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On Modified Integral Inequalities for a Generalized Class of Convexity and Applications

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Abstract: In this paper, we concentrate on and investigate the idea of a novel family of modified p-convex functions. We elaborate on some of this newly proposed idea's attractive algebraic characteristics to support it. This is used to study some novel integral inequalities in the frame of the Hermite–Hadamard type. A unique equality is established for differentiable mappings. The Hermite–Hadamard inequality is extended and estimated in a number of new ways with the help of this equality to strengthen the findings. Finally, we investigate and explore some applications for some special functions. We think the approach examined in this work will further pique the interest of curious researchers.

Keywords: convexity theory; p-convex function; m-convex function; Hermite–Hadamard inequality

MSC: 26A51; 26A33; 26D10

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1. Introduction

Since more than a century ago, convexity has been the focus of intense investigation, and it has an amazing history in mathematics. This theory and its generalizations have significant advantages for the investigation of extremum problems. In addition to its fascinating and in-depth findings in numerous disciplines of applied and engineering sciences, this theory is widely accepted and provides a numerical setup and framework for scientists to analyze a wide range of unrelated problems. The term "convexity" has attracted a lot of attention and has become a fruitful source of research and ideas. Interested readers can see the following literature about convex analysis, convex functions, and their applications [1]; s-convex functions [2]; n-polynomial harmonically s-type convex functions [3]; convex functions with applications to means [4]; GA-convex functions [5]; p–harmonic exponential type convexity [6]; and generalized exponential-type convex functions [7].

The term convexity played a significant and vital role in the generalizations and extensions of inequalities throughout the past few decades. The theories of inequality and convexity are strongly related to one another. Information technology, statistics, stochastic processes, probability, integral operator theory, optimization theory, and numerical integration all make use of the integral inequalities. Many mathematicians and academics have focused their considerable efforts and contributions over the past few decades on the study of inequalities. Interested readers can go through the following articles about different type of inequalities, i.e. Hermite–Hadamard integral inequalities [8], Ostrowski type inequalities [9], weighted Chebysev–Ostrowski type inequalities [10], Ostrowski type integral inequalities using hypergeometric functions [11], reverse Minkowski's inequality [12], reverse Hermite–Hadamard's inequalities [13], and Minkowski's inequalities [14].

The primary goal and uniqueness of this article is that it discusses Hermite–Hadamard inequalities and their refinements for modified p-convex functions using a new identity with the aid of power mean and Hölder inequalities.

We organized this article in the following manner: we discuss some fundamental definitions and ideas in Section 2. In Section 3, we elaborate the concept and properties of the modified p-convex function. For a modified p-convex function, we examine a new generalization of the Hermite–Hadamard type inequality in Section 4. The Hermite–Hadamard type inequality is then improved in Section 5 using a modified p-convex. In Section 6, we investigate some applications involving modified Bessel functions via modified p-convex. Lastly, in Section 7, a conclusion and future directions of the newly introduced idea are expressed.

2. Preliminaries

It is advisable to explore and expound on a few definitions, theorems, and notes in the first part for the sake of thoroughness, quality, and reader interest. This section's main goal is to explain and examine certain familiar terms and definitions that we require for our examination in subsequent sections. The first concepts we discuss are convex function, Hermite–Hadamard type inequality, h-convex function, s-type convex function, p-convex function, and m-convex function. A few theorems related to the p-convex function are also included.

In the year 1905, Jensen [15], for the first time, presented the meaning of a convex function, which reads as follows:

Definition 1. *Assume that $\mathtt{X}$ is a convex subset of a real vector space $\mathtt{R}$ and $\mathcal{D}:\mathtt{X}\to\mathtt{R}$ is a real valued function. A real valued function $\mathcal{D}$ is convex if*

$$\mathcal{D}(\mathtt{k}\mathtt{b}_1+(1-\mathtt{k})\mathtt{b}_2)\leq \mathtt{k}\mathcal{D}(\mathtt{b}_1)+(1-\mathtt{k})\mathcal{D}(\mathtt{b}_2), \tag{1}$$

holds $\forall\ \mathtt{b}_1,\mathtt{b}_2\in\mathtt{X}$, and $\mathtt{k}\in[0,1]$.

Theorem 1 ([16]). *Assume that $\mathtt{X}$ is an interval in $\mathtt{R}$ and $\mathcal{D}:\mathtt{X}\to\mathtt{R}$ is convex. Then, $\mathcal{D}$ is Lipschitz on any closed interval $\mathtt{X}$.*

No one can deny the Hermite–Hadamard inequality's astonishing and spectacular significance in literature due to its importance in various fields. Since that time, scholars have continued to be interested in the aforementioned inequality, and as a result, numerous generalizations and enhancements have been made. Due to the extensive perception and uses of this kind of inequality in the scope of pure and applied analysis, it has continued to be a topic of significant interest. This inequality states that if real-valued function $\mathcal{D}$ is convex for $\mathtt{b}_1,\mathtt{b}_2\in\mathtt{X}$, and $\mathtt{b}_1<\mathtt{b}_2$, then

$$\mathcal{D}\left(\frac{\mathtt{b}_1+\mathtt{b}_2}{2}\right)\leq\frac{1}{\mathtt{b}_2-\mathtt{b}_1}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\mathcal{D}(\chi)d\chi\leq\frac{\mathcal{D}(\mathtt{b}_1)+\mathcal{D}(\mathtt{b}_2)}{2}. \tag{2}$$

Interested readers can refer to (see [17–20]).

The term m-convexity was investigated and explored by G. Toader in (see [21]).

Definition 2 (see [21])**.** *A real valued function $\mathcal{D} : [0, \mathtt{b}_2] \to \mathtt{R}$, $\mathtt{b}_2 > 0$ is m-convex if*

$$\mathcal{D}(\mathtt{k}\mathtt{b}_1 + m(1-\mathtt{k})\mathtt{b}_2) \le \mathtt{k}\mathcal{D}(\mathtt{b}_1) + m(1-\mathtt{k})\mathcal{D}(\mathtt{b}_2) \tag{3}$$

holds $\forall\, \mathtt{b}_1, \mathtt{b}_2 \in [0, \mathtt{b}_2]$, $m \in [0,1]$, *and* $\mathtt{k} \in [0,1]$.

Definition 3 (see [22])**.** *Assume that $\mathtt{X}$ and $\mathtt{J}$ are an interval in $\mathtt{R}$, $(0,1) \subset \mathtt{J}$ and let $h : \mathtt{J} \to \mathtt{R}$ be a non-negative function, $h \neq 0$. A non-negative function $\mathcal{D} : \mathtt{X} \to \mathtt{R}$ is h-convex if*

$$\mathcal{D}(\mathtt{k}\mathtt{b}_1 + (1-\mathtt{k})\mathtt{b}_2) \le h(\mathtt{k})\mathcal{D}(\mathtt{b}_1) + h(1-\mathtt{k})\mathcal{D}(\mathtt{b}_2) \tag{4}$$

holds for all $\mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, *and* $\mathtt{k} \in (0,1)$.

Remark 1. *Choosing $h(\mathtt{k}) = \mathtt{k}$, then the above function collapses to the classical convex function (see [23,24]).*

Definition 4 (see [25])**.** *A function $\mathcal{D} : \mathtt{X} \subset (0, +\infty) \to \mathtt{R}$ is p-convex if*

$$\mathcal{D}\left(\left[\mathtt{k}\mathtt{b}_1^p + (1-\mathtt{k})\mathtt{b}_2^p\right]^{\frac{1}{p}}\right) \le \mathtt{k}\mathcal{D}(\mathtt{b}_1) + (1-\mathtt{k})\mathcal{D}(\mathtt{b}_2), \tag{5}$$

$\forall\, \mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, $\mathtt{k} \in [0,1]$, *and* $p \in \mathtt{R}\backslash 0$.

Remark 2. *Choosing $p = 1$, then the above function collapse to ordinary convex function.*

Definition 5 (see [26])**.** *A function $\mathcal{D} : \mathtt{X} \to \mathtt{R}$ is s–type convex, if*

$$\mathcal{D}(\mathtt{k}\mathtt{b}_1 + (1-\mathtt{k})\mathtt{b}_2) \le [1 - s(1-\mathtt{k})]\mathcal{D}(\mathtt{b}_1) + [1 - s\mathtt{k}]\mathcal{D}(\mathtt{b}_2), \tag{6}$$

holds $\forall\, \mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, $s \in [0,1]$ *and* $\mathtt{k} \in [0,1]$.

Theorem 2 (see [16])**.** *Assume that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are real functions defined on $[a,b]$ and if $|\mathcal{D}_1|^p$ and $|\mathcal{D}_2|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |\mathcal{D}_1(x)\mathcal{D}_2(x)|dx \le \left(\int_a^b |\mathcal{D}_1(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |\mathcal{D}_2(x)|^q dx\right)^{\frac{1}{q}}, \tag{7}$$

with equality holding if and only if $A|\mathcal{D}_1|^p = B|\mathcal{D}_2|^q$, almost everywhere, where A and B are constants.

Theorem 3 (see [27])**.** *Assume that $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are real functions defined on $[a,b]$ and if $|\mathcal{D}_1|$ and $|\mathcal{D}_1||\mathcal{D}_2|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |\mathcal{D}_1(x)\mathcal{D}_2(x)|dx \le \left(\int_a^b |\mathcal{D}_1(x)|dx\right)^{1-\frac{1}{q}} \left(\int_a^b |\mathcal{D}_1(x)|dx \int_a^b |\mathcal{D}_2(x)|^q dx\right)^{\frac{1}{q}}. \tag{8}$$

3. Modified p-Convex Functions and Its Algebraic Properties

Due to the theory of convexity's numerous applications in applied sciences and optimization, it has undergone a remarkable development during the past few decades. Even while convexity has yielded a variety of conclusions, the majority of the problems

in the real world are nonconvex in nature. Studying nonconvex functions, which are roughly close to convex functions, is therefore always worthwhile. Convex functions have received acclaim from numerous well-known mathematicians during the twentieth century, including Jensen, Hermite, Holder, and Stolz. An unprecedented amount of research was conducted throughout the 20th century, yielding significant findings in the fields of convex analysis, geometric functional analysis, and nonlinear programming.
We will provide our basic definition of the modified p-convex function and its corresponding features as the main topic of this section.

Definition 6. *A function $\mathcal{D} : \mathtt{X} \to \mathtt{R}$ is said to be modified p-convex if*

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right)^{\frac{1}{p}}\right) \le (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b}_1) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b}_2) \tag{9}$$

holds for all $\mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, $m \in [0,1]$, $s \in [0,1]$, $\mathtt{k} \in [0,1]$, and $p \in \mathtt{R}\backslash 0$.

Remark 3. *(i) Choosing $s = m = 1$, we obtain Definition 4.*
(ii) Choosing $p = m = 1$ in Definition 6, we obtain Definition 5.
(iii) Choosing $p = -1$ and $m = 1$ in Definition 6, then

$$\mathcal{D}\left(\frac{\mathtt{b}_1\mathtt{b}_2}{\mathtt{k}\mathtt{b}_2 + (1-\mathtt{k})\mathtt{b}_1}\right) \le (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b}_1) + (1-s\mathtt{k})\mathcal{D}(\mathtt{b}_2). \tag{10}$$

(iv) Choosing $p = m = 1$ and $s = 1$ in Definition 6, we obtain Definition 1.
(v) Suppose $m = s = 1$ and $p = -1$ in Definition 6, we obtain Definition (2.1) in [28].

These are the amazing advantages of this newly investigated concept; if we choose the value of p, m, and s, then we attain new extended inequalities and also obtain some inequalities, which are associated with some previously published results.

Lemma 1. *The following inequalities $(1-(s(1-\mathtt{k}))) \ge \mathtt{k}$ and $m(1-s\mathtt{k}) \ge m(1-\mathtt{k})$ are held, if for all $m \in [0,1]$, $s \in [0,1]$, and $\mathtt{k} \in [0,1]$.*

Proof. The proof is obvious. □

Remark 4. *Assume that $m = 1$ in the above Lemma 1, then we attain the following inequalities $(1-(s(1-\mathtt{k}))) \ge \mathtt{k}$ and $(1-s\mathtt{k}) \ge (1-\mathtt{k})$.*

Proposition 1. *Every p-convex function on a p-convex set, i.e., $\mathtt{X} \subset (0,+\infty)$, is a modified p-convex function.*

Proof. Using the Definition 6, we have

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right)^{\frac{1}{p}}\right) \le \mathtt{k}\mathcal{D}(\mathtt{b}_1) + (1-\mathtt{k})\mathcal{D}(\mathtt{b}_2).$$

From the Remark 4, since $\mathtt{k} \le (1-(s(1-\mathtt{k})))$ and $(1-\mathtt{k}) \le m(1-s\mathtt{k})$ for all $m \in [0,1]$, $s \in [0,1]$, and $\mathtt{k} \in [0,1]$, we have

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right)^{\frac{1}{p}}\right) \le (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b}_1) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b}_2).$$

□

Proposition 2. *Every (m,p)-convex function on a p-convex set, i.e., $\mathtt{X} \subset (0,+\infty)$, is a modified p-convex function.*

Proof. Using the Definition 6, we have

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \leq \mathtt{k}\mathcal{D}(\mathtt{b}_1) + m(1-\mathtt{k})\mathcal{D}(\mathtt{b}_2).$$

From the Lemma 1, since $\mathtt{k} \leq (1-(s(1-\mathtt{k})))$ and $m(1-\mathtt{k}) \leq m(1-s\mathtt{k})$ for all $m \in [0,1]$, $s \in [0,1]$, and $\mathtt{k} \in [0,1]$, we have

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \leq (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b}_1) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b}_2).$$

□

Proposition 3. *Every modified p-convex function with the mentioned condition $h(\mathtt{k}) = (1-(s(1-\mathtt{k})))$ is an (h,m)-convex.*

Proof. Using the Definition 6, we have

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b_1}^p + (1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \leq (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b}_1) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b}_2).$$

Using the condition $h(\mathtt{k}) = (1-(s(1-\mathtt{k})))$, we have

$$\mathcal{D}\left(\left(\mathtt{k}\mathtt{b_1}^p + (1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \leq h(\mathtt{k})\mathcal{D}(\mathtt{b}_1) + mh(1-\mathtt{k})\mathcal{D}(\mathtt{b}_2).$$

□

Now, we present some examples regarding the newly introduced definition, i.e., modified p-convex function.

Example 1. *If $m = 1$, $p \in (-\infty,0) \cup [1,\infty)$, and $\mathcal{D}(x) = x^p$ is a (m,p)-convex function $\forall x > 0$ [29]; then, by employing Proposition 2, it is a modified p-convex function.*

Example 2. *Let $p \geq 1$, $m = 1$, $\mathcal{D} : (0,\infty) \to \mathtt{R}$, and $\mathcal{D}(x) = x^{-p}$; then, $\mathcal{D}$ is a (m,p)-convex function [29], so by employing Proposition 2, it is a modified p-convex function.*

Example 3. *Let $m = 1$, $p \geq 1$, $\mathcal{D} : (0,\infty) \to \mathtt{R}$, and $\mathcal{D}(x) = -\ln x$; then, $\mathcal{D}$ is a (m,p)-convex function [29], so by employing Proposition 2, it is a modified p-convex function.*

Now, we will discuss and prove some of its properties here.

Theorem 4. *Assume that $\mathcal{D}$ and H are two modified p-convex functions; then, $\mathcal{D}$ + H is also a modified p-convex function.*

Proof. Let $\mathcal{D}$ and H be a modified p-convex, $s \in [0,1]$, $\mathtt{k} \in [0,1]$, and $m \in [0,1]$; then,

$$\begin{aligned}
&(\mathcal{D}+\mathsf{H})\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \\
&= \mathcal{D}\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) + \mathsf{H}\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \\
&\leq (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b_1}) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b_2}) \\
&+ (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b_1}) + m(1-s\mathtt{k})\mathsf{H}(\mathtt{b_2}) \\
&= (1-(s(1-\mathtt{k})))(\mathcal{D}(\mathtt{b_1}) + \mathsf{H}(\mathtt{b_1})) + m(1-s\mathtt{k})(\mathcal{D}(\mathtt{b_2}) + \mathsf{H}(\mathtt{b_2})) \\
&= (1-(s(1-\mathtt{k})))(\mathcal{D}+\mathsf{H})(\mathtt{b_1}) + m(1-s\mathtt{k})(\mathcal{D}+\mathsf{H})(\mathtt{b_2}),
\end{aligned}$$

which completes the proof. □

Theorem 5. *If $\mathcal{D}$ is a modified p-convex function, then for non-negative real number c, $c\mathcal{D}$ is a modified p-convex function.*

Proof. Let $\mathcal{D}$ be a modified p-convex function, $s \in [0,1]$, $\mathtt{k} \in [0,1]$, and $m \in [0,1]$; then,

$$\begin{aligned}
&(c\mathcal{D})\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \\
&\leq c\Big((1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b_1}) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b_2})\Big) \\
&= (1-(s(1-\mathtt{k})))c\mathcal{D}(\mathtt{b_1}) + m(1-s\mathtt{k})c\mathcal{D}(\mathtt{b_2}) \\
&= \ (1-(s(1-\mathtt{k})))(c\mathcal{D})(\mathtt{b_1}) + m(1-s\mathtt{k})(c\mathcal{D})(\mathtt{b_2}),
\end{aligned}$$

which completes the proof. □

Theorem 6. *Let a function $\mathsf{H} : \mathtt{X} \to \mathbb{J}$ be p-convex, and increasing function $\mathcal{D} : \mathbb{J} \to \mathtt{R}$ is an s-type m-convex function. Then, $\mathcal{D} \circ \mathsf{H} : \mathtt{X} \to \mathtt{R}$ is a modified p-convex function.*

Proof. $\forall\ \mathtt{b_1}, \mathtt{b_2} \in \mathtt{X}$, $s \in [0,1]$, $\mathtt{k} \in [0,1]$, and $m \in [0,1]$, we have

$$\begin{aligned}
&(\mathcal{D}\circ\mathsf{H})\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right) \\
&= \mathcal{D}\left(\mathsf{H}\left(\left(\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right)^{\frac{1}{p}}\right)\right) \\
&\leq \mathcal{D}(\mathtt{k}\mathsf{H}(\mathtt{b_1}) + m(1-\mathtt{k})\mathsf{H}(\mathtt{b_2})) \\
&\leq (1-(s(1-\mathtt{k})))\mathcal{D}(\mathsf{H}(\mathtt{b_1})) + m(1-s\mathtt{k})\mathcal{D}(\mathsf{H}(\mathtt{b_2})) \\
&= \ (1-(s(1-\mathtt{k})))(\mathcal{D}\circ\mathsf{H})(\mathtt{b_1}) + m(1-s\mathtt{k})(\mathcal{D}\circ\mathsf{H})(\mathtt{b_2}).
\end{aligned}$$

This is the required proof. □

Theorem 7. *Let $\mathcal{D}_i : [\mathtt{b_1}, \mathtt{b_2}] \to \mathtt{R}$ be an arbitrary family of modified p-convex functions, and let $\mathcal{D}(\mathtt{b}) = \sup_i \mathcal{D}_i(\mathtt{b})$. If $O = \{\mathtt{b} \in [\mathtt{b_1}, \mathtt{b_2}] : \mathcal{D}(\mathtt{b}) < +\infty\} \neq \emptyset$, then O is an interval and $\mathcal{D}$ is a modified p-convex function on O.*

Proof. $\forall\, \mathtt{b}_1, \mathtt{b}_2 \in O$, $s \in [0,1]$, $\mathtt{k} \in [0,1]$, and $m \in [0,1]$, then we have

$$\begin{aligned}
&\mathcal{D}\left(\left(\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right)^{\frac{1}{p}}\right) \\
&= \sup_i \mathcal{D}_i\left(\left(\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right)^{\frac{1}{p}}\right) \\
&\leq (1-(s(1-\mathtt{k})))\sup_j \mathcal{D}_j(\mathtt{b}_1) + m(1-s\mathtt{k})\sup_j \mathcal{D}_j(\mathtt{b}_2) \\
&= \ (1-(s(1-\mathtt{k})))\mathcal{D}(\mathtt{b}_1) + m(1-s\mathtt{k})\mathcal{D}(\mathtt{b}_2) < +\infty,
\end{aligned}$$

This shows simultaneously that O is an interval since it contains every point between any two of its points, and $\mathcal{D}$ is a modified p-convex function on O. $\square$

4. New Generalization of $(\mathrm{H}-\mathrm{H})$ Type Inequality Using Modified p-Convex Function

Massive generalizations of mathematical inequalities for multiple functions have significantly influenced traditional research. Numerous fields, including linear programming, combinatorics, theory of relativity, optimization theory, quantum theory, number theory, dynamics, and orthogonal polynomials, are affected by and use integral inequalities. This issue has received much attention from researchers. The Hermite–Hadamard inequality is widely used and a popular inequality in the literature pertaining to convexity theory. The main focus of this part is to derive a new generalization of $(\mathrm{H}-\mathrm{H})$ type integral inequality via a modified p-convex function.

Theorem 8. *Let $\mathcal{D} : [\mathtt{b}_1, \mathtt{b}_2] \to \mathtt{R}$ be a modified p-convex function. If $\mathcal{D} \in L_1([\mathtt{b}_1, \mathtt{b}_2])$, then*

$$\begin{aligned}
&\frac{2}{2-s}\mathcal{D}\left[\frac{\mathtt{b}_1^p + m\mathtt{b}_2^p}{2}\right]^{\frac{1}{p}} \leq \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\left[\int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx + m^2\int_{\frac{\mathtt{b}_2}{m}}^{\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx\right] \\
&\leq \left[\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2) + m\left(\frac{\mathcal{D}(\mathtt{b}_1)}{m} + m\mathcal{D}(\mathtt{b}_2)\right)\right]\left(\frac{2-s}{2}\right).
\end{aligned} \tag{11}$$

Proof. Let

$$x = \left(\mathtt{k}\mathtt{b}_1^p + m(1-\mathtt{k})\mathtt{b}_2^p\right)^{\frac{1}{p}} \Rightarrow x^p = \mathtt{k}\mathtt{b}_1^p + m(1-\mathtt{k})\mathtt{b}_2^p,$$

and

$$y = \left((1-\mathtt{k})\frac{\mathtt{b}_1^p}{m} + \mathtt{k}\mathtt{b}_2^p\right)^{\frac{1}{p}} \Rightarrow y^p = (1-\mathtt{k})\frac{\mathtt{b}_1^p}{m} + \mathtt{k}\mathtt{b}_2^p.$$

Since $\mathcal{D}$ is modified p-convexity, we have

$$\mathcal{D}\left(\left[\mathtt{k}x^p + m(1-\mathtt{k})y^p\right]^{\frac{1}{p}}\right) \leq [1-(s(1-\mathtt{k}))]\mathcal{D}(x) + m[1-(s\mathtt{k})]\mathcal{D}(y), \tag{12}$$

which leads to

$$\mathcal{D}\left(\left[\frac{x^p + my^p}{2}\right]^{\frac{1}{p}}\right) \leq [1-(\frac{s}{2})]\mathcal{D}(x) + m[1-(\frac{s}{2})]\mathcal{D}(y)$$

$$\Rightarrow \mathcal{D}\left(\left[\frac{\mathtt{b}_1^p + m\mathtt{b}_2^p}{2}\right]^{\frac{1}{p}}\right) \leq [1-(\frac{s}{2})]\mathcal{D}(\mathtt{b}_1) + m[1-(\frac{s}{2})]\mathcal{D}(\mathtt{b}_2)$$

Using the change of variables, we obtain

$$\mathcal{D}\left(\left[\frac{\mathtt{b}_1^p+\mathtt{b}_2^p}{2}\right]^{\frac{1}{p}}\right)\leq\left[1-(\frac{s}{2})\right]\left\{\mathcal{D}\left(\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)+m\mathcal{D}\left(\left[(1-\mathtt{k})\frac{\mathtt{b}_1{}^p}{m}+\mathtt{k}\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)\right\}.$$

Integrate the above inequality with respect to $\mathtt{k}$ on $[0,1]$, and we attain

$$\frac{2}{2-s}\mathcal{D}\left[\frac{\mathtt{b}_1^p+m\mathtt{b}_2^p}{2}\right]^{\frac{1}{p}}\leq\frac{p}{m\mathtt{b}_2^p-\mathtt{b}_1^p}\left[\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx+m^2\int_{\frac{\mathtt{b}_1}{m}}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right].$$

Here, we prove the first half of the desired inequality.

For the next half, suppose $x=\left(\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)$, and using the Definition 6, we obtain

$$\begin{aligned}
&\frac{p}{m\mathtt{b}_2^p-\mathtt{b}_1^p}\left[\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx+m^2\int_{\frac{\mathtt{b}_2}{m}}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right]\\
&=\int_0^1\mathcal{D}\left(\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)d\mathtt{k}+m\int_0^1\mathcal{D}\left(\left[(1-\mathtt{k})\frac{\mathtt{b}_2{}^p}{m}+\mathtt{k}\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)d\mathtt{k}\\
&\leq\int_0^1\left[[1-(s(1-\mathtt{k}))]\mathcal{D}(\mathtt{b}_1)+m[1-(s\mathtt{k})]\mathcal{D}(\mathtt{b}_2)\right]d\mathtt{k}\\
&+\int_0^1 m\left[[1-s\mathtt{k}]\frac{\mathcal{D}(\mathtt{b}_1)}{m}+m[1-(s(1-\mathtt{k}))]\mathcal{D}(\mathtt{b}_2)\right]d\mathtt{k}\\
&=\mathcal{D}(\mathtt{b}_1)\int_0^1[1-(s(1-\mathtt{k}))]d\mathtt{k}+\mathcal{D}(\mathtt{b}_2)\int_0^1[1-s\mathtt{k}]d\mathtt{k}\\
&+\mathcal{D}(\mathtt{b}_1)\int_0^1[1-s\mathtt{k}]d\mathtt{k}+m^2\mathcal{D}(\mathtt{b}_2)\int_0^1[1-(s(1-\mathtt{k}))]d\mathtt{k}\\
&=\left[\mathcal{D}(\mathtt{b}_1)+m\mathcal{D}(\mathtt{b}_2)+m\left(\frac{\mathcal{D}(\mathtt{b}_1)}{m}+m\mathcal{D}(\mathtt{b}_2)\right)\right]\left(\frac{2-s}{2}\right).
\end{aligned}$$

This concludes the proof. □

Corollary 1. *Assume that $p=1$ in the above Theorem 8, then*

$$\begin{aligned}
\frac{2}{2-s}\mathcal{D}\left[\frac{\mathtt{b}_1+m\mathtt{b}_2}{2}\right]&\leq\frac{1}{m\mathtt{b}_2-\mathtt{b}_1}\left[\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\mathcal{D}(x)dx+m^2\int_{\frac{\mathtt{b}_2}{m}}^{\mathtt{b}_2}\mathcal{D}(x)dx\right]\\
&\leq\left[\mathcal{D}(\mathtt{b}_1)+m\mathcal{D}(\mathtt{b}_2)+m\left(\frac{\mathcal{D}(\mathtt{b}_1)}{m}+m\mathcal{D}(\mathtt{b}_2)\right)\right]\left(\frac{2-s}{2}\right). \qquad (13)
\end{aligned}$$

Corollary 2. *Assume that $s=1$ in Theorem 8, then*

$$\begin{aligned}
2\mathcal{D}\left[\frac{\mathtt{b}_1^p+m\mathtt{b}_2^p}{2}\right]^{\frac{1}{p}}&\leq\frac{p}{m\mathtt{b}_2^p-\mathtt{b}_1^p}\left[\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx+m^2\int_{\frac{\mathtt{b}_2}{m}}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right]\\
&\leq\frac{\left[\mathcal{D}(\mathtt{b}_1)+m\mathcal{D}(\mathtt{b}_2)+m\left(\frac{\mathcal{D}(\mathtt{b}_1)}{m}+m\mathcal{D}(\mathtt{b}_2)\right)\right]}{2}. \qquad (14)
\end{aligned}$$

Corollary 3. *Assume that $m = 1$ in Theorem 8, then*

$$\frac{2}{2-s}\mathcal{D}\left[\frac{\mathtt{b}_1^p + \mathtt{b}_2^p}{2}\right]^{\frac{1}{p}} \le \frac{2p}{\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx \le [\mathcal{D}(\mathtt{b}_1) + \mathcal{D}(\mathtt{b}_2)](2-s). \tag{15}$$

Corollary 4. *Assume that $s = p = 1$ in Theorem 8, then*

$$2\mathcal{D}\left[\frac{\mathtt{b}_1 + m\mathtt{b}_2}{2}\right] \le \frac{1}{m\mathtt{b}_2 - \mathtt{b}_1}\left[\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\mathcal{D}(x)dx + m^2\int_{\frac{\mathtt{b}_2}{m}}^{\mathtt{b}_2}\mathcal{D}(x)dx\right]$$
$$\le \frac{\left[\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2) + m\left(\frac{\mathcal{D}(\mathtt{b}_1)}{m} + m\mathcal{D}(\mathtt{b}_2)\right)\right]}{2}. \tag{16}$$

Corollary 5. *Assume that $m = s = 1$ in Theorem 8, then*

$$2\mathcal{D}\left[\frac{\mathtt{b}_1^p + \mathtt{b}_2^p}{2}\right]^{\frac{1}{p}} \le \frac{2p}{\mathtt{b}_2 - \mathtt{b}_1}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx \le \Big(\mathcal{D}(\mathtt{b}_1) + \mathcal{D}(\mathtt{b}_2)\Big). \tag{17}$$

Corollary 6. *Assume that $p = m = 1$ in Theorem 8, then*

$$\frac{2}{2-s}\mathcal{D}\left[\frac{\mathtt{b}_1 + \mathtt{b}_2}{2}\right] \le \frac{2}{\mathtt{b}_2 - \mathtt{b}_1}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\mathcal{D}(x)dx \le [\mathcal{D}(\mathtt{b}_1) + \mathcal{D}(\mathtt{b}_2)](2-s). \tag{18}$$

Remark 5. *Assume that $p = s = m = 1$ in Theorem 8, then we retrieve inequality (2).*

5. Refinements of $(\mathrm{H}-\mathrm{H})$ Type Inequality via Modified p-Convex Function

First, we prove a new lemma. On the basis of the new lemma, with the help of Holder and power mean inequality using newly introduced definition, we obtained some refinements of the $(\mathrm{H}-\mathrm{H})$ inequality. For the comprehensiveness of this section, some corollaries are presented.

Lemma 2. *Let $\mathcal{D} : \mathtt{X} \to \mathtt{R}$ be differentiable mapping on $\mathtt{X}^\circ$ with $\mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, and $\mathtt{b}_1 < \mathtt{b}_2$. If $\mathcal{D}' \in L_1[\mathtt{b}_1, \mathtt{b}_2]$, $\mathtt{k} \in [0,1]$, and $m \in [0,1]$, then*

$$\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx$$
$$= \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\int_0^1 \frac{1-2\mathtt{k}}{\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]\right)^{1-\frac{1}{p}}}\mathcal{D}'\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)d\mathtt{k}. \tag{19}$$

Proof.

$$\int_0^1 \frac{1-2\mathtt{k}}{\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]\right)^{1-\frac{1}{p}}}\mathcal{D}'\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)d\mathtt{k}$$
$$= \frac{p(\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2))}{m\mathtt{b}_2^p - \mathtt{b}_1^p} - \frac{2p^2}{(m\mathtt{b}_2^p - \mathtt{b}_1^p)^2}\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx,$$

multiplies both sides by $\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}$, then we obtain the required result. □

Theorem 9. *Let $\mathcal{D} : \mathtt{X} \to \mathtt{R}$ be a differentiable function on $\mathtt{X}^\circ$ with $\mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, and $\mathtt{b}_1 < \mathtt{b}_2$. If $\mathcal{D}' \in L_1[\mathtt{b}_1, \mathtt{b}_2]$ and $|\mathcal{D}'|^q$ are modified p-convex on $[\mathtt{b}_1, \mathtt{b}_2]$ for $q \geq 1$, $\mathtt{k} \in [0,1]$, $s \in [0,1]$, and $m \in [0,1]$, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p} \int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}} dx\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)(\mathtt{B}_1)^{1-\frac{1}{q}} \left[\mathtt{B}_2|\mathcal{D}'(\mathtt{b}_1)|^q + m\mathtt{B}_3|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}}, \tag{20}$$

where

$$\mathtt{B}_1 = \int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}} d\mathtt{k}, \quad \mathtt{B}_2 = \int_0^1 \frac{|1-2\mathtt{k}|[1-(s(1-\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}} d\mathtt{k},$$

$$\mathtt{B}_3 = \int_0^1 \frac{|1-2\mathtt{k}|[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}} d\mathtt{k}.$$

Proof. Employing Lemma 2 and property of modulus, we have

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p} \int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}} dx\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right) \int_0^1 \left|\frac{1-2\mathtt{k}}{\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]\right)^{1-\frac{1}{p}}}\right| \left|\mathcal{D}'\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)\right| d\mathtt{k}$$

Using power mean inequality, we have

$$\begin{aligned}\left|\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p} \int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}} dx\right| &\leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}} d\mathtt{k}\right)^{1-\frac{1}{q}} \\ &\times \left(\int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}} \left|\mathcal{D}'\left(\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)\right|^q d\mathtt{k}\right)^{\frac{1}{q}}\end{aligned}$$

Using modified p-convexity of $|\mathcal{D}'|^q$, we have

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p} \int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}} dx\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}} d\mathtt{k}\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}\left\{[1-(s(1-\mathtt{k}))]|\mathcal{D}(\mathtt{b}_1)|^q + m[1-(s\mathtt{k})]|\mathcal{D}(\mathtt{b}_2)|^q\right\}d\mathtt{k}\right]^{\frac{1}{q}}$$
$$= \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1 \frac{|1-2\mathtt{k}|[1-(s(1-\mathtt{k}))]}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}|\mathcal{D}'(\mathtt{b}_1)|^q d\mathtt{k} + \int_0^1 \frac{m|1-2\mathtt{k}|[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}|\mathcal{D}'(\mathtt{b}_2)|^q d\mathtt{k}\right]^{\frac{1}{q}}$$
$$= \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)(\mathtt{B}_1)^{1-\frac{1}{q}}\left[\mathtt{B}_2|\mathcal{D}'(\mathtt{b}_1)|^q + m\mathtt{B}_3|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}}.$$

This is the required proof. □

Corollary 7. *Choosing $m = 1$ in the above Theorem 9, we have*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + \mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)(\mathtt{B}_4)^{1-\frac{1}{q}}\left[\mathtt{B}_5|\mathcal{D}'(\mathtt{b}_1)|^q + \mathtt{B}_6|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where

$$\mathtt{B}_4 = \int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}, \quad \mathtt{B}_5 = \int_0^1 \frac{|1-2\mathtt{k}|[1-(s(1-\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},$$
$$\mathtt{B}_6 = \int_0^1 \frac{|1-2\mathtt{k}|[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}.$$

Corollary 8. *If we put $s = 1$ in Theorem 9, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)(\mathtt{B}_7)^{1-\frac{1}{q}}\left[\mathtt{B}_8|\mathcal{D}'(\mathtt{b}_1)|^q + m\mathtt{B}_9|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where

$$\mathtt{B}_7 = \int_0^1 \frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}, \quad \mathtt{B}_8 = \int_0^1 \frac{|1-2\mathtt{k}|\mathtt{k}}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},$$
$$\mathtt{B}_9 = \int_0^1 \frac{|1-2\mathtt{k}|[1-\mathtt{k}]}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}.$$

Corollary 9. *If we put $s = m = 1$ in Theorem 9, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1)+\mathcal{D}(\mathtt{b}_2)}{2}-\frac{p}{\mathtt{b}_2^p-\mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq\left(\frac{\mathtt{b}_2^p-\mathtt{b}_1^p}{2p}\right)(\mathtt{B}_{10})^{1-\frac{1}{q}}\left[\mathtt{B}_{11}|\mathcal{D}'(\mathtt{b}_1)|^q+\mathtt{B}_{12}|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where

$$\mathtt{B}_{10}=\int_0^1\frac{|1-2\mathtt{k}|}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},\quad \mathtt{B}_{11}=\int_0^1\frac{|1-2\mathtt{k}|\mathtt{k}}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},$$

$$\mathtt{B}_{12}=\int_0^1\frac{|1-2\mathtt{k}|[1-\mathtt{k}]}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}.$$

Theorem 10. *Let $\mathcal{D}:\mathtt{X}\to\mathtt{R}$ be a differentiable function on $\mathtt{X}^\circ$ with $\mathtt{b}_1,\mathtt{b}_2\in\mathtt{X}$, and $\mathtt{b}_1<\mathtt{b}_2$. If $\mathcal{D}'\in L_1[\mathtt{b}_1,\mathtt{b}_2]$ and $|\mathcal{D}'|^q$ are modified p-convex on $[\mathtt{b}_1,\mathtt{b}_2]$ for $q>1$, $\frac{1}{l}+\frac{1}{q}=1$, $\mathtt{k}\in[0,1]$, $s\in[0,1]$, and $m\in[0,1]$, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1)+m\mathcal{D}(\mathtt{b}_2)}{2}-\frac{p}{m\mathtt{b}_2^p-\mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq\left(\frac{m\mathtt{b}_2^p-\mathtt{b}_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\left[\mathtt{C}_1|\mathcal{D}'(\mathtt{b}_1)|^q+m\mathtt{C}_2|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},\tag{21}$$

where

$$\mathtt{C}_1=\int_0^1\frac{[1-(s(1-\mathtt{k}))]}{\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}}d\mathtt{k}$$

and

$$\mathtt{C}_2=\int_0^1\frac{[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}}d\mathtt{k}.$$

Proof. Employing Lemma 2, Hölder's inequality and modified p-convexity of $|\mathcal{D}'|^q$, we have

$$\left|\frac{\mathcal{D}(\mathtt{b}_1)+m\mathcal{D}(\mathtt{b}_2)}{2}-\frac{p}{m\mathtt{b}_2^p-\mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq\left(\frac{m\mathtt{b}_2^p-\mathtt{b}_1^p}{2p}\right)\left(\int_0^1|1-2\mathtt{k}|^l d\mathtt{k}\right)^{\frac{1}{l}}$$
$$\times\left(\int_0^1\frac{1}{\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q(1-\frac{1}{p})}}\left|\mathcal{D}'\left(\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{\frac{1}{p}}\right)\right|^q d\mathtt{k}\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}$$
$$\times \left(\int_0^1 \frac{[1-(s(1-\mathtt{k}))]|\mathcal{D}(\mathtt{b}_1)|^q + m[1-(s\mathtt{k})]|\mathcal{D}(\mathtt{b}_2)|^q}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}\right)^{\frac{1}{q}}$$
$$= \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}} \times \left[\mathtt{C}_1|\mathcal{D}'(\mathtt{b}_1)|^q + m\mathtt{C}_2|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}}.$$

This is the required proof. □

Corollary 10. *Choosing $m = 1$ in the above Theorem 10, we have*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + \mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\left[\mathtt{C}_3|\mathcal{D}'(\mathtt{b}_1)|^q + \mathtt{C}_4|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where

$$\mathtt{C}_3 = \int_0^1 \frac{[1-(s(1-\mathtt{k}))]}{\left[\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}$$

and

$$\mathtt{C}_4 = \int_0^1 \frac{[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p + (1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}.$$

Corollary 11. *If we put $s = 1$ in Theorem 10, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\left[\mathtt{C}_5|\mathcal{D}'(\mathtt{b}_1)|^q + m\mathtt{C}_6|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where

$$\mathtt{C}_5 = \int_0^1 \frac{\mathtt{k}}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}$$

and

$$\mathtt{C}_6 = \int_0^1 \frac{1-\mathtt{k}}{\left[\mathtt{k}\mathtt{b}_1{}^p + m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}.$$

Corollary 12. *Choosing $s = m = 1$ in the above Theorem 10, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1) + \mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\left[\mathtt{C}_7|\mathcal{D}'(\mathtt{b}_1)|^q + \mathtt{C}_8|\mathcal{D}'(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where

$$\mathtt{C}_7 = \int_0^1 \frac{\mathtt{k}}{\left[\mathtt{k}\mathtt{b_1}^p + (1-\mathtt{k})\mathtt{b_2}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}$$

and

$$\mathtt{C}_8 = \int_0^1 \frac{1-\mathtt{k}}{\left[\mathtt{k}\mathtt{b_1}^p + (1-\mathtt{k})\mathtt{b_2}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\mathtt{k}.$$

Theorem 11. *Let $\mathcal{D} : \mathtt{X} \to \mathtt{R}$ be differentiable mapping on $\mathtt{X}^\circ$ with $\mathtt{b}_1, \mathtt{b}_2 \in \mathtt{X}$, and $\mathtt{b}_1 < \mathtt{b}_2$. If $\mathcal{D}' \in L_1[\mathtt{b}_1, \mathtt{b}_2]$ and $|\mathcal{D}'|$ are modified p-convex on $[\mathtt{b}_1, \mathtt{b}_2]$, $\mathtt{k} \in [0,1]$, $s \in [0,1]$, and $m \in [0,1]$, then*

$$\begin{aligned} &\left| \frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p} \int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}} dx \right| \\ &\le \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p} \right) \times \left(\mathtt{E}_1 |\mathcal{D}'(\mathtt{b}_1)|^q + m\mathtt{E}_2 |\mathcal{D}'(\mathtt{b}_2)|^q \right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \mathtt{E}_1 &= \int_0^1 \frac{|1-2\mathtt{k}|[1-(s(1-\mathtt{k}))]}{\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right]^{1-\frac{1}{p}}} d\mathtt{k}, \\ \mathtt{E}_2 &= \int_0^1 \frac{|1-2\mathtt{k}|[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p\right]^{1-\frac{1}{p}}} d\mathtt{k}. \end{aligned}$$

Proof. Employing Lemma 2 and modified p-convexity of $|\mathcal{D}'|^q$, we have

$$\begin{aligned} &\left| \frac{\mathcal{D}(\mathtt{b}_1) + m\mathcal{D}(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p} \int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{D}(x)}{x^{1-p}} dx \right| \\ &\le \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p} \right) \int_0^1 \left| \frac{1-2\mathtt{k}}{\left(\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p \right] \right)^{1-\frac{1}{p}}} \right| \left| \mathcal{D}' \left(\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p \right]^{\frac{1}{p}} \right) \right| d\mathtt{k} \\ &\le \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p} \right) \int_0^1 \left| \frac{1-2\mathtt{k}}{\left(\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p \right] \right)^{1-\frac{1}{p}}} \right| \\ &\times \left[[1-(s(1-\mathtt{k}))] |\mathcal{D}'(\mathtt{b}_1)| + m[1-(s\kappa)] |\mathcal{D}'(\mathtt{b}_2)| \right] d\mathtt{k} \\ &= \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p} \right) \left[|\mathcal{D}'(\mathtt{b}_1)| \int_0^1 \frac{|1-2\mathtt{k}|[1-(s(1-\mathtt{k}))]}{\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p \right]^{1-\frac{1}{p}}} d\mathtt{k} \right. \\ &\left. + m|\mathcal{D}'(\mathtt{b}_2)| \int_0^1 \frac{|1-2\mathtt{k}|[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b_1}^p + m(1-\mathtt{k})\mathtt{b_2}^p \right]^{1-\frac{1}{p}}} d\mathtt{k} \right] \\ &= \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p} \right) \left(\mathtt{E}_1 |\mathcal{D}'(\mathtt{b}_1)| + m\mathtt{E}_2 |\mathcal{D}'(\mathtt{b}_2)| \right). \end{aligned}$$

This is the required proof. □

Corollary 13. *Suppose $m = 1$ in Theorem 11, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1)+\mathcal{D}(\mathtt{b}_2)}{2}-\frac{p}{\mathtt{b}_2^p-\mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right| \leq \left(\frac{\mathtt{b}_2^p-\mathtt{b}_1^p}{2p}\right)\left(\mathtt{E}_3|\mathcal{D}'(\mathtt{b}_1)|^q+\mathtt{E}_4|\mathcal{D}'(\mathtt{b}_2)|^q\right),$$

where

$$\mathtt{E}_3 = \int_0^1 \frac{|1-2\mathtt{k}|[1-(s(1-\mathtt{k}))]}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},$$
$$\mathtt{E}_4 = \int_0^1 \frac{|1-2\mathtt{k}|[1-(s\mathtt{k})]}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}.$$

Corollary 14. *If we put $s = 1$ in Theorem 11, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1)+m\mathcal{D}(\mathtt{b}_2)}{2}-\frac{p}{m\mathtt{b}_2^p-\mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{m\mathtt{b}_2^p-\mathtt{b}_1^p}{2p}\right)\left(\mathtt{E}_5|\mathcal{D}'(\mathtt{b}_1)|^q+m\mathtt{E}_6|\mathcal{D}'(\mathtt{b}_2)|^q\right),$$

where

$$\mathtt{E}_5 = \int_0^1 \frac{|1-2\mathtt{k}|\mathtt{k}}{\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},$$
$$\mathtt{E}_6 = \int_0^1 \frac{|1-2\mathtt{k}|[1-\mathtt{k}]}{\left[\mathtt{k}\mathtt{b}_1{}^p+m(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}.$$

Corollary 15. *Suppose $s = m = 1$ in Theorem 11, then*

$$\left|\frac{\mathcal{D}(\mathtt{b}_1)+\mathcal{D}(\mathtt{b}_2)}{2}-\frac{p}{\mathtt{b}_2^p-\mathtt{b}_1^p}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\frac{\mathcal{D}(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{\mathtt{b}_2^p-\mathtt{b}_1^p}{2p}\right)\left(\mathtt{E}_7|\mathcal{D}'(\mathtt{b}_1)|^q+\mathtt{E}_8|\mathcal{D}'(\mathtt{b}_2)|^q\right),$$

where

$$\mathtt{E}_7 = \int_0^1 \frac{|1-2\mathtt{k}|\mathtt{k}}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k},$$
$$\mathtt{E}_8 = \int_0^1 \frac{|1-2\mathtt{k}|[1-\mathtt{k}]}{\left[\mathtt{k}\mathtt{b}_1{}^p+(1-\mathtt{k})\mathtt{b}_2{}^p\right]^{1-\frac{1}{p}}}d\mathtt{k}.$$

6. Application for Some Special Functions

This section involves some applications to the estimations of some special functions, namely, modified Bessel functions. These functions can be found in transmission line studies, non-uniform beams, and the statistical treatment of relativistic gas in statistical mechanics. In order to find the applications of these special functions regarding the newly introduced idea, first, we remember the following remark, which is proved and discussed by İşcan (see [29], p. 142).

Remark 6. *Let $I \subset (0,\infty)$ be a real interval, $p \in \mathbb{R}\setminus\{0\}$ and $\mathcal{D} : I \to \mathbb{R}$ be a function, then*
(i) If $p \leq 1$ and $\mathcal{D}$ is a convex and non-decreasing function, then $\mathcal{D}$ is p-convex.
(ii) If $p \geq 1$ and $\mathcal{D}$ is a convex and non-increasing function, then $\mathcal{D}$ is p-convex.

Modified Bessel functions:

First of all, one thing to have in mind, throughout such types of applications, MBF represents a modified Bessel function.

Recall that the series representation of the first kind of MBF is represented by $I_\rho(\mathtt{b})$ (see [30], p. 77) and is given by

$$I_\rho(\mathtt{b}) = \sum_{n\geq 0} \frac{(\frac{\mathtt{b}}{2})^{\rho+2n}}{n!\Gamma(\rho+n+1)}, \quad \forall \mathtt{b} \in \mathbb{R}, \tag{23}$$

while the second kind of MBF is represented by $K_\rho(\mathtt{b})$ (see [30], p.78) and is given by

$$K_\rho(\mathtt{b}) = \frac{\pi}{2}\frac{I_{-\rho}(\mathtt{b}) + I_\rho(\mathtt{b})}{\sin \rho\pi}. \tag{24}$$

For this, we assume that $\mathcal{I}_\rho : R \to [1, \infty)$, which is defined by

$$\mathcal{I}_\rho(\mathtt{b}) = 2^\rho \Gamma(\rho+1)\mathtt{b}^{-\rho} I_\rho(\mathtt{b}). \tag{25}$$

Proposition 4. *For $\rho > -1$ and $0 < \mathtt{b}_1 < \mathtt{b}_2$, then*

$$\left|\frac{\mathcal{I}_\rho(\mathtt{b}_1) + m\mathcal{I}_\rho(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{I}_\rho(\mathtt{b})}{\mathtt{b}^{1-p}} d\mathtt{b}\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{4p(\rho+1)}\right)(\mathtt{B}_1)^{1-\frac{1}{q}} \tag{26}$$
$$\times\left[\mathtt{b}_1^q \mathtt{B}_2 |\mathcal{I}_{\rho+1}(\mathtt{b}_1)|^q + m\mathtt{b}_2^q \mathtt{B}_3 |\mathcal{I}_{\rho+1}(\mathtt{b}_2)|^q\right]^{\frac{1}{q}}.$$

In particular, ($\rho = -\frac{1}{2}$), then we obtain the following inequality

$$\left|\frac{\cosh(\mathtt{b}_1) + m\cosh(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\cosh(\mathtt{b})}{\mathtt{b}^{1-p}} d\mathtt{b}\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)(\mathtt{B}_1)^{1-\frac{1}{q}} \tag{27}$$
$$\times\left[\mathtt{B}_2 |\sinh(\mathtt{b}_1)|^q + m\mathtt{B}_3 |\sinh(\mathtt{b}_2)|^q\right]^{\frac{1}{q}},$$

where $\mathtt{B}_1$, $\mathtt{B}_2$, and $\mathtt{B}_3$ are defined in Theorem 9.

Proof. Applying inequality (20) to the mapping $\mathcal{D}(\mathtt{b}) = \mathcal{I}_\rho(\mathtt{b}), \quad \mathtt{b} > 0$ and $\mathcal{I}_\rho'(\mathtt{b}) = \frac{\mathtt{b}}{2(p+1)}\mathcal{I}_{\rho+1}(\mathtt{b})$, but Agarwal proved in (see [31]) that $\mathcal{I}_\rho'(\mathtt{b})$ is convex on $[0, \infty)$ since the power series only has positive coefficients. It is obvious that if we fix the value of $\rho > -1$ throughout the interval $\mathtt{b} \in (0, \infty)$, then $\mathcal{I}_\rho'(\mathtt{b})$ is positive and non-decreasing. So, this implies that $\mathcal{I}_\rho'(\mathtt{b})$ is convex and non-decreasing. Further, this implies that $|\mathcal{I}_\rho'(\mathtt{b})|^q$ is convex and non-decreasing. If $p \leq 1$, then by using Remark 6(i), it is a p-convex. Finally, according to Proposition 1, it is a modified p-convex function. So, we deduce the inequality (26). Now, we have used the fact that $I_{-\frac{1}{2}}(\mathtt{b}) = \cosh(\mathtt{b})$ and $I_{\frac{1}{2}}(\mathtt{b}) = \frac{\sinh(\mathtt{b})}{\mathtt{b}}$; then, the inequality (26) reduces to the inequality (27). □

Proposition 5. *For $\rho > -1$ and $0 < \mathtt{b}_1 < \mathtt{b}_2$, then*

$$\left|\frac{\mathcal{I}_\rho(\mathtt{b}_1) + m\mathcal{I}_\rho(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2} \frac{\mathcal{I}_\rho(\mathtt{b})}{\mathtt{b}^{1-p}} d\mathtt{b}\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{4p(\rho+1)}\right)\left(\frac{1}{l+1}\right)^{\frac{1}{l}} \tag{28}$$
$$\times\left[\mathtt{b}_1^q \mathtt{C}_1 |\mathcal{I}_{\rho+1}(\mathtt{b}_1)|^q + m\mathtt{b}_2^q \mathtt{C}_2 |\mathcal{I}_{\rho+1}(\mathtt{b}_2)|^q\right]^{\frac{1}{q}}.$$

In particular, choosing $\rho = -\frac{1}{2})$, then we obtain the following inequality,

$$\left|\frac{\cosh(\mathtt{b}_1) + m\cosh(\mathtt{b}_2)}{2} - \frac{p}{m\mathtt{b}_2^p - \mathtt{b}_1^p}\int_{\mathtt{b}_1}^{m\mathtt{b}_2}\frac{\cosh(\mathtt{b})}{\mathtt{b}^{1-p}}d\mathtt{b}\right| \leq \left(\frac{m\mathtt{b}_2^p - \mathtt{b}_1^p}{2p}\right)\left(\frac{1}{l+1}\right)^{\frac{1}{l}} \tag{29}$$

$$\times\left[\mathtt{C}_1\left|\sinh(\mathtt{b}_1)\right|^q + m\mathtt{C}_2\left|\sinh(\mathtt{b}_2)\right|^q\right]^{\frac{1}{q}}$$

is true, where $\mathtt{C}_1$ and $\mathtt{C}_2$ are defined in Theorem 10.

Proof. Applying inequality (21) to the mapping $\mathcal{D}(\mathtt{b}) = \mathcal{I}_\rho(\mathtt{b}), \quad \mathtt{b} > 0$, and $\mathcal{I}_\rho'(\mathtt{b}) = \frac{\mathtt{b}}{2(\rho+1)}\mathcal{I}_{\rho+1}(\mathtt{b})$, we deduce the inequality (28). Now, we have used the fact that $I_{-\frac{1}{2}}(\mathtt{b}) = \cosh(\mathtt{b})$ and $I_{\frac{1}{2}}(\mathtt{b}) = \frac{\sinh(\mathtt{b})}{\mathtt{b}}$, then the inequality (28) reduces to the inequality (29). □

7. Conclusions

Convexity is important and crucial in many branches of pure and applied sciences. For a novel class of convexity known as the modified p-convex function, we proposed new assessments of the $(\mathrm{H}-\mathrm{H})$ type inequality. We also reviewed and investigated some of its algebraic properties. We demonstrated that our novel class of modified p-convex functions are far larger than known function classes such as convex and harmonically convex. We have enhanced the Hermite–Hadamard inequality for functions whose first derivative in absolute form at a given power is a modified p-convex. Our recent findings are expected to have applications in convex theory, quantum calculus, special functions, and post-quantum calculus. They may also serve as catalysts for further research in a variety of unrelated pure and applied fields.

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