1}}. \tag{0.0.1}$$

As a consequence most of the GGS-groups over the p^n-adic tree are not periodic. We extend some of Vovkivsky's results to the non-periodic GGS-groups acting over the p^n-adic tree.

More specifically let G be a GGS-group over the p^n-adic tree, let $\mathbf{e}$ be its defining vector and let

$$\mathcal{F}(p^n) = (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1} \setminus (p\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}.$$

If $\mathbf{e}$ does not belong to $\mathcal{F}(p^n)$, the corresponding GGS-group cannot act transitively on each level and by definition it is not a branch group. Thus we reduce our study to the GGS-groups with defining vector $e \in \mathcal{F}(p^n)$.

Given $\mathbf{e} \in \mathcal{F}(p^n)$, we define

$$Y := \{1 \leq i \leq p^n - 1 \mid e_i \not\equiv 0 \bmod p\}, \tag{0.0.2}$$

and

$$t := \max\{s \in \mathbb{Z} \mid s \geq 0 \text{ and } p^s \mid i \text{ for all } i \in Y\}.$$

Then we have $Y \subseteq \{p^t, 2p^t, \ldots, p^n - p^t\}$, and we say that Y is *maximal* if the equality holds. Also we define

$$\mathcal{E}(p^n) = \{\mathbf{e} \in \mathcal{F}(p^n) \mid e_{ip^t} \equiv e_{jp^t} \bmod p \text{ for all } 1 \leq i, j \leq p^{n-t} - 1\},$$

that is, the set of vectors that have the same values modulo p for the set of indices $\{p^t, 2p^t, \ldots, p^n - p^t\}$. Note that if $\mathbf{e} \in \mathcal{E}(p^n)$ then Y is maximal. Finally, we denote by $\mathcal{E}'(2^n)$ the following set:

$$\mathcal{E}'(2^n) = \{e \in \mathcal{F}(2^n) \mid t = n-1\}.$$

We proved that if G is a GGS-group acting over the p^n-adic tree with defining vector belonging to $\mathcal{F}(p^n) \setminus \mathcal{E}(p^n)$ then it is a regular branch group over G' or over $\gamma_3(G)$. In particular, for periodic GGS-groups this result improves Vovkivsky's result.

For the groups in $\mathcal{E}(p^n)$ the problem is still open. Only for a particular class of GGS-groups with defining vector inside $\mathcal{E}(p^n)$ that we define as *partially constant* we prove that they are regular branch.

For the GGS-groups with defining vector in $\mathcal{F}(p^n) \setminus \mathcal{E}'(2^n)$ we get the following result.

Theorem. *A GGS-group G over the p^n-adic tree with defining vector $\mathbf{e} \in \mathcal{F}(p^n) \setminus \mathcal{E}'(2^n)$ is weakly regular branch over G''.*

The GGS-group with constant defining vector acting on the p^n-adic tree is a groups with defining vector belonging to $\mathcal{E}(p^n)$ and it has a different structure. It is proved in [20] that the GGS-group with constant defining vector acting on the p-adic tree is weakly regular branch over the subgroup K', where $K = \langle ba^{-1}\rangle^G$, but it is not branch. We extend this result to the GGS-groups over the p^n-adic tree with constant defining vector.

The results of this chapter are collected in the article [18].

Chapter 3 is devoted to the study of a generalization of the Basilica group to a family of groups acting over the p-adic tree, called the p-Basilica groups. The Basilica group acts on the binary tree and is generated by two elements, a and b, which are recursively defined as follows:

$$\psi(a) = (1, b) \quad \text{and} \quad \psi(b) = (1, a)\sigma,$$

where σ is the cyclic permutation $(1\,2)$, which swaps the two maximal subtrees. For every prime p, the p-Basilica group is a natural generalisation of the Basilica group that acts on the p-adic tree. Such a group G is generated by two automorphisms, a and b, defined as follows:

$$\psi(a) = (1, \overset{p-1}{\ldots}, 1, b) \quad \text{and} \quad \psi(b) = (1, \overset{p-1}{\ldots}, 1, a)\sigma,$$

where σ is the cyclic permutation $(1\,2\,\cdots\,p)$. We note that for the prime $p = 2$ the 2-Basilica group coincides with the Basilica group. For every prime p the generators have infinite order and the groups are torsion free, self-similar, spherically transitive and super strongly fractal. These groups are weakly regular branch over the derived subgroups, but they are not branch since the first rigid stabilizer has infinite index in the group. These results give the first example of weakly branch, but not branch, groups that are super strongly fractal.

We determine the structures of some subgroups and quotients of the p-Basilica group G for any prime p. In particular $G/\gamma_3(G)$ is isomorphic to the integral Heisenberg group, and this result is used to prove that for every prime p the p-Basilica group does not have the weak congruence subgroup property. This property is a different weaker version of the congruence subgroup property and asks whether any normal subgroup of finite index contains the derived subgroup of some level stabilizer.

Since for any prime p the p-Basilica group is contained in the pro-p subgroup Γ, the group G does not have the congruence subgroup property as all quotients of G by level stabilizers are p-groups and G contains subgroups with arbitrary index as its abelianization is $\mathbb{Z} \times \mathbb{Z}$. This provides the first examples of weakly branch groups with the p-congruence subgroup property but not the congruence subgroup property nor the weak congruence subgroup property.

The determination of the orders of the congruence quotients of the p-Basilica groups lets us compute the Hausdorff dimension of the closure of the p-Basilica group G in the group Γ. For a subgroup $\mathcal{G}$ of Γ, the Hausdorff dimension of the closure of $\mathcal{G}$ in Γ is given by

$$\mathrm{hdim}_{\Gamma}(\overline{\mathcal{G}}) = \varliminf_{n\to\infty} \frac{\log|\mathcal{G} : \mathrm{st}_{\mathcal{G}}(n)|}{\log|\Gamma : \mathrm{st}_{\Gamma}(n)|} \in [0,1], \tag{0.0.3}$$

where $\varliminf$ represents the lower limit. The Hausdorff dimension of $\overline{\mathcal{G}}$ is a measure of how dense $\overline{\mathcal{G}}$ is in Γ. This concept was first applied by Abercrombie [1] and by Barnea and Shalev [3] in the more general setting of profinite groups.

Theorem. *Let G be a p-Basilica group, for p a prime. Then:*

1. *The orders of the congruence quotients of G are given by*

$$\log_p |G : \mathrm{st}_G(n)| = \begin{cases} p^{n-1} + p^{n-3} + \cdots + p^3 + p + \frac{n}{2} & \text{for } n \text{ even,} \\ p^{n-1} + p^{n-3} + \cdots + p^4 + p^2 + \frac{n+1}{2} & \text{for } n \text{ odd.} \end{cases}$$

2. *The Hausdorff dimension of the closure of G in Γ is*

$$\mathrm{hdim}_\Gamma(\overline{G}) = \frac{p}{p+1}.$$

Francoeur [24, Thm. 4.28] proved that the Basilica group does not possess maximal subgroups of infinite index, thus providing the first example of a weakly branch but not branch group without maximal subgroups of infinite index. Also, the Basilica group has non-normal maximal subgroups [23, Cor. 8.3.2]. We extend these results to p-Basilica groups for all primes p, likewise giving another infinite family of weakly branch groups with such properties. Note that the first infinite family of weakly branch, but not branch, groups without maximal subgroups of infinite index was given by Francoeur and Thillaisundaram in [26], namely the GGS-groups over the p-adic tree defined by the constant vector.

Theorem. *Let G be a p-Basilica group, for p a prime. Then all maximal subgroups of G have finite index, and G has infinitely many non-normal maximal subgroups.*

The results about the p-Basilica groups are collected in the article [19].

Both papers that collect the results of this thesis were carried out in collaboration with other authors. The research work was done through numerous face-to-face and online discussions, and the author of this thesis participated actively in obtaining all of the results.

Chapter 1

Preliminaries

In this chapter we collect definitions and known results about groups acting on regular rooted trees.

1.1 Automorphisms of regular rooted trees

A tree is a connected graph with no cycles. For $m \geq 2$, a *regular rooted tree* of degree m, also called the m-adic tree, is a tree with a distinguished vertex called the *root* and such that each vertex has m descendants.

More precisely, let X be a finite set, also called alphabet, and X^* be the free monoid of the words with letters in X where the operation is juxtaposition. If $m \geq 2$ is an integer, the m-adic tree $\mathcal{T}$ is the rooted tree whose vertices are words of X^* in the alphabet $X = \{x_1, \dots, x_m\}$. The root of $\mathcal{T}$ is the empty word $\emptyset$. The descendants of a word $u \in X^*$ are the words v of the form $v = uz$ where $z \in X^*$. The vertex v is said to be an immediate descendant of u if $z \in X$. Two vertices u and v are joined by an edge if any of the two is an immediate descendant of the other one. We denote by $\mathcal{L}_n$ the n-th level of $\mathcal{T}$, that is the set of words of length n in X^*. (See Figure 1.1)

A sequence of consecutive vertices which starts at the root and such that each vertex occurs at most once is called *path*. An *end* is an infinite path.

An automorphism f of $\mathcal{T}$ is a bijective map of the set of the vertices of $\mathcal{T}$ that preserves incidence, i.e. an automorphism of $\mathcal{T}$ as a graph. As a consequence an automorphism fixes the root since it is the unique vertex connected with m

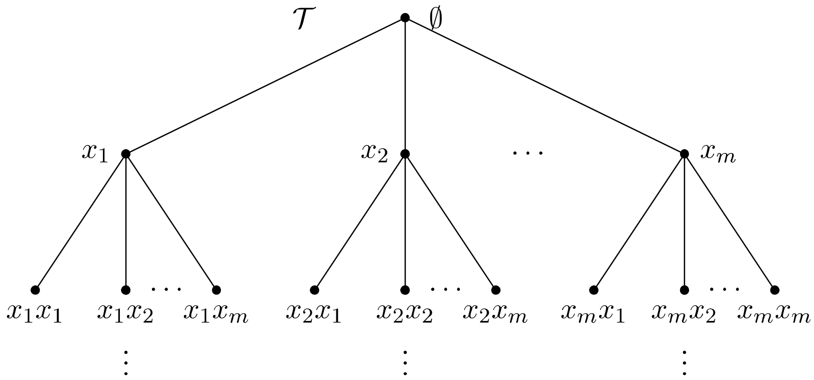


FIGURE 1.1: The m-adic tree.

vertices, sends vertices of $\mathcal{L}_n$ to vertices of the same level, and sends *paths* and *ends* respectively to paths and ends.

The set $\operatorname{Aut}\mathcal{T}$ of all automorphisms of a tree $\mathcal{T}$ forms a group under composition. We will write the composition as juxtaposition by writing fg instead of $g \circ f$.

The set of all words of length less than or equal to a given $n \in \mathbb{N}$ can be identified with the finite subtree of $\mathcal{T}$ consisting of its vertices up to the level n. We denote this finite subtree by $\mathcal{T}_n$ and we refer to it as the *tree truncated at level n*. Considering the restriction map:

$$\begin{array}{rccc} \Phi_n: & \operatorname{Aut}\mathcal{T} & \to & \operatorname{Aut}\mathcal{T}_n \\ & f & \to & f|_{\mathcal{T}_n} \end{array}$$

the group $\operatorname{Aut}\mathcal{T}_n$ can be seen as a quotient of the whole group $\operatorname{Aut}\mathcal{T}$. More precisely the kernel of this restriction map is the set of all automorphisms that fix all the vertices of $\mathcal{T}$ up to the level n. Actually this kernel coincides with the *stabilizer* of the level n that we denote by $\operatorname{st}(n)$ and is formally defined by

$$\operatorname{st}(n) = \{f \in \mathcal{T} \mid f(u) = u \ \forall u \in \mathcal{L}_n\}. \tag{1.1.1}$$

Indeed, by definition of automorphism of a rooted tree, if we know the image $f(u)$ of an automorphism $f \in \operatorname{Aut}\mathcal{T}$ at the vertex $u \in \mathcal{L}_n$, we automatically know the images under f of all vertices above u in the path connecting u to the root. As a consequence, the action of f on $\mathcal{T}_n$ is completely determined by the action of f on $\mathcal{L}_n$. This means that an automorphism f fixes all the vertices of $\mathcal{T}$ up to the level n if and only if it fixes all the vertices of $\mathcal{L}_n$. Since Φ_n is surjective, for all $n \in \mathbb{N}$ we have

$$\mathrm{Aut}\,\mathcal{T}_n \cong \mathrm{Aut}\,\mathcal{T}/\,\mathrm{st}(n).$$

We observe that for $n = 0$, the level $\mathcal{L}_0$ contains only the root. Thus $\mathrm{st}(0)$ coincides with the full group $\mathrm{Aut}\,\mathcal{T}$.

For a vertex $u \in X^*$, we denote by $\mathrm{st}(u)$ the stabilizer of the vertex u defined as

$$\mathrm{st}(u) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u\}.$$

From (1.1.1) we observe that $\mathrm{st}(n)$ coincides with the intersection of the stabilizers of all vertices of $\mathcal{L}_n$.

An automorphism f of a regular rooted tree $\mathcal{T}$ can be described recursively as follows. The root vertex is always fixed and, if we assume that f is already defined on $\mathcal{L}_n$ and that $v = ux$ is an immediate descendant of $u \in \mathcal{L}_n$, we have

$$f(v) = f(u)f_{(u)}(x) \tag{1.1.2}$$

where $f_{(u)}$ is a permutation in $\mathrm{Sym}(X)$ and is said to be the *label* of f at u. The set of all these labels is called the *portrait* of f. Thus giving a portrait is equivalent to giving an automorphism.

The subtree $\mathcal{T}_u$ of $\mathcal{T}$ hanging from $u \in X^*$ is formed by all the descendants of u and is isomorphic to $\mathcal{T}$. As a consequence $\mathrm{Aut}\,\mathcal{T}_u \cong \mathrm{Aut}\,\mathcal{T}$.

As noted before, if $f \in \mathrm{Aut}\,\mathcal{T}$ and we know the image $f(u)$ for all $u \in \mathcal{L}_n$, we know the images under f of all vertices in $\mathcal{T}_n$. Thus an automorphism $f \in \mathrm{Aut}\,\mathcal{T}$ can be also described by knowing for all $u \in \mathcal{L}_n$ the image $f(u)$ and the action of f on $\mathcal{T}_u$ called the *section* of f in u and denoted by f_u. Formally the section $f_u \in \mathrm{Aut}\,\mathcal{T}$ is defined as follows for all $v \in X^*$:

$$f(uv) = f(u)f_u(v). \tag{1.1.3}$$

Since uv is a descendant of u, then $f(uv)$ is a descendant of $f(u)$, i.e. there exists $w \in X^*$ such that $f(uv) = f(u)w$. Thus the section of f at the vertex u is the automorphism of $\mathcal{T}$ that sends v to w. In particular if $f \in \mathrm{Aut}\,\mathcal{T}$ fixes a vertex $u \in X^*$ then the section of f at the vertex u can be identified with the restriction of f to the subtree hanging from u.

We have the following rules for the sections that can be easily proved making use of the definition. All of them, with the exception of the last one, are also satisfied by the labels. Let $f, g \in \operatorname{Aut}\mathcal{T}$ and $u, v \in X^*$, we have

$$\begin{aligned}
(fg)_u &= f_u g_{f(u)} \\
(f^{-1})_u &= (f_{f^{-1}(u)})^{-1} \\
(f^g)_u &= (g_{g^{-1}(u)})^{-1} f_{g^{-1}(u)} g_{f(g^{-1}(u))} \\
f_{uv} &= (f_u)_v.
\end{aligned} \tag{1.1.4}$$

We observe that when $f \in \operatorname{st}(u)$ for the vertex $g(u) \in X^*$ we can write the third formula in a more elegant way

$$(f^g)_{g(u)} = g_u^{-1} f_u g_{f(u)} = (f_u)^{g_u}. \tag{1.1.5}$$

1.2 On the structure of $\operatorname{Aut}\mathcal{T}$

In this section we analyze properties of the group $\operatorname{Aut}\mathcal{T}$ useful for describing its structure.

First of all we observe that for all $n \in \mathbb{N}$ the stabilizer of the n-th level is a normal subgroup of $\operatorname{Aut}\mathcal{T}$ of finite index. This follows from the fact that $\operatorname{Aut}\mathcal{T}_n$ is a finite group since it is a subgroup of the symmetric group of $\mathcal{T}_n$. Thus the stabilizers of the levels form a chain of normal subgroups with finite index in $\operatorname{Aut}\mathcal{T}$ with trivial intersection and this proves that $\operatorname{Aut}\mathcal{T}$ is a residually finite group.

Moreover the family $\mathcal{F} = \{\operatorname{st}(n)\}_{n=1}^{\infty}$ with the inclusion is a directed set and the family of quotients $\{\operatorname{Aut}\mathcal{T}/\operatorname{st}(n)\}_{n=1}^{\infty}$ with the homomorphisms

$$\begin{array}{rccc}
\pi_{mn}: & \operatorname{Aut}\mathcal{T}/\operatorname{st}(m) & \to & \operatorname{Aut}\mathcal{T}/\operatorname{st}(n) \\
 & f\operatorname{st}(m) & \to & f\operatorname{st}(n)
\end{array} \tag{1.2.1}$$

for all $m \geq n$ forms an inverse system of finite groups. Then the group $\operatorname{Aut}\mathcal{T}$ is a profinite group since it is isomorphic to the inverse limit of this quotients.

$$\operatorname{Aut}\mathcal{T} \cong \varprojlim_{n\to\mathbb{N}} \{\operatorname{Aut}\mathcal{T}/\operatorname{st}(n)\}_{n=1}^{\infty}.$$

The elements belonging to $\operatorname{st}(n)$ can be characterized in terms of their portrait as follows. If $f \in \operatorname{st}(n)$ then necessarily $f_{(u)} = 1$ for all $u \in \mathcal{T}_{n-1}$. Indeed if for a certain $u \in \mathcal{T}_{n-1}$ the label $f_{(u)} \neq 1$ then the descendants of u are permuted according to the permutation $f_{(u)}$. By the property of the automorphisms of preserving the incidence, it follows that the subtrees hanging from the descendants of u are permuted according to that permutation and as a consequence the vertices of the n-th level that belong to these subtrees are not fixed by f.

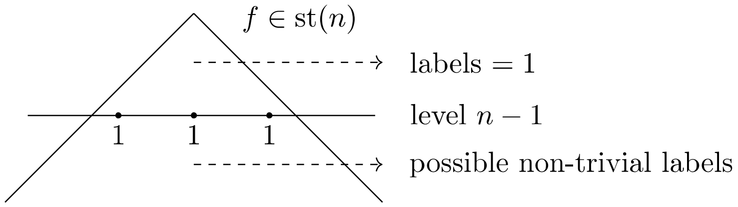


FIGURE 1.2: The portrait of $f \in \mathrm{st}(n)$.

Thus if $f \in \mathrm{st}(n)$ we can identify f with a vector whose components are the sections of f hanging from the vertices of the n-th level, because all labels until the level $n-1$ are equal to 1. For all $n \in \mathbb{N}$ we can define the following map

$$\begin{array}{rccc} \psi_n: & \mathrm{st}(n) & \rightarrow & \mathrm{Aut}\,\mathcal{T} \times \overset{m^n}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & f & \rightarrow & (f_u)_{u \in \mathcal{L}_n} \end{array} \tag{1.2.2}$$

When $n = 1$ for simplicity we will write ψ instead of ψ_1. From (1.1.4) it is easy to show that this map is a homomorphism. It is also injective because $f \in \ker(\psi_n)$ if and only if $f_u = 1$ for all $u \in \mathcal{L}_n$ and from the characterization of the elements in $\mathrm{st}(n)$ it follows that each label in f must be trivial and so f is the trivial automorphism. Thus for all $n \in \mathbb{N}$ we have the following isomorphism

$$\mathrm{st}(n) \cong \mathrm{Aut}\,\mathcal{T} \times \overset{m^n}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \tag{1.2.3}$$

which means that $\mathrm{Aut}\,\mathcal{T}$ has the property of containing direct products of itself.

In a similar way, for every positive integer n and for all $k \leq n$, we can define the following isomorphism when we work in the quotient group $\mathrm{Aut}\,\mathcal{T}_n$:

$$\begin{array}{rccc} \phi_{n,k}: & \mathrm{st}_{\mathrm{Aut}\,\mathcal{T}_n}(k) & \rightarrow & \mathrm{Aut}\,\mathcal{T}_{n-k} \times \overset{m^k}{\cdots} \times \mathrm{Aut}\,\mathcal{T}_{n-k} \\ & f & \rightarrow & (f_u)_{u \in \mathcal{L}_k} \end{array} \tag{1.2.4}$$

Also in this case when $n = 1$ we will write ϕ_n instead of $\phi_{n,1}$.

From the characterization of the portrait of an element f in $\mathrm{st}(n)$ we observe that the sections of such an element in the vertices of the first level are automorphisms that stabilize the level $\mathcal{L}_{n-1}$. Thus from (1.2.2) we have an isomorphism

$$\begin{array}{rccc} \psi: & \mathrm{st}(n) & \to & \mathrm{st}(n-1) \times \overset{m}{\cdots} \times \mathrm{st}(n-1) \\ & f & \to & (f_u)_{u \in X} \end{array} \tag{1.2.5}$$

and in general for $r \in \{1, \dots, n\}$ we have an isomorphism

$$\begin{array}{rccc} \psi_r: & \mathrm{st}(n) & \to & \mathrm{st}(n-r) \times \overset{m^r}{\cdots} \times \mathrm{st}(n-r) \\ & f & \to & (f_u)_{u \in \mathcal{L}_{n-r}} \end{array}$$

Similarly, when we work in the group $\mathrm{Aut}\,\mathcal{T}_n$, for $k \leq n$ we have an isomorphism

$$\begin{array}{rccc} \phi_n: & \mathrm{st}_{\mathrm{Aut}\,\mathcal{T}_n}(k) & \to & \mathrm{st}_{\mathrm{Aut}\,\mathcal{T}_{n-1}}(k-1) \times \overset{m}{\cdots} \times \mathrm{st}_{\mathrm{Aut}\,\mathcal{T}_{n-1}}(k-1) \\ & f & \to & (f_u)_{u \in X} \end{array} \tag{1.2.6}$$

For each vertex $u \in \mathcal{T}$ we can define the following map

$$\begin{array}{rccc} \psi_u: & \mathrm{st}(u) & \to & \mathrm{Aut}\,\mathcal{T} \\ & f & \to & f_u \end{array} \tag{1.2.7}$$

and from (1.1.4) it is easy to show that this map is a homomorphism.

The following proposition shows that the group $\mathrm{Aut}\,\mathcal{T}$ is a semidirect product.

Proposition 1.1. *Let $\mathcal{T}$ be the m-adic tree, and for $n \in \mathbb{N}$ let H_n be the subgroup defined as follows:*

$$H_n = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f_u = 1 \ \forall u \in \mathcal{L}_n\}.$$

Then $\mathrm{Aut}\,\mathcal{T} = H_n \ltimes \mathrm{st}(n)$.

Proof. From the definition of H_n and the characterization of the elements in $\mathrm{st}(n)$ it is easy see that H_n is a subgroup of $\mathrm{Aut}\,\mathcal{T}$ that intersects $\mathrm{st}(n)$ trivially. The statement is proved noting that each element $f \in \mathrm{Aut}\,\mathcal{T}$ can be written as a product of two automorphisms g and h, respectively belonging to $\mathrm{st}(n)$ and H_n, whose portraits are represented in Figure 1.3. Indeed by the formulas in (1.1.4) for every $u \in \mathcal{L}_m$ for $m < n$ we have

$$(gh)_{(u)} = g_{(u)} h_{(g(u))} = 1 \cdot h_{(g(u))} = f_{(g(u))} = f_{(u)}$$

and for every $u \in \mathcal{L}_m$ for $m \geq n$ we have

$$(gh)_{(u)} = g_{(u)} h_{(g(u))} = g_{(u)} \cdot 1 = f_{(u)}.$$

□

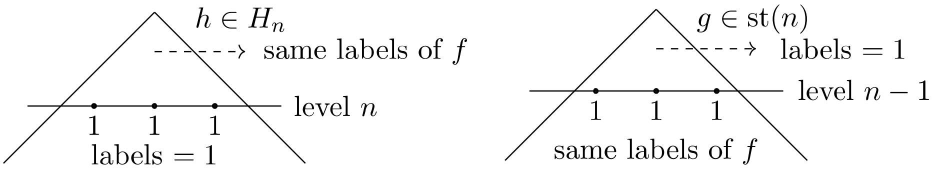


FIGURE 1.3: Decomposition of $f = gh \in \mathrm{Aut}\,\mathcal{T}$ with $g \in \mathrm{st}(n)$ and $h \in H_n$.

Definition 1.2. An automorphism $f \in \mathrm{Aut}\,\mathcal{T}$ is called

- *finitary* if there exists $n \in \mathbb{N}$ such that $f \in H_n$,
- *directed* if there exists a path $\{\emptyset, u_1, u_1u_2, u_1u_2u_3, \ldots\}$ with $u_i \in X$ for all $i \geq 1$, such that the unique possible non-trivial labels of f are those in the vertices of the form $u_1 \cdots u_n x$ for $x \in X$, $x \neq u_{n+1}$ and $n \geq 0$.
- *bounded* if the sets $\{w \in X^n \mid f_w \neq 1\}$ have uniformly bounded cardinalities over all n.

We observe that for $n = 1$ the subgroup H_1 is the set of the *rooted automorphisms*. Formally $f \in \mathrm{Aut}\,\mathcal{T}$ is a rooted automorphism if f_u is the identity map for every $u \in \mathcal{L}_1$. Actually a rooted automorphism f permutes rigidly the subtrees hanging from the vertices of $\mathcal{L}_1$. Indeed if $x \in \mathcal{L}_1$ and $u \in X^*$ then $f(xu) = \sigma(x)u$ where $\sigma \in \mathrm{Sym}(X)$ is the permutation at the root. So H_1 is isomorphic to $\mathrm{Sym}(X)$ and from the previous proposition we have the following structure for $\mathrm{Aut}\,\mathcal{T}$.

$$\begin{aligned}\mathrm{Aut}\,\mathcal{T} = \mathrm{Sym}(X) \ltimes \mathrm{st}(1) &\cong \mathrm{Sym}(X) \ltimes (\mathrm{Aut}\,\mathcal{T} \times \overset{m}{\cdots} \times \mathrm{Aut}\,\mathcal{T}) \\ &\cong \mathrm{Aut}\,\mathcal{T} \wr \mathrm{Sym}(X).\end{aligned} \tag{1.2.8}$$

This structure suggests a more compact way of representing an automorphism. For an automorphism $f \in \mathrm{Aut}\,\mathcal{T}$ we can write

$$f = g\tau$$

where $\tau \in \mathrm{Sym}(X)$ and g is an automorphism in st(1) whose sections are $\psi(g) = (g_1, \ldots, g_m)$. Thus we can extend the isomorphism ψ defined in (1.2.2) to an isomorphism between the full group $\mathrm{Aut}\,\mathcal{T}$ and the wreath product $\mathrm{Aut}\,\mathcal{T} \wr \mathrm{Sym}(X)$ as follows

$$\begin{array}{rccc} \psi: & \mathrm{Aut}\,\mathcal{T} & \to & \mathrm{Aut}\,\mathcal{T} \wr \mathrm{Sym}(X) \\ & f & \to & \psi(f) = (g_1, \ldots, g_m)\tau. \end{array} \tag{1.2.9}$$

By iterating the formula in (1.2.8), the group $\mathrm{Aut}\,\mathcal{T}$ can be also seen as the iterated permutational wreath product

$$\mathrm{Aut}\,\mathcal{T} \cong (\cdots (\mathrm{Sym}(X) \wr (\mathrm{Sym}(X) \wr \mathrm{Sym}(X))) \cdots).$$

In the following sections we will consider groups acting on primary trees, i.e. regular rooted trees of degree a power of a prime number. Most of the groups that we will analyse are subgroups of the group $\Gamma^n \leq \mathrm{Aut}\,\mathcal{T}$ defined as follows.

Definition 1.3. Let $\mathcal{T}$ be a regular rooted tree of degree p^n for a prime p and a positive integer n. The subgroup Γ^n of $\mathrm{Aut}\,\mathcal{T}$ is the set of all automorphisms whose labels are powers of the cycle $\sigma = (1\,2\,\cdots\,p^n)$.

In particular Γ^n coincides with the following iterated wreath product

$$\Gamma^n \cong \varprojlim_{m \in \mathbb{N}} C_{p^n} \wr \overset{m}{\cdots} \wr C_{p^n}. \tag{1.2.10}$$

We observe that only for $n = 1$, the group Γ^1, that we will indicate by Γ for ease of notation, is a Sylow pro-p subgroup of $\mathrm{Aut}\,\mathcal{T}$ corresponding to the p-cycle $(1\,2\,\cdots\,p)$.

1.3 Self-similar and branch groups

In this section we collect definitions and properties of the subgroups of $\mathrm{Aut}\,\mathcal{T}$ that, in a certain sense, tell us how similar the structure of these groups is to the group $\mathrm{Aut}\,\mathcal{T}$.

Definition 1.4. A subgroup G of $\mathrm{Aut}\,\mathcal{T}$ is said to be *spherically transitive* if it acts transitively on each level $\mathcal{L}_n$ of $\mathcal{T}$.

Let G be any subgroup of $\mathrm{Aut}\,\mathcal{T}$. We denote by $\mathrm{st}_G(n) = \mathrm{st}(n) \cap G$ the stabilizer of the n-th level in G and by $\mathrm{st}_G(u) = \mathrm{st}(u) \cap G$ the stabilizer of the vertex $u \in X^*$

in G. Considering the restriction maps

$$\begin{array}{rcccc} \psi_n : & \mathrm{st}_G(n) & \to & \mathrm{Aut}\,\mathcal{T} \times \overset{m^n}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & g & \to & (g_u)_{u \in \mathcal{L}_n} \end{array}$$

and

$$\begin{array}{rcccc} \psi_u : & \mathrm{st}_G(u) & \to & \mathrm{Aut}\,\mathcal{T} \\ & g & \to & g_u, \end{array}$$

in general the images of these maps are not contained in $G \times \overset{m^n}{\cdots} \times G$ and in G respectively. A family of groups for which this happens is the following.

Definition 1.5. A subgroup G of $\mathrm{Aut}\,\mathcal{T}$ is said to be *self-similar* if for all $g \in G$ and all $u \in X^*$ the section g_u belongs to G.

By using induction on the levels and the formulas (1.1.4), a group $G \leq \mathrm{Aut}\,\mathcal{T}$ is self-similar if the previous condition is satisfied for a set of generators of the group G and for the vertices of the first level.

Lemma 1.6. *Let S be a generating set for a group $G \leq \mathrm{Aut}\,\mathcal{T}$. Then G is self-similar if and only if $s_x \in G$ for all $s \in S$ and $x \in X$.*

When the group G is the full group $\mathrm{Aut}\,\mathcal{T}$, the maps ψ_n and ψ_u defined respectively in (1.2.2) and (1.2.7) are surjective. This might not be true when G is a proper subgroup of $\mathrm{Aut}\,\mathcal{T}$. In this case we have the following definitions.

Definition 1.7. A self-similar subgroup G of $\mathrm{Aut}\,\mathcal{T}$ is said to be

- *fractal* if $\psi_u(\mathrm{st}_G(u)) = G$ for all $u \in X^*$.
- *strongly fractal* if $\psi_x(\mathrm{st}_G(1)) = G$ for all $x \in X$.
- *super strongly fractal* if $\psi_u(\mathrm{st}_G(n)) = G$ for all $u \in \mathcal{L}_n$ and for all $n \in \mathbb{N}$.

Actually G is fractal if the property $\psi_u(\mathrm{st}_G(u)) = G$ holds for all $u \in X$, since by the self-similarity this property extends to all the vertices of the other levels. Moreover if the group is transitive on the first level, the following result shows that it is sufficient to check the condition for a specific vertex $x \in X$ (see also [32, Sec.3]).

Lemma 1.8. *[55, Lem. 2.7] If $G \leq \mathrm{Aut}\,\mathcal{T}$ is self-similar and transitive on the first level and $\psi_x(\mathrm{st}_G(x)) = G$ for some $x \in X$, then G is fractal and spherically transitive.*

The next lemma is a corollary of [55, Lem. 2.12] and it is useful to prove that a group G is super strongly fractal. For completeness we give the proof here.

Lemma 1.9. *If $G \leq \operatorname{Aut}\mathcal{T}$ is self-similar and transitive on the first level and for each $n \in \mathbb{N}$ there exists $u_n \in \mathcal{L}_n$ such that $\psi_{u_n}(\mathrm{st}_G(n)) = G$, then G is super strongly fractal.*

Proof. Let v be a vertex in $\mathcal{L}_n$ and, for simplicity, let us write u for u_n. Since G is self-similar, the inclusion $\psi_v(\mathrm{st}_G(n)) \subseteq G$ holds. By the previous lemma, the group G is spherically transitive, so there exists $g \in G$ such that $v = g(u)$. Let $h \in G$ and observe that $h^{g_u^{-1}} \in G$ as G is self-similar. By hypothesis there exists $f \in \mathrm{st}_G(n)$ such that

$$f_u = \psi_u(f) = h^{g_u^{-1}}.$$

Then by (1.1.5) we have

$$\psi_v(f^g) = (f^g)_{g(u)} = (f_u)^{g_u} = (h^{g_u^{-1}})^{g_u} = h.$$

This proves the reverse inclusion and the proof is complete. □

We observe that most of the groups acting over the p-adic tree studied in the literature are subgroups of the the Sylow pro-p subgroup Γ defined in (1.2.10), thus each element in such group G permutes the vertices of the first level according to a power of the cycle $(12\cdots p)$. According to this permutation, if a vertex of the first level is fixed, then all the other vertices must be fixed. This implies that for such group G the stabilizer of a vertex of the first level coincides with the first level stabilizer $\mathrm{st}_G(x) = \mathrm{st}_G(1)$ for all $x \in X$. Hence for a self-similar group G contained in Γ, the conditions of being fractal and strongly fractal are equivalent.

Obviously, every super strongly fractal group is also strongly fractal, and every strongly fractal group is fractal. In [55] the author shows that these inclusions are strict, indeed for a prime p, the GGS-group acting over the p-adic tree with constant defining vector is strongly fractal but not super strongly fractal, and a certain subgroup of the Hanoi tower group is fractal but not strongly fractal. For more details see [55]. Examples of super strongly fractal groups are given by the GGS-groups acting over the p-adic tree with non-constant defining vector, and an example of a self-similar but non-fractal group is given by the group G acting over the m-adic tree generated by two automorphisms a and b, where $\psi(a) = (1, \dots, 1)\sigma$

is a rooted automorphism defined by the permutation $\sigma = (1\,2 \cdots m)$, and b is the so called *adding machine*, i.e. an automorphism defined by $\psi(b) = (1, \ldots, 1, b)\sigma$. Indeed $a \notin \psi_x(\mathrm{st}_G(x))$ for all $x \in X$.

We observe that when the group G is self-similar we have

$$\psi_n(\mathrm{st}_G(n)) \subseteq G \times \overset{m^n}{\cdots} \times G.$$

When the group is also strongly fractal, this map is surjective in each component. But we remark that this does not imply that the stabilizer $\mathrm{st}_G(n)$ maps under ψ_n onto $G \times \overset{m^m}{\cdots} \times G$ as happens when G is the full group $\mathrm{Aut}\,\mathcal{T}$, neither need $\mathrm{st}_G(n)$ map onto a natural direct product inside $G \times \overset{m^n}{\cdots} \times G$, where by natural direct product we mean that there exist $H_1, \ldots, H_{m^n}$ subgroups of G such that $\psi_n(\mathrm{st}_G(n)) = H_1 \times \cdots \times H_{m^n}$. The biggest subgroup of $\mathrm{st}_G(n)$ that maps under ψ_n onto a natural direct product is the *rigid stabilizer* of the level n denoted by $\mathrm{rst}_G(n)$ and defined as follows

$$\mathrm{rst}_G(n) = \langle \mathrm{rst}_G(u) \mid u \in \mathcal{L}_n \rangle = \prod_{u \in \mathcal{L}_n} \mathrm{rst}(u)$$

where by $\mathrm{rst}_G(u)$ we denote the *rigid stabilizer in* G of a vertex u, that is defined as the subgroup of G that consists of those automorphisms of $\mathcal{T}$ that fix all vertices not having u as a prefix. In other words an automorphism g is in the rigid vertex stabilizer of u if all labels of g outside $\mathcal{T}_u$ are trivial (see Figure 1.4).

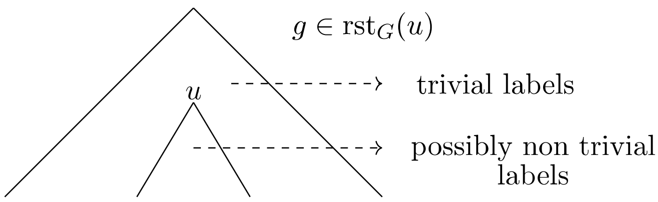


FIGURE 1.4: Portrait of $g \in \mathrm{rst}_G(u)$.

Definition 1.10. A spherically transitive group $G \leq \mathrm{Aut}\,\mathcal{T}$ is said to be *weakly branch* if $\mathrm{rst}_G(n) \neq 1$ for all n, and it is said to be *branch* if the index $|G : \mathrm{rst}_G(n)|$ is finite for all n.

Definition 1.11. A self-similar and spherically transitive group $G \le \operatorname{Aut} \mathcal{T}$ is said to be *weakly regular branch over* K, where $K \le \operatorname{st}_G(1)$, if

$$K \times \overset{m}{\cdots} \times K \subseteq \psi(K).$$

It is called *regular branch over* K if it is weakly regular branch over K and the index $|G : K|$ is finite. The group K is called *branching subgroup*.

One of the problems we investigate regards the branch structures of the GGS-groups acting over regular rooted trees of degree a power of a prime and of the p-Basilica groups. The next lemma is the main tool for finding branch structures in a subgroup of $\operatorname{Aut} \mathcal{T}$. This result is given in [22, Prop. 2.18] for GGS-groups acting over a p-adic tree, but the same proof works more generally for spherically transitive fractal groups.

Lemma 1.12. *Let G be a spherically transitive fractal subgroup of $\operatorname{Aut} \mathcal{T}$, and let L and N be two normal subgroups of G. Suppose that $L = \langle S \rangle^G$ and that $(1, \ldots, 1, s, 1, \ldots, 1) \in \psi(N)$ for every $s \in S$, where s appears always at the same position in the tuple. Then $L \times \overset{m}{\cdots} \times L \subseteq \psi(N)$.*

In order to find the branch structures of groups acting on trees, it is useful to identify isomorphic groups where the branch structure is preserved. We show now that for a group G acting on a tree $\mathcal{T}$ of degree m and for an automorphism $f \in \operatorname{Aut} \mathcal{T}$ with *constant portrait*, i.e. an automorphism such that there exists $\tau \in \operatorname{Sym}(X)$ such that $f_{(u)} = \tau$ for all $u \in X^*$, the branch structure of G can be found from the branch structure of G^f. The following lemma is useful for proving this result.

Lemma 1.13. *Let $f \in \operatorname{Aut} \mathcal{T}$ be an automorphism with constant portrait. Then $f_u = f$ for all $u \in X^*$.*

Proof. Let $\tau \in \operatorname{Sym}(X)$ be the permutation such that $\tau = f_{(z)}$ for any vertex $z \in X^*$. We prove that $(f_u)_{(z)} = \tau$ for all $u, z \in X^*$. This implies that $f_u = f$, since they both have the same portrait. From (1.1.2) and (1.1.3) for every $x \in X$ we have

$$f(uzx) = f(uz) f_{(uz)}(x) = f(uz)\tau(x)$$

and

$$f(uzx) = f(u) f_u(zx) = f(u) f_u(z) (f_u)_{(z)}(x) = f(uz)(f_u)_{(z)}(x).$$

This implies that $(f_u)_{(z)} = \tau$, as desired. □

Lemma 1.14. *Let $G \leq \operatorname{Aut}\mathcal{T}$ and $f \in \operatorname{Aut}\mathcal{T}$. Then $\operatorname{rst}_G(u)^f = \operatorname{rst}_{G^f}(f(u))$ for all $u \in X^*$ and $\operatorname{rst}_G(\ell)^f = \operatorname{rst}_{G^f}(\ell)$ for every $\ell \in \mathbb{N}$.*

Proof. Let v be a vertex of $\mathcal{T}$ and let $w \in \mathcal{T}$ be such that $f(w) = v$. Then v is not a descendant of $f(u)$ if and only if w is not a descendant of u. Let $g \in G$ then

$$g^f(v) = g^f(f(w)) = (f^{-1}gf)(f(w)) = f(g(w)).$$

Thus $g^f(v) = v$ if and only if $f(g(w)) = f(w)$, that is equivalent to $g(w) = w$. This implies that $\operatorname{rst}_G(u)^f = \operatorname{rst}_{G^f}(f(u))$ for all $u \in X^*$ and $\operatorname{rst}_G(\ell)^f = \operatorname{rst}_{G^f}(\ell)$ for every $\ell \in \mathbb{N}$. □

The following corollary (see [22, Prop. 2.17]) is a consequence of the previous result. For completeness we give the full proof here.

Corollary 1.15. *Let $G \leq \operatorname{Aut}\mathcal{T}$ be a spherically transitive group and let $f \in \operatorname{Aut}\mathcal{T}$. Then*

(i) G is weakly branch (branch) if and only if G^f is weakly branch (branch).

(ii) G is weakly regular branch (regular branch) over K if and only if G^f is weakly regular branch (regular branch) over K^f.

Proof. For the first item we observe that

$$\begin{aligned}\operatorname{rst}_G(n)^f &= \langle \operatorname{rst}_G(u)^f \mid u \in \mathcal{L}_n \rangle \\ &= \langle \operatorname{rst}_{G^f}(f(u)) \mid u \in \mathcal{L}_n \rangle = \operatorname{rst}_{G^f}(n).\end{aligned}$$

Thus $\operatorname{rst}_G(n)$ is trivial for some $n \in \mathbb{N}$ if and only if $\operatorname{rst}_{G^f}(n)$ is. Moreover $|G : \operatorname{rst}_G(n)| = |G^f : \operatorname{rst}_G(n)^f| = |G^f : \operatorname{rst}_{G^f}(n)|$ and the result follows.

(ii) The group G is weakly regular branch over K if $K \times \overset{m}{\cdots} \times K \subseteq \psi(K)$. For a given $x \in X$, let $L_x \leq K$ be defined by

$$\psi(L_x) = 1 \times \cdots \times 1 \times K \times 1 \times \cdots \times 1,$$

where K appears at the position x. Then $L_x \subseteq \operatorname{rst}_G(x)$ and from the previous lemma we have $L_x^f \subseteq \operatorname{rst}_{G^f}(f(x))$. Let $l \in L_x$ and let $k \in K$ be such that $\psi(l) =$

$(1,\dots,1,k,1,\dots,1)$. Let $h \in \mathrm{st}(1)$ be such that $\psi(h) = (f,\dots,f)$ and let $\tau \in \mathrm{Sym}(X)$ be the permutation such that $f_{(u)} = \tau$ for all $u \in X^*$. Then we can write $f = h\tau$, and we have

$$\begin{aligned}\psi(l^f) &= (1,\dots,1,k,1,\dots,1)^{h\tau} = (1,\dots,1,k^f,1,\dots,1)^{\tau}\\ &= (1,\dots,1,k^f,1,\dots,1)\end{aligned}$$

where the last equality follows from the fact that $\tau(x) = f(x)$, and k^f is at the position $f(x)$. This proves that

$$\psi(L_x^f) = 1 \times \cdots \times 1 \times K^f \times 1 \times \cdots \times 1$$

As $L_x^f \leq K^f$, it follows that $K^f \times \cdots \times K^f \subseteq \psi(K^f)$. The reverse implication follows by noting that f^{-1} is of constant portrait too. The fact that $|G : K| = |G^f : K^f|$ completes the proof. □

From the definition of $\mathrm{st}(n)$, for a self-similar group G the image under ψ of $\mathrm{st}_G(n)$ is actually contained in $\mathrm{st}_G(n-1) \times \overset{m}{\cdots} \times \mathrm{st}_G(n-1)$. The reverse inclusion does not hold since an automorphism f of the form $\psi(f) = (f_1,\dots,f_m)$ with $f_i \in \mathrm{st}_G(n-1)$ for all $i \in \{1,\dots,m\}$ need not be an element in the group G. However we have the following result.

Lemma 1.16. *Let $G \leq \operatorname{Aut}\mathcal{T}$ be a self-similar group acting over the m-adic tree. Then*

$$\psi(\mathrm{st}_G(n)) = \mathrm{st}_G(n-1) \times \overset{m}{\cdots} \times \mathrm{st}_G(n-1) \cap \psi(\mathrm{st}_G(1)). \tag{1.3.1}$$

We indicate by G_n the quotient $G_n = G/\mathrm{st}_G(n)$ that can be seen as a group acting over the truncated tree $\mathcal{T}_n$ and we refer to this quotient as the *n-th congruence quotient* of G. Since G_n can be seen as a subgroup of $\operatorname{Aut}\mathcal{T}_n$, we can consider the restriction of the map ϕ_n defined in (1.2.6) and we have the following result.

Lemma 1.17. *Let $G \leq \operatorname{Aut}\mathcal{T}$ be a self-similar group acting over the m-adic tree. Then*

$$\phi_n(\mathrm{st}_{G_n}(k)) = \mathrm{st}_{G_{n-1}}(k-1) \times \overset{m}{\cdots} \times \mathrm{st}_{G_{n-1}}(k-1) \cap \phi_n(\mathrm{st}_{G_n}(1)). \tag{1.3.2}$$

The following theorem enables us to determine whether a self-similar group is torsion-free by looking at an appropriate quotient.

Theorem 1.18. *Let G be a self-similar subgroup of* $\operatorname{Aut}\mathcal{T}$ *and suppose that there exists a torsion-free quotient G/N with $N \leq \operatorname{st}_G(1)$. Then G is torsion-free.*

Proof. For every $n \in \mathbb{N}\cup\{0\}$, we indicate by S_n the set of torsion elements in $\operatorname{st}_G(n)\setminus\operatorname{st}_G(n+1)$. Then our goal is to prove that these sets are all empty. By way of contradiction, suppose that $S_n \neq \varnothing$ for some n, which we choose as small as possible.

Since G/N is torsion-free and $N \leq \operatorname{st}_G(1)$, it is clear that $n \geq 1$. Let $g \in S_n$ be a torsion element. If $\psi(g) = (g_1, \ldots, g_p)$ then some g_i belongs to $\operatorname{st}_G(n-1)\setminus\operatorname{st}_G(n)$ and, of course, the element g_i is of finite order. So $g_i \in S_{n-1}$, which is a contradiction. □

In particular, we will use this result to prove that the p-Basilica groups are torsion-free (see Section 3.2).

1.4 Contracting property, growth and amenability

In this section we will review the concept of growth of a group and we will prove some general results. If G is an arbitrary group generated by a symmetric subset S, i.e. a set for which $S = S^{-1}$, then for every $g \in G$,

$$|g| = \min\{n \geq 0 \mid g = s_1 \cdots s_n, \text{ for } s_1, \ldots, s_n \in S\}$$

is called the *length* of g with respect to S. Now assume that G is a self-similar subgroup of $\operatorname{Aut}\mathcal{T}$. Then for every $g \in G$ and every $n \in \mathbb{N}\cup\{0\}$ we define

$$\ell_n(g) = \max\{|g_u| \mid u \in \mathcal{L}_n\}.$$

From the rule $(gh)_u = g_u h_{g(u)}$ we get that the function ℓ_n is subadditive, i.e. that

$$\ell_n(gh) \leq \ell_n(g) + \ell_n(h) \quad \text{for every } g, h \in G, \tag{1.4.1}$$

and from $(g^{-1})_u = (g_{g^{-1}(u)})^{-1}$, that

$$\ell_n(g^{-1}) = \ell_n(g) \quad \text{for every } g \in G.$$

If there exist $\lambda < 1$ and $C, L \in \mathbb{N}$ such that

$$\ell_n(g) \leq \lambda|g| + C, \quad \text{for every } n > L \text{ and every } g \in G,$$

then we say that the group G is *contracting* with respect to S.

The following lemma is straightforward, but very useful in proving that a subgroup of $\operatorname{Aut}\mathcal{T}$ is contracting.

Lemma 1.19. *Let $G = \langle S \rangle$ be a self-similar subgroup of $\operatorname{Aut}\mathcal{T}$, where S is symmetric and suppose that $\ell_1(s) \leq 1$ for all $s \in S$. Then $\ell_n(g) \leq \ell_{n-1}(g)$ for every $n \in \mathbb{N}$ and $g \in G$.*

Proof. Observe that the condition $\ell_1(s) \leq 1$ for all $s \in S$, together with (1.4.1), imply that $\ell_1(g) \leq |g|$ for every $g \in G$. Now let $u \in \mathcal{L}_n$ and write $u = vx$ with $v \in \mathcal{L}_{n-1}$ and $x \in X$. Then for every $g \in G$ we have

$$|g_u| = |(g_v)_x| \leq \ell_1(g_v) \leq |g_v| \leq \ell_{n-1}(g),$$

and the result follows. □

We recall now some preliminary definitions about growth of groups and amenability.

Let G be a group generated by a finite symmetric subset S. The length function on G is a metric on G and therefore one can define the ball of radius n:

$$B(n) = \{g \in G : |g| \leq n\}.$$

We say that the map $\gamma : \mathbb{N}_0 \longrightarrow [0, \infty)$ where $\gamma(n) = |B(n)|$, is the *growth function* of G.

If we consider two growth functions γ_1, γ_2, we say that γ_2 *dominates* γ_1 and we write $\gamma_1 \preceq \gamma_2$ if there exist $C, \alpha > 0$ such that $\gamma_1(n) \leq C\gamma_2(\alpha n)$ for every $n \in \mathbb{N}$. If $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$, we write $\gamma_1 \sim \gamma_2$. It is easy to see that this is an equivalence relation.

As proved in [5, Lem. 2.10] (see also [39, Prop. 1.3])), the growth functions of G with respect to two different generating sets are equivalent, for this reason we will refer simply to the growth function of G and we will not specify the generating set.

If $\gamma(n) \preceq n^a$ for some $a \in \mathbb{N}$, we say that G has *polynomial growth.* Instead G is said to have *exponential growth* if $\lim_{n\to\infty} \gamma(n)^{1/n} > 1$ (notice that such a limit always exists [5, Lem. 8.1]). Finally $\gamma(n)$ has *intermediate growth* if $\gamma(n)$ is equivalent to neither of the above. Notice that it is also common to say that a group G has *subexponential growth* if $\lim_{n\to\infty} \gamma(n)^{1/n} = 0$, and *superpolynomial growth* if $\lim_{n\to\infty} \ln(\gamma(n))/\ln(n) = \infty$. Groups of polynomial and exponential growth are common. Examples of groups with exponential growth are the free groups. Examples of groups with polynomial growth are the abelian groups and more generally the virtually nilpotent groups. Actually, Gromov proved in [35] the following celebrated result.

Theorem 1.20. *[35] A finitely generated group G is of polynomial growth if and only if it is virtually nilpotent.*

We recall that for a property $\mathcal{P}$, a group is said to be *virtually* $\mathcal{P}$ if it contains a subgroup of finite index that has the property $\mathcal{P}$.

In [34] Grigorchuk and Żuk show that the Basilica group has exponential growth. We will also see in Chapter 3 that the p-Basilica groups have exponential growth. In 1968 Milnor asked whether the growth of a group is necessarily equivalent to either polynomial or exponential [14]. This problem is known as the Milnor problem and it remained open until 1983 when Grigorchuk proved in [29] that the first Grigorchuk group has intermediate growth. Other examples of groups with intermediate growth are a family of groups generalising the first Grigorchuk group defined by Bartholdi and Sunic in [9].

Next, we say that a group G is *amenable* if there is a finitely additive left-invariant measure μ on the subsets of G such that $\mu(G) = 1$, where μ is said to be left-invariant if $\mu(gA) = \mu(A)$ for $A \subseteq G$ and $g \in G$. (See [43] for details).

We denote the class of amenable groups by AG. The class of amenable groups was introduced in 1929 by Von Neumann in [41] to explain why the Banach-Tarsky Paradox occurs only for dimensions greater than of equal to three. In the same paper the author proved that the finite groups and the abelian groups are amenable and that the class AG is closed under taking subgroups, quotients, extensions, and direct unions. Other examples are the solvable groups and consequently also the nilpotent groups and the p-groups. On the other hand, the free group of rank 2 is an example of non-amenable group.

The class EG of *elementary amenable* groups, introduced in 1957 by Mahlon Day in [17], is the smallest class of groups that contains all abelian and finite groups and it is closed under taking quotients, subgroups, extensions and direct unions. For many years the problem posed by Mahlon Day in [17] about the existence of amenable but not elementary amenable groups remained open. In 1980 Chou showed in [16] that all elementary amenable groups have either polynomial or exponential growth. Finally the question was answered by Grigorchuk in 1984 when he proved in [30] that the first Grigorchuk group is an example of group that is amenable since it has intermediate growth but not elementary amenable by the result of Chou in [16]. Thus the inclusion $EG \subset AG$ is strict.

So, a natural generalization of EG is the class SG of *subexponentially amenable* groups defined in [15], i.e. the smallest class of groups which contains all groups of subexponential growth and is closed under taking subgroups, quotients, extensions, and direct unions. Of course, the class SG contains the class EG.

The first example of amenable but not subexponentially amenable group with exponential growth is the Basilica group. It was proved by Grigorchuk and Żuk in [34] that this group does not belong to the class of subexponentially amenable groups, whereas it was proved by Bartholdi and Virág in [10] that the Basilica group is amenable. This proves that also the inclusion $SG \subset AG$ is strict.

In Section 3.6 we show that also the p-Basilica group for a prime p is amenable but not subexponentially amenable. To this end we use the following result that appears as [37, Cor. 3].

Lemma 1.21. *Let $G \leq \operatorname{Aut}\mathcal{T}$ be a finitely generated, non-abelian, infinite group. If G is weakly regular branch over a subgroup K and there exists some vertex $u \in X^*$ such that $\psi_u(\mathrm{st}_K(u))$ contains G, then G is not elementary amenable. Moreover, if G is of exponential growth, then G is not subexponentially amenable.*

1.5 Congruence subgroup property

In this section we introduce the congruence subgroup problem, that represents one of the most studied problems concerning groups acting on trees. We start by recalling that for a group G the *profinite topology* is the topology given by the basis of open sets

$$\mathcal{B} = \{gH \mid g \in G, H \leq G \text{ and } |G : H| < \infty\}.$$

Since every subgroup of G of finite index contains a normal subgroup of G of finite index, the topology given by the basis

$$\mathcal{B}' = \{gN \mid g \in G, N \trianglelefteq G \text{ and } |G : N| < \infty\}$$

coincides with the profinite topology. Indeed the inclusion $\mathcal{B}' \subseteq \mathcal{B}$ is trivial. For the reverse inclusion, let $H \in \mathcal{B}$ and let N be a normal subgroup of G with finite index such that $N \leq H$. Then we can write $H = \bigcup_{h \in H} hN$, thus H is open in the topology defined by $\mathcal{B}'$.

Inside the profinite group $\operatorname{Aut}\mathcal{T}$, the family $\{\mathrm{st}(n) \mid n \in \mathbb{N}\}$ forms a system of neighborhoods of the identity. For a subgroup $G \leq \operatorname{Aut}\mathcal{T}$, equipped with the subspace topology, we can take as system of neighborhoods of the identity the family $\{G \cap \mathrm{st}(n) \mid n \in \mathbb{N}\} = \{\mathrm{st}_G(n) \mid n \in \mathbb{N}\}$, and the topology given by the basis

$$\{g\, \mathrm{st}_G(n) \mid g \in G \text{ and } n \in \mathbb{N}\}$$

is called the *congruence topology.* Since $\mathrm{st}_G(n)$ are normal subgroups of G with finite index for every $n \in \mathbb{N}$, the profinite topology is finer that the congruence topology. The congruence subgroup problem asks whether these two topologies coincide.

Definition 1.22. A subgroup $G \leq \operatorname{Aut}\mathcal{T}$ is said to have the *congruence subgroup property* if every finite-index subgroup of G contains a level stabiliser $\mathrm{st}_G(n)$ for some $n \in \mathbb{N}$.

One can reformulate this property in terms of completions. More precisely, let $\hat{G}$ and $\overline{G}$ be respectively the profinite completion and the congruence competion of G, i.e. the completion of G with respect to the profinite topology and the congruence topology. If $(\mathcal{I}, \succeq)$ is a directed set and $\mathcal{I}'$ is a subset of $\mathcal{I}$, the set $\mathcal{I}'$ is said to be *cofinal* in $\mathcal{I}$ if for every $i \in \mathcal{I}$ there is some $i' \in \mathcal{I}'$ such that $i \succeq i'$. Now $\mathcal{N} = \{N \trianglelefteq G \mid |G : N| < \infty\}$ and $\mathcal{N}' = \{\mathrm{st}_G(n) \mid n \in \mathbb{N}\}$ are directed sets with respect to the reverse inclusion. If the group G has the congruence subgroup property, that is for all $N \in \mathcal{N}$ there exists $n_N \in \mathbb{N}$ such that $\mathrm{st}_G(n_N) \leq N$, it follows that $\mathcal{N}'$ is cofinal in $\mathcal{N}$. In this case from [46, Lem. 1.1.9] the completions of G with respect to the profinite topology and the congruence topology are isomorphic.

This property holds for some well-known examples of groups acting on rooted trees, such as the Grigorchuk group [6] and GGS-groups over the p-adic tree with non-constant defining vectors [20].

Most of the well known subgroups of $\operatorname{Aut}\mathcal{T}$ are groups acting over the p-adic tree for a prime p, and they are often subgroups of the pro-p group Γ defined in Definition 1.3. For a group $G \leq \Gamma$, the index $|G : \mathrm{st}_G(n)|$ is a power of p for all $n \in \mathbb{N}$. Thus when G does not have the congruence subgroup property it is natural to ask if G has the *p-congruence subgroup property*, i.e. if every normal subgroup of G with index a power of p contains some level stabilizer. This weaker version of the congruence subgroup property was introduced by Garrido and Uria-Albizuri [27]. In this paper, examples of weakly branch, but not branch, groups with the p-congruence subgroup property and not the congruence subgroup property were provided. For p odd, their examples were the GGS-groups defined by the constant vector, and for $p = 2$, their example was the Basilica group. We will extend this result to p-Basilica groups, for all odd primes p (see Section 3.4 for details).

Thinking of the congruence subgroup property from a topological point of view, by allowing other completions, in [27] the authors introduce a further generalization of the congruence subgroup property, i.e. the *$\mathcal{C}$-congruence subgroup property*, where $\mathcal{C}$ is a pseudo-variety, a class of groups satisfying the following definition.

Definition 1.23. Let $\mathcal{C}$ be a class of finite groups. We say that $\mathcal{C}$ is a pseudo-variety of finite groups if the following properties are satisfied:

(i) $\mathcal{C}$ is closed under taking subgroups, that is, if $G \in \mathcal{C}$ and $H \leq G$ then $H \in \mathcal{C}$,

(ii) $\mathcal{C}$ is closed under taking quotients, that is, if $G \in \mathcal{C}$ and $N \trianglelefteq G$ then $G/N \in \mathcal{C}$,

(iii) $\mathcal{C}$ is closed under taking finite direct products, that is, if $G_1, \ldots, G_k \in \mathcal{C}$ for $k \in \mathbb{N}$ then $\prod_{i=1}^{k} G_i \in \mathcal{C}$.

Definition 1.24. Given a pseudo-variety $\mathcal{C}$ and a subgroup $G \leq \operatorname{Aut}\mathcal{T}$ such that $G/\mathrm{st}_G(n) \in \mathcal{C}$ for all $n \in \mathbb{N}$, the group G satisfies the $\mathcal{C}$-congruence subgroup property, or $\mathcal{C}$-CSP for short, if every quotient of G lying in $\mathcal{C}$ is a quotient of some $G/\mathrm{st}_G(n)$.

Remark 1.25. We observe that when $\mathcal{C}$ represents the class of finite groups or of finite p-groups, that is in the case of the congruence subgroup property or of the p-congruence subgroup property, the class $\mathcal{C}$ is also closed under taking extensions.

Since we often deal with groups that are subgroups of $\operatorname{Aut}\mathcal{T} \times \cdots \times \operatorname{Aut}\mathcal{T}$ we reformulate the last definition in a more general context as follows.

Definition 1.26. Let K be a group and let $\mathcal{C}$ be a pseudo-variety of finite groups. If $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ is a descending series of normal subgroups of K with $K/K_n \in \mathcal{C}$ for every $n \in \mathbb{N}$ then we say that K has the $\mathcal{C}$-congruence subgroup property (the $\mathcal{C}$-CSP, for short) relative to $\mathcal{K}$ if the following property holds: if $L \trianglelefteq K$ and $K/L \in \mathcal{C}$ then there exists $n \in \mathbb{N}$ such that $K_n \leq L$.

Thus the $\mathcal{C}$-CSP for a subgroup G of $\operatorname{Aut}\mathcal{T}$ is nothing but the $\mathcal{C}$-CSP relative to the series given by $S_n = \{\mathrm{st}_G(n)\}_{n\in\mathbb{N}}$. Observe that the classical congruence subgroup property of matrix groups also fits into this definition.

If we have a group K with a series $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ then every subgroup H and every factor group K/N have naturally induced series $\{H\cap K_n\}_{n\in\mathbb{N}}$ and $\{K_nN/N\}_{n\in\mathbb{N}}$. Then as in [27] we have the following.

Lemma 1.27. *Let K be a group, let $\mathcal{C}$ be a pseudo-variety of finite groups, and let $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ be a descending series of normal subgroups of K with $K/K_n \in \mathcal{C}$ for every $n \in \mathbb{N}$. Then, given $N \trianglelefteq K$, the following hold:*

(i) If K has the $\mathcal{C}$-CSP relative to $\mathcal{K}$ then K/N has the $\mathcal{C}$-CSP relative to the induced series.

(ii) If both N and K/N have the $\mathcal{C}$-CSP relative to the induced series, then K has the $\mathcal{C}$-CSP relative to $\mathcal{K}$.

(iii) Assume that $\mathcal{C}$ is closed under extensions. If K has the $\mathcal{C}$-CSP relative to $\mathcal{K}$ and $K/N \in \mathcal{C}$ then N has the $\mathcal{C}$-CSP relative to the induced series.

Proof. (i) Let H be a normal subgroup of K containing N such that $H/N \trianglelefteq K/N$ and $\frac{K/N}{H/N} \cong K/H \in \mathcal{C}$. Since K has the $\mathcal{C}$-CSP relative to $\mathcal{K}$ and H is a normal subgroup of K such that $K/H \in \mathcal{C}$, there exists $n \in \mathbb{N}$ such that $K_n \leq H$. Hence $K_nN/N \leq HN/N = H/N$.

(ii) Let $L \trianglelefteq K$ such that $K/L \in \mathcal{C}$. We consider the intersection $L \cap N \trianglelefteq N$ and we note that

$$\frac{N}{L\cap N} \cong \frac{NL}{L} \leq \frac{K}{L} \in \mathcal{C}.$$

Since the class $\mathcal{C}$ is closed under subgroups, it follows that $N/(L\cap N) \in \mathcal{C}$. Thus there exists $n \in \mathbb{N}$ such that $K_n \cap N \leq L \cap N$.

We observe that $(L \cap K_n)N$ is normal in K. Moreover the map sending $\overline{g} \in K/(L\cap K_n)$ to $(gL, gK_n) \in K/L \times K/K_n$ is an injective homomorphism, so $K/(L\cap$

K_n) is isomorphic to a subgroup of $K/L \times K/K_n$. Since both quotients K/L and K/K_n belong to $\mathcal{C}$, from the properties (i) and (iii) in Definition 1.23 it follows that $K/(L \cap K_n) \in \mathcal{C}$. The quotient $K/(L \cap K_n)N$ is isomorphic to a quotient of $K/(L \cap K_n)$ and by (ii) in Definition 1.23 it belongs to $\mathcal{C}$. By hypothesis, K/N has the $\mathcal{C}$-CSP. Since $(L \cap K_n)N/N \trianglelefteq K/N$, there exists $l \in \mathbb{N}$ such that $K_l N/N \leq (L \cap K_n)N/N$, so $K_l N \leq (L \cap K_n)N$. Let m be the maximun between n and l. By the Dedekind law we have

$$K_m = K_n \cap K_l \leq K_n \cap (L \cap K_n)N = (L \cap K_n)(K_n \cap N) \leq L$$

and the result follows.
(ii) Let $L \trianglelefteq N$ such that $N/L \in \mathcal{C}$. Since $K/N \in \mathcal{C}$ and $\mathcal{C}$ is closed under extensions it follows that $K/L \in \mathcal{C}$. So there exists n such that $K_n \leq L$ so $K_n \cap N \leq L$. This completes the proof. □

For a group K with a series $\mathcal{K} = \{K_n\}_{n \in \mathbb{N}}$ of normal subgroups, we let $K^{(d)}$ denote the dth cartesian power of K, and $\mathcal{K}^{(d)} = \{K_n^{(d)}\}_{n \in \mathbb{N}}$ the series naturally induced by $\mathcal{K}$ in $K^{(d)}$. We have the following result.

Lemma 1.28. *Let K be a group, let $\mathcal{C}$ be a pseudo-variety of finite groups, and let $\mathcal{K} = \{K_n\}_{n \in \mathbb{N}}$ be a descending series of normal subgroups of K with $K/K_n \in \mathcal{C}$ for every $n \in \mathbb{N}$. If K has the $\mathcal{C}$-CSP relative to $\mathcal{K}$, then $K^{(d)}$ has the $\mathcal{C}$-CSP relative to the induced series for every $d \in \mathbb{N}$.*

Proof. Let N be a normal subgroup of $K^{(d)}$ such that $K^{(d)}/N \in \mathcal{C}$. For every $i \in \{1, \ldots, d\}$, let

$$K_i = \{(1, \overset{i-1}{\ldots}, k, 1, \overset{d-i}{\ldots}) \mid k \in K\}$$

and $N_i = N \cap K_i$. Then $K_i \cong K$ has the $\mathcal{C}$-CSP relative to the series

$$K_{i,n} = \{(1, \overset{i-1}{\ldots}, k, 1, \overset{d-i}{\ldots}) \mid k \in K_n\}.$$

Since $K_i/N_i \cong K_i N/N \leq K^{(d)}/N$ it follows that $K_i/N_i \in \mathcal{C}$ and consequently there exists $n_i \in \mathbb{N}$ such that $K_{i,n_i} \leq N_i$. If $n = \max_{i=1,\ldots,d} n_i$ then $K_n^{(d)} \leq N$, which completes the proof. □

We reformulate [27, Lem. 6] as follows.

Lemma 1.29. *Let G be a subgroup of* $\operatorname{Aut}\mathcal{T}$ *and* $N \trianglelefteq M \trianglelefteq G$*. If* G/M *has the* $\mathcal{C}$*-CSP and* M/N *has the* $\mathcal{C}$*-CSP, then* G/N *has the* $\mathcal{C}$*-CSP.*

We will focus on the case of the variety of finite p-groups, thus in the following we refer to the p-CSP.

The following lemma is a slight generalisation of [27, Thm. 1], which corresponds to the case when N is chosen so that $L \leq K'$. For completeness we give the full proof here.

Lemma 1.30. *Let G be a subgroup of* $\operatorname{Aut}\mathcal{T}$ *that is weakly branch over a normal subgroup* K*. Let N be a normal subgroup of G such that:*

(i) $K' \leq N \leq K$.

(ii) If $L = \psi^{-1}(N \times \cdots \times N)$ *then* $G/N, N/L$ *and* N/K' *have the p-CSP.*

Then G has the p-CSP.

Proof. Set $L_k = \psi_k^{-1}(N \times \overset{m^k}{\cdots} \times N)$ for every $k \in \mathbb{N}$. We first prove by induction on k that $L_k \trianglelefteq G$. Let $g \in G$. Since G is self-similar, there exist $h_1, \dots h_m \in G$ such that $\psi(g) = (h_1, \dots, h_m)\tau$ where $\tau \in \operatorname{Sym}(X)$. The subgroup N is normal in G, thus we have

$$\psi(L^g) = (N^{h_1} \times \cdots \times N^{h_m})^\tau = (N \times \cdots \times N)^\tau = \psi(L)$$

where the last equality holds since the conjugation by τ just permutes the components. Assume the result true for $k-1$. We observe that

$$\psi(L_k) = L_{k-1} \times \overset{m}{\cdots} \times L_{k-1}.$$

Hence by induction we have

$$\psi(L_k^g) = ((L_{k-1})^{h_1} \times \cdots \times (L_{k-1})^{h_m})^\tau = (L_{k-1} \times \overset{m}{\cdots} \times L_{k-1})^\tau = \psi(L_k)$$

and the result follows.

Now we are going to prove by induction on k that G/L_k has the p-CSP. Since $L_1 = L$ and both G/N and N/L have the p-CSP, the result for $k = 1$ follows by Lemma 1.29. Now we suppose that it holds for k and we prove it for $k+1$. Observe that it suffices to prove that L_k/L_{k+1} satisfies the p-CSP. The series we are considering in this quotient is the one induced by $\{\mathrm{st}_{L_k}(n)\}_{n\in\mathbb{N}}$. We have

$$\psi_k(L_k) = N \times \overset{m^k}{\cdots} \times N \tag{1.5.1}$$

and

$$\psi_k(L_{k+1}) = L \times \overset{m^k}{\cdots} \times L,$$

so that ψ_k induces an isomorphism between L_k/L_{k+1} and $N/L \times \overset{m^k}{\cdots} \times N/L$. Also

$$\psi_k(\mathrm{st}_{L_k}(n)) = \mathrm{st}_N(n-k) \times \overset{m^k}{\cdots} \times \mathrm{st}_N(n-k), \qquad \text{for every } n \geq k. \tag{1.5.2}$$

Since N/L has the p-CSP with respect to $\{\mathrm{st}_N(n)L/L\}_{n\in\mathbb{N}}$, it follows that

$$N/L \times \overset{m^k}{\cdots} \times N/L \cong (N \times \overset{m^k}{\cdots} \times N)/(L \times \overset{m^k}{\cdots} \times L)$$

has the p-CSP with respect to the series induced by (1.5.2). It follows that L_k/L_{k+1} has the p-CSP, as desired.

In order to conclude that G has the p-CSP, let $J \trianglelefteq G$ be such that G/J is a finite p-group. Since G is spherically transitive by the definition of weakly branch group, we have $\mathrm{rst}_G(i)' \leq J$ for some $i \in \mathbb{N}$ by [27, Lem. 4]. Define the subgroup K_i of the i-th rigid stabiliser by the condition

$$\psi_i(K_i) = K \times \overset{m^i}{\cdots} \times K. \tag{1.5.3}$$

Since $K' \leq N$ it follows that $K_i' \leq L_i$. Now taking into account that N/K' has the p-CSP, the same argument as above yields that L_i/K_i' has the p-CSP as well. Thus from Lemma 1.29 it follows that G/K_i' has the p-CSP for every $i \in \mathbb{N}$. Since $K_i' \leq \mathrm{rst}_G(i)' \leq J$, this proves that J contains some level stabiliser in G and consequently G has the p-CSP. ☐

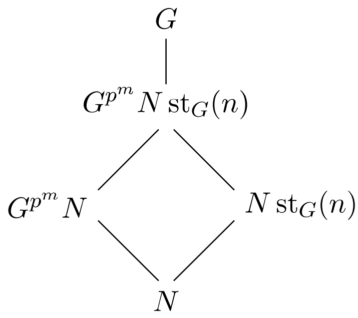
The next lemmas are useful to prove that a group has the p-CSP.

Lemma 1.31. *Let N and G be subgroups of $\mathrm{Aut}\,\mathcal{T}$ with $N \trianglelefteq G$ and G/N free abelian of rank r, for some $r \in \mathbb{N}$. Suppose that, for large enough $n \in \mathbb{N}$, we have*

$$G/N\,\mathrm{st}_G(n) \cong C_{p^{\lambda_1(n)}} \times \cdots \times C_{p^{\lambda_r(n)}}, \tag{1.5.4}$$

with $\lim_{n\to\infty} \lambda_i(n) = \infty$ for $1 \leq i \leq r$. Then G/N has the p-CSP.

Proof. Let $N \leq J \trianglelefteq G$, where $|G : J| = p^m$, for some $m \in \mathbb{N}$. Then $G^{p^m} \leq J$. Now choose an integer n such that $\lambda_i(n) \geq m$ for $1 \leq i \leq r$. We observe that



and by (1.5.4) we have

$$\begin{aligned} |G/N \operatorname{st}_G(n) : (G/N \operatorname{st}_G(n))^{p^m}| &= |G/N \operatorname{st}_G(n) : G^{p^m} N \operatorname{st}_G(n)/N \operatorname{st}_G(n)| = p^{rm} \\ &= |G/N : (G/N)^{p^m}|, \end{aligned}$$

which implies $G^{p^m} N \operatorname{st}_G(n) = G^{p^m} N$. Hence $\operatorname{st}_G(n) \leq G^{p^m} N \leq J$, and G/N has the p-CSP. □

We recall that a group $G \leq \operatorname{Aut} \mathcal{T}$ has the *weak congruence subgroup property* if every finite-index subgroup contains the derived subgroup of some level stabiliser; cf. [50]. As we will see, the p-Basilica groups are the first examples of weakly regular branch groups with the p-congruence subgroup property but not the weak congruence subgroup property.

1.6 Maximal subgroups and prodense subgroups

The study of maximal subgroups of finitely generated branch groups started in 2000 when Pervova proved in [44] that the Grigorchuk groups and the periodic GGS-groups acting over the p-adic tree do not contain maximal subgroups of infinite index. In [7] Bartholdi, Grigorchuk and Sunic asked if every maximal subgroup in a finitely generated branch group is necessarily of finite index. This problem was solved in 2010 by Bondarenko who constructed in [12] the first example of a finitely generated branch group that does have maximal subgroups of infinite index. His method does not apply to groups acting on the binary and ternary trees. Francoeur and Garrido showed in [25] that the non-torsion Sunic groups have maximal subgroups of infinite index, providing the first examples of finitely generated branch groups acting on the binary tree with such property. In [26] the authors extend the result of Pervova by showing that all the non-periodic GGS-groups acting over the p-adic tree have maximal subgroups only of finite index. In [24] the author proves that every maximal subgroup of the Basilica group is of finite index. In Section 3.7 we will see that this result extends to the p-Basilica groups.

The existence of maximal subgroups of infinite index is related to the existence of proper prodense subgroups.

Definition 1.32. Let G be a group. A subgroup H of G is called *prodense* if $HN = G$ for all non-trivial normal subgroups N of G.

Following the notation in [24] we will denote by $\mathcal{MF}$ the class of groups whose maximal subgroups are all of finite index.

Proposition 1.33. *[24, Prop. 2.22] Let G be a finitely generated infinite group such that every proper quotient of G belongs to $\mathcal{MF}$. Then G admits a proper prodense subgroup if and only if G admits a maximal subgroup of infinite index.*

We observe that when the group G is finite, then G belongs to $\mathcal{MF}$ and every proper quotient of G is in $\mathcal{MF}$. This is not the case for an infinite group, but for a finitely generated branch group the condition to be in $\mathcal{MF}$ for every proper quotient of G is always satisfied.

Proposition 1.34. *[24, Prop. 2.23] Let G be a finitely generated branch group. Then, G is infinite and every proper quotient of G is in $\mathcal{MF}$. In particular, G admits a maximal subgroup of infinite index if and only if it admits a proper prodense subgroup.*

We will see in Chapter 3 that for every prime p, the p-Basilica group is not a branch group, so the previous proposition cannot be applied directly. As proved for the Basilica group in [24, Sec. 4] we will see that every proper quotient of the p-Basilica group is virtually nilpotent. More specifically we will use the following results.

Theorem 1.35. *[24, Thm. 4.10] Let $\mathcal{T}$ be an m-adic tree for $m \geq 2$, and let $G \leq \operatorname{Aut} \mathcal{T}$ be a weakly regular branch group over a subgroup K. Let $\mathcal{P}$ be a property of groups that is preserved under taking finite direct products, quotients and subgroups. Then, every proper quotient of G is virtually $\mathcal{P}$ if and only if G/K' is virtually $\mathcal{P}$.*

In Chapter 3 we will show that for a prime p, every proper quotient of the p-Basilica group is virtually nilpotent. Since every finitely generated virtually nilpotent group is in $\mathcal{MF}$ we can use the following result.

Theorem 1.36. *[24, Lem. 3.1, Thm. 3.2 and Thm. 3.3] Let $\mathcal{T}$ be a regular rooted tree and let $G \leq \operatorname{Aut}\mathcal{T}$ be a weakly branch group. Suppose that every proper quotient of G is in $\mathcal{MF}$. If H is a (proper) prodense subgroup of G, then $\psi_u(\mathrm{st}_H(u))$ is a (proper) prodense subgroup of $\psi_u(\mathrm{st}_G(u))$, for every vertex $u \in X^*$. Furthermore, if $M < G$ is a maximal subgroup of G of infinite index, then $\psi_u(\mathrm{st}_M(u))$ is a maximal subgroup of $\psi_u(\mathrm{st}_G(u))$, for every vertex $u \in X^*$.*

1.7 Automata groups

In this section we define automata groups. The automata groups are constructed from automata. We will not define automata in all generality, we limit ourselves to define the class of *invertible synchronous finite state automata.* We refer to [33] and [8] for more details.

Definition 1.37. An *invertible synchronous finite state automaton* is a set $\mathcal{A} = (Q, X, \pi, \lambda)$ where

(i) X is a finite set called *the alphabet*,

(ii) Q is a finite set called *the set of states*,

(iii) π is a map $\pi : X \times Q \to Q$ called *the transition function*,

(iv) λ is a map $\lambda : X \times Q \to X$, called *the output function*, such that the function $\lambda_q : X \to X$ defined as $\lambda_q(x) = \lambda(x, q)$ is a permutation of X for all $q \in Q$.

In the sequel we will refer to the invertible synchronous finite state automata simply as finite automata.

The maps π and λ can be extended to $X^* \times Q$ according to the following rules:

$$\begin{array}{lcl} \pi(\emptyset, q) = q & \text{and} & \pi(xw, q) = \pi(w, \pi(x, q)) \\ \lambda(\emptyset, q) = \emptyset & & \lambda(xw, q) = \lambda(x, q)\lambda(w, \pi(x, q)) \end{array} \tag{1.7.1}$$

where $x \in X$, $q \in Q$ and $w \in X^*$.

From (1.7.1) for all $q \in Q$ the function λ_q can be extended to an automorphism of X^* that fixes the empty word and preserves the incidence, i.e. it defines an automorphism of an m-adic tree where $m = |X|$.

If the automorphism λ_q is bounded for all $q \in Q$, the automaton $\mathcal{A}$ is called *bounded*.

Definition 1.38. The automorphisms λ_q for $q \in Q$ defined by an automaton $\mathcal{A}$ generate a group with respect to the composition called *automaton group generated by $\mathcal{A}$* or *group generated by the automaton $\mathcal{A}$*.

A finite automaton can be also represented by using directed labelled graphs whose vertices correspond to the elements of Q and there exists an edge from the state q_1 to q_2 if and only if $\pi(x, q_1) = q_2$, for some $x \in X$. In this case the edge is labelled by $x|\lambda(x, q_1)$. By starting from the node corresponding to a state q and following the edges it is possible to find the images of a word $u \in X^*$ under the action of λ_q. In the next figure we have an example.

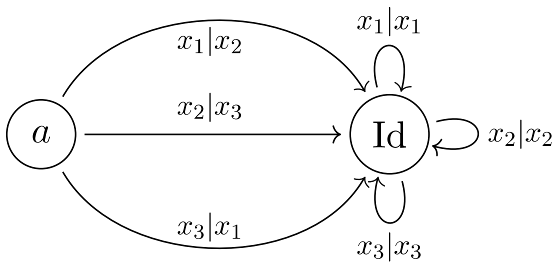


FIGURE 1.5: An example of a directed labelled graph.

In this case $Q = \{\mathrm{Id}, a\}$, $X = \{x_1, x_2, x_3\}$ and the functions π and λ are defined as follows

$$\begin{array}{rccl} \pi: & Q \times X & \to & Q \\ & (\mathrm{Id}, x_i) & \to & \mathrm{Id} \quad \text{for } i \in \{1,2,3\} \\ & (a, x_i) & \to & \mathrm{Id} \quad \text{for } i \in \{1,2,3\} \end{array}$$

and

$$\begin{array}{rcccl} \lambda: & Q \times X & \to & Y & \\ & (\mathrm{Id}, x_i) & \to & x_i & \text{for } i \in \{1,2,3\} \\ & (a, x_i) & \to & x_{i+1} & \text{for } i \in \{1,2\} \\ & (a, x_3) & \to & x_1 & \end{array}$$

For example, if f is the action of a in the automaton represented in Figure 1.5 the image of the word $x_2x_1x_1x_3$ under f is the following

$$f(x_2x_1x_1x_3) = \lambda_a(x_2x_1x_1x_3) = x_3\lambda_{\mathrm{Id}}(x_1x_1x_3) = x_3x_1x_1x_3$$

and it is easy to note that for a general word $x_{i_1}\cdots x_{i_n}$ the image is $f(x_{i_1}\cdots x_{i_n}) = \overline{x}x_{i_2}\cdots x_{i_n}$, where $\overline{x}$ coincides with x_{i_1+1} when $i_1 \in \{1,2\}$ and $\overline{x} = x_1$ when $i_1 = 3$, i.e. it represents a rooted automorphism in $\operatorname{Aut}\mathcal{T}$ where $\mathcal{T}$ is the 3-adic tree.

Now suppose that we are given a subgroup $G = \langle S\rangle$ of $\operatorname{Aut}\mathcal{T}$, where S is finite. If we have the property that $s_x \in S$ for all $s \in S$ and all $x \in X$, then G is an automaton group with $Q = S$. Indeed, it suffices to consider the automaton corresponding to the graph whose vertices are the elements of S and having an edge from s to s_x for every $x \in X$, labelled with $x|s(x)$. By using this, one can see that all groups defined in the next section are automata groups.

1.8 Some groups of automorphisms of regular rooted trees

In this section we define and describe the properties of some well known groups of automorphisms of regular rooted trees.

1.8.1 The first and the second Grigorchuk group

The first Grigorchuk group Γ is a group acting on the binary tree generated by 4 automorphisms $\Gamma = \langle a, b, c, d\rangle$ where a is the rooted automorphism corresponding to the transposition $\sigma = (1,2)$, and the other generators are directed automorphisms defined recursively as follows

$$\psi(b) = (a, c)$$
$$\psi(c) = (a, d)$$
$$\psi(d) = (1, b)$$

The portraits of the generators b, c and d are represented in the Figure 1.6 where the only non-trivial labels until the third level are represented in the picture and from the fourth level the non-trivial labels appear with the same sequence.

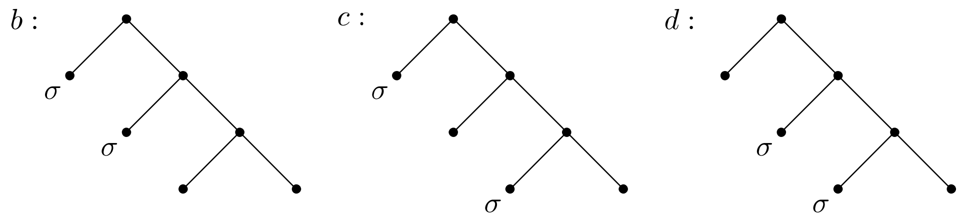


FIGURE 1.6: Directed generators of the first Grigorchuk group.

This group was introduced by Grigorchuk in 1980 [28] as a new counterexample to the General Burnside Problem, indeed it is an infinite finitely generated periodic group. Moreover it is the first known group with intermediate growth, is just infinite, is regular branch over the subgroup $K = \langle [a,b] \rangle^{\Gamma}$ and has the congruence subgroup property.

The second Grigorchuk group was introduced in the same paper. It is a group acting over the 4-adic tree and it is generated by two automorphisms a and b where a is the rooted automorphism corresponding to the cycle $\sigma = (1\,2\,3\,4)$ and b is a directed automorphism defined recursively as follows

$$\psi(b) = (a, 1, a, b).$$

In Figure 1.7 we show the portraits of the generators a and b.

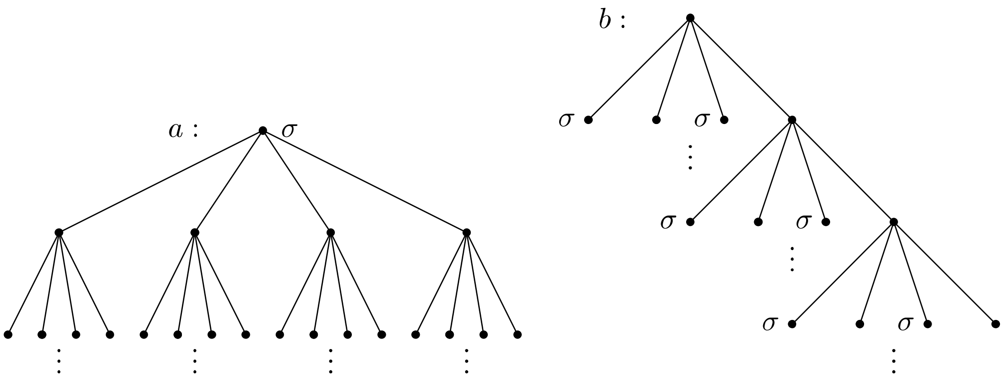


FIGURE 1.7: Generators of the second Grigorchuk group.

As the first Grigorchuk group, the second Grigochuk group is a counterexample to the General Burnside Problem, is just infinite, is regular branch over $\gamma_3(G)$ and possesses the congruence subgroup property.

1.8.2 The Gupta-Sidki groups

For an odd prime p, the Gupta-Sidki p-group is a group acting over the p-adic tree generated by two automorphisms a and b where a is the rooted automorphism corresponding to the cycle $\sigma = (1\,2\,\cdots\,p)$ and b is a directed automorphism defined recursively as follows

$$\psi(b) = (a, a^{-1}, 1, \dots, 1, b).$$

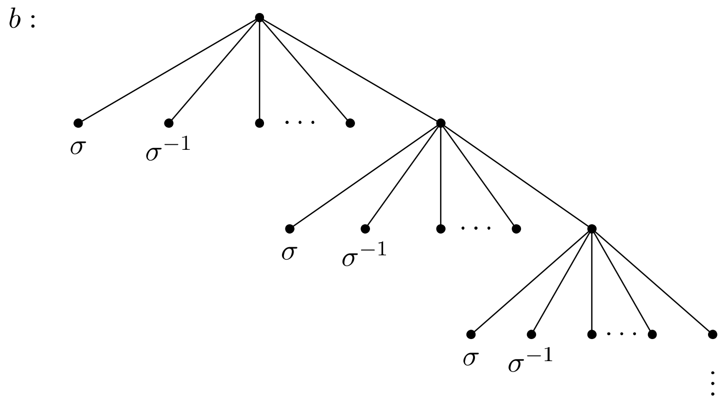


Figure 1.8: Directed generator of the Gupta-Sidki p-group.

These groups were defined by Gupta and Sidki in 1983 [36]. For every prime p the Gupta-Sidki p-group is a counterexample to the General Burnside Problem, is just infinite, is regular branch over its derived subgroup and has the congruence subgroup property.

1.8.3 The GGS-groups

The GGS-groups are a generalization of the second Grigorchuk group and the Gupta-Sidki groups. Given $\mathbf{e} = (e_1, \dots, e_{m-1}) \neq (0, \dots, 0)$ in $(\mathbb{Z}/m\,\mathbb{Z})^{m-1}$, *the GGS-group* G corresponding to the *defining vector* $\mathbf{e}$ is the subgroup $G = \langle a, b\rangle$ of $\operatorname{Aut}\mathcal{T}$, where a denotes the rooted automorphism corresponding to the cyclic permutation $\sigma =$

$(1, 2, \ldots, m)$ and b is a directed automorphism belonging to $\mathrm{st}(1)$ defined recursively by

$$\psi(b) = (a^{e_1}, \ldots, a^{e_{m-1}}, b).$$

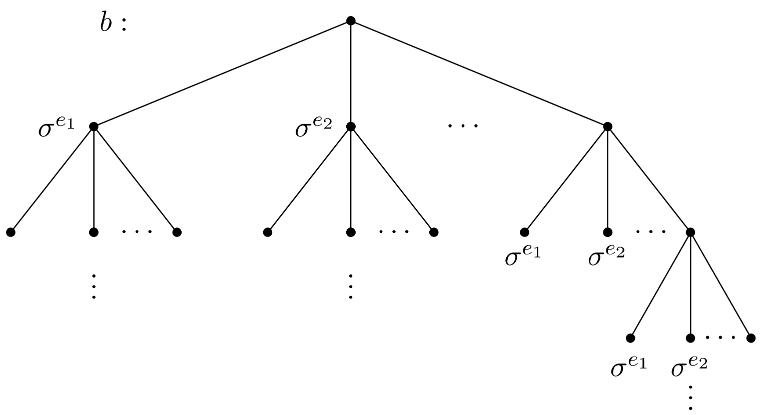


FIGURE 1.9: Directed generator of a general GGS-group.

From the recursive definition of b it is clear the reason for the exclusion of the defining vector $\mathbf{e} = (0, \ldots, 0)$, since in this case the automorphism b would be trivial and G would be isomorphic to the cyclic group C_m. We note that a has order m and the order of b is m/d where $d = \gcd(e_1, \ldots, e_{m-1}, m)$.

The labels in the portrait of b are all trivial except the labels in the vertices of the form ux where $u \in \{\emptyset, x_m, x_m x_m, x_m x_m x_m, \ldots\}$ and $x \in \{x_1, x_2, \ldots, x_{m-1}\}$ that are equal to σ^{e_x}.

We observe that the second Grigorchuk group is the GGS-group acting over the 4-adic tree with defining vector $\mathbf{e} = (1, 0, 1)$ and for a prime p the Gupta-Sidki p-group is the GGS-group acting over the p-adic tree with defining vector given by $\mathbf{e} = (1, -1, 0, \overset{p-3}{\ldots}, 0)$.

1.8.4 The Basilica group

The Basilica group is a group acting over the binary tree introduced by Grigorchuk and Żuk in [34]. The Basilica group is torsion-free, of exponential growth, is weakly regular branch over its derived subgroup but not branch, does not have the congruence subgroup property and is not just-infinite. It was the first example of an amenable but not subexponentially amenable group.

The Basilica group $\mathcal{B}$ is generated by two automorphisms $\mathcal{B} = \langle a, b \rangle$ defined recursively as follows:

$$\psi(a) = (1, b) \quad \text{and} \quad \psi(b) = (1, a)\sigma$$

where σ denotes the transposition $(1, 2)$. The portraits of the generators are the following.

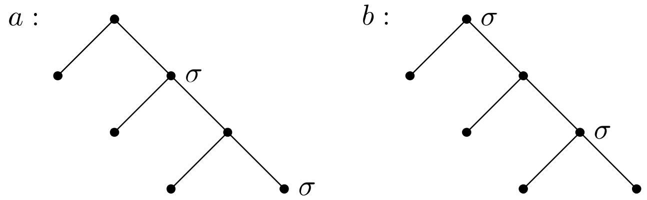


FIGURE 1.10: Generators of the Basilica group.

The labels in the portraits are all trivial except the labels in the vertices $u \in \{x_2 \overset{2n+1}{\cdots} x_2 \mid n \in \mathbb{N}\}$ for the automorphism a and in vertices $u \in \{\emptyset, x_2 \overset{2n}{\cdots} x_2 \mid n \in \mathbb{N}\}$ for the automorphism b that are equal to σ.

Chapter 2

GGS-groups over primary trees: branch structures

In this chapter we study some properties of the GGS-groups acting over the p^n-adic tree that let us describe the branch structures of most of these groups.

We already mentioned that Vovkivsky gave in [56] a criterion for an infinite and periodic GGS-group G over the p^n-adic tree to be a regular branch group. He actually showed that a periodic GGS-group G with defining vector $\mathbf{e} = (e_1, \dots, e_{p^n-1})$ is a regular branch group over G'' if and only if there exists $k \in \{1, \dots, p^n - 1\}$ such that $e_k \not\equiv 0 \bmod p$. He also proved two criteria for these groups to be respectively infinite and periodic. In particular G is periodic if and only if for each $k = 0, \dots, n-1$ the following conditions hold

$$S_k := e_{p^k} + e_{2p^k} + \cdots + e_{p^n - p^k} \equiv 0 \pmod{p^{k+1}} \tag{2.0.1}$$

and it is infinite if and only if there exists an $i \geq 0$ such that

$$R_0 \leq R_1 \leq \cdots \leq R_i = R_{i+1} = \cdots < n \tag{2.0.2}$$

where the sequence R_j is defined recursively as follows: R_0 is the largest integer such that $p^{R_0} \mid e_s$ for all $s \in \{1, \dots, p^n - 1\}$, and for $j \geq 0$, and while $R_j < n$, R_{j+1} is defined inductively as the largest integer such that $p^{R_{j+1}}$ divides e_s for all $s \in \{p^{R_j}, 2p^{R_j}, \dots, p^n - p^{R_j}\}$.

By Vovkivsky's result it follows that if $n = 1$ a GGS-group G acting over the p-adic tree is always infinite and it is periodic if and only if the sum of the components of the defining vector is 0 modulo p. In [22] and [20] it has been proved that a GGS-group over the p-adic tree is always regular branch with the sole exception when the defining vector is constant, in which case it is shown that G is not branch although it is weakly regular branch.

From (2.0.1) we note that most of the GGS-groups over the p^n-adic tree are not periodic. In this chapter we extend some of the previous results to the general case of a p^n-adic tree in many instances.

2.1 Preliminary results

In this section we collect some properties and preliminary results about the GGS-groups acting over the p^n-adic tree.

It is convenient to recall the definition. The GGS-group G corresponding to the defining vector $\mathbf{e} = (e_1, \dots, e_{p^n-1}) \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1} \setminus \{0\}$ is the subgroup of $\operatorname{Aut}\mathcal{T}$ generated by two automorphisms $G = \langle a, b\rangle$, where a is the rooted automorphism corresponding to the cycle $\sigma = (1\,2\,\cdots\,p^n)$ and b is the directed automorphism defined recursively as follows

$$\psi(b) = (a^{e_1}, \dots, a^{e_{p^n-1}}, b).$$

For every integer $i \geq 0$ let $b_i := b^{a^i}$. From the property of the section (1.1.4) we have

$$\begin{aligned}
\psi(b) &= (a^{e_1}, \dots, a^{e_{p^n-1}}, b)\\
\psi(b_1) &= (b, a^{e_1}, \dots, a^{e_{p^n-1}})\\
&\vdots\\
\psi(b_{p^n-1}) &= (a^{e_2}, \dots, a^{e_{p^n-1}}, b, a^{e_1}).
\end{aligned}$$

As already mentioned, we let p^{R_0} be the highest power of p dividing all entries of $\mathbf{e}$. We note that the order of the generator a is p^n and from the general expression for the order of b over the m-adic tree given in Section 1.8.3, it follows that the order of b is p^{n-R_0}.

Proposition 2.1. *Let G be a GGS-group acting over the p^n-adic tree. Then*

(i) $\operatorname{st}_G(1) = \langle b\rangle^G = \langle b, b^a, \dots, b^{a^{p^n-1}}\rangle$.

(ii) $G = \langle a \rangle \ltimes \mathrm{st}_G(1)$.

Proof. Clearly $\langle b \rangle^G = \langle b, b^a, \ldots, b^{a^{p^n-1}} \rangle$ and also the inclusion $\langle b, b^a, \ldots, b^{a^{p^n-1}} \rangle \leq \mathrm{st}_G(1)$ is trivial. For the reverse inclusion we note that an element $g \in G$ can be written as $g = a^i h$ where $i \in \{0, 1, \ldots, p^n - 1\}$ and $h \in \langle b, b^a, \ldots, b^{a^{p^n-1}} \rangle$. In particular $g \in \mathrm{st}_G(1)$ if and only if $i = 0$ and (i) is proved. Since $\mathrm{st}_G(1) \trianglelefteq G$ and the intersection $\langle a \rangle \cap \mathrm{st}_G(1)$ is trivial, the proof is completed. □

Our first goal is to determine the abelianisation of G. Following the notation in [2], let $H = \langle c, d \mid c^{p^n} = d^{p^{n-R_0}} = 1 \rangle$ be the free product $\langle c \rangle * \langle d \rangle$ of two cyclic groups of order respectively p^n and p^{n-R_0}. We prove that the abelianisation of G coincides with the abelianisation of H, i.e. $G/G' \cong H/H' \cong C_{p^n} \times C_{p^{n-R_0}}$ (see [47, Exercise 6.2.4]).

We consider the natural projection $\pi : H \to G$ such that $c \mapsto a$ and $d \mapsto b$. Thus $G \cong H/K$ where $K = \ker \pi$. The description of the kernel K can be deduced from [48, Proposition 1] where it was proved for Aleshin type groups. These groups can be seen as spherically transitive groups acting over spherically homogeneous rooted trees generated by rooted automorphisms and directed automorphisms. However the condition to be spherically transitive is not required in the proof of [48, Proposition 1]. Thus it applies to all GGS-groups acting over the p^n-adic tree. Such proposition can be rephrased, as done in [2, Proposition 4.2] for multi-edge spinal groups, in the case of GGS-groups acting on primary trees as follows. For this purpose, let us consider the homomorphism

$$\begin{array}{rccc} \Phi : & \langle d \rangle^H & \to & H \times \overset{p^n}{\cdots} \times H \\ & d^{c^k} & \mapsto & (c^{e_1}, \ldots, c^{e_{p^n-1}}, d)^{\sigma^k}, \end{array}$$

where σ permutes the components of a tuple in $H \times \overset{p^n}{\cdots} \times H$ according to the cycle $(1\,2 \ldots p^n)$.

Proposition 2.2. *Let $G = \langle a, b \rangle$ be a GGS-group acting on the p^n-adic tree, and let H be as above. Consider the subgroup $K = \cup_{n=0}^{\infty} K_n$ where $K_0 = 1$ and $K_n = \Phi^{-1}(K_{n-1} \times \overset{p^n}{\cdots} \times K_{n-1})$ for $n \in \mathbb{N}$. Then the epimorphism $\pi : H \to G$ given by $c \mapsto a$ and $d \mapsto b$ has kernel $\ker(\pi) = K$. In particular $G \cong H/K$.*

Now, following the idea in [2, Proposition 4.3] we determine the abelianisation G/G'.

Theorem 2.3. *Let G be a GGS-group acting on the p^n-adic tree. Then $G/G' \cong C_{p^n} \times C_{p^{n-R_0}}$.*

Proof. First of all we prove that

$$\Phi^{-1}(H' \times \overset{p^n}{\cdots} \times H') \leq H'. \tag{2.1.1}$$

For this purpose, let $h \in \langle d \rangle^H$, and so $h = (d^{i_1})^{c^{j_1}} \cdots (d^{i_m})^{c^{j_m}}$ for $m \geq 0$, $i_1, \dots, i_m \in \{1, \dots, p^{n-R_0} - 1\}$, and $j_1, \dots, j_m \in \mathbb{Z}/p^n\mathbb{Z}$ with $j_s \neq j_{s+1}$ for $1 \leq s \leq m-1$. We write $\Phi(h) = (h_1, \dots, h_{p^n})$. Let $\epsilon_{d,k}(h)$ the sum of i_l for $l \in \{1, \dots, m\}$ such that $j_l = k$, which coincides with the exponent sum of h with respect to d^{c^k}. Then we have

$$\epsilon_d(h) = \sum_{l=1}^{m} i_l = \sum_{k=1}^{p^n} \epsilon_{d,k}(h) = \sum_{k=1}^{p^n} \epsilon_d(h_k).$$

If $h \notin H'$, since $\epsilon_c(h) \equiv 0 \pmod{p^n}$ and $H/H' \cong C_{p^n} \times C_{p^{n-R_0}}$, it follows that $\epsilon_d(h) \not\equiv 0 \pmod{p^{n-R_0}}$, thus $\epsilon_d(h_k) \not\equiv 0 \pmod{p^{n-R_0}}$ for some $k \in \{1, \dots, p^n\}$. It follows that $\Phi(h) \notin H' \times \overset{p^n}{\cdots} \times H'$ and (2.1.1) holds.

Let $K = \cup K_n$ as above. We show now that $K \leq H'$ by proving by induction that $K_n \leq H'$ for all n. Trivially the result holds for $n = 0$, and the case $n = 1$ follows from (2.1.1). Assume by induction that $K_{n-1} \leq H'$, then again from (2.1.1) we have

$$K_n = \Phi^{-1}(K_{n-1} \times \overset{p^n}{\cdots} \times K_{n-1}) \leq \Phi^{-1}(H' \times \overset{p^n}{\cdots} \times H') \leq H',$$

as desired. Since $G \cong H/K$ from the previous proposition, it follows that $G/G' \cong H/H'K = H/H'$, which completes the proof. ☐

It readily follows from the definition that all GGS-groups are self-similar. Consequently we can consider the group homomorphism $\psi_u : \mathrm{st}_G(u) \to G$ given by $g \mapsto g_u$. Recall that a subgroup G of $\mathrm{Aut}\,\mathcal{T}$ is said to be *fractal* if it is self-similar and ψ_u is onto for every vertex u of the tree.

We denote by $\mathcal{F}(p^n)$ the set of defining vectors $\mathbf{e} \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1} \backslash (p\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}$. We observe that the condition $\mathbf{e} \in \mathcal{F}(p^n)$ is equivalent to requiring that $R_0 = 0$.

The following lemma shows that the condition $\mathbf{e} \in \mathcal{F}(p^n)$ is a necessary and sufficient condition for the group G to be fractal and spherically transitive.

Lemma 2.4. *Let G be a GGS-group over the p^n-adic tree with defining vector* $\mathbf{e}$*. Then the following conditions are equivalent:*

(i) G *is spherically transitive.*

(ii) G *is fractal.*

(iii) The composition of ψ *with the projection on any component is surjective from* $\mathrm{st}_G(1)$ *onto* G*.*

(iv) $\mathbf{e} \in \mathcal{F}(p^n)$*.*

Proof. We start by proving that $\mathbf{e} \in \mathcal{F}(p^n)$ implies that G is spherically transitive, fractal and that the composition of ψ with the projection on any component is surjective from $\mathrm{st}_G(1)$ onto G. By [55, Lemma 2.7], in order to prove that G is spherically transitive and fractal, it suffices to see that G acts transitively on the vertices of the first level of $\mathcal{T}$ and that $\psi_x(\mathrm{st}_G(x)) = G$ for some $x \in X$. The former is obvious, since $a \in G$, and for the latter observe that since $\mathbf{e} \in \mathcal{F}(p^n)$ we have $p \nmid e_i$ for some i and then $\psi_{p^n}(b_{-i}) = a^{e_i}$ and $\psi_{p^n}(b) = b$ generate G. This, together with the fact that G is spherically transitive, implies (iii).

Now let $\mathbf{e} \notin \mathcal{F}(p^n)$. Since p divides all components of $\mathbf{e}$, for every $x \in X$ and $g \in \mathrm{st}_G(x)$ the section g_x is a word in $\{a^p, b\}$. Thus G is not fractal and ψ_x cannot be surjective from $\mathrm{st}_G(1)$ onto G since we cannot obtain sections of the form $a^i h$ for i coprime with p and $h \in \mathrm{st}_G(1)$. Now assume for a contradiction that G is spherically transitive. Then there exists $g \in G$ such that $g(x_1x_1) = x_1x_2$. However, from (1.1.3) and by the above we have $g(x_1x_1) = g(x_1)g_{x_1}(x_1) = x_1x_j$ for some $j \equiv 1 \pmod p$, which is a contradiction. □

As a consequence, since branch groups are spherically transitive by definition, in the remainder we will always assume that $\mathbf{e} \in \mathcal{F}(p^n)$, unless otherwise stated. Then $R_0 = 0$ and both a and b have order p^n.

The next lemma is the main tool for finding a branch structure in a GGS-group and it is the generalization of [22, Prop. 2.18] to the case of a GGS-group G acting on a p^n-adic tree with defining vector in $\mathcal{F}(p^n)$. Indeed the only required conditions on G in the proof on the aforementioned proposition are that G is spherically transitive and that the map ψ is surjective on each component.

Lemma 2.5. *Let G be a GGS-group over the p^n-adic tree with defining vector in $\mathcal{F}(p^n)$, and let L and N be two normal subgroups of G. If $L = \langle X \rangle^G$, and $(x, 1, \ldots, 1) \in \psi(N)$ for every $x \in X$, then*

$$\underbrace{L \times \cdots \times L}_{p^n} \subseteq \psi(N).$$

We observe that if $G = \langle a, b \rangle$ is a GGS-group with defining vector $\mathbf{e}$, then for an integer $\lambda \not\equiv 0 \bmod p$ the set $\{a, b^\lambda\}$ is also a generating set for G, where b^λ can be thought of as a directed automorphism defined recursively by $\psi(b^\lambda) = (a^{\lambda e_1}, \ldots, a^{\lambda e_{p^n-1}}, b^\lambda)$. Since we reduced our study to the GGS-groups with defining vector $\mathbf{e} \in \mathcal{F}(p^n)$, there exists an invertible component, say e_k. Thus up to multiplying the defining vector by the inverse of e_k modulo p^n we can always assume that one of the components of $\mathbf{e}$ is 1. The next lemma, that generalizes [22, Thm. 2.16], tells us also more. It shows that if a component e_k of the defining vector of a GGS-group G is invertible mod p then there exists another GGS-group which coincides with G up conjugation in $\operatorname{Aut}\mathcal{T}$ whose defining vector has the p^s-th component equal to 1, where p^s is the highest power of p dividing k.

Lemma 2.6. *Let G be a GGS-group over the p^n-adic tree with defining vector $\mathbf{e} \in \mathcal{F}(p^n)$, and assume that $e_k \not\equiv 0 \bmod p$. If p^s is the highest power of p dividing k then there exist $\alpha \in \operatorname{Sym}(p^n - 1)$ and $f \in \operatorname{Aut}\mathcal{T}$ such that:*

(i) $\alpha(p^s) = k$,

(ii) $\alpha(p^n - i) = p^n - \alpha(i)$ *for all* $i = 1, \ldots, p^n - 1$.

(iii) G^f *is the GGS-group with defining vector* $\mathbf{e}' = e_k^{-1}(e_{\alpha(1)}, \ldots, e_{\alpha(p^n-1)})$. *In particular,* $e'_{p^s} = 1$.

Proof. By hypothesis we can write $k = hp^s$ where $h \not\equiv 0 \bmod p$. If r is a solution to the congruence $hr \equiv 1 \bmod p^{n-s}$, then the permutation $\delta \in \operatorname{Sym}(p^n)$ given by $\delta(i) \equiv ri \bmod p^n$ for every i satisfies that $\sigma^\delta = \sigma^r$ and $\delta(k) = p^s$.

Let us define $f \in \operatorname{Aut}\mathcal{T}$ recursively by $f = dh$, where d is the rooted automorphism corresponding to δ, and $h \in \operatorname{st}(1)$ is defined via $\psi(h) = (f, \ldots, f)$. Note that h commutes with any rooted automorphism, since its components under ψ are all the same. Then

$$a^f = (a^d)^h = (a^r)^h = a^r, \tag{2.1.2}$$

since a^d is the rooted automorphism corresponding to the permutation $\sigma^\delta = \sigma^r$.

Now let $\alpha = \delta^{-1}$ and observe that α satisfies (i) and (ii). Also

$$\psi(b^f) = \psi(b^d)^{\psi(h)} = (a^{e_{\alpha(1)}}, \dots, a^{e_{\alpha(p^n-1)}}, b)^{\psi(h)} = (a^{re_{\alpha(1)}}, \dots, a^{re_{\alpha(p^n-1)}}, b^f),$$

by using (2.1.2). Since $r \not\equiv 0 \pmod p$, it follows that $G^f = \langle a^r, b^f \rangle = \langle a, b^f \rangle$ is the GGS-group with defining vector $(re_{\alpha(1)}, \dots, re_{\alpha(p^n-1)})$. If we multiply this vector by the inverse of re_k modulo p^n then we see that G^f is also the GGS-group with defining vector $\mathbf{e}'$, which proves (iii). □

Using the notation in [20] we define $y_0 = ba^{-1}$ and for all integers i we define $y_i = y_0^{a^i}$. For a GGS-group G over the p^n-adic tree the subgroup $K = \langle y_0 \rangle^G$ will play an important role in the study of the structure of G, so we collect here some of its properties.

Lemma 2.7. *Let G be a GGS-group over the p^n-adic tree and let $K = \langle y_0 \rangle^G$. Then*

(i) $G' \leq K$.

(ii) $K = \langle y_0, \dots, y_{p^n-1} \rangle$.

(iii) $K' = \langle [y_0, y_i] \mid i \in \{1, \dots, p^n - 1\} \rangle^G = \langle [y_0^g, y_1^h] \mid g, h \in G \rangle^G$.

(iv) $y_{p^n-1} y_{p^n-2} \cdots y_1 y_0 = 1$.

Proof. We observe that the set $\{a, y_0\}$ generates G, thus $G' = \langle [a, y_0] \rangle^G$ and (i) follows since K is normal in G. The second item follows from the fact that a has order p^n and $y_i^b = y_i^{aa^{-1}b} = y_{i+1}^{y_1}$. Since K' is normal in G we have

$$K' = \langle [y_i, y_j] \mid i, j \in \{1, \dots, p^n - 1\} \rangle^G.$$

The second equality in (iii) follows from the fact that $K = \langle y_0 \rangle^G = \langle y_1 \rangle^G$. For the first equality we observe that $[y_i, y_j] = [y_0, y_{j-i}]^{a^i}$ and this completes the proof of (iii). An easy calculation shows that (iv) holds. □

2.2 Regular branch GGS-groups over primary regular rooted trees

As remarked in the previous section, in order to find a branch structure for a GGS-group G acting over the p^n-adic tree, we can assume that G has a defining vector in

$\mathcal{F}(p^n)$. This is equivalent to asking that $R_0 = 0$ and this implies that the order of both generators a and b of G is p^n.

Given $\mathbf{e} \in \mathcal{F}(p^n)$, we define

$$Y(\mathbf{e}) := \{1 \le i \le p^n - 1 \mid e_i \not\equiv 0 \bmod p\} \subseteq X, \tag{2.2.1}$$

and

$$t(\mathbf{e}) := \max\{s \in \mathbb{Z} \mid s \ge 0 \text{ and } p^s \mid i \text{ for all } i \in Y(\mathbf{e})\}. \tag{2.2.2}$$

If there is no confusion about $\mathbf{e}$ then we simply write Y and t for $Y(\mathbf{e})$ and $t(\mathbf{e})$. Then we have $Y \subseteq \{p^t, 2p^t, \ldots, p^n - p^t\}$, and we say that Y is *maximal* if the equality holds. Also we define

$$\mathcal{E}(p^n) = \{\mathbf{e} \in \mathcal{F}(p^n) \mid e_{ip^t} \equiv e_{jp^t} \bmod p \text{ for all } 1 \le i, j \le p^{n-t} - 1\}, \tag{2.2.3}$$

that is, the set of vectors that have the same values modulo p for the set of indices $\{p^t, 2p^t, \ldots, p^n - p^t\}$. Note that if $\mathbf{e} \in \mathcal{E}(p^n)$ then Y is maximal.

According to Lemma 1.14, G has one of the four types of branch structure that we have defined in Definition 1.10 and Definition 1.11 if and only if G^f does. Thus, by part (i) of Lemma 2.6, in order to study branch properties in a GGS-group, we may assume without loss of generality that $e_{p^t} = 1$, where $t = t(\mathbf{e})$ is as in (2.2.2). In the remainder of this section, we fix the notation $k := p^t$.

Definition 2.8. Suppose that $\mathbf{e}$ is a defining vector for a GGS-group G. We shall say that $\mathbf{e}$ is invertible-symmetric, IS for short, if for all i the component e_i is invertible if and only if e_{p^n-i} is. Then we also say that G is IS.

We start our analysis of branch structures in GGS-groups by dealing with the case when G is not IS. By Lemma 2.6 we may assume that $e_k = 1$ and $q := e_{p^n-k} \equiv 0 \bmod p$. (Note that here part (ii) of the lemma is also needed.) We define a sequence $\{g_i\}_{i \ge 0}$ of automorphisms of $\mathcal{T}$ by means of

$$\psi(g_i) = (1, \ldots, 1, [a, b], 1, \ldots, 1) \cdot (1, \ldots, 1, [b^{q^i}, a^{q^{i+1}}], 1, \ldots, 1), \tag{2.2.4}$$

where the non-trivial components appear in the k-th position and in the $(p^n - 2ik)$-th position, respectively, the latter being understood modulo p^n.

Lemma 2.9. *The sequence $\{g_i\}_{i \ge 0}$ defined in (2.2.4) is contained in $\mathrm{st}_G(1)'$.*

Proof. Since $\psi([b,b_k]) = (1,\dots,1,[a,b],1,\dots,1,[b,a^q])$, where the non-trivial components are at positions k and p^n, we have $g_0 = [b,b_k]$. Similarly,

$$\psi([b^{q^i}, b_k^{q^{i-1}}]^{a^{-(2i-1)k}}) = (1,\dots,1,[b^{q^i},a^{q^i}],1,\dots,1)\cdot(1,\dots,1,[a^{q^i},b^{q^{i-1}}],1,\dots,1), \tag{2.2.5}$$

where the non-trivial components appear at positions $p^n-(2i-1)k$ and $p^n-(2i-2)k$, respectively, and

$$\psi([b^{q^i}, b_k^{q^i}]^{a^{-2ik}}) = (1,\dots,1,[b^{q^i},a^{q^{i+1}}],1,\dots,1)\cdot(1,\dots,1,[a^{q^i},b^{q^i}],1,\dots,1), \tag{2.2.6}$$

with non-trivial components at positions p^n-2ik and $p^n-(2i-1)k$. By combining (2.2.4), (2.2.5), and (2.2.6), one can readily check that

$$g_i = g_{i-1}[b^{q^i}, b_k^{q^{i-1}}]^{a^{-(2i-1)k}}[b^{q^i}, b_k^{q^i}]^{a^{-2ik}}$$

for all $i \geq 1$. Thus $g_i \in \mathrm{st}_G(1)'$ by induction on i. □

The following result is a consequence of the previous lemma.

Theorem 2.10. *If G is not IS then*

$$\psi(\mathrm{st}_G(1)') = G' \times \cdots \times G'.$$

In particular, G is regular branch over G'.

Proof. The inclusion $\subseteq$ is obvious since G is self-similar, so we only need to prove the reverse inclusion. Let g_i be defined as in (2.2.4). Since $o(b) = p^n$ and q is divisible by p, we have $[b^{q^n}, a^{q^{n+1}}] = 1$. Thus $\psi(g_n)$ has all components equal to 1 with the exception of the component at position k, which is equal to $[a,b]$. Since $g_n \in \mathrm{st}_G(1)'$ by Lemma 2.9 and $G' = \langle [a,b]\rangle^G$, the desired inclusion follows from Lemma 2.5. □

In the remainder of this section we shall assume that G is IS. We continue our analysis of branch structures by considering the case when Y is not maximal. Let h be the smallest integer in $\{1,\dots,p^{n-t}-1\}$ such that $hk \notin Y$. Note that $h \geq 2$.

Then we set

$$q := e_{p^n - hk}$$
$$y := e_{p^n-(h-1)k}$$
$$z := e_{p^n-k},$$

in other words, q, y and z are the symmetrical components of e_{hk}, $e_{(h-1)k}$ and e_k. Thus p divides q, and y and z are invertible modulo p. In this case we define a sequence $\{g_i\}_{i\geq 0}$ of automorphisms of $\mathcal{T}$ as follows:

$$\psi(g_i) = (1,\ldots,1,[a,b,a],1,\ldots,1)\cdot(1,\ldots,1,[b^{z^i},a^{z^{i+1}},a^{q^{2i+1}y^{-(2i+1)}}],1,\ldots,1), \tag{2.2.7}$$

where the non-trivial components are the k-th and the (p^n-2ik)-th.

Lemma 2.11. *The sequence* $\{g_i\}_{i\geq 0}$ *defined in* (2.2.7) *is contained in* $\gamma_3(\mathrm{st}_G(1))$.

Proof. It is easy to see that $g_0 = [b, b_k, b_{hk}^{y^{-1}}]$. We claim that $g_i = g_{i-1}c_1c_2$, where

$$c_1 = ([b_k^{z^{i-1}}, b^{z^i}, b_{hk}^{q^{2i-1}y^{-2i}}]^{-1})^{a^{-(2i-1)k}}$$

and

$$c_2 = [b^{z^i}, b_k^{z^i}, b_{hk}^{q^{2i}y^{-(2i+1)}}]^{a^{-2ik}}.$$

Then g_i belongs to $\gamma_3(\mathrm{st}_G(1))$ by induction on i.

The claim follows immediately from (2.2.7) by taking into account that

$$\psi(c_1) = (1,\ldots,1,[a^{z^i},b^{z^i},a^{q^{2i}y^{-2i}}]^{-1},1,\ldots,1)$$
$$\cdot(1,\ldots,1,[b^{z^{i-1}},a^{z^i},a^{q^{2i-1}y^{-(2i-1)}}]^{-1},1,\ldots,1),$$

where the non-trivial components appear in positions $p^n-(2i-1)k$ and $p^n-(2i-2)k$, and that

$$\psi(c_2) = (1,\ldots,1,[b^{z^i},a^{z^{i+1}},a^{q^{2i+1}y^{-(2i+1)}}],1,\ldots,1)$$
$$\cdot(1,\ldots,1,[a^{z^i},b^{z^i},a^{q^{2i}y^{-2i}}],1,\ldots,1),$$

with non-trivial components p^n-2ik and $p^n-(2i-1)k$. □

As a consequence of the previous lemma we have the following result.

Theorem 2.12. *If G is IS and Y is not maximal, then*

$$\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \cdots \times \gamma_3(G).$$

In particular, G is regular branch over $\gamma_3(G)$.

Proof. Since p divides q and a has order p^n, for large enough i we have

$$\psi(g_i) = (1, \ldots, 1, [a, b, a], 1, \ldots, 1),$$

where $[a, b, a]$ appears in the k-th component. Also $g_i \in \gamma_3(\mathrm{st}_G(1)))$ by Lemma 2.11. Moreover, since z is invertible modulo p,

$$\psi(b_k^{z^{-1}} b_{-k}^{-1}) = (\ldots, b^{z^{-1}} a^{-e_{2k}}, \ldots, 1)$$

where the first displayed component is the k-th one. Hence

$$\psi([b, b_k, b_k^{z^{-1}} b_{-k}^{-1}]) = (1, \ldots, 1, [a, b, b^{z^{-1}} a^{-e_{2k}}], 1, \ldots, 1)$$

where the non-trivial component appears in position k. Since $G = \langle a, b^{z^{-1}} a^{-e_{2k}} \rangle$ then

$$\gamma_3(G) = \langle [a, b, a], [a, b, b^{z^{-1}} a^{-e_{2k}}] \rangle^G,$$

and the result follows from Lemma 2.5. □

We now consider the case when Y is maximal. In this case G is trivially IS.

Theorem 2.13. *Suppose that Y is maximal. If there exists $m \in Y \setminus \{k, p^n - k\}$ such that*

$$\delta_m := \det \begin{pmatrix} e_{m-k} & e_m \\ e_m & e_{m+k} \end{pmatrix} \not\equiv 0 \bmod p$$

then $\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \cdots \times \gamma_3(G)$. In particular, G is regular branch over $\gamma_3(G)$.

Proof. In the following formulas the displayed components are the k-th and the last one. We observe that

$$\psi(b_{-m}^{e_{m-k}} b_{-m+k}^{-e_m}) = (\ldots, a^{\delta_m}, \ldots, 1).$$

Since by hypothesis δ_m is invertible modulo p^n, there exists a suitable power g of $b_{-m}^{e_{m-k}} b_{-m+k}^{-e_m}$ such that

$$\psi(g) = (\dots, a, \dots, 1).$$

On the other hand,

$$\psi(b_k b_{-k}^{-e_{p^n-k}}) = (\dots, ba^{-e_{2k}e_{p^n-k}}, \dots, 1)$$

and so by multiplying $b_k b_{-k}^{-e_{p^n-k}}$ by a suitable power of g we can find an element $h \in \mathrm{st}_G(1)$ such that

$$\psi(h) = (\dots, b, \dots, 1).$$

Since all components of $\psi([b, b_k])$ are trivial except for the k-th one and the last one, the former being equal to $[a, b]$, we get

$$\psi([b, b_k, g]) = (1, \dots, 1, [a, b, a], 1, \dots, 1),$$
$$\psi([b, b_k, h]) = (1, \dots, 1, [a, b, b], 1, \dots, 1).$$

Hence $\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \cdots \times \gamma_3(G)$ by Lemma 2.5. □

Theorem 2.14. *Suppose that Y is maximal and that for all $m \in Y \setminus \{k, p^n - k\}$ we have*

$$\delta_m := \det \begin{pmatrix} e_{m-k} & e_m \\ e_m & e_{m+k} \end{pmatrix} \equiv 0 \bmod p.$$

If $\mathbf{e} \notin \mathcal{E}(p^n)$ then

$$\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \cdots \times \gamma_3(G).$$

In particular, G is regular branch over $\gamma_3(G)$.

Proof. We first observe that when $p = 3$ and $k = 3^{n-1}$, the set Y coincides with $\{k, p^n - k\}$ since it is maximal by hypothesis. Thus the condition on the determinants vacuously holds. This special case needs to be studied separately. Assume first that $Y \setminus \{k, p^n - k\} \neq \emptyset$. From the condition $\delta_m \equiv 0 \pmod p$ for all $m \in Y \setminus \{k, p^n - k\}$ we get

$$e_{ik} \equiv e_{2k}^{i-1} \bmod p \tag{2.2.8}$$

for all $i = 2, \dots, p^{n-t} - 1$. Since $\mathbf{e}$ is not constant modulo p for the indices in Y, it follows that $e_{2k} \not\equiv 1 \bmod p$.

We observe that

$$\psi([b, b_k, b_k b_{-k}^{-e_{p^n-k}}]) = (1, \dots, 1, [a, b, ba^{-e_{2k}e_{p^n-k}}], 1, \dots, 1), \tag{2.2.9}$$

where the displayed component is the k-th one.

Now, taking into account that $k \neq 3^{n-1}$, we set $g := b^{e_{p^n-3k}} b_{2k}^{-e_{p^n-k}}$. Then

$$\psi(g) = (\dots, a^{e_{p^n-3k} - e_{p^n-k}^2}, \dots, 1, \dots),$$

where the displayed components are the k-th one and the $(p^n - k)$-th one. If y is the inverse of e_{2k} modulo p^n, we get

$$\psi([b_{p^n-k}^y, b_k, g]) = (1, \dots, 1, [a, b, a^{e_{p^n-3k} - e_{p^n-k}^2}], 1, \dots, 1), \tag{2.2.10}$$

where the only non-trivial component is the k-th one.

Let $s := p^{n-t}$. By (2.2.8) and Fermat's Little Theorem, we have

$$e_{p^n-3k} - e_{p^n-k}^2 = e_{(s-3)k} - e_{(s-1)k}^2 \equiv e_{2k}^{s-4} - e_{2k}^{2(s-2)} \equiv e_{2k}^{-3}(1 - e_{2k}) \not\equiv 0 \bmod p,$$

since $e_{2k} \not\equiv 1 \pmod p$. Hence $\{a^{e_{p^n-3k} - e_{p^n-k}^2}, ba^{-e_{2k}e_{p^n-k}}\}$ is a generating set of G. Since $G' = \langle [a, b] \rangle^G$, we have

$$\gamma_3(G) = \langle [a, b, a^{e_{p^n-3k} - e_{p^n-k}^2}], [a, b, ba^{-e_{2k}e_{p^n-k}}] \rangle^G.$$

From (2.2.9) and (2.2.10) we get $\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \cdots \times \gamma_3(G)$, once again by Lemma 2.5.

Assume now that $p = 3$ and $Y = \{k, p^n - k\} = \{3^{n-1}, 3^n - 3^{n-1}\}$. By hypothesis $e_i \bmod 3$ is not constant for $i \in Y$, thus $e_{3^n-k} \equiv 2 \bmod 3$. Let $x = e_{3^n-k}$ and let y be such that $xy \equiv 1 \bmod 3^n$. In the next formulas the displayed components are those in positions $k, 3^n - k$ and 3^n. Then we have

$$\begin{aligned}
\psi(b) &= (\dots, a, \dots, a^x, \dots, b) \\
\psi(b_k b_{3^n-k}^{-x}) &= (\dots, ba^{-x^2}, \dots, ab^{-x}, \dots, 1) \\
\psi(b_k^y b_{3^n-k}^{-1}) &= (\dots, b^y a^{-x}, \dots, a^y b^{-1}, \dots, 1)
\end{aligned}$$

where the non-displayed components are powers of a with exponent divisible by 3. Then

$$\psi([b, b_k, b_k b_{3^n-k}^{-x}]) = (1, \dots, 1, [a, b, ba^{-x^2}], 1, \dots, 1) \tag{2.2.11}$$

and

$$\psi([b_k^y, b, (b_k^y b_{3^n-k}^{-1})^{a^k}]) = (1, \dots, 1, [a, b, a^y b^{-1}]). \tag{2.2.12}$$

We observe that $\langle ba^{-x^2}, a^y b^{-1}\rangle = \langle ba^{-x^2}, a^{y-x^2}\rangle = G$, indeed $y - x^2 \equiv 1 \bmod 3$ as both x and y are congruent to 2 modulo 3. Thus from Lemma 2.5 and from (2.2.11) and (2.2.12) it follows that $\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \cdots \times \gamma_3(G)$. This completes the proof. □

Remark 2.15. The previous theorems show that all GGS-groups over the p^n-adic tree are regular branch over G' or over $\gamma_3(G)$ except for the ones with defining vector with Y maximal and such that $e_i \equiv e_j \bmod p$ for all $i, j \in Y$, i.e. the ones whose defining vector belongs to the set $\mathcal{E}(p^n)$ defined as follows

$$\mathcal{E}(p^n) = \{\mathbf{e} \in \mathcal{F}(p^n) \mid e_{ip^t} \equiv e_{jp^t} \bmod p \text{ for all } 1 \le i, j \le p^{n-t} - 1\}. \tag{2.2.13}$$

We denote by $\mathcal{E}'(2^n)$ the following set:

$$\mathcal{E}'(2^n) = \{\mathbf{e} \in \mathcal{F}(2^n) \mid t = n - 1\}.$$

For the class of GGS-groups with defining vector in $\mathcal{F}(p^n) \setminus \mathcal{E}'(2^n)$, we can show that they are weakly regular branch. The problem is still open for the GGS-groups with defining vector in $\mathcal{E}'(2^n)$.

Theorem 2.16. *A GGS-group G over the p^n-adic tree with defining vector $\mathbf{e} \in \mathcal{F}(p^n) \setminus \mathcal{E}'(2^n)$ is weakly regular branch over G''.*

Proof. From Lemma 2.4 for every $g_1, g_2 \in G$ there exist $h_1, h_2 \in \mathrm{st}_G(1)$ such that the k-th components of $\psi(h_1)$ and $\psi(h_2)$ are g_1 and g_2, respectively. If G does not have an IS defining vector, then from Theorem 2.10 it is a regular branch group over G' and in particular it is weakly regular branch over G''. Indeed let $f \in \mathrm{st}_G(1)'$ be such that $\psi(f) = (1, \dots, 1, [a, b], 1, \dots, 1)$ where the non-trivial component appears in position k. Then $\psi([f^{h_1}, f^{h_2}]) = (1, \dots, 1, [[a, b]^{g_1}, [a, b]^{g_2}], 1, \dots, 1)$, and the result follows from Lemma 2.5 since $G'' = \langle [[a, b]^{g_1}, [a, b]^{g_2}] \mid g_1, g_2 \in G\rangle^G$.

Thus we can assume in the remaining part of the proof that G has an IS defining vector belonging to $\mathcal{F}(p^n)$. An easy calculation shows that when $\mathbf{e} \notin \mathcal{E}'(2^n)$

$$\psi([[b, b_k]^{h_1}, [b_k, b_{2k}]^{h_2}]) = (1, \ldots, 1, [[a, b]^{g_1}, [b, a^{e_{p^n-k}}]^{g_2}], 1, \ldots, 1),$$

where the non-trivial component is at the k-th position. Now since G is IS, the component e_{p^n-k} is not divisible by p. Thus the sets $\{[a, b]^{g_1} \mid g_1 \in G\}$ and $\{[b, a^{e_{p^n-k}}]^{g_2} \mid g_2 \in G\}$ are generating sets for G'. Then $G'' = \langle [[a, b]^{g_1}, [b, a^{e_{p^n-k}}]^{g_2}] \mid g_1, g_2 \in G \rangle^G$ and thus G is weakly regular branch over G'' by Lemma 2.5. □

Remark 2.17. When $\mathbf{e} \in \mathcal{E}'(2^n)$ the problem is still open. In this case $2k$ coincides with 2^n, so we have

$$\psi([b, b_k]) = (1, \ldots, 1, [a, b], 1, \ldots, 1, [b, a]),$$

where the non-trivial components are the k-th and the last ones. By using the same notation of the previous theorem we have

$$\begin{aligned}\psi([[b, b_k]^{h_1}, [b_k, b_{2k}]^{h_2}]) &= \psi([[b, b_k]^{h_1}, [b_k, b]^{h_2}]) \\ &= (1, \ldots, 1, [[a, b]^{g_1}, [b, a]^{g_2}], 1, \ldots, 1, [[b, a]^{g_3}, [a, b]^{g_4}]),\end{aligned}$$

where we denote by $g_3, g_4 \in G$ the last components of h_1 and h_2 respectively. Since the last component need not be trivial, the argument in the previous theorem cannot be applied.

We note that if G is periodic the quotient G/G'' is a finitely generated soluble group of derived length less then or equal to 2, thus it is finite and as consequence of the previous theorem G is a regular branch group over G'', as proved by Vovkivsky in [56]. Actually we can deduce an improved version of Vovkivsky's result from Remark 2.15.

Corollary 2.18. *Let G be a periodic GGS-group over the p^n-adic tree. Then G is regular branch over $\gamma_3(G)$.*

Proof. From Remark 2.15 it suffices to show that $\mathbf{e} \notin \mathcal{E}(p^n)$. Otherwise all e_{ip^t} have the same (non-zero) value modulo p, for $i = 1, \ldots, p^{n-t} - 1$. Now since G is periodic, from (2.0.1) we get in particular that

$$S_t = e_{p^t} + e_{2p^t} + \cdots + e_{p^n - p^t} = (p^{n-t} - 1)e_{p^t} \equiv 0 \pmod{p}.$$

This implies that $e_{p^t} \equiv 0 \pmod{p}$, which is a contradiction. □

For the GGS-groups with defining vector in $\mathcal{E}(p^n)$ we don't have a complete characterization. However we can describe the branch structure of a particular class of such GGS-groups characterized by a special defining vector that belongs to the set $\mathcal{P}$ of the *partially constant defining vectors* defined formally as follows.

Definition 2.19. Suppose that $\mathbf{e}$ is a defining vector of a GGS-group such that $\mathbf{e} \in \mathcal{E}(p^n)$. We say that $\mathbf{e}$ is *partially constant* if it is constant on Y and constant equal to 0 outside Y.

We have the following result for the GGS-groups with partially constant defining vector.

Theorem 2.20. *Let G be a GGS-group with partially constant, but not constant, defining vector. Then G is a regular branch group over $\gamma_3(G)$.*

Proof. We start by noting that the element $[b, a, b]$ is trivial since each of the components of $\psi([b, a, b])$ is. Indeed

$$\psi([b, a]) = (b, 1, \ldots, 1, a^{-1}, a, 1, \ldots, 1, a^{-1}, a, 1, \ldots, 1, b^{-1})$$

where the components equal to a^{-1} are those in positions multiple of k, in the position immediately on the right appear the ones equal to a, and all the other components, with the exception of the first and the last ones, are trivial. Since the defining vector is not constant, the component e_1 is 0, and an easy calculation shows that $[b, a, b]$ is trivial. This implies that $\gamma_3(G) = \langle [a, b, a] \rangle^G$. Now we observe that

$$\begin{aligned} \psi([b, b_k, b]) &= (1, \ldots, 1, [a, b, a], 1, \ldots, 1, [b, a, b]) \\ &= (1, \ldots, 1, [a, b, a], 1, \ldots, 1). \end{aligned}$$

Thus the result follows again by Lemma 2.5. □

2.3 GGS-groups over primary trees with constant defining vector

The GGS-groups with constant defining vector constitute a special class of GGS-groups inside the set $\mathcal{E}(p^n)$ that have a different structure as for the p-adic case. By Theorem 2.16 these groups are weakly regular branch, in this section we prove that they are not branch.

Let G be a GGS-group acting over the p^n-adic tree with constant defining vector. As remarked in the previous section, up to multiplying by an invertible element the defining vector of G, we can assume that G is the GGS-group with defining vector $\mathbf{e} = (1, \overset{p^n-1}{\ldots}, 1)$, thus the directed automorphism b is defined as follows

$$\psi(b) = (a, \ldots, a, b).$$

Throughout this section, we let $\mathcal{G}$ denote the GGS-group defined on the p^n-adic tree by the vector $\mathbf{e} = (1, \ldots, 1)$. As in the previous section, we denote by K the subgroup of $\mathcal{G}$ defined by $K = \langle ba^{-1} \rangle^{\mathcal{G}}$ and we set

$$y_0 = ba^{-1} \quad \text{and} \quad y_i = y_0^{a^i} \quad \text{for all} \quad i \in \mathbb{Z}\,.$$

We observe that

$$y_i^a = y_{i+1} \tag{2.3.1}$$
$$y_i^b = y_{i+1}^{y_1} \tag{2.3.2}$$

and $y_i = y_j$ if $i \equiv j \pmod{p^n}$.

The following lemmas are generalisations of [22, Lem. 4.1 and Lem. 4.2] to the case $n > 1$. For completeness we give the full proofs here.

Lemma 2.21. *Let $\mathcal{G}$ be the GGS-group with constant defining vector. Then*

(i) If z_i is the tuple whose only non-trivial components are the $i-2$-nd and the $i-1$-st ones corresponding respectively to y_2 and y_1^{-1}, then

$$\psi([y_i, y_j]) = z_i z_j^{-1}, \quad \textit{for every } i \textit{ and } j. \tag{2.3.3}$$

(ii) We have

$$[y_i, y_j] = [y_i, y_{i-1}][y_{i-1}, y_{i-2}] \cdots [y_{j+1}, y_j], \quad \textit{for every } i > j. \tag{2.3.4}$$

Proof. (i) Clearly, it is enough to prove the result for $i > j$. Since both sequences $\{y_i\}$ and $\{z_i\}$ are periodic of period p^n, we can assume that i and j lie in the set $\{3, \dots, p^n + 2\}$. We observe that when $r = j - 3$ and $k = i - r$ we have

$$[y_i, y_j] = [y_k^{a^r}, y_3^{a^r}] = [y_k, y_3]^{a^r}.$$

Thus it suffices to prove the result for $[y_k, y_3]$ with $4 \le k \le p^n + 2$ since the vector $\psi([y_i, y_j])$ can be obtained from $\psi([y_k, y_3])$ by shifting every component r positions to the right. Since $y_i = a^{-i}ba^{i-1} = a^{-1}b_{i-1}$ for every i, we have

$$[y_k, y_3] = b_{k-1}^{-1}ab_2^{-1}aa^{-1}b_{k-1}a^{-1}b_2 = b_{k-1}^{-1}b_1^{-1}b_{k-2}b_2 = (b_1^{-1}b_{k-2})^{b_{k-1}}b_{k-1}^{-1}b_2.$$

When $4 \le k \le p^n + 1$, we have

$$\begin{aligned}\psi((b_1^{-1}b_{k-2})^{b_{k-1}}) &= ((b^{-1}, a^{-1}, \dots, a^{-1})(a, \overset{k-3}{\dots}, a, b, a, \dots, a))^{(a, \overset{k-2}{\dots}, a, b, a, \dots, a)} \\ &= (y_1^{-1}, 1, \overset{k-4}{\dots}, 1, y_1, 1, \dots, 1)^{(a, \overset{k-2}{\dots}, a, b, a, \dots, a)} \\ &= (y_2^{-1}, 1, \dots, 1, y_2, 1, \dots, 1),\end{aligned}$$

and when $k = p^n + 2$ we have

$$\begin{aligned}\psi((b_1^{-1}b)^{b_1}) &= ((b^{-1}, a^{-1}, \dots, a^{-1})(a, \dots, a, b))^{(b, a, \dots, a)} \\ &= (y_1^{-1}, 1, \overset{p^n-2}{\dots}, 1, y_1)^{(b, a, \dots, a)} \\ &= (y_1^{-1}y_2^{-1}y_1, 1, \dots, 1, y_2)\end{aligned}$$

where the last equality holds since $y_1^b = y_2^{y_1}$. Similarly we have

$$\psi(b_{k-1}^{-1}b_2) = \begin{cases} (1, y_1, 1, \overset{k-4}{\dots}, 1, y_1^{-1}, 1, \dots, 1) & \text{if } 4 \le k \le p^n + 1 \\ (y_1^{-1}, y_1, 1, \dots, 1) & \text{if } k = p^n + 2 \end{cases}$$

Thus for every k we obtain

$$\psi([y_k, y_3]) = z_k z_3^{-1}$$

as desired.

(ii) From (i) we have

$$\begin{aligned}\psi([y_i,y_j]) &= (z_iz_{i-1}^{-1})(z_{i-1}z_{i-2}^{-1})\cdots(z_{j+1}z_j^{-1})\\ &= \psi([y_i,y_{i-1}])\psi([y_{i-1},y_{i-2}])\cdots\psi([y_{j+1},y_j])\\ &= \psi([y_i,y_{i-1}][y_i,y_{i-1}]\cdots[y_{j+1},y_j])\end{aligned}$$

and the proof is completed. □

We have seen in Lemma 2.7 some general properties of the subgroups K and K' that hold for a GGS-group over the p^n-adic tree. In the next lemma we will see other properties that hold when the GGS-group has constant defining vector.

Lemma 2.22. *Let $\mathcal{G}$ be the GGS-group with constant defining vector. Then the following hold:*

(i) $\mathcal{G}' \leq K$ and $|\mathcal{G} : K| = |K : \mathcal{G}'| = p^n$.

(ii) $K = \langle y_0, \dots, y_{p^n-1}\rangle$ and $K' = \langle [y_1, y_0]\rangle^{\mathcal{G}}$.

(iii) $K' \times \cdots \times K' \subseteq \psi(K') \subseteq \psi(\mathcal{G}') \subseteq K \times \cdots \times K$; in particular, $\mathcal{G}$ is weakly regular branch over K'.

(iv) $\psi(\mathcal{G}'') = K' \times \cdots \times K'$.

(v) If $L = \psi^{-1}(K' \times \cdots \times K')$ then the conjugates $[y_{i+1}, y_i]^{b^j}$, where $0 \leq i, j \leq p-1$, generate K' modulo L.

Proof. (i) We observe that $\mathcal{G}/K = \langle aK\rangle = \langle bK\rangle$ so $\mathcal{G}' \leq K$. Hence $K = \langle ba^{-1}, \mathcal{G}'\rangle$ and from Theorem 2.3 we get $|\mathcal{G} : K| = |K : \mathcal{G}'| = p^n$.

(ii) From (2.3.1) and (2.3.2), clearly $K = \langle y_0, \dots, y_{p^n-1}\rangle$. Hence K' coincides with the following subgroup

$$K' = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1\rangle^{G}$$

and the result follows from (ii) of Lemma 2.21.

(iii) We observe that

$$\begin{aligned}\psi([b,a]) &= (a^{-1},\dots,a^{-1},b^{-1})(b,a,\dots,a)\\ &= (a^{-1}b,1,\dots,1,b^{-1}a) \in K \times \overset{p^n}{\cdots} \times K.\end{aligned}$$

Since $\mathcal{G}' = \langle [b,a] \rangle^{\mathcal{G}}$, the inclusion $\psi(\mathcal{G}') \subseteq K \times \cdots \times K$ follows from Lemma 1.12. To prove the other non-trivial inclusion we observe that

$$\psi([b,a]) = (y_1, 1, \ldots, 1, y_1^{-1})$$

and

$$\psi([y_3, y_4]) = z_3 z_4^{-1} = (y_2, y_1^{-1} y_2^{-1}, y_1, 1, \ldots, 1).$$

Thus for $p^n \geq 3$ we have

$$\psi([[y_3, y_4], [b,a]]) = ([y_1, y_2], 1, \ldots, 1)$$

and the result follows from Lemma 2.5 and from (ii) since $K' = \langle [y_1, y_2] \rangle^G$.

When $p^n = 2$ the result is still true since $K = \langle y_0, y_1 \rangle$ and K' is trivial. Indeed we have

$$[y_0, y_1] = ab^{-1}b^{-1}aba^{-1}a^{-1}b = 1,$$

where the last equality holds since both generators have order 2. We recall that for $p^n = 2$, the GGS group acting over the binary tree is isomorphic to the infinite dihedral group.

This completes the proof of (iii) and also (iv) is proved since by (iii) we have $\psi(G') \subseteq K \times \cdots \times K$, which implies that $\psi(G'') \subseteq K' \times \cdots \times K'$.

(v) Let $g \in G$. We can write $g = ha^i b^j$ for some $i, j \in \mathbb{Z}$ and $h \in G'$. Then

$$[y_1, y_0]^g = ([y_1, y_0]^h)^{a^i b^j} = ([y_1, y_0][y_1, y_0, h])^{a^i b^j}$$

By (iv) we have $\psi([y_1, y_0, h]) \in \psi(G'') \subseteq K' \times \cdots \times K'$ so $[y_1, y_0, h] \in L$. This proves that

$$[y_1, y_0]^g \equiv [y_1, y_0]^{a^i b^j} \mod L$$

that is

$$[y_1, y_0]^g \equiv [y_{i+1}, y_i]^{b^j} \mod L.$$

Since the conjugates $[y_1, y_0]^g$ generate K' the result follows. □

Proposition 2.23. *Let $\mathcal{G}$ be the GGS-group with constant defining vector. Then* $\mathrm{st}_{\mathcal{G}}(1)' = \mathrm{st}_{\mathcal{G}}(2)$.

Proof. We have $\mathrm{st}_{\mathcal{G}}(1) = \langle b_i \mid i = 1, \dots, p^n \rangle$, where $b_i = b^{a^i}$. Hence $\mathrm{st}_{\mathcal{G}}(1)' = \langle [b_i, b_j] \mid i, j = 1, \dots, p^n \rangle^{\mathcal{G}}$. Now for $1 \leq i < j \leq p^n$ we have

$$\psi([b_i, b_j]) = (1, \dots, 1, \overset{i}{[b, a]}, 1, \dots, 1, \overset{j}{[a, b]}, 1, \dots, 1) \in \mathrm{st}_{\mathcal{G}}(1) \times \cdots \times \mathrm{st}_{\mathcal{G}}(1),$$

and consequently $[b_i, b_j] \in \mathrm{st}_{\mathcal{G}}(2)$. This proves that $\mathrm{st}_{\mathcal{G}}(1)' \leq \mathrm{st}_{\mathcal{G}}(2)$.

For the reverse inclusion, consider an element $g \in \mathrm{st}_{\mathcal{G}}(2)$. By looking at it as an element of $\mathrm{st}_{\mathcal{G}}(1)$, we can write $g = hg'$, where $g' \in \mathrm{st}_{G}(1)'$ and h is of the form

$$h = b_1^{k_1} b_2^{k_2} \cdots b_{p^n}^{k_{p^n}},$$

for some integers k_i. Observe that $\psi(h)$ is given by the vector

$$(b^{k_1} a^{k_2 + \cdots + k_{p^n}}, a^{k_1} b^{k_2} a^{k_3 + \cdots + k_{p^n}}, \dots, a^{k_1 + \cdots + k_{p^n - 2}} b^{k_{p^n - 1}} a^{k_{p^n}}, a^{k_1 + \cdots + k_{p^n - 1}} b^{k_{p^n}}).$$

Now since $h = g(g')^{-1} \in \mathrm{st}_{\mathcal{G}}(2)$ and a has order p^n modulo $\mathrm{st}_{\mathcal{G}}(1)$, if we set $S = k_1 + \cdots + k_{p^n}$, the following congruences hold:

$$\begin{cases} S - k_1 \equiv 0 \mod p^n \\ S - k_2 \equiv 0 \mod p^n \\ \vdots \\ S - k_{p^n - 1} \equiv 0 \mod p^n \\ S - k_{p^n} \equiv 0 \mod p^n \end{cases}$$

It follows that $k_i \equiv S \mod p^n$ for all $i = 1, \dots, p^n$, in particular $k_i \equiv k_j \bmod p^n$ for all $i, j \in \{1, \dots, p^n\}$. Since $S = k_1 + \cdots + k_{p^n}$ it follows that $k_i \equiv k_1 + \cdots + k_{p^n} \bmod p^n$, i.e. $(p^n - 1)k_i \equiv 0 \mod p^n$ for all $i \in \{1, \dots, p^n\}$. Hence $k_i \equiv 0 \mod p^n$ for all i, and consequently $g = g' \in \mathrm{st}_{\mathcal{G}}(1)'$, as desired. □

The following result generalizes [22, Lem. 4.4].

Proposition 2.24. *Let $g \in \mathrm{st}_{\mathcal{G}}(1)$ and write $\psi(g) = (g_1, \dots, g_{p^n})$. Then the following hold:*

(i) If $g \in \mathcal{G}'$ then $\prod_{i=1}^{p^n} g_i \in K'$.

(ii) If $g \in K'$ then $\prod_{i=1}^{p^n - 1} g_i^a g_i^{a^2} \cdots g_i^{a^i} \in K'$.

Proof. (i) For every $h \in \mathrm{st}_{\mathcal{G}}(1)$ we define $\pi(h) = \prod_{i=1}^{p^n} h_i$, where $\psi(h) = (h_1, \ldots, h_{p^n})$. Since $\mathrm{st}_{\mathcal{G}}(1) = \langle b_i \mid i = 1, \ldots, p^n \rangle$ and

$$\pi(b_i) = a^{i-1} b a^{p^n - i} = (ba^{-1})^{a^{-i+1}} = y_{-i+1} \in K,$$

it follows that $\pi(\mathrm{st}_{\mathcal{G}}(1)) \subseteq K$. Then the map $\overline{\pi} : \mathrm{st}_{\mathcal{G}}(1) \to K/K'$ given by $\overline{\pi}(h) = \pi(h)K'$ is a group homomorphism, and since $\ker \overline{\pi}$ is clearly invariant under conjugation by a, we have $\ker \overline{\pi} \trianglelefteq \mathcal{G}$. Now observe that $\psi([a,b]) = (b^{-1}a, 1, \ldots, 1, a^{-1}b)$ implies that $[a,b] \in \ker \overline{\pi}$. Hence $\mathcal{G}' = \langle [a,b] \rangle^{\mathcal{G}} \leq \ker \overline{\pi}$ and (i) follows.

(ii) This can be proved exactly as in [22, Lem. 4.4]. For completeness we give the full proof here. We consider the following map:

$$\begin{array}{cccc} Q: & K \times \overset{p^n}{\cdots} \times K & \to & K/K' \\ & (g_1, \ldots, g_{p^n}) & \to & \prod_{i=1}^{p^n - 1} g_i^{a + a^2 + \cdots + a^i} K' \end{array}$$

Clearly Q is a homomorphism. From Lemma 2.22, $\psi(K')$ is contained in the domain of Q. We prove now that $\psi(K')$ is contained in the kernel of the map. Since $Q(K' \times \cdots \times K') = \overline{1}$, it suffices to see that $\psi(g) \in \ker(Q)$ for every system of generators of K' mod L where $L = \psi^{-1}(K' \times \cdots \times K')$. By (v) of Lemma 2.22 such system of generators is formed by the conjugates $[y_{i+1}, y_i]^{b^j}$, where $0 \leq i, j \leq p^n - 1$.

Let $c \in \Gamma$ be defined by $\psi(c) = (a, \ldots, a)$. We prove now that

$$g^b \equiv g^c \bmod L \text{ for every } g \in K'. \tag{2.3.5}$$

We observe that $\psi(b) = \psi(c)(1, \ldots, 1, a^{-1}b)$, and so

$$\begin{aligned} \psi(g^b) &= \psi(g^c)^{(1,\ldots,1,a^{-1}b)} = \psi(g^c)[\psi(g^c), (1, \ldots, 1, a^{-1}b)] \\ &\equiv \psi(g^c) \bmod K' \times \overset{p^n}{\cdots} \times K' \end{aligned}$$

since $\psi(g^c) \in K \times \overset{p^n}{\cdots} \times K$ and $a^{-1}b \in K$. As a consequence of (2.3.5), it suffices to see that $\psi([y_{i+1}, y_i]^{c^j})$ lies in $\ker(Q)$. Actually it is enough to prove that $\psi([y_{i+1}, y_i])$ lies in $\ker(Q)$ since for every i we have

$$\begin{aligned} Q(\psi([y_{i+1}, y_i]^{c^j})) &= Q(\psi([y_{i+1}, y_i]^{(a^j, \ldots, a^j)})) \\ &= Q(\psi([y_{i+1}, y_i]))^{a^j}. \end{aligned}$$

From (i) of Lemma 2.21 we have $\psi([y_{i+1}, y_i]) = z_{i+1}z_i^{-1}$. Thus, in order to prove the result it is sufficient to prove that $Q(z_i) = Q(z_j)$ for all i and j. Now, for $i \geq 3$ we have

$$\begin{aligned} Q(z_i) &= Q((1, \overset{i-3}{\ldots}, 1, y_2, y_1^{-1}, 1, \ldots, 1)) \\ &= y_2^{a+a^2+\cdots+a^{i-2}}(y_1^{-1})^{a+a^2+\cdots+a^{i-1}}K' \\ &= y_3y_4\cdots y_iy_2^{-1}y_3^{-1}\cdots y_i^{-1}K' \\ &= y_2^{-1}K'. \end{aligned}$$

For $i = 2$ and $i = 1$ we get the same result, indeed we have

$$\begin{aligned} Q(z_2) &= Q((y_1^{-1}, 1, \ldots, 1, y_2)) \\ &= (y_1^{-1})^aK' = y_2^{-1}K' \end{aligned}$$

and

$$\begin{aligned} Q(z_1) &= Q(1, \ldots, 1, y_2, y_1^{-1}) \\ &= y_2^{a+a^2+\ldots+a^{p^n-1}}K' \\ &= y_3y_4\cdots y_{p^n-1}y_0y_1K' = y_2^{-1}K' \end{aligned}$$

where the last equality holds since $y_{p^n-1}\cdots y_2y_1 = 1$. This proves that $\psi([y_{i+1}, y_i])$ lies in $\ker(Q)$ as desired. □

Corollary 2.25. *Let $g \in K' \operatorname{st}_{\mathcal{G}}(m)$ for some $m \in \mathbb{N}$. If $\psi(g) = (x, 1, \ldots, 1, y)$ then both x and y lie in $K' \operatorname{st}_{\mathcal{G}}(m-1)$.*

Proof. Since $\psi(\operatorname{st}_{\mathcal{G}}(m)) \subseteq \operatorname{st}_{\mathcal{G}}(m-1) \times \cdots \times \operatorname{st}_{\mathcal{G}}(m-1)$, part (i) of Proposition 2.24 implies that $xy \in K' \operatorname{st}_{\mathcal{G}}(m-1)$, and part (ii) implies that $x^a \in K' \operatorname{st}_{\mathcal{G}}(m-1)$. Thus $x, y \in K' \operatorname{st}_{\mathcal{G}}(m-1)$. □

Lemma 2.26. *For every $m \geq 2$ the quotient $Q_m = \mathcal{G}/K' \operatorname{st}_{\mathcal{G}}(m)$ is a p-group of class m and order $p^{(m+1)n}$.*

Proof. It is obvious that Q_m is a finite p-group, since $\mathcal{G}/\operatorname{st}_{\mathcal{G}}(m)$ is so. The lemma will be proved if we show that $|Q_m : Q_m'| = p^{2n}$, that $|\gamma_i(Q_m) : \gamma_{i+1}(Q_m)| = p^n$ for $2 \leq i \leq m$, and that $\gamma_{m+1}(Q_m) = 1$.

First of all, observe that

$$Q_m/Q'_m \cong \mathcal{G}/\mathcal{G}' \operatorname{st}_{\mathcal{G}}(m) = \mathcal{G}/\mathcal{G}' \cong C_{p^n} \times C_{p^n},$$

by using that $\operatorname{st}_{\mathcal{G}}(2) \le \mathcal{G}'$ from Proposition 2.23, and Theorem 2.3. Hence

$$\exp \gamma_i(Q_m)/\gamma_{i+1}(Q_m) \mid \exp Q_m/Q'_m = p^n \tag{2.3.6}$$

for every $i \ge 2$.

Let us use the bar notation modulo $K' \operatorname{st}_{\mathcal{G}}(m)$. Then $Q_m = \langle \overline{a}, \overline{b} \rangle$ and consequently $Q'_m = \langle [\overline{b}, \overline{a}], \gamma_3(Q_m) \rangle$. Since $A_m = K/K' \operatorname{st}_{\mathcal{G}}(m)$ is an abelian normal subgroup of Q_m and $Q_m = \langle \overline{a}, A_m \rangle = \langle \overline{b}, A_m \rangle$, it follows that

$$\gamma_i(Q_m) = \langle [\overline{b}, \overline{a}, \overline{b}, \overset{i-2}{\ldots}, \overline{b}], \gamma_{i+1}(Q_m) \rangle = \langle [\overline{b}, \overline{a}, \overset{i-1}{\ldots}, \overline{a}], \gamma_{i+1}(Q_m) \rangle \tag{2.3.7}$$

for every $i \ge 2$. From (2.3.6) we get $|\gamma_i(Q_m) : \gamma_{i+1}(Q_m)| \le p^n$ for $i \ge 2$. Hence the proof will be complete if we show that:

(1) $[b, a, b, \overset{m-1}{\ldots}, b] \in K' \operatorname{st}_{\mathcal{G}}(m)$.

(2) $[b, a, b, \overset{m-2}{\ldots}, b]^{p^{n-1}} \notin K' \operatorname{st}_{\mathcal{G}}(m)$.

for every $m \ge 2$. Indeed (1) then shows that $\gamma_{m+1}(Q_m) = 1$, while (2) shows that $|\gamma_i(Q_m) : \gamma_{i+1}(Q_m)| \ge |\gamma_i(Q_i) : \gamma_{i+1}(Q_i)| \ge p^n$, by applying it with i in the place of m. Note that, according to (2.3.7), (1) is equivalent to $[b, a, \overset{m}{\ldots}, a] \in K' \operatorname{st}_{\mathcal{G}}(m)$ and (2) is equivalent to $[b, a, \overset{m-1}{\ldots}, a]^{p^{n-1}} \notin K' \operatorname{st}_{\mathcal{G}}(m)$.

Now we prove (1) and (2) by induction on $m \ge 2$. Suppose first that $m = 2$. We have

$$\psi([b, a]) = \psi(b^{-1} b^a) = (a^{-1} b, 1, \ldots, 1, b^{-1} a)$$

and

$$\psi([b, a, b]) = ([a^{-1} b, a], 1, \ldots, 1, [b^{-1} a, b]) = ([b, a], 1, \ldots, 1, [a, b]). \tag{2.3.8}$$

Since $\mathcal{G}' \le \operatorname{st}_{\mathcal{G}}(1)$ the latter shows that $[b, a, b] \in \operatorname{st}_{\mathcal{G}}(2)$ and (1) holds for $m = 2$. On the other hand, if $[b, a]^{p^{n-1}} \in K' \operatorname{st}_{\mathcal{G}}(2)$ then Corollary 2.25 implies that $(b^{-1} a)^{p^{n-1}} \in K' \operatorname{st}_{\mathcal{G}}(1) = \operatorname{st}_{\mathcal{G}}(1)$. This is a contradiction, since

$$(b^{-1} a)^{p^{n-1}} \equiv a^{p^{n-1}} \not\equiv 1 \pmod{\operatorname{st}_{\mathcal{G}}(1)}.$$

This completes the proof of (1) and (2) for $m = 2$.

Now we assume that $m \geq 3$. From (2.3.8), we get

$$\psi([b, a, b, \overset{i}{\ldots}, b]) = ([b, a, \overset{i}{\ldots}, a], 1, \ldots, 1, [a, b, \overset{i}{\ldots}, b]) \tag{2.3.9}$$

for every $i \geq 1$. Thus, by the induction hypothesis,

$$\psi([b, a, b, \overset{m-1}{\ldots}, b]) \in (K' \operatorname{st}_{\mathcal{G}}(m-1) \times \cdots \times K' \operatorname{st}_{\mathcal{G}}(m-1)) \cap \operatorname{Im}\psi.$$

Now observe that

$$\begin{aligned}
(K' \operatorname{st}_{\mathcal{G}}(m-1) \times \cdots \times K' \operatorname{st}_{\mathcal{G}}(m-1)) \cap \operatorname{Im}\psi \\
&= (K' \times \cdots \times K')(\operatorname{st}_{\mathcal{G}}(m-1) \times \cdots \times \operatorname{st}_{\mathcal{G}}(m-1)) \cap \operatorname{Im}\psi \\
&= (K' \times \cdots \times K')(\operatorname{st}_{\mathcal{G}}(m-1) \times \cdots \times \operatorname{st}_{\mathcal{G}}(m-1) \cap \operatorname{Im}\psi) \\
&\subseteq \psi(K')\psi(\operatorname{st}_{\mathcal{G}}(m)) = \psi(K' \operatorname{st}_{\mathcal{G}}(m)),
\end{aligned}$$

where the second equality follows from Dedekind's Law and the inclusion from $\mathcal{G}$ being weakly regular branch over K'. Thus

$$[b, a, b, \overset{m-1}{\ldots}, b] \in K' \operatorname{st}_{\mathcal{G}}(m)$$

and (1) holds. On the other hand, if $[b, a, b, \overset{m-2}{\ldots}, b]^{p^{n-1}} \in K' \operatorname{st}_{\mathcal{G}}(m)$ then from (2.3.9) and Corollary 2.25 we obtain that $[b, a, \overset{m-2}{\ldots}, a]^{p^{n-1}} \in K' \operatorname{st}_{\mathcal{G}}(m-1)$, contrary to the induction hypothesis. This proves (2). $\square$

Now we can determine the structure of the factor group $\mathcal{G}/K'$. Let M be the companion matrix of the polynomial $X^{p^n-1} + X^{p^n-2} + \cdots + X + 1$, that is

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \in M_{p^n-1}(\mathbb{Z}).$$

We have the following result.

Theorem 2.27. *The quotient $\mathcal{G}/K'$ is isomorphic to $\mathbb{Z}^{p^n-1} \rtimes \langle x \rangle$, where the element x is of order p^n and acts on $\mathbb{Z}^{p^n-1}$ via M.*

Proof. Let P be the semidirect product in the statement of the theorem. We first study the lower central series of P. Set $V = \mathbb{Z}^{p^n-1}$ and write $(v_1, \ldots, v_{p^n-1})$ for the canonical basis of V. Since we use right actions of groups, if M is the companion matrix of $f(X) = X^{p^n-1} + X^{p^n-2} + \cdots + X + 1$, then $v^x = vM$ for every $v \in V$.

Let us define the map ρ in the following way:

$$\begin{array}{rccc} \rho: & V & \to & V \\ & v & \to & [v,x] = -v + v^x \end{array}$$

The matrix associated to ρ is $M - I$, where I stands for the identity matrix of order $p^n - 1$, that is the matrix

$$M - I = \begin{pmatrix} -1 & 0 & \cdots & 0 & -1 \\ 1 & -1 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

and the matrix associated to $\rho_i : v \to [v, x, \overset{i}{\cdots}, x]$ is $(M - I)^i$ since ρ_i is the composition of the map ρ with itself i times.

For every $W \leq V$ that is normal in P, we have $[W, P] = \{[w, x] \mid w \in W\}$, since V is abelian and the map $\rho : v \to [v, x]$ is a homomorphism on V. Since P/V is cyclic, we have $P' = [V, P]$ and consequently

$$\gamma_i(P) = \{[v, x, \overset{i-1}{\ldots}, x] \mid v \in V\} = \{v(M - I)^{i-1} \mid v \in V\}.$$

Hence the rows of $(M - I)^{i-1}$ are generators of $\gamma_i(P)$. Since V is a free abelian group of finite rank, it follows that

$$|P : \gamma_i(P)| = p^n \, |V : \gamma_i(P)| = p^n \, \det(M - I)^{i-1} = p^{in} \quad \text{for every } i \geq 2, \qquad (2.3.10)$$

by taking into account that $\det(M - I) = f(1) = p^n$, since $f(X)$ is the characteristic polynomial of M. Observe also that $\cap_{i\geq 1} \gamma_i(P) = 1$, since P is residually a finite p-group.

Now recall that $K = \langle y_j \mid j = 0, \dots, p^n - 1\rangle$ with $y_j^a = y_{j+1}$ for all j. In particular $y_{p^n-2}^a = y_{p^n-1} = y_0^{-1} \dots y_{p^n-2}^{-1}$. Hence the assignments $x \mapsto a$ and $v_i \mapsto y_{i-1}$ define a surjective homomorphism α from P onto $Q = \mathcal{G}/K'$. Suppose that $1 \neq w \in \ker\alpha$ and let $m \geq 1$ be such that $w \notin \gamma_{m+1}(P)$. Then α induces an epimorphism from $P/\gamma_{m+1}(P)$ onto $Q/\gamma_{m+1}(Q)$ whose kernel is not trivial, and consequently

$$|P : \gamma_{m+1}(P)| > |Q : \gamma_{m+1}(Q)| \geq |Q_m : \gamma_{m+1}(Q_m)|,$$

where $Q_m = \mathcal{G}/K' \operatorname{st}_{\mathcal{G}}(m)$. This is a contradiction, since $|P : \gamma_{m+1}(P)| = p^{(m+1)n}$ by (2.3.10) and $|Q_m : \gamma_{m+1}(Q_m)| = |Q_m| = p^{(m+1)n}$ by Lemma 2.26. Thus $\ker\alpha = 1$ and we conclude that $P \cong \mathcal{G}/K'$, as desired. □

Now we can generalise [20, Thm. 3.7] and show that $\mathcal{G}$ is not a branch group.

Theorem 2.28. *Let $\mathcal{G}$ be a GGS-group with constant defining vector. Then $\mathcal{G}$ is not a branch group.*

Proof. Let $L = \psi^{-1}(K' \times \overset{p^n}{\cdots} \times K')$. By (iii) of Lemma 2.22 and by the definition of rigid stabilizer, we have $L \leq \operatorname{rst}_{\mathcal{G}'}(1)$. Let us see that the equality holds. To this purpose, let $g \in \operatorname{rst}_{\mathcal{G}'}(v)$ for some vertex v of the first level. Then all components of $\psi(g)$ are trivial, except possibly that corresponding to the position of v, call it h. By using (i) of Proposition 2.24 we get $h \in K'$. This proves that $g \in L$, and consequently $\operatorname{rst}_{\mathcal{G}'}(1) = L$.

Now assume by way of contradiction that the group $\mathcal{G}$ is branch. Then $|\mathcal{G} : \operatorname{rst}_{\mathcal{G}}(1)|$ is finite and from [20, Lem. 3.6] also $|\mathcal{G}' : \operatorname{rst}_{\mathcal{G}'}(1)|$ is finite. Thus $|\mathcal{G} : L|$ is finite, and since $L \leq K'$ by Lemma 2.22, also $|\mathcal{G} : K'|$ is finite. This is a contradiction, since Theorem 2.27 shows that the factor group $\mathcal{G}/K'$ is infinite. □

Chapter 3

A generalization of the Basilica group

In this chapter we analyze the structure and some properties of a new class of groups introduced in [19] that generalize the Basilica group, a group acting over the binary tree introduced for the first time by Grigorchuk and Żuk in [34] and whose principal properties have been introduced in Section 1.8.4.

3.1 Definition and basic properties

For a prime p, the *p-Basilica group* is a group acting on the p-adic tree and generated by two automorphisms a and b defined as follows:

$$\psi(a) = (1, \overset{p-1}{\ldots}, 1, b) \quad \text{and} \quad \psi(b) = (1, \overset{p-1}{\ldots}, 1, a)\sigma,$$

where σ is the cyclic permutation $(1\,2\,\cdots\,p)$. Clearly for $p = 2$ the 2-Basilica group coincides with the Basilica group. This generalisation of the Basilica group mirrors Sidki and Silva's generalisation of the Brunner-Sidki-Vieira group; (see [51] and [13]). A different generalisation of the Basilica group to the p-adic tree, with p generators, was first investigated by Sasse in her Master thesis [49], and Sasse's work has been developed further by Petschick and Rajeev [45]. Our 2-generator p-Basilica groups are also investigated in [45]. Some of their results agree with ours even if our and their works were developed independently.

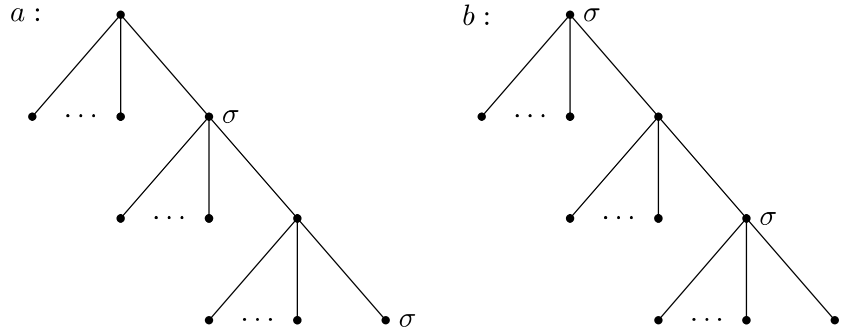


FIGURE 3.1: Generators of the p-Basilica group

3.2 First properties

In this section we prove some basic properties of the p-Basilica groups. For any prime p both generators of the p-Basilica group are bounded automorphisms (see Definition 1.2). Indeed they both have at most one section different from the indentity in each level. From the last paragraph of Section 1.7, for every prime p the p-Basilica group is a group generated by a finite bounded automaton with set of states $\{\mathrm{Id}, a, b\}$. The directed labelled graph for the 3-Basilica group is represented in Figure 3.2.

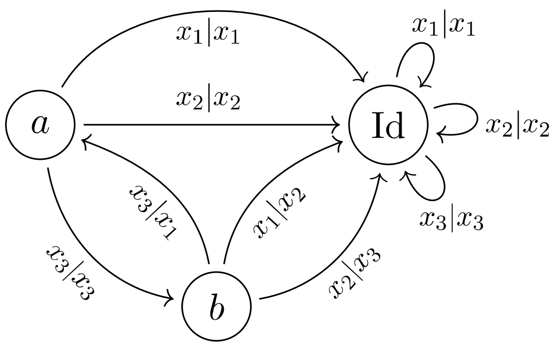


FIGURE 3.2: Directed labelled graph of the 3-Basilica group.

For every prime p the p-Basilica group is a self-similar group. This follows trivially from the first level decomposition of the generators a and b and from Lemma 1.6. The following is an elementary but essential result.

Lemma 3.1. *Let G be a p-Basilica group, for a prime p. Then G is fractal and spherically transitive.*

Proof. By Lemma 1.8, it suffices to show that G acts transitively on the first level and that $\psi_x(\mathrm{st}_G(x)) = G$ for some $x \in X$. This is straightforward since b acts transitively on the first level and since $\psi(a) = (1, \ldots, 1, b)$ and $\psi(b^p) = (a, \ldots, a)$. □

Next we consider the stabilizers in G of the first two levels. Recall that we indicate by $G_n = G/\mathrm{st}_G(n)$ the congruence quotients of G. For convenience, we set

$$A = \langle a \rangle^G = \langle a \rangle G' \quad \text{and} \quad B = \langle b \rangle^G = \langle b \rangle G', \tag{3.2.1}$$

and we observe that $G = AB$.

In the following for a prime p and a group $\mathcal{G}$ we will denote by $W_p(\mathcal{G})$ the wreath product of $\mathcal{G}$ with a cyclic group of order p.

Lemma 3.2. *Let G be a p-Basilica group, for a prime p. Then:*

(i) $\mathrm{st}_G(1) = A\langle b^p \rangle = \langle a, a^b, \ldots, a^{b^{p-1}}, b^p \rangle$ *and* $G_1 = \langle b\, \mathrm{st}_G(1) \rangle \cong C_p$.

(ii) $G_2 \cong C_p \wr C_p$ *is a p-group of maximal class of order p^{p+1}.*

Proof. (i) Observe that $a \in \mathrm{st}_G(1)$, and for $q, r \in \mathbb{N}$ with $0 \leq r < p$ such that $n = pq + r$ we have

$$\psi(b^n) = (a^q, \ldots, a^q, a^{q+1}, \overset{r}{\ldots}, a^{q+1})\sigma^r.$$

Thus $b^n \in \mathrm{st}_G(1)$ if and only if $p \mid n$ and for $i \in \mathbb{N}$ we have

$$\psi(b^{ip}) = (a^i, \ldots, a^i). \tag{3.2.2}$$

Since $\mathrm{st}_G(1) \trianglelefteq G$ we get $A\langle b^p \rangle \leq \mathrm{st}_G(1)$. This inclusion is an equality, since $G/A\langle b^p \rangle$ has order p and $\mathrm{st}_G(1)$ is a proper subgroup of G. Now the result follows by observing that $\langle a, a^b, \ldots, a^{b^{p-1}}, b^p \rangle \trianglelefteq G$.

(ii) Since $\psi(b^p) = (a, \ldots, a)$ and $a \in \mathrm{st}_G(1)$, we have $b^p \in \mathrm{st}_G(2)$. By (i), if we use the bar notation in G_2, we have

$$\mathrm{st}_{G_2}(1) = \overline{\langle a, a^b, \ldots, a^{b^{p-1}} \rangle}.$$

Since $\overline{a}$ has order p in G_2 and the tuples

$$\psi(a) = (1, \ldots, 1, b) \tag{3.2.3}$$
$$\psi(a^{b^i}) = (1, \overset{i-1}{\ldots}, 1, a^{-1}ba, 1, \ldots, 1), \text{ for } 1 \leq i \leq p-1, \tag{3.2.4}$$

commute with each other, $\mathrm{st}_{G_2}(1)$ is elementary abelian of order p^p. Thus $|G_2| = p^{p+1}$. To complete the proof, observe that since $G \leq \Gamma$, the quotient $G_2 = G/\mathrm{st}_G(2)$ embeds in $\Gamma_2 = \Gamma/\mathrm{st}_\Gamma(2) \cong C_p \wr C_p$, and that the latter is a p-group of maximal class of order p^{p+1}. □

Our next goal is to study the abelianisation of G. Observe that $\psi(a) = (1, \ldots, 1, b)$ and G being self-similar imply

$$\psi(A) \subseteq B \times \cdots \times B. \tag{3.2.5}$$

On the other hand, the map

$$\begin{array}{rccc} \pi: & W_p(G) & \to & G/G' \\ & (g_1, \ldots, g_p)\sigma^i & \to & g_1 \cdots g_p G' \end{array}$$

is clearly a group homomorphism. Since $\psi(b) = (1, \ldots, 1, a)\sigma$, it follows that

$$(\pi \circ \psi)(B) \subseteq A/G'. \tag{3.2.6}$$

Theorem 3.3. *Let G be a p-Basilica group, for a prime p. Then:*

(i) $G/A = \langle bA \rangle$ and $G/B = \langle aB \rangle$ are infinite cyclic. In particular, the elements a and b have infinite order in G.

(ii) $A \cap B = G'$.

(iii) $G/G' = \langle aG' \rangle \times \langle bG' \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Proof. (i) We prove that G/A and G/B are infinite simultaneously. Assume for a contradiction that, for some $n \in \mathbb{N}$, we have either $a^n \in B$ or $b^n \in A$, and let us choose n as small as possible. If $b^n \in A \subseteq \mathrm{st}_G(1)$ then $n = pm$ for some m, and consequently

$$\psi(b^n) = (a^m, \ldots, a^m) \in \psi(A) \subseteq B \times \cdots \times B,$$

by (3.2.5). Hence $a^m \in B$, which is impossible since $m < n$. On the other hand, if $a^n \in B$ then

$$b^n G' = (\pi \circ \psi)(a^n) \in (\pi \circ \psi)(B) \subseteq A/G'$$

by (3.2.6). Thus $b^n \in A$, which we just proved is not the case. This completes the proof of (i).

(ii) Since $[a, b] = a^{-1}a^b = (b^{-1})^a b$, from the definition of A and B it follows that $G' \leq A \cap B$. For the reverse inclusion let us take $g \in A \cap B$. As $g \in A$, from the previous part of the proof, the exponent sum of b in g must be zero, and as $g \in B$, also the exponent sum of a in g must be zero. This proves that g is trivial modulo G' so $A \cap B \leq G'$.

Since $G/G' = A/G' \cdot B/G'$, the last item follows trivially from (i) and (ii). □

The next result follows from Theorem 1.18 noting that for a p-Basilica group G we have $G' \leq \mathrm{st}_G(1)$ and from (iii) of Theorem 3.3 the quotient G/G' is torsion-free.

Corollary 3.4. *Let G be a p-Basilica group, for a prime p. Then G is torsion-free.*

Next we study rigid stabilizers and the branch structure of G.

Theorem 3.5. *Let G be a p-Basilica group, for a prime p. Then:*

(i) $\mathrm{rst}_G(1) = A$ *with* $\psi(A) = B \times \cdots \times B$*. In particular, the group G is not branch.*

(ii) $\psi(\mathrm{st}_G(1)') = G' \times \cdots \times G'$*. As a consequence, the group G is weakly regular branch over G'.*

Proof. (i) We already know from (3.2.5) that $\psi(A) \subseteq B \times \cdots \times B$, and the reverse inclusion follows from Lemma 1.12, since $\psi(a) = (1, \ldots, 1, b)$. Hence $A \leq \mathrm{rst}_G(1) \leq \mathrm{st}_G(1) = A\langle b^p \rangle$ and so $\mathrm{rst}_G(1) = A\langle b^{pn} \rangle$ for some $n \geq 0$. It follows that

$$\psi(\mathrm{rst}_G(1)) = \psi(A\langle b^{pn} \rangle) = (B \times \cdots \times B) \rtimes \langle (a^n, \ldots, a^n) \rangle,$$

where the product is semidirect since B is normal in G and a has infinite order modulo B. Assume $n \neq 0$, then there exists $i \neq 0$ such that $(a^{in}, \ldots, a^{in}) \in \psi(\mathrm{rst}_G(1))$, and by definition of rigid stabilizer we have $(a^{in}, 1, \ldots, 1) \in \psi(\mathrm{rst}_G(1))$. Since $(B \times \cdots \times B) \cap (\langle a \rangle \times \cdots \times \langle a \rangle) = \{(1, \ldots, 1)\}$ it follows that $(a^{in}, 1, \ldots, 1) \in \langle (a^n, \ldots, a^n) \rangle$, which is a contradiction. Hence $n = 0$ and $\mathrm{rst}_G(1) = A$, which has infinite index in G, so G is not branch.

(ii) As $\psi(\mathrm{st}_G(1)) \subseteq G\times\cdots\times G$, the inclusion $\subseteq$ is clear. For the reverse inclusion, observe that $\psi([b^p,a]) = (1,\ldots,1,[a,b])$. Since $G' = \langle [a,b]\rangle^G$, the result follows from Lemma 1.12. □

3.3 Commutator subgroup structure

In Section 3.2 we determined the abelianisation of a p-Basilica group G and proved that G is weakly regular branch over G'. Now we study further properties of G' and of other subgroups obtained by taking commutators, and obtain some consequences.

In the following we will denote by C and D the following subgroups of $G\times\overset{p}{\cdots}\times G$:

$$\begin{aligned} C &= \{(b^{i_1},\ldots,b^{i_p}) \mid i_1+\cdots+i_p = 0\},\\ D &= \{(g_1,\ldots,g_p) \in G'\times\cdots\times G' \mid g_1\cdots g_p \in \gamma_3(G)\}. \end{aligned}$$

Since b is of infinite order, the subgroup C is a free abelian group of rank $p-1$ generated by the elements

$$c_i = (1,\ldots,1,b^{-1},b,1,\overset{i}{\ldots},1), \quad \text{for } i \in \{0,1,\ldots,p-2\}. \tag{3.3.1}$$

Note further that for all $i \in \{0,\ldots,p-2\}$ we have

$$c_i = \psi([b^{-1},a]^{b^{-i}}), \tag{3.3.2}$$

indeed

$$\begin{aligned} \psi([b^{-1},a]^{b^{-i}}) &= (1,\ldots,1,a,\overset{i}{\ldots},a)\sigma^i\psi([b^{-1},a])\sigma^{-i}(1,\ldots,1,a^{-1},\overset{i}{\ldots},a^{-1})\\ &= (1,\ldots,1,a,\overset{i}{\ldots},a)\sigma^i(1,\ldots,1,b^{-1},b)\sigma^{-i}(1,\ldots,1,a^{-1},\overset{i}{\ldots},a^{-1})\\ &= (1,\ldots,1,b^{-1},b,1,\overset{i}{\ldots},1), \end{aligned}$$

so C is a subgroup of $\psi(G')$. Also since

$$c_0c_1\cdots c_{p-2} = (b^{-1},1,\overset{p-2}{\ldots},1,b), \tag{3.3.3}$$

it is clear that C is normalised by σ in the wreath product $W_p(G) = G \wr \langle\sigma\rangle$.

We start our study of commutators by identifying the images of G', $\gamma_3(G)$ and G'' under ψ.

Theorem 3.6. *Let G be a p-Basilica group, for p a prime. The following hold:*

(i) $\psi(G') = (G' \times \cdots \times G') \rtimes C$.

(ii) $\psi(G'') = \gamma_3(G) \times \cdots \times \gamma_3(G)$.

(iii) $\psi(\gamma_3(G)) = \langle y_0, \ldots, y_{p-2}\rangle \ltimes D$, *where* $y_0 = c_0^{-p}([a, b^{-1}], 1, \ldots, 1)$ *and* $y_i = c_{i-1}c_i^{-1}$ *for* $1 \le i \le p-2$.

Proof. (i) The inclusion $\supseteq$ is clear, taking into account that G is weakly regular branch over G', together with (3.3.2). For the reverse inclusion, observe that $G' = \langle [b^{-1}, a]\rangle^G$ implies $\psi(G') \le \langle c_0\rangle^{W_p(G)}$. Moreover $c_0 \in C$, so it suffices to show that

$$C^{W_p(G)} = C^{(G\times\cdots\times G)\rtimes\langle\sigma\rangle} \le (G' \times \cdots \times G') \rtimes C. \tag{3.3.4}$$

We observe that

$$C^{(G\times\cdots\times G)\rtimes\langle\sigma\rangle} \le (C[C, G\times\cdots\times G])^{\langle\sigma\rangle} \le \big(C(G'\times\cdots\times G')\big)^{\langle\sigma\rangle}.$$

where the last inclusion follows since $C \le G \times \cdots \times G$. As σ normalises both C and $(G' \times \cdots \times G')$ the inclusion in (3.3.4) follows. Finally, since b has infinite order modulo G' by Theorem 3.3(iii), we observe that the intersection of $G' \times \cdots \times G'$ and C is trivial, and so their product is semidirect.

(ii) We first show that $\gamma_3(G) \times \cdots \times \gamma_3(G) \le \psi(G'')$. First of all, taking into account (3.2.3) and (3.2.4) we have

$$\psi([b, a, a]) = \psi([(a^{-1})^b a, a]) = 1,$$

as a and a^b commute by (3.2.3) and (3.2.4). So $\gamma_3(G) = \langle [b, a, b]\rangle^G$. Now since G is weakly regular branch over G', we have $(1, \ldots, 1, [b, a]) \in \psi(G')$, and on the other hand $(1, \ldots, 1, b^{-1}, b) = c_0 \in \psi(G')$. Hence $(1, \ldots, 1, [b, a, b]) \in \psi(G'')$ and the desired inclusion follows from Lemma 1.12.

To show the other inclusion, it is sufficient to prove that $\psi(G')/(\gamma_3(G) \times \cdots \times \gamma_3(G))$ is abelian. This is obvious from the expression for $\psi(G')$ obtained in (i) indeed

$$\frac{\psi(G')}{\gamma_3(G)\times\cdots\times\gamma_3(G)} = \frac{(G'\times\cdots\times G')C}{\gamma_3(G)\times\cdots\times\gamma_3(G)}.$$

As $G'/\gamma_3(G)$ and C are both abelian, the result follows.

(iii) We have $\gamma_3(G) = [G', a][G', b]G''$ and consequently

$$\begin{aligned}\psi(\gamma_3(G)) &= [\psi(G'), \psi(a)][\psi(G'), \psi(b)](\gamma_3(G) \times \cdots \times \gamma_3(G))\\ &= [\psi(G'), \psi(b)](\gamma_3(G) \times \cdots \times \gamma_3(G))\end{aligned}$$

since $\psi(a) = (1, \ldots, 1, b)$ clearly centralises $\psi(G') = (G' \times \cdots \times G')C$ modulo $\psi(G'') = \gamma_3(G) \times \cdots \times \gamma_3(G)$. Now we observe that the product

$$[G' \times \cdots \times G', \psi(b)](\gamma_3(G) \times \cdots \times \gamma_3(G)) = [G' \times \cdots \times G', \sigma](\gamma_3(G) \times \cdots \times \gamma_3(G)),$$

corresponds to the commutator subgroup of the wreath product $W_p(G'/\gamma_3(G))$, i.e. with the elements of the base group whose component-wise product is 1 in $G'/\gamma_3(G)$. Hence this subgroup coincides with D:

$$D = [G' \times \cdots \times G', \psi(b)](\gamma_3(G) \times \cdots \times \gamma_3(G)) \leq \psi(\gamma_3(G)) \tag{3.3.5}$$

and we have $\psi(\gamma_3(G)) = [\psi(G'), \psi(b)]D$.

Now the factor group $\psi(G')/D$ is generated by $z = ([a, b^{-1}], 1, \ldots, 1)$ and by $c_0, \ldots, c_{p-2}$. Indeed $\psi(G')/D$ is a quotient of $(G' \times \cdots \times G')C/(\gamma_3(G) \times \cdots \times \gamma_3(G))$ whose generators are $z_i = (1, \overset{i}{\ldots}, 1, [a, b^{-1}], 1, \ldots, 1)$ for $i \in \{0, \ldots, p-1\}$ and $c_0, \ldots, c_{p-2}$. For $i \in \{0, \ldots, p-2\}$ we have

$$z_i^{\psi(b)} = (1, \overset{i}{\ldots}, 1, [a, b^{-1}], 1, \ldots, 1)^{(1, \ldots, 1, a)\sigma} = z_{i+1},$$

and from (3.3.5) for all $i \in \{0, \ldots, p-1\}$ the element z_i coincides with z modulo D. It is clear that $[z, \psi(b)] \in D$ and that $[c_i, \psi(b)] = y_i$ for $1 \leq i \leq p-2$. For $i = 0$ we have

$$[c_0, \psi(b)] = c_0^{-1}(b, 1, \ldots, 1, b^{-1})z = c_0^{-2}c_1^{-1} \cdots c_{p-3}^{-1}c_{p-2}^{-1}z,$$

by using (3.3.3). Hence

$$\psi(\gamma_3(G)) = \langle y_1, \ldots, y_{p-2}, c_0^{-2}c_1^{-1} \cdots c_{p-3}^{-1}c_{p-2}^{-1}z\rangle D.$$

Since

$$c_0^{-2}c_1^{-1} \cdots c_{p-3}^{-1}c_{p-2}^{-1}y_{p-2}^{-1}y_{p-3}^{-2} \cdots y_1^{-(p-2)} = c_0^{-p},$$

it follows that $\psi(\gamma_3(G)) = \langle y_0, \ldots, y_{p-2}\rangle D$. To prove that the product is semidirect, we observe that $y_i = (1, \ldots, 1, b, b^{-2}, b, 1, \overset{i-1}{\ldots}, 1)$ for $i \in \{1, \ldots, p-2\}$ and $y_0 = ([a, b^{-1}], 1, \ldots, 1, b^p, b^{-p})$. Since $D \leq G' \times \cdots \times G'$ and b has infinite order modulo G' the result follows. □

Corollary 3.7. *Let G be a p-Basilica group, for p a prime. Then the centraliser of G' in G, and hence the centre of G, is trivial.*

Proof. Suppose that $g \in C_G(G')$ and let $\psi(g) = (g_1, \ldots, g_p)\sigma^k$. Since g commutes with $[b^p, a]$ and $\psi([b^p, a]) = (1, \ldots, 1, [a, b])$, it follows that $p \mid k$. Hence $g \in \mathrm{st}_G(1)$. In view of Theorem 3.6(i), the product $G' \times \cdots \times G' \subseteq \psi(G')$, so all components g_i also centralise G' and consequently belong to $\mathrm{st}_G(1)$. Repeating this process yields $g \in \mathrm{st}_G(n)$ for every $n \in \mathbb{N}$, and so $g = 1$. □

For a group property $\mathcal{P}$, recall that a group H is just non-$\mathcal{P}$ if H does not have property $\mathcal{P}$ but every proper quotient of H has $\mathcal{P}$.

The next lemma, that appears as [34, Lem. 10], is useful for proving the next corollary.

Lemma 3.8. *Let G be weakly regular branch over K. If G/K and $\psi(K)/(K \times \cdots \times K)$ are solvable, then G is just non-solvable.*

Corollary 3.9. *For a prime p, a p-Basilica group is just non-solvable.*

Proof. The group G is weakly regular branch over G'. By the previous lemma it suffices to show that $\psi(G')/(G' \times \cdots \times G')$ is solvable. Now from Theorem 3.6(i) the quotient $\psi(G')/(G' \times \cdots \times G') \cong C$ is abelian. Thus the result follows. □

In the following we give important information about the congruence quotient $G_n = G/\mathrm{st}_G(n)$, namely the orders of the images of a and b, and the structure of its abelianisation. In the remainder, let $\beta(n) = \lceil n/2 \rceil$, i.e. $\beta(n) = n/2$ for n even and $\beta(n) = (n+1)/2$ for n odd.

Theorem 3.10. *Let G be a p-Basilica group, for a prime p. Then, for every $n \in \mathbb{N}$, we have:*

(i) The orders of a and b modulo $\mathrm{st}_G(n)$ are $p^{\beta(n-1)}$ and $p^{\beta(n)}$, respectively.

(ii) We have $|G_n : G_n'| = p^n$ and $G_n/G_n' \cong C_{p^{\beta(n-1)}} \times C_{p^{\beta(n)}}$, where the first factor corresponds to a and the second to b.

Proof. First of all, we prove by induction on $n \geq 0$ the following result: that if n is even then

$$b^{p^{n/2}} \in \mathrm{st}_G(n) \backslash G' \, \mathrm{st}_G(n+1), \tag{3.3.6}$$

and that if n is odd then

$$a^{p^{(n-1)/2}} \in \mathrm{st}_G(n) \backslash G' \, \mathrm{st}_G(n+1). \tag{3.3.7}$$

Note that this already implies (i), and furthermore that the orders of the images of a and b in G_n/G'_n are $p^{\beta(n-1)}$ and $p^{\beta(n)}$, respectively.

The result is obvious for $n = 0$. Now we suppose it holds for $n-1$. If $n = 2m+1$ is odd, we have $\psi(a^{p^m}) = (1, \ldots, 1, b^{p^m})$, and since $b^{p^m} \in \mathrm{st}_G(n-1)$ by the induction hypothesis, we get $a^{p^m} \in \mathrm{st}_G(n)$. Assume, by way of contradiction, that $a^{p^m} \in G' \, \mathrm{st}_G(n+1)$. By applying ψ and taking Theorem 3.6(i) into account, we get

$$\psi(a^{p^m}) = (1, \ldots, 1, b^{p^m}) \in (G' \, \mathrm{st}_G(n) \times \cdots \times G' \, \mathrm{st}_G(n))C.$$

Thus for some $i_1, \ldots, i_p$ summing up to 0, we have

$$(b^{i_1}, \ldots, b^{i_{p-1}}, b^{i_p + p^m}) \in G' \, \mathrm{st}_G(n) \times \cdots \times G' \, \mathrm{st}_G(n).$$

By multiplying together all components, we get $b^{p^m} \in G' \, \mathrm{st}_G(n)$, contrary to the induction hypothesis.

Let us now consider the case when $n = 2m$ is even. Note that $\psi(b^{p^m}) = (a^{p^{m-1}}, \ldots, a^{p^{m-1}})$. Since $a^{p^{m-1}} \in \mathrm{st}_G(n-1)$, we get $b^{p^m} \in \mathrm{st}_G(n)$. As before, suppose that $b^{p^m} \in G' \, \mathrm{st}_G(n+1)$. Applying ψ as above and looking at the first component, we get

$$b^i a^{p^{m-1}} \in G' \, \mathrm{st}_G(n) \tag{3.3.8}$$

for some integer i. Hence $b^i \in G' \, \mathrm{st}_G(n-1)$. Since by induction $b^{p^{m-1}} \notin G' \, \mathrm{st}_G(n-1)$, necessarily p^m divides i. Then by (3.3.8) we obtain that $a^{p^{m-1}} \in G' \, \mathrm{st}_G(n)$, contrary to the induction hypothesis.

We now prove (ii). The abelian group G_n/G'_n is generated by the images of a and b, of orders $p^{\beta(n-1)}$ and $p^{\beta(n)}$, respectively. Hence $|G_n : G'_n| \leq p^{\beta(n-1)+\beta(n)} = p^n$, and all assertions in (ii) immediately follow if we show that $|G_n : G'_n| \geq p^n$. To this purpose, observe that (3.3.6) and (3.3.7) imply that the index $|G' \, \mathrm{st}_G(n) :$

$G' \operatorname{st}_G(n+1)|$ is non-trivial for every n. Thus

$$|G_n : G_n'| = |G : G' \operatorname{st}_G(n)| = \prod_{i=0}^{n-1} |G' \operatorname{st}_G(i) : G' \operatorname{st}_G(i+1)| \geq p^n,$$

as desired. □

Now we can give a stronger form of Lemma 3.1.

Theorem 3.11. *A p-Basilica group G, for p a prime, is super strongly fractal.*

Proof. Let $u_n = x_p \overset{n}{\cdots} x_p$ for every $n \in \mathbb{N}$. Since G is self-similar and spherically transitive, by Lemma 1.9 it suffices to show that $\psi_{u_n}(\operatorname{st}_G(n)) = G$ for all n. Since

$$\psi_{u_{2n-1}}(b^{p^n}) = a, \quad \psi_{u_{2n-1}}(a^{p^{n-1}}) = b, \quad \psi_{u_{2n}}(b^{p^n}) = b, \quad \text{and} \quad \psi_{u_{2n}}(a^{p^n}) = a,$$

the result follows from Theorem 3.10. □

The above result gives the first examples of weakly branch, but not branch, groups that are super strongly fractal; cf. [54, Prop. 3.11] and [55, Prop. 4.3].

Definition 3.12. Let $\mathcal{G}$ be a subgroup of $\operatorname{Aut}\mathcal{T}$. The group $\mathcal{G}$ is said *saturated* if for any $n \in \mathbb{N}$ there exists a subgroup $H_n \leq \operatorname{st}_{\mathcal{G}}(n)$ that is characteristic in $\mathcal{G}$ and $\psi_v(H_n)$ is spherically transitive for all vertices $v \in \mathcal{L}_n$.

If the group $\mathcal{G}$ is spherically transitive and super strongly fractal, then it suffices to show this last property for a single vertex $u \in \mathcal{L}_n$, since if we write $v = g(u)$ with $g \in \mathcal{G}$ then, taking into account that $H_n \subseteq \operatorname{st}_{\mathcal{G}}(n)$ and $u \in \mathcal{L}_n$ so $h(u) = u$ for all $h \in H_n$, by (1.1.4) we have

$$\psi_v(H_n) = \psi_{g(u)}(H_n^g) = \psi_u(H_n)^{g_u} = \psi_u(H_n),$$

where last equality follows from $\psi_u(H_n) \trianglelefteq \psi_u(\operatorname{st}_{\mathcal{G}}(n)) = \mathcal{G}$, since $\mathcal{G}$ is super strongly fractal.

Corollary 3.13. *Let G be a p-Basilica group for a prime p. Then* $\operatorname{Aut}(G) = N_{\operatorname{Aut}\mathcal{T}}(G)$.

Proof. From [38, Prop. 7.5] a saturated weakly branch group has its automorphism group equal to its normalizer in $\operatorname{Aut}\mathcal{T}$, thus it suffices to show that G is saturated.

We set $H_0 = G$, $H_1 = G'$, and $H_n = [H_{n-1}, H_{n-2}^p]$ for all $n \geq 2$, which are characteristic subgroups of G. We prove now by induction that $H_n \leq \mathrm{st}_G(n)$ for every $n \in \mathbb{N}$. We first observe that the quotient of two consecutive level stabilizers in G is an elementary abelian p-group. Indeed G is a subgroup of Γ defined in (1.2.10) and $\mathrm{st}_\Gamma(n)/\,\mathrm{st}_\Gamma(n+1) \cong C_p \times \overset{p^n}{\cdots} \times C_p$.

For $i = 0, 1$ trivially we have $H_i \leq \mathrm{st}_G(i)$. Assume the result true until n. Then we have

$$H_{n+1} = [H_n, H_{n-1}^p] \leq [\mathrm{st}_G(n), \mathrm{st}_G(n-1)^p] \leq \mathrm{st}_G(n)' \leq \mathrm{st}_G(n+1). \tag{3.3.9}$$

Set $u_n = x_p \overset{n}{\cdots} x_p$ for every $n \in \mathbb{N}$. As explained above, we only need to show that $\psi_{u_n}(H_n)$ is spherically transitive. This will follow if we prove that $\psi_{u_{n-1}}(H_n) = G'$, since $\psi([b^{-1}, a]) = (1, \ldots, 1, b^{-1}, b)$, with b acting transitively on the first level vertices, and since $G' \times \cdots \times G' \leq \psi(G')$.

Let us then prove that $\psi_{u_{n-1}}(H_n) = G'$ for every $n \in \mathbb{N}$. We use induction on n, the case $n = 1$ being obvious. Assume now that $n \geq 2$. From (3.3.9) we have

$$H_n \leq \mathrm{st}_G(n-1)'$$
$$\psi_{u_{n-1}}(H_n) \leq \psi_{u_{n-1}}(\mathrm{st}_G(n-1)') = G'$$

where the equality follows from G being super strongly fractal. Thus we only have to show that $G' \leq \psi_{u_{n-1}}(H_n)$. Observe first that $b \in \psi_{u_{n-1}}(H_{n-1})$ by the induction hypothesis. Since also $b \in \psi_{u_{n-2}}(H_{n-2})$ and $\psi(b^p) = (a, \ldots, a)$, we have $a \in \psi_{u_{n-1}}(H_{n-2}^p)$. Consequently $[b, a] \in \psi_{u_{n-1}}([H_{n-1}, H_{n-2}^p]) = \psi_{u_{n-1}}(H_n)$. Since $G' = \langle [b, a] \rangle^G$, in order to complete the proof it remains to show that $\psi_{u_{n-1}}(H_n) \trianglelefteq G$. We first observe that H_n is normal in G since it is characteristic. Moreover $H_n \subseteq \mathrm{st}_G(n) \subseteq \mathrm{st}_G(n-1)$, thus $H_n \trianglelefteq \mathrm{st}_G(n-1)$. Since G is super strongly fractal, it follows that the map

$$\begin{array}{rccc} \psi_{u_{n-1}}: & \mathrm{st}_G(n-1) & \to & G \\ & g & \to & g_{u_{n-1}} \end{array}$$

is a surjective homomorphism and $\psi_{u_{n-1}}(H_n) \trianglelefteq \mathrm{Im}(\psi_{u_{n-1}}) = G$, and the result follows. □

The previous result implies that every automorphism of the group G is actually a conjugation by an automorphism of $\operatorname{Aut}\mathcal{T}$. Examples of other groups acting on rooted trees with automorphism group equal to its normaliser in $\operatorname{Aut}\mathcal{T}$ are the Grigorchuk group and the Brunner-Sidki-Vieira group [38], and the branch multi-EGS groups [54].

Next we generalise Theorem 3.5(ii) and give a relation between rigid stabilizers and level stabilizers.

Theorem 3.14. *Let G be a p-Basilica group, for a prime p. Then the following hold:*

(i) $\psi_n(\mathrm{st}_G(n)') = G' \times \overset{p^n}{\cdots} \times G'$ *for every* $n \in \mathbb{N}$.

(ii) $\mathrm{rst}_G(n) = \mathrm{st}_G(n)'$ *for every* $n \geq 2$.

Proof. (i) Since G is self-similar, the inclusion $\subseteq$ is obvious. For the reverse inclusion, we first prove by induction on n that

$$(1, \overset{p^n-1}{\ldots}, 1, [b,a]) = \begin{cases} \psi_n([b^{p^{n/2}}, a^{p^{n/2}}]) & \text{if } n \text{ is even,} \\ \psi_n([a^{p^{(n-1)/2}}, b^{p^{(n+1)/2}}]) & \text{if } n \text{ is odd.} \end{cases} \tag{3.3.10}$$

The case $n = 1$ follows trivially since $\psi([a, b^p]) = (1, \overset{p-1}{\ldots}, 1, [b,a])$. For $n = 2$, since $\psi([b^p, a^p]) = (1, \overset{p-1}{\ldots}, 1, [a, b^p])$, it follows that

$$\psi_2([b^p, a^p]) = (1, \overset{p^2-p}{\ldots}, 1, \psi([a, b^p])) = (1, \overset{p^2-1}{\ldots}, 1, [b,a]). \tag{3.3.11}$$

Assume the result true until n and assume that $n+1$ is even. We first observe that for $k \in \mathbb{N}$ we have

$$\psi(b^{p^k}) = (a^{p^{k-1}}, \ldots, a^{p^{k-1}}) \qquad \text{and} \qquad \psi(a^{p^k}) = (1, \ldots, 1, b^{p^k}).$$

Thus

$$\psi([b^{p^{(n+1)/2}}, a^{p^{(n+1)/2}}]) = (1, \overset{p-1}{\ldots}, 1, [a^{p^{(n-1)/2}}, b^{p^{(n+1)/2}}])$$

and by induction we have

$$\begin{aligned} \psi_{n+1}([b^{p^{(n+1)/2}}, a^{p^{(n+1)/2}}]) &= (\psi_n(1), \overset{p-1}{\ldots}, \psi_n(1), \psi_n([a^{p^{(n-1)/2}}, b^{p^{(n+1)/2}}])) \\ &= (1, \overset{p^{n+1}-1}{\ldots}, 1, [a,b]). \end{aligned}$$

If $n+1$ is odd the proof is analogous noting that

$$\psi([a^{p^{n/2}}, b^{p^{(n+2)/2}}]) = (1, \overset{p-1}{\ldots}, 1, [b^{p^{n/2}}, a^{p^{n/2}}]).$$

By Theorem 3.10, it follows that $(1, \overset{p^n-1}{\ldots}, 1, [b,a]) \in \psi_n(\mathrm{st}_G(n)')$. Now the result follows from Lemma 1.12.

(ii) It suffices to show that $\psi_n(\mathrm{rst}_G(n)) = G' \times \overset{p^n}{\cdots} \times G'$ for every $n \geq 2$. For $n = 2$ we have

$$\begin{aligned}\psi(\mathrm{rst}_G(2)) &\subseteq (\mathrm{rst}_G(1) \times \overset{p}{\cdots} \times \mathrm{rst}_G(1)) \cap \psi(\mathrm{rst}_G(1))\\ &= (A \times \overset{p}{\cdots} \times A) \cap (B \times \overset{p}{\cdots} \times B)\\ &= G' \times \overset{p}{\cdots} \times G',\end{aligned}$$

by using first Theorem 3.5 and then Theorem 3.3(ii). Hence

$$\psi_2(\mathrm{rst}_G(2)) \subseteq \psi(G') \times \overset{p}{\cdots} \times \psi(G') = (G' \times \overset{p}{\cdots} \times G')C \times \overset{p}{\cdots} \times (G' \times \overset{p}{\cdots} \times G')C, \quad (3.3.12)$$

by Theorem 3.6. Now since G is weakly regular branch over G', by definition of rigid stabilizer we have

$$G' \times \overset{p^2}{\cdots} \times G' \subseteq \psi_2(\mathrm{rst}_G(2)).$$

If this inclusion is strict then we may assume that for $m, l \in \mathbb{N}$, not both equal to zero, the element $(a^l b^m, \ldots)$ belongs to $\psi_2(\mathrm{rst}_G(2))$. By the definition of rigid stabiliser, it follows that $(a^l b^m, 1, \ldots, 1) \in \psi_2(\mathrm{rst}_G(2))$, which contradicts (3.3.12). This proves the case $n = 2$.

Assume now the result true for $n-1$. Then

$$\begin{aligned}\psi_{n-1}(\mathrm{rst}_G(n)) &\subseteq (\mathrm{rst}_G(1) \times \overset{p^{n-1}}{\cdots} \times \mathrm{rst}_G(1)) \cap \psi_{n-1}(\mathrm{rst}_G(n-1))\\ &= (A \times \overset{p^{n-1}}{\cdots} \times A) \cap (G' \times \overset{p^{n-1}}{\cdots} \times G')\\ &= G' \times \overset{p^{n-1}}{\cdots} \times G',\end{aligned}$$

and

$$\psi_n(\mathrm{rst}_G(n)) \subseteq (G' \times \overset{p}{\cdots} \times G')C \times \overset{p^{n-1}}{\cdots} \times (G' \times \overset{p}{\cdots} \times G')C. \quad (3.3.13)$$

We note that $G' \times \overset{p^n}{\cdots} \times G' \subseteq \psi_n(\mathrm{rst}_G(n))$ since G is weakly regular branch over

G'. Assuming the inclusion strict and following a similar argument as for the case $n = 2$, there would exist $l, m \in \mathbb{N}$, where at least one between l and m is different from zero, such that $(a^l b^m, 1, \overset{p^n}{\ldots}, 1) \in \psi_n(\mathrm{rst}_G(n))$, which contradicts (3.3.13). $\square$

We close this section by determining the structure of the quotients $G'/\gamma_3(G)$, G'/G'', and $\gamma_3(G)/G''$, which is key for Section 3.4. We need a couple of lemmas.

Lemma 3.15. *Let G be a p-Basilica group, for a prime p. For every $n \in \mathbb{N}$, we have*

$$\psi(\mathrm{st}_{G'}(n)) = \Big(\mathrm{st}_{G'}(n-1) \times \cdots \times \mathrm{st}_{G'}(n-1)\Big)\, C^{p^{\beta(n-1)}}.$$

Proof. The inclusion $\supseteq$ is obvious from Theorem 3.6(i) and Theorem 3.10(i). For the other direction, let $g \in \mathrm{st}_{G'}(n)$. From Theorem 3.6(i) we can write $\psi(g) = (w_1, \ldots, w_p)(b^{i_1}, \ldots, b^{i_p})$, where the first factor is in $G' \times \cdots \times G'$ and the second is in C. Fix an index $j \in \{1, \ldots, p\}$. Since $w_j b^{i_j} \in \mathrm{st}_G(n-1)$ we have $b^{i_j} \in G' \,\mathrm{st}_G(n-1)$, and then $p^{\beta(n-1)}$ divides i_j by Theorem 3.10. Thus $b^{i_j} \in \mathrm{st}_G(n-1)$ and it follows that also $w_j \in \mathrm{st}_G(n-1)$. This proves the result. $\square$

Lemma 3.16. *Let G be a p-Basilica group, for p a prime. Then the order of $[a, b]$ modulo $\gamma_3(G)\,\mathrm{st}_{G'}(n)$ is at least $p^{\beta(n-1)}$ for every $n \in \mathbb{N}$.*

Proof. Since $[a, b]$ and $[a, b^{-1}]$ are inverse conjugate, we prove the result for the order of $[a, b^{-1}]$. Observe that $\psi([a, b^{-1}]) = (1, \ldots, 1, b, b^{-1}) = c_0^{-1}$. We use induction on n. The result is obvious if $n = 1$, so we suppose that $n \geq 2$. If $[a, b^{-1}]^{p^m} \in \gamma_3(G)\,\mathrm{st}_{G'}(n)$, we want to show that $m \geq \beta(n-1)$. By way of contradiction, we assume that $m < \beta(n-1)$. From Theorem 3.6(iii) and Lemma 3.15, we get

$$\begin{aligned}\psi([a, b^{-1}]^{p^m}) &\in \psi(\gamma_3(G)\,\mathrm{st}_{G'}(n)) \\ &= \langle y_0, \ldots, y_{p-2}\rangle D\Big(\mathrm{st}_{G'}(n-1) \times \cdots \times \mathrm{st}_{G'}(n-1)\Big)\, C^{p^{\beta(n-1)}},\end{aligned}$$

thus by applying ψ to $[a, b^{-1}]^{p^m}$ we get

$$c_0^{-p^m} = ydgc, \tag{3.3.14}$$

where $y = y_0^{k_0} \ldots y_{p-2}^{k_{p-2}}$ for some $k_0, \ldots, k_{p-2} \in \mathbb{Z}$, $d \in D$, $g \in \mathrm{st}_{G'}(n-1) \times \cdots \times \mathrm{st}_{G'}(n-1)$, and $c \in C^{p^{\beta(n-1)}}$. If we reduce (3.3.14) modulo $G' \times \cdots \times G'$ and use that y_0 reduces to c_0^{-p}, we get

$$c_0^{p^m-pk_0+k_1} c_1^{-k_1+k_2} \cdots c_{p-3}^{-k_{p-3}+k_{p-2}} c_{p-2}^{-k_{p-2}} \in C^{p^{\beta(n-1)}}. \tag{3.3.15}$$

Since $c_0, \dots, c_{p-2}$ form a basis for the free abelian group C, it follows that all exponents in (3.3.15) are divisible by $p^{\beta(n-1)}$ and, as a consequence, so is $p^m - pk_0$. Since $m < \beta(n-1)$, it follows that the p-part of pk_0 is p^m. Now since $B = \langle b \rangle G'$ is abelian modulo $\gamma_3(G)$, the map

$$\begin{aligned} \tau \quad : \quad B \times \cdots \times B &\longrightarrow B/\gamma_3(G) \\ (g_1, \dots, g_p) &\longmapsto g_1 \cdots g_p \gamma_3(G) \end{aligned}$$

is a group homomorphism. Observe that both C and D lie in the kernel of τ, and that $\tau(y_0) = [a, b^{-1}]\gamma_3(G)$. Hence by applying τ to (3.3.14), we get

$$[a, b^{-1}]^{k_0} \in \gamma_3(G) \operatorname{st}_{G'}(n-1).$$

Since the p-part of k_0 is p^{m-1}, by the induction hypothesis we have $m-1 \geq \beta(n-2)$, and so $m \geq \beta(n-2) + 1 \geq \beta(n-1)$, contrary to our assumption. This completes the proof. □

Theorem 3.17. *Let G be a p-Basilica group, for p a prime. Then:*

(i) $G'/\gamma_3(G) \cong \mathbb{Z}$.

(ii) $G'/G'' \cong \mathbb{Z}^{2p-1}$.

(ii) $\gamma_3(G)/G'' \cong \mathbb{Z}^{2p-2}$.

Proof. (i) We observe that $G'/\gamma_3(G)$ is cyclic and generated by the image of $[a, b]$. From Lemma 3.16, the order of $[a, b]$ tends to infinity modulo $\gamma_3(G) \operatorname{st}_{G'}(n)$ as n goes to infinity. Hence the statement immediately follows.

(ii) The result follows from (i) and from Theorem 3.6 since we have

$$\begin{aligned} \frac{G'}{G''} \cong \frac{\psi(G')}{\psi(G'')} &\cong \left(\frac{G'}{\gamma_3(G)} \times \overset{p}{\cdots} \times \frac{G'}{\gamma_3(G)} \right) \times C \\ &\cong \mathbb{Z} \times \overset{p}{\cdots} \times \mathbb{Z} \times \mathbb{Z}^{p-1}. \end{aligned}$$

This completes the proof since (iii) is a straightforward consequence of (i) and (ii).

□

As a corollary of the previous result we can prove that the quotient $G/\gamma_3(G)$ is isomorphic to the integral Heisenberg group. In the next section we will use this result to prove that the p-Basilica group does not have the weak congruence subgroup property for any prime p.

The integral Heisenberg group is the group defined by

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c, \in \mathbb{Z} \right\}.$$

It is a 2-generator nilpotent group of class 2 whose presentation is the following

$$H_3(\mathbb{Z}) = \langle x, y \mid [x, y] = z, [x, z] = [y, z] = 1 \rangle,$$

where

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $H_3(\mathbb{Z}) \cong \langle x \rangle \ltimes \langle y, z \rangle \cong \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z})$.

For q an odd prime, the integral Heisenberg group modulo q is the group defined by

$$H_3(q) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c, \in \mathbb{Z}/q\mathbb{Z} \right\}.$$

It is a group of order q^3 whose presentation is the following

$$\begin{aligned} H_3(q) &= \langle x, y \mid [x, y] = z, [x, z] = [y, z] = 1, x^q = y^q = z^q = 1 \rangle \\ &\cong \langle x \rangle \ltimes \langle y, z \rangle \cong \frac{\mathbb{Z}}{q\mathbb{Z}} \ltimes \left(\frac{\mathbb{Z}}{q\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}} \right). \end{aligned}$$

For $q = 2$ the integral Heisenberg group modulo 2 is isomorphic to the dihedral group D_4.

Corollary 3.18. *Let G be a p-Basilica group, for p a prime. Then $G/\gamma_3(G)$ is isomorphic to the integral Heisenberg group $H_3(\mathbb{Z})$.*

Proof. The proof is analogous to [27, Prop. 23], where it was proved for the case $p = 2$. In [24, Prop. 4.8] the author proved the same result using a different approach, also for $p = 2$. For completeness we give the full proof here. From Theorem 3.3 and Theorem 3.17 we know that

$$\frac{B}{G'} = \langle bG' \rangle \cong \mathbb{Z} \quad \text{and} \quad \frac{G'}{\gamma_3(G)} = \langle [a,b]\gamma_3(G) \rangle \cong \mathbb{Z}\,.$$

Since G' is a normal subgroup of B and B/G' is cyclic, it follows that $B' = [B, G'] \leq \gamma_3(G)$, so $B/\gamma_3(G)$ is abelian. Moreover $B/\gamma_3(G) = \langle b\gamma_3(G)\rangle \times \langle [a,b]\gamma_3(G)\rangle \cong \mathbb{Z} \times \mathbb{Z}$. Each element g in the group G can be written as $g = a^i h$ where $i \in \mathbb{N}$ and h is a product of conjugates of b, i.e. $G = \langle a \rangle \ltimes B$. Thus modulo $\gamma_3(G)$ we have

$$\frac{G}{\gamma_3(G)} \cong \langle a\gamma_3(G)\rangle \ltimes \frac{B}{\gamma_3(G)} \cong \langle a\gamma_3(G)\rangle \ltimes (\langle b\gamma_3(G)\rangle \times \langle [a,b]\gamma_3(G)\rangle) \cong \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z}),$$

and it is clear that this group has the same presentation as $H_3(\mathbb{Z})$. □

As noted in [24, Cor. 4.9], the previous result yields an alternative proof that the p-Basilica groups are not branch. As every proper quotient of a branch group is virtually abelian and the integral Heisenberg group is not virtually abelian, the result follows.

3.4 Congruence subgroup properties

Let G be a p-Basilica group, for p a prime. Since $G/G' \cong \mathbb{Z} \times \mathbb{Z}$, the group G does not have the congruence subgroup property as all quotients of G by level stabilizers are p-groups. In this subsection we show that G has the p-congruence subgroup property (p-CSP for short) but not the weak congruence subgroup property.

Theorem 3.19. *Let G be a p-Basilica group, for a prime p. Then G has the p-CSP.*

Proof. We apply Lemma 1.30 with $K = N = G'$. Thus if $L = \psi^{-1}(G' \times \cdots \times G')$ and we prove that G/G', G'/L and L/G'' have the p-CSP, from Lemma 1.29 also G'/G'' has the p-CSP and the result follows from Lemma 1.30 .

First of all, the factor group G/G' has the p-CSP by Lemma 1.31, since

$$G/G' \operatorname{st}_G(n) \cong C_{p^{\beta(n-1)}} \times C_{p^{\beta(n)}}$$

with $\beta(n) = \lceil n/2 \rceil$, according to Theorem 3.10(ii).

Next we deal with G'/L, which is free abelian of rank $p-1$ since by Theorem 3.6(i) it is isomorphic to the subgroup C. By Lemma 3.15, we have

$$\psi(L \operatorname{st}_{G'}(n)) = (G' \times \cdots \times G')C^{p^{\beta(n-1)}}$$

and consequently $G'/L \operatorname{st}_{G'}(n) \cong \psi(G')/\psi(L \operatorname{st}_{G'}(n)) \cong C/C^{p^{\beta(n-1)}}$. Hence this case also follows from Lemma 1.31.

Let us finally consider the case of L/G''. From (ii) of Theorem 3.6 and from (i) of Theorem 3.17 we have

$$L/G'' \cong (G' \times \cdots \times G')/(\gamma_3(G) \times \cdots \times \gamma_3(G)) \cong \mathbb{Z}^p.$$

Since

$$\begin{aligned}
\psi(\operatorname{st}_L(n)) &= \psi(L \cap \operatorname{st}(n)) \\
&= (G' \times \cdots \times G') \cap (\operatorname{st}(n-1) \times \cdots \times \operatorname{st}(n-1)) \\
&= \operatorname{st}_{G'}(n-1) \times \cdots \times \operatorname{st}_{G'}(n-1),
\end{aligned}$$

it follows that

$$\psi(G'' \operatorname{st}_L(n)) = \gamma_3(G) \operatorname{st}_{G'}(n-1) \times \cdots \times \gamma_3(G) \operatorname{st}_{G'}(n-1),$$

and we have

$$L/G'' \operatorname{st}_L(n) \cong G'/(\gamma_3(G) \operatorname{st}_{G'}(n-1)) \times \cdots \times G'/(\gamma_3(G) \operatorname{st}_{G'}(n-1)).$$

Since $G'/(\gamma_3(G) \operatorname{st}_{G'}(n-1))$ is generated by $[a, b]$ modulo $\gamma_3(G) \operatorname{st}_{G'}(n-1)$ whose order is at least $\beta(n-2)$ from Lemma 3.16, there exists $m \geq \beta(n-2)$ such that

$$L/G'' \operatorname{st}_L(n) \cong C_{p^m} \times \cdots \times C_{p^m}.$$

and by applying once again Lemma 1.31 we get the result. □

Theorem 3.20. *Let G be a p-Basilica group, for a prime p. Then G does not have the weak congruence subgroup property.*

Proof. Let $q \neq p$ be a prime, and let $N = \langle a^q, b^q, [a,b]^q \rangle \gamma_3(G)$, which is normal and of finite index in G. By Corollary 3.18, we have $G/N \cong H_3(q)$. We claim that $\mathrm{st}_G(n)' \not\leq N$ for every odd n, which is enough to prove the theorem. Arguing by way of contradiction, since by Theorem 3.14 we have $\psi_n(\mathrm{st}_G(n)') = G' \times \overset{p^n}{\cdots} \times G'$, and according to (3.3.10)

$$\psi_n([a^{p^{(n-1)/2}}, b^{p^{(n+1)/2}}]) = (1, \overset{p^n-1}{\ldots}, 1, [b,a]) \in G' \times \overset{p^n}{\cdots} \times G'$$

for odd n, it follows that $[a^{p^{(n-1)/2}}, b^{p^{(n+1)/2}}] \in N$. As $\gamma_3(G) \leq N$, we get $[a,b]^{p^n} \in N$. Since also $[a,b]^q \in N$ and $q \neq p$, we conclude that $[a,b] \in N$ and G/N is abelian. This contradicts the fact that $G/N \cong H_3(q)$. □

3.5 Hausdorff dimension

In this section we determine the orders of the congruence quotients of the p-Basilica groups. This enables us to compute the Hausdorff dimension of the closure of the p-Basilica group G in the group Γ of p-adic automorphisms of $\mathcal{T}$. We recall that

$$\Gamma \cong \varprojlim_{n \in \mathbb{N}} C_p \wr \overset{n}{\cdots} \wr C_p$$

is a Sylow pro-p subgroup of $\mathrm{Aut}\,\mathcal{T}$ corresponding to the p-cycle $(1\,2\,\cdots\,p)$.

For a subgroup $\mathcal{G}$ of Γ, according to a result of Abercrombie, Barnea and Shalev, the Hausdorff dimension of the closure of $\mathcal{G}$ in Γ is given by

$$\mathrm{hdim}_\Gamma(\overline{\mathcal{G}}) = \varliminf_{n\to\infty} \frac{\log |\mathcal{G} : \mathrm{st}_{\mathcal{G}}(n)|}{\log |\Gamma : \mathrm{st}_\Gamma(n)|} \in [0,1], \tag{3.5.1}$$

where $\varliminf$ represents the lower limit. The Hausdorff dimension of $\overline{\mathcal{G}}$ is a measure of how dense $\overline{\mathcal{G}}$ is in Γ. This concept was first applied by Abercrombie [1] and by Barnea and Shalev [3] in the more general setting of profinite groups. We note that the Hausdorff dimension of the closures of several prominent weakly branch groups, such as the first [31] and second [42] Grigorchuk groups, the siblings of the first Grigorchuk group [53], the GGS-groups [22], the branch path groups [21], and generalisations of the Hanoi tower groups [52], have been computed.

Theorem 3.21. *Let G be a p-Basilica group, for p a prime. Then:*

(i) The orders of the congruence quotients of G are given by

$$\log_p |G : \mathrm{st}_G(n)| = \begin{cases} p^{n-1} + p^{n-3} + \cdots + p^3 + p + \frac{n}{2}, & \text{for } n \text{ even,} \\ p^{n-1} + p^{n-3} + \cdots + p^4 + p^2 + \frac{n+1}{2}, & \text{for } n \text{ odd.} \end{cases}$$

(ii) The Hausdorff dimension of the closure of G in Γ is

$$\mathrm{hdim}_\Gamma(\overline{G}) = \frac{p}{p+1}.$$

Proof. (i) We argue by induction on n. The case $n = 1$ is clear, so we assume $n \geq 2$. Write $n = 2m + e$, with $e = 0$ or 1. We need to establish that

$$\log_p |G : \mathrm{st}_G(n)| = \frac{p^{n+1} - p^{1+e}}{p^2 - 1} + m + e.$$

Note that, by Theorem 3.10,

$$|G : \mathrm{st}_G(n)| = |G : G' \,\mathrm{st}_G(n)|\, |G' \,\mathrm{st}_G(n) : \mathrm{st}_G(n)| = p^n \,|G' : \mathrm{st}_{G'}(n)|$$

and that $|G' : \mathrm{st}_{G'}(n)|$ coincides with

$$\begin{aligned} |\psi(G') : \psi(\mathrm{st}_{G'}(n))| &= |(G' \times \overset{p}{\cdots} \times G')C : (\mathrm{st}_{G'}(n-1) \times \overset{p}{\cdots} \times \mathrm{st}_{G'}(n-1))C^{p^{\beta(n-1)}}| \\ &= p^{(p-1)\beta(n-1)} \,|G' : \mathrm{st}_{G'}(n-1)|^p \\ &= p^{(p-1)\beta(n-1) - p(n-1)} \,|G : \mathrm{st}_G(n-1)|^p, \end{aligned}$$

where we have used Lemma 3.15 and the fact that C is free abelian of rank $p - 1$. Here $\beta(n-1) = \lceil (n-1)/2 \rceil$ as before. Thus

$$\begin{aligned} \log_p |G : \mathrm{st}_G(n)| &= p \log_p |G : \mathrm{st}_G(n-1)| + (p-1)(\beta(n-1) - n + 1) + 1 \\ &= p \log_p |G : \mathrm{st}_G(n-1)| - (p-1)(m + e - 1) + 1, \end{aligned}$$

since $\beta(n-1) = m$. Now the result follows from the induction hypothesis.

(ii) In order to get the Hausdorff dimension of $\overline{G}$ in Γ, we just need to take into account formula (3.5.1) and the fact that

$$\log_p |\Gamma : \mathrm{st}_\Gamma(n)| = 1 + p + \cdots + p^{n-1} = \frac{p^n - 1}{p - 1}. \qquad \square$$

We remark that the Hausdorff dimension of the Basilica group was given by Bartholdi in [4].

3.6 Growth and amenability

Following the notation in Section 1.4, we prove in the next part that the p-Basilica groups are contracting.

We remark that the contracting property is a tool to prove that a certain group is torsion-free, by using induction on the length of a word, as it decreases in the subsequent levels. The p-Basilica groups, like the Basilica group (see [34, Thm. 1]), are examples of contracting groups that are also torsion-free.

Observe that the condition $\ell_1(s) \leq 1$ for all $s \in S$ of Lemma 1.19 is satisfied by the set of generators $S = \{a, b, a^{-1}, b^{-1}\}$ in a p-Basilica group. Thus we have the following result.

Theorem 3.22. *For p a prime, the p-Basilica group G is contracting with respect to the set of generators $S = \{a, b, a^{-1}, b^{-1}\}$, with $\lambda = \frac{2}{3}$ and $C = L = 1$.*

Proof. By Lemma 1.19, it suffices to prove that

$$\ell_2(g) \leq \frac{2}{3}|g| + 1, \quad \text{for every } g \in G. \tag{3.6.1}$$

We start by proving that $\ell_2(h) \leq 2$ for every element $h \in G$ of length 3. We observe that

$$\begin{aligned} \psi(b^2) &= (1, \ldots, 1, a, a)\sigma^2 \\ \psi(b^{-2}) &= (a^{-1}, a^{-1}1, \ldots, 1)\sigma^{-2} \\ \psi(ba) &= (1, \ldots, 1, b, a)\sigma \\ \psi(ba^{-1}) &= (1, \ldots, 1, b^{-1}, a)\sigma \\ \psi(ab^{-1}) &= (a^{-1}, 1, \ldots, 1, b)\sigma^{-1} \\ \psi(a^{-1}b^{-1}) &= (a^{-1}, 1, \ldots, 1, b^{-1})\sigma^{-1}, \end{aligned} \tag{3.6.2}$$

so $\ell_1(g) \leq 1$ for $g \in \{b^2, b^{-2}, ba, ba^{-1}, ab^{-1}, a^{-1}b^{-1}\}$. The components of all other words of length 2, with the exception of $a^{-1}b$ and $b^{-1}a$, are trivial or equal to b^2, b^{-2}, ba or $a^{-1}b^{-1}$ indeed

$$\begin{aligned} \psi(a^2) &= (1, \dots, 1, b^2) \\ \psi(a^{-2}) &= (1, \dots, 1, b^{-2}) \\ \psi(ab) &= (1, \dots, 1, ba)\sigma \\ \psi(b^{-1}a^{-1}) &= (a^{-1}b^{-1}, 1, \dots, 1)\sigma^{-1} \end{aligned} \tag{3.6.3}$$

while for $a^{-1}b$ and $b^{-1}a$ the first level decomposition is given by

$$\begin{aligned} \psi(a^{-1}b) &= (1, \dots, 1, b^{-1}a)\sigma \\ \psi(b^{-1}a) &= (a^{-1}b, 1, \dots, 1)\sigma^{-1}. \end{aligned} \tag{3.6.4}$$

This implies that $\ell_2(g) \leq 1$ for every g of length 2 other than $a^{-1}b$ and $b^{-1}a$. Since h has length 3, one of the elements h or h^{-1}, call it h^*, does not contain $a^{-1}b$ or $b^{-1}a$ as a prefix. If we write $h^* = gs$ with g of length 2, then

$$\ell_2(h) = \ell_2(h^*) \leq \ell_2(g) + \ell_2(s) \leq 2,$$

as claimed.

Let now g be an arbitrary element in G. Then we write $|g| = 3j + k$ with $k \in \{0, 1, 2\}$ and $g = h_1 \cdots h_j f$ with $|h_1| = \cdots = |h_j| = 3$ and $|f| = k$, and (3.6.1) immediately follows from the subadditivity of ℓ_2 (1.4.1) and the fact that $\ell_2(f) \leq k$. This completes the proof. □

Corollary 3.23. *Let G be a p-Basilica group, for a prime p. Then G has solvable word problem.*

Proof. This follows from Theorem 3.22 and [40, Prop. 2.13.8] where it is proved that a contracting finitely generated subgroup of $\operatorname{Aut}\mathcal{T}$ has solvable word problem. □

In the following we determine the growth of a p-Basilica group G, for p an odd prime, and we prove that G is amenable but not subexponentially amenable. The corresponding versions of Theorem 3.27 and Lemma 3.28 for $p = 2$ were proved in [34, Lem. 4 and Prop. 4] and in [34, Prop. 13] and [37, Cor. 10], respectively.

In order to get these results we need some facts about semigroups. Let S and T be two semigroups and let S^T be the direct product of copies of S indexed by T. The *regular wreath product* of S and T denoted by $S \wr T$ is the semigroup whose elements are the formal expressions $\left(s_t\right)_{t\in T} z$ with $s_t \in S$ for all $t \in T$ and $z \in T$ and whose operation is defined as in the regular wreath product of two groups, namely

$$\left(s_t\right)_{t\in T} z \cdot \left(s'_t\right)_{t\in T} z' = \left(s_t s'_{tz}\right)_{t\in T} zz'$$

for all $\left(s_t\right)_{t\in T}, \left(s'_t\right)_{t\in T} \in S^T$ and $z, z' \in T$.

Let x and y be two symbols and let S be the free semigroup generated by x and y. Let σ be the cycle $(1\,2\,\cdots\,p)$ and let η be the semigroup homomorphism defined as follows:

$$\begin{aligned} \eta:\ S &\to S \wr \langle\sigma\rangle \\ x &\to (1,\dots,1,y) \\ y &\to (1,\dots,1,x)\sigma \end{aligned} \tag{3.6.5}$$

For each word u in S, we denote by $u_1, \dots, u_p$ the elements in S such that

$$\eta(u) = (u_1, \dots, u_p)\sigma^r$$

for a certain $r \in \{0, \dots, p-1\}$ depending on u. We have the following result.

Lemma 3.24. *Let S be the free semigroup generated by x and y. If u and v are two different words in S, then there exists $i \in \{1, \dots, p\}$ such that $u_i \neq v_i$.*

Proof. Since $u \neq v$, we can assume without loss of generality that $u = wxc$ and $v = wyd$ for certain $w, c, d \in S$. Let $w_1, \dots, w_p \in S$ and $r \in \{0, \dots, p-1\}$ be such that $\eta(w) = (w_1, \dots, w_p)\sigma^r$. Then

$$\begin{aligned} \eta(u) &= (w_1, \dots, w_p)\sigma^r(1, \dots, 1, y)\eta(c) \\ &= (w_1, \dots, w_{p-r-1}, w_{p-r}y, w_{p-r+1}, w_p)\sigma^r\eta(c) \end{aligned}$$

and

$$\begin{aligned} \eta(v) &= (w_1, \dots, w_p)\sigma^r(1, \dots, 1, x)\sigma\eta(d) \\ &= (w_1, \dots, w_{p-r-1}, w_{p-r}x, w_{p-r+1}, w_p)\sigma^{r+1}\eta(d) \end{aligned}$$

Thus $u_{p-r} = w_{p-r}y\overline{u}$ and $v_{p-r} = w_{p-r}x\overline{v}$ for certain $\overline{u}$ and $\overline{v}$ in S. Since S is the free semigroup in x and y, it follows that $u_{p-r} \neq v_{p-r}$ and the proof is complete. □

Remark 3.25. We observe that if u and v are two different words in the semigroup S generated by x and y then, by applying the previous lemma two times, there exist $i, j \in \{1, \ldots, p\}$ such that $(u_i)_j \neq (v_i)_j$.

In the next lemma we denote by $|\cdot|$ the length with respect to the set $\{x, y\}$.

Lemma 3.26. *Let S be the free semigroup generated by x and y and let u be a word in S such that $|u| \geq 2$. Then there exist $i, j \in \{1, \ldots, p\}$ such that $|(u_i)_j| < |u|$.*

Proof. Assume first that $i \in \{1, \ldots, p-1\}$. Let $z \in \{x, y\}$ and $w \in S$ be such that $u = zw$. Then

$$\begin{aligned} \eta(u) &= (1, \ldots, 1, y)\eta(w) \qquad &\text{if} \quad z = x \\ \eta(u) &= (1, \ldots, 1, x)\sigma\eta(w) \qquad &\text{if} \quad z = y. \end{aligned}$$

From (3.6.5) one can note that for each word $\omega \in S$ we have

$$\max\{|\omega_i| \mid i \in \{1, \ldots, p\}\} \leq |\omega|. \tag{3.6.6}$$

Thus for $i \in \{1, \ldots, p-1\}$ and for all $j \in \{1, \ldots, p\}$ we have

$$|(u_i)_j| \leq |u_i| \leq |w| < |w| + 1 = |u|$$

and the result follows.

It remains to check the result for $i = p$. Let us assume first that u contains y^2. So there exist $v, w \in S$ such that $u = vy^2w$. Since $\eta(y^2) = (1, \ldots, 1, x, x)\sigma^2$, it follows that $|(y^2)_i| < |y^2|$ for all $i \in \{1, \ldots, p\}$. Since $\eta(u) = \eta(v)\eta(y^2)\eta(w)$ from (3.6.6) for all $j \in \{1, \ldots, p\}$ we have that

$$\begin{aligned} |(u_p)_j| \leq |u_p| &\leq |v_p| + 1 + \max\{|w_i| \mid i \in \{1, \ldots, p\}\} \\ &< |v| + |y^2| + |w| = |u|. \end{aligned}$$

The same happens if u contains yx since $\eta(yx) = (1, \ldots, 1, y, x)\sigma$ and $|(yx)_i| < |yx|$ for all $i \in \{1, \ldots, p\}$.

It remains to check the cases $u = x^l$ for $l \geq 2$ and $u = x^m y$ for $m \geq 1$. We observe that

$$\eta(x^l) = (1, \ldots, 1, y^l)$$
$$\eta(x^m y) = (1, \ldots, 1, y^m x)\sigma.$$

By the first part of the proof, the lengths of all components of $\eta(y^l)$ and $\eta(y^m x)$ are strictly less than $|y^l| = |x^l|$ and $|y^m x| = |x^m y|$, respectively. This completes the proof. □

Theorem 3.27. *Let $G = \langle a, b\rangle$ be a p-Basilica group, for p an odd prime. Then the semigroup generated by a and b is free. Consequently, the group G is of exponential growth.*

Proof. Let $R \subseteq G$ be the subsemigroup generated by a and b and consider the following commutative diagram of semigroup homomorphisms

$$\begin{array}{ccc} S & \xrightarrow{\ \xi\ } & R \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ S \wr \langle\sigma\rangle & \xrightarrow{\ \delta\ } & R \wr \langle\sigma\rangle \end{array} \tag{3.6.7}$$

where ξ is the surjective homomorphism defined by $\xi(x) = a$ and $\xi(y) = b$, the homomorphism δ is the natural extension of ξ to the wreath product $S \wr \langle\sigma\rangle$, i.e. for all $r \in \{0, \ldots, p-1\}$ and $(s_1, \ldots, s_p)\sigma^r \in S \wr \langle\sigma\rangle$ we have

$$\delta((s_1, \ldots, s_p)\sigma^r) = (\xi(s_1), \ldots, \xi(s_p))\sigma^r,$$

the map η is the homomorphism defined in (3.6.5), and ψ is the restriction of the group homomorphism $\psi : G \to G \wr \langle\sigma\rangle$ to R.

In order to prove that R is free, assume by way of contradiction that there exist two different words in a and b representing the same element f in R. By the surjectivity of ξ, there exist u and v different words in the semigroup S such that $\xi(u) = \xi(v) = f$. Without loss of generality we can assume that $\rho = \max(|u|, |v|)$ is minimal, i.e. for all words $\overline{u} \neq \overline{v}$ whose images under ξ represent the same element in R we have $\max(|\overline{u}|, |\overline{v}|) \geq \rho$. We first observe that $\rho \geq 2$, indeed if this is not the case then u and v are two different words in $\{1, x, y\}$ and their images under ξ cannot represent the same element in R.

Without loss of generality we can assume that $\rho = |u|$. Since $u \neq v$, from Remark 3.25 there exist $i, j \in \{1, \dots, p\}$ such that $(u_i)_j \neq (v_i)_j$. Since $|u| \geq 2$, by Lemma 3.26 we know that $|(u_i)_j| < |u| = \rho$. If $|v| < \rho$ then $|(v_i)_j| \leq |v| < \rho$ for all $i, j \in \{1, \dots, p\}$. If $|v| = \rho \geq 2$, from Lemma 3.26 we have $|(v_i)_j| < |v| = \rho$ for all $i, j \in \{1, \dots, p\}$. Since the diagram in (3.6.7) commutes, the elements $(u_i)_j$ and $(v_i)_j$ are two different words whose images under ξ represent the same element in R and such that $\max(|(u_i)_j|, |(v_i)_j|) < \rho$. This contradicts the minimality of ρ and the proof is complete. □

Lemma 3.28. *For p a prime, the p-Basilica group G is amenable but not subexponentially amenable. In particular it is not elementary amenable.*

Proof. For the Basilica group the result was proved by Grigorchuk and Żuk in [34, Prop. 13] and [10]. Hence we assume that p is odd. As remarked in Section 3.2 the group G is a group generated by a finite bounded automaton. From the main result in [8], any group generated by a finite bounded automaton is amenable. It follows that G is amenable, so it suffices to show that G is not subexponentially amenable. Since G is weakly regular branch over G', from Lemma 1.21 (see [37, Cor. 3]) the result follows provided that $\psi_u(\mathrm{st}_{G'}(u))$ contains G for some vertex u. We observe that

$$\psi([b^{-1}, a]^p) = (1, \overset{p-2}{\dots}, 1, b^{-p}, b^p) \quad \text{and} \quad \psi([a, b^p]) = (1, \overset{p-1}{\dots}, 1, [b, a]),$$

thus $\psi_u([b^{-1}, a]^p) = a$ and $\psi_u([a, b^p]) = b$, where $u = x_p x_p$. This completes the proof. □

3.7 Virtually nilpotent quotients and maximal subgroups

In this final section we study nilpotency and virtual nilpotency of quotients of a p-Basilica group G, and we prove that all maximal subgroups of G have finite index, and that G has infinitely many non-normal maximal subgroups. The following lemma will be useful for both purposes.

Lemma 3.29. *Let G be a p-Basilica group, for a prime p. Then G has a proper quotient isomorphic to $W_p(\mathbb{Z})$.*

Proof. Let $L = \psi^{-1}(G' \times \cdots \times G')$. We have $G = A\langle b \rangle$, and on the other hand $\psi(A) = B \times \cdots \times B$ by Theorem 3.5(i). Hence ψ induces an isomorphism between G/L and the semidirect product $(B/G' \times \cdots \times B/G') \ltimes \langle \psi(b) \rangle$. Observe that $\psi(b) = (1, \ldots, 1, a)\sigma$ acts as σ on the direct product of p copies of B/G', and that $\psi(b^p) = (a, \ldots, a)$ acts trivially. If we set $N = L\langle b^p \rangle$ then it is clear that $N \trianglelefteq G$ and that $G/N \cong W_p(\mathbb{Z})$, since $B/G' \cong \mathbb{Z}$ by Theorem 3.3(iii). □

Recall from Corollary 3.9 that the p-Basilica groups are just non-solvable. In [23, Sec. 8.3] it was shown that the Basilica group is not just non-nilpotent. On the other hand, by [23, Lem. 8.3.5 and Prop. 8.3.6], all proper quotients of the Basilica group are virtually nilpotent. We extend these results to the p-Basilica groups for all primes p. To this purpose we will use the following result proved by Baumslag in [11].

Theorem 3.30. *Let H and K be two nilpotent groups and let $W = H \wr K$. The group W is nilpotent if and only if H and K are p-groups with H of finite exponent and K finite.*

Theorem 3.31. *Let G be a p-Basilica group, for a prime p. Then:*

(i) The group G is not just non-nilpotent.

(ii) Every proper quotient of G is virtually nilpotent, but G itself is not virtually nilpotent.

Proof. (i) By Lemma 3.29, the group G has a proper quotient isomorphic to $W_p(\mathbb{Z})$. From Theorem 3.30, this wreath product is not nilpotent. Hence G is not just non-nilpotent.

(ii) From Theorem 3.6(ii), the map ψ induces an embedding of G/G'' into the wreath product $W_p(G/\gamma_3(G))$. Since the latter is virtually nilpotent, also is G/G''.

Now since G is weakly regular branch over G' and G/G'' is virtually nilpotent, it follows that every proper quotient of G is also virtually nilpotent by Theorem 1.35. On the other hand, the group G is not virtually nilpotent by Gromov's celebrated theorem 1.20, in light of Theorem 3.27. □

Let us now consider the maximal subgroups of G. We first prove that G does not possess maximal subgroups of infinite index. The proof is analogous to that of [24, Sec. 4.4], however with a necessary change to the end of [24, Prop. 4.27]. Due to the proof being so similar, we refer the reader to [24, Sec. 4.4], and only record here the part that needs to be changed.

Recall from Definition 1.32 that a subgroup H of a group $\mathcal{G}$ is *prodense* if $HN = \mathcal{G}$ for all non-trivial normal subgroups N of $\mathcal{G}$. As noted in Section 1.6, finitely generated virtually nilpotent groups belong to the class $\mathcal{MF}$, i.e. the class of groups whose maximal subgroups are all of finite index. The previous theorem implies that every proper quotient of a p-Basilica group in $\mathcal{MF}$. Thus from Proposition 1.34, in order to prove that there are no maximal subgroups of infinite index in a p-Basilica group it suffices to show that there are no proper prodense subgroups.

For H a proper prodense subgroup of G, by Theorem 1.36, for all vertices $u \in T$, the subgroup $\psi_u(\mathrm{st}_H(u))$ is a proper prodense subgroup of G. We consider a prodense subgroup H of G, and seek a vertex u such that $\psi_u(\mathrm{st}_H(u)) = G$, which then proves the theorem.

As in [24, Prop. 4.27], there is a vertex v such that either $ab, b^{-1}a \in \psi_v(\mathrm{st}_H(v))$ or $ba, b^{-1}a \in \psi_v(\mathrm{st}_H(v))$. In the former case, we obtain $a^2 \in \psi_v(\mathrm{st}_H(v))$. Since p is an odd prime, it follows that $b^{-1}a^p \in \psi_v(\mathrm{st}_H(v))$. Now

$$\begin{aligned} \psi((b^{-1}a^p)^p) &= (a^{-1}b^p, \dots, a^{-1}b^p), \\ \psi(a^{-1}b^p) &= (a, \dots, a, b^{-1}a), \end{aligned}$$

and

$$\begin{aligned} \psi((ab)^p) &= (ba, \dots, ba), \\ \psi((ba)^p) &= (ba, \dots, ba, ab). \end{aligned} \tag{3.7.1}$$

Therefore, for $u = vx_px_1$, we have $\psi_u((b^{-1}a^p)^p) = a$ and $\psi_u((ab)^{p^2}) = ba$. Since both $(b^{-1}a^p)^p$ and $(ab)^{p^2}$ belong to $\mathrm{st}_H(u)$, it follows that $\psi_u(\mathrm{st}_H(u)) = \langle a, ba\rangle = G$ and the result holds in this case.

In the latter case, we have $ba, b^{-1}a \in \psi_v(\mathrm{st}_H(v))$, and so $b^2 \in \psi_v(\mathrm{st}_H(v))$. As before, we obtain $b^pa = \psi^{-1}((a, \dots, a, ab)) \in \psi_v(\mathrm{st}_H(v))$. Setting $u = vx_1$, and taking into account (3.7.1), we see that $a, ba \in \psi_u(\mathrm{st}_H(u))$ and the result follows.

We conclude by showing the existence of non-normal maximal subgroups in the p-Basilica groups.

Proposition 3.32. *Let G be a p-Basilica group, for p an odd prime. Then for every* *prime q such that p divides $q-1$, the group G has a non-normal subgroup of index q.*

Proof. By Lemma 3.29, the group G has a quotient isomorphic to $W_p(\mathbb{Z})$, and so also a quotient isomorphic to $W_p(\mathbb{Z}/q\mathbb{Z})$. Thus it suffices to find a non-normal subgroup of index q in the latter group.

Let $V = \mathbb{Z}/q\mathbb{Z} \times \overset{p}{\cdots} \times \mathbb{Z}/q\mathbb{Z}$ be the base group of $W_p(\mathbb{Z}/q\mathbb{Z})$. The characteristic polynomial corresponding to the action of σ on V is $X^p - 1$, which by the condition that p divides $q-1$, has p different roots in $\mathbb{Z}/q\mathbb{Z}$. Let $\lambda \neq 1$ be one of these roots, and let $U = \langle u \rangle$ be the eigenspace of λ in V. Then we can write $V = U \times K$ for a suitable subgroup K. If we set $H = K\langle \sigma \rangle$ then H has index q in $W_p(\mathbb{Z}/q\mathbb{Z})$. At the same time, H is not a normal subgroup of $W_p(\mathbb{Z}/q\mathbb{Z})$, since otherwise $[u, \sigma] = u^{\lambda-1} \neq 1$ belongs to $U \cap H = 1$. $\square$

Observe that there are actually infinitely many non-normal maximal subgroups in a p-Basilica group, due to Dirichlet's theorem about primes in arithmetic progressions.

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